Math 235 Notes

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These are my 2nd year Linear Algebra 2 notes at the University of Waterloo (MATH 235). They are pretty similar to the content you may see in the course notes by D. Wolczuk.

You will find that these aren't very useful as notes, in the sense that they are not significantly shorter than the content in the course notes, they're really just a way for me to type down the content I am learning and absorb it.

Thanks to Prof. Dan Wolczuk for providing me with the macros to typeset this LaTeX document.

Week 1

7. Fundamental Subspaces

Let A be an $m \times n$ matrix. The **four fundamental subspaces** of A are

1. $\text{Col}(A) = \{A\vec{x} \in \mathbb{R}^m | \vec{x} \in \mathbb{R}^n\}$, called the **column space**

2. $\text{Row}(A)\{A^T\vec{x} \in \mathbb{R}^n | \vec{x} \in \mathbb{R}^m\}$, called the **row space**

3. $\text{Null}(A) = \{\vec{x} \in \mathbb{R}^n | A\vec{x} = \vec{0}\}$, called the **null space**

4. $\text{Null}(A^T) = \{\vec{x} \in \mathbb{R}^m | A^T\vec{x} = \vec{0}\}$, called the **left nullspace**

If A is an $m \times n$ matrix, then $\text{Col}(A)$ and $\text{Null}(A^T)$ are subspaces of $\mathbb{R}^m$, and $\text{Row}(A)$ and $\text{Null}(A)$ are subspaces of $\mathbb{R}^n$

Theorem 7.1.1

If A is an $m \times n$ matrix, then the colums of A which correspond to leading ones in the RREF of A form a basis for $\text{Col}(A)$. Moreover,

Theorem 7.1.2

$$\dim \text{Col}(A) = \text{rank } A$$

If R is an $m \times n$ matrix and E is an $n \times n$ invertible matrix, then

Theorem 7.1.3

$$\{RE\vec{x} | \vec{x} \in \mathbb{R}^n\} = \{R\vec{y} | \vec{y} \in \mathbb{R}^n\}$$

If A is an $m \times n$ matrix, then the non-zero rows in the reduced row echelon form of A form a basis for $\text{Row}(A)$. Hence,

Theorem 7.1.4

$$\dim \text{Row}(A) = \text{rank } A$$

For any $m \times n$ matrix A we have $\text{rank } A = \text{rank } A^T$

Corollary 7.1.5

If A is an $m \times n$ matrix, then

Dimension Theorem

$$\text{rank } A + \dim \text{Null}(A) = n$$

8. Linear Mappings

Some review

A mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ is called **linear** if

$$L(s\vec{x} + t\vec{y}) = sL(\vec{x}) + tL(\vec{y})$$

for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$
Two linear mappings $L : \mathbb{R}^n \to \mathbb{R}^m$ and $M : \mathbb{R}^n \to \mathbb{R}^m$ are said to be **equal**
if $L(\vec{v}) = M(\vec{v}), \forall \vec{v} \in \mathbb{R}^n$

The **range** of a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ is defined by

$$\text{Range}(L) = \{L(\vec{x}) | \vec{x} \in \mathbb{R}^n\}$$

The **kernel** of a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ is defined by

$$\text{Ker}(L) = \{\vec{x} \in \mathbb{R}^n | L(\vec{x}) = \vec{0}\}$$

The **standard matrix** of a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ is defined by

$$[L] = [L(\vec{e}_1) \ \ldots \ L(\vec{e}_n)]$$

It satisfies

$$L(\vec{x}) = [L]\vec{x}$$

for all $\vec{x} \in \mathbb{R}^n$

If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping, then $\text{Range}(L) = \text{Col}([L])$

If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping, then $\text{Ker}(L) = \text{Null}([L])$

Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping. Then,

$$\dim(\text{Range}(L)) + \dim(\text{Ker}(L)) = \dim(\mathbb{R}^n)$$

8.1 General Linear Mappings

Linear Mappings $L : \mathbb{V} \to \mathbb{W}$
We will extend our definition of a linear mapping to the case where the domain and codomain are general vector spaces instead of just $\mathbb{R}^n$

Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces. A mapping $L : \mathbb{V} \to \mathbb{W}$ is called **linear** if

$$L(s\vec{x} + t\vec{y}) = sL(\vec{x}) + tL(\vec{y})$$

for all $\vec{x}, \vec{y} \in \mathbb{V}$ and $s, t \in \mathbb{R}$
Two linear mappings $L : \mathbb{V} \to \mathbb{W}$ and $M : \mathbb{V} \to \mathbb{W}$ are said to be **equal**
if $L(\vec{v}) = M(\vec{v}), \forall \vec{v} \in \mathbb{V}$

Definition
Linear
Mapping

Note: A linear mapping $L : \mathbb{V} \to \mathbb{V}$ is called a **linear operator**

Let $L : \mathbb{V} \to \mathbb{W}$ and $M : \mathbb{V} \to \mathbb{W}$ be linear mappings. We define $L + M : \mathbb{V} \to \mathbb{W}$ by

$$(L + M)(\vec{v}) = L(\vec{v}) + M(\vec{v})$$

and for any $t \in \mathbb{R}$ we define $tL : \mathbb{V} \to \mathbb{W}$ by

$$(tL)(\vec{v}) = tL(\vec{v})$$

Definition
Addition
Scalar Multi-
plication

Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces. The set $\mathbb{L}$ of all linear mappings $L : \mathbb{V} \to \mathbb{W}$ with standard addition and scalar multiplication of mappings is a vector space

Theorem
8.1.1

Let $L : \mathbb{V} \to \mathbb{W}$ and $M : \mathbb{W} \to \mathbb{U}$ be linear mappings. We define $M \circ L : \mathbb{V} \to \mathbb{U}$ by

$$(M \circ L)(\vec{v}) = M(L(\vec{v})), \ \forall \vec{v} \in \mathbb{V}$$

Definition
Composition

If $L : \mathbb{V} \to \mathbb{W}$ and $M : \mathbb{W} \to \mathbb{U}$ are linear mappings, then $M \circ L : \mathbb{V} \to \mathbb{U}$ is a linear mapping

Theorem
8.1.2

Let $L : \mathbb{V} \to \mathbb{W}$ and $M : \mathbb{V} \to \mathbb{W}$ be linear mappings. If $(M \circ L)(\vec{v}) = \vec{v}, \forall \vec{v} \in \mathbb{V}$ and $(L \circ M)(\vec{w}) = \vec{w}, \forall \vec{w} \in \mathbb{W}$, then L and M are said to be **invertible**.
We write $M = L^{-1}$ and $L = M^{-1}$

Definition
Invertible
Mapping

8.2 Rank-Nullity Theorem

Let's extend the definitions of the range and kernel to general linear mappings.

For a linear mapping $L : \mathbb{V} \to \mathbb{W}$ the **kernel** of L is defined to be

$$\text{Ker}(L) = \{\vec{v} \in \mathbb{V} | L(\vec{v}) = \vec{0}_{\mathbb{W}}\}$$

and the **range** of L is defined to be

$$\text{Range}(L) = \{L(\vec{v}) \in \mathbb{W} | \vec{v} \in \mathbb{V}\}$$

If $\mathbb{V}$ and $\mathbb{W}$ are vector spaces and $L : \mathbb{V} \to \mathbb{W}$ is a linear mapping, then

$$L(\vec{0}_{\mathbb{V}}) = \vec{0}_{\mathbb{W}}$$

If $L : \mathbb{V} \to \mathbb{W}$ is a linear mapping, then $\text{Ker}(L)$ is a subspace of $\mathbb{V}$ and $\text{Range}(L)$ is a subspace of $\mathbb{W}$

Let $L : \mathbb{V} \to \mathbb{W}$ be a linear mapping. We define the **rank** of L to be

$$\text{rank}(L) = \dim(\text{Range}(L))$$

We define the **nullity** of L to be

$$\text{nullity}(L) = \dim(\text{Ker}(L))$$

Let $\mathbb{V}$ be an n-dimensional vector space and let $\mathbb{W}$ be a vector space. If $L : \mathbb{V} \to \mathbb{W}$ is linear, then

$$\text{rank}(L) + \text{nullity}(L) = n$$

Remark: The proof for the Rank-Nullity Theorem is identical to the proof for the dimension theorem, only that this time it is a generalisation for general vector spaces. You could see the Rank-Nullity theorem as an analog of the Dimension theorem for general linear mappings.

8.3 Matrix of a Linear Mapping

We will now show that every linear mapping $l : \mathbb{V} \to \mathbb{W}$ can also be represented as a matrix mapping. However, we must be careful when dealing with general vector spaces as our domain and codomain. For example, it is certainly impossible to represent a linear mapping that maps polynomials to matrices, since we cannot multiply a matrix by a polynomial.

Thus, if we are going to convert vectors in $\mathbb{V}$ to vectors in $\mathbb{R}^n$ in order to define matrix representations of general linear mappings. Recall the coordinate vector of $\vec{x} \in \mathbb{V}$ with respect to a basis $\mathcal{B}$ is a vector in $\mathbb{R}^n$. In particular, if $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ and $\vec{x} = b_1\vec{v}_1 + \cdots + b_n\vec{v}_n$ then the coordinate vector of $\vec{x}$ with respect to $\mathcal{B}$ is defined to be

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Using coordinates, we can write a matrix mapping representaion of a linear mapping $L : \mathbb{V} \to \mathbb{W}$. We want to find a matrix A such that

$$[L(\vec{x})]_{\mathcal{C}} = A[\vec{x}]_{\mathcal{B}}$$

for every $\vec{x} \in \mathbb{V}$, where $\mathcal{B}$ is a basis for $\mathbb{V}$ and $\mathcal{C}$ is a basis for $\mathbb{W}$

Considering $[L(\vec{x})]_{\mathcal{C}}$ and using the properties of linear mappings and coordinates, we get

$$[L(\vec{x})]_{\mathcal{C}} = [L(b_1\vec{v}_1 + \cdots + b_n\vec{v}_n)]_{\mathcal{C}} = b_1[L(\vec{v}_1)]_{\mathcal{C}} + \cdots + b_n[L(\vec{v}_n)]_{\mathcal{C}}$$

$$= \begin{bmatrix} [L(\vec{v}_1)]_{\mathcal{C}} & \cdots & [L(\vec{v}_n)]_{\mathcal{C}} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

This, we have the matrix $\begin{bmatrix} [L(\vec{v}_1)]_{\mathcal{C}} & \cdots & [L(\vec{v}_n)]_{\mathcal{C}} \end{bmatrix}$ being matrix-vector multiplied by the vector $[\vec{x}]_{\mathcal{B}}$ as desired.

Suppose $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is a basis for a vector space $\mathbb{V}$ and $\mathcal{C}$ is any basis for a finite dimensional vector space $\mathbb{W}$. For a linear mapping $L : \mathbb{V} \to \mathbb{W}$, the **matirx of L with respect to basis $\mathcal{B}$ and $\mathcal{C}$** is defined by

$$_{\mathcal{C}}[L]_{\mathcal{B}} = \begin{bmatrix} [L(\vec{v}_1)]_{\mathcal{C}} & \cdots & [L(\vec{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

and satisfies

$$[L(\vec{x})]_{\mathcal{C}} = {}_{\mathcal{C}}[L]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}$$

for all $\vec{x} \in \mathbb{V}$ In the special case of a linear operator L acting on a finite dimensional vector space $\mathbb{V}$ with basis $\mathcal{B}$, we often wish to find the matrix $\mathcal{B}[L]_{\mathcal{B}}$

Definition
Matrix of a Linear Mapping

Suppose $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ is any basis for an n-dimensional vector space $\mathbb{V}$ and let $L : \mathbb{V} \to \mathbb{V}$ be a linear operator. The **$\mathcal{B}$-matrix of L** (or the matrix of L with respect to the basis $\mathcal{B}$) is defined by

$$[L]_{\mathcal{B}} = \begin{bmatrix} [L(\vec{v}_1)]_{\mathcal{B}} & \cdots & [L(\vec{v}_n)]_{\mathcal{B}} \end{bmatrix}$$

and satisfies

$$[L(\vec{x})]_{\mathcal{B}} = [L]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}$$

for all $\vec{x} \in \mathbb{V}$

Definition
Matrix of a Linear Operator

(Makes more sense if you refer to the course notes) In example 4 we found that the matrix of L with respect to basis $\mathcal{B}$ is diagonal since the vectors in $\mathcal{B}$ are eigenvectors of L. Such a basis is called a **geometrically natural basis**