Math 237 Notes

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These are my 2nd year Calculus 3 notes at the University of Waterloo (MATH 237). They are pretty similar to the content you may see in the course notes by J. Wainwright, J. West, D. Wolczuk.

You will find that these aren't very useful as notes, in the sense that they are not significantly shorter than the content in the course notes, they're really just a way for me to type down the content I am learning and absorb it.

Thanks to Prof. Dan Wolczuk for providing me with the macros to typeset this LaTeX document.

Week 1

Unit 1: Graphs of Scalar Functions

Scalar Functions

A review of basic vocabulary about functions in general

• A function $f : A \to B$ associates with each element $a \in A$ a unique element in $f(a) \in B$ called the **image** of a under f

• The set A is called the **domain** of f and is denoted by $D(f)$

• The set B is called the **codomain** of f

• The subset of B consisting of all $f(a)$ is called the **range** of f and is denoted by $R(f)$

We will usually look at functions whose domain is a subset of $\mathbb{R}^2$ and whose codomain is $\mathbb{R}$. I.e. we consider functions f which map points $(x, y) \in \mathbb{R}^2$ to a real scalar $f(x, y) \in \mathbb{R}$. We write $z = f(x, y)$. We will also consider more general functions $f(x_1, ..., x_n)$ which map subsets of $\mathbb{R}^n$ to $\mathbb{R}$

A **scalar function** $f(x_1, ..., x_n)$ of n variables is a functions whose domain is a subset of $\mathbb{R}^n$ and whose range is a subset of $\mathbb{R}$ — Definition: Scalar Function

May also be denoted by $f(\mathbf{x})$ or $f(\vec{x})$

Geometric Interpretation of $\mathbf{z = f(x, y)}$

When we graph a function $y = f(x)$, we plot points $(a, f(a))$ in the xy-plane. Observe that we can think of $f(a)$ as representing the height of the graph $y = f(x)$ above (or below if negative) the x-axis at $x = a$

We define the **graph** of a function $f(x, y)$ as the set of all points

$$\{(a, b, f(a, b)) \in \mathbb{R}^3 : (a, b) \in D(f)\}$$

We think of $f(a, b)$ as representing the height of the graph $z = f(x, y)$ above (or below if negative) the xy-plane at the point $(x, y) = (a, b)$

◦ When f is defined as $f(x, y) = c_1x + c_2y + c_3$, where $c_1, c_2, c_3 \in \mathbb{R}$, the graph of $z = f(x, y)$ is a **plane**. (Note: it is of the form $z - c_1x + c_2y = c_3$)

Level curves are 2-dimensional slices of a surface, sort of like a top-down view of what the curve looks like for a fixed z value.

The level curves of a function $f(x, y)$ are the curves — Definition: Level Curves

$$f(x, y) = k, \ k \in \mathbb{R}$$

The level curve of $f(x,y)=k$ is the intersection of $z=f(x,y)$ and the horizontal plane $z=k$. In our family of curves, each value of k represents a height above the xy-plane. Thus, the family of level curves is often called a **contour map** or a **topographic map**
(A little general info: Weather maps which show regions of constant temperatures are called **isotherms**, in barometric pressure charts curves of constant pressure are called **isobars**. Another example would be an MRI scan)

◦ A level curve that behaves unusually compared to other members of the family is called an **exceptional level curve**

A **cross section** of a surface $z=f(x,y)$ is the intersection of $z=f(x,y)$ with a plane

Definition
Cross Sections

For sketching purposes, it is useful to consider cross sections formed by intersection $z=f(x,y)$ with the *vertical* planes $x=c$ and $y=d$, where c,d can take on multiple values as k did in level curves
◦ $f(x,y)=x^2+y^2$ gives a **paraboloid** surface (level curves are circles; cross sections are parabolas)
◦ $f(x,y)=x^2-y^2$ gives a **saddle surface** (level curves are hyperbolae, about x-axis above/below a certain k value, about y-axis below/above a cetain k value; cross sections are parabolas `(if im not wrong)`)
◦ $f(x,y)=x^2$ gives a **parabolic cylinder** (level curves are straight lines, $x=\pm\sqrt{k}$; cross section is a parabola. Since it has the same cross section for all planes $y=d$, it is called a cylinder by definition)

A **level surface** of a scalar function $f(x,y,z)$ is defined by

$$f(x,y,z)=k,\ \ k\in R(f)$$

Definition
Level Surfaces

A level surface, is the analogy case of a surface having a level curve, but one-dimension up. So, we have a 4-dimensional shape (instead of a surface), which is made up of (layers of) surfaces, level surfaces. Each level surface can be found by parameterzing $f(x,y,z)$

A **level set** of a scalar function $f(\vec{x}),\vec{x}\in\mathbb{R}^n$ is defined by

$$\{\vec{x}\in\mathbb{R}^n|f(\vec{x})=k,\ \ for\ k\in R(f)\}$$

Definition
Level Sets

So a level set is a generalisation of what we learned for level curves (case of $f:\mathbb{R}^2\to\mathbb{R}$) and level surfaces (case of $f:\mathbb{R}^3\to\mathbb{R}$) for $f:\mathbb{R}^n\to\mathbb{R}$
◦ For an f defined by: $f(x_1,...,x_n)=x_1^2+...+x_n^2$, the level sets for $f(\vec{x})=k,k>0\in\mathbb{R}^n$ are called **$(\mathbf{n-1})$-spheres**, denoted by S^{n-1} (e.g. for $n=3$ we get a 2-sphere denoted by S^2)

Week 2

Unit 2: Limits

Definition of a Limit for One Variable

For a real-valued function $f(x)$ we defined $\lim_{x \to a} f(x) = L$ to mean that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently close to a.
More precisely,

For every $\epsilon > 0$ there exists a $\delta > 0$:

$$|f(x) - L| < \epsilon \textit{ whenever } 0 < |x - a| < \delta \ (*)$$

and $\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

Definition
Single Variable definition of a Limit

This means no matter what $\epsilon > 0$ value we choose, we can always find a corresponding $\delta > 0$ value that would satisfy the condition $(*)$

Definition of a Limit for Functions of Two Variables

We define the limit for functions of two variables in a very similar way to the limit of functions of a single variable. For a scalar function $f(x, y)$, we want $\lim_{(x,y) \to (a,b)} f(x, y) = L$, to mean that the values of $f(x, y)$ can be made arbitrarily close to L by taking (x, y) sufficiently close to (a, b)

For a single variable we could approach the limit from either the left or the right.
For multivariable scalar functions our domain is multidimensional and so we can approach it from infinitely many directions, moreover, we aren't even restricted to straight lines either; we can approach (a, b) along any smooth curve.

An **open interval** is defined as

$$(-r, r) = \{x : |x| < r\}$$

where $r \in \mathbb{R}$

Definition
Open Interval

Euclidian distance in $\mathbb{R}^2$ is defined as

$$||(x, y) - (a, b)|| = \sqrt{(x - a)^2 + (y - b)^2}$$

Definition
Euclidian Distance

An **r-neighbourhood** of a point $(a,b) \in \mathbb{R}^2$ is a set

Definition
Neighbourhood

$$N_r(a,b) = \{(x,y) \in \mathbb{R}^2 |\ ||(x,y)-(a,b)|| < r,\ r \in \mathbb{R}\}$$

You may notice that the r-neighbourhood of (a,b) is simply a locus of distance r or less from the point (a,b)

Assume $f(x,y)$ is defined in a neighbourhood of (a,b), except possibly at (a,b). If, for every $\epsilon > 0$ there exists a $\delta > 0$ such that

Definition
Limit

$$0 < ||(x,y)-(a,b)|| < \delta \implies |f(x,y) - L| < \epsilon$$

Then

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

Although we said that the we can approach the limits from infinitely many directions, note that the limit definition does not refer to any direction at all, and refers only to the distance between (x,y) and (a,b)

Limit Theorems

If $\lim_{(x,y)\to(a,b)} f(x,y)$ and $\lim_{(x,y)\to(a,b)} g(x,y)$ both exist, then

Limit
Theorem 1

1. $$\lim_{(x,y)\to(a,b)} [f(x,y)+g(x,y)] = \lim_{(x,y)\to(a,b)} f(x,y) + \lim_{(x,y)\to(a,b)} g(x,y)$$

2. $$\lim_{(x,y)\to(a,b)} [f(x,y)g(x,y)] = \left[\lim_{(x,y)\to(a,b)} f(x,y)\right]\left[\lim_{(x,y)\to(a,b)} g(x,y)\right]$$

3. $$\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{\lim_{(x,y)\to(a,b)} f(x,y)}{\lim_{(x,y)\to(a,b)} g(x,y)},\ \textit{provided}\ \lim_{(x,y)\to(a,b)} g(x,y) \neq 0$$

If $\lim_{(x,y)\to(a,b)} f(x,y)$ exists, then the limit is unique

Limit
Theorem 2

Proving a Limit Does Not Exist

For a single variable function, we often showed a limit did not exist by showing the left-hand and right-hand limit did not equal each other, and used the fact that the limit is supposed to be unique. For multivariable functions, we will essentially do the same thing, only now we have to remember that we can approach (a, b) along any smooth curve.

One can approach a question like this by taking the equation $y = mx$ or $x = my$ (for any real coefficient m) and if the limit turns out to be dependent on m, then we know that the limit is not unique.
Though, this approach does not always work, as $y = mx$ does not describe all the lines, (it cannot represent vertical lines).

Sometimes trying out several straight lines will give the same limit, but using a continuous curve will show that the limit in-fact does not exist. The trick to use here would be to choose a curve in such a way that (if the function is a fraction) the numerator and denominator cancel out. (e.g of forms $y = mx^k$ or $y = mx^{p/q}$ etc)
Caution: Be sure to use lines or curves that actually approach the limit point in question.

Note: Finding two paths that show that a limit does not exist does indeed mean that it doesn't exist. But being unable to find a contradictory value for a limit does not necessarily mean that a limit exists. We then use other methods such as the Squeeze theorem to test if this consistently occuring value of L is the actual limit or not.

Proving a Limit Exists

If there exists a function $B(x, y)$ such that Squeeze Theorem

$$|f(x, y) - L| \leq B(x, y), \ \textit{for all } (x, y) \neq (a, b)$$

in some neighbourhood of (a, b) and $\lim_{(x,y)\to(a,b)} B(x, y) = 0$ then

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

Proof:
(Our hypothesis says that $B(x, y) \geq 0$ for all $(x, y) \neq (a, b)$)
Let $\epsilon > 0$

Since $\lim_{(x,y)\to(a,b)} B(x,y) = 0$, by definition of limit, there exists $\delta > 0$ such that

$$0 < ||(x,y) - (a,b)|| < \delta \Longrightarrow |B(x,y) - 0| < \epsilon$$

Hence, if $0 < ||(x,y) - (a,b)|| < \delta$, then we have

$$|f(x,y) - L| \leq B(x,y) = |B(x,y)| < \epsilon$$

as our hypothesis requires that $B(x,y) \geq 0$ for all $(x,y) \neq (a,b)$ i the neighbourhood of (a,b). Therefore, by definition of a limit, we have

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

Generalizations

The concept of neighbourhood, the definition of a limit, the Squeeze Theorm and limit theorems are all valid for scalar functions $f(\mathbf{x}), \mathbf{x} \in \mathbb{R}^n$. In fact, to generalise these concepts, one only needs to know that $\mathbf{x} = (x_1, ..., x_n)$ and $\mathbf{a} = (a_1, ..., a_n)$ are in $\mathbb{R}^n$, then the Euclidean distance from $\mathbf{x}$ and $\mathbf{a}$ is

$$||\mathbf{x} - \mathbf{a}|| = \sqrt{(x_1 - a_1)^2 + ... + (x_n - a_n)^2}$$

With this adjustment and some more rephrasing, the previous section can be generalised for $\mathbb{R}^n$

Appendix: Inequalities

Trichotomy Property: For any real numbers a and b, one and only one of the follow holds

$$a = b, \;\; a < b, \;\; b < a$$

Transitivity Property: If $a < b$ and $b < c$, then $a < c$
Addition Property: If $a < b$ then for all c, $a + c < b + c$
Multiplication Property: If $a < b$ and $c < 0$, then $bc < ac$
Absolute value of a real number a is defined by

$$|a| = \begin{cases} a & if\, a \geq 0 \\ -a & if\, a < 0 \end{cases}$$

A few useful results

1. $|a| = \sqrt{a^2}$
2. $|a| < b \iff -b < a < b$
3. The Triangle Inequality: $|a + b| \leq |a| + |b|$, $\forall a, b\mathbb{R}$

When using the Squeeze Theorem, the most commonly used inequalities are:

1. Triangle Inequality
2. If $c > 0$, then $a < a + c$
3. The cosine inequality $2|x||y| \leq x^2 + y^2$

Unit 3: Continuous Functions

Definition of a Continuous Function

A quick review of the definition of a continuous function in one variable

A function of a single variable $f(x)$ is continuous at $x = a$ if and only if

Definition

Continuity

Single

Variable

Function

1. f is defined at $x = a$
2. $\lim_{x \to a} f(x)$ exists, which means that
   (a) $\lim_{x \to a^-} f(x)$ exists; and
   (b) $\lim_{x \to a^+} f(x)$ exists and;
   (c) $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$
3. $\lim_{x \to a} f(x) = f(a)$

A function $f(x, y)$ is **continuous** at (a, b) if and only if

Definition

Continuous

$$\lim_{(x,y) \to (a,b)} f(x, y) = f(a, b)$$

Additionally, if f is continuous at every point in a set $D \subset \mathbb{R}^2$, then we say that f is continuous on D

Remark: Just like in single variable calculus, there are three requirements in this definition:

1. $\lim_{(x,y) \to (a,b)} f(x, y)$ exists
2. f is defined at (a, b), and
3. $\lim_{(x,y) \to (a,b)} f(x, y) = f(a, b)$

Basic functions

To make the process of verifying if a function is continuous, we will employ the use of simpler or "basic" functions, which we know are continuous and view functions we inspect as being made up of these basic functions

In this course, we can take the continuity of these functions on their domain as a given

1. the constant function $f(x,y) = k$
2. the power functions $f(x,y) = x^n$, $f(x,y) = y^n$
3. the logarithm function $\ln(\cdot)$
4. the exponential function $e^{(\cdot)}$
5. the trignometric functions, $\sin(\cdot), \cos(\cdot)$, etc.
6. the inverse trigonometric functions, $\arcsin(\cdot)$, etc.
7. the absolute value function $|\cdot|$

Definition
Operations on Functions

if $f(x,y)$ and $g(x,y)$ are scalar functions and $(x,y) \in D(f) \cap D(g)$, then:

1. the **sum** $f+g$ is defined by
$$(f+g)(x,y) = f(x,y) + g(x,y)$$
2. the **product** fg is defined by
$$(fg)(x,y) = f(x,y)g(x,y)$$
3. the **quotient** $\frac{f}{g}$ is defined by
$$\left(\frac{f}{g}\right)(x,y) = \frac{f(x,y)}{g(x,y)}, \text{ if } g(x,y) \neq 0$$

Definition
Composite Functions

For scalar functions $g(t)$ and $f(x,y)$ the **composite function** $g \circ f$ is defined by

$$(g \circ f)(x,y) = g(f(x,y))$$

for all $(x,y) \in D(f)$ for which $f(x,y) \in D(g)$

Remark: When composing multivariable functions, it is very important to make sure that the range of the inner function is a subset of the domain of the outer function.

Continuity Theorems

With basic functions and operations on functions discussed, we now state some thorems that will be of use. (Most proofs in course notes)

Continuity Theorem 1

If f and g are both continuous at (a,b), then $f+g$ and fg are continuous at (a,b)

If f and g are both continuous at (a, b) and $g(a, b) \neq 0$, then the quotient $\frac{f}{g}$ is continuous at (a, b) — Continuity Theorem 2

If $f(x, y)$ is continuous at (a, b) and $g(t)$ is continuous at $f(a, b)$, then the composition $g \circ f$ is continuous at (a, b) — Continuity Theorem 3

Week 3

Unit 4: The Linear Approximation

Partial Derivatives

A scalar function $f(x, y)$ can be differentiated in two natural ways, by treating y as a constant and differentiating with respect to x to get $\frac{\partial f}{\partial x}$ or treating x as constant and differentiating with respec to y to get $\frac{\partial f}{\partial y}$. These are called the (first) **partial derivatives** of f

Definition

Partial

Derivatives

The **partial derivatives** of $f(x, y)$ are defined by

$$\frac{\partial f}{\partial x}(x, y) = f_x(x, y) = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$\frac{\partial f}{\partial y}(x, y) = f_y(x, y) = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

provided these limits exist

Sometimes it's convenient to use the **operator notation** D_{1f} and D_{2f} for the partial derivates of $f(x, y)$, where D_{1f} means to differentiate wrt the variable in the first position, holding the others fixed. Sometimes $\frac{\partial f}{\partial x}(x, y)$ is simply written as $\frac{\partial f}{\partial x}$

Higher Order Partial Derivatives

Partial derivatives of a scalar function of two variables are also a scalar function of two variables, so we can take partial derivatives of the partial derivatives of any scalar function. There are four possible second partial derivatives of f

- $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$
- $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$
- $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$
- $\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$

Remark: It is often convenient to use the subscript notation or the operator notation:
$\frac{\partial^2 f}{\partial x^2} = f_{xx} = D_1^2 f$, $\frac{\partial^2 f}{\partial y \partial x} = f_{xy} = D_2 D_1 f$,
$\frac{\partial^2 f}{\partial x \partial y} = f_{yx} = D_1 D_2 f$, $\frac{\partial^2 f}{\partial y^2} = f_{yy} = D_2^2 f$
You will notice that sometimes $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$, this is in fact a general property of parital derivatives, subject to a continuity requirement, as follows.

Theorem

Caliraut's

Theorem

If f_{xy} and f_{yx} are defined in some neighbourhood of (a, b) and are both continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b)$$

We can take higher-order partial derivatives in the expected way. $f(x, y)$ has eight third partial derivatives.
Clairaut's theorem also extends to higher-order partial derivatives: if the higher-order partial derivatives are define in a neighbourhood of a point (a, b) and are continuous at (a, b), then $f_{i_1,\ldots,i_k} = f_{j_1,\ldots,j_k}$, whenever $(i_1, \ldots, i_k)$ and $(j_1, \ldots, j_k)$ are tuples (ordered sets/sequences) of indices (variable symbols) which are arrangements of each other.
E.g., If the partial derivatives of f satisfy Caliraut's theorem, then

$$f_{xxy}(a, b) = f_{xyx}(a, b) = f_{yxx}(a, b)$$

In many situations, we will want to require that a function have continuous partial derivatives of some order. Some terminology;
If the k-th partial derivatives of $f(x_1, \ldots, x_n) = f(\mathbf{x})$ are continuous, then we write

$$f \in C^k$$

and say "f is in class C^k"
Having $f(x, y) \in C^2$, for example, means that f has continuous second partial derivatives, and therefore, by Clairaut's theorem, that $f_{xy} = f_{yx}$. More generally, $f(x, y) \in C^k$ means that f has continuous k-th partial derivatives and that the mixed higher-order partial derivatives are equal regardless of the order in which they are taken

The Tangent Plane

The **tangent plane** to $z = f(x, y)$ at the point $(a, b, f(a, b))$ is

$$z = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b)$$

Definition
Tangent Plane

Linear Approximation

In the One-Dimensional case, for a function $f(x)$ the tangent line to $y = f(x)$ at the point $(a, f(a))$ is $y = f(a) + f'(a)(x - a)$. The function L_a defined by $L_a(x) = f(a) + f'(a)(x - a)$ is called the **linearization** of f at a since $L_a(x)$ approximates $f(x)$ for sufficiently close to a. For x sufficiently close to a, the approximation $f(x) \approx L_a(x)$, is called the **linear approximation** of f at a

For a multivariable function $f(x,y)$, we can use the tangent plane to approximate the surface $z = f(x,y)$ near a point of tangency $P(a,b,f(a,b))$.

For a function $f(x,y)$ we define the **linearization** $L_{(a,b)}(x,y)$ of f at (a,b) by

Definition

Linearization and Linear Approximation

$$L_{(a,b)}(x,y) = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(y-b)$$

We call the approximation

$$f(x,y) \approx L_{(a,b)}(x,y)$$

the **linear approximation** of $f(x,y)$ at (a,b)

In the case we want to know the change in the value of $f(x,y)$ due to a change $(\Delta x, \Delta y)$ away from the point (a,b) , where $\Delta x = x-a$ and $\Delta y = y-b$, we can manipulate the linear approximation to get the **increment form** of the linear approximation

$$\Delta f \approx \frac{\partial f}{\partial x}(a,b)\Delta x + \frac{\partial f}{\partial y}(a,b)\Delta y$$

Linear Approximation in $\mathbb{R}^3$

By analogy with the case of a function with two variablesm we can define the linearization of a function $f(x,y,z)$ at $\vec{a} = (a,b,c)$ by

$$L_{\vec{a}}(x,y,z) = f(\vec{a}) + f_x(\vec{a})(x-a) + f_y(\vec{a})(y-b) + f_z(\vec{a})(z-c)$$

To simplify the notation we can represent the final three terms as the dot product of the vectors

$$(x-a, y-b, z-c) = (x,y,z) - (a,b,c), \ \ \textit{and} \ \nabla f(\vec{a}) = (f_x(\vec{a}), f_y(\vec{a}), f_z(\vec{a}))$$

since

$$(x-a, y-b, z-c) \cdot (f_x(\vec{a}), f_y(\vec{a}), f_z(\vec{a})) = f_x(\vec{a})(x-a) + f_y(\vec{a})(y-b) + f_z(\vec{a})(z-c)$$

The vector $\nabla f(\vec{a})$ is called the **gradient** oof f at $\vec{a}$

Suppose that $f(a,b,z)$ has partial derivatives at $\vec{a} \in \mathbb{R}^3$. The **gradient** of f at $\vec{a}$ is defined by

Definition

Gradient

$$\nabla f(\vec{a}) = (f_x(\vec{a}), f_y(\vec{a}), f_z(\vec{a}))$$

Supoose that $f(\vec{x}), \mathbf{x} \in \mathbb{R}^3$, has partial derivatives at $\vec{a} \in \mathbb{R}^3$. The **linearization** of f at $\vec{a}$ is defined by

$$L_{\vec{a}}(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$$

The **linear approximation** of f at $\vec{a}$ is

$$f(\vec{x}) \approx f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$$

Definition

Linearizaton and Linear Approximation

Generalization for $\mathbb{R}^n$

The advantage of using vector notation is that the equations for **linearization** and **linear approximation** hold for a function of n variables $f(\vec{x}), \vec{x} \in \mathbb{R}^n$. For an arbitrary vector $\vec{a} \in \mathbb{R}^n$, we have

$$\Delta\vec{x} = \vec{x} - \vec{a} = (x_1 - a_1, \ldots, x_n - a_n$$

And we define the gradient of f at $\vec{a}$ to be

$$\nabla f(\vec{a}) = (D_1 f(\vec{a}), D_2 f(\vec{a}), \ldots, D_n f(\vec{a}))$$

Then, the increment form of the linear approximation for $f(\vec{x})$ is

$$\Delta f \approx \nabla f(\vec{a}) \cdot \Delta\vec{x}$$

This is a true generalization as using the formula for $n = 1$ we find out familiar equation (in increment form) $\Delta g \approx \nabla g(a) \cdot \Delta\vec{x} = g'(a)(x - a)$
And for $n = 2$, we get $\Delta f \approx \nabla f(a, b) \cdot \Delta(x, y) = f_x(a, b)(x - a) + f_y(a, b)(y - b)$