# Math 235 Notes

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These are my 2nd year Linear Algebra 2 notes at the University of Waterloo (MATH 235). They are pretty similar to the content you may see in the course notes by D. Wolczuk.
You will find that these aren't very useful as notes, in the sense that they are not significantly shorter than the content in the course notes, they're really just a way for me to type down the content I am learning and absorb it.
Thanks to Prof. Dan Wolczuk for providing me with the macros to typeset this LaTeX document.

# Week 1

## Fundamental Subspaces

Let $A$ be an $m \times n$ matrix. The **four fundamental subspaces** of $A$ are

Definition
Fudamental Subspaces

1. $\text{Col}(A) = \{A\vec{x} \in \mathbb{R}^m | \vec{x} \in \mathbb{R}^n\}$, called the **column space**

2. $\text{Row}(A)\{A^T\vec{x} \in \mathbb{R}^n | \vec{x} \in \mathbb{R}^m\}$, called the **row space**

3. $\text{Null}(A) = \{\vec{x} \in \mathbb{R}^n | A\vec{x} = \vec{0}\}$, called the **null space**

4. $\text{Null}(A^T) = \{\vec{x} \in \mathbb{R}^m | A^T\vec{x} = \vec{0}\}$, called the **left nullspace**

Theorem 7.1.1

If $A$ is an $m \times n$ matrix, then $\text{Col}(A)$ and $\text{Null}(A^T)$ are subspaces of $\mathbb{R}^m$, and $\text{Row}(A)$ and $\text{Null}(A)$ are subspaces of $\mathbb{R}^n$

Theorem 7.1.2

If $A$ is an $m \times n$ matrix, then the colums of $A$ which correspond to leading ones in the RREF of $A$ form a basis for $\text{Col}(A)$. Moreover,

$$\dim \text{Col}(A) = \text{rank } A$$

Theorem 7.1.3

If $R$ is an $m \times n$ matrix and $E$ is an $n \times n$ invertible matrix, then

$$\{RE\vec{x} | \vec{x} \in \mathbb{R}^n\} = \{R\vec{y} | \vec{y} \in \mathbb{R}^n\}$$

Theorem 7.1.4

If $A$ is an $m \times n$ matrix, then the non-zero rows in the reduced row echelon form of $A$ form a basis for $\text{Row}(A)$. Hence,

$$\dim \text{Row}(A) = \text{rank } A$$

Corollary 7.1.5

For any $m \times n$ matrix $A$ we have $\text{rank } A = \text{rank } A^T$

Theorem
Dimension Theorem

If $A$ is an $m \times n$ matrix, then

$$\text{rank } A + \dim \text{Null}(A) = n$$

# Linear Mappings

## General Linear Mappings

**Linear Mappings** $L : \mathbb{V} \to \mathbb{W}$
We will extend our definition of a linear mapping to the case where the domain and codomain are general vector spaces instead of just $\mathbb{R}^n$

Definition — Linear Mapping

Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces. A mapping $L : \mathbb{V} \to \mathbb{W}$ is called **linear** if

$$L(s\vec{x} + t\vec{y}) = sL(\vec{x}) + tL(\vec{y})$$

for all $\vec{x}, \vec{y} \in \mathbb{V}$ and $s, t \in \mathbb{R}$
Two linear mappings $L : \mathbb{V} \to \mathbb{W}$ and $M : \mathbb{V} \to \mathbb{W}$ are said to be **equal** if $L(\vec{v}) = M(\vec{v}), \forall \vec{v} \in \mathbb{V}$

**Note**: A linear mapping $L : \mathbb{V} \to \mathbb{V}$ is called a **linear operator**

Definition — Addition Scalar Multiplication

Let $L : \mathbb{V} \to \mathbb{W}$ and $M : \mathbb{V} \to \mathbb{W}$ be linear mappings. We define $L + M : \mathbb{V} \to \mathbb{W}$ by

$$(L + M)(\vec{v}) = L(\vec{v}) + M(\vec{v})$$

and for any $t \in \mathbb{R}$ we define $tL : \mathbb{V} \to \mathbb{W}$ by

$$(tL)(\vec{v}) = tL(\vec{v})$$

Theorem 8.1.1

Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces. The set $\mathbb{L}$ of all linear mappings $L : \mathbb{V} \to \mathbb{W}$ with standard addition and scalar multiplication of mappings is a vector space

Definition — Composition

Let $L : \mathbb{V} \to \mathbb{W}$ and $M : \mathbb{W} \to \mathbb{U}$ be linear mappings. We define $M \circ L : \mathbb{V} \to \mathbb{U}$ by

$$(M \circ L)(\vec{v}) = M(L(\vec{v})),\ \forall \vec{v} \in \mathbb{V}$$

Theorem 8.1.2

If $L : \mathbb{V} \to \mathbb{W}$ and $M : \mathbb{W} \to \mathbb{U}$ are linear mappings, then $M \circ L : \mathbb{V} \to \mathbb{U}$ is a linear mapping

Definition — Invertible Mapping

Let $L : \mathbb{V} \to \mathbb{W}$ and $M : \mathbb{V} \to \mathbb{W}$ be linear mappings. If $(M \circ L)(\vec{v}) = \vec{v}, \forall \vec{v} \in \mathbb{V}$ and $(L \circ M)(\vec{w}) = \vec{w}, \forall \vec{w} \in \mathbb{W}$, then $L$ and $M$ are said to be **invertible**. We write $M = L^{-1}$ and $L = M^{-1}$