Math 235 Notes

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These are my 2nd year Linear Algebra 2 notes at the University of Waterloo (MATH 235). They are pretty similar to the content you may see in the course notes by D. Wolczuk.
You will find that these aren't very useful as notes, in the sense that they are not significantly shorter than the content in the course notes, they're really just a way for me to type down the content I am learning and absorb it.
Thanks to Prof. Dan Wolczuk for providing me with the macros to typeset this LaTeX document.

Week 1

7. Fundamental Subspaces

Let A be an $m \times n$ matrix. The **four fundamental subspaces** of A are

Definition
Fudamental Subspaces

1. $\text{Col}(A) = \{A\vec{x} \in \mathbb{R}^m | \vec{x} \in \mathbb{R}^n\}$, called the **column space**

2. $\text{Row}(A)\{A^T\vec{x} \in \mathbb{R}^n | \vec{x} \in \mathbb{R}^m\}$, called the **row space**

3. $\text{Null}(A) = \{\vec{x} \in \mathbb{R}^n | A\vec{x} = \vec{0}\}$, called the **null space**

4. $\text{Null}(A^T) = \{\vec{x} \in \mathbb{R}^m | A^T\vec{x} = \vec{0}\}$, called the **left nullspace**

If A is an $m \times n$ matrix, then $\text{Col}(A)$ and $\text{Null}(A^T)$ are subspaces of $\mathbb{R}^m$, and $\text{Row}(A)$ and $\text{Null}(A)$ are subspaces of $\mathbb{R}^n$

Theorem 7.1.1

If A is an $m \times n$ matrix, then the colums of A which correspond to leading ones in the RREF of A form a basis for $\text{Col}(A)$. Moreover,

Theorem 7.1.2

$$\dim \text{Col}(A) = \text{rank } A$$

If R is an $m \times n$ matrix and E is an $n \times n$ invertible matrix, then

Theorem 7.1.3

$$\{RE\vec{x} | \vec{x} \in \mathbb{R}^n\} = \{R\vec{y} | \vec{y} \in \mathbb{R}^n\}$$

If A is an $m \times n$ matrix, then the non-zero rows in the reduced row echelon form of A form a basis for $\text{Row}(A)$. Hence,

Theorem 7.1.4

$$\dim \text{Row}(A) = \text{rank } A$$

For any $m \times n$ matrix A we have $\text{rank } A = \text{rank } A^T$

Corollary 7.1.5

If A is an $m \times n$ matrix, then

Theorem
Dimension Theorem

$$\text{rank } A + \dim \text{Null}(A) = n$$

8. Linear Mappings

Some review

If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping, then Range$(L) = \text{Col}([L])$ — Theorem 7.2.1

If $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear mapping, then Ker$(L) = \text{Null}([L])$ — Theorem 7.2.2

Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping. Then, — Theorem 7.2.3

$$\dim(\text{Range}(L)) + \dim(\text{Ker}(L)) = \dim(\mathbb{R}^n)$$

A mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ is called **linear** if — Definition Linear Mapping

$$L(s\vec{x} + t\vec{y}) = sL(\vec{x}) + tL(\vec{y})$$

for all $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $s, t \in \mathbb{R}$
Two linear mappings $L : \mathbb{R}^n \to \mathbb{R}^m$ and $M : \mathbb{R}^n \to \mathbb{R}^m$ are said to be **equal** if $L(\vec{v}) = M(\vec{v}), \forall \vec{v} \in \mathbb{R}^n$

The **range** of a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ is defined by — Definition Range

$$\text{Range}(L) = \{L(\vec{x}) | \vec{x} \in \mathbb{R}^n\}$$

The **kernel** of a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ is defined by — Definition Kernel

$$\text{Ker}(L) = \{\vec{x} \in \mathbb{R}^n | L(\vec{x}) = \vec{0}\}$$

The **standard matrix** of a linear mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ is defined by — Definition Standard Matrix

$$[L] = [L(\vec{e}_1) \ \ldots \ L(\vec{e}_n)]$$

It satisfies

$$L(\vec{x}) = [L]\vec{x}$$

for all $\vec{x} \in \mathbb{R}^n$

8.1 General Linear Mappings

Linear Mappings $L : \mathbb{V} \to \mathbb{W}$
We will extend our definition of a linear mapping to the case where the domain and codomain are general vector spaces instead of just $\mathbb{R}^n$

Definition — Linear Mapping

Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces. A mapping $L : \mathbb{V} \to \mathbb{W}$ is called **linear** if

$$L(s\vec{x} + t\vec{y}) = sL(\vec{x}) + tL(\vec{y})$$

for all $\vec{x}, \vec{y} \in \mathbb{V}$ and $s, t \in \mathbb{R}$
Two linear mappings $L : \mathbb{V} \to \mathbb{W}$ and $M : \mathbb{V} \to \mathbb{W}$ are said to be **equal** if $L(\vec{v}) = M(\vec{v}), \forall \vec{v} \in \mathbb{V}$

Note: A linear mapping $L : \mathbb{V} \to \mathbb{V}$ is called a **linear operator**

Definition — Addition Scalar Multiplication

Let $L : \mathbb{V} \to \mathbb{W}$ and $M : \mathbb{V} \to \mathbb{W}$ be linear mappings. We define $L + M : \mathbb{V} \to \mathbb{W}$ by

$$(L + M)(\vec{v}) = L(\vec{v}) + M(\vec{v})$$

and for any $t \in \mathbb{R}$ we define $tL : \mathbb{V} \to \mathbb{W}$ by

$$(tL)(\vec{v}) = tL(\vec{v})$$

Theorem 8.1.1

Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces. The set $\mathbb{L}$ of all linear mappings $L : \mathbb{V} \to \mathbb{W}$ with standard addition and scalar multiplication of mappings is a vector space

Definition — Composition

Let $L : \mathbb{V} \to \mathbb{W}$ and $M : \mathbb{W} \to \mathbb{U}$ be linear mappings. We define $M \circ L : \mathbb{V} \to \mathbb{U}$ by

$$(M \circ L)(\vec{v}) = M(L(\vec{v})), \ \forall \vec{v} \in \mathbb{V}$$

Theorem 8.1.2

If $L : \mathbb{V} \to \mathbb{W}$ and $M : \mathbb{W} \to \mathbb{U}$ are linear mappings, then $M \circ L : \mathbb{V} \to \mathbb{U}$ is a linear mapping

Definition — Invertible Mapping

Let $L : \mathbb{V} \to \mathbb{W}$ and $M : \mathbb{V} \to \mathbb{W}$ be linear mappings. If $(M \circ L)(\vec{v}) = \vec{v}, \forall \vec{v} \in \mathbb{V}$ and $(L \circ M)(\vec{w}) = \vec{w}, \forall \vec{w} \in \mathbb{W}$, then L and M are said to be **invertible**.
We write $M = L^{-1}$ and $L = M^{-1}$

8.2 Rank-Nullity Theorem

Let's extend the definitions of the range and kernel to general linear mappings.

Definition
Range
Kernel

For a linear mapping $L : \mathbb{V} \to \mathbb{W}$ the **kernel** of L is defined to be

$$\text{Ker}(L) = \{\vec{v} \in \mathbb{V} | L(\vec{v}) = \vec{0}_{\mathbb{W}}\}$$

and the **range** of L is defined to be

$$\text{Range}(L) = \{L(\vec{v}) \in \mathbb{W} | \vec{v} \in \mathbb{V}\}$$

Theorem 8.2.1

If $\mathbb{V}$ and $\mathbb{W}$ are vector spaces and $L : \mathbb{V} \to \mathbb{W}$ is a linear mapping, then

$$L(\vec{0}_{\mathbb{V}}) = \vec{0}_{\mathbb{W}}$$

Theorem 8.2.2

If $L : \mathbb{V} \to \mathbb{W}$ is a linear mapping, then $\text{Ker}(L)$ is a subspace of $\mathbb{V}$ and $\text{Range}(L)$ is a subspace of $\mathbb{W}$

Definition
Rank Nullity

Let $L : \mathbb{V} \to \mathbb{W}$ be a linear mapping. We define the **rank**of L to be

$$\text{rank}(L) = \dim(\text{Range}(L))$$

We define the **nullity** of L to be

$$\text{nullity}(L) = \dim(\text{Ker}(L))$$

Theorem Rank-Nullity Theorem

Let $\mathbb{V}$ be an n-dimensional vector space and let $\mathbb{W}$ be a vector space. If $L : \mathbb{V} \to \mathbb{W}$ is linear, then

$$\text{rank}(L) + \text{nullity}(L) = n$$

Remark: The proof for the Rank-Nullity Theorem is identical to the proof for the dimension theorem, only that this time it is a generalisation for general vector spaces. You could see the Rank-Nullity theorem as an analog of the Dimension theorem for general linear mappings.

8.3 Matrix of a Linear Mapping

We will now show that every linear mapping $l : \mathbb{V} \to \mathbb{W}$ can also be represented as a matrix mapping. However, we must be careful when dealing with general vector spaces as our domain and codomain. For example, it is certainly impossible to represent a linear mapping that maps polynomials to matrices, since we cannot multiply a matrix by a polynomial.
Thus, if we are going to convert vectors in $\mathbb{V}$ to vectors in $\mathbb{R}^n$ in order to define matrix representations of general linear mappings. Recall the coordinate vector of $\vec{x} \in \mathbb{V}$ with respect to a basis $\mathcal{B}$ is a vector in $\mathbb{R}^n$. In particular, if $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ and $\vec{x} = b_1\vec{v}_1 + \cdots + b_n\vec{v}_n$ then the coordinate vector of $\vec{x}$ with respect to $\mathcal{B}$ is defined to be

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

Using coordinates, we can write a matrix mapping representaion of a linear mapping $L : \mathbb{V} \to \mathbb{W}$. We want to find a matrix A such that

$$[L(\vec{x})]_{\mathcal{C}} = A[\vec{x}]_{\mathcal{B}}$$

for every $\vec{x} \in \mathbb{V}$, where $\mathcal{B}$ is a basis for $\mathbb{V}$ and $\mathcal{C}$ is a basis for $\mathbb{W}$
Considering $[L(\vec{x})]_{\mathcal{C}}$ and using the properties of linear mappings and coordinates, we get

$$\begin{aligned} [L(\vec{x})]_{\mathcal{C}} = [L(b_1\vec{v}_1 + \cdots + b_n\vec{v}_n)]_{\mathcal{C}} &= b_1[L(\vec{v}_1)]_{\mathcal{C}} + \cdots + b_n[L(\vec{v}_n)]_{\mathcal{C}} \\ &= \begin{bmatrix} [L(\vec{v}_1)]_{\mathcal{C}} & \dots & [L(\vec{v}_n)]_{\mathcal{C}} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \end{aligned}$$

This, we have the matrix $\begin{bmatrix} [L(\vec{v}_1)]_{\mathcal{C}} & \dots & [L(\vec{v}_n)]_{\mathcal{C}} \end{bmatrix}$ being matrix-vector multiplied by the vector $[\vec{x}]_{\mathcal{B}}$ as desired.

Definition Matrix of a Linear Mapping

Suppose $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is a basis for a vector space $\mathbb{V}$ and $\mathcal{C}$ is any basis for a finite dimensional vector space $\mathbb{W}$. For a linear mapping $L : \mathbb{V} \to \mathbb{W}$, the **matirx of L with respect to basis $\mathcal{B}$ and $\mathcal{C}$** is defined by

$${}_{\mathcal{C}}[L]_{\mathcal{B}} = \begin{bmatrix} [L(\vec{v}_1)]_{\mathcal{C}} & \dots & [L(\vec{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

and satisfies

$$[L(\vec{x})]_{\mathcal{C}} = {}_{\mathcal{C}}[L]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}$$

for all $\vec{x} \in \mathbb{V}$ In the special case of a linear operator L acting on a finite dimensional vector space $\mathbb{V}$ with basis $\mathcal{B}$, we often wish to find the matrix $\mathcal{B}[L]_{\mathcal{B}}$

Definition Matrix of a Linear Operator

Suppose $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is any basis for an n-dimensional vector space $\mathbb{V}$ and let $L : \mathbb{V} \to \mathbb{V}$ be a linear operator. The **$\mathcal{B}$-matrix of** L (or the matrix of L with respect to the basis $\mathcal{B}$) is defined by

$$[L]_{\mathcal{B}} = \begin{bmatrix} [L(\vec{v}_1)]_{\mathcal{B}} & \dots & [L(\vec{v}_n)]_{\mathcal{B}} \end{bmatrix}$$

and satisfies

$$[L(\vec{x})]_{\mathcal{B}} = [L]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}$$

for all $\vec{x} \in \mathbb{V}$
(Makes more sense if you refer to the course notes) In example 4 we found that the matrix of L with respect to basis $\mathcal{B}$ is diagonal since the vectors in $\mathcal{B}$ are eigenvectors of L. Such a basis is called a **geometrically natural basis**