GAMES, COOPERATIVE BEHAVIOR, AND ECONOMICS

Peter Borm

CONTENTS

CHAPTER 1 TU-games

Interactive decision theory, or *game theory,* is a mathematical framework for modeling and analyzing conflict situations that involve economic agents with possibly diverging interests. For a given economic problem one extracts the essential features, these are integrated in a stylized model of a *game*, the game is analyzed and the result is translated back into economic terms. It may be noted that the construction of an appropriate game is not a matter of routine but is an essential part of the analysis. Game theory thus provides a language and framework which allows for a systematic study of various features of behavioural interaction. It helps to relate economic situations which at first sight may seem very different and to recognize common elements.

Cooperative game theory focusses on cooperative behaviour by analyzing the negotiation process within a group of players in establishing a contract on a joint plan of activities, including an allocation of the correspondingly generated revenues. In particular, the possible joint revenues of each possible *coalition* (a subgroup of cooperating players) are taken into account so as to allow for a better comparison of each player's role and impact within the group as a whole, and to settle on a compromise allocation (a solution) in an objectively justifiable way. Depending on the exact underlying context the coalitional revenues can be viewed as the actual result of optimal cooperation or, if partly cooperation is infeasible or if the joint revenues depend on specific assumptions on behavior outside a coalition, as the result of a consistent thought experiment for comparative purposes only. The most basic format of a cooperative game is the model of TU-games introduced below.

Let N be a finite set of players and denote by 2^N the collection of all subsets of N. Elements of 2^N are called *coalitions*. A transferable utility (TU) game (or a game in coalitional form) assigns to each coalition S a real number. In principle this number specifies the monetary amount the coalition S can jointly generate on itself by means of optimal cooperation without any help of players in $N\backslash S$. Formally, a TU-game is a pair (N, v) where $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. The function v is called the *characteristic function*; for $S \subset N$, $v(S)$ is called the *worth* or *value* of the coalition S.

Example 1.1 (A glove market game)
Let N be divided into two disjoint subgroups L and $R : N = L \cup R$, $L \cap R = \emptyset$.
Members of L each have one left hand glove, members of R one right hand glove. A single glove is worth nothing, a (right-left) pair 10 Euro. This situation can be described by a TU-game (N, v) where

$$v(S) = 10 \min\{|L \cap S|,\ |R \cap S|\}$$

for all $S \in 2^N$. $\diamond$

A TU-game (N, v) is called *monotonic* if $v(S) \leq v(T)$ for all $S, T \in 2^N$ with $S \subset T$.
A monotonic game (N, v) with $v(S) \in \{0, 1\}$ for all $S \in 2^N$ and $v(N) = 1$ is called *simple*.

Example 1.2 (A voting game)
Consider a parliament of 150 seats with four parties 1, 2, 3 and 4, party 1 having 60 seats and the other parties each 30. The treshold on the number of seats to pass bills is 76. This voting situation can be described by the simple TU-game (N, v) given by

$$v(S) = \begin{cases} 1 & \text{if } |S| \geq 2 \text{ and } 1 \in S, \text{ or } S = \{2, 3, 4\}, \\ 0 & \text{otherwise.} \end{cases}$$

In this case, the game (N, v) does not reflect monetary gains but voting power instead. A coalition is assigned a value of 1 if and only if this coalition has a majority in parliament and therefore is able to pass bills. $\diamond$

Most TU-games (N, v) derived from practical situations satisfy *superadditivity*, i.e.

$$v(S \cup T) \geq v(S) + v(T)$$

for all $S, T \in 2^N$ with $S \cap T = \emptyset$. In a superadditive game breaking up a coalition into parts does not pay. A game (N, v) is called *additive* if

$$v(S \cup T) = v(S) + v(T)$$

for all $S, T \in 2^N$ with $S \cap T = \emptyset$. Note that an additive game is determined by a vector $a \in \mathbb{R}^N$ with $a_i = v(\{i\})$, $i \in N$, since $v(S) = \sum_{i \in S} a_i$ for all $S \in 2^N$.

An important notion is strategic equivalence of games: two TU-games (N, v) and (N, w) are called *S-equivalent* if there is a real number $k > 0$ and a vector $a \in \mathbb{R}^N$ (an additive game) such that

$$w(S) = kv(S) + \sum_{i \in S} a_i$$

for all $S \in 2^N$, or shortly, such that $w = kv + a$.
The positive number k reflects a rescaling of monetary units while adding the vector a boils down to giving each player a fixed amount of money (in the new units) independent of the coalition under consideration. Clearly, if v can be used to model a cooperative situation, also w can, and the other way around.

Example 1.3 (A spanning tree game)
Consider three communities 1, 2 and 3 (the players) and a power source 0. For all possible links the connection costs are depicted in Figure 1.1.

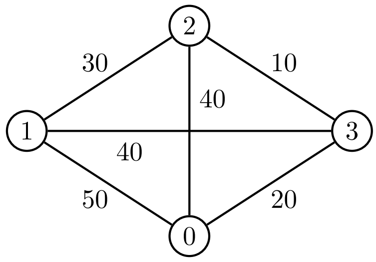


Figure 1.1: The spanning tree problem of Example 1.3.

Assuming that each player has to be connected to the source and that a coalition can not use links with an endpoint being a player outside this coalition, the minimal costs of each coalition to connect each of its members to the source is given by the function $c : 2^N \to \mathbb{R}$ with

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$c(S)$	0	50	40	20	70	60	30	60

Formally, we need to translate the costs into rewards to obtain a TU-game. Obviously this can be done by considering (N, v_0) with $N = \{1, 2, 3\}$ and $v_0 = -c$. A more standard way to do is to consider the *cost savings game* (N, v) defined by

$$v(S) = \sum_{i \in S} c(\{i\}) - c(S)$$

for all $S \in 2^N$. Since $v = v_0 + a$ with $a \in \mathbb{R}^N$ such that $a_i = c(\{i\})$ for all $i \in N$, v and v_0 are S-equivalent. For the cost savings game we find

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	0	20	10	30	50

We will come back to this type of games in Chapter 3. $\diamond$

Let TU^N denote the class of all TU-games with player set N. A game $v \in \mathrm{TU}^N$ (now we can omit the player set in the description of the game) is called *zero-normalized* if $v(\{i\}) = 0$ for every $i \in N$. It is called *zero-one-normalized* if it is zero-normalized and $v(N) = 1$.

Clearly a game $v \in \mathrm{TU}^N$ is determined by providing $2^{|N|} - 1$ real numbers (the values of the coalitions, recall that $v(\emptyset) = 0$ by definition), and hence corresponds to a vector in $\mathbb{R}^{2^{|N|}-1}$. Since also the converse is true, TU^N can be identified with $\mathbb{R}^{2^{|N|}-1}$ and thus constitutes a $2^{|N|} - 1$ dimensional linear space. A useful basis of TU^N is provided by the collection of *unanimity games*.
For each $T \in 2^N \backslash \{\emptyset\}$, the *unanimity game* $u_T \in \mathrm{TU}^N$ is defined by

$$u_T(S) = \begin{cases} 1 & \text{if } T \subset S, \\ 0 & \text{otherwise,} \end{cases}$$

for all $S \in 2^N$.

Theorem 1.1
Every game $v \in TU^N$ can be written in a unique way as a linear combination of unanimity games:

$$v = \sum_{T \in 2^N \backslash \{\emptyset\}} c_T u_T$$

with $c_T \in \mathbb{R}$, $T \in 2^N \backslash \{\emptyset\}$, uniquely determined. In fact, the coefficients c_T, $T \in 2^N \backslash \{\emptyset\}$, satisfy the following recursive formula:

$$c_T = v(T) - \sum_{S \in 2^T \backslash \{\emptyset, T\}} c_S.$$

Proof

For the first part, by the main theorem of linear algebra it is sufficient to prove that the collection of unanimity games is linearly independent. Hence, assume that

$$\sum_{T\in 2^N\setminus\{\emptyset\}} c_T u_T = 0.$$

We have to prove that $c_T = 0$ for all $T \in 2^N \setminus\{\emptyset\}$. Take $S = \{i\}$ with $i \in N$. Then,

$$0 = \sum_{T\in 2^N\setminus\{\emptyset\}} c_T u_T(\{i\}) = c_{\{i\}}$$

implies that $c_{\{i\}} = 0$. Hence, $c_T = 0$ for all $T \in 2^N$ with $|T| = 1$. Now, take $S = \{i, j\}$ with $i, j \in N, i \neq j$. Then,

$$0 = \sum_{T\in 2^N\setminus\{\emptyset\}} c_T u_T(\{i, j\}) = c_{\{i\}} + c_{\{j\}} + c_{\{i,j\}}$$

and hence $c_{\{i,j\}} = 0$. Therefore, $c_T = 0$ for all $T \in 2^N$ with $|T| = 2$. By recursive application of this argument with increasing size $|T|$ it follows that $c_T = 0$ for all $T \in 2^N \setminus\{\emptyset\}$.
For the second part, let $T \in 2^N\setminus\{\emptyset\}$. Then

$$v(T) = \sum_{S\in 2^N\setminus\{\emptyset\}} c_S u_S(T) = \sum_{S\in 2^N\setminus\{\emptyset\}, S\subset T} c_S = c_T + \sum_{S\in 2^T\setminus\{\emptyset, T\}} c_S.$$

□

Exercises

Exercise 1.1

Determine the coalitional values for a glove game a la Example 1.1 where each member of L has 3 left hand gloves and each member of R has 2 right hand gloves.

Exercise 1.2

(a) The security council of the United Nations consists of 5 permanent members and 10 non-permanent members. To pass a resolution, at least 9 (out of 15) member votes to pass are needed, with all 5 permanent members voting to pass the resolution. Represent this situation by means of a simple game.

(b) A simple game $v \in TU^N$ is called a weighted majority game if there exists a treshold $q \geq 0$ and a collection $\{w_i\}_{i\in N}$ of non-negative weights such that

$$w(S) = \begin{cases} 1 & \text{if } \sum_{i\in S} w_i \geq q, \\ 0 & \text{otherwise.} \end{cases}$$

(i) Represent, if possible, the UN-security council game of part (a) as a weighted majority game.

(ii) Construct a 4-person simple game that is not a weighted majority game.

(c) A simple game $v \in TU^N$ is called proper if $\min\{v(S), v(T)\} = 0$ for all $S, T \in 2^N$ with $S \cap T = \emptyset$.

(i) Prove that a simple game is proper if and only if it is superadditive.

For a simple game $v \in TU^N$, let $WC(v) = \{S \in 2^N \setminus \{\emptyset\} | v(S) = 1\}$ be the set of winning coalitions and $BC(v) = \{S \in 2^N \setminus \{\emptyset\} \mid v(N \setminus S) = 0\}$ the set of blocking coalitions.

(ii) Prove that a simple game v is proper if and only if $WC(v) \subset BC(v)$.

Exercise 1.3

(a) Show that the game (N, w) with $N = \{1, 2, 3\}$ and w as given by

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$w(S)$	0	1	2	3	43	24	65	106

is S-equivalent to the cost savings game (N, v) of Example 1.3.

(b) Prove that S-equivalence is an equivalence relation on TU^N (i.e., show reflexivity, symmetry and transitivity).

(c) Prove that each game $v \in \text{TU}^N$ is S-equivalent to a zero-normalized game.

(d) Prove that each game $v \in \text{TU}^N$, with $v(N) > \sum_{i\in N} v(\{i\})$, is S-equivalent to a zero-one normalized game. Where do you use the condition $v(N) > \sum_{i\in N} v(\{i\})$?

Exercise 1.4

(a) Let $N = \{1,2,3,4\}$, $v = 3u_{12} - 2u_{124} + 5u_{34}$ and $w = 3u_{12} - 2u_{124} + 5u_{13.}$ Determine all coalitional values of both v and w.

(b) Let $v \in TU^N$ with $N = \{1,2,3,4\}$ be determined by

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$
$v(S)$	0	0	0	0	6	3	0	0	3	3

S	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	N
$v(S)$	6	6	5	3	11

Decompose v into unanimity games.

(c) Show that each unanimity game is simple and superadditive.

Exercise 1.5 (a digraph competition)
Consider a directed graph $D \subset N \times N$ that is irreflexive (i.e., $(i,i) \notin D$ for all $i \in N$). Such a digraph can be interpreted as reflecting the results of a sports competition in which several teams play matches against each other. In this case the nodes in N correspond to the teams while $(i,j) \in D$ means that team i has won the match played against team j.
Denoting the set of predecessors of i in D by$P_D(i) = \{j \in N | (j,i) \in D\}$, we introduce an associated score game $v_D \in \mathrm{TU}^N$ by

$$v_D = \sum_{j \in N} u_{P_D(j) \cup \{j\}}.$$

Calculate the score game v_D for the diagraph competition D (cf. Figure 1.2), given by

$$N = \{1,2,3,4\} \text{ and } D = \{(1,2),(1,3),(4,1),(2,3),(2,4),(3,4)\}.$$

Suppose that this particular D indeed reflects the results of a sports competition. Can you provide an interpretation for the coalitional values provided by v_D? How would you rank the teams on the basis of the results? In particular, indicate how the game v_D could help to solve this ranking problem.

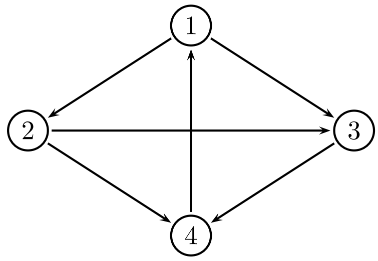


Figure 1.2: The directed graph (N, D) of Exercise 1.5.

Some relevant literature

von Neumann, J. and O. Morgenstern (1944). Theory of Games and Economic Behavior, Princeton University Press, Princeton.

Harsanyi, J.C. (1959). A bargaining model for cooperative n-person games. In: Contributions to the theory of games IV (eds. Tucker, A.W. and R.D. Luce), Princeton University Press, Princeton, pp. 325-355.

Taylor, A. and W. Zwicker (1993). Weighted voting, multicameral representation, and power. *Games and Economic Behavior*, 5, 170-181.

Borm, P. van den Brink, R. and M. Slikker (2002). An iterative procedure for evaluating digraph competitions. *Annals of Operations Research*, 109, 61-75.

CHAPTER 2

Core and balancedness

This section will make a start with analyzing ("solving") TU-games. From now on we are assuming that the players are negotiating about the formation of the grand coalition N and that in the process they are trying to allocate $v(N)$ in a fair and justifiable way among themselves, in particular taking into account the values $v(S)$ of every possible coalition $S \in 2^N$.

Two obvious requirements of an allocation $x \in \mathbb{R}^N$ for a game $v \in \mathrm{TU}^N$ are

(i) *Efficiency:* $\sum_{i \in N} x_i = v(N)$.

(ii) *Individual rationality*: $x_i \geq v(\{i\})$ for all $i \in N$.

Allocations satisfying (i) and (ii) are called *imputations*. The set of all imputations of a game $v \in \mathrm{TU}^N$ is denoted by $I(v)$. Clearly we have

$$I(v) \neq \emptyset \Leftrightarrow v(N) \geq \sum_{i \in N} v(\{i\})$$

Moreover, it is easy to verify that $I(v) = \mathrm{Conv}(\{r^i\}_{i \in N})$, where for each $i \in N$, $r^i \in \mathbb{R}^N$ is defined by

$$r_k^i = \begin{cases} v(\{k\}) & \text{if } k \neq i, \\ v(N) - \sum\limits_{j \in N \setminus \{k\}} v(\{j\}) & \text{if } k = i, \end{cases}$$

for all $k \in N$.

Example 2.1

Let (N, v) be such that $N = \{1, 2, 3\}$, $v(\{1\}) = v(\{3\}) = 0$, $v(\{2\}) = 3$, $v(\{1, 2\}) =$

$v(\{2,3\}) = 4$, $v(\{1,3\}) = 1$ and $v(\{1,2,3\}) = 6$. Then $r^1 = (3,3,0)$, $r^2 = (0,6,0)$ and $r^3 = (0,3,3)$. The imputation set $I(v)$ is given by

$$I(v) = \mathrm{Conv}(\{(3,3,0),(0,6,0),(0,3,3)\}). \qquad \diamond$$

Next we introduce one of the most fundamental concepts within the theory of cooperative games. The *core* $\mathrm{C}(v)$ of a game $v \in \mathrm{TU}^N$ is defined by

$$\mathrm{C}(v) = \left\{ x \in \mathbb{R}^N \,\middle|\, \sum_{i\in N} x_i = v(N), \sum_{i\in S} x_i \geq v(S) \text{ for all } S \in 2^N \right\}$$

So, core elements are imputations (i.e. efficient and individually rational) which are stable against coalitional deviations. No coalition can rightfully object to a proposal $x \in \mathrm{C}(v)$ because what this coalition is allocated in total according to x (i.e. $\sum_{i\in S} x_i$) is at least what it can obtain by splitting off from the grand coalition (i.e. $v(S)$). In particular, if $\sum_{i\in S} x_i > v(S)$, then in any division of $v(S)$ among the members of S, at least one player gets strictly less then what he gets according to x.

Example 2.2
For the game (N, v) of Example 2.1 the core is given by

$$\mathrm{C}(v) = \mathrm{Conv}(\{(2,4,0),(1,5,0),(0,5,1),(0,4,2),(1,3,2),(2,3,1)\})$$

and represented in Figure 2.1. $\diamond$

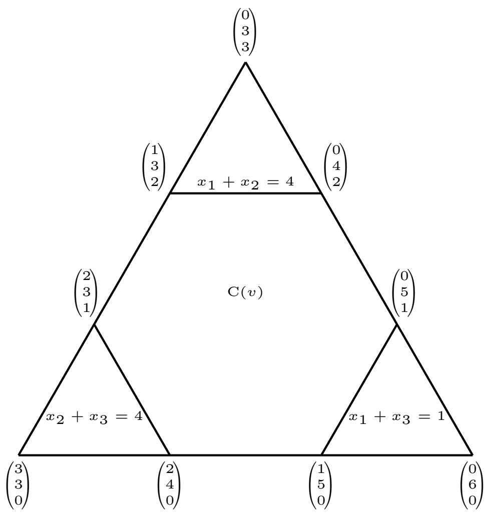


Figure 2.1: The core of the three-person game of Example 2.1.

In general, since the core is bounded (why exactly?) and is determined by a finite system of (weak) linear inequalities, it is a *polytope*: a convex hull of finitely many points. Moreover, it is not difficult to check that the core is representation-independent. More specifically the core satisfies *relative invariance* with respect to S-equivalence, i.e. if $w = kv + a$ $(k > 0, a \in \mathbb{R}^N)$, then $x \in \mathrm{C}(v)$ implies that $kx + a \in \mathrm{C}(w)$.

Unfortunately, a game can have an empty core as can be seen in Exercise 2.1. A general characterization of games with a non-empty core is provided by the notion of balancedness. For this we first need to define balanced maps and balanced collections. A map $\lambda : 2^N \backslash \{\emptyset\} \to [0, \infty)$ is called *balanced for N* if

$$\sum_{S \in 2^N \backslash \{\emptyset\}} \lambda(S) e^S = e^N$$

where, for each $S \subset N$, $S \neq \emptyset$, $e^S \in \mathbb{R}^N$ is such that

$$e_i^S = \begin{cases} 1 & \text{if } i \in S, \\ 0 & \text{otherwise,} \end{cases}$$

for all $i \in N$. A balanced map assigns non-negative weights to coalitions in such a way that every player has a total weight of exactly 1. A collection $\mathcal{B} \subset 2^N \backslash \{\emptyset\}$ of coalitions is called *balanced for N* if there is a balanced map λ on N such that $\mathcal{B}(\lambda) = \mathcal{B}$, where $\mathcal{B}(\lambda)$ is defined by

$$\mathcal{B}(\lambda) = \{S \in 2^N \backslash \{\emptyset\} | \lambda(S) > 0\}.$$

For example, the collection $\{\{1,2\}, \{1,3\}, \{2,3\}\}$ is balanced for $\{1,2,3\}$ since the map $\lambda : 2^N \backslash \{\emptyset\} \to \mathbb{R}$ defined by

$$\lambda(S) = \begin{cases} \frac{1}{2} & \text{if } S \in \{\{1,2\}, \{1,3\}, \{2,3\}\}, \\ 0 & \text{otherwise} \end{cases}$$

is balanced because $\frac{1}{2}(1,1,0) + \frac{1}{2}(1,0,1) + \frac{1}{2}(0,1,1) = (1,1,1)$, and $\lambda(S) > 0$ if and only if $S \in \{\{1,2\}, \{1,3\}, \{2,3\}\}$.

A game $v \in \mathrm{TU}^N$ is called *balanced* if for every balanced map λ for N we have that

$$\sum_{S \in 2^N \backslash \{\emptyset\}} \lambda(S) v(S) \leq v(N)$$

Theorem 2.1 (Bondareva/Shapley).
For $v \in TU^N$:

$\mathrm{C}(v) \neq \emptyset \Leftrightarrow v$ *is balanced.*

In the proof of this theorem we use the following duality result.

Lemma 2.2
Let A be an $m \times n$ matrix, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}^m$.
If $\{y \in \mathbb{R}^n | Ay = c,\ y \geq 0\} \neq \emptyset$ and $\{x \in \mathbb{R}^m | x^T A \geq b^T\} \neq \emptyset$, then

$$\min\{x^T c | x \in \mathbb{R}^m, x^T A \geq b^T\} = \max\{b^T y | y \in \mathbb{R}^n, Ay = c, y \geq 0\}.$$

Proof of Theorem 2.1

$$\begin{aligned}
\mathrm{C}(v) \neq \emptyset \quad &\Leftrightarrow \quad v(N) = \min\{\sum_{i\in N} x_i | x \in \mathbb{R}^N, \sum_{i \in S} x_i \geq v(S) \text{ for all } S \in 2^N \backslash \{\emptyset\}\} \\
&\Leftrightarrow \quad v(N) = \max\{\sum_{S \in 2^N \backslash \{\emptyset\}} \lambda(S) v(S) | \lambda(S) \geq 0 \text{ for all } S \in 2^N \backslash \{\emptyset\}, \\
&\qquad\qquad \sum_{S \in 2^N \backslash \{\emptyset\}} \lambda(S) e^S = e^N\} \\
&\Leftrightarrow \quad v \text{ is balanced}
\end{aligned}$$

At the second equivalence we use Lemma 2.2 with A being the $|N| \times (2^{|N|} - 1)$-matrix in which the column corresponding to coalition S equals e^S, $c = e^N$ and b is a $2^{|N|} - 1$-vector with the entry corresponding to coalition S equal to $v(S)$. □

For a game $v \in \mathrm{TU}^N$ and $T \in 2^N \setminus \{\emptyset\}$, the *subgame* $v|_T \in \mathrm{TU}^T$ is defined by

$$v|_T(S) = v(S)$$

for all $S \in 2^T \backslash \{\emptyset\}$. Note that $v|_N = v$. A game $v \in \mathrm{TU}^N$ is called *totally balanced* if every subgame of v is balanced, i.e. if every subgame has a non-empty core.

Next define Λ^N as the collection of all balanced maps for N. Note that a balanced map $\lambda \in \Lambda^N$ can be identified with a vector of length $2^{|N|} - 1$. Since Λ^N is bounded (all values are between 0 and 1) and determined by a finite number of (weak) linear inequalities, Λ^N is a polytope. Moreover, a balanced collection $\mathcal{B}$ is called *minimally balanced* for N if there does not exist a balanced collection $\mathcal{B}^1$ for N with $\mathcal{B}^1 \subsetneq \mathcal{B}$. Correspondingly, a balanced map λ is called *minimally balanced* for N if $\mathcal{B}(\lambda)$ is a minimally balanced collection for N.

Theorem 2.3

(i) $\Lambda^N = \mathrm{Conv}(\{\lambda \in \Lambda^N \mid \lambda$ *is minimally balanced for* $N\})$.

(ii) A game $v \in TU^N$ is balanced if and only if

$$\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S)v(S) \leq v(N)$$

for each minimally balanced map λ for N.

In the proof of this theorem we use the following result about compact and convex sets.

Lemma 2.4 (Krein-Milman)
If $C \subset \mathbb{R}^m$ is compact and convex, then $C = \mathrm{Conv}(ext(C))$. Here, $ext(C)$ denotes the set of extreme points of C, i.e. $ext(C)$ is the set of all $c \in C$ for which there are no $c_1, c_2 \in C$ with $c_1 \neq c_2$ such that c is a convex combination of c_1 and c_2.

Proof of Theorem 2.3
Let $v \in TU^N$. We will first prove the following claim.

Claim 1: If $\mathcal{B}$ is a minimally balanced collection for N, then there exists a unique balanced map λ for N such that $\mathcal{B}(\lambda) = \mathcal{B}$.

Proof of Claim 1
Let $\mathcal{B}$ be a minimally balanced collection for N. Suppose $\lambda_1, \lambda_2 \in \Lambda^N$ are such that $\mathcal{B}(\lambda_1) = \mathcal{B}$, $\mathcal{B}(\lambda_2) = \mathcal{B}$ and $\lambda_1 \neq \lambda_2$. Take $T \in \mathcal{B}$ such that $\frac{\lambda_1(T)}{\lambda_2(T)} \leq \frac{\lambda_1(S)}{\lambda_2(S)}$ for all $S \in \mathcal{B}$. Note that $0 < \frac{\lambda_1(T)}{\lambda_2(T)} < 1$. Define $\bar{\lambda}(S)$ by

$$\bar{\lambda}(S) = \frac{\lambda_2(T)}{\lambda_2(T) - \lambda_1(T)}\lambda_1(S) - \frac{\lambda_1(T)}{\lambda_2(T) - \lambda_1(T)}\lambda_2(S)$$

for all $S \in 2^N \setminus \{\emptyset\}$. One readily verifies that $\bar{\lambda} \in \Lambda^N$ while $\mathcal{B}(\bar{\lambda}) \subset \mathcal{B}$. However, $\bar{\lambda}(T) = 0$ and hence $\mathcal{B}(\bar{\lambda}) \subsetneqq \mathcal{B}$, contradicting the fact that $\mathcal{B}$ is minimally balanced.

Next we prove the following claim

Claim 2: $\mathrm{Ext}(\Lambda^N) = \{\lambda \in \Lambda^N | \lambda \text{ is minimally balanced for } N\}$.
Note that Claim 2 is stronger than Theorem 2.3(i) by Lemma 2.4.

Proof of Claim 2
For the "$\supset$"-part of Claim 2, let $\lambda \in \Lambda^N$ be minimally balanced for N. Suppose $\lambda \notin \mathrm{Ext}(\Lambda^N)$. Then, there exist $\lambda_1, \lambda_2 \in \Lambda^N$ with $\lambda_1 \neq \lambda_2$ such that $\lambda = \alpha\lambda_1 + (1-\alpha)\lambda_2$ with $\alpha \in (0, 1)$. Clearly, this implies $\mathcal{B}(\lambda_1) \subset \mathcal{B}(\lambda)$ and $\mathcal{B}(\lambda_2) \subset \mathcal{B}(\lambda)$. Consequently, since λ is minimally

balanced, we have that $\mathcal{B}(\lambda_1) = \mathcal{B}(\lambda_2) = \mathcal{B}(\lambda)$. Then however, using Claim 1, it should be the case that $\lambda = \lambda_1 = \lambda_2$, a contradiction.

For the "$\subset$"-part of Claim 2, let $\lambda \in \text{Ext}(\Lambda^N)$. Suppose λ is not minimally balanced for N. Then, $\mathcal{B}(\lambda)$ is not a minimally balanced collection for N and hence there exists a balanced collection $\mathcal{B}$ for N such that $\mathcal{B} \subsetneqq \mathcal{B}(\lambda)$. Choose a balanced map μ such that $\mathcal{B}(\mu) = \mathcal{B}$. Take $T \in \mathcal{B}(\mu)$ such that $\frac{\lambda(T)}{\mu(T)} \leq \frac{\lambda(S)}{\mu(S)}$ for all $S \in \mathcal{B}(\mu)$. Note that $0 < \frac{\lambda(T)}{\mu(T)} < 1$ (why exactly?). Define $\tilde{\lambda}(S)$ by

$$\tilde{\lambda}(S) = \frac{\mu(T)}{\mu(T) - \lambda(T)}\lambda(S) - \frac{\lambda(T)}{\mu(T) - \lambda(T)}\mu(S)$$

for all $S \in 2^N \setminus \{\emptyset\}$. One readily checks that $\tilde{\lambda} \in \Lambda^N$ while $\tilde{\lambda}(T) = 0$, $\mu(T) > 0$, and

$$\lambda = \frac{\lambda(T)}{\mu(T)}\mu + \left(1 - \frac{\lambda(T)}{\mu(T)}\right)\tilde{\lambda},$$

contradicting the fact that $\lambda \in \text{Ext}(\lambda^N)$.

With respect to Theorem 2.3 (ii), the "only if" part is obvious. For the "if" part, take an arbitrary balanced map λ for N, not necessarily minimally balanced. According to Theorem 2.3 (i), there exists a $t \in \mathbb{N}$, $\alpha_1, \alpha_2, \ldots, \alpha_t \in [0,1]$ with $\sum_{i=1}^t \alpha_i = 1$ and minimally balanced maps $\lambda_1, \lambda_2, \ldots, \lambda_t$ for N such that $\lambda = \sum_{i=1}^t \alpha_i \lambda_i$. Hence,

$$\begin{aligned}
\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda(S)v(S) &= \sum_{S \in 2^N \setminus \{\emptyset\}} \left(\sum_{i=1}^t \alpha_i \lambda_i(S)v(S)\right) \\
&= \sum_{i=1}^t \alpha_i \left(\sum_{S \in 2^N \setminus \{\emptyset\}} \lambda_i(S)v(S)\right) \\
&\leq \sum_{i=1}^t \alpha_i \left(v(N)\right) \\
&= v(N).
\end{aligned}$$

□

Exercises

Exercise 2.1
Calculate the core of

(a) the glove market game of Example 1.1 with $N = \{1, 2, 3\}$, $L = \{1\}$, and $R = \{2, 3\}$,

(b) the unanimity game (N, u_T) with $N = \{1, 2, 3\}$ and $T = \{1, 2\}$,

(c) the spanning tree game of Example 1.3,

(d) the game (N, v) with $N = \{1, 2, 3\}$ and $v(S) = \begin{cases} 1 & \text{if } |S| \geq 2, \\ 0 & \text{otherwise.} \end{cases}$

(e) Consider the game $v \in \mathrm{TU}^N$ with $N = \{1, 2, 3\}$ given by

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	0	10	10	30	a

with $a \in R$.

(i) Calculate $\mathrm{C}(v)$ for $a = 20$, $a = 30$, and $a = 40$.

(ii) Determine $\mathrm{C}(v)$ as a polytope by giving its extreme points for arbitrary a.

Exercise 2.2

Let $N = \{1, 2, 3, 4\}$.

(a) Calculate $\mathrm{C}(v)$ for $v \in TU^N$ given by

$$v(S) = \begin{cases} 0 & \text{if } |S| \leq 2, \\ 6 & \text{if } S = \{1,2,3\} \text{ or } S = \{1,2,4\}, \\ 5 & \text{if } S = \{1,3,4\}, \\ 3 & \text{if } S = \{2,3,4\}, \\ 11 & \text{if } S = N. \end{cases}$$

(b) Calculate $\mathrm{C}(v_1)$ for $v_1 \in TU^N$ given by

$$v_1(S) = \begin{cases} 6 & \text{if } S = \{1,2\}, \\ v(S) & \text{otherwise.} \end{cases}$$

(c) Calculate $\mathrm{C}(v_2)$ for $v_2 \in TU^N$ given by

$$v_2(S) = \begin{cases} 3 & \text{if } S = \{1,3\}, \\ v_1(S) & \text{otherwise.} \end{cases}$$

Make pictures!

Exercise 2.3

(a) Let $v, w \in \mathrm{TU}^N$ be two balanced games. Let $x \in \mathrm{C}(v)$ and $y \in \mathrm{C}(v)$. Show that $x + y \in \mathrm{C}(v + w)$.

(b) Provide an example of two balanced games $v, w \in \mathrm{TU}^N$ and an allocation $z \in \mathrm{C}(v+w)$ for which there does not exist $x \in \mathrm{C}(v)$ and $y \in \mathrm{C}(w)$ such that $z = x + y$.

Exercise 2.4

(a) Give a 3-person TU-game that is superadditive but not balanced.

(b) Give a 3-person TU-game that is balanced but not superadditive.

(c) Let $v \in TU^N$ be balanced. Show that v is N-superadditive, i.e.

$$v(N) \geq v(S) + v(N \setminus S)$$

for all $S \in 2^N$.

(d) Show that each totally balanced game is superadditive.

Exercise 2.5 (Simple games)
Let (N, v) be a simple game and define the set veto(v) of veto-players by

$$\mathrm{veto}(v) = \bigcap \{S \in 2^N | v(S) = 1\}.$$

E.g. in the UN-security council the 5 permanent members are veto-players, the other members are not.

(a) Give a 3-person simple game w such that $\mathrm{veto}(w) = \emptyset$. Calculate the core of this game.

(b) Given a 3-person simple game w such that $\mathrm{veto}(w) \neq \emptyset$. Calculate the core of this game.

(c) Prove that $\mathrm{Conv}(\{e^{\{i\}} | i \in \ \mathrm{veto}(v)\}) = \mathrm{C}(v)$.

Hence, in particular, v is balanced if and only if $\mathrm{veto}(v) \neq \emptyset$.

(d) Prove that v is balanced if and only if v is totally balanced.

Exercise 2.6

(a) Determine all 21 balanced collections for $N = \{1,2,3\}$ which do not contain N.

(b) How many balanced collections for $N = \{1,2,3\}$ are there in total?

(c) Determine all minimally balanced collections for $N = \{1,2,3\}$.

(d) Determine all balanced collections $\mathcal{B}$ for $N = \{1,2,3,4\}$ with $|\mathcal{B}| = 2$ or $|\mathcal{B}| = 3$.

Exercise 2.7
Let $\mathcal{B}$ be balanced for N such that $N \notin \mathcal{B}$. Let λ be a balanced function such that $\mathcal{B}(\lambda) = \mathcal{B}$.

(a) Show that $\sum_{S \in \mathcal{B}} \lambda(S) > 1$.

(b) Prove that $\mathcal{B}^c = \{S \in 2^N \backslash \{\emptyset\} | N \backslash S \in \mathcal{B}\}$ is also balanced for N. Hint: show that for μ defined by

$$\mu(S) = \frac{\lambda(N \backslash S)}{\sum\limits_{T \in \mathcal{B}} \lambda(T) - 1} \quad (S \in \mathcal{B}^c),$$

we have that $\mu \in \Lambda^N$ and $\mathcal{B}^c = \mathcal{B}(\mu)$.

Exercise 2.8
Let $\mathcal{B}$ be a minimally balanced collection for N. Show that $|\mathcal{B}| \leq |N|$.

Exercise 2.9 (Dominance core)
Let $v \in \mathrm{TU}^N$. The *dominance core* $\mathrm{DC}(v)$ is defined by

$$\mathrm{DC}(v) = \{x \in I(v) |\text{ there are no } y \in I(v) \text{ and } S \in 2^N \backslash \{\emptyset\} \text{ such that } \\ y_i > x_i \text{ for all } i \in S, \text{ and } \sum_{i \in S} y_i \leq v(S)\}.$$

(a) Let $N = \{1,2,3\}$, $v(N) = 1$, $v(\{1,2\}) = 2$ and $v(S) = 0$ otherwise. Show that $\mathrm{C}(v) = \emptyset$ and $\mathrm{DC}(v) = \mathrm{Conv}(\{(1,0,0),(0,1,0)\})$.

(b) Let $N = \{1,2,3\}$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$ and $v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\})$ $= v(N) = 1$. Show that $\mathrm{DC}(v) = \emptyset$.

(c) Prove that $\mathrm{C}(v) \subset \mathrm{DC}(v)$.

(d) Let $v \in \mathrm{TU}^N$ be such that $v(N) \geq v(S) + \sum_{i \in N \setminus S} v(\{i\})$ for all $S \in 2^N$, which is weaker than superadditivity. Show that under this condition $\mathrm{C}(v) = \mathrm{DC}(v)$.

Exercise 2.10 (Stable sets)

Let $v \in \mathrm{TU}^N$. A subset M of $I(v)$ is called *stable* (or a *von Neumann-Morgenstern solution)* if $M \cap Dom(M) = \emptyset$ *(internal stability)* and $I(v) \backslash M \subset Dom(M)$ *(external stability)*, where

$$\mathrm{Dom}(M) = \{x \in I(v) \,\big|\, \text{There are } y \in M \text{ and } S \in 2^N \backslash \{\emptyset\} \text{ such that } y_i > x_i \text{ for all } i \in S \text{ and } \sum_{i \in S} y_i \leq v(S)\}.$$

(a) Verify that $\mathrm{DC}(v) = I(v) \backslash Dom(I(v))$, cf. Exercise 2.9.

(b) Show that $\mathrm{DC}(v) \subset M$ for any stable set M.

(c) Prove the following statement: if $\mathrm{DC}(v)$ is a stable set, then there are no other stable sets.

Consider the game $v \in \mathrm{TU}^N$ with $N = \{1, 2, 3\}$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$ and $v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = v(N) = 1$.

(d) Show that $M = \{(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2}), (0, \frac{1}{2}, \frac{1}{2})\}$ is stable for v.

(e) Show that $M = \{x \in I(v) | x_1 = \frac{3}{4}\}$ is not stable for v.

(f) Show that $M = \{x \in I(v) | x_1 = \frac{1}{4}\}$ is stable for v.

We just remark that there exist TU-games without stable sets.

Some relevant literature

Bondareva, O. (1963). Some applications of linear programming methods to the theory of cooperative games. *Problemy Kibernet*, 10, 119-139 (In Russian).

Shapley, L.S. (1967). On balanced sets and cores. *Naval Research Logistics Quarterly*, 14, 453-460.

Peleg, B. (1965). An inductive method for constructing minimal balanced collections of finite sets. *Naval Research Logistics Quarterly*, 12, 155-162.

CHAPTER 3

Operations Research games

Since the early developments of operations research and game theory there has been a strong interplay between the two disciplines. Especially the interrelation between operations research and noncooperative game theory is well-known: between duality results in mathematical programming and minimax results for zero-sum games, between linear complementary and bimatrix games, between Markov decision processes and stochastic games, and between optimal control theory and differential games.

The interrelation between operations research and cooperative game theory is of a more recent date and is summarised under the heading of operations research games. One can say that an important part of the interplay between cooperative games and operations research stems from the basic (discrete) structure of a graph, network or system that underlies various types of combinatorial optimisation problems. If one assumes that at least two players are located at or control parts (e.g., vertices, edges, resource bundles, jobs) of the underlying system, then a cooperative game can be associated with this type of optimisation problem. In working together, the players can possibly create extra gains or save costs compared to the situation in which everybody optimises individually. Hence the question arises how to share the extra revenues or cost savings.

One way to analyse this question is to study the general properties (e.g., balancedness) of all games arising from that specific type of operations research problem and to apply a suitable existing game theoretic solution concept (e.g., core) to this class. Another way is to create a context specific allocation rule. Such a rule can be based either on desirable properties in this specific context or on a kind of decentralised mechanism that prescribes an allocation on the basis of the algorithmic process along which a jointly optimal combinatorial structure is

established.
As an illustration this section will consider four specific classes of operations research games: spanning tree games, flow games, sequencing games and linear production games. Other examples found in the literature are travelling salesman games, Chinese postman (or delivery) games, assignment games and inventory games.

3.1 Spanning Tree games

Consider a group of villages, each of which needs to be connected to some source, either directly or via other villages. Every possibly connection has some (nonnegative) costs associated to it and the problem is how to connect every village to the source such that the total joint costs of the created network are minimal. Since connection costs are nonnegative, a minimal cost graph that connects all villages to the source is indeed a tree, which explains the name spanning tree problem. The OR literature provides several algorithms (Prim, Kruskal) for solving this kind of problem. Constructing a minimum cost spanning tree, however, is only part of the problem. In addition to minimising total joint costs, a cost allocation problem has to be addressed as well.

Formally, a *spanning tree problem* is a triple $(N, 0, t)$, where $N = \{1, \ldots, n\}$ is the player set, 0 is the source and $t : E_{N\cup\{0\}} \to \mathbb{R}_+$ is a nonnegative cost function. E_S is defined as the set of all edges between pairs of elements of $S \subset N \cup \{0\}$, so that (S, E_S) is the complete graph on S :

$$E_S = \{\{i, j\} | i, j \in S,\ i \neq j\}.$$

Given a spanning tree problem $(N, 0, t)$ and a minimum cost spanning tree $(N \cup \{0\}, R)$ for the grand coalition N, *Bird's tree allocation* β is constructed by assigning to each player $i \in N$ the cost of the first edge e_i on the unique path in $(N \cup \{0\}, R)$ from player i to the source 0. The computation of this allocation can be integrated into the Prim algorithm, which, starting from the source, constructs an minimum cost spanning tree by consecutively adding edges with the lowest cost, without introducing cycles.

Algorithm (Bird's cost allocation)
Input: a spanning tree problem $(N, 0, t)$.
Output: an edge set $R \subset E_{N\cup\{0\}}$ of a minimum cost spanning tree and a corresponding Bird allocation $\beta \in \mathbb{R}^N$.

1. Choose the source 0 as root.

2. Initialise $R = \emptyset$.

3. Find a minimal cost edge $\{i, j\} \in E_{N\cup\{0\}}\backslash R$ incident on 0 or any of the vertices present in one of the edges in R in such a way that joining $\{i, j\}$ to R does not introduce a cycle.

4. With $e = \{i, j\}$ being the edge of step 3, one of i and j, say j, was previously connected to the source and the other vertex i is a player who was not yet connected to the source. Define $e_i = \{i, j\}$ and assign the cost $t(e_i)$ to agent i, i.e. $\beta_i = t(e_i)$.

5. Join e_i to R.

6. If not all vertices are connected to the source in the graph $(N \cup \{0\}, R)$, go back to step 3.

The following example illustrates the algorithm.

Example 3.1
Consider the spanning tree problem with $N = \{1, 2, 3\}$ as presented in Figure 3.1, where the numbers on the edges represent the costs.

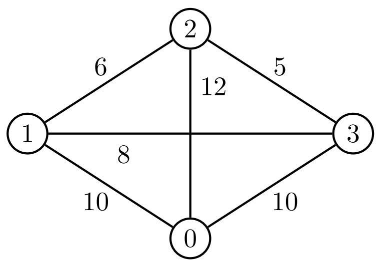


Figure 3.1: The spanning tree problem of Example 3.1.

When we apply the algorithm to this problem, the first edge we join to R is either $\{0, 1\}$ or $\{0, 3\}$. Suppose we choose the first one, then $e_1 = \{0, 1\}$ and $\beta_1 = 10$. Subsequently, we add $\{1, 2\}$ to R, set $e_2 = \{1, 2\}$ and define $\beta_2 = 6$. Next we add $\{2, 3\}$ to R, set $e_3 = \{2, 3\}$ and $\beta_3 = 5$. This gives us a Bird cost allocation vector of (10,6,5). On the other hand, suppose we start with $\{0, 3\}$. Then we end up with a Bird cost allocation vector of $(6, 5, 10)$. The corresponding two minimum cost spanning trees are drawn in Figure 3.2. $\diamond$

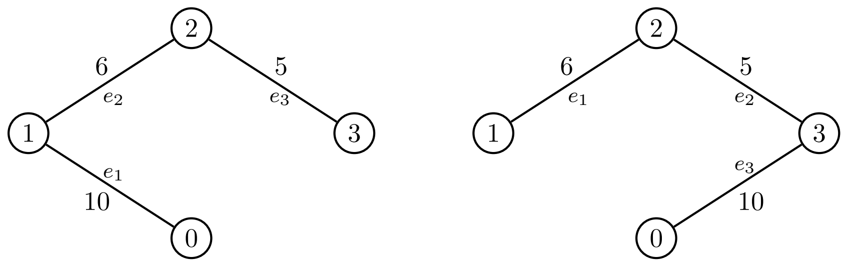


Figure 3.2: The two minimum cost spanning trees of Example 3.1.

With each spanning tree problem $(N, 0, t)$ we can associate a cooperative cost game (N, c) in which $c(S)$ represents the minimal costs of a spanning tree on $S \cup \{0\}$:

$$c(S) := \min\{\sum_{e\in R} t(e) | R \subset E_{S\cup\{0\}} \text{ and } (S \cup \{0\}, R) \text{ is a tree}\}$$

for all $S \subset N$, $S \neq \emptyset$.

Theorem 3.1

Let $(N, 0, t)$ be a minimum cost spanning tree problem and (N, c) the corresponding cost game. Then, for every minimum cost spanning tree $(N \cup \{0\}, R)$, the corresponding Bird's allocation vector β satisfies

$$\sum_{i\in N} \beta_i = c(N)$$

and

$$\sum_{i\in S} \beta_i \leq c(S) \text{ for all } S \in 2^N \backslash \{\emptyset\}.$$

Consequently the vector $(c(\{i\}) - \beta_i)_{i\in N}$ is an element of the core of corresponding cost savings game $v \in TU^N$ defined by $v(S) = \sum_{i\in S} c(\{i\}) - c(S)$ for all $S \in 2^N$.

Proof

Let β be a Bird's allocation vector corresponding to a minimum cost spanning tree $(N \cup \{0\}, R)$. By construction

$$\sum_{i\in N} \beta_i = \sum_{i\in N} t(e_i) = \sum_{e\in R} t(e) = c(N).$$

Let $S \in 2^N \backslash \{\emptyset\}$. Consider a tree $(S \cup \{0\}, F)$ such that $F \subset E_{S\cup\{0\}}$ and $c(S) = \sum_{e\in F} t(e)$. Define $G := \{e_i \in R | i \in N\backslash S\}$. Clearly, $F \cap G = \emptyset$ and $(N \cup \{0\},\ F \cup G)$ is a tree (why exactly?).

Hence,

$$c(N) \leq \sum_{e\in F\cup G} t(e) = \sum_{e\in F} t(e) + \sum_{e\in G} t(e) = c(S) + \sum_{i\in N\backslash S} \beta_i$$

and, consequently,

$$\sum_{i \in S} \beta_i = c(N) - \sum_{i \in N \setminus S} \beta_i \leq c(S).$$

The rest is obvious. □

3.2 Flow games

Consider a flow problem between specific nodes (called source and sink) via a given network of arcs, each having its own maximal capacity per time unit. The corresponding optimization problem is to find an optimal flow in which the total inflow for each intermediate node equals the total outflow in such a way that the total flow from source to sink per time unit is maximized. According to the theorem of Ford-Fulkerson, the maximal total flow equals the minimal total capacity of a cut, where a cut is a collection of arcs such that no flow is possible from source to sink within the subnetwork of all arcs outside the cut. If one assumes that a set of players governs the arcs, i.e. if the use of an arc is restricted to specific coalitions only, then an allocation problem arises. How to allocate the revenues corresponding to a maximal total flow among the players?

Formally, a *flow problem* is a tuple $(N, V, A, t, \{w^a\}_{a \in A})$ where $N = \{1, 2, \ldots, n\}$ is a set of players, $V = \{v_0, v_1, \ldots, v_m\}$ is a finite set of nodes with v_0 representing the source and v_m representing the sink,

$$A \subset \{(v_k, v_l) | k, l \in \{0, 1, \ldots, m\}, k \neq l\}$$

a set of arcs, $t : A \to \mathbb{R}_+$ a capacity function where $t(a)$ represents the maximal capacity of arc a, and, for each $a \in A$, $w^a \in \mathrm{TU}^N$ is a *control game* with $w^a(S) \in \{0, 1\}$ for all $S \in 2^N$ and $w^a(N) = 1$. The interpretation of the control games is that coalition S can use arc a to establish a flow from source to sink if and only if $w^a(S) = 1$. Note that a control game generalizes the notion of a simple game in the sense that the monotonicity requirement of simple games is dropped.

Given a flow problem $(N, V, A, t, \{w^a\}_{a \in A})$ and defining $A(S) = \{a \in A | w^a(S) = 1\}$ as the set of arcs that coalition S can use, the associated *flow game* $v \in \mathrm{TU}^N$ is determined by defining $v(S)$ to be equal to the maximal total flow from source to sink via the network $(V, A(S))$. Note that by the Theorem of Ford-Fulkerson $v(S)$ also equals the minimal total capacity of a cut in the network $(V, A(S))$. This definition implicitly assumes that the revenues of a flow depend linearly on the amount transported from source to sink.

Example 3.2

Consider the 3-person flow problem as represented in Figure 3.3. On each arc, the capacity and corresponding control game are depicted. For the associated flow game v, we find the following coalitional values.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	6	7	0	10

Note that $\mathrm{C}(v) = \mathrm{Conv}(\{(10,0,0),(6,0,4),(7,3,0),(3,3,4)\})$.

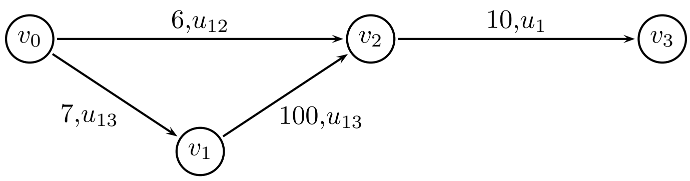


Figure 3.3: The flow problem of Example 3.2.

$\diamond$

Example 3.3

Consider the 3-person flow problem as represented in Figure 3.4.

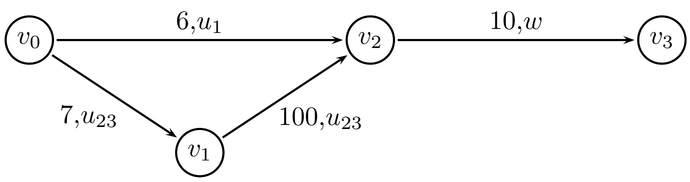


Figure 3.4: The flow problem of Example 3.3.

The control game w is given by $w(S) = 1$ for all $S \in 2^N \setminus \{\emptyset\}$, representing public use of the arc (v_2, v_3). The associated flow game v is given below.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	6	0	0	6	6	7	10

Note that $\mathrm{C}(v) = \emptyset$. $\diamond$

Examples 3.2 and 3.3 indicate that balancedness of a flow game depends on the underlying control games. In particular, the flow game of Example 3.2 is balanced while all underlying control games are balanced too. The flow game of Example 3.3 however is not balanced while one of the underlying control games is not balanced either.

Note that in analogy to Exercise 2.5 on simple games one can show that a control game w is balanced if and only if there is at least one player $i \in N$ such that $i \in S$ for all S with $w(S) = 1$ (a veto player). Moreover, core elements of w allocate $w(N) = 1$ in an arbitrary (non-negative) way among the veto players, while non-veto players obtain 0.

Theorem 3.2
Let $(N, V, A, t, \{w^a\}_{a\in A})$ be a flow problem and let $v \in TU^N$ be the corresponding flow game. If all control games w^a, $a \in A$, are balanced, then the flow game v is balanced too.

Proof
Let w^a be balanced for all $a \in A$. Choose $x^a \in \mathrm{C}(w^a)$ for each $a \in A$. Determine a cut $C \subset A$ such that $v(N) = \sum_{a\in C} t(a)$. Define $x \in \mathbb{R}^N$ by $x_i = \sum_{a\in C} x_i^a t(a)$ for all $i \in N$. We show that $x \in \mathrm{C}(v)$.
Efficiency follows from

$$\sum_{i\in N} x_i = \sum_{i\in N}\sum_{a\in C} x_i^a t(a) = \sum_{a\in C} t(a) \sum_{i\in N} x_i^a = \sum_{a\in C} t(a) = v(N).$$

Stability follows from the fact that for all $S \in 2^N$,

$$\sum_{i\in S} x_i = \sum_{i\in S}\sum_{a\in C} x_i^a t(a) = \sum_{a\in C} t(a) \sum_{i\in S} x_i^a \geq \sum_{a\in C} t(a) w^a(S) = \sum_{a\in C: w^a(S)=1} t(a) \geq v(S),$$

where the last inequality holds because $\{a \in C | w^a(S) = 1\}$ is a cut in the network $(V, A(S))$.
$\square$

For the flow problem of Example 3.2 there is a unique cut $\{(v_2, v_3)\}$ with total minimum capacity. Since the corresponding control game u_1 has a unique core element $(1, 0, 0)$, it follows from the construction in the proof of Theorem 3.2 that $(10, 0, 0)$ belongs to the core of the corresponding flow game. Note that none of the other core elements can be reached in this way. In fact, the reverse of Theorem 3.2 is also true.

Theorem 3.3
Every non-negative and balanced TU-game is a flow game.

Proof

Let $v \in \mathrm{TU}^N$ be such that v is balanced and $v \geq 0$. Decompose v into finitely many balanced control games, i.e. find a natural number $s \in \mathbb{N}$, positive numbers $t_1, t_2, \ldots, t_s$, and balanced control games $w_1, w_2, \ldots, w_s$ such that $v = \sum_{k=1}^{s} t_k w_k$. In Exercise 3.4 we will see that it is possible. Consider the flow problem represented in Figure 3.5. Clearly, the flow game associated to this problem equals v.

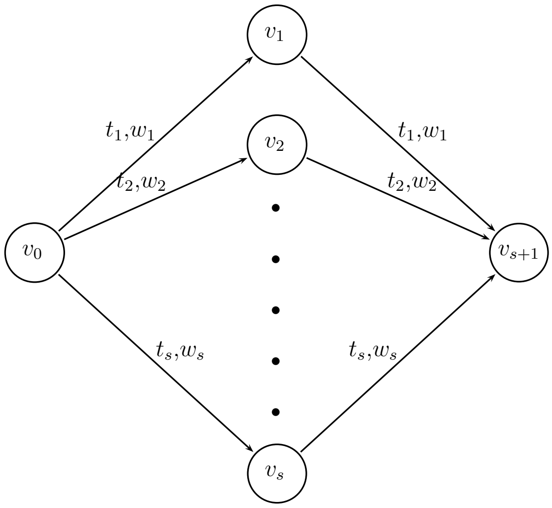


Figure 3.5: The flow problem used in the proof of Theorem 3.3.

□

3.3 Sequencing games

In a deterministic one-machine sequencing problem a finite collection of jobs has to be processed on a single machine. Each job is specified by its processing time, the time the machine needs to handle the job, and its cost function which, for our purposes , is supposed to depend linearly on the completion time of the job. Further restrictive assumptions as e.g. ready times or due dates are not present in the basic sequencing model discussed in this section.

Within this setting it is well-known that the total costs are minimized by processing the jobs in order of decreasing *urgency*, the quotient of the linear cost parameter and the processing time. Such an order will be called a *Smith order*.

Associating each job to a player and assuming there exists an initial order on the jobs (representing processing rights), an optimal Smith order can only be reached via cooperation and the joint cost savings that can be obtained by switching from the initial order to an optimal one, have to be allocated among the players.

A *sequencing problem* can be summarized by a tuple $(N, \sigma_0, \{p_i\}_{i\in N}, \{\alpha_i\}_{i\in N})$. N is the finite set of n players, each with one job to be processed on the single machine. A processing order is described by a bijection $\sigma : \{1,\ldots,n\} \to N$. More specifically $\sigma(k) = i$ means that the job of player i is processed in position k. Let $\Pi(N)$ denote the set of all such processing orders. The order $\sigma_0 \in \Pi(N)$ specifies the initial processing order. For each $i \in N$, $p_i > 0$ specifies the processing time of the job of player i. Finally, for each $i \in N$, the cost of player i of spending t time units in the system is assumed to be determined by a linear cost function $c_i : [0,\infty) \to \mathbb{R}$, given by $c_i(t) = \alpha_i t$ with $\alpha_i > 0$.
More specifically, given a processing order $\sigma \in \Pi(N)$, the completion time $C_i(\sigma)$ of the job of player $i \in N$ is given by

$$C_i(\sigma) = p_i + \sum_{j\in N:\sigma^{-1}(j)<\sigma^{-1}(i)} p_j,$$

and the corresponding costs are given by $\alpha_i C_i(\sigma)$.
A *Smith order* $\hat{\sigma} \in \Pi(N)$ is defined to be such that

$$\frac{\alpha_{\hat{\sigma}(k)}}{p_{\hat{\sigma}(k)}} \geq \frac{\alpha_{\hat{\sigma}(k+1)}}{p_{\hat{\sigma}(k+1)}}$$

for all $k \in \{1, 2, \ldots, n-1\}$. It turns out that Smith orders minimize total joint costs.

Lemma 3.4
Let $\hat{\sigma} \in \Pi(N)$ be a Smith order. Then

$$\sum_{i\in N} \alpha_i C_i(\hat{\sigma}) \leq \sum_{i\in N} \alpha_i C_i(\sigma)$$

for all $\sigma \in \Pi(N)$.

Proof
Let $\sigma \in \Pi(N)$. Assume that σ is *not* a Smith order. Then there exists an $l \in \{1,\ldots,n-1\}$ such that

$$\frac{\alpha_{\hat{\sigma}(l)}}{p_{\hat{\sigma}(l)}} < \frac{\alpha_{\hat{\sigma}(l+1)}}{p_{\hat{\sigma}(l+1)}}. \tag{3.1}$$

Define $\tau \in \Pi(N)$ by

$$\tau(k) = \begin{cases} \sigma(l+1) & \text{if } k = l, \\ \sigma(l) & \text{if } k = l+1, \\ \sigma(k) & \text{otherwise.} \end{cases}$$

Set $\sigma(l) = i$ and $\sigma(l+1) = j$. Then

$$\sum_{p\in N} \alpha_p C_p(\sigma) - \sum_{p\in N} \alpha_p C_p(\tau) = \alpha_i C_i(\sigma) + \alpha_j C_j(\sigma) - \alpha_i C_i(\tau) - \alpha_j C_j(\tau)$$
$$= \alpha_j (C_j(\sigma) - C_j(\tau)) + \alpha_i (C_i(\sigma) - C_i(\tau)) = \alpha_j p_i - \alpha_i p_j > 0$$

because of inequality (3.1). We may conclude that τ has lower total costs than σ. The order τ however need not be a Smith order (yet). Obviously, by repeating the argument above starting from τ and by consecutively removing "misplaced" neighbours we obtain a Smith order with lower total cost than σ. Moreover, one readily establishes that any two Smith orders have equal total costs. This finishes the proof. □

Given the initial order $\sigma_0 \in \Pi(N)$, define the set of *misplaced (ordered) pairs* $\mathrm{MP}(\sigma_0)$ by

$$\mathrm{MP}(\sigma_0) := \{(i,j) \in N \times N | \sigma_0^{-1}(i) < \sigma_0^{-1}(j) \text{ and } \frac{\alpha_i}{p_i} < \frac{\alpha_j}{p_j}\}.$$

Moreover, for all $(i,j) \in \mathrm{MP}(\sigma_0)$, define

$$g_{ij} := \alpha_j p_i - \alpha_i p_j \ (> 0).$$

As established in the proof of Lemma 3.4, g_{ij} corresponds to the gains obtained from switching the jobs i and j when they are neighbours in any order σ with $\sigma^{-1}(j) = \sigma^{-1}(i) + 1$. Moreover, with $\hat{\sigma} \in \Pi(N)$ being a Smith order it directly follows that the maximal cost savings with respect to the initial order σ_0 are given by

$$\sum_{i\in N} \alpha_i (C_i(\sigma_0) - C_i(\hat{\sigma})) = \sum_{(i,j)\in \mathrm{MP}(\sigma_0)} g_{ij}.$$

Since a Smith order can be reached from the initial order from consecutive neighbour switches of a misplaced pair (i,j) with gains g_{ij}, it seems a natural idea to share these gains g_{ij} equally between players i and j. This leads to the definition of the Equal Gain Splitting Rule EGS. For a sequencing problem $(N, \sigma_0, \{p_i\}_{i\in N}, \{\alpha_i\}_{i\in N})$ the *Equal Gain Splitting Rule* is defined by

$$\mathrm{EGS}_i(N, \sigma_0, \{p_i\}_{i\in N}, \{\alpha_i\}_{i\in N}) := \frac{1}{2} \sum_{(i,j)\in \mathrm{MP}(\sigma_0)} g_{ij} + \frac{1}{2} \sum_{(j,i)\in \mathrm{MP}(\sigma_0)} g_{ji}$$

for all $i \in N$.

Example 3.4

Consider the sequencing problem with $N = \{1,2,3,4\}$ given by

i	1	2	3	4
p_i	2	1	3	2
α_i	2	3	1	8

and $\sigma_0 = (1\ 2\ 3\ 4)$. Then the total joint costs with respect to the initial order σ_0 are given by

$$\sum_{i \in N} \alpha_i C_i(\sigma_0) = 4 + 9 + 6 + 64 = 83.$$

The unique Smith order $\hat{\sigma}$ that minimizes total joint costs is given by $\hat{\sigma} = (4\ 2\ 1\ 3)$ with costs

$$\sum_{i \in N} \alpha_i C_i(\hat{\sigma}) = 16 + 9 + 10 + 8 = 43,$$

leading to a maximal cost savings of $83 - 43 = 40$. Clearly, the set of misplaced pairs $\mathrm{MP}(\sigma_0)$ is given by

$$\mathrm{MP}(\sigma_0) = \{(1,2), (1,4), (2,4), (3,4)\},$$

while

$$g_{12} = 4,\ \ g_{14} = 12,\ \ g_{24} = 2,\ \ g_{34} = 22.$$

Hence the EGS-rule distributes the maximal cost savings in the following way:

$$(2 + 6, 2 + 1, 11, 6 + 1 + 11) = (8, 3, 11, 18). \qquad \diamond$$

Let $(N, \sigma_0, \{p_i\}_{i \in N}, \{\alpha_i\}_{i \in N})$ be a sequencing problem. To define a corresponding sequencing game we have to make an assumption on the possible reorderings of a coalition $S \subset N$. We will assume that processing orders where members of the coalition S under consideration pass agents outside S are not allowed. More formally, $\sigma \in \Pi(N)$ is called *admissible for* S if

$$\{j \in N | \sigma^{-1}(j) < \sigma^{-1}(i)\} = \{j \in N | \sigma_0^{-1}(j) < \sigma_0^{-1}(i)\}$$

for all $i \in N \backslash S$. Let $\mathcal{A}(S)$ denote the set of all admissible orders for S. Clearly $\mathcal{A}(N) = \Pi(N)$. Now define the *sequencing game* $v \in \mathrm{TU}^N$ by

$$v(S) := \max_{\sigma \in \mathcal{A}(S)} \sum_{i \in S} \alpha_i (C_i(\sigma_0) - C_i(\sigma))$$

for all $S \subset N$.

The definition of admissibility implies that reordering of jobs for S is only allowed within maximally connected components of S. Here $T \subset N$ is called *connected* if $\sigma_0(k) \in T$, $\sigma_0(l) \in T$ with $k < l$ implies that $\{\sigma_0(k+1), \ldots, \sigma_0(l-1)\} \subset T$. A connected coalition $T \subset S$ is called a (maximally connected) *component* of S if $T \subset T' \subset S$ and T' connected imply that $T' = T$. The partition of S into components is denoted by S/σ_0. From the definition of the coalitional values of a sequencing game it directly follows that

$$v(S) = \sum_{T \in S/\sigma_0} v(T) \tag{3.2}$$

for all $S \subset N$. Moreover, for each connected coalition T the optimal admissible order $\hat{\sigma}_T$ will correspond to a restricted Smith order where only the misplaced pairs within T are reordered. Hence,

$$v(T) = \sum_{(i,j) \in \mathrm{MP}(\sigma_0)} g_{ij}$$

for every connected coalition $T \subset N$.

Example 3.5

Let $v \in \mathrm{TU}^N$ be the sequencing game corresponding to the sequencing problem of Example 3.4. Then the coalitional values of v are given by

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$
$v(S)$	0	0	0	0	4	0	0	0	0	22

S	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	N
$v(S)$	4	4	22	24	40

Note that the allocation $(8, 3, 11, 18)$ prescribed by the EGS-rule belongs to the core of this sequencing game. $\diamond$

Theorem 3.5

Let $(N, \sigma_0, \{p_i\}_{i \in N}, \{\alpha_i\}_{i \in N})$ be a sequencing problem and let $v \in TU^N$ be the corresponding sequencing game. Then,

$$EGS(N, \sigma_0, \{p_i\}_{i \in N}, \{\alpha_i\}_{i \in N}) \in \mathrm{C}(v).$$

Proof

Clearly the EGS-rule provides an efficient allocation of $v(N)$. Let $S \subset N$. Using (3.2) it suffices to prove that

$$\sum_{i \in T} \mathrm{EGS}_i(N, \sigma_0, \{p_i\}_{i \in N}, \{\alpha_i\}_{i \in N}) \geq v(T)$$

for all $T \in S/\sigma_0$.
Let $T \in S/\sigma_0$. Then

$$\begin{aligned}
\sum_{i \in T} \mathrm{EGS}_i(N, \sigma_0, \{p_i\}_{i \in N}, \{\alpha_i\}_{i \in N}) &= \sum_{i \in T} \left(\frac{1}{2} \sum_{j \in N : (i,j) \in \mathrm{MP}(\sigma_0)} g_{ij} + \frac{1}{2} \sum_{j \in N : (j,i) \in MP(\sigma_0)} g_{ji} \right) \\
&\geq \sum_{i \in T} \left(\frac{1}{2} \sum_{j \in T : (i,j) \in MP(\sigma_0)} g_{ij} + \frac{1}{2} \sum_{j \in T : (j,i) \in MP(\sigma_0)} g_{ji} \right) \\
&= \sum_{(i,j) \in MP(\sigma_0)} g_{ij} \\
&= v(T).
\end{aligned}$$

$\square$

3.4 Linear Production

A *linear production problem* (or shortly LP-problem) can be described by a tuple (R, P, A, b, c) with

- R : a finite set of resources,
- P : a finite set of products,
- A : a (linear) technology matrix with the entry A_{rp}, for $r \in R$ and $p \in P$, representing the number of units of resource r needed to produce one unit of product p,
- b : a vector in $\mathbb{R}^R$ representing the supply bundle of resources,
- c : a vector in $\mathbb{R}^P$ denoting the (exogenous) market prices per unit of product.

The corresponding LP-optimization problem reflects the related profit-maximization issue on feasible bundles of products:

$$\max c^T x \qquad \text{s.t. } Ax \leq b \text{ with } x \in \mathbb{R}^P,\ x \geq 0.$$

Implicitly the description of an LP-problem assumes the technology to be publicly available and the resources to be owned by just one decision maker. Assuming resources to be owned by various players, all having access to the same public production technology we arrive at LP-processes and corresponding LP-games.

In an *LP-process* $(N, R, P, A, \{b^i\}_{i\in N}, c)$, R, P, A and c are as before but now we have a finite set N of players and for each player a resource bundle $b^i \in \mathbb{R}^R$, with b^i_r representing the number of units of resource r available to player i. To evaluate LP-processes we consider several LP-problems, i.e. one for every possible coalition $S \in 2^N$. With $L = (N, R, P, A, \{b^i\}_{i\in N}, c)$ this leads to the corresponding *LP-game* $v_L \in \mathrm{TU}^N$ defined by

$$\begin{array}{rll} v_L(S) := & \max & c^T x \\ & \text{s.t.} & Ax \le \sum_{i\in S} b^i \text{ with } x \in \mathbb{R}^P,\ x \ge 0,\ (\text{or shortly } x \in F(S)) \end{array} \tag{3.3}$$

for every $S \in 2^N$. Note that $\sum_{i\in S} b^i$ represents the total resource bundle available to coalition S. From duality theory we know that

$$\begin{array}{rll} v_L(S) = & \min & y^T(\sum_{i\in S} b^i) \\ & \text{s.t.} & y^T A \ge c^T \text{ with } y \in \mathbb{R}^R,\ y \ge 0,\ (\text{or shortly } y \in F^*) \end{array} \tag{3.4}$$

since the feasible regions $F(S)$ and F^* can be shown to be non-empty (under rather weak conditions). Note that the feasible region F^* of the dual program does not depend on the coalition S one is considering and hence can be readily used to determine v_L just by changing the objective function.

Example 3.6

Consider the LP-process L with $N = \{1, 2, 3\}$, two resources, two products,

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix},\ b^1 = \begin{bmatrix} 28 \\ 28 \end{bmatrix},\ b^2 = \begin{bmatrix} 42 \\ 0 \end{bmatrix},\ b^3 = \begin{bmatrix} 0 \\ 35 \end{bmatrix} \text{ and } c^T = \begin{bmatrix} 6 & 8 \end{bmatrix}.$$

The feasible region F^* (for any coalition) is given by

$$2y_1 + y_2 \ge 6,\ y_1 + 4y_2 \ge 8,\ y_1 \ge 0,\ y_2 \ge 0$$

From this we readily derive (e.g. by comparing the value of the objective function in the corner points $(8, 0)$, $(0, 6)$, and $(\frac{16}{7}, \frac{10}{7})$ of F^*) that the corresponding LP-game v_L is given by

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v_L(S)$	104	0	0	168	154	146	250

Note that $\min_{y\in F^*} y^T(\sum_{i\in N} b^i) = 250$ is (uniquely) attained in $(\frac{16}{7}, \frac{10}{7})$. ◇

For an LP-process $L = (N, R, P, A, \{b^i\}_{i \in N}, c)$ we define the *Owen set* Owen(L) by

$$\text{Owen}(L) := \{(y^T b^i)_{i \in N} \in \mathbb{R}^N | y \in F^*,\ v_L(N) = y^T \sum_{i \in N} b^i\}$$

So to determine an element of the Owen set, an *Owen vector*, we first have to determine an optimal solution y of the dual program for the grand coalition N (cf. (3.4) with $S = N$). For each $r \in R$, y_r is to be interpreted as the shadowprice for resource r. Then, for each $i \in N$, $y^T b^i$ represents the shadowprice of the initial resource bundle of player i.

Example 3.7
Reconsider the LP-process L of Example 3.6. As we have seen the unique optimal solution of the dual program for N equals $y^T = \left(\frac{16}{7}, \frac{10}{7}\right)$.
Hence the Owen set of this LP-process consists of one point z with

$$z = (y^T b^i)_{i \in N} = (104, 96, 50),$$

where for example $96 = \frac{16}{7}42 + \frac{10}{7}0$ reflects the shadowvalue of the initial resource bundle of player 2. Note that $z \in C(v_L)$. ◇

Interestingly, each Owen vector belongs to the core of the corresponding LP-game.

Theorem 3.6
For any LP-process L we have

$$\textit{Owen}(L) \subset C(v_L).$$

Proof
Let $L = (N, R, P, A, \{b^i\}_{i \in N}, c)$ be an LP-process. Take $z \in \text{Owen}(L)$ and let $y \in F^*$ with $y^T \sum_{i \in N} b^i = v_L(N)$ be such that $z = (y^T b^i)_{i \in N}$. Then

$$\sum_{i \in N} z_i = \sum_{i \in N} y^T b^i = y^T \sum_{i \in N} b^i = v_L(N)$$

and, for all $S \in 2^N$,

$$\sum_{i \in S} z_i = y^T \sum_{i \in S} b^i \geq v_L(S)$$

since $y \in F^*$ and thus y is feasible for the dual program (3.4) corresponding to S. □

In particular Theorem 3.6 implies that every LP-game is balanced. In fact, since each subgame of an LP-game is also an LP-game itself (corresponding to a natural "sub" LP-process), LP-games are totally balanced and also non-negative (by definition). In fact, also the reverse of this statement is true.

Theorem 3.7
Every non-negative and totally balanced TU-game is an LP-game.

Proof
Let $v \in \mathrm{TU}^N$, $v \geq 0$, be totally balanced. Define the LP-process $D(v) = (N, R, P, A, \{b^i\}_{i \in N}, c)$ by

$$R = N,\ P = 2^N \backslash \{\emptyset\},\ A = \begin{array}{c} S \\ \begin{bmatrix} \ldots & e^S & \ldots \end{bmatrix} \end{array},\ b^i = e^{\{i\}} \text{ for all } i \in N, \text{ and } c^T = \begin{array}{c} S \\ \begin{bmatrix} \ldots & v(S) & \ldots \end{bmatrix} \end{array}.$$

It is readily checked that $v_{D(v)} = v$ (where exactly do we use the fact that v is totally balanced?).

□

The LP-process $D(v)$ in the proof of Theorem 3.7 is called the *direct* LP-process corresponding to the TU-game v. Here, players are the resources (think of labourers), coalitions can be produced, each player has only himself to offer on the labour market and, finally, the price of a coalition (a product) is determined by the coalitional values of the underlying game v.

Exercises

Exercise 3.1
Consider Example 3.1 with the modification that $t(\{0, 1\}) = a$ with $a > 0$. Calculate the corresponding cost game and all Bird allocation vectors.

Exercise 3.2
Another type of cost allocation for spanning tree problems can be obtained in the following way.
Input: a spanning tree problem $(N, 0, t)$ and a bijection $\pi : \{1, \ldots, |N|\} \to N$ which is to be thought of as an order on the players.
Output: a cost allocation $\gamma^\pi \in \mathbb{R}^N$.

Initialize: $R^0 = \emptyset$ (an empty set of players).
Step $k \in \{1, \ldots, |N|\}$: Find an edge e_k of minimal cost incident on the component of the graph (N, R^{k-1}) that contains player $\pi(k)$ such that $(N, R^{k-1} \cup \{e_k\})$ does not contain a cycle. Define $R^k := R^{k-1} \cup \{e_k\}$ and allocate $t(e_k)$ to player $\pi(k)$, i.e., define $\gamma^{\pi}_{\pi(k)} = t(e_k)$.

Consider the spanning tree problem $(N, 0, t)$ of Example 3.1.

(a) Calculate γ^{π} for each possible order π, and their average γ.

(b) Do all γ^{π}'s (and γ) belong to the core of the corresponding spanning tree game?

(c) Which γ^{π}'s do not correspond to Bird allocations?

Exercise 3.3

(a) Reconsider the flow problem of Example 3.2 but put the capacity of (v_2, v_3) equal to $c > 0$.

Determine the corresponding flow game. For which c is this game balanced?

(b) Reconsider the flow problem of Example 3.3 but put the capacity of (v_2, v_3) equal to $c > 0$.

Determine the corresponding flow game. For which c is this game balanced?

Exercise 3.4 (Decomposition in control games)
Let $v \in \mathrm{TU}^N$ be such that $N = \{1, 2, \ldots, n\}$, $v \geq 0$, $v \neq 0$, and v is balanced. Choose $x \in \mathrm{C}(v)$. Define

$$\begin{aligned}
i &:= \min\{j \in \{1, \ldots, n\} | x_j > 0\}, \\
t &:= \min\{x_i, \min\{v(S) | i \in S,\, v(S) > 0\}\}, \\
w(S) &:= \begin{cases} 1 & \text{if } i \in S, v(S) > 0, \\ 0 & \text{otherwise,} \end{cases} \\
v' &:= v - tw
\end{aligned}$$

Clearly $i \in N$ and $t > 0$ are well defined and w is a control game (since $v \geq 0$, $v \neq 0$, and v is balanced it follows that $v(N) > 0$) with core elements (i is a veto player of w). Moreover, $v' \geq 0$ and $x - te^{\{i\}} \in \mathrm{C}(v')$.

(a) Verify the last two assertions.

We may conclude that $v = tw + v'$. If $v' = 0$, the decomposition is there. If $v' \neq 0$, then we can repeat the same procedure as above starting from the core element $x - te^{\{i\}}$ for v'. Repeating this procedure will lead in finitely many steps to a decomposition of v. With respect to finiteness note that for the "next" game there is either at least one more coalitional value equal to zero, or in the newly found core element at least one more coordinate is equal to zero.

(b) Decompose the 3-person game v given by

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	2	0	0	7	8	5	10

into balanced control games.

(c) Consider the 3-person game v given by

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	1	2	3	7

Show that the above procedure leads to two different decompositions starting from $x^1 = (0, 1, 6)$ and $x^2 = (2, 5, 0)$.

(d) Consider the 3-person game v given by

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	0	3	0	0	4

Show that the above procedure starting from $x^1 = (2, 1, 1)$ and $x^2 = (4, 0, 0)$ leads to the same decomposition but in a different number of steps.

Exercise 3.5

Reconsider the sequencing problem and the associated sequencing game v of Example 3.4 and Example 3.5.

(a) Decompose v into unanimity games. Do you recognize a general structure?

Without changing anything else, set $p_4 = a$ (> 0).

(b) Determine all Smith orders and calculate the EGS-rule for this sequencing problem.

(c) Determine the corresponding sequencing game.

Exercise 3.6 (gain splitting rules)

Let $(N, \sigma_0, \{p_j\}_{j\in N}, \{\alpha_j\}_{j\in N})$ be a sequencing problem. For each misplaced pair $(i,j) \in \mathrm{MP}(\sigma_0)$, pick $\lambda_{ij} \in [0,1]$ and define the outcome of the *gain splitting rule* GS^λ by

$$\mathrm{GS}_i^\lambda(N, \sigma_0, \{p_j\}_{j\in N}, \{\alpha_j\}_{j\in N}) = \sum_{(i,j)\in\mathrm{MP}(\sigma_0)} \lambda_{ij} g_{ij} + \sum_{(j,i)\in\mathrm{MP}(\sigma_0)} (1-\lambda_{ji}) g_{ji}$$

for all $i \in N$. Note that, with $\lambda_{ij} = \frac{1}{2}$ for all $(i,j) \in\mathrm{MP}(\sigma_0)$, $\mathrm{GS}^\lambda = \mathrm{EGS}$.

(a) Let $v \in \mathrm{TU}^N$ be the corresponding sequencing game. Show that

$$\mathrm{GS}^\lambda(N, \sigma_0, \{p_j\}_{j\in N}, \{\alpha_j\}_{j\in N}) \in \mathrm{C}(v).$$

Consider the sequencing problem with $N = \{1,2,3\}$ given by

i	1	2	3
p_i	2	2	1
α_i	4	6	5

and $\sigma_0 = (1\ 2\ 3)$. Clearly the unique Smith order $\hat{\sigma}$ is given by $\hat{\sigma} = (3\ 2\ 1)$ and $\mathrm{MP}(\sigma_0) = \{(1,2),(1,3),(2,3)\}$.

(b) Define $\lambda_{12} = \frac{3}{4}$, $\lambda_{13} = \frac{1}{3}$, and $\lambda_{23} = 1$. Calculate GS^λ.

(c) Let $v \in \mathrm{TU}^N$ be the corresponding sequencing game. Determine $\mathrm{C}(v)$. Find a core element that cannot be constructed as the outcome of any gain splitting rule GS^λ.

Exercise 3.7

(a) Consider the LP-process L with $N = \{1,2\}$, 2 resources, 3 products, technology matrix $A = \begin{bmatrix} 12 & 1 & 3 \\ 40 & 10 & 90 \end{bmatrix}$, resource bundles $b^1 = \begin{bmatrix} 100 \\ 240 \end{bmatrix}$ and $b^2 = \begin{bmatrix} 0 \\ 480 \end{bmatrix}$, and price vector $c^T = \begin{bmatrix} 200 & 30 & 150 \end{bmatrix}$.

(i) Calculate the corresponding LP-game v_L.

(ii) Determine $\mathrm{C}(v_L)$.

(iii) Determine $\mathrm{Owen}(L)$.

(b) Consider the LP-process D with $N = \{1,2\}$, 2 resources, 3 products, technology matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, resource bundles $b^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $b^2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and price vector $c^T = \begin{bmatrix} 1200 & 0 & 2440 \end{bmatrix}$.

   (i) Calculate the corresponding LP-game v_D.

   (ii) Determine $\mathrm{C}(v_D)$.

   (iii) Determine Owen(D).

Exercise 3.8

(a) Consider the (glove market) game v with $N = \{1,2,3\}$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{2,3\}) = 0$, $v(\{1,2\}) = v(\{1,3\}) = v(\{N\}) = 10$. Construct a non-direct LP-process L such that the corresponding game v_L coincides with v.

(b) Consider an LP-process L with player set $N = \{1,2,3\}$ and with resource bundles

$$b^1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \; b^2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \; b^3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Suppose that $(10,5,5)^T \in \text{Owen}(L)$. Determine $v(\{1,2,3\})$ and an optimal solution y of the dual program (3.4) of coalition N.

Some relevant literature

Borm, P., Hamers, H. and R. Hendrickx (2001). Operations research games: a survey. *TOP*, 9, 139-199.

Bird, C. (1976). On cost allocation for a spanning tree: a game theoretic approach. *Networks*, 6, 335-350.

Claus, A. and D. Kleitman (1973). Cost allocation for a spanning tree. *Networks*, 3, 289-304.

Feltkamp, V. (1995). Cooperation in controlled network structures. PhD thesis, Tilburg University, Tilburg, The Netherlands.

Kalai, E. and Zemel (1982). Totally balanced games and games of flow. *Mathematics of Operations Research*, 7, 476-478.

Curiel, I., Derks, J. and S. Tijs (1989). On balanced games and flow games with committee control. *OR Spektrum*, 11, 83-88.

Curiel, I., Pederzoli, G. and S. Tijs (1989). Sequencing games. *European Journal of Operational Research*, 40, 344-351.

Hamers, H. (1995). Sequencing and delivery situations: a game theoretic approach. PhD thesis, Tilburg University, Tilburg, The Netherlands.

Hamers, H., Borm, P., Suijs, J., and S. Tijs (1998). The split core for sequencing games. *Games and Economic Behavior*, 15, 165-176.

Owen, G. (1975). On the core of linear production games. *Mathematical Programming*, 9, 358-370.

Gellekom, A. van, Potters, J., Reijnierse, J., Tijs, S. and M. Engel (2000). Characterization of the Owen set of linear production processes. *Games and Economic Behavior*, 32, 139-156.

CHAPTER 11

Strategic games and Nash equilibria

We have seen how to construct a cooperative game to an economic problem as a production situation with a publicly available production technique and multiple agents with private resource bundles. The aim of this construction was to get insight in how possible cooperation between economic agents can take place and to find reasonable proposals for allocating the extra revenues or cost savings resulting from cooperation. The analysis was oriented towards coalitions. This is clearly reflected in the definition a core element in which the incentives for an arbitrary subgroup of agents to withdraw (and to form a separate collaborating collective) are taken away. In such a cooperative approach one implicity assumes that binding agreements (contracts) between agents, for example about what everyone should do and how the eventual revenues should be divided, are not only possible but are also verifiable and enforceable.

In various competitive economic settings the agents however do not (or can not) bother about possible cooperation but are interested in reaching the highest possible individual payoff by means of a purely individual effort. Since the payoff or revenue of a certain agent in this type of situations typically is not only dependent on his own choice of action but also of the actions taken by (some of) the other agents, the situation should also be considered from the viewpoint of the other agents to reach an adequate private choice of action. Such an analysis falls within the scope of non-cooperative game theory. Generally speaking non-cooperative models are excessively detailed in describing the exact rules of the game. Moreover, the ultimate predictions may be highly sensitive to these details. On the other hand, cooperative modelling might be viewed as incomplete or more abstract, while attempting to obtain more robust conclusions.
Implicity one assumes within non-cooperative theory that binding agreements are not possi-

ble. Put differently, before the actual play of the game starts, the players could communicate and make agreements but afterwards, in actual play, nobody can be enforced to adhere to these agreements. The quality of an agreement on individual strategy choices will therefore depend on the individual incentives to actually adhere to them. These considerations will lead to the concept of a Nash equilibrium. Within the theory on non-cooperative games the Nash equilibrium plays a central role comparable to the role of the core within cooperative game theory.

A *strategic game* G with a finite set N of players is given by

$$G = \{(X_i, \pi_i)\}_{i \in N}$$

where for each $i \in N$, X_i denotes the *strategy space* and $\pi_i : \prod_{j \in N} X_j \to \mathbb{R}$ the monetary *payoff function* on the space $\prod_{j \in N} X_j$ of all *strategy combinations*. The game G is called *finite* if $|X_i| < \infty$ for all $i \in N$.
It is assumed that the players choose their strategies simultaneously and independently. *Extensive games* with sequential moves over time will be the topic of a later chapter. Pre-play communication is allowed but binding agreements can not be made. Please note that the payoff to a player in a strategic game may not only depend on his own strategy choice but also on the strategy choices of others. We assume *monetary* payoffs to avoid going into an extensive treatment of utility function representations of complete and transitive preference relations on the space of all strategy combinations.

Let $G = \{(X_i, \pi_i)\}_{i \in N}$ be a strategic game.
A strategy combination $\hat{x} \in \prod_{j \in N} X_j$ is called a *Nash equilibrium* of G if

$$\pi_i(\hat{x}) \geq \pi_i(x_i, \hat{x}_{N \setminus \{i\}})$$

for all $x_i \in X_i$ and all $i \in N$. The set of all Nash equilibria of G is denoted by $E(G)$.
A Nash equilibrium is intrinsically stable in the sense that every player exhibits local optimal behaviour against the combination of strategy choices of the other players within the equilibrium. Put differently, unilateral deviation from a Nash equilibrium does not pay.

A strategy $\hat{x}_i \in X_i$ is called *dominant* for player i if

$$\pi_i(\hat{x}_i, x_{N \setminus \{i\}}) \geq \pi_i(x_i, x_{N \setminus \{i\}})$$

for all $x_i \in X_i$ and all strategy combinations $x_{N \setminus \{i\}} \in \prod_{j \in N \setminus \{i\}} X_j$.

A dominant strategy exhibits global optimal behaviour against any possible combination of strategy choices of the other players. It will be clear that if each player has a dominant strategy, then the combination of these strategies will be a Nash equilibrium.

Example 11.1 (Prisoners' Dilemma)
Probably the most famous example of a two-person strategic game is the so-called Prisoners' Dilemma. Two suspects of a jointly committed offense are interrogated separately and simultaneously. Each of the suspects (player 1 and player 2) has two possible actions: he can testify against his partner (T) or not (NT). The fine for the offense, if proved by means of a testimony of the other, is $10,000$ Euro. A testimony against the other gives a reduction to the final fine of $1,000$ Euro. Moreover, if there is no proof against a certain suspect the fine that can be charged is $1,000$ Euro.
This situation can be summarized by means of a so-called bimatrix (a matrix with two numbers in each of its cells)

$$\begin{array}{cc} & \begin{array}{cc} T & NT \end{array} \\ \begin{array}{c} T \\ NT \end{array} & \begin{bmatrix} -9,-9 & 0,-10 \\ -10,0 & -1,-1 \end{bmatrix} \end{array}$$

This matrix should be read in the following way. Player 1 chooses between the rows T and NT, player 2 between the columns T and NT. If they both choose NT, then both players get a fine of $1,000$ Euro, which is represented in the bimatrix by a payoff of -1. If player 1 chooses to testify against player 2 (choosing row T) and player 2 chooses not to testify against player 1 (choosing column NT), then player 2 can be fined for $10,000$ Euro, corresponding with -10 in the second coordinate in the cell (T, NT). Player 1, however, in this case gets a fine of $1,000$ Euro but also gets reduction of $1,000$ Euro, represented by the 0 in the first coordinate of the cell (T, NT). The payoffs in the other cells of the bimatrix can be obtained in a similar way.
The related finite, two-person game $G = \{(X_i, \pi_i)\}_{i \in N}$ with $N = \{1, 2\}$ is given by $X_1 = X_2 = \{T, NT\}$ while e.g. $\pi_1((T, T)) = -9$ and $\pi_2((T, NT)) = -10$, as represented in the bimatrix.
What should the players do? Actually it is very simple: each of the players has a dominant strategy. T is always preferred to NT, irrespective of the other player's choice! Therefore, one could say that the natural prediction for the outcome of this game is (T, T) leading to a net fine of $9,000$ Euro for each player. Note that the outcome corresponding to the strategy combination (NT, NT) is actually preferred by both players but this strategy combination is unstable.

The Prisoners' Dilemma is a role model for many economic situations. We restrict ourselves to just one possible illustration. Consider a situation in which two competing firms (player 1 and player 2) produce certain washing powders of the same quality. Suppose that the consumer market has a value of $1,000,000$ Euro. If further nothing happens this value is equally distributed between the two firms, $500,000$ Euro each. Each firm however could decide to start advertising (action A) for its washing powder. The advantage of advertising is that if the other firm does not advertise (action NA), 80% of the market value is gathered. If on the contrary the other firm also decides to advertise, again the market value is shared equally. Let us suppose for the sake of convenience that the costs of adequate advertising for each firm equals $100,000$ Euro. In a bimatrix the corresponding game (with payoffs in multiples of $1,000$ Euro) boils down to a variant of Prisoners' Dilemma.

$$\begin{array}{cc} & \begin{array}{cc} A \quad\quad & \quad\quad NA \end{array} \\ \begin{array}{c} A \\ NA \end{array} & \begin{bmatrix} 400,\ 400 & 700,\ 200 \\ 200,\ 700 & 500,\ 500 \end{bmatrix} \end{array}$$

◇

Example 11.2 (Purification)
Three firms located at the same lake drain their residuals on the lake either in purified form (as legally required) or not. For convenience we assume that the costs of purification are equal to $10,000$ Euro for each firm. The authorities perform monthly tests on the water of the lake and as soon as two or three firms within the same month do not purify their residuals this will be observed in the tests and legal action will be taken. The authorities however can not find out exactly which of the firms did not purify (and a firm is not able to prove its "innocence"). In case of a legal offense, each firm will be charged a fine of $30,000$ Euro. In units of $10,000$ Euro this situation can be represented as a three-person finite strategic game as follows:

$$\begin{array}{cc} & \begin{array}{cc} P \quad\quad & \quad\quad NP \end{array} \\ \begin{array}{c} P \\ NP \end{array} & \begin{bmatrix} -1,-1,-1 & -1,\ 0,-1 \\ 0,-1,-1 & -3,-3,-4 \end{bmatrix} \\ & P \end{array} \qquad \begin{array}{cc} & \begin{array}{cc} P \quad\quad & \quad\quad NP \end{array} \\ \begin{array}{c} P \\ NP \end{array} & \begin{bmatrix} -1,-1,\ 0 & -4,-3,-3 \\ -3,-4,-3 & -3,-3,-3 \end{bmatrix} \\ & NP \end{array}$$

In this (trimatrix) representation firm 1 picks a row, firm 2 a column and firm 3 a matrix ($P =$ purify, $NP =$ do not purify). In each cell there are three numbers representing the payoffs to firm 1, 2 and 3, respectively. Please make sure you understand this representation and check all payoffs yourself. Clearly there are 4 Nash equilibria: (P, NP, P), (NP, P, P), (P, P, NP) and (NP, NP, NP) ◇

Example 11.3 (Cournot duopoly)
Consider the following market for e.g. mineral water. No individual firm directly controls the market price, all outputs appear on the market simultaneously and all firms produce mineral water of identical quality in the sense that they receive exactly the same market price for their products. Moreover, the market price is a decreasing function of the total output.
To fix ideas, consider the following *duopoly* with two firms, where the market price $p(x)$ per liter with x representing total production in liters of mineral water per day, is given by

$$p(x) = \begin{cases} 2 - \frac{1}{1000}x & \text{if } 0 \leq x \leq 2000 \\ 0 & \text{if } x > 2000 \end{cases}$$

Further we assume that the costs $c_i(y)$ for individually producing y liters per day for firm $i \in \{1, 2\}$ are given by

$$c_1(y) = y \text{ and } c_2(y) = \tfrac{4}{5}y.$$

This duopoly for a specific day can be modelled as a two-person strategic game $\{(X_i, \pi_i)\}_{i \in N}$ with $N = \{1, 2\}$, $X_1 = X_2 = [0, \infty)$,

$$\pi_1(x_1, x_2) = x_1(2 - \tfrac{1}{1000}(x_1 + x_2)) - x_1 = -\tfrac{1}{1000}x_1^2 + (1 - \tfrac{1}{1000}x_2)x_1$$

and

$$\pi_2(x_1, x_2) = x_2(2 - \tfrac{1}{1000}(x_1 + x_2) - \tfrac{4}{5}x_2 = -\tfrac{1}{1000}x_2^2 + (\tfrac{6}{5} - \tfrac{1}{1000}x_1)x_2$$

for all combinations of production levels (x_1, x_2) such that $x_1 + x_2 \leq 2000$.
If $x_1 + x_2 > 2000$, $\pi_1(x_1, x_2) = -x_1$ and $\pi_2(x_1, x_2) = -\frac{4}{5}x_2$.
How to determine the Nash equilibria of this game? This can be done by looking at the so-called *reaction curves* which describe what the firms should optimally produce as a function of the production level of the other firm. Consider firm 1 and fix $x_2 \in [0, \infty)$. Using some elementary differential calculus it is clear that reaction curve $R_1(x_2)$ is given by

$$R_1(x_2) = \begin{cases} 500 - \frac{1}{2}x_2 & \text{if } 0 \leq x_2 \leq 1000 \\ 0 & \text{otherwise} \end{cases}$$

Similarly the reaction curve $R_2(x_1)$ is given by

$$R_2(x_1) = \begin{cases} 600 - \frac{1}{2}x_1 & \text{if } 0 \leq x_1 \leq 1200 \\ 0 & \text{otherwise} \end{cases}$$

In Figure 11.1 the graph of R_1 and the mirrored graph of R_2 are drawn in the same (x_1, x_2) plane. By definition Nash equilibria exactly correspond to combinations of production levels

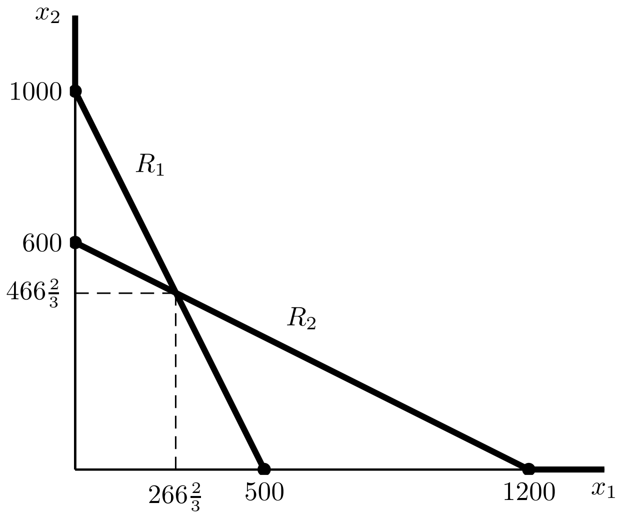


Figure 11.1: Reaction curves in Example 11.3.

that lie on both reaction curves simultaneously. It follows that there is a unique Nash equilibrium $(\hat{x}_1, \hat{x}_2)$ with $\hat{x}_1 = 266\frac{2}{3}$ and $\hat{x}_2 = 466\frac{2}{3}$. $\diamond$

Example 11.4 (Extensive games)
Often the description of a non-cooperative game is provided rather in a sequential format over time than in a simultaneous one. E.g. in chess white moves first, then black etc. *Extensive games* (or *tree games*) explicitly take account this sequential time-related aspect. As an illustration, we consider the following game. First player 1 has to decide between L(eft) and R(ight). Then player 2 is informed of the choice of player 1. If player 1 has choosen L, then player 2 has to decide between T(op) and B(ottom). If player 1 has choosen R, then player 2 has to choose between t(op) and b(ottom). The combination (L, T) leads to a payoff vector $(6, 7)$: 6 to player 1, 7 to player 2, (L, B) to $(4, 3)$, (R, t) to $(2, 1)$ and (R, b) to $(7, 8)$. The corresponding game tree is depicted below.

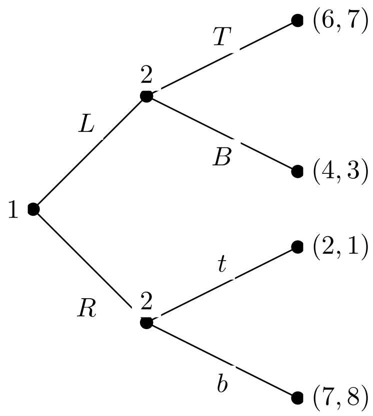


Figure 11.2: Game tree in Example 11.4.

An extensive game can be translated back into a strategic game by letting the strategies in the strategic game represent "complete plans of actions". A strategy should prescribe what to do at any particular decision node in the tree (no matter if in actual play it is reached or not).

Here, player 1 obviously has two strategies: L and R. Player 2 has four: Tt, Tb, Bt and Bb. Here e.g. Bt means, "play B" in case you are asked to choose between T and B and "play t"whenever you are asked to choose between t and b. Of course, in actual play only one of these choices has to be made but a strategy should tell what to play in any possible eventuality.

Adopting the bimatrix representation, the corresponding strategic game is given by

$$\begin{array}{cc} & \begin{array}{cccc} Tt & Tb & Bt & Bb \end{array} \\ \begin{array}{c} L \\ R \end{array} & \begin{bmatrix} 6,7 & 6,7 & 4,3 & 4,3 \\ 2,1 & 7,8 & 2,1 & 7,8 \end{bmatrix} \end{array}$$

One readily checks that this finite game has 3 Nash equilibria: (L, Tt), (R, Tb) and (R, Bb). One of them however has a special flavour in the original sequential setting. If player 2 is asked to choose between T and B, he will pick T because the corresponding payoff of 7 is higher than the payoff of 3 w.r.t. B. Similarly, if player 2 is asked to choose between t and b, he will pick b. Reasoning backwards player 1 at the starting node will prefer R over L with a final payoff of 7 for himself. In this way the so-called *subgame perfect* equilibrium (R, Tb) is selected.

Note that in the strategic form, so without knowledge of the underlying game tree, this type of reasoning can not be applied to discriminate between the three Nash equilibria. $\diamond$

Example 11.5 (Mixed extensions)

Consider the finite two-person strategic game represented by the bimatrix

$$\begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{bmatrix} 1,0 & 0,2 \\ 0,2 & 1,0 \end{bmatrix} \end{array}$$

Clearly, none of the four possible strategy combinations (T, L), (T, R), (B, L) and (B, R) is a Nash equilibrium. This means that this game is without Nash equilibria. This feature, however, will disappear by allowing randomization. Basically this means that players can do more than just pick a row or column. In particular we will allow player 1 to choose row T with probability $\frac{1}{2}$ and (therefore) row B with probability $\frac{1}{2}$. In this particular example

this in fact does not seem to be a bad choice. To realize this *randomized* or *mixed* strategy, his row choice could e.g. depend on the outcome of the throw of a fair coin. Obviously we will not only allow equal probabilities on both rows but arbitrary probability distributions: probability p_T on T and $1-p_T$ on B (with $0 \leq p_T \leq 1$). Similarly player 2 will be allowed to randomize too: probability q_L on L and $1-q_L$ on R (with $0 \leq q_L \leq 1$).
However, by extending the strategy spaces one also has to extend the payoff functions. This will be done using expected payoffs. Given the mixed strategy combination $(p,q) = ((p_T, 1-p_T),(q_L, 1-q_L))$ the probabilities to end up in each of the four cells in the bimatrix are given by

$$\begin{bmatrix} p_T q_L & p_T(1-q_L) \\ (1-p_T)q_L & (1-p_T)(1-q_L) \end{bmatrix}$$

The extended expected payoff functions can be derived from this. For our numerical example we find

$$\begin{aligned}\pi_1(p,q) &= 1p_T q_L + 0p_T(1-q_L) + 0(1-p_T)q_L + 1(1-p_T)(1-q_L) \\ &= p_T q_L + (1-p_T)(1-q_L)\end{aligned}$$

and

$$\pi_2(p,q) = 2p_T(1-q_L) + 2(1-p_T)q_L.$$

Now, one readily checks that this so-called *mixed extension* of the original game has a Nash equilibrium $(\hat{p},\hat{q})$ with $\hat{p}_T = \frac{1}{2}$ and $\hat{q}_L = \frac{1}{2}$ since

$$\hat{\pi}_1(\hat{p},\hat{q}) = \tfrac{1}{2} = \hat{\pi}_1(p,\hat{q})$$

for all possible mixed strategies p and

$$\hat{\pi}_2(\hat{p},\hat{q}) = 1 = \hat{\pi}_2(\hat{p},q)$$

for all possible mixed strategies q. ◇

As we have seen in Example 11.5, existence of Nash equilibria is not guaranteed for every strategic game. Theorem 11.1 provides general conditions on a strategic game such that Nash equilibria do exist.

Theorem 11.1 *(Existence of Nash equilibria)*
Let $G = \{(X_i, \pi_i)\}_{i\in N}$ be a strategic game such that the following conditions are satisfied:

(i) $X_i \subset \mathbb{R}^{m_i}$ *for all* $i \in N$,

(ii) X_i *is convex and compact for all* $i \in N$,

(iii) *For all* $i \in N$ *and all* $x_{N\setminus\{i\}} \in \prod_{j \in N\setminus\{i\}} X_j$, *the function* $g : X_i \to \mathbb{R}$ *defined by*

$$g(x_i) = \pi_i(x_i, x_{N\setminus\{i\}})$$

for all $x_i \in X_i$, *is concave,*

(iv) $\pi_i : \prod_{j \in N} X_j \to \mathbb{R}$ *is continuous for all* $i \in N$.

Then G *has at least one Nash equilibrium.*

Proof

Define the best reply correspondence B_i for a player $i \in N$ by

$$B_i(x_{N\setminus\{i\}}) = \{\hat{x}_i \in X_i \mid \pi_i(\hat{x}_i, x_{N\setminus\{i\}}) \geq \pi_i(x_i, x_{N\setminus\{i\}}) \text{ for all } x_i \in X_i\}$$

for all $x_{N\setminus\{i\}} \in \prod_{j \in N\setminus\{i\}} X_j$.

From the conditions (i)-(iv) it readily follows that $B_i(x_{N\setminus\{i\}})$ is convex and compact for all $x_{N\setminus\{i\}}$ and that the graph $G(B_i)$ defined by

$$G(B_i) = \{x \in \prod_{j \in N} X_j | x_i \in B_i(x_{N\setminus\{i\}}) \text{ for all } i \in N\}$$

is closed. Next define the correspondence B by $B(x) = \prod_{i \in N} B_i(x_{N\setminus\{i\}})$ for all $x \in \prod_{j \in N} X_j$. Clearly *fixed points* of B, i.e., $\hat{x} \in \prod_{j \in N} X_j$ such that $\hat{x} \in B(\hat{x})$, correspond to Nash equilibria of G.

Since $\prod_{j \in N} X_j$ is convex and compact, $B(x)$ is non-empty, convex and compact for every $x \in \prod_{j \in N} X_j$, fixed points of B (and thus Nash equilibria of G) exist if the graph $G(B)$ is closed. Here we use Kakutani's fixed point theorem for correspondences which states the following:

Let $C \subset \mathbb{R}^t$ *be convex and compact and let* F *be a correspondence on* C *such that for all* $c \in C$, $F(c)$ *is a non-empty, convex and compact subset of* C. *If the graph* $G(F)$ *is closed, then* F *has a fixed point* $\hat{c} \in C$ *such that* $\hat{c} \in F(\hat{c})$.

Take a sequence $\{(x^n, y^n)\}_{n \in \mathbb{N}} \subset G(B)$ such that (x^n, y^n) converges to (x, y). To establish closedness of $G(B)$ it suffices to prove that $(x, y) \in G(B)$

Clearly $(x, y) \in \prod_{j\in N} X_j \times \prod_{j\in N} X_j$ and it suffices to show that $y \in B(x)$, or equivalently, that $y_i \in B_i(x_{N\setminus\{i\}})$ for all $i \in N$.
Take $i \in N$. Since $(x^n, y^n) \in G(B)$ we have that $y_i^n \in B_i(x^n_{N\setminus\{i\}})$ for all $i \in N$, which means that

$$\pi_i(y_i^n, x^n_{N\setminus\{i\}}) \geq \pi_i(z, x^n_{N\setminus\{i\}})$$

for all $z \in X_i$. Taking limits and using the continuity of π_i we find that

$$\pi_i(y_i, x_{N\setminus\{i\}}) \geq \pi_i(z, x_{N\setminus\{i\}})$$

for all $z \in X_i$, which implies that $y_i \in B_i(x_{N\setminus\{i\}})$. $\square$

Our focal point of attention will be mixed extensions of two-person finite strategic games for which existence of Nash equilibria is guaranteed. For this class of games we now introduce some standard notation and terminology which we will use from now on.
Let A and B be two real $m\times n$ matrices. The rows of A and B will be denoted by $e_1, e_2, \ldots e_m$ and the columns by $f_1, f_2, \ldots, f_n$. In the corresponding finite two-person game G represented by the bimatrix (A, B), if player 1 chooses row e_i and player 2 chooses cloumn f_j, player 1 receives a payoff of A_{ij} and player 2 a payoff of B_{ij}.
The associated mixed extension of G is called an $m \times n$ *bimatrix game*. This game is denoted by (A, B), and is formally given by $\{(\Delta_m, \pi_1), (\Delta_n, \pi_2)\}$. Here,

$$\Delta_m = \{p \in \mathbb{R}^m | p \geq 0, \sum_{i=1}^{m} p_i = 1\} \quad \text{and} \quad \Delta_n = \{q \in \mathbb{R}^n | q \geq 0, \sum_{j=1}^{n} q_j = 1\}$$

represent the mixed or randomized strategy spaces. With $p \in \Delta_m(q \in \Delta_n)$, $p_i(q_j)$ is interpreted as the probability to select $e_i(f_j)$.
Moreover a mixed strategy combination $(p, q) \in \Delta_m\times\Delta_n$ is evaluated by means of its expected payoffs:

$$\pi_1(p, q) = p^T Aq = \sum_{i=1}^{m}\sum_{j=1}^{n} p_i q_j A_{ij} \quad \text{and} \quad \pi_2(p, q) = p^T Bq = \sum_{i=1}^{m}\sum_{j=1}^{n} p_i q_i B_{ij}$$

for all $(p, q) \in \Delta_m \times \Delta_n$.

The rows e_i (columns f_j) are identified with the unit probability vectors in Δ_m (Δ_n) and are called *pure strategies*. Obviously for all $(p, q) \in \Delta_m \times \Delta_n$ we have

$$\pi_1(p, q) = \sum_{i=1}^{m} p_i(e_i^T Aq) = \sum_{j=1}^{n} q_j(p^T Af_j)$$

while

$$\pi_2(p,q) = \sum_{i=1}^{m} p_i(e_i^T Bq) = \sum_{j=1}^{n} q_j(p^T Bf_j).$$

For convenience, we will omit the cumbersome transpose signs from the notation from now on. Moveover, a strategy $p \in \Delta_m$ will often be written as $p = p_1e_1 + \cdots + p_me_m$. Similarly, for $q \in \Delta_n$ we write $q = q_1f_1 + \cdots + q_nf_n$. From Theorem 11.1 it is readily concluded that every bimatrix game has at least one Nash equilibrium.

Theorem 11.2
Let (A, B) be an $m \times n$ bimatrix game. Then $E(A, B) \neq \emptyset$.

To check if a mixed strategy combination is a Nash equilibrium of a bimatrix game only unilateral deviations to pure strategies have to be considered.

Theorem 11.3
Let (A, B) be an $m \times n$ bimatrix game with $(\hat{p}, \hat{q}) \in \Delta_m \times \Delta_n$.
Then $(\hat{p}, \hat{q}) \in E(A, B)$ if and only if

$$\hat{p}A\hat{q} \geq e_iA\hat{q} \quad \textit{for all } i \in \{1, \ldots, m\}$$

and

$$\hat{p}B\hat{q} \geq \hat{p}Bf_j \quad \textit{for all } j \in \{1, \ldots, n\}$$

Proof
It suffices to prove that

$$\hat{p}A\hat{q} \geq pA\hat{q} \quad \text{for all}\, p \in \Delta_m$$

if and only if

$$\hat{p}A\hat{q} \geq e_iA\hat{q} \quad \text{for all}\, i \in \{1, \ldots, m\}$$

The "only if" part is obvious. Now assume $\hat{p}A\hat{q} \geq e_iA\hat{q}$ for all $i \in \{1, \ldots, m\}$ and take $p \in \Delta_m$. Then

$$pA\hat{q} = \sum_{i=1}^{m} p_i(e_iA\hat{q}) \leq \sum_{i=1}^{m} p_i(\hat{p}A\hat{q}) \leq \hat{p}A\hat{q}. \qquad \square$$

A bimatrix game (A, B) with $B = -A$ is called a *matrix game* and is usually just denoted by A. From Theorem 11.3 one readily derives

Theorem 11.4

Let A be an $m \times n$ matrix game with $(\hat{p}, \hat{q}) \in \Delta_m \times \Delta_n$. Then $(\hat{p}, \hat{q}) \in E(A)$ if and only if

$$\hat{p}A\hat{q} \geq e_i A\hat{q} \quad \text{for all } i \in \{1, \dots, m\}$$

and

$$\hat{p}A\hat{q} \leq \hat{p}Af_j \quad \text{for all } j \in \{1, \dots, n\}.$$

Exercises

Exercise 11.1

Determine, if possible, all dominant strategies (for each player) and all Nash equilibria for the following four two-person finite strategic games represented by the bimatrices below.

a)
$$\begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{bmatrix} 2,1 & 0,0 \\ 0,0 & 1,2 \end{bmatrix} \end{array}$$

b)
$$\begin{array}{cc} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{bmatrix} 2,1 & 0,0 \\ 0,0 & a,2 \end{bmatrix} \end{array} \quad \text{with } a \in \mathbb{R}$$

c)
$$\begin{array}{cc} & \begin{array}{ccc} L & M & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{bmatrix} 1,1 & 0,1 & 7,0 \\ 0,0 & -1,0 & 8,0 \end{bmatrix} \end{array}$$

d)
$$\begin{array}{cc} & \begin{array}{ccc} L & M & R \end{array} \\ \begin{array}{c} T \\ C \\ B \end{array} & \begin{bmatrix} 11,10 & 6,9 & 10,9 \\ 11,6 & 6,6 & 9,6 \\ 12,10 & 6,9 & 9,11 \end{bmatrix} \end{array}$$

Exercise 11.2

Reconsider the purification situation of Example 11.2.

(a) Construct the corresponding finite strategic game if the fine is $40,000$ Euro instead of $30,000$ Euro. Determine all Nash equilibria of this game.

(b) Construct the corresponding finite strategic game if the fine is $5,000$ Euro instead of $30,000$ Euro. Determine all Nash equilibria of this game.

(c) Is it possible to find a fine such that (P, P, P) becomes a Nash equilibrium of the corresponding finite strategic game?

Exercise 11.3

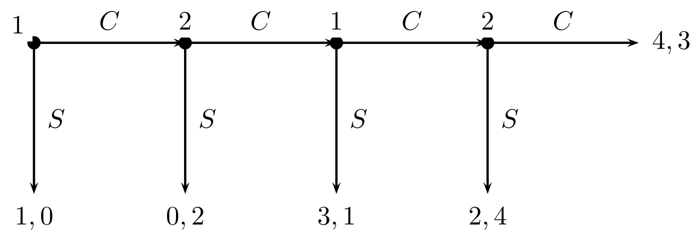
Determine all Nash equilibria of the two-person strategic game $\{(X_i, \pi_i)\}_{i \in N}$ with $N = \{1, 2\}$ and $X_1 = X_2 = [0, 1]$ if

a) $\pi_1(x_1, x_2) = x_1 x_2 - x_1^2 + x_1$ and $\pi_2(x_1, x_2) = x_1 x_2 - 5x_2^2 + x_1$ for all $(x_1, x_2) \in X_1 \times X_2$.

b) $\pi_1(x_1, x_2) = -x_1^2 + 4x_1 x_2$ and $\pi_2(x_1, x_2) = -x_2^2 + (2 - 2x_1)x_2$ for all $(x_1, x_2) \in X_1 \times X_2$.

Exercise 11.4

Consider the following two-person extensive "centipede-game" with game tree given by



with S representing "stop" and C representing "continue".

(a) Determine the corresponding strategic game.

(b) Determine all Nash equilibria of the strategic game in (a).

(c) Determine all subgame perfect equilibria of the centipede game.

Exercise 11.5 (A game of timing)

Consider the following noisy duel between two players. Each player has at most one firing option within the time interval $[0, T]$. For both players the hitting probability depends on the moment of firing and is given by $p(t) = \frac{t}{T}, t \in [0, T]$. So the longer you wait, the higher the actual hitting probability. The player who has been hit first pays 100 Euro to the other player. If a player fires but misses, the other player is aware of this ("noisy") and can wait until the moment $t = T$ to fire and have a hit with probability 1. Finally, if the players fire at exactly the same time moment and both have a successful hit, no payment takes place. Model this noisy duel as a strategic game and determine a Nash equilibrium of this game.

Exercise 11.6

Let $v \in \mathrm{TU}^N$. A way to construct a corresponding finite strategic game $G^v = \{(X_i, \pi_i)\}_{i \in N}$ is the following.

Define

$$X_i = \{S \in 2^N | i \in S\}$$

and

$$\pi_i(\{S_j\}_{j \in N}) = \begin{cases} \frac{v(S_i)}{|S_i|} & \text{if } S_j = S_i \text{ for all } j \in S_i \\ v(\{i\}) & \text{otherwise} \end{cases}$$

for all $\{S_j\}_{j \in N} \in \prod\limits_{j \in N} X_j$.

(a) Determine the corresponding three-person finite strategic game G^v if $N = \{1, 2, 3\}$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1, 2\}) = v(\{1, 3\}) = 10$, $v(\{2, 3\}) = 0$ and $v(N) = 10$. Calculate all Nash equilibria of G^v.

Reversely, let $G = \{(X_i, \pi_i)\}_{i \in N}$ be a finite strategic game. Define the corresponding TU-game v^G by

$$v^G(S) = \max_{x_S \in \prod\limits_{i \in S} X_i} \; \min_{x_{N \setminus S} \in \prod\limits_{i \in N \setminus S} X_i} \sum_{i \in S} \pi_i(x_S, x_{N \setminus S})$$

for all $S \in 2^N \setminus \{\emptyset\}$.

(b) Calculate v^G for the following finite two-person strategic game in bimatrix representation

$$\begin{array}{c c} & \begin{array}{cc} L & R \end{array} \\ \begin{array}{c} T \\ B \end{array} & \begin{bmatrix} 1,0 & 0,2 \\ 0,2 & 1,0 \end{bmatrix} \end{array}$$

(c) Calculate v^G for the finite three-person purification game of Example 11.2.

(d) Show that v^G is superadditive.

(e) Calculate $v^{(G^v)}$ for the game G^v of part (a).

(f) Show that for any $v \in TU^N$ we have that $v \leq v^{(G^v)}$.

(g) Show for any $v \in TU^N$ we have that $v = v^{(G^v)}$ if and only if v is superadditive.

Exercise 11.7

The exercise provides an alternative proof of Theorem 11.2, the existence of Nash equilibria for bimatrix games.

This alternative proof uses the fixed point theorem of Brouwer:

Let $C \subset \mathbb{R}^+$ be convex and compact and let $f : C \to C$ be a continuous function. Then there exists a fixed point $\hat{c} \in C$ such that $F(\hat{c}) = \hat{c}$.

Let (A, B) be an $m \times n$-bimatrix game. For $i \in \{1, 2, \ldots, m\}$, define $s_i : \Delta_m \times \Delta_n \to \mathbb{R}$ by

$$s_i(p, q) = \max\{0, e_i Aq - pAq\}$$

for all $(p, q) \in \Delta_m \times \Delta_n$.

Similarly, for $j \in \{1, 2, \ldots, n\}$, let the function $t_j : \Delta_m \times \Delta_n \to \mathbb{R}$ be defined by

$$t_j(p, q) = \max\{0, pBf_j - pBq\}$$

for all $(p, q) \in \Delta_m \times \Delta_n$.

Consider $s = (s_1, s_2, \ldots, s_m) : \Delta_m \times \Delta_n \to \mathbb{R}^m$ and $t = (t_1, t_2, \ldots, t_n) : \Delta_m \times \Delta_n \to \mathbb{R}^n$ and define $f : \Delta_m \times \Delta_n \to \Delta_m \times \Delta_n$ by

$$f(p, q) = \left(\frac{p + s(p, q)}{1 + \sum_{i=1}^{m} s_i(p, q)}, \frac{q + t(p, q)}{1 + \sum_{j=1}^{n} t_j(p, q)} \right)$$

for all $(p, q) \in \Delta_m \times \Delta_n$.

(a) Show that f is well-defined and continuous.

(b) Show that a fixed point of f is a Nash equilibrium of (A, B).

(c) Show that f has at least one fixed point.

Exercise 11.8

(a) Show that the strategy combination $(\frac{1}{2}e_1 + \frac{1}{2}e_2, f_3)$ is a Nash equilibrium of the bimatrix game (A, B) given by

$$(A,B) = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \begin{bmatrix} 0,8 & 1,0 & 0,6 \\ 2,0 & 0,8 & 0,6 \end{bmatrix} \end{array}$$

(b) Calculate the set $E(A)$ of Nash equilibria of the matrix game A given by

$$A = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \begin{bmatrix} 3 & -6 & -1 \\ 4 & 4 & 7 \end{bmatrix} \end{array}$$

Exercise 11.9

Let $\mathcal{G}$ be a class of strategic games in which the games possibly have different sets of players. A solution ϕ on $\mathcal{G}$ specifies for each strategic game $G \in \mathcal{G}$ a subset of the set of strategy combinations of G. A solution ϕ satisfies *individual rationality* on $\mathcal{G}$ if for every one-person game $G = \{(X_1, \pi_1)\} \in \mathcal{G}$ we have that

$$\phi(G) = \{\hat{x}_1 \in X_1 | \pi_1(\hat{x}_1) \geq \pi_1(x_1) \text{ for all } x_1 \in X_1\}$$

Clearly the Nash equilibrium correspondence E is a solution (on any class $\mathcal{G}$) which is indiviudally rational.

For $G = \{(X_i, \pi_i)\}_{i \in N} \in \mathcal{G}$, $\hat{x} \in \prod_{i \in N} X_i$ and $S \in 2^N \backslash \{\emptyset, N\}$ define the *reduced strategic game* $G^{\hat{x},S} = \{(X_i, \pi_i^{\hat{x},S})\}_{i \in S}$ by

$$\pi_i^{\hat{x},S}(x_S) = \pi_i(x_S, \hat{x}_{N \backslash S})$$

for all $x_S \in \prod_{j \in S} x_j$ and all $i \in S$.

(a) With G the bimatrix game given by

$$\begin{array}{c c} & \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \left[\begin{array}{ccc} 0,8 & 2,0 & 0,6 \\ 1,0 & 0,8 & 0,6 \end{array}\right] \end{array}$$

and $\hat{x} = (\frac{1}{2}e_1 + \frac{1}{2}e_2, \frac{2}{3}f_2 + \frac{1}{3}f_3)$, determine both $G^{\hat{x},\{1\}}$ and $G^{\hat{x},\{2\}}$.

(b) With G the finite three-person game of Example 11.2 and $\hat{x} = (P, NP, P)$, determine $G^{\hat{x},\{1,2\}}$.

A class $\mathcal{G}$ of strategic games is called *closed* if for all $G = \{(X_i, \pi_i)\}_{i \in N} \in \mathcal{G}$, all $\hat{x} \in \prod_{i \in N} X_i$ and all $S \in 2^N \backslash \{\emptyset, N\}$ we have that $G^{\hat{x},S} \in \mathcal{G}$. This means that a class $\mathcal{G}$ is closed if for all games within the class also all its reduced games belong to the class.

(c) Verify that the class of all finite strategic games is closed.

(d) Verify that the class of all mixed extensions of finite strategic games is closed.

Next we define the notion of consistency for a solution on a closed class $\mathcal{G}$ of strategic games. A solution ϕ on a closed class $\mathcal{G}$ is called *consistent* if for all $G = \{(X_i, \pi_i)\}_{i \in N} \in \mathcal{G}$, for all $\hat{x} \in \phi(G)$ and all $S \in 2^N \backslash \{\emptyset, N\}$ we have that $\hat{x}_S \in \phi(G^{\hat{x},S})$.

(e) Prove that the Nash equilibrium correspondence E satisfies consistency on any closed class $\mathcal{G}$ of strategic games.

(f) Let ϕ be a solution on a closed class $\mathcal{G}$ of strategic games that satisfies both individual rationality and consistency. Prove that $\phi(G) \subset E(G)$ for all $G \in \mathcal{G}$.

From (f) we may conclude that the Nash equilibrium correspondence E is the "largest" solution satisfying individual rationality and consistency on an arbitrary closed class $\mathcal{G}$ of strategic games.

Some relevant literature

von Neumann, J. and O. Morgenstern (1944). Theory of Games and Economic Behavior, Princeton University Press, Princeton.

Nash, J. (1951). Non-cooperative games. *Annals of Mathematics*, 54, 286-295.

Rosen, J. (1965). Existence and uniqueness of equilibrium points for n-person games. *Econometrica*, 33, 520-534.

Kakutani, S. (1941). A generalization of Brouwer's fixed point theorem. *Duke Mathematical Journal*, 8, 457-459.

Brouwer, L. (1912). Uber Abbildungen von Mannigfaltigkeiten. *Mathematische Annalen*, 71, 97-115.

Serrano, R. (2008). Nash program. In:The New Palgrave Dictionary of Economics. (Eds. Blume, L. and S. Durlauf), McMillan, London.

Hammond, P., Seidl, C. and S. Barbera (2004). Handbook of utility theory. Kluwer Academic Publishers, The Netherlands.

Borm, P. and S. Tijs (1992). Strategic claim games corresponding to an NTU-game. *Games and Economic Behavior*, 4, 58-71.

CHAPTER 12

Matrix games

An $m \times n$ matrix game A is an example of a strictly competitive game in the sense that the payoff functions of the two players are strictly opposed: for any mixed strategy combination $(p, q) \in \Delta_m \times \Delta_n$ player 1 receives a payoff of pAq while player 2 receives a payoff of $-pAq$. In other words: one player's gains are the other player's losses.
From Theorem 11.2 we know that each matrix game has at least one Nash equilibrium. We will show that Nash equilibria of matrix games exhibit special and rather compelling features. In particular, we will see that all Nash equilibria lead to the same payoffs. Moreover, the choice of an equilibrium strategy can indeed be viewed as the result of optimal behavior. To illustrate the underlying ideas we first analyze a specific example.

Example 12.1
Consider the 3×4 matrix game A given by

$$A = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array} \begin{array}{c} \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{bmatrix} 0 & 3 & 1 & 2 \\ 3 & -3 & 1 & 1 \\ -4 & -4 & -4 & -4 \end{bmatrix} \end{array}$$

Clearly $(\hat{p}, \hat{q}) = (\frac{2}{3}e_1 + \frac{1}{3}e_2, f_3) \in E(A)$ since $\hat{p}A\hat{q} = 1$ while

$$\hat{p}Af_1 = \hat{p}Af_2 = \hat{p}Af_3 = 1,\ \hat{p}Af_4 = \tfrac{5}{3}$$

which implies that $\hat{p}A\hat{q} \le \hat{p}Aq$ for all $q \in \Delta_4$, and

$$e_1A\hat{q} = e_2A\hat{q} = 1,\ e_3A\hat{q} = -4$$

which implies that $\hat{p}A\hat{q} \geq pA\hat{q}$ for all $p \in \Delta_3$.
Using a similar reasoning one can verify that also $(\hat{p}, \bar{q}) \in E(A)$ with $\bar{q} = \frac{2}{3}f_1 + \frac{1}{3}f_2$. Note that the expected payoff $\hat{p}A\bar{q}$ to player 1 in this equilibrium also equals 1. In fact we have that

$$E(A) = \{\tfrac{2}{3}e_1 + \tfrac{1}{3}e_2\} \times \text{Conv}(\{\tfrac{2}{3}f_1 + \tfrac{1}{3}f_2, f_3\})$$

and all Nash equilibria lead to an expected payoff of 1 for player 1.
So, combined with any equilibrium strategy of player 2, the strategy $\hat{p} = \frac{2}{3}e_1 + \frac{1}{3}e_2$ leads to an expected payoff of 1 for player 1. In fact it does more than that: $\hat{p}$ will *guarantee* an expected payoff of *at least* 1 whatever player 2 does:

$$\hat{p}Aq = (1 - q_4)1 + q_4\tfrac{5}{3} \geq 1 \quad \text{for all}\, q \in \Delta_4.$$

Moreover, there is no strategy p that will guarantee an expected payoff higher than 1: if $p_1 > \frac{2}{3}$, then $pAf_1 < 1$ and, if $p_1 < \frac{2}{3}$, then $pAf_2 < 1$. We may conclude that from a guaranteeing perspective $\hat{p}$ is optimal.
Similarly, any equilibrium strategy of player 2 guarantees the expected payoff to player 1 to be at most 1 whatever player 1 does, e.g.

$$pAf_3 = (1 - p_3)1 + p_3(-4) \leq 1 \quad \text{for all}\, p \in \Delta_3. \qquad \diamond$$

Theorem 12.1
Let A be an $m \times n$ matrix game with $(p^1, q^1) \in E(A)$ and $(p^2, q^2) \in E(A)$.
Then $p^1Aq^1 = p^2Aq^2$.

Proof
Using equilibrium conditions we have

$$p^1Aq^1 \geq p^2Aq^1 \geq p^2Aq^2 \geq p^1Aq^2 \geq p^1Aq^1$$

and therefore we have equalities everywhere. $\square$

We may conclude that all Nash equilibria of a matrix game lead to the same payoff to player 1. This unique equilibrium payoff to player 1 in a matrix game A is called the *value* of A and is denoted by $v(A)$.

An alternative interpretation of the value of a matrix game is obtained by considering optimally guaranteed expected payoff levels for both players.

The highest expected payoff level player 1 can guarantee himself is given by the number $\underline{v}(A) = \max_{p \in \Delta_m} \min_{q \in \Delta_n} pAq$.
This number reflects a rather pessimistic point of view for player 1. Each strategy $p \in \Delta_m$ is evaluated via a worst-case scenario. Given p a strategy of player 2 is selected that minimizes the expected payoff to player 1. Denoting this lowest payoff level by $l(p)$ we have that $l(p) = \min_{q \in \Delta_n} pAq$.
Subsequently, player 1 selects a strategy p for which the worst-case evaluation $l(p)$ is the best for him. The corresponding highest expected payoff level that player 1 can guarantee himself equals $\max_{p \in \Delta_m} l(p) = \max_{p \in \Delta_m} \min_{q \in \Delta_n} pAq = \underline{v}(A)$.
Similarly, the lowest expected payoff level to player 1 that player 2 can guarantee is given by the number $\bar{v}(A) = \min_{q \in \Delta_n} \max_{p \in \Delta_m} pAq$. Here, each strategy $q \in \Delta_n$ is pessimistically evaluated by the worst possible expected payoff to player 2 given by $h(q) = \max_{p \in \Delta_m} pAq$. Selecting a strategy for player 2 such that this evaluation is best for player 2 boils down to considering $\min_{q \in \Delta_n} h(q) = \min_{q \in \Delta_n} \max_{p \in \Delta_m} pAq = \bar{v}(A)$.

Note that both $\underline{v}(A)$ and $\bar{v}(A)$ are well-defined: all minima and maxima under consideration indeed exist. We illustrate this for $\underline{v}(A)$. The level $l(p)$ exists because it corresponds to a minimum of the continuous function $q \longmapsto pAq$ over the compact set Δ_n. Moreover, since $l(p) = \min_{q \in \Delta_n} pAq = \min_{j \in \{1,\ldots,n\}} pAf_j$ is the minimum over a finite number of continuous functions, it is continuous itself and $\underline{v}(A) = \max_{p \in \Delta_m} l(p)$ exists because Δ_m is compact.

Theorem 12.2

Let A be an $m \times n$ matrix game. Then $v(A) = \underline{v}(A) = \bar{v}(A)$.

Proof

First of all we prove that $\underline{v}(A) \le \bar{v}(A)$.
Obviously, for all $p^0 \in \Delta_m$ and $q^0 \in \Delta_n$ we have that

$$p^0 A q^0 \le \max_{p \in \Delta_m} pAq^0.$$

So, for each $p^0 \in \Delta_m$, the function $q^0 \longmapsto p^0Aq^0$ lies below the function $q^0 \longmapsto \max_{p \in \Delta_m} pAq^0$ and consequently, the minimum of the first function lies below the minimum of the second. Hence, for all $p^0 \in \Delta_m$, we have that

$$\min_{q \in \Delta_n} p^0Aq \le \min_{q \in \Delta_n} \max_{p \in \Delta_m} pAq = \bar{v}(A).$$

So, the function $p^0 \longmapsto \min_{q \in \Delta_n} p^0Aq$ lies below the constant function $p^0 \longmapsto \bar{v}(A)$. Consequently this is also true for the maximum of this function

$$\bar{v}(A) \ge \max_{p \in \Delta_m} \min_{q \in \Delta_n} pAq = \underline{v}(A).$$

Secondly, take $(\hat{p}, \hat{q}) \in E(A)$. This is possible because of Theorem 11.2. Then, on the one hand

$$v(A) = \hat{p}A\hat{q} = \max_{p \in \Delta_m} pA\hat{q} \geq \min_{q \in \Delta_n} \max_{p \in \Delta_m} pAq = \bar{v}(A),$$

while on the other hand

$$v(A) = \hat{p}A\hat{q} = \min_{q \in \Delta_n} \hat{p}Aq \leq \max_{p \in \Delta_m} \min_{q \in \Delta_n} pAq = \underline{v}(A).$$

Since, however, $\underline{v}(A) \leq \bar{v}(A)$ both inequalities above must be equalities and we may conclude that $v(A) = \underline{v}(A) = \bar{v}(A)$. $\square$

Example 12.2

Although $v(A) = \max_{p \in \Delta_m} \min_{q \in \Delta_n} pAq = \max_{p \in \Delta_m} \min_{j \in \{1,\dots,n\}} pAf_j$, it is not necessarily true that $v(A) = \max_{i \in \{1,\dots,m\}} \min_{j \in \{1,\dots,n\}} e_iAf_j$.
For this, reconsider the 3×4 matrix game A of Example 12.1.
We know that $v(A) = 1$ while

$$\max_{i \in \{1,\dots,3\}} \min_{j \in \{1,\dots,4\}} A_{ij} = \max\{0, -3, -4\} = 0. \qquad \diamond$$

Next we define optimal strategies to be such that these strategies guarantee the payoff of $v(A)$, for player 1 from above and for player 2 from below. Formally, the sets $O_1(A)$ and $O_2(A)$ of *optimal strategies* for the players are defined by

$$O_1(A) = \{\hat{p} \in \Delta_m | \hat{p}Aq \geq v(A) \quad \text{for all}\, q \in \Delta_n\}$$

while

$$O_2(A) = \{\hat{q} \in \Delta_n | pA\hat{q} \leq v(A) \quad \text{for all}\, p \in \Delta_m\}.$$

Clearly for determining optimal strategies, we can restrict to the levels of guarantee w.r.t. pure strategies of the opponent:

$$O_1(A) = \{\hat{p} \in \Delta_m | \hat{p}Af_j \geq v(A) \quad \text{for all}\, j \in \{1, \dots, n\}\}$$

and

$$O_2(A) = \{\hat{q} \in \Delta_n | e_iA\hat{q} \leq v(A) \quad \text{for all}\, i \in \{1, \dots, m\}\}.$$

From this, it is clear that both optimal strategy sets are polytopes. In particular $O_1(A)$ and $O_2(A)$ are convex: any convex combination of two optimal strategies is optimal too.
The relation between nash equilibria and optimal strategies is described in

Theorem 12.3

Let A be an $m \times n$ matrix game. Then $E(A) = O_1(A) \times O_2(A)$.

Proof

"$\subset$" : Let $(\hat{p}, \hat{q}) \in E(A)$. Then $v(A) = \hat{p}A\hat{q} \leq \hat{p}Af_j$ for all $j \in \{1, \ldots, n\}$. Consequently $\hat{p} \in O_1(A)$. Similarly, it follows that $\hat{q} \in O_2(A)$.

"$\supset$" : Let $\hat{p} \in O_1(A)$ and $\hat{q} \in O_2(A)$. Since $\hat{p} \in O_1(A)$, we have that

$$v(A) \leq \min_{q \in \Delta_n} \hat{p}Aq \leq \hat{p}A\hat{q}.$$

Similarly, since $\hat{q} \in O_2(A)$, we have that

$$v(A) \geq \max_{p \in \Delta_m} pA\hat{q} \geq \hat{p}A\hat{q}.$$

Hence, the four inequalities above are equalities and thus $(\hat{p}, \hat{q}) \in E(A)$. $\square$

From Theorem 12.3 it immediately follows that there is no coordination problem in selecting equilibrium strategies.

Corollary 12.4

Let A be an $m \times n$ matrix game with $(p^1, q^1) \in E(A)$ and $(p^2, q^2) \in E(A)$. Then both $(p^1, q^2) \in E(A)$ and $(p^2, q^1) \in E(A)$.

We now discuss a geometric method to determine the set $E(A)$ of all Nash equilibria (or equivalently the value $v(A)$ and the optimal strategy sets $O_1(A)$ and $O_2(A)$) for an arbitrary $2 \times n$ matrix game.
This method will be generalized to $2 \times n$ bimatrix games in Chapter 14.

Solution method for $2 \times n$ matrix games

(1) *Draw the n lines $p \longmapsto pAf_j, j \in \{1, \ldots, n\}$.*

(2) *Determine the (piecewise linear) minimum function.*

$$p \longmapsto l(p) = \min_{j \in \{1,\ldots,n\}} pAf_j.$$

(3) *$v(A)$ equals the maximum value of the minimum function $l(p)$.*

(4) *$O_1(A)$ exactly corresponds to the set of maximum locations of the minimum function $l(p)$.*

(5) *$\hat{q} \in O_2(A)$ if and only if the line $p \longmapsto pA\hat{q}$ lies (weakly) below the horizontal line $p \longmapsto v(A)$.*
(Note that the line $p \longmapsto pA\hat{q}$ is a convex combination of the lines $p \longmapsto pAf_j$, $j \in \{1, \ldots, n\}$, via the probabilities on the pure strategies f_j prescribed by $\hat{q}$.)

This method is clarified in the next example.

Example 12.3

Consider the 2×4 matrix game A given by

$$A = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{bmatrix} 2 & 0 & 1 & 2 \\ 0 & 3 & 1 & 3 \end{bmatrix} \end{array}.$$

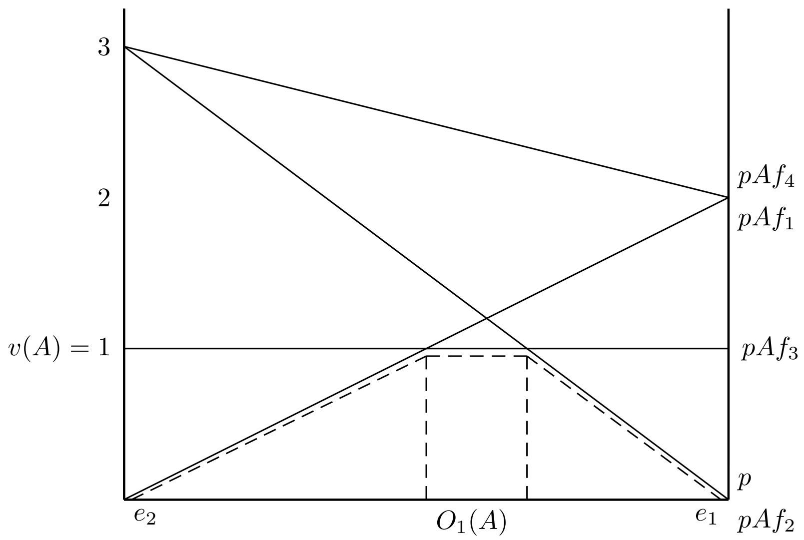


Figure 12.1: Solving the $2 \times n$ matrix game in Example 12.3.

From Figure 12.1 it can be deduced that

$$v(A) = 1,\ O_1(A) = \text{Conv}(\{\tfrac{1}{2}e_1 + \tfrac{1}{2}e_2,\ \tfrac{2}{3}e_1 + \tfrac{1}{3}e_2\}) \text{ and } O_2(A) = \{f_3\}.$$

Consequently,

$$E(A) = \text{Conv}(\{\tfrac{1}{2}e_1 + \tfrac{1}{2}e_2,\ \tfrac{2}{3}e_1 + \tfrac{1}{3}e_2\}) \times \{f_3\}. \qquad \diamond$$

Exercises

Exercise 12.1

Using reaction curves (best reply curves), determine the value and all Nash equilibria for the folllowing 2×2 matrix games.

(a)
$$A = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{cc} f_1 & f_2 \end{array} \\ \begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix} \end{array}.$$

(b)
$$A = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{cc} f_1 & f_2 \end{array} \\ \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \end{array}.$$

Exercise 12.2

Determine the value and all Nash equilibria for the following 3×3 matrix games.

(a)
$$A = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array} \begin{array}{c} \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}.$$

(b) (Rock, Paper, Scissors)
$$A = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array} \begin{array}{c} \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \end{array}.$$

(c)
$$A = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array} \begin{array}{c} \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \begin{bmatrix} 37 & 0 & 0 \\ 0 & 38 & 0 \\ 0 & 0 & 39 \end{bmatrix} \end{array}.$$

Exercise 12.3

An attacker (player 1) with 3 armies aims to defeat a defender (player 2) with 2 armies on two different locations A and B. Both the attacker and the defender simultaneously have to decide upon an allocation of their armies over the two locations.

At each location the player with the higher number of armies wins the battle and his payoff at this location equals 1 (for conquering the location) or 0 (for maintaining the location) plus the number of armies of the opponent that are defeated. If, at a certain location, both the attacker and the defender have the same number of armies we have a standoff with zero payoffs. The total payoff of a player for a specific engagement is the sum of the two payoffs per location.
Model this conflict as a matrix game, calculate the value of this game, and determine all optimal strategies for each player.

Exercise 12.4
Let A and A' be two $m \times n$ matrix games.

(a) Show that

$$O_1(A) \cap O_1(A') \neq \emptyset \Rightarrow v(A + A') \geq v(A) + v(A').$$

(b) Provide A and A' such

$$v(A + A') < v(A) + v(A').$$

Exercise 12.5
For an $m \times n$ matrix game A let (as before)

$$\underline{v}(A) = \max_{p \in \Delta_m} \min_{j \in \{1,\ldots,n\}} pAf_j,$$

and

$$\overline{v}(A) = \min_{q \in \Delta_n} \max_{i \in \{1,\ldots,m\}} e_i Aq.$$

This exercise provides a direct (induction) proof of the fact that $\underline{v}(A) = \overline{v}(A)$.

(a) Show that $m + n = 2$ implies $\underline{v}(A) = \overline{v}(A)$.

Take a fixed $m \times n$ matrix game A with $m + n = t, t > 2$ and assume that for all $r \times s$ matrix games B with $r + s < t$ it holds that $\underline{v}(B) = \overline{v}(B)$. Take $\hat{p} \in \Delta_m$ and $\hat{q} \in \Delta_n$ such that

$$\underline{v}(A) = \min_{j \in \{1,\ldots,n\}} \hat{p}Af_j \quad \text{and} \quad \overline{v}(A) = \max_{i \in \{1,\ldots,m\}} e_i A\hat{q}.$$

(b) Show that $\underline{v}(A) = \overline{v}(A)$ if

$$\hat{p}Af_j = \underline{v}(A), \; e_i A\hat{q} = \overline{v}(A)$$

for all $i \in \{1, \ldots, m\}$ and all $j \in \{1, \ldots, n\}$.

Because of (b) assume (without loss of generality) that there is an $k \in \{1, \ldots, m\}$ such that $e_k Aq < \overline{v}(A)$. Then, obviously, $m > 1$ and define A^{-k} as the $(m-1) \times n$ submatrix of A where the k-th row is deleted.

(c) Show that $\underline{v}(A) \geq \underline{v}(A^{-k})$ and $\overline{v}(A) \geq \overline{v}(A^{-k})$.

Now it is sufficient to show that $\overline{v}(A) = \bar{v}(A^{-k})$.

(d) Prove that this is indeed sufficient.

Suppose on the contrary that $\overline{v}(A) > \overline{v}(A^{-k})$ (in view of (c)). Take $q^* \in \Delta_n$ such that $\max_{i \in \{1,\ldots,m\} \setminus \{k\}} e_i A^{-k} q^* = \overline{v}(A^{-k})$ and define $q(\varepsilon) = (1-\varepsilon)\hat{q} + \varepsilon q^*$ for $\varepsilon \in (0,1)$.

(e) Show that $e_i Aq(\varepsilon) < \overline{v}(A)$ for all $i \in \{1, \ldots, m\},\ i \neq k$

(f) Show that $e_k Aq(\varepsilon) < \overline{v}(A)$ for ε small enough

(g) Arrive at a contradiction, thus finishing the proof.

Exercise 12.6

(a) Let A be an $m \times n$ matrix game. Prove that $v(-A^T) = -v(A)$.

(b) An $m \times n$ matrix game A is called *symmetric* if $A = -A^T$.
Prove that $v(A) = 0$ for a symmetric matrix game.

Exercise 12.7

Determine the value and the set of all Nash equilibria for the following matrix games.

(a)
$$A = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \end{bmatrix} \end{array}.$$

(b)
$$A = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array} \begin{array}{c} \begin{array}{cc} f_1 & f_2 \end{array} \\ \begin{bmatrix} -1 & -2 \\ -2 & -3 \\ -3 & -2 \end{bmatrix} \end{array}.$$

(c)
$$A = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 0 \end{bmatrix} \end{array}.$$

(d)
$$A = \begin{array}{c c} & \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{bmatrix} 8 & 0 & 2 \\ 0 & 8 & 6 \end{bmatrix} \end{array}.$$

Exercise 12.8
For all $a \in \mathbb{R}$ determine $v(A)$ and $E(A)$ for the matrix game A given by

$$A = \begin{array}{c c} & \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{bmatrix} 0 & 2 & 4 & 2 \\ 4 & 2 & 2 & a \end{bmatrix} \end{array}.$$

Exercise 12.9
Let (A, B) be an $m \times n$ bimatrix game such there exists a constant c for which $A_{ij} + B_{ij} = c$ for all $i \in \{1, \ldots, m\}$ and all $j \in \{1, \ldots, n\}$. Prove that $E(A, B) = E(A)$.

Exercise 12.10 (A characterization of the *value function* for matrix games)
Show the following four assertions:

(i) If $A = [A_{11}]$ is a 1×1 matrix game, then $v(A) = A_{11}$.

(ii) If A^1 and A^2 are two arbitrary $m \times n$ matrix games with $A^1 \geq A^2$, then $v(A^1) \geq v(A^2)$.

(iii) For an arbitrary $m \times n$ matrix game we have (cf. Exercise 12.6)
$$v(A) = -v(-A^T).$$

(iv) If A is an arbitrary $m \times n$ matrix game and B is an $(m+1) \times n$ matrix game derived from A by adding one row e_{m+1} which is inferior to (or equivalent to) a convex combination of the rows of A (i.e., there exists a $p \in \Delta_m$ such that $pA \geq e_{m+1}B$), then $v(B) = v(A)$.

It can be shown that the value function is the only real-valued function on the class of all matrix games satisfying the properties (i) to (iv).

Exercise 12.11 (General two-person zero-sum games)
Consider a general two-person *zero-sum* game G of the form

$$G = \{(X_1, \pi), (X_2, -\pi)\}$$

(a) Let $(\hat{x}_1, \hat{x}_2) \in E(G)$ and $(\hat{y}_1, \hat{y}_2) \in E(G)$.

(i) Prove that $\pi(\hat{x}_1, \hat{x}_2) = \pi(\hat{y}_1, \hat{y}_2)$.

(ii) Prove that $(\hat{x}_1, \hat{y}_2) \in E(G)$ and $(\hat{y}_1, \hat{x}_2) \in E(G)$.

Define the *lower value* $\underline{v}(G)$ by $\underline{v}(G) = \sup_{x_1 \in X_1} \inf_{x_2 \in X_2} \pi(x_1, x_2)$ and the *upper value* $\overline{v}(G)$ by $\overline{v}(G) = \inf_{x_2 \in X_2} \sup_{x_1 \in X_1} \pi(x_1, x_2)$.

(b) Determine $\overline{v}(G)$ and $\underline{v}(G)$ for the finite two-person zero-sum game G represented by

$$G = \begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{cc} L & R \end{array} \\ \begin{bmatrix} 0,0 & 3,-3 \\ 3,-3 & -3,3 \end{bmatrix} \end{array}$$

(c) Determine $\overline{v}(G)$ and $\underline{v}(G)$ for the two-person zero-sum game $G = \{(X_1, \pi), (X_2, -\pi)\}$ with $X_1 = \{T, B\}$ and $X_2 = \{a_1, a_2, a_3, \dots\}$ where the payoff function π is represented by the following semi-infinite matrix

$$G = \begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{ccccc} a_1 & a_2 & a_3 & a_4 & \dots \end{array} \\ \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \dots \\ 0 & 0 & 0 & 0 & \dots \end{bmatrix} \end{array}$$

(d) Prove that $\underline{v}(G) \leq \overline{v}(G)$.

Define the set of *optimal strategies* $O_1(G)$ for player 1 by

$$O_1(G) = \{\hat{x}_1 \in X_1 | \pi(\hat{x}_1, x_2) \geq \underline{v}(G) \quad \text{for all} \;\; x_2 \in X_2\}$$

and the set $O_2(G)$ of optimal strategies for player 2 by

$$O_2(G) = \{\hat{x}_2 \in X_1 | \pi(x_1, \hat{x}_2) \leq \overline{v}(G) \quad \text{for all} \;\; x_1 \in X_1\}.$$

(e) Determine the sets of optimal strategies for the games of part (b) and (c).

(f) Consider the mixed extension of the semi-infinite zero-sum game represented by

$$G = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{ccccc} f_1 & f_2 & f_3 & f_4 & \dots \end{array} \\ \begin{bmatrix} 2 & 0 & 0 & 0 & \dots \\ 0 & 2\frac{1}{2} & 2\frac{1}{3} & 2\frac{1}{4} & \dots \end{bmatrix} \end{array}$$

Here the mixed strategy space of player 2 is defined by

$$\begin{aligned} \Delta_\infty = \{ & (q_1, q_2, q_3, \dots) \mid \sum_{j=1}^{\infty} q_j = 1, q_j \geq 0 \quad \text{for all} \quad j \in \{1, 2, \dots\} \\ & \text{and there is an integer} \;\; N \text{ such that } q_n = 0 \text{ for all } n \geq N\} \end{aligned}$$

to avoid problems in calculating expected payoffs.

Show that $\underline{v}(G) = \overline{v}(G) = 1$ and that $O_2(G) = \emptyset$.

(g) Determine $\underline{v}(G)$ and $\overline{v}(G)$ for the two-person zero-sum game $G = \{(X_1, \pi), (X_2, -\pi)\}$ where π is represented by

$$\begin{array}{cc} & \begin{array}{cc} f_1 & \quad f_2 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{bmatrix} 0,0 & 3,-3 \\ 3,-3 & -3,3 \end{bmatrix} \end{array}$$

if

(i) $X_1 = \{e_1, e_2\}, X_2 = \Delta_2$.

(ii) $X_1 = \Delta_2, X_2 = \{f_1, f_2\}$.

(g) Prove that for any two-person zero-sum game G

$$E(G) \neq \emptyset \Leftrightarrow \underline{v}(G) = \overline{v}(G), O_1(G) \neq \emptyset \,\text{and}\, O_2(G) \neq \emptyset.$$

Some relevant literature

von Neumann, J. (1928). Zur Theorie der Gesellschaftsspiele. *Mathematische Annalen*, 100, 295-300.

Owen, G. (1967). An elementary proof of the minimax theorem. *Management Science*, 13, 765.

Parthasarathy, T. and T. Raghavan (1971). Some topics in two-person games. Elsevier.

Vilkas, E. I. (1963). Axiomatic definition of the value of a matrix game. *Theory of Probability and its Applications*, 8, 304-307.

Karlin, S. (2003) Mathematical methods and theory in games, programming and economics: two volumes bound as one. Dover Phoenix Editions, Courier Dover Publications.

CHAPTER 13
Applications

This section discusses two application fields of matrix games. The first application deals with casino games with special emphasis on the distinction between games of skill and games of chance. The second application analyzes the role of a minimax approach within the field of statistical decision theory.

13.1 Skill in games

Does one really need skill to play the dice game backgammon or is it basically just a game of chance? Perhaps the nice thing about this question is the fact that almost everybody will arrive at a specific opinion rather quickly. For those who play backgammon frequently it will definitely be a game of skill, in this way implicitly justifying the amount of time spent to this game. For other people, for example for fanatic chess players, the fact that a player at every move is dependent on the throwing of dices will be unbearable and for this reason alone these people will be inclined to qualify backgammon as a game of chance. This is typical for many games: opposite opinions about the nature or qualification of a game. One person will judge the level of skill of a casino game high if already one individual can beat the casino in the long run. Another person will only be convinced of the presence of skill if a vast majority of players actually displays skill. Still another person will say that by definition every casino game is a game of chance.

The aim of this section is to present a basic and introductory treatment of the concept of relative skill. We restrict ourselves to a juridical point of view as reflected by gaming acts in several European countries, such as in the Dutch Gaming Act:

> *.... it is not allowed to: exploit games with monetary prizes if the participants in general do not have a predominant influence on the winning possibilities, unless in compliance to this act, a license is granted ...*

In practice the Dutch state only grants such a license to Holland Casino's foundation. It is important to note that the Gaming Act only relates to games with monetary prizes. Moreover, the formulation of the Gaming Act clearly implies that skill should be considered *relatively* with respect to chance. In the remainder of this section we call a game that is covered by the Gaming Act a *game of chance*. A game which is not a game of chance will be called a *game of skill*. So, in particular, to commercialize games of skill no license within the meaning of the Gaming Act is needed. Unfortunately, judgements about the role of chance can be rather subjective, as illustrated above. If it would be possible to order a broad class of casino games with chance elements by means of an operational and objective criterium which quantifies the level of skill relative to chance, e.g. on a scale from zero to one, the legislator would be able to decide on a certain *skill threshold* on the level of relative skill, below which a game should be considered as a game of chance. To this aim this section proposes a specific definition to measure relative skill. Approximations of this measure for simple variations of the game of Poker are provided.

The Gaming Act makes a distinction between a player's actions as a determinant of his winning possibilities (measured in terms of monetary gain) and extraneous factors. The jurisprudence therefore distinguishes between a *learning effect*, which is the effect of a player's strategy on the outcome, and a *random effect*, which is the effect of the chance elements on the outcome. Apparently, given the term *learning* effect, the elementary level of a beginner forms the starting point of the discussion.

To quantify both effects, we consider three player types. A *beginner* is a player who has just mastered the rules of the game and is endowed with a particular but typically naive strategy, that possibly involves randomization. An *optimal player* is a player who has completely mastered the rules of the game and picks a strategy that maximizes his expected gains. A *fictive player* is a player who chooses a gain-maximizing strategy, whilst knowing (but not being able to influence) in advance the realizations of *all* random factors. We will come back to this in more detail later on.

For each of the three player types, the player's strategy may depend on his position or *role* in the game (asymmetry between the players as caused by the game's rules). In a poker game, players behave differently in the first position at the table than they do at the last position. The strategy of a player typically depends on which position at the table he occupies.

In a one-player game, like roulette or blackjack, the expected gain of a player in each role is determined by the descriptions above. Both roulette and blackjack are of course played with more players, but a particular player's gain does not depend on the other players' strategy choices. So, from a strategic point of view, these games can be viewed as a series of parallel one-player games. For a beginner, you simply compute the expected gain of his endowed strategy, while for the optimal and fictive player, you solve a constrained optimization problem to determine their optimal behavior.
In a game with more players, however, a player's gain is ambiguous, because you should specify *against which* (strategies of the) opponents the gain should be computed. We will make the assumption that all opponents are beginners, each endowed with a predetermined strategy. This assumption can be motivated as follows: First of all, beginners form a natural benchmark. The learning effect is supposed to measure the effect of mastering all the strategic intricacies of the game and the most natural way to measure this is to keep all other things (the opponents' strategies) constant. Secondly, and most importantly, since game trees are large even in the most basic variant of poker, fixing the beginner's strategy for each opponent is a necessity from a computational perspective.
The beginner's strategy thus provides a benchmark against which the gains of all player types are measured. It is important to stress that we do *not* provide a detailed prescription for coming up with an appropriate beginner's strategy. Rather, our methodology provides a general framework for determining the relative skill level of a particular game once a suitable beginner's strategy has been agreed upon. Note that this beginner's strategy in itself is of no interest in view of the Gaming Act, only the resulting gains of the three opposing player types.

We denote the finite set of roles by R and the gain of the beginner, optimal player and fictive player in role $r \in R$ by g_r^b, g_r^o and g_r^f, respectively. In the analysis of poker, the player's expected payoff is taken as gain. The obvious alternative would be to consider the expected return. Since betting heights however are a strategic choice in poker, the expected payoff seems more appropriate. Moreover, in the underlying analysis of strategic behavior, expected payoffs seem the most natural choice to evaluate the quality of strategies.

The *learning effect* in role $r \in R$ is defined as the difference in gain between the optimal player and the beginner

$$LE_r = g_r^o - g_r^b$$

and the *random effect* is defined as the difference between the gain of the fictive player and the optimal player

$$RE_r = g_r^f - g_r^o.$$

It follows from the definitions of the player types that both effects are always non-negative: an optimal player maximizes his gain over a strategy space which includes the (fixed) beginner's strategy, and the fictive player maximizes his gain over an even larger strategy space.
Next, we average the effects over all player roles (implicitly making the natural assumption that each player is assigned each role with equal probability, or alternatively that all players take each role in turn):

$$LE = \frac{1}{|R|} \sum_{r \in R} LE_r, \quad RE = \frac{1}{|R|} \sum_{r \in R} RE_r.$$

If the learning effect is not predominant, i.e., small compared to the random effect, the game is deemed a game of chance. This leads to the following definition of relative skill S:

$$S = \frac{LE}{LE + RE}.$$

The minimal relative skill level of 0 indicates a *pure game of chance* in the sense that the beginner and the optimal player have the same gain. The maximal relative skill level of 1 indicates a *pure game of skill*, because apparently the fictive player cannot obtain additional gains compared to the optimal player using the extra information he has about all random factors. In particular, the latter will typically occur in games that possess no chance elements.

We argue that the fictive player should be endowed with knowledge of the realisations of internal chance moves (an opponent randomizing between various pure strategies) as well as external chance moves (for instance, the dealing of cards).
Consider the well-known game of stone-paper-scissors. Two players simultaneously choose either stone, paper or scissors by means of a gesture. A player choosing stone beats his opponent if that player chooses scissors, scissors beats paper and paper beats stone. If both players make the same choice, the game ends in a tie. The winner receives 1 Euro from the loser, and in case of a tie no payment is made. Because of the cyclical winning conditions, there is no *a priori* distinction between the three pure strategies. As a result, it seems reasonable to take the mixed strategy in which stone, paper and scissors are all played with probability $\frac{1}{3}$ as the beginner's strategy. The optimal player cannot play this game any better than the beginner, both having an expected gain of 0. We assume that the fictive player knows the realisations of both the external chance moves and the opponents' internal chance moves beforehand. Since there are no external chance moves in stone-paper-scissors, he does not

gain anything on the optimal player on that basis. However, knowing the realization of the internal chance moves, he will win every single game. Hence, the random effect is 1, leading to a relative skill level of 0, a rather suitable conclusion.

To give an impression of the magnitude of S, the relative skill level of various one-player games like standard roulette, American roulette, and Golden Ten are 0, 0.004, and 0.012, respectively. Depending on the exact variant at hand, the level of relative skill for Blackjack ranges between 0.06 and 0.2.

A recent legal case with far-reaching consequences involved Grand Prix Manager 2003, which was deemed to be a game of skill. GPM 2003 is a so-called management game, in which a participant acts as the manager of a fictive sports team. The goal is to assemble a racing team (in terms of car components and personnel) that performs well in a simulated season of Formula One racing. Note that a management game is not a strategic game, since the probabilities involved in the external chance moves are not explicitly known to the players as argued in Van der Genugten et al. (2004). Because the game has many participants and many rounds, however, one can use statistical techniques to analyze the relative skill level of such a management game as if it were a strategic game. Using a prize scheme in which the prizes are not restricted to only the top few players in the final ranking, the relative skill level of GPM 2003 equals approximately 0.3. Comparing this with earlier verdicts on games that were judged to be games of chance, a reasonable skill threshold area above which a game should be considered a game of skill would be $0.2 - 0.3$. Arnhem District Court accepted the arguments in full (2 February 2005, nr.105364), thereby setting a legal precedent for the skill level threshold area of $0.2 - 0.3$.

As an illustration we compute the relative skill level for a basic poker variant.

Example 13.1 (Two-person Mini Poker)
Two-person Mini Poker is a game of cards played by two players, named player 1 and player 2, and with three cards of which only the numeric value is important. These values are 10, 20 and 30, respectively.
Before playing, both players donate 1 Euro to the stakes. After shuffling the deck of cards each player is dealt one card. Each player knows his own card but not the card of this opponent. Thus, the one card which remains in the deck is not shown to either of the players. Player 1 starts the play and has to decide between "checking" (Ch) or "betting"(B). If he decides to check, a "showdown" follows. If player 1 decides to bet, he has to add 1 Euro extra to the stakes. Subsequently, player 2 has to decide between "folding"(F) or "calling"(Ca). If

he decides to fold, player 1 gets the stakes. If player 2 decides to call, he also has to add 1 Euro to the stakes and a "showdown" follows. If the players reach a "showdown", then both cards are compared and the player with the highest card value gets the stakes.

Obviously, already in this basic game of Poker both chance (dealing cards) and skill (a good betting strategy) play a role.

First we describe the possible strategies of both players. For each of the three possible cards, both player 1 and player 2 have to choose between two possible actions, leading to a total of 8 pure strategies for each player. These pure strategies are represented in Table 13.1 and Table 13.2.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
10	Ch	Ch	Ch	Ch	B	B	B	B
20	Ch	Ch	B	B	Ch	Ch	B	B
30	Ch	B	Ch	B	Ch	B	Ch	B

Table 13.1: Pure strategies of player 1 in Example 13.1.

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8
10	F	F	F	F	Ca	Ca	Ca	Ca
20	F	F	Ca	Ca	F	F	Ca	Ca
30	F	Ca	F	Ca	F	Ca	F	Ca

Table 13.2: Pure strategies of player 2 in Example 13.1.

For example, the strategy e_4 of player 1 should be interpreted as

"Check with 10, bet with 20, bet with 30",

and the strategy f_6 of player 2 as

"Call with 10, fold with 20, call with 30",

should player 2 be called upon to act, i.e., after a bet of player 1. Note that, after a check of player 1, player 2 does not have to make a choice at all.
Allowing for mixed strategies, each player may randomize between his pure strategies. The strategy $\frac{1}{2}e_2 + \frac{1}{2}e_4$ of player 1 will indicate that player 1 will choose the pure strategy e_2 with probability $\frac{1}{2}$ and the pure strategy e_4 with probability $\frac{1}{2}$: "Check with 10, bet with 30, and with 20: check with probability $\frac{1}{2}$ and bet with probability $\frac{1}{2}$"

We first analyze this game from a purely game theoretic perspective. To this aim we first calculate the corresponding 8×8 matrix game A. The result is presented in Table 13.3. More specifically, in Table 13.3 the result of a specific strategy combination is the mean of the outcomes w.r.t. the six possible card combinations.

$$A = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \\ e_6 \\ e_7 \\ e_8 \end{array} \begin{array}{c} \begin{array}{cccccccc} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 & f_7 & f_8 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1/6 & 1/6 & 1/6 & 1/6 & 1/3 & 1/3 \\ 1/3 & -1/6 & 1/3 & -1/6 & 1/2 & 0 & 1/2 & 0 \\ 1/3 & -1/6 & 1/2 & 0 & 2/3 & 1/6 & 5/6 & 1/3 \\ 2/3 & 1/6 & 1/6 & -1/3 & 2/3 & 1/6 & 1/6 & -1/3 \\ 2/3 & 1/6 & 1/3 & -1/6 & 5/6 & 1/3 & 1/2 & 0 \\ 1 & 0 & 1/2 & -1/2 & 7/6 & 1/6 & 2/3 & -1/3 \\ 1 & 0 & 2/3 & -1/3 & 4/3 & 1/3 & 1 & 0 \end{bmatrix} \end{array}$$

Table 13.3: The two-person Mini Poker in Example 13.1 as a matrix game.

By iteratively deleting *dominated* pure strategies (e.g. the strategy e_1 of player 1 is dominated by strategy e_2, because $e_2A \geq e_1A$ and $e_2A \neq e_1A$), the 8×8-matrix game of Table 13.3 can be reduced to a 2×2 matrix game A^R with

$$A^R = \begin{array}{c} \\ e_2 \\ e_6 \end{array} \begin{array}{c} \begin{array}{cc} f_2 & f_4 \end{array} \\ \begin{bmatrix} 0 & 1/6 \\ 1/6 & -1/6 \end{bmatrix} \end{array}$$

Consequently, $v(A^R) = \frac{1}{18}$. Moreover, the optimal strategies of the players in A^R are uniquely determined: $\frac{2}{3}e_2 + \frac{1}{3}e_6$ for player 1 and $\frac{2}{3}f_2 + \frac{1}{3}f_4$ for player 2. It is readily seen that these strategies are also optimal in A but in theory there could be more optimal strategies in A. The two optimal strategies boil down to the following optimal behavior:

For player 1:

With 10: Ch with probability $\frac{2}{3}$, B with probability $\frac{1}{3}$

With 20: Ch

With 30: B

For player 2 (if called upon to act):

With 10: F

With 20: F with probability $\frac{2}{3}$, Ca with probability $\frac{1}{3}$

With 30: Ca.

Parts of these optimal strategies we could have determined beforehand. Knowing that he will lose more by calling with 10 than by folding, player 2 will fold with 10. Similarly he will win more by calling with 30 than by folding, so player 2 will call with 30.
With respect to player 1 having 30, there is no disadvantage in betting:
player 1 is sure to win and by betting he can only win more. Anticipating the behavior of player 2, player 1 does also know what is best to do with 20. With probability $\frac{1}{2}$ player 2 has 10 and will fold after a bet. In this case both Ch and B lead to a payoff of 1. With probability $\frac{1}{2}$ however player 2 has 30 and will call after a bet. In this case Ch leads to a payoff of -1 while B leads to -2. Consequently, with 20 player 1 should check.

Moreover, note that the optimal strategy of player 1 provided above involves the aspect of *strategic bluffing*. With 10 he bets with probability $\frac{1}{3}$, thus exploiting the uncertainty of player 2 in case he should have 20. Already in this basic variant of poker bluffing is an essential part of optimal rational behavior.

Next, we determine the relative skill level for two-person Mini Poker. For this we need to make a choice for the strategy of a beginner in both the role of player 1 and player 2.

We assume a naive player 1 to check with 10 and to bet with 30. How to act with 20? Probably it is wise to vary between checking and raising, and using symmetry arguments, to check and raise with equal probability. In fact, the same reasoning applies w.r.t. a naive player 2. If he is called upon to act, we assume he folds with 10, calls with 30 and folds and calls with equal probability with 20.

Using the notation introduced above, this boils down to the beginner's strategy $\frac{1}{2}e_2 + \frac{1}{2}e_4$ for player 1 and to $\frac{1}{2}f_2 + \frac{1}{2}f_4$ for player 2.

From Table 13.3 it readily follows that

$$g_1^b = (\tfrac{1}{2}e_2 + \tfrac{1}{2}e_4)A(\tfrac{1}{2}f_2 + \tfrac{1}{2}f_4) = \tfrac{1}{4}(0 + \tfrac{1}{6} + (-\tfrac{1}{6}) + 0) = 0.$$

Consequently $g_2^b = -g_1^b = 0$.

Furthermore, since

$$A(\tfrac{1}{2}f_2 + \tfrac{1}{2}f_4) = \begin{bmatrix} 0 \\ \frac{1}{12} \\ -\frac{1}{6} \\ -\frac{1}{12} \\ -\frac{1}{12} \\ 0 \\ -\frac{1}{4} \\ -\frac{1}{6} \end{bmatrix},$$

$g_1^o = \frac{1}{12}$. Similarly, from the fact that

$$(\tfrac{1}{2}e_2 + \tfrac{1}{2}e_4)(-A) = \begin{bmatrix} -\frac{1}{6} & \frac{1}{12} & -\frac{1}{3} & -\frac{1}{12} & -\frac{5}{12} & -\frac{1}{6} & -\frac{7}{12} & -\frac{1}{3} \end{bmatrix}$$

one derives that $g_2^o = \frac{1}{12}$.

Note that in determining g_1^o and g_2^o we can not in general restrict attention to A^R. If player 2's beginner's strategy would be f_7, then e_8 would be the optimal thing to do for player 1 and not a choice between e_2 and e_6.

To determine g_1^f, we have to consider a fictive player in the role of player 1.

By assumption, a fictive player will know the precise card combination (C_1, C_2), with $C_i \in \{10, 20, 30\}, C_1 \neq C_2$, and the outcome of the randomization invoked by the mixed strategy $\frac{1}{2}f_2 + \frac{1}{2}f_4$ of player 2 as a beginner.

For each card combination and randomization outcome a fictive player can choose the action with maximal gains for him. The number g_1^f then will be the mean of the six corresponding gains per card combination where for each card combination the gains are computed as the expected results via the probabilities induced by player 2's mixed strategy. This computation, which is illustrated in Table 13.4, leads to

$$g_1^f = \tfrac{1}{6}(0 - 1 + 1 - 1 + 1 + \tfrac{3}{2}) = \tfrac{1}{4}.$$

Using Table 13.5 (where the numbers represent payoffs to player 2) one derives in an analogous way that $g_2^f = \frac{1}{12}$.

We may conclude that

$$\begin{array}{llll} LE_1 = g_1^o - g_1^b = \frac{1}{12}, & LE_2 = g_2^o - g_2^b = \frac{1}{12} & \text{and} & LE = \frac{1}{12} \\ RE_1 = g_1^f - g_1^o = \frac{2}{12}, & RE_2 = g_2^f - g_2^o = 0 & \text{and} & RE = \frac{1}{12} \end{array}$$

while

$$S = \frac{LE}{LE + RE} = 0.5 \ . \qquad \diamond$$

	player 2: 10 F 20 $\frac{1}{2}$F, $\frac{1}{2}$Ca 30 Ca					
card combination randomization	$(10,20)$ F Ca	$(10,30)$	$(20,10)$	$(20,30)$	$(30,10)$	$(30,20)$ F Ca
Check	-1 $\boxed{-1}$	$\boxed{-1}$	$\boxed{1}$	$\boxed{-1}$	$\boxed{1}$	$\boxed{1}$ 1
Bet	$\boxed{1}$ -2	-2	$\boxed{1}$	-2	$\boxed{1}$	$\boxed{1}$ $\boxed{2}$
Fictive payoff	0	-1	1	-1	1	$\frac{3}{2}$

Table 13.4: The computation of g_1^f in Example 13.1.

	player 1: 10 Ch 20 $\frac{1}{2}$Ch, $\frac{1}{2}$B 30 B					
card combination randomization	$(10,20)$	$(10,30)$	$(20,10)$ Ch B	$(20,30)$ Ch B	$(30,10)$	$(30,20)$
Fold	$\boxed{1}$	$\boxed{1}$	$\boxed{-1}$ $\boxed{-1}$	$\boxed{1}$ -1	$\boxed{-1}$	$\boxed{-1}$
Call	$\boxed{1}$	$\boxed{1}$	$\boxed{-1}$ -2	$\boxed{1}$ $\boxed{2}$	-2	-2
Fictive payoff	1	1	-1	$\frac{3}{2}$	-1	-1

Table 13.5: The computation of g_2^f in Example 13.1.

13.2 Statistical decision theory

Statistical decision theory analyzes decision making on the basis of statistical information. Among others, it provides an alternative view on standard statistical problems as estimation problems. Formally, a statistical decision problem (SDP) is described by $(\Theta, \mathcal{A}, L, \mathcal{X})$ where

(i) Θ is the set of possible parameters. The decision maker does not know the exact parameter corresponding to the true state of nature, only the set to which it belongs.

(ii) $\mathcal{A}$ is the set of possible actions the decision maker can choose from.

(iii) $L : \Theta \times \mathcal{A} \longmapsto \mathbb{R}$ is the loss function. With $\theta \in \Theta$ and $a \in \mathcal{A}$, $L(\theta, a)$ represents the monetary loss of choosing action a if θ is the true parameter.

(iv) $\mathcal{X}$ is the outcome space of a stochastic variable X (corresponding to a statistical experiment) with probability distribution $f(x|\theta)$ for $x \in \mathcal{X}$ depending on θ, such that the occurrence of a specific outcome x provides statistical information about the true state of nature. Usually $f(x|\theta)$ is provided in the form of a point mass function (for discrete distributions) on a density function (for absolutely continuous distributions).

To focus ideas, we consider two running examples.

Example 13.2

A teacher has to decide if a specific student is an economics or an econometrics student, e.g., in order to put his exam on the right pile of mathematics exams. He will incur no loss if he takes the right decision and a loss of 1 Euro if he takes the wrong decision. All information the teacher has about the student is what mathematical background this student has. Before entering the university he has had training in either Mathematics A, in Mathematics B or in both. Information about the general distribution of mathematical background among economics and econometrics students is available. How to classify the student in each of three possible cases with respect to his mathematical background?

This situation can be modeled as an SDP in the following way:

- The parameter space $\Theta = \{\theta_0, \theta_1\}$, with

  θ_0 : the student is an economics student.

  θ_1 : the student is an econometrics student.

- The action space $\mathcal{A} = \{a_0, a_1\}$, with

  a_0 : classify the student as being an economics student.

  a_1 : classify the student as being an econometrics student.

- The loss function L, defined by the following table:

	a_0	a_1
θ_0	0	1
θ_1	1	0

- The outcome space $\mathcal{X} = \{\mathrm{A}, \mathrm{B}, \mathrm{A{+}B}\}$ where the general distribution $f(x|\theta)$ is given by

	A	B	$A+B$
θ_0	$\frac{3}{8}$	$\frac{4}{8}$	$\frac{1}{8}$
θ_1	0	$\frac{6}{8}$	$\frac{2}{8}$

  This means that from the general population of all economics students 37.5% has only mathematics A as a background, 50% only mathematics B and 12.5% both mathematics A and B. For the general population of econometrics students, these fractions are 0%, 75% and 25%, respectively. ◇

Example 13.3

A statistician wants to estimate the mean θ of a normal distribution with known variance

$\sigma^2 = 1$ on the basis of a sample of size 1. Choosing a (standard) quadratic loss function, this situation could be modeled as an SDP in the following way:
$\Theta = \mathbb{R}, \mathcal{A} = \mathbb{R}, L(\theta, a) = (\theta - a)^2$ and $\mathcal{X} = \mathbb{R}$, the outcome space of a stochastic variable X with $X \sim N(\theta, 1)$. $\diamond$

Since the decision maker in an SDP can base his action on the observed outcome $x \in X$, he can in fact choose between *decision rules* $\delta : \mathcal{X} \mapsto \mathcal{A}$, as actions may vary depending on the observation at hand. Let us denote the set of all possible decision rules by D. Obviously, if both $\mathcal{A}$ and $\mathcal{X}$ are finite, then $|D| = |\mathcal{A}|^{|\mathcal{X}|}$.
To evaluate a decision rule δ, we consider its *risk* $R(\theta, \delta)$ which depends on the actual parameter, and is defined as the expected loss:

$$R(\theta, \delta) = E^X\{L(\theta, \delta(X))\} = \int_{\mathcal{X}} f(x|\theta)L(\theta, \delta(x))\, dx$$

Obviously, for finite $\mathcal{X}$, the integral just boils down to a sum.

Example 13.4
Reconsider the SDP of Example 13.2. Since $|\mathcal{X}| = 3$ and $|\mathcal{A}| = 2$, we have $|D| = 8$. The 8 decision rules are given by

	δ^1	δ^2	δ^3	δ^4	δ^5	δ^6	δ^7	δ^8
A	a_0	a_0	a_0	a_1	a_1	a_1	a_0	a_1
B	a_0	a_0	a_1	a_0	a_1	a_0	a_1	a_1
$A+B$	a_0	a_1	a_0	a_0	a_0	a_1	a_1	a_1

For each pure decision rule δ we can compute the risk $R(\theta, \delta)$ for all $\theta \in \Theta$. For example,

$$\begin{aligned} R(\theta_1, \delta^3) &= f(A|\theta_1) \cdot L(\theta_1, \delta^3(A)) + f(B|\theta_1) \cdot L(\theta_1, \delta^3(B)) \\ &+ f(A+B|\theta_1) \cdot L(\theta_1, \delta^3(A+B)) \\ &= 0 \cdot L(\theta_1, a_0) + \tfrac{6}{8} \cdot L(\theta_1, a_1) + \tfrac{2}{8} \cdot L(\theta_1, a_0) \\ &= \tfrac{6}{8} \cdot 0 + \tfrac{2}{8} \cdot 1 = \tfrac{2}{8}. \end{aligned}$$

All risks are displayed in Table 13.6.

$\diamond$

$R(\theta,\delta)$	δ^1	δ^2	δ^3	δ^4	δ^5	δ^6	δ^7	δ^8
θ_0	0	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{3}{8}$	$\frac{7}{8}$	$\frac{4}{8}$	$\frac{5}{8}$	1
θ_1	1	$\frac{6}{8}$	$\frac{2}{8}$	1	$\frac{2}{8}$	$\frac{6}{8}$	0	0

Table 13.6: Table of risks in Example 13.4.

Example 13.5

Reconsider the estimation problem of Example 13.3.

Since $\mathcal{X} = \mathbb{R}$ and $\mathcal{A} = \mathbb{R}$, all functions from $\mathbb{R}$ to $\mathbb{R}$ can be viewed as decision rules. We will restrict our attention to the subclass D defined by

$$D = \{\delta_c : \mathbb{R} \to \mathbb{R} \mid \delta_c(x) = cx, c \in \mathbb{R}\}$$

containing the "obvious" estimator δ_1 defined by $\delta(x) = x$ for all $x \in \mathbb{R}$. With respect to the risk of an estimator $\delta_c \in D$ we find

$$\begin{aligned}
R(\theta,\delta_c) &= E^X\{L(\theta,\delta_c(X))\} \\
&= E^X\{(\theta - cX)^2\} \\
&= E^X\{\theta^2 - 2c\theta X + c^2X^2\} \\
&= \theta^2 - 2c\theta E^X\{X\} + c^2E^X\{X^2\} \\
&= \theta^2 - 2c\theta^2 + c^2(1+\theta^2) \\
&= c^2 + (c-1)^2\theta^2,
\end{aligned}$$

where the one but last equality follows from the fact that $X \sim N(\theta, 1)$ and $Var\{X\} = E\{X^2\} - (E\{X\})^2$.

Note that for all $\theta \in \mathbb{R}$, $R(\theta,\delta_1) = 1$, while $R(\theta,\delta_c) > 1$ for all c with $|c| > 1$.

This means that δ_1 is a better estimator than any δ_c with $|c| > 1$ based on this SDP modelling. We just want to note than in any SDP with a quadratic loss function which corresponds to an estimation problem, the risk $R(\theta,\delta)$ of a *pure* estimator δ (i.e. for which $E\{\delta(X)\} = \theta$) equals the variance $Var\{\delta(X)\}$, thus providing a statistical justification for a quadratic loss function and its corresponding risk. ◇

One way to solve an SDP is by means of a Bayesian approach. For this we need one more ingredient in the model, a *prior* probability distribution $\pi \in \Delta(\Theta)$ on the set of all possible parameters. To evaluate decision rules $\delta \in D$ one subsequently uses the *Bayes-risk* $r(\pi,\delta)$ defined by

$$r(\pi,\delta) = E^\pi\{R(\theta,\delta)\} = \int_\Theta \pi(\theta)R(\theta,\delta)\, d\theta$$

A *Bayes-rule* δ^B within D with respect to π is such that

$$r(\pi, \delta^B) = \min_{\delta \in D} r(\pi, \delta)$$

Another way to solve an SDP is by means of a minimax approach. For this one does not need a prior distribution and one evaluates each decision rule $\delta \in D$ on the basis of a worst case scenario. Obviously, for each $\delta \in D$, the highest possible risk $h(\delta)$ is determined by

$$h(\delta) = \sup_{\theta \in \Theta} R(\theta, \delta).$$

A *minimax rule* δ^M within D is such that

$$h(\delta^M) = \min_{\delta \in D} h(\delta)$$

and $h(\delta^M)$ is called the *minimax risk* within D.

Example 13.6

Reconsider the SDP of the Examples 13.2 and 13.4.

If e.g. on the basis of the total numbers of economics and econometrics students, one chooses $\pi \in \Delta(\Theta)$ with $\pi(\theta_0) = \frac{4}{5}$ and $\pi(\theta_1) = \frac{1}{5}$ as a prior distribution, the Bayes-risks of the 8 decision rules are given by

δ	δ^1	δ^2	δ^3	δ^4	δ^5	δ^6	δ^7	δ^8
$r(\pi, \delta)$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{9}{20}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{11}{20}$	$\frac{1}{2}$	$\frac{4}{5}$

Clearly the Bayes-rule δ^B w.r.t. π is given by $\delta^M = \delta^1$. Following a minimax approach instead, it is seen that the highest possible risks for the 8 decision rules are given by

δ	δ^1	δ^2	δ^3	δ^4	δ^5	δ^6	δ^7	δ^8
$h(\delta)$	1	$\frac{6}{8}$	$\frac{4}{8}$	1	$\frac{7}{8}$	$\frac{6}{8}$	$\frac{5}{8}$	1

From this it is readily concluded that the minimax rule δ^M within D is given by $\delta^M = \delta^3$. $\diamond$

Example 13.7

Reconsider the SDP of the Examples 13.3 and 13.5.

Suppose the prior $\pi \in \Delta(\Theta)$ is such that it has a normal distribution with mean 0 and (known) variance τ^2.

For each decision rule $\delta_c \in D$ the Bayes-risk w.r.t. π is computed in the following way:

$$\begin{aligned} r(\pi, \delta_c) &= E^{\pi}\{R(\theta, \delta_c)\} \\ &= E^{\pi}\{c^2 + (c-1)^2\theta^2\} \\ &= c^2 + (c-1)^2 E^{\pi}\{\theta^2\} \\ &= c^2 + (c-1)^2 Var^{\pi}\{\theta\} \\ &= c^2 + (c-1)^2(r^2) \\ &= (1+r^2)c^2 - 2r^2c + r^2. \end{aligned}$$

To find the corresponding Bayes rule δ^B we have to minimize over c, which obviously leads to the conclusion that $\delta^M = \delta_{c^*}$with

$$c^* = \frac{\tau^2}{1+\tau^2}.$$

Following a minimax approach instead, it is seen that the highest possible risks for decision rules δ_c within D are given by

$$h(\delta_c) = \begin{cases} 1 & \text{for } c = 1. \\ \infty & \text{for } c \neq 1. \end{cases}$$

Hence the minimax rule δ^M within D is given by $\delta^M = \delta_1$. ◇

In the remainder of this section we will restrict our attention to the SDPs $(\Theta, \mathcal{A}, L, \mathcal{X})$ with both $|\mathcal{A}| < \infty$ and $|\mathcal{X}| < \infty$. In this setting, we will also allow the decision maker to randomize between his decision rules in D. This leads to *randomized decision rules* $\mu \in \Delta(D)$, with $\mu = \{\mu(\delta) \mid \delta \in D\}$ such that $\sum_{\delta \in D} \mu(\delta) = 1$ and $\mu(\delta) \geq 0$ for all $\delta \in D$.
Within game theory randomization is a widely accepted phenomenon, within statistics this is not the case. It e.g. can be justified from the fact that if you want to base your action on *sufficient* variables only, the analysis will not lose strength only if you allow randomization. For $\mu \in \Delta(D)$, the risk $R(\theta, \mu)$ is determined in the natural way by defining, for all $\theta \in \Theta$,

$$R(\theta, \mu) = E^{\mu}\{R(\theta, \delta)\} = \sum_{\delta \in D} \mu(\delta) R(\theta, \delta).$$

Next, following a minimax approach, define the highest possible risk $h(\mu)$ of a randomized decision rule μ by

$$h(\mu) = \sup_{\theta \in \Theta} R(\theta, \mu)$$

and a *randomized minimax* rule μ^M within $\Delta(D)$ such that

$$h(\mu^M) = \min_{\mu \in \Delta(D)} R(\theta, \mu)$$

where $h(\mu^M)$ is called the *randomized minimax risk* within $\Delta(D)$. Randomization could lead to improvements with respect to minimax risks: with $\delta^M \in D$ minimax within D and $\mu^M \in \Delta(D)$ minimax within $\Delta(D)$ it could be that

$$h(\mu^M) < h(\delta^M)$$

as illustrated in the following example.

Example 13.8

Reconsider the SDP of the Examples 13.2, 13.4, and 13.6.

To determine a randomized minimax rule μ^M graphically, we first depict the risk vectors (see Figure 13.1) for all 8 (pure) decision rules $\delta \in D$. Next project each vector onto the lower right of the line $\theta_0 = \theta_1$ onto the θ_0-axis and each vector to the upper left of this 45°-line onto the θ_1-axis. These projections indicate the maximum risk for the corresponding decision rule. Minimizing over both coordinates gives the (pure) minimax rule. We see (again) that θ^3 is the minimax rule within D. However, if we allow randomization, we can reach all risk vectors in the risk set

$$S = \{(R(\theta_0, \mu), R(\theta_1, \mu)) \mid \mu \in \Delta(D)\} = \text{Conv}\{(R(\theta_0, \delta), R(\theta_1, \delta)) \mid \delta \in D\}$$

and it is easily seen (using the same projection method) that the randomized rule μ^M is better w.r.t. the minimax criterion than δ^3, and in fact also better than any other randomized decision rule. The rule θ^M corresponds to the middle of the line segment between δ^2 and δ^7. Therefore $\mu^M = \frac{1}{2}\delta^2 + \frac{1}{2}\delta^7$, $R(\theta_0, \mu^M) = R(\theta_1, \mu^M) = \frac{3}{8}$ and hence $h(\mu^M) = \frac{3}{8}$ while $h(\delta^M) = h(\delta^3) = \frac{4}{8}$. Note that μ^M can also be determined as

	μ^M
A	a_0
B	$\frac{1}{2}a_0, \frac{1}{2}a_1$
$A+B$	a_1

$\diamond$

As you will have probably realized by now, the minimax approach can be viewed as determining an optimal strategy of player 2 in a specific matrix game. Formally, in this matrix

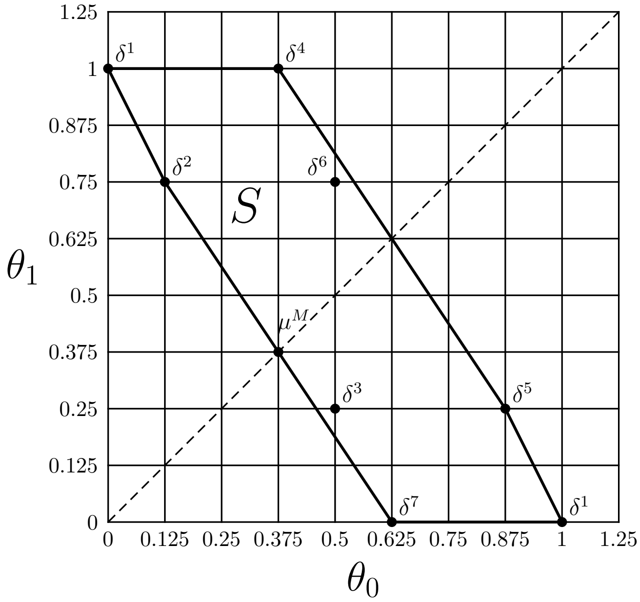


Figure 13.1: The risk set in Example 13.8.

game A, the pure strategies of player 1 (nature) correspond to the selection of a parameter $\theta \in \Theta$, the pure strategies of player 2 (the statistical decision maker) correspond to selecting a (pure) decision rule. Furthermore, the payoff (to player 1) in the cell of A corresponding to θ and δ equals $R(\theta, \delta)$. Thus, we implicitly assume that nature acts strictly competitive in aiming to maximize the risk for the statistical decision maker.

It is no problem that in the formal setting of the matrix game A also nature is allowed to use a randomized strategy, i.e., a probability distribution $\pi \in \Delta(\theta)$ by defining

$$r(\pi, \mu) = E^{\pi}\{R(\theta, \mu)\} = \sum_{\theta \in \Theta} \sum_{\delta \in D} \pi(\theta)\mu(\delta)R(\theta, \delta)$$

for all $\pi \in \Delta(\Theta)$ and $\mu \in \Delta(\Theta)$ as the expected Bayes-risk corresponding to expected payoffs in the matrix game A since the value $v(A)$ equals the minimax risk $h(\mu^M)$ because

$$\begin{aligned} v(A) = \min_{\mu \in \Delta(D)} \max_{\pi \in \Delta(\Theta)} r(\pi, \mu) &= \min_{\mu \in \Delta(D)} \max_{\theta \in \Theta} R(\theta, \mu) \\ &= \min_{\mu \in \Delta(D)} h(\mu) = h(\mu^M) \end{aligned}$$

This implies that randomized minimax rules exactly correspond to optimal strategies of player 2.

How about optimal strategies of player 1 (i.e. of nature) in the matrix game A? Such strategies can be interpreted as *least favourable priors*. For, if $\pi^L \in O_1(A)$, then (in game theoretic terms) $\pi^L A\mu \geq v(A)$ for all $\mu \in \Delta(D)$ and, consequently

$$\min_{\mu \in \Delta(D)} r(\pi^L, \mu) \geq \min_{\mu \in \Delta(D)} r(\pi, \mu).$$

for all $\pi \in \Delta(\Theta)$ because $v(A) = \max_{\pi\in\Delta(\Theta)} \min_{\mu\in\Delta(D)} r(\pi,\mu)$.
In other words, the lowest possible attainable Bayes-risk by selecting a corresponding Bayes-rule is the highest for π^L, and thus the least favourable from the perspective of the decision maker.
Note that any minimax rule μ^M will be a Bayes-rule w.r.t. any least favourable prior π^L since, in game theoretic terms, any combination of optimal strategies is a Nash equilibrium, or in statistical terms

$$r(\pi^L,\mu^M) \leq \max_{\pi\in\Delta(\Theta)} r(\pi,\mu^M) = \min_{\mu\in\Delta(D)} \max_{\pi\in\Delta(\Theta)} r(\pi,\mu)(= v(A))$$

and

$$r(\pi^L,\mu^M) \geq \min_{\mu\in\Delta(D)} r(\pi^L,\mu) = \max_{\pi\in\Delta(D)} \min_{\mu\in\Delta(\Theta)} \tau(\pi,\mu)(= v(A)),$$

where the two equalities follow from the fact that $\mu^M \in O_2(A)$ and $\pi^L \in O_1(A)$, respectively, This implies in particular that

$$r(\pi^L,\mu^M) = \min_{\mu\in\Delta(D)} r(\pi^L,\mu).$$

If $|\Theta| = 2$, then randomized minimax decision rules and least favorable priors can be determined using the graphical method for $2 \times n$ matrix games as presented in Chapter 12. Alternatively, further exploiting Figure 13.1, one could also use the following approach.

Method for an SDP with $\Theta = \{\theta_0, \theta_1\}$ and $|D| < \infty$

(1) Determine the risk set

$$S = \text{Conv}\{(R(\theta_0,\delta), R(\theta_1,\delta)) \mid \delta \in D\}).$$

(2) Determine $v = \min\{\alpha \mid S \cap Q_\alpha \neq \emptyset\}$, with $Q_\alpha = \{x \in \mathbb{R}^2 \mid x_1 \leq \alpha \text{ and } x_2 \leq \alpha\}$ a quadrant with right upper point (α, x) on the main diagonal.

(3) $\mu \in \Delta(D)$ is minimax if and only if $(R(\theta_0,\mu), R(\theta_1,\mu)) \in S \cap Q_v$,

(4) $\pi \in \Delta(\Theta)$ is a least favourable prior if and only if π corresponds to the normal of a separating line between Q_v and S.

Example 13.9

Reconsider the SDP of Example 13.8.
Figure 13.2 contains a visualization of the results of the various steps in the method to determine minimax rules and least favourable priors:

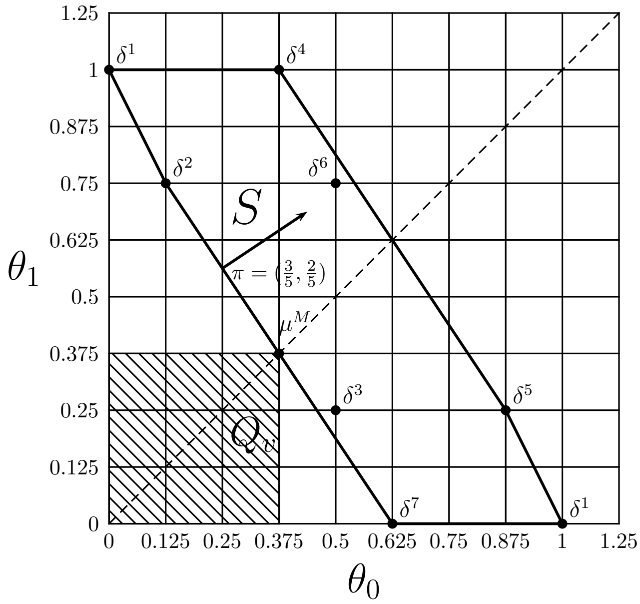


Figure 13.2: Determining randomized minimax rules and least favourable priors in Example 13.9.

(1) We plot the risk vectors of the 8 pure decision rules and determine the risk set S as the convex hull of these 8 risk vectors

(2) For $\alpha \geq \frac{3}{8}$, the intersection of S and Q_α is nonempty. For $\alpha < \frac{3}{8}$, this intersection is empty. Therefore, $v = \frac{3}{8}$.

(3) $S \cap Q_{\frac{3}{8}}$ consists of a single risk vector. The unique corresponding randomized decision rule μ^M corresponds to $\frac{1}{2}\delta^2 + \frac{1}{2}\delta^7$.

(4) Note that there is a unique line separating $Q_{\frac{3}{8}}$ from S. Hence the unique least favourable prior can be determined by computing the normal that corresponds to this line through δ^2 and δ^7. Writing $(x, y) = (R(\theta_0, \delta), R(\theta_1, \delta))$ the equation of this line is determined by

$$\begin{aligned} & y = -\tfrac{3}{2}x + c, \\ & (\tfrac{3}{2}, 1) \cdot (x, y) = c, \\ \text{or} \quad & (\tfrac{3}{5}, \tfrac{2}{5}) \cdot (x, y) = \tfrac{2}{5}c. \end{aligned}$$

From which we can conclude that $\pi^L = (\frac{3}{5}, \frac{2}{5})$ is the unique least favourable prior.

Note, for any $\pi \in \Delta(\Theta)$ and $\mu \in \Delta(D)$, we can easily recognize the corresponding Bayes-risk $r(\pi, \mu)(= \pi(\theta_0) \cdot R(\theta_0, \mu) + \pi(\theta_1) \cdot R(\theta_1, \mu))$ drawing a line with normal π through $(R(\theta_0, \mu), R(\theta_1, \mu))$ and looking at one of the coordinates of the intersection point of this line with the main diagonal. Using this fact, one easily understands that for any other prior the lowest attainable Bayes-risk will be lower than $\frac{3}{8}$, the lowest attainable Bayes-risk with respect to π^L. $\diamond$

Exercises

Exercise 13.1
Reconsider two-person Mini Poker. Calculate the relative skill S if the beginners' strategies are given by

(a) e_2 for player 1 and f_2 for player 2 (playing "safe"),

(b) e_8 for player 1 and f_4 for player 2 (playing "aggressive"),

(c) for both players: equal probability $\frac{1}{2}$ whenever a choice between two actions has to be made (full "gambling").

Exercise 13.2
This exercise analyzes a variant of two-person Mini Poker, the only difference being that betting and calling (upon a bet) means adding b Euro (with $b > 0$) extra to the stakes. So $b = 1$ is the case elaborated upon in Example 13.1.

(a) Determine the corresponding 8×8 matrix game $A(b)$. Please use the same notation for the pure strategies as in Example 13.1.

(b) Show that $A(b)$ can be reduced to

$$A^R(b) = \begin{array}{c} \\ e_2 \\ e_6 \end{array} \begin{array}{c} \begin{array}{cc} \quad f_2 \quad & \quad f_4 \quad \end{array} \\ \begin{bmatrix} 0 & b/6 \\ 1/3 - b/6 & -b/6 \end{bmatrix} \end{array}.$$

(c) Determine $v(A(b))$ and an optimal strategy for both player 1 and 2 in $A(b)$.

To provide an analysis with respect to relative skill we assume that a beginner's strategy does not depend on b, so we (still) take $\frac{1}{2}e_2 + \frac{1}{2}e_4$ for player 1 and $\frac{1}{2}f_2 + \frac{1}{2}f_4$ for player 2.

(d) Calculate the relative skill $S(b)$ as a function of b.

Exercise 13.3

This exercise analyzes the variant of two-person Mini Poker where player 1 has to choose between checking (Ch), betting with 1 Euro (B_1) or betting with 3 Euro (B_3).

(a) Show that the corresponding matrix game A is of size 27×64.

(b) Show that the number of pure strategies of player 2 can be reduced from 64 to 4.

(c) Subsequently, show that the number of pure strategies for player 1 can be reduced from 27 to 6.

(d) Calculate the remaining 6×4 matrix game.

(e) Further reduce the 6×4 matrix game to a 2×2 matrix game A^R. Calculate $v(A)$ and determine optimal strategies for both players in A. Explain how the optimal strategy of player 1 exhibits a "sandbagging" feature.

(f) Determine the relative skill S if we assume that the beginners' strategies are given by

$$\begin{array}{lll} \text{For player 1:} & \text{with } 10: & \text{Ch} \\ & \text{with } 20: & \text{B}_1 \\ & \text{with } 30: & \text{B}_3 \\ \text{For player 2:} & \text{with } 10: & \text{F} \\ & \text{with } 20: & \begin{cases} \text{Ca in case of B}_1 \\ \text{F in case of B}_3 \end{cases} \\ & \text{with } 30: & \text{Ca} \end{array}$$

Exercise 13.4

Reconsider the econometrics/economics SDP of Example 13.2 with the following change. Now the loss function $L(\theta, a)$ is given by

	a_0	a_1
θ_0	0	1
θ_1	3	0

(a) Calculate $R(\theta, \delta)$ for every $\theta \in \{\theta_0, \theta_1\}$ and $\delta \in D$.

(b) Let the prior $\pi \in \Delta(\Theta)$ be such that $\pi(\theta_0) = \frac{4}{5}$. Determine all Bayes-rules within D with respect to π.

(c) Determine a prior π with at least two different Bayes rules within D corresponding to it.

(d) Determine all mimimax rules δ^M and the corresponding minimax risk.

(e) Determine all randomized minimax rules μ^M and the corresponding randomized minimax risk.

(f) Determine all least favourable priors π^L.

Exercise 13.5

A statistician wants to analyze the following testing problem from the perspective of statistical decision theory. On the basis of a single observation from a binomial distribution $X \sim Bin(n, \theta)$, he wants to test $H_0 : \theta \in [0, \frac{1}{2}]$ against the alternative $H_1 : \theta \in (\frac{1}{2}, 1]$ knowing that a wrong decision leads to a loss of 1, and a correct decision to a loss of 0.

(a) Model this situation as an SDP $(\Theta, \mathcal{A}, L, \mathcal{X})$.

(b) What is a natural class of decision rules D to restrict attention to?

(c) Calculate the risk of an arbitrary rule within D. Does this risk adequately reflect the statistical considerations in the testing problem?

Exercise 13.6

A casino offers a simplified form or Roulette in which there are only 3 possible numbers as outcomes: $0, 1$ and 2. Each player only participates once and can bet either 900 Euro on 1 or 900 Euro on 2. If he predicts the outcome 1 (or 2) of the spinning of the Roulette wheel correctly, then he gets 1800 Euro (or 2700 Euro) back leading to a net profit of 900 Euro (or 1800 Euro). If he is wrong, he loses his bet.

Furthermore, it is known that the casino has two Roulette wheels available. One (θ_0) which is fair in the sense that the probability for each of the three outcomes is $\frac{1}{3}$ and another one (θ_1) where the probability of the outcome 1 is $\frac{2}{3}$ while the outcomes 0 and 2 each occur with probability $\frac{1}{6}$.

Before actual play, a player does not know which Roulette wheel is used but he is allowed to observe the outcome of one previous spinning of the wheel.

(a) Model this situation as an SDP $(\Theta, \mathcal{A}, L, \mathcal{X})$.

(b) Determine $R(\theta, \delta)$ for all $\theta \in \Theta$ and all pure decision rules δ.

(c) Determine all possible Bayes rules within D with respect to some prior.

(d) Determine all randomized minimax rules μ^M and the corresponding randomized minimax risk.

(e) Determine a least favourable prior π^L and a corresponding pure Bayes rule.

Exercise 13.7

(a) Reformulate the method for SDPs with $|\Theta| = 2$ and $|D| < \infty$ into a method to determine the value and all optimal strategies for both players for $2 \times n$ matrix games.

(b) Apply the method described in (a) to determine the value and all optimal strategies for both players for the following four matrix games

(i) $$\begin{array}{cc} & \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{bmatrix} 0 & 12 & 8 & 2 \\ 6 & 0 & 3 & 10 \end{bmatrix} \end{array}$$

(ii) $$\begin{array}{cc} & \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{bmatrix} 2 & 4 & 1 \\ 2 & 2 & 3 \end{bmatrix} \end{array}$$

(iii) $$\begin{array}{cc} & \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 2 \end{bmatrix} \end{array}$$

(iv) $$\begin{array}{cc} & \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{bmatrix} 16 & 0 & 4 \\ 0 & 16 & 12 \end{bmatrix} \end{array}$$

Exercise 13.8

Reconsider the SDP $(\Theta, \mathcal{A}, L, \mathcal{X})$ of the Examples 13.2, 13.4, 13.6, 13.8 and 13.9 with prior distribution π as given in Example 13.6. Given an outcome $x \in \mathcal{X}$, one can calculate the *posterior distribution* $\pi(\theta|x)$ using Bayes' Rule:

$$\pi(\theta|x) = \frac{\pi(\theta) f(x|\theta)}{\sum\limits_{\theta' \in \theta} \pi(\theta') f(x|\theta')}$$

(a) Calculate the posterior distribution $\pi(\theta|x)$ for all $x \in \mathcal{X}$.

Subsequently, a *posterior Bayes action* $a^{PB}(x)$ given outcome x is such that the expected posterior loss is minimized, i.e.

$$\sum_{\theta\in\Theta} \pi(\theta|x) \cdot L(\theta, a^{PB}(x)) = \min_{a\in A} \sum_{\theta\in\Theta} \pi(\theta|x) \cdot L(\theta, a)$$

(b) Determine the posterior Bayes actions $a^{PB}(x)$ for all $x \in \mathcal{X}$.

Next, define a *posterior Bayes rule* $\delta^{PB} \in D$ as a decision rule that for every outcome $x \in \mathcal{X}$ chooses a posterior Bayes action.

(c) Calculate all posterior Bayes rules. How does your answer relate to the (prior) Bayes rule found in Example 13.6?

We just want to note that the posterior Bayes approach can be extended to general SDPs, that in estimation problems with quadratic loss functions a posterior Bayes estimator corresponds to the expected value of the posterior distribution and, finally, that posterior Bayes rules and (prior) Bayes rules coincide (almost everywhere).

Some relevant literature

Genugten, B. van der and P. Borm (2012). Texas Hold'em: a game of skill. Report, Department of Econometrics and OR, Tilburg University, The Netherlands.
Available at: http://www.tilburguniversity.edu/webwijs/show/?uid=p.e.m.borm & uid=p.e.m.borm

Genugten, B. van der and P. Borm (2012). Texas Hold'em cash games: a mathematical analysis. Report, Department of Econometrics and OR, Tilburg University, The Netherlands.
Available at: http://www.tilburguniversity.edu/webwijs/show/?uid=p.e.m.borm & uid=p.e.m.borm

Borm, P. and B. van der Genugten (2009). Cash- en toernooi poker: behendigheidsspelen? Report, Department of Econometrics and OR, Tilburg University, The Netherlands.
Available at: http://www.tilburguniversity.edu/webwijs/show/?uid=p.e.m.borm & uid=p.e.m.borm

Maaten, R., Borm, P., Genugten, B. van der and R. Hendrickx (2010). The relative skill of Texas Hold'em: An experiment in cooperation with VARA Nieuwslicht. Report, Department of Econometrics and OR, Tilburg University, The Netherlands.
Available at: http://www.tilburguniversity.edu/webwijs/show/?uid=p.e.m.borm & uid=p.e.m.borm

Genugten, B. van der, Das, M. and P. Borm (1999). An analysis of Blackjack variants. Report, CentER of Applied Research, Tilburg University, The Netherlands.
Available at: http://www.tilburguniversity.edu/webwijs/show/?uid=p.e.m.borm & uid=p.e.m.borm

Borm, P. and B. van der Genugten (2001). On a relative measure of skill for games with chance elements. *TOP*, 9, 91-114.

Genugten, B. van der, Das, J. and P. Borm (2001). Behendig gokken in en rond het casino, Academic Service, Schoonhoven.

Dreef, M., Borm, P. and B. van der Genugten (2004). A new relative skill measure for games with chance elements. *Managerial and Decision Economics*, 25, 255-264.

Genugten, B. van der, Borm, P. and M. Dreef (2004). De toepassing van de wet op de kansspelen op de managementspelen Competition Manager end Grand Prix manager. Report, Department of Econometrics and OR, Tilburg University, The Netherlands.

Hendrickx, R., Borm, P. and B. van der Genugten (2008). Measuring skill in more person games with applications to poker. CentER DP 2008-106, Tilburg University.

Berger, J. (1980). Statistical decision theory and Bayesian analysis, Springer, New York.

CHAPTER 14

Bimatrix games

Let (A, B) be an $m \times n$ bimatrix game. Recall that a strategy combination $(p, q) \in \Delta_m \times \Delta_n$ is a Nash equilibrium of (A, B) if a unilateral deviation by one of the two players will not lead to a strictly higher payoff for this player, i.e., if

$$pAq \geq p'Aq \text{ and } pBq \geq pBq'$$

for all $p' \in \Delta_m$ and $q' \in \Delta_n$. To verify if $(p, q) \in E(A, B)$ it suffices to check all possible unilateral deviations to pure strategies by Theorem 11.3. Moreover, Theorem 11.2 guarantees that $E(A, B) \neq \emptyset$. For strategies $p \in \Delta_m$ and $q \in \Delta_n$ we introduce the following notations. The *carriers*

$$C(p) = \{e_i \mid i \in \{1, ..., m\}, p_i > 0\} \,\text{and}\, C(q) = \{f_j \mid j \in \{1, ..., n\}, q_j > 0\}$$

contain all pure strategies that are played with positive probability according to p and q, respectively. The sets of *pure best replies* of player 1 (against q) and of player 2 (against p) are given by

$$PB_1(q) = \{e_i \mid i \in \{1, ..., m\}, e_iAq \geq e_kAq \,\text{for all}\, k \in \{1, ..., m\}\}$$

and

$$PB_2(p) = \{f_j \mid j \in \{1, ..., n\}, pBf_j \geq pBf_l \,\text{for all}\, l \in \{1, ..., n\}\}$$

Finally, the sets of *best replies* of player 1 (against q) and of player 2 (against p) are given by (cf. the notations in the proof of Theorem 11.1)

$$B_1(q) = \{p \in \Delta_m \mid pAq \geq p'Aq \,\text{for all}\, p' \in \Delta_m\}$$

and

$$B_2(p) = \{q \in \Delta_n \mid pBq \geq pBq' \text{ for all } q' \in \Delta_n\}$$

Using the linearity of the expected payoff functions it is readily observed that

$$B_1(q) = \mathrm{Conv}(PB_1(q)) \text{ and } B_2(p) = \mathrm{Conv}(PB_2(p)).$$

This implies the following characterization of Nash equilibria.

Lemma 14.1
Let (A, B) be an $m \times n$ bimatrix game and $(p, q) \in \Delta_m \times \Delta_n$. Then the following three statements are equivalent:

(i) $(p, q) \in E(A, B)$.

(ii) $C(p) \subset PB_1(q)$ *and* $C(q) \subset PB_2(p)$.

(iii) $p \in B_1(q)$ *and* $q \in B_2(p)$.

From Lemma 14.1 we can conclude that the Nash equilibria of a bimatrix game are the intersection points of the graphs of the best reply correspondences B_1 and B_2 (i.e. of the two reaction curves, cf. Example 11.3). This observation is useful to determine all Nash equilibria for 2×2 bimatrix games graphically in a rectangle since each of the strategy sets Δ_2 can be represented by a line segment $[0, 1]$ by identifying a strategy with its first coordinate. This procedure is illustrated in the following example.

Example 14.1
Consider the 2×2 bimatrix game (A, B) given by

$$(A, B) = \begin{array}{cc} & \begin{array}{cc} f_1 & f_2 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{bmatrix} 1,2 & 0,0 \\ 0,0 & 3,1 \end{bmatrix} \end{array}$$

Since for $p \in \Delta_2$, $pBf_1 = 2p_1$ and $pBf_2 = 1 - p_1$ it follows that

$$B_2(p) = \begin{cases} \{f_1\} & \text{if } \frac{1}{3} < p_1 \leq 1 \\ \Delta_2 & \text{if } p_1 = \frac{1}{3} \\ \{f_2\} & \text{if } 0 \leq p_1 < \frac{1}{3} \end{cases}$$

Similarly, $e_1Aq = q_1, e_2Aq = 3(1-q_1)$ for $q \in \Delta_2$, and consequently

$$B_1(q) = \begin{cases} \{e_1\} & \text{if } \frac{3}{4} < q_1 \leq 1 \\ \Delta_2 & \quad q_1 = \frac{3}{4} \\ \{e_2\} & \text{if } 0 \leq q_1 < \frac{3}{4} \end{cases}$$

From Figure 14.1 one may conclude that

$$E(A,B) = \{(e_1, f_1), (e_2, f_2), (\tfrac{1}{3}e_1 + \tfrac{2}{3}e_2, \tfrac{3}{4}f_1 + \tfrac{1}{4}f_2)\}$$

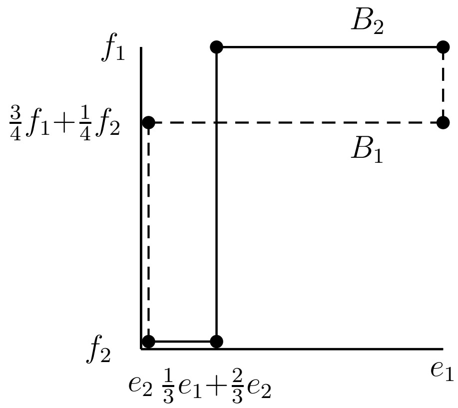


Figure 14.1: Solving the 2×2 bimatrix game in Example 14.1.

$\diamond$

Example 14.2
Consider the 2×2 bimatrix game (A, B) given by

$$(A,B) = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{cc} f_1 & \quad f_2 \end{array} \\ \begin{bmatrix} 1,0 & -1,2 \\ 0,2 & 3,0 \end{bmatrix} \end{array}.$$

Using a picture of the best reply correspondences one readily verifies that this game has a unique Nash equilibrium $(\frac{1}{2}e_1 + \frac{1}{2}e_2, \frac{4}{5}f_1 + \frac{1}{5}f_2)$ leading to an expected payoff of $\frac{3}{5}$ for player 1 and an expected payoff of 1 for player 2.
This example exhibits a rather curious feature. In equilibrium player 1 will receive a payoff of $\frac{3}{5}$. His equilibrium strategy $\frac{1}{2}e_1 + \frac{1}{2}e_2$, however, does not *guarantee* $\frac{3}{5}$. This payoff is reached *in equilibrium* only. In particular, should player 2 decide to play f_1 instead, $\frac{1}{2}e_1 + \frac{1}{2}e_2$ leads to the lower payoff of $\frac{1}{2}$. The interesting thing is that player 1 does have a strategy available that

guarantees a payoff of at least $\frac{3}{5}$. A way to find this strategy is to look at the (corresponding) 2×2 matrix game A from the perspective of the row player. Since

$$A = \begin{array}{c c} & \begin{array}{cc} f_1 & f_2 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \end{array},$$

one finds that

$$v(A) = \tfrac{3}{5} \text{ and } O_1(A) = \{\tfrac{3}{5}e_1 + \tfrac{2}{5}e_2\}.$$

This means that the strategy $\frac{3}{5}e_1 + \frac{2}{5}e_2$ does guarantee a payoff of $\frac{3}{5}$.
How about player 2? In equilibrium his expected payoff equals 1. To calculate how much player 2 can guarantee himself we can consider the (corresponding) 2×2 matrix game B^T from the perspective of the row player (why exactly?)
Since

$$B^T = \begin{array}{c c} & \begin{array}{cc} e_1 & e_2 \end{array} \\ \begin{array}{c} f_1 \\ f_2 \end{array} & \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \end{array},$$

one finds that $v(B^T) = 1$ and $O_1(B^T) = \{\frac{1}{2}f_1 + \frac{1}{2}f_2\}$.
This means that also from the perspective of player 2 the equilibrium payoff of 1 is not guaranteed by his unique equilibrium strategy while at the same time there exists another strategy that does guarantee the equilibrium payoff. $\diamond$

The feature described in Example 14.2 is in some sense only a boundary case. Any equilibrium payoff for a player lies (weakly) above the payoff level this player can guarantee himself. This is formalized in the following proposition.

Proposition 14.2
Let (A, B) be an $m \times n$ bimatrix game. If $(p, q) \in E(A, B)$, then

$$pAq \geq v(A) \text{ and } pBq \geq v(B^T).$$

Proof
Let $(p, q) \in E(A, B)$. Clearly it suffices to show that $pAq \geq v(A)$.
From Theorem 12.2 it follows that

$$v(A) = \max_{p' \in \Delta_m} \min_{q' \in \Delta_n} p'Aq' \leq \max_{p' \in \Delta_m} p'Aq = pAq. \qquad \square$$

Next we discuss a graphical method to determine all Nash equilibria for an arbitrary $2 \times n$ bimatrix game. Let (A, B) be a $2 \times n$ bimatrix game.
The method starts by drawing the n lines $p \longmapsto pBf_j, j \in \{1, ..., n\}$ representing all possible payoffs to player 2 corresponding to the pure strategy f_j as a function of $p_1 \in [0, 1]$. In the picture $p_1 = 0$ on the left hand side corresponds to e_2, $p_1 = 1$ on the right hand side to e_1. The important part of the picture is the piecewise linear maximum function and all lines present at this maximum function for some $p \in \Delta_2$. The reason is that the maximum function fully describes the best reply correspondence of player 2.
The next step is to attach to each of the n lines one out of 3 possible *labels*:

(i) $[1]_j$ if $PB_1(f_j) = \{e_1\}$.

(ii) $[2]_j$ if $PB_1(f_j) = \{e_2\}$.

(iii) $[12]_j$ if $PB_1(f_j) = \{e_1, e_2\}$.

These labels provide information about the best reply correspondence of player 1 w.r.t all pure strategies of player 2. Although the information is only partial we will see that from the picture obtained now, the format of the set of Nash equilibria is readily determined.
Clearly, the piecewise linear maximum function has t line segments for some $t \in \{1, 2, ..., n\}$. Denote by $c_0, c_1, ..., c_t$ the numbers that correspond to the end points of these segments ordered such that $0 = c_0 < c_1 < ... < c_{t-1} < c_t = 1$.
Moreover define, for each $k \in \{0, 1, ..., t\}$,

$$p_k = c_k e_1 + (1 - c_k)e_2$$

as the corresponding strategies of player 1 and, for each $k \in \{1, ..., t\}$, define

$$I_k = \{\alpha e_1 + (1 - \alpha)e_2 \mid c_{k-1} < \alpha < c_k\}$$

as the open interval of strategies corresponding to the k$^{\text{th}}$ line segment of the maximum function (ordered from left to right).
For $p \in \Delta_2$ let $PB_2(p, [1])$, $PB_2(p, [2])$ and $PB_2(p, [12])$ denote the sets of pure best replies to p with the corresponding label.
Since $PB_2(p') = PB_2(p'')$ for all $p', p'' \in I_k$, $k \in \{1, ..., t\}$, we can unambigously define $PB_2(I_k)$ and also $PB_2(I_k, [1])$, $PB_2(I_k, [2])$ and $PB_2(I_k, [12])$.
For each $p \in \Delta_2$, let $S(p)$ be the set of solutions to p, i.e., the set of all strategies $q \in \Delta_n$ of player 2 such that $(p, q) \in E(A, B)$.

Since $S(p)$ is a bounded set determined by a finite system of linear inequalities, $S(p)$ is a polytope. Of course, for specific p, $S(p)$ can be empty.
The extreme points of $S(p)$ are provided by the set $PS(p)$ of *pure solutions* given by

$$PS(p) = \{q \in S(p) \mid |C(q)| = 1\}$$

and the set $CS(p)$ of *coordination solutions* given by

$$CS(p) = \{q \in S(p) \mid |C(q)| = 2, C(q) \not\subset PS(p)\}$$

such that $S(p) = \text{Conv}\{PS(p) \cup CS(p)\}$.
Pure solutions are easily recognized since

$$PS(p) = \begin{cases} PB_2(p,[12]) & \text{if} \quad p \in \Delta_2 \backslash \{e_1, e_2\} \\ PB_2(p,[12]) \cup PB_2(p,[2]) & \text{if} \quad p = e_2 \\ PB_2(p,[12]) \cup PB_2(p,[1]) & \text{if} \quad p = e_1 \end{cases}$$

Further, with respect to coordination solutions it holds that $CS(p)$ consist of all strategies $q(j,l) \in \Delta_n$ with $j \in PB_2(p,[1])$ and $l \in PB_2(p,[2])$ such that $C(q(j,l)) = \{f_j, f_l\}$ and $e_1Aq(j,l) = e_2Aq(j,l)$. Note that given a pure strategy f_j with label $[1]_j$ and a pure strategy f_k with label $[2]_j$, it is readily verified that there is a unique strategy $q(j,l)$ of player 2 that puts zero probability on all pure strategies different from f_j and f_l while at the same time it makes player 1 indifferent between his strategies.
Since that both $PS(p') = PS(p'')$ and $CS(p') = CS(p'')$ for all $p', p'' \in I_k$ with $k \in \{1, ..., t\}$ also $PS(I_k)$, $CS(I_k)$ and $S(I_k) = \text{Conv}\{PS(I_k) \cup CS(I_k)\}$ are defined properly.
Consequently, the set $E(A, B)$ of Nash equilibria can be determined in $2t + 1$ steps:

$$E(A,B) = \bigcup_{k=0}^{t} \{p_k\} \times S(p_k) \cup \bigcup_{k=1}^{t} I_k \times S(I_k)$$

By noting that, for all $k \in \{1, 2, ..., t\}$,

$$S(I_k) \subset S(p_{k-1}) \quad \text{and} \quad S(I_k) \subset S(p_k)$$

it follows that

$$E(A,B) = \bigcup_{k=0}^{t} \{p_k\} \times S(p_k) \cup \bigcup_{k=1}^{t} \bar{I}_k \times S(I_k)$$

with $\bar{I}_k = \{\alpha e_1 + (1-\alpha)e_2 | c_{k-1} \leq \alpha \leq c_k\}$, $k \in \{1, \ldots, t\}$. As a general rule, one can best determine all possible equilibria w.r.t. the line segments I_k, $k \in \{1, ..., t\}$, include the corresponding end points in the description of the equilibria and only to consider the endpoints $p_0, p_1, ..., p_t$ separately if "something extra happens", i.e., if a new pure or coordination solution arises as compared to a line segment the endpoint belongs to, or alternatively, if both line segments the end point belongs to, have solutions which can be combined in the end point itself. Moreover, one can conclude that $E(A,B)$ is the union of at most $2t+1$ polytopes.

The above method is illustrated in

Example 14.3

Consider the 2×6 bimatrix game (A,B) given by

$$(A,B) = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{cccccc} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \end{array} \\ \begin{bmatrix} 2,0 & 0,6 & 0,8 & 0,6 & 3,8 & -1,3 \\ 0,8 & 0,6 & 1,0 & 2,6 & 1,-8 & 0,7 \end{bmatrix} \end{array}$$

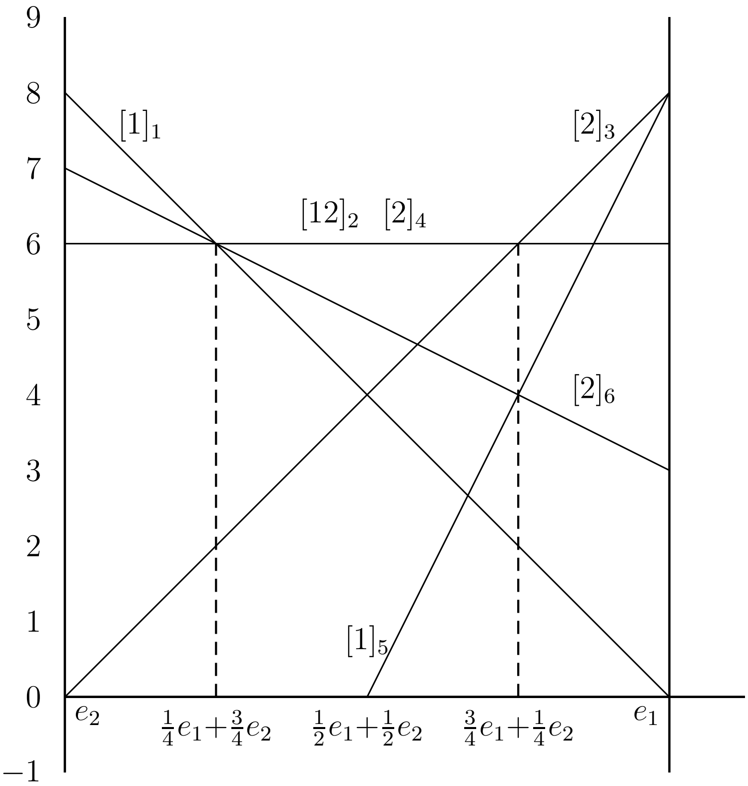


Figure 14.2: Solving the $2 \times n$ bimatrix game in Example 14.3.

From Figure 14.2 one readily derives that $t = 3, p_0 = e_2, p_1 = \frac{1}{4}e_1 + \frac{3}{4}e_2$, $p_2 = \frac{3}{4}e_1 + \frac{1}{4}e_2$ and $p_3 = e_1$.

With respect to the three line segments I_1, I_2 and I_3 we find

$$PS(I_1) = CS(I_1) = \emptyset,\ PS(I_3) = CS(I_3) = \emptyset$$

and

$$PS(I_2) = \{f_2\},\ CS(I_2) = \emptyset.$$

Hence, $\mathrm{Conv}\{\frac{1}{4}e_1 + \frac{3}{4}e_2, \frac{3}{4}e_1 + \frac{1}{4}e_2\} \times \{f_2\} \subset E(A,B)$.
In the endpoints $p_0 = e_2$ and $p_2 = \frac{3}{4}e_1 + \frac{1}{4}e_2$ nothing extra happens. For the other two endpoints we find that

$$PS(\tfrac{1}{4}e_1 + \tfrac{3}{4}e_2) = \{f_2\},\ CS(\tfrac{1}{4}e_1 + \tfrac{3}{4}e_2) = \{q(1,4), q(1,6)\}$$

and

$$PS(e_1) = \{f_5\},\ CS(e_1) = \{q(5,3)\}$$

Since $q(1,4) = \frac{1}{2}f_1 + \frac{1}{2}f_4$, $q(1,6) = \frac{1}{3}f_1 + \frac{2}{3}f_6$ and $q(5,3) = \frac{2}{3}f_3 + \frac{1}{3}f_5$, it can be concluded that $E(A,B)$ is the union of three polytopes:

$$\begin{aligned} E(A,B) &= \{\tfrac{1}{4}e_1 + \tfrac{3}{4}e_2\} \times \mathrm{Conv}\{f_2, \tfrac{1}{2}f_1 + \tfrac{1}{2}f_4, \tfrac{1}{3}f_1 + \tfrac{2}{3}f_6\} \\ &\cup \ \mathrm{Conv}\{\tfrac{1}{4}e_1 + \tfrac{3}{4}e_2, \tfrac{3}{4}e_1 + \tfrac{1}{4}e_2\} \times \{f_2\} \\ &\cup \ \{e_1\} \times \mathrm{Conv}\{f_5, \tfrac{2}{3}f_3 + \tfrac{1}{3}f_5\} \end{aligned}$$

◇

Using the graphical solution method we concluded that the set of Nash equilibria of a $2 \times n$ bimatrix game is the union of finitely many polytopes. This result can be generalized to arbitrary bimatrix games by considering the notion of a Nash component.

Let (A,B) be an $m \times n$ bimatrix game. A Nash component C of (A,B) is a maximal convex subset of $E(A,B)$, i.e., there does not exist another convex $C' \subset E(A,B)$ with $C \subsetneq C'$. Lemma 14.3 below shows that Nash components also correspond to maximal exchangeable subsets of $E(A,B)$.

Lemma 14.3
Let (A,B) be an $m \times n$ bimatrix game.

(i) Let $C \subset E(A,B)$ be convex. Defining

$$P_1 = \{p \in \Delta_m \mid \text{ there is a } q \in \Delta_n \text{ such that } (p,q) \in C\}$$

and

$$P_2 = \{q \in \Delta_n \mid \text{ there is a } p \in \Delta_n \text{ such that } (p,q) \in C'\}$$

as the projections of C on the individual strategy spaces and defining

$$P = P_1 \times P_2$$

we have that $P \subset E(A,B)$ and that P is convex.

(ii) Let $D_1 \subset \Delta_m$ *and* $D_2 \subset \Delta_n$ *be such that* $D_1 \times D_2 \subset E(A,B)$.

Then $\mathrm{Conv}(D_1) \times \mathrm{Conv}(D_2) \subset E(A,B)$.

Proof

(i) First we show that P_1 is convex. For this, let $p, p' \in P_1$ and $\alpha \in [0,1]$. By definition, there exist $q, q' \in \Delta_n$ with $(p,q) \in C$ and $(p',q') \in C$. By convexity of C, we have that $(\alpha p + (1-\alpha)p', \alpha q + (1-\alpha)q') \in C$ which implies that $\alpha p + (1-\alpha)p' \in P_1$. Hence P_1 is convex. Along the same lines one sees that P_2 is convex and consequently that $P_1 \times P_2$ is convex.

Secondly we show that $P \subset E(A,B)$. For this, let $p \in P_1$ and $q \in P_2$. To show that $(p,q) \in E(A,B)$ it suffices to prove that $C(p) \subset PB_1(q)$. Take $p' \in \Delta_m$ and $q' \in \Delta_n$ such that $(p,q') \in C$ and $(p',q) \in C$. For $0 < t < 1$, define $p(t) = tp + (1-t)p'$ and $q(t) = tq' + (1-t)q$.

Obviously, by convexity of P, $(p(t), q(t)) \in P$. Hence $(p(t), q(t)) \in E(A,B)$ for all $0 < t < 1$. Moreover, for all $0 < t < 1$, $C(p(t)) = C(p) \cup C(p')$ and, consequently

$$C(p) \cup C(p') \subset PB_1(q(t))$$

Furthermore, for t close enough to 0,

$$PB_1(q(t)) \subset PB_1(q),$$

and hence $C(p) \subset PB_1(q)$.

(ii) Let $p, p' \in D_1$, $q, q' \in D_2$, $\alpha \in [0,1]$ and $\beta \in [0,1]$.

Define $p(\alpha) = \alpha p + (1-\alpha)p'$ and $q(\beta) = \beta q + (1-\beta)q'$.

Since $D_1 \times D_2 \subset E(A,B)$ we have that $(p(\alpha), q(\beta)) \in E(A,B)$ for any $\alpha \in \{0,1\}$ and $\beta \in \{0,1\}$. Now let $0 < \alpha < 1$ and $0 < \beta < 1$. To prove that $(p(\alpha), q(\alpha)) \in E(A,B)$ it suffices to prove that $C(p(\alpha)) \subset PB_1(q(\beta))$.

Since $C(p(\alpha)) = C(p) \cup C(p')$, $C(p) \subset PB_1(q)$, $C(p) \subset PB_1(q')$, $C(p') \subset PB_1(q)$ and $C(p') \subset PB_1(q')$ we have that

$$C(p(\alpha)) \subset PB_1(q) \cup PB_1(q').$$

Hence, with $e_i \in C(p(\alpha))$ and $k \in \{1, ..., m\}$, we have

$$\begin{aligned} e_i A q(\beta) &= e_i A(\beta q + (1-\beta)q') \\ &= \beta e_i A q + (1-\beta) e_i A q' \\ &\geq \beta(e_k A q) + (1-\beta) e_k A q' \\ &= e_k A(\beta q + (1-\beta)q') = e_k A q(\beta) \end{aligned}$$

Consequently $C(p(\alpha)) \subset PB_1(q(\beta))$. □

As a direct consequence of Theorem 12.3 and Lemma 14.3 we have

Theorem 14.4
For an $m \times n$ matrix game A, $E(A) = O_1(A) \times O_2(A)$ is the unique Nash component.

For general bimatrix games it can be shown that each Nash component is a polytope and that the number of Nash components is finite.

Theorem 14.5
Let (A, B) be an $m \times n$ bimatrix game. Then $E(A, B)$ is the union of the Nash components of (A, B). Moreover, each Nash component of (A, B) is a polytope and there are finitely many Nash components.

For the idea of a proof of Theorem 14.5 we refer to Exercise 14.9.

To conlcude this section we discuss a special subclass of bimatrix games, the class of competition games. This class exhibits the typical two features of matrix games: the set of Nash equilibria consists of exactly one Nash component (and hence the Nash equilibria are exchangeable, cf. Lemma 14.3) while, moreover, all Nash equilibria lead to the same payoff vector. For a matrix game A this is the vector $\begin{pmatrix} v(A) \\ -v(A) \end{pmatrix}$.

Let (A, B) be an $m \times n$ bimatrix game. For $\hat{p} \in \Delta_m$ and $\hat{q} \in \Delta_n$ define the sets $A_2(\hat{p})$ and $A_1(\hat{p})$ of *antagonistic replies* by

$$A_2(\hat{p}) = \{q \in \Delta_n \mid \hat{p}Aq \leq \hat{p}Aq' \text{ for all } q' \in \Delta_n\}$$

and

$$A_1(\hat{q}) = \{p \in \Delta_m \mid pB\hat{q} \leq p'B\hat{q} \text{ for all } p' \in \Delta_m\}$$

A strategy combination $(\hat{p}, \hat{q}) \in \Delta_m \times \Delta_n$ is called a *twisted equilibrium* of (A, B) if $\hat{p} \in A_1(\hat{q})$ and $\hat{q} \in A_2(\hat{p})$. The set of all twisted equilibria of (A, B) is denoted by $TE(A, B)$. It is readily verified that $TE(A, B) = E(-B, -A)$.
If the set of Nash equilibria $E(A, B)$ equals the set of twisted equilibria $TE(A, B)$, then (A, B) is called a *competition game*. Obviously, every matrix game is competition game.

Example 14.4
Two examples of competition games are the Prisoners' dilemma game (A, B) given by

$$
\begin{array}{ccc} & & \begin{array}{cc} f_1 & f_2 \end{array} \\ (A,B) = & \begin{array}{c} e_1 \\ e_2 \end{array} & \left[\begin{array}{cc} 4,4 & 7,2 \\ 2,7 & 5,5 \end{array} \right] \end{array}
$$

with $E(A, B) = TE(A, B) = \{(e_1, f_1)\}$, and the 2×3 bimatrix game (A, B) given by

$$
\begin{array}{ccc} & & \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ (A,B) = & \begin{array}{c} e_1 \\ e_2 \end{array} & \left[\begin{array}{ccc} 4,1 & 1,5 & 5,3 \\ 1,5 & 4,1 & 2,2 \end{array} \right] \end{array}
$$

with $E(A, B) = TE(A, B) = \{\frac{1}{2}e_1 + \frac{1}{2}e_2\} \times \{\frac{1}{2}f_1 + \frac{1}{2}f_2\}$.
The 2×2 bimatrix game (A, B) given by

$$
\begin{array}{ccc} & & \begin{array}{cc} f_1 & f_2 \end{array} \\ (A,B) = & \begin{array}{c} e_1 \\ e_2 \end{array} & \left[\begin{array}{cc} 0,2 & 1,0 \\ 1,0 & 0,3 \end{array} \right] \end{array}
$$

is not a competition game since $E(A, B) = \{\frac{3}{5}e_1 + \frac{2}{5}e_2\} \times \{\frac{1}{2}f_1 + \frac{1}{2}f_2\}$ while $TE(A, B) = \{\frac{1}{2}e_1 + \frac{1}{2}e_2\} \times \{\frac{3}{5}f_1 + \frac{2}{5}f_2\}$. $\diamond$

Theorem 14.6
Let (A, B) be a competition game. Then

(i) $(\hat{p}, \hat{q}) \in E(A, B)$ *if and only if both* $(\hat{p}, \hat{q}) \in E(A)$ *and* $(\hat{q}, \hat{p}) \in E(B^T)$

(ii) there is a unique Nash component and all Nash equilibria lead to the payoff vector $\begin{pmatrix} v(A) \\ v(B^T) \end{pmatrix}$.

Proof
(i) First let $(\hat{p}, \hat{q}) \in E(A, B)$. Since (A, B) is a competition game we also have that $(\hat{p}, \hat{q}) \in TE(A, B)$. Since $\hat{q} \in A_2(\hat{p})$ and $\hat{p} \in B_1(\hat{q})$ we find that

$$\hat{p}Aq \geq \hat{p}A\hat{q} \text{ and } \hat{p}A\hat{q} \geq pA\hat{q}$$

for all $p \in \Delta_m$ and $q \in \Delta_n$. This implies that $(\hat{p}, \hat{q}) \in E(A)$. Further, since $\hat{p} \in A_1(\hat{q})$ and $\hat{q} \in B_2(\hat{p})$ we find that

$$pB\hat{q} \geq \hat{p}B\hat{q} \text{ and } \hat{p}B\hat{q} \geq \hat{p}Bq,$$

and thus

$$\hat{q}B^T p \geq \hat{q}B^T \hat{p} \text{ and } \hat{q}B^T \hat{p} \geq qB^T \hat{p}$$

for all $p \in \Delta_m$ and $q \in \Delta_n$. This implies that $(\hat{q}, \hat{p}) \in E(B^T)$.
Conversely, let $(\hat{p}, \hat{q}) \in E(A)$ and $(\hat{q}, \hat{p}) \in E(B^T)$. Since $(\hat{p}, \hat{q}) \in E(A)$ we have that $\hat{p}A\hat{q} \geq pA\hat{q}$ for all $p \in \Delta_m$. Since $(\hat{q}, \hat{p}) \in E(B^T)$ we find that $\hat{p}B\hat{q} = \hat{q}B^T\hat{p} \geq qB^T\hat{p} = \hat{p}Bq$ for all $q \in \Delta_n$. Hence, $(\hat{p}, \hat{q}) \in E(A, B)$.

(ii) Using Theorem 14.5 and the definition of a Nash component, to establish that there is a unique Nash component, it suffices to prove that $E(A, B)$ is convex.
Theorem 12.3 and part (i) imply that

$$E(A, B) = (O_1(A) \cap O_2(B^T)) \times (O_2(A) \cap O_1(B^T))$$

Since $O_1(A)$ and $O_2(B^T)$ are convex, $O_1(A) \cap O_2(B^T)$ is convex, too. Similarly, we find that $O_2(A) \cap O_1(B^T)$ is convex.
Now consider $(\hat{p}, \hat{q}) \in E(A, B)$. Using part (i) we have $(\hat{p}, \hat{q}) \in E(A)$ and $(\hat{q}, \hat{p}) \in E(B^T)$. Hence

$$\hat{p}A\hat{q} = v(A) \text{ and } \hat{p}B\hat{q} = \hat{q}B^T\hat{p} = v(B^T). \qquad \square$$

Exercises

Exercise 14.1

(i) Using best reply correspondences, determine all Nash equilibria for the following four 2×2 bimatrix games

(a) $\begin{array}{cc} & \begin{array}{cc} f_1 & f_2 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{bmatrix} 2,0 & 1,0 \\ 0,0 & 3,0 \end{bmatrix} \end{array}$,

(b) $\begin{array}{cc} & \begin{array}{cc} f_1 & f_2 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{bmatrix} 3,0 & 3,1 \\ 1,1 & 0,0 \end{bmatrix} \end{array}$,

(c) $\begin{array}{cc} & \begin{array}{cc} f_1 & f_2 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{bmatrix} 1,1 & a,0 \\ 0,0 & 2,1 \end{bmatrix} \end{array}$, with $a \in \mathbb{R}$,

$$\begin{array}{lccc} & & f_1 & f_2 \\ \text{(d)} & \begin{array}{c} e_1 \\ e_2 \end{array} & \multicolumn{2}{c}{\left[\begin{array}{cc} 1,1 & 0,0 \\ 0,0 & b,b \end{array}\right]} \end{array}, \text{ with } b \in \mathbb{R}.$$

(ii) How many different types of best reply correspondences for a 2×2 bimatrix game would you discriminate between?

(iii) Use a graphical representation of best reply correspondences in a prism $(\Delta_2 \times \Delta_3)$ to determine all Nash equilibria for the following four 2×3 bimatrix games (A, B)

$$\text{(a)} \quad \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \left[\begin{array}{ccc} 1,0 & 1,1 & 1,1 \\ 0,1 & 0,0 & 0,0 \end{array}\right] \end{array},$$

$$\text{(b)} \quad \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \left[\begin{array}{ccc} 1,0 & 1,1 & 0,1 \\ 0,1 & 0,0 & 0,0 \end{array}\right] \end{array},$$

$$\text{(c)} \quad \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \left[\begin{array}{ccc} 1,0 & 1,1 & 0,1 \\ 0,1 & 0,0 & 1,0 \end{array}\right] \end{array},$$

$$\text{(d)} \quad \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \left[\begin{array}{ccc} 0,0 & 1,1 & 0,1 \\ 1,1 & 0,0 & 0,0 \end{array}\right] \end{array}.$$

Exercise 14.2

Reconsider the 3-person finite strategic purification game of Example 11.2. Now we allow each player to randomize between his two actions P and NP while a mixed strategy combination is evaluated by means of expected payoffs. In this way we obtain a so-called $2\times2\times2$ trimatrix game.

(a) Explicitly determine the expected payoffs for each of the players corresponding to the mixed strategy combination $(p, q, r) \in \Delta_2 \times \Delta_2 \times \Delta_2$, where p_1, q_1 and r_1 denote the probabilities to choose the action P.

(b) Draw the graphs of the three best reply correspondences in a cube (representing $\Delta_2 \times \Delta_2 \times \Delta_2$) and from this derive the 9 Nash equilibria of this game.

Exercise 14.3

(a) Let (A, B) be a $2 \times n$ bimatrix game and let $j, l \in \{1, ..., m\}$ be such that $PB_1(f_j) = \{e_1\}$ and $PB_1(f_l) = \{e_2\}$. Provide an explicit formula for $q(j, l)$ in terms of the entries of A.

(b) Using the graphical solution method for $2 \times n$ bimatrix games, (re)calculate all Nash equilibria for the four 2×2 bimatrix games of Exercise 14.1 (i) and for the four 2×3 bimatrix games of Exercise 14.1 (iii).

Exercise 14.4

(a) Determine $E(A, B)$ for the 2×4 bimatrix game given by

$$(A, B) = \begin{array}{c c} & \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{bmatrix} 0,3 & 0,0 & 1,1 & 2,2 \\ 0,0 & 2,2 & 0,2 & 0,2 \end{bmatrix} \end{array}$$

(b) For all $a \in \mathbb{R}$, determine $E(A, B)$ for the 2×4 bimatrix game given by

$$(A, B) = \begin{array}{c c} & \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{bmatrix} 0,3 & 0,0 & 1,1 & 2,a \\ 0,0 & 2,2 & 0,2 & 0,a \end{bmatrix} \end{array}$$

Exercise 14.5

Reconsider the matrix games of Exercise 12.7 (c), (d) and of Exercise 12.8. Using the graphical solution method for $2 \times n$ bimatrix games, recalculate the set of Nash equilibria for these three matrix games.

Is there a connection between the labels of the lines corresponding to pure strategies of player 2 in a matrix game and the slope of these lines in the picture?

Exercise 14.6

(a) (Re)prove the existence of Nash equilibria for a $2 \times n$ bimatrix game using a combinatorial argument w.r.t. the labels on the maximum function.

(b) Let (A, B) be a 2×37 bimatrix game. Suppose $|E(A, B)| < \infty$.

How many Nash equilibria can (A, B) have at most?

How is this for a 2×38 bimatrix game?

(c) How to use the graphical procedure for a $2 \times n$ bimatrix game to determine all Nash equilibria of an $m \times 2$ bimatrix game?

Exercise 14.7

For each of the four cases below, construct a 2×4 bimatrix game (A, B) with

(a) $E(A,B) = \{\frac{1}{2}e_1 + \frac{1}{2}e_2\} \times \{\frac{1}{2}f_1 + \frac{1}{2}f_4\}$.

(b) $E(A,B) = \{e_1\} \times \mathrm{Conv}\{f_2, \frac{1}{2}f_2 + \frac{1}{2}f_3\}$.

(c) $|E(A,B)| = 2$.

(d) $|E(A,B)| = 4$.

Exercise 14.8

(a) Determine all Nash components for the 2×4 bimatrix game of Exercise 14.4 (a).

(b) What is the maximal number of Nash components for a $2 \times n$ bimatrix game?

(c) Construct a $2 \times 2 \times 2$ trimatrix game for which the set of Nash equilibria is not a finite union of polytopes.

Exercise 14.9

Let (A, B) be an $m \times n$ bimatrix game. With notations as in Lemma 14.3, let $C = C_1 \times C_2$ be a Nash component.

Take $p^0 \in C_1$ such that for every $p \in C_1 \backslash \{p_0\}$ there is an $\varepsilon > 1$ such that $(1-\varepsilon)p + \varepsilon p^0 \in C_1$. This boils down to picking p^0 in the relative interior of the convex set C_1.

Define $S_2(p^0) = \{q \in \Delta_n \mid (p^0, q) \in E(A,B)\}$ as the set of all strategies q of player 2 such that (p^0, q) constitutes a Nash equilibrium.

(a) Show that $C_2 \subset S_2(p^0)$.

(b) Prove that $C_2 = S_2(p^0)$ by proving that $C_1 \times S_2(p^0) \subset E(A,B)$.

It will be clear that with q^0 in the relative interior of the convex set C_2 and $S_1(q^0) = \{p \in \Delta_m \mid (p, q^0) \in E(A,B)\}$ we have that $C_1 = S_1(q^0)$ and, consequently, that $C = S_1(q^0) \times S_2(p^0)$. Hence C is a polytope.

(c) Provide an argument why -the number of Nash components has to be finite.

Exercise 14.10

Check for each of the following six bimatrix games if it is a competition game or not. Explicitly calculate the set of Nash equilibria and the set of twisted equilibria.

(a) $$(A,B) = \begin{array}{c c} & \begin{array}{cc} f_1 & f_2 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{bmatrix} 2,3 & 0,0 \\ 0,0 & 2,3 \end{bmatrix} \end{array}.$$

(b) $$(A,B) = \begin{array}{c c} & \begin{array}{cc} f_1 & f_2 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{bmatrix} 3,1 & 0,1 \\ 0,0 & 2,3 \end{bmatrix} \end{array}.$$

(c) $$(A,B) = \begin{array}{c c} & \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{bmatrix} 2,0 & 0,6 & 0,8 \\ 0,8 & 0,6 & 1,0 \end{bmatrix} \end{array}.$$

(d) $$(A,B) = \begin{array}{c c} & \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{bmatrix} 3,0 & 0,6 & -1,8 \\ -1,8 & 0,6 & 3,0 \end{bmatrix} \end{array}.$$

Exercise 14.11 (Rivalry games)

Let (A,B) be an $m \times n$ bimatrix game. (A,B) is called a *rivalry game* if both $A_1(q) = B_1(q)$ and $A_2(p) = B_2(p)$ for all $p \in \Delta_m$ and $q \in \Delta_n$.

(a) Show that any rivalry game is a competition game.

(b) Verify that the 2×3 competition game of Example 14.4 is not a rivalry game.

(c) Show that for a rivalry game (A,B), $(p,q) \in E(A,B)$ if and only if $(p,q) \in E(A)$. Hint: use Theorem 14.6 (i).

Exercise 14.12 (Strict equilibria)

A strategy combination $(p,q) \in \Delta_m \times \Delta_n$ is called a *strict* equilibrium for an $m \times n$ bimatrix game (A,B) if $B_1(q) = \{p\}$ and $B_2(p) = \{q\}$.

(a) Show that a strict equilibrium must consist of pure strategies.

(b) Provide an example of a 2×2 bimatrix game without strict equilibria.

(c) Provide an example of a 2×2 bimatrix game with exactly two strict equilibria.

(d) How many strict equilibria does a 2×37 bimatrix game have at most?

Exercise 14.13 (Correlated equilibria)
Let (A, B) be an $m \times n$ bimatrix game. A *correlated strategy* is an overall probability distribution $P \in \Delta_{m \times n}$ over the set of all pure strategy combinations $\{(e_i, f_j) \mid i \in \{1, ..., m\}, j \in \{1, ..., n\}\}$. Note that a correlated strategy P can be represented by an $m \times n$ matrix with non-negative entries summing up to 1. The expected payoffs $\pi_1(P)$ and $\pi_2(P)$ to the players for a correlated strategy P are given by

$$\pi_1(P) = \sum_{i=1}^{m} \sum_{j=1}^{n} P_{ij} A_{ij} \text{ and } \pi_2(P) = \sum_{i=1}^{m} \sum_{j=1}^{n} P_{ij} B_{ij}.$$

Clearly any mixed strategy combination $(p, q) \in \Delta_m \times \Delta_n$ corresponds naturally to correlated strategy $P(p, q)$ with $P_{ij}(p, q) = p_i q_j$ for all $i \in \{1, ..., m\}$ and $j \in \{1, ..., n\}$.

Now consider $(A, B) = \begin{array}{cc} & \begin{array}{cc} f_1 & \;\; f_2 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{bmatrix} 5, 1 & 0, 0 \\ 0, 0 & 1, 5 \end{bmatrix} \end{array}$

(a) Let $P^1 = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$ and $P^2 = \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}$ be two correlated strategies. Calculate the vectors $\pi(P^1)$ and $\pi(P^2)$ of expected payoffs.

(b) Show that P^1 corresponds to a mixed strategy combination while P^2 does not.

(c) Determine the set of all payoff vectors $\pi(P)$ that can be obtained via some correlated strategy P.

(d) Determine the set of all payoff vectors (c, c) with equal payoff c to both players that can be reached via some mixed strategy combination $(p, q) \in \Delta_m \times \Delta_n$.

A correlated strategy P can be implemented in the following way. First, an arbiter draws a pure strategy combination (e_i, f_j) according to the distribution P (known to both players). Secondly, the arbiter advises player 1 to play e_i (without knowing f_j) and player 2 to play f_j (without knowing e_i).

A correlated strategy P is a *correlated equilibrium* if for *each* possible realization (e_i, f_j) of the draw by the arbiter, playing the corresponding advised strategy is incentive compatible for both players, i.e., if

$$\sum_{l=1}^{n} P_{il}A_{il} \geq \sum_{l=1}^{n} P_{il}A_{kl} \text{ for all } k \in \{1, ..., m\}$$

and

$$\sum_{k=1}^{m} P_{kj}B_{kj} \geq \sum_{k=1}^{m} P_{kj}A_{kl} \text{ for all } l \in \{1, ..., n\}.$$

The set of all correlated equilibria of (A, B) is denoted by $CE(A, B)$.
It is readily verified that $CE(A, B)$ is a polytope.

(e) Prove that for any bimatrix game (A, B) we have that

$$\text{Conv}\{P(p, q) \mid (p, q) \in E(A, B)\} \subset CE(A, B)$$

(f) Show that $\hat{P}_1 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ and $\hat{P}_2 = \begin{bmatrix} \frac{5}{11} & 0 \\ \frac{1}{11} & \frac{5}{11} \end{bmatrix}$ are correlated equilibria for the 2×2 bimatrix game (A, B) analyzed above.

In fact it can be shown that for this game $CE(A, B)$ equals

$$\text{Conv}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{5}{36} & \frac{25}{36} \\ \frac{1}{36} & \frac{5}{36} \end{bmatrix}, \begin{bmatrix} \frac{1}{7} & \frac{5}{7} \\ 0 & \frac{1}{7} \end{bmatrix}, \begin{bmatrix} \frac{5}{11} & 0 \\ \frac{1}{11} & \frac{5}{11} \end{bmatrix} \right\}$$

Note that $\begin{bmatrix} \frac{5}{36} & \frac{25}{36} \\ \frac{1}{36} & \frac{5}{36} \end{bmatrix}$ corresponds to the mixed strategy Nash equilibrium $(\frac{1}{6}e_1 + \frac{5}{6}e_2, \frac{5}{6}e_1 + \frac{1}{6}e_2)$.

(g) For the 2×2 bimatrix game analyzed above, what are all expected payoff vectors that can be obtained from correlated equilibria? What are all payoff vectors obtained by randomizing the 3 Nash equilibrium payoff vectors?

(h) Now consider $(A, B) = \begin{array}{c|cc} & f_1 & f_2 \\ \hline e_1 & & \\ e_2 & & \end{array}$ $\begin{array}{cc} & \\ e_1 & \\ e_2 & \end{array}\begin{bmatrix} 5, 1 & 0, 0 \\ 4, 4 & 1, 5 \end{bmatrix}$ where $CE(A, B)$ equals

$$\text{Conv}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}, \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} \end{bmatrix} \right\}$$

Answer the two questions posed in (g) for this game.

Some relevant literature

Borm, P. (1987). A classification of 2×2 bimatrix games. *Cahiers du CERO*, 29, 69-84.

Gonzalez-Alcon, C., Hendrickx, R., Kuijk, K. van and P. Borm (2007). A taxonomy of best-reply multifunctions in $2 \times 2 \times 2$ trimatrix games, *TOP*, 15, 297-306.

Borm, P., Gijsberts, A. and S.Tijs (1989). A geometric-combinatorial approach to bimatrix games. *Methods of Operations Research*, 59, 199-209.

Jansen, M. (1981). Maximal Nash subsets for bimatrix games. *Naval Research Logistics Quarterly*, 28, 147-152.

Aumann, R. (1961). Almost strictly competitive games. *Journal of the Society of Industrial and Applied Mathematics*, 9, 544-550.

Kats, A. and J. Thisse (1992). Unilaterally competitive games. *International Journal of Game Theory*, 21, 291-299.

Rauhut, B., Schmitz, N. and E. Zachow (1979). Spieltheorie. Teubner, Stuttgart.

Borm, P., Estévez-Fernández, A. and G. Fiestras-Janeiro (2008). Competitive environments and protective behavior. *Games and Economic Behavior*, 67, 245-252.

Hendrickx, R., Peeters, R. and J. Potters (2002). A relation between Nash equilibria and correlated equilibria. *International Game Theory Review*, 4, 405-413.

CHAPTER 15

Refinements: perfect and proper equilibria

The set of Nash equilibria of a bimatrix game can be quite large and may contain rather unsuitable outcomes. One can try to reduce the set of Nash equilibria by imposing extra conditions in the hope to remove unreasonable outcomes. Specific thought experiments testing the robustness of Nash equilibria with respect to slight perturbations of the game parameters form the basis of several refinements.

Probably the most important refinement is the concept of perfect equilibria. In the underlying thought experiment, one allows for a slight imperfection of rationality of the players in the sense that players, after having decided which action to choose, can make mistakes in executing the action (a trembling hand), so there is a small but positive probability that any other action will be executed. Perfect equilibria are defined as the limits of sequences of Nash equilibria of the perturbed games within the thought experiment when the trembling probabilities tend to 0.
The idea of the trembling hand thought experiment is formalized below.

Let (A, B) be an $m \times n$ bimatrix game. If $\varepsilon^1 \in \mathbb{R}^m, \varepsilon^2 \in \mathbb{R}^n$ are such that

$$\varepsilon^1 > 0, \varepsilon^2 > 0, \sum_{i=1}^{m} \varepsilon_i^1 \leq 1, \sum_{j=1}^{n} \varepsilon_j^2 \leq 1$$

then $(\varepsilon^1, \varepsilon^2)$ is called a *mistake vector.*
The $(\varepsilon^1, \varepsilon^2)$-*perturbed game* $(A, B;\ \varepsilon^1, \varepsilon^2)$ of (A, B) is the game in strategic form given by $(\Delta_m^{\varepsilon^1}, \Delta_n^{\varepsilon^2}, \pi_1, \pi_2)$ where

$$\Delta_m^{\varepsilon^1} = \left\{ p \in \Delta_m \mid p_i \geq \varepsilon_i^1 \text{ for all } i \in \{1, 2, \ldots, m\} \right\}$$

$$\Delta_n^{\varepsilon^2} = \{q \in \Delta_n \mid q_j \geq \varepsilon_j^2 \text{ for all } j \in \{1, 2, \ldots, n\}\}$$

while

$$\pi_1(p, q) = pAq \text{ and } \pi_2(p, q) = pBq$$

for all $(p, q) \in \Delta_m^{\varepsilon^1} \times \Delta_n^{\varepsilon^2}$. So the only difference between a bimatrix game and a corresponding perturbed game lies in the fact that the strategy spaces in the perturbed game are restricted: every pure strategy has to be played with some minimal positive probability determined by the mistake vector.
Note that, although a perturbed game is *not* a bimatrix game, Theorem 11.4 guarantees the existence of Nash equilibria in such a game. Moreover, in order to obtain Nash equilibria of a perturbed game it is required to play a pure strategy which is not a best reply with minimal probability. This is formalized in

Lemma 15.1
Let (A, B) be an $m \times n$ bimatrix game, $(\varepsilon^1, \varepsilon^2)$ a mistake vector and let $(p, q) \in \Delta_m^{\varepsilon_1} \times \Delta_n^{\varepsilon_2}$. Then $(p, q) \in E(A, B; \varepsilon^1, \varepsilon^2)$ if and only if the following conditions are satisfied

$$\textit{for all } i \in \{1, \ldots, m\} : e_i \notin PB_1(q) \Rightarrow p_i = \varepsilon_i^1$$

$$\textit{for all } j \in \{1, \ldots, n\} : f_j \notin PB_2(p) \Rightarrow q_j = \varepsilon_j^2$$

Now we can define the notion of perfection.
Let (A, B) be an $m \times n$ bimatrix game. A strategy combination $(p, q) \in \Delta_m \times \Delta_n$ is called *perfect* if there is a sequence of mistake vectors $\{(\varepsilon^1(k), \varepsilon^2(k))\}_{k \in \mathbb{N}}$ converging to 0 and a sequence $\{(p^k, q^k)\}_{k \in \mathbb{N}} \subset \Delta_m \times \Delta_n$ converging to (p, q) such that $(p^k, q^k) \in E(A, B; \varepsilon^1(k), \varepsilon^2(k))$ for all $k \in \mathbb{N}$. The set of all perfect strategy combinations for (A, B) is denoted by $PE(A, B)$.

The following theorem shows that perfect strategy combinations are Nash equilibria and that each bimatrix game possesses at least one perfect Nash equilibrium.

Theorem 15.2
Let (A, B) be an $m \times n$ bimatrix game. Then $PE(A, B) \subset E(A, B)$ and $PE(A, B) \neq \emptyset$.

Proof
Let $(p, q) \in PE(A, B)$. Take sequences $\{(\varepsilon^1(k), \varepsilon^2(k))\}_{k \in \mathbb{N}}$ and $\{(p^k, q^k)\}_{k \in \mathbb{N}}$ as in the definition of perfectness. To prove that (p, q) is a Nash equilibrium we use the characterization given in Lemma 14.1.

Let $i \in C(p)$. Then $p_i^k > \varepsilon_i^1(k)$ for large k because p_i^k converges to $p_i > 0$ and $\varepsilon_i^1(k)$ converges to 0 if k tends to infinity. According to Lemma 15.1 this implies that $e_i \in B_1(q^k)$ for large k. Consequently, for all $r \in \{1, \ldots, m\}$, $e_iAq^k \geq e_rAq^k$ for large k and thus $e_iAq \geq e_rAq$. Hence, $e_i \in PB_1(q)$, and, consequently, $C(p) \subset PB_1(q)$. Similarly, it follows that $C(q) \subset PB_2(p)$ and hence $(p, q) \in E(A, B)$.
To prove existence of a perfect equilibrium, take an arbitrary sequence $\{(A, B; \varepsilon^1(k), \varepsilon^2(k))\}_{k\in\mathbb{N}}$ of perturbed games with $(\varepsilon^1(k), \varepsilon^2(k))$ converging to 0 if k tends to infinity. By Theorem 11.1 there exists a sequence $\{(p^k, q^k)\}_{k\in\mathbb{N}}$ with $(p^k, q^k) \in E(A, B; \varepsilon^1(k), \varepsilon^2(k))$ for all $k \in \mathbb{N}$. Since $\Delta_m \times \Delta_n$ is compact, this sequence has a convergent subsequence. By definition the limit of this subsequence is perfect. $\square$

Example 15.1
Consider the 2×2 bimatrix game (A, B) given by

$$(A,B) = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{cc} f_1 & \quad f_2 \end{array} \\ \begin{bmatrix} 1,1 & 0,0 \\ 0,0 & 0,0 \end{bmatrix} \end{array}$$

Clearly, $E(A, B) = \{(e_1, f_1), (e_2, f_2)\}$ but (e_2, f_2) seems an undesirable equilibrium. We will see that (e_2, f_2) is ruled out by perfection. Consider a mistake vector $(\varepsilon^1, \varepsilon^2) \in \mathbb{R}^2 \times \mathbb{R}^2$. The restricted best reply correspondences as drawn in Figure 15.1 (using Lemma 15.1) readily imply that each perturbed game $(A, B; \varepsilon^1, \varepsilon^2)$ has a unique Nash equilibrium given by

$$((1 - \varepsilon_2^1)e_1 + \varepsilon_2^1 e_2, (1 - \varepsilon_2^2)f_1 + \varepsilon_2^2 f_2)$$

Hence, for *any* sequence $\{(\varepsilon^1(k), \varepsilon^2(k))\}_{k\in\mathbb{N}}$ of mistake vectors converging to 0, a corresponding sequence of Nash equilibria is uniquely determined with limit (e_1, f_1). It follows that $PE(A, B) = \{(e_1, f_1)\}$. In fact the “any" here is more than strictly necessary: for perfection it suffices to find just *one particular* sequence of mistake vectors with the required property. $\diamond$

Let (A, B) be an $m \times n$ bimatrix game. A strategy $p \in \Delta_m$ is called *completely mixed* if $C(p) = \{e_1, e_2, \ldots, e_m\}$, i.e., if all pure strategies are played with positive probability. The set of all completely mixed strategies for player 1 is denoted by $\mathring{\Delta}_m$. Similarly one defines the set $\mathring{\Delta}_n$ of completely mixed strategies for player 2.
Theorem 15.3 offers two alternative characterizations of the perfectness concept.

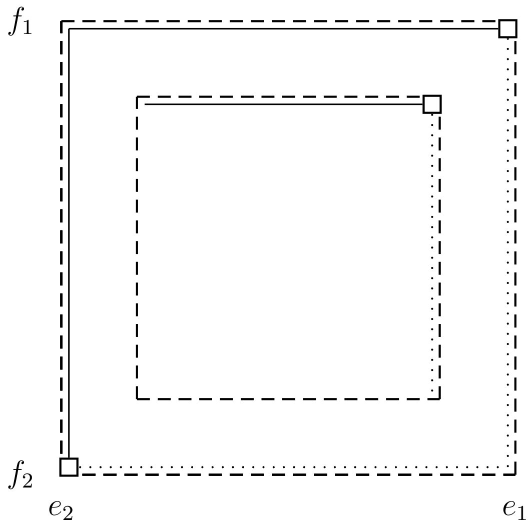


Figure 15.1: Restricted best reply correspondences in Example 15.1.

Theorem 15.3

Let (A, B) be an $m \times n$ bimatrix game and let $(p, q) \in \Delta_m \times \Delta_n$. The following three assertions are equivalent.

(i) *(p, q) is perfect.*

(ii) *There is a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0, \infty)$ of positive real numbers converging to zero and a sequence $\{(p^k, q^k)\}_{k\in\mathbb{N}} \subset \mathring{\Delta}_m \times \mathring{\Delta}_n$ of completely mixed strategy combinations with limit (p, q) such that (p^k, q^k) is ε_k-perfect for each $k \in \mathbb{N}$, i.e.,*

$$e_i A q^k < e_r A q^k \Longrightarrow p_i^k \le \varepsilon_k \quad (i, r \in \{1, \ldots, m\})$$

$$p^k B f_j < p^k B f_s \Longrightarrow q_j^k \le \varepsilon_k \quad (j, s \in \{1, \ldots, n\}).$$

(iii) *There is a sequence $\{(p^k, q^k)\}_{k\in\mathbb{N}} \subset \mathring{\Delta}_m \times \mathring{\Delta}_n$ converging to (p, q) such that*

$$p \in B_1(q^k) \text{ and } q \in B_2(p^k)$$

for all $k \in \mathbb{N}$.

Proof

We prove (i)⇒(ii),(ii)⇒(iii) and (iii)⇒(i).

(i)⇒(ii): Let $(p, q) \in \Delta_m \times \Delta_n$ be perfect. Take sequences $\{(\varepsilon^1(k), \varepsilon^2(k))\}_{k\in\mathbb{N}}$ of mistake vectors and $\{(p^k, q^k)\}_{k\in\mathbb{N}}$ of Nash equilibria of the corresponding perturbed games as in the definition of perfection. Define

$$\varepsilon_k = \max\{\varepsilon_1^1(k), \varepsilon_2^1(k), ..., \varepsilon_m^1(k), \varepsilon_1^2(k), \varepsilon_2^2(k), ..., \varepsilon_n^2(k)\}$$

for all $k \in \mathbb{N}$. Clearly $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,\infty)$, $\{\varepsilon_k\}_{k\in\mathbb{N}}$ tends to zero if k tends to infinity, $(p^k, q^k) \in \mathring{\Delta}_m \times \mathring{\Delta}_n$ for all $k \in \mathbb{N}$ while $\{(p^k, q^k)\}_{k\in\mathbb{N}}$ tends to (p, q) if k tends to infinity.
It is sufficient to show that (p^k, q^k) is ε_k-perfect for each $k \in \mathbb{N}$. For this, let $k \in \mathbb{N}$ and $i, r \in \{1, ..., m\}$ be such that $e_i A q^k < e_r A q^k$. Then $e_i \notin PB_1(q^k)$ and Lemma 15.1 implies that $p_i^k = \varepsilon_i^1(k)$. Since $\varepsilon_i^1(k) \le \varepsilon_k$ we find that $p_i^k \le \varepsilon_k$. The remaining part is shown in a similar way.

(ii)$\Rightarrow$(iii): Let $(p, q) \in \Delta_m \times \Delta_n$, $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ converging to 0, $\{(p^k, q^k)\}_{k\in\mathbb{N}} \subset \mathring{\Delta}_m \times \mathring{\Delta}_n$ converging to (p, q) such that (p^k, q^k) is ε_k-perfect for all $k \in \mathbb{N}$.
It suffices to prove that $p \in B_1(q^k)$. For this, let $e_i \in C(p)$. The fact that $p_i > 0$ implies that $p_i^k > 0$ and hence that $p_i^k > \varepsilon_k$ for large k. Using the definition of ε_k-perfectness of (p^k, q^k), this implies that $e_i \in PB_1(q^k)$.

(iii)$\Rightarrow$(i): Let $(p, q) \in \Delta_m \times \Delta_n$ and $\{(p^k, q^k)\}_{k\in\mathbb{N}} \subset \mathring{\Delta}_m \times \mathring{\Delta}_n$ converging to (p, q) such that $p \in B_1(q^k)$ and $q \in B_2(p^k)$ for all $k \in \mathbb{N}$.
It suffices to construct a sequence $\{(\varepsilon^1(k), \varepsilon^2(k))\}_{k\in\mathbb{N}}$ of mistake vectors converging to 0 such that $(p^k, q^k) \in E(A, B; \varepsilon^1(k), \varepsilon^2(k))$ for large k.
For $i \in \{1, ..., m\}$ and $k \in \mathbb{N}$, define $\varepsilon_i^1(k)$ by

$$\varepsilon_i^1(k) = \begin{cases} p_i^k & \text{if } e_i \notin C(p) \\ \frac{1}{k} & \text{if } e_i \in C(p) \end{cases}$$

For $j \in \{1, ..., n\}$ and $k \in \mathbb{N}$, define $\varepsilon_j^2(k)$ by

$$\varepsilon_j^2(k) = \begin{cases} q_j^k & \text{if } f_j \notin C(q) \\ \frac{1}{k} & \text{if } f_j \in C(q) \end{cases}$$

Clearly, for large k, $(\varepsilon^1(k), \varepsilon^2(k))$ is a mistake vector and $(p^k, q^k) \in \Delta_m^{\varepsilon^1(k)} \times \Delta_n^{\varepsilon^2(k)}$ while $\{(\varepsilon^1(k), \varepsilon^2(k))\}_{k\in\mathbb{N}}$ converges to 0.
It remains to prove that $(p^k, q^k) \in E(A, B; \varepsilon^1(k), \varepsilon^2(k))$ for large k. For this we use Lemma 15.1. Let $k \in \mathbb{N}$ be large and let $i \in \{1, ..., m\}$ be such that $e_i \notin B_1(q^k)$. Since $p \in B_i(q^k)$ we have that $C(p) \subset PB_1(q^k)$ and hence that $e_i \notin C(p)$. Consequently $p_i^k = \varepsilon_i^1(k)$. The rest is obvious. $\square$

Next we will show that perfect equilibria correspond to Nash equilibria that consist of two undominated strategies.
Before defining undominated strategies we consider undominated elements of a convex set. Let $C \subset \mathbb{R}^t$ be a convex set and let $z \in C$. Then z is called *undominated in C* if there does

not exist a $c \in C$ such that $c \geq z$ and $c \neq z$. Let $U(C)$ denote the set of all undominated elements of C. Undominated elements of a polytope are characterized below.

Lemma 15.4
Let $P \subset \mathbb{R}^t$ be a polytope and let $z \in U(P)$. Then there exists a vector $u \in \mathbb{R}^t$, $u > 0$ such that $p.u \leq z.u$ for all $p \in P$.

Now consider an $m \times n$ bimatrix game (A, B). A *strategy* $p \in \Delta_m$ is called *undominated* if the corresponding payoff vector pA is an undominated element of the polytope $\text{Conv}\{e_1A, \ldots, e_mA\}$ or, equivalently, if for all $\bar{p} \in \Delta_m$,

$$\bar{p}A \geq pA \Rightarrow \bar{p}A = pA.$$

The set of all undominated strategies for player 1 is denoted by $U_1(A)$. Similarly, $q \in \Delta_n$ is undominated if Bq is undominated in $\text{Conv}\{Bf_1, \ldots, Bf_n\}$, or equivalently, if for all $\bar{q} \in \Delta_n$

$$B\bar{q} \geq Bq \Rightarrow B\bar{q} = Bq.$$

The set of all undominated strategies for player 2 is denoted by $U_2(B)$.

Example 15.2
For the 3×2 payoff matrix A^1 given by

$$A^1 = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array} \begin{array}{c} \begin{array}{cc} f_1 & f_2 \end{array} \\ \begin{bmatrix} 1 & 1 \\ 4 & 0 \\ 0 & 4 \end{bmatrix} \end{array}$$

it follows from Figure 15.2 that $U_1(A^1) = \text{Conv}\{e_2, e_3\}$. For the 3×2 payoff matrix A^2 given by

$$A^2 = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array} \begin{array}{c} \begin{array}{cc} f_1 & f_2 \end{array} \\ \begin{bmatrix} 3 & 3 \\ 4 & 0 \\ 0 & 4 \end{bmatrix} \end{array},$$

however, it follows from Figure 15.2 that $U_1(A^2) = \text{Conv}\{e_1, e_2\} \cup \text{Conv}\{e_1, e_3\}$.
Note that although the pure strategy e_1 is not dominated by any of the two other pure strategies with respect to A^1, it is dominated by a mixed strategy (e.g. by $\frac{1}{2}e_2 + \frac{1}{2}e_3$). Moreover, although both e_2 and e_3 are undominated with respect to A^2 no (real) convex combination of e_2 and e_3 is undominated.

$\diamond$

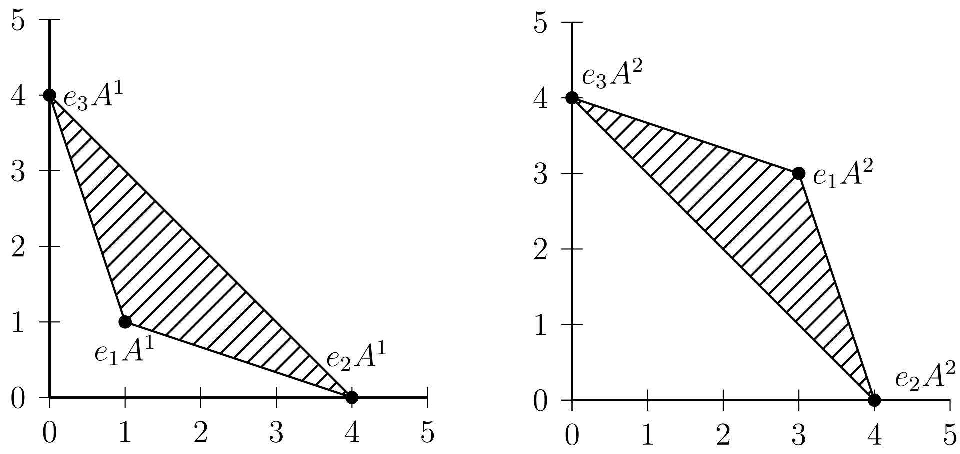


Figure 15.2: Determining undominated strategies in Example 15.2.

Theorem 15.5

Let (A, B) be an $m \times n$ bimatrix game. Then

$$PE(A, B) = E(A, B) \cap (U_1(A) \times U_2(B)).$$

Proof

We first show that each perfect equilibrium consists of undominated strategies. Let $(p, q) \in PE(A, B)$. According to Theorem 15.3 there exists a sequence $\{(p^k, q^k)\}_{k \in \mathbb{N}} \subset \mathring{\Delta}_m \times \mathring{\Delta}_n$ converging to (p, q) such that $p \in B_1(q^k)$ and $q \in B_2(p^k)$ for all $k \in \mathbb{N}$. Suppose $p \notin U_1(A)$. Then there exists a $\bar{p} \in \Delta_m$ such that

$$\bar{p}A \geq pA, \ \bar{p}A \neq pA$$

Since q^k is completely mixed this would imply that

$$\bar{p}Aq^k > pAq^k$$

for all $k \in \mathbb{N}$, contradicting the fact that $p \in B_1(q^k)$. Hence $p \in U_1(A)$. Similarly, one shows that $q \in U_2(B)$.

Next, let $(p, q) \in E(A, B)$ be such that $p \in U_1(A)$ and $q \in U_2(B)$. We will show that $(p, q) \in PE(A, B)$ by constructing a sequence $\{(p^k, q^k)\}_{k \in \mathbb{N}} \subset \mathring{\Delta}_m \times \mathring{\Delta}_n$ converging to (p, q) such that $p \in B_1(q^k)$ and $q \in B_2(p^k)$ for all $k \in \mathbb{N}$. Since $p \in U_1(A)$, pA is an undominated element of the polytope $\text{Conv}\{e_1A, \ldots, e_mA\}$. According to Lemma 15.4 there exists a vector $u \in \mathbb{R}^n$, $u > 0$ such that, for all $\bar{p} \in \Delta_m$,

$$pAu \geq \bar{p}Au.$$

Next, define $q^* \in \Delta_n$ by $q^* = \frac{u}{\sum_{j=1}^{n} u_j}$. Clearly $q^* \in \mathring{\Delta}_n$ and $p \in B_1(q^*)$.

Finally, define $q^k \in \Delta_m$ by

$$q^k = (1 - \frac{1}{k+1})q + \frac{1}{k+1}q^*.$$

for all $k \in \mathbb{N}$. One readily checks that the sequence $\{q^k\}_{k\in\mathbb{N}}$ converges to q and that $q^k \in \mathring{\Delta}_n$ for all $k \in \mathbb{N}$. Moreover, since $p \in B_1(q)$ and $p \in B_1(q^*)$, p is a best reply against any convex combination of q and q^* and therefore, in particular, $p \in B_1(q^k)$ for all $k \in \mathbb{N}$. This finishes the construction of an adequate sequence $\{q^k\}_{k\in\mathbb{N}}$. Using similar arguments, starting from the fact that $q \in U_2(B)$, one can construct an adequate sequence $\{p^k\}_{k\in\mathbb{N}}$, and combine the two. □

With rather straightforward generalizations of the various concepts we note that Lemma 15.1, Theorem 15.2 and Theorem 15.3 can be extended to mixed extensions of n-person finite games. With respect to Theorem 15.5 only “half" of it can be extend. For mixed extensions of n-person games every perfect equilibrium consists of undominated strategies only but the reverse need not be true for $n \geq 3$. For an example we refer to Exercise 15.3.

Example 15.3

Consider the 2×4 bimatrix game (A, B) given by

$$(A,B) = \begin{array}{c c} & \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{bmatrix} 2,0 & 0,6 & 0,8 & 3,8 \\ 0,8 & 0,6 & 1,0 & 1,-8 \end{bmatrix} \end{array}$$

With the techniques from Chapter 14 it readily follows that

$$\begin{aligned} E(A,B) &= \text{Conv}\{\tfrac{1}{4}e_1 + \tfrac{3}{4}e_2, \tfrac{3}{4}e_1 + \tfrac{1}{4}e_2\} \times \{f_2\} \\ &\cup \{e_1\} \times \text{Conv}\{f_4, \tfrac{2}{3}f_3 + \tfrac{1}{3}f_4\}. \end{aligned}$$

Since $U_1(A) = \Delta_2$ and $U_2(B) = \text{Conv}\{f_1, f_2\} \cup \text{Conv}\{f_2, f_3\}$, Theorem 15.5 implies that

$$PE(A,B) = \text{Conv}\{\tfrac{1}{4}e_1 + \tfrac{3}{4}e_2, \tfrac{3}{4}e_1 + \tfrac{1}{4}e_2\} \times \{f_2\}. \qquad \diamond$$

The structure of the set of perfect equilibria of a bimatrix game is quite similar to the structure of the set of Nash equilibria itself (cf. Theorem 14.5): it consists of finitely many polytopes. More precisely we have

Theorem 15.6

The set of perfect equilibria of a bimatrix game is the union of faces of Nash components.

Next we discuss a natural strengthening of the perfectness concept to strict perfection.
Let (A,B) be an $m \times n$ bimatrix game. A strategy combination $(p,q) \in \Delta_m \times \Delta_n$ is called *strictly perfect* if for *each* sequence $\{(\varepsilon^1(k), \varepsilon^2(k))\}_{k\in\mathbb{N}}$ of mistake vectors converging to 0, there is a sequence $\left\{\left(p^k, q^k\right)\right\}_{k\in\mathbb{N}} \subset \Delta_m \times \Delta_n$ converging to (p,q), such that $(p^k, q^k) \in E(A,B;\varepsilon^1(k),\varepsilon^2(k))$ for all $k \in \mathbb{N}$.

The set of all strictly perfect strategy combinations for (A,B) is denoted by $SPE(A,B)$.

Obviously, by comparing the definitions, a strictly perfect strategy combination is also perfect, so $SPE(A,B) \subset PE(A,B)$. Contrary to a perfect Nash equilibrium, a strictly perfect Nash equilibrium is stable against arbitrary slight imperfections of rationality regarding the execution of strategy choice.
Unfortunately it may happen that $SPE(A,B) = \emptyset$.

Example 15.4
Let (A,B) be the 2×3 bimatrix game given by

$$(A,B) = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{ccc} \quad f_1 \quad & \quad f_2 \quad & \quad f_3 \quad \end{array} \\ \begin{bmatrix} (1,1) & (1,0) & (0,0) \\ (1,1) & (0,0) & (1,0) \end{bmatrix} \end{array}$$

Consider (for large k) the sequence $\{(\varepsilon^1(k), \varepsilon^2(k))\}_{k\in\mathbb{N}}$ of mistake vectors with

$$\varepsilon^1(k) = (\tfrac{1}{k}, \tfrac{1}{k}) \text{ and } \varepsilon^2(k) = \begin{cases} (\frac{1}{k}, \frac{1}{k}, \frac{2}{k}) \text{ if } k \text{ even.} \\ (\frac{1}{k}, \frac{2}{k}, \frac{1}{k}) \text{ if } k \text{ odd.} \end{cases}$$

Clearly the sequence $\{(\varepsilon^1(k), \varepsilon^2(k))\}_{k\in\mathbb{N}}$ converges to 0. The unique Nash equilibrium (p^k, q^k) of $(A,B;\varepsilon^1(k),\varepsilon^2(k))$ is given by

$$(p^k, q^k) = \begin{cases} ((\frac{1}{k}, 1-\frac{1}{k}), (1-\frac{3}{k}, \frac{1}{k}, \frac{2}{k})) \text{ if } k \text{ is even.} \\ ((1-\frac{1}{k}, \frac{1}{k}), (1-\frac{3}{k}, \frac{2}{k}, \frac{1}{k})) \text{ if } k \text{ is odd.} \end{cases}$$

The sequence $\{(p^k, q^k)\}_{k\in\mathbb{N}}$ does not converge and hence $SPE(A,B) = \emptyset$. Note that $PE(A,B) = E(A,B) = \Delta_2 \times \{f_1\}$. $\diamond$

Another strengthening of perfection is provided by the notion of properness.
Let (A,B) be an $m \times n$ bimatrix game. A strategy combination (p,q) is called *proper* if there is a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ of positive real numbers converging to 0 and a sequence

$\{(p^k,q^k)\}_{k\in\mathbb{N}} \subset \mathring{\Delta}_m \times \mathring{\Delta}_n$ of completely mixed strategies converging to (p,q) such that (p^k,q^k) is ε_k-proper for each $k \in \mathbb{N}$, i.e.,

$$e_iAq^k < e_rAq^k \Longrightarrow p_i^k \leq \varepsilon_k p_r^k \qquad (i,r \in \{1,\ldots,m\})$$

$$p^kBf_j < p^kBf_s \Longrightarrow q_j^k \leq \varepsilon_k q_s^k \qquad (j,s \in \{1,\ldots,n\}).$$

The set of all proper strategy combinations for (A,B) is denoted by $PR(A,B)$.

Since any ε_k-proper strategy combination is also ε_k-perfect, Theorem 15.3 directly implies that $PR(A,B) \subset PE(A,B)$.
The additional idea behind the properness concept is that a player will assure that more costly mistakes will occur with significantly smaller probability (as measured by a factor ε_k) than less costly mistakes.

Example 15.5
For the 2×2 bimatrix game of Example 15.1 we have seen that (e_1,f_1) is a perfect equilibrium while (e_2,f_2) is a non-perfect equilibrium. This situation however will change if we add a (strictly) dominated row and column, obtaining the 3×3 bimatrix game (A,B) given by

$$(A,B) = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array} \begin{array}{c} \begin{array}{ccc} \quad f_1 \quad & \quad f_2 \quad & \quad f_3 \quad \end{array} \\ \begin{bmatrix} 1,1 & 0,0 & -9,-9 \\ 0,0 & 0,0 & -7,-7 \\ -9,-9 & -7,-7 & -9,-9 \end{bmatrix} \end{array}$$

Since now both e_1 and e_2 belong to $U_1(A)$ (and f_1 and f_2 to $U_2(B)$), both (e_1,f_1) and (e_2,f_2) are perfect equilibria. We will show that the unattractive equilibrium (e_2,f_2) is not proper. Take a sequence $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ converging to 0 and a sequence $\{(p^k,q^k)\}_{k\in\mathbb{N}} \subset \mathring{\Delta}_m \times \mathring{\Delta}_n$ converging to some (p,q), such that (p^k,q^k) is ε_k-proper for all $k \in \mathbb{N}$.
Since $e_1Aq^k > e_3Aq^k$ and $p^kBf_1 > p^kBf_3$ it follows that

$$p_3^k \leq \varepsilon_k p_1^k \text{ and } q_3^k \leq \varepsilon_k q_1^k$$

Consequently, for large k,

$$p^kBf_1 > p^kBf_2 > p^kBf_3 \text{ and } e_1Aq^k > e_2Aq^k > e_3Aq^k,$$

which implies that

$$p_3^k \leq \varepsilon_k p_2^k \leq \varepsilon_k^2 p_1^k \text{ and } q_3^k \leq \varepsilon_k q_2^k \leq \varepsilon_k^2 q_1^k.$$

Since $1 = p_1^k + p_2^k + p_3^k \leq p_1^k(1+\varepsilon_k+\varepsilon_k^2)$, we have for the limit p that $p_1 = 1$ and hence that $p = e_1$. Similarly, $q = f_1$. Hence, $PR(A,B) = \{(e_1,f_1)\}$. $\diamond$

Contrary to strictly perfect equilibria, proper equilibria always exist.

Theorem 15.7

Every bimatrix game has at least one proper equilibrium.

Proof

Let (A, B) be an $m \times n$ bimatrix game. The proof will establish the existence of a $\frac{1}{k}$-proper equilibrium (p^k, q^k) for (large) $k \in \mathbb{N}$. Then, since $\Delta_m \times \Delta_n$ is compact, the corresponding sequence $\{(p^k, q^k)\}_{k\in\mathbb{N}}$ then has a convergent subsequence and the limit of this subsequence is by definition a proper equilibrium of (A, B).

Fix $k \in \mathbb{N}$ (large). Define $t = \max\{m, n\}$, $\alpha_k = \frac{1}{t}(\frac{1}{k})^t$ and let

$$\Delta_m^{\alpha_k} = \{p \in \Delta_m \mid p_i \geq \alpha_k \text{ for all } i \in \{1, 2, ..., m\}\}$$

and

$$\Delta_n^{\alpha_k} = \{q \in \Delta_n \mid q_j \geq \alpha_k \text{ for all } j \in \{1, 2, ..., n\}\}.$$

Next consider the correspondence F from $\Delta_m^{\alpha_k} \times \Delta_n^{\alpha_k}$ into itself defined by

$$F(p, q) = (F_1(p, q), F_2(p, q))$$

with

$$F_1(p, q) = \{p' \in \Delta_m^{\alpha_k} \mid e_i A q < e_r A q \Rightarrow p'_i \leq \frac{1}{k} p'_r\}$$

and

$$F_2(p, q) = \{q' \in \Delta_n^{\alpha_k} \mid pBf_j < pBf_s \Rightarrow q'_j \leq \frac{1}{k} q'_s\}$$

for all $(p, q) \in \Delta_m^{\alpha_k} \times \Delta_n^{\alpha_k}$.

We first verify that $F_1(p, q) \neq \emptyset$ for all $(p, q) \in \Delta_m^{\alpha_k} \times \Delta_n^{\alpha_k}$.

Fix $(p, q) \in \Delta_m^{\alpha_k} \times \Delta_n^{\alpha_k}$. For all $i \in \{1, ..., m\}$, let

$$m_i = |\{r \in \{1, ..., m\} \mid e_i A q < e_r A q\}|$$

be the number of pure strategies e_r that perform better against q than e_i. Now define $p' \in \Delta_m$ by

$$p'_i = \frac{(\frac{1}{k})^{m_i}}{\sum_{r=1}^{m} (\frac{1}{k})^{m_r}}$$

for all $i \in \{1, ..., m\}$. Now, it is readily verfied that $p' \in F_1(p,q)$. In a similar way, one can verify that $F_2(p,q) \neq \emptyset$ for all $(p,q) \in \Delta_m^{\alpha_k} \times \Delta_n^{\alpha_k}$. Moreover, using some (technical) routine skills, one can show that F satisfies the conditions of Kakutani's Theorem. Hence there is a fixed point $(p^k, q^k) \in \Delta_m^{\alpha_k} \times \Delta_n^{\alpha_k}$ such that $(p^k, q^k) \in F(p^k, q^k)$ which exactly boils down to (p^k, q^k) being $\frac{1}{k}$-proper. □

For a long time it was conjectured that each strictly perfect equilibrium is also proper. As we will see this is indeed the case for any $2 \times n$ bimatrix game. In general however it need not be true: cf. Exercise 15.4.

Interestingly, proper equilibria of bimatrix games can be alternatively motivated by a less passive thought experiment than the original one, in the sense that the alternative thought experiment requires explicit secondary strategic choices by the players.
Let (A, B) be an $m \times n$ bimatrix game. Let $(\varepsilon_1, \varepsilon_2)$ be a vector of positive numbers both smaller than 1. Here, ε_1 represents the probability that an action chosen by player 1 (which could of course be the result of a mixed strategy) will be *blocked* in the underlying thought experiment. To anticipate on this eventuality, player 1's pure strategies in the corresponding perturbed fall back proper game will be an order on his m pure strategies $e_1, e_2, ..., e_m$ which will be represented by a bijection $g : \{1, 2, ..., m\} \to \{e_1, e_2, ..., e_m\}$. Allowing for *repeated* blocking occurrences the strategy g will lead to the action e_i with probability $(1-\varepsilon_1)\varepsilon_1^{g^{-1}(e_i)-1}$: all pure strategies before e_i (the number of these strategies is $g^{-1}(e_i) - 1$) are blocked independently with probability ε_1 while e_i is not blocked (with probability $1 - \varepsilon_1$). If all pure strategies of player 1 are blocked (this happens with probability ε_1^m) we assume that the game is not played with a payoff of 0 to both players. With similar interpretations ε_2 represents the blocking probability for an action chosen by player 2 while pure strategies of player 2 in the corresponding perturbed fall back proper game will be orders on the pure strategies $f_1, f_2, ..., f_n$ and represented by bijections $h : \{1, 2, ..., n\} \to \{f_1, f_2, ..., f_n\}$.

With the interpretation provided above, the *perturbed fall back proper game* $(A^{FB}(\varepsilon_1, \varepsilon_2), B^{FB}(\varepsilon_1, \varepsilon_2))$ can be viewed as an $m! \times n!$ bimatrix game with payoff entries corresponding to the pure strategy combination (g, h) given by

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(1-\varepsilon_1)\varepsilon_1^{g^{-1}(e_i)-1}(1-\varepsilon_2)\varepsilon_2^{h^{-1}(f_j)-1}A_{ij}$$

and

$$\sum_{i=1}^{m}\sum_{j=1}^{n}(1-\varepsilon_1)\varepsilon_1^{g^{-1}(e_i)-1}(1-\varepsilon_2)\varepsilon_2^{h^{-1}(f_j)-1}B_{ij}$$

for player 1 and player 2, respectively. Mixed strategies in this perturbed game will be denoted by ρ and σ for player 1 and player 2, respectively.
A strategy combination $(p,q) \in \Delta_m \times \Delta_n$ is called *fall back proper* for (A,B) if there exists a sequence $\{(\varepsilon_1^k, \varepsilon_2^k)\}_{k\in\mathbb{N}}$ of positive blocking probabilities converging to 0 and a sequence $\{(\rho^k,\sigma^k)\}_{k\in\mathbb{N}}$ of strategy combinations in the corresponding perturbed fall back proper game such that

$$(\rho^k,\sigma^k) \in E(A^{FB}(\varepsilon_1,\varepsilon_2), B^{FB}(\varepsilon_1,\varepsilon_2))$$

for all $k \in \mathbb{N}$ while $\{(\rho^k,\sigma^k)\}_{k\in\mathbb{N}}$ converges to (ρ,σ) if k tends to infinity such that

$$p_i = \sum_{g:g^{-1}(e_i)=1} \rho(g) \text{ and } q_j = \sum_{h:h^{-1}(f_j)=1} \sigma(h)$$

for all $i \in \{1,...,m\}$ and $j \in \{1,...,n\}$.

Theorem 15.8
For a bimatrix game a strategy combination is proper if and only if it is fall back proper.

The definition of fall back proper strategy combinations can be easily generalized to mixed extensions of finite n-person games. It turns out that any fall back proper strategy profile is also proper but the reverse need not hold for $n \geq 3$.

The final part of this section describes how the labelling method for the determination of Nash equilibria for $2 \times n$ bimatrix games as presented in Section 14 can also be used to determine the sets of perfect, proper and strictly perfect equilibria.

Let (A,B) be a $2 \times n$ bimatrix game. Let $J([1])$, $J([2])$ and $J([12])$ denote the sets of pure strategies of player 2 with label [1], [2] and [12], respectively. Using these notations, Lemma 15.9 below describes the sets $U_1(A)$ and $U_2(B)$ of undominated strategies for both players. It constitutes the basic ingredient in understanding the procedure for determining perfect equilibria.

Lemma 15.9
Let (A,B) be $2 \times n$ bimatrix game. Then

$$U_1(A) = \begin{cases} \{e_1\} & \textit{if} \quad J([2]) = \emptyset \textit{ and } J([1]) \neq \emptyset, \\ \{e_2\} & \textit{if} \quad J([1]) = \emptyset \textit{ and } J([2]) \neq \emptyset, \\ \Delta_2 & \textit{otherwise} \end{cases}$$

and

$$U_2(B) = \bigcup_{k=1}^{t} \mathrm{Conv}(PB_2(I_k)) \cup \bigcup_{k=1}^{t-1} \mathrm{Conv}(PB_2(p_k))$$

Proof

The expression for $U_1(A)$ is straightforward. With respect to $U_2(B)$ it suffices to prove that $U_2(B) = \bigcup_{p \in \mathring{\Delta}_2} B_2(p)$. We first show that $\bigcup_{p \in \mathring{\Delta}_2} B_2(p) \subset U_2(B)$. Take $q \in B_2(p)$ for some $p \in \mathring{\Delta}_2$. Suppose $q \notin U_2(B)$. Then there exists a $\bar{q} \in \Delta_n$ such that $B\bar{q} \geq Bq$ and $B\bar{q} \neq Bq$. Since p is completely mixed, this would imply that $pB\bar{q} > pBq$, contradicting the fact that $q \in B_2(p)$. To show that $U_2(B) \subset \bigcup_{p \in \mathring{\Delta}_2} B_2(p)$, it suffices to prove that any $q \in \Delta_n$ which does not belong to $\bigcup_{p \in \mathring{\Delta}_2} B_2(p)$ is dominated. Consider such a q. Note that the line $p \longmapsto pBq$ will not touch the maximum function $p \longmapsto \max_{j \in \{1,\ldots,n\}} pBf_j$ perhaps with the exception of one of the boundaries corresponding to e_1 or e_2. It suffices to construct a strategy $\bar{q} \in \Delta_n$ such that the line $p \longmapsto pB\bar{q}$ lies strictly above the line $p \longmapsto pBq$, again perhaps with the exception for one of the boundaries.

In case $p \longmapsto pBq$ does not touch the maximum function, one can verify graphically that there will exist a $\bar{q} \in \Delta_n$ which is a best reply to one of the intermediate end points $p_1, p_2, \ldots, p_{t-1}$ such that the line $p \longmapsto pB\bar{q}$ is above and parallel to $p \longmapsto pBq$.

In case $p \longmapsto pBq$ touches the maximum function in $e_2(e_1)$ choosing $\bar{q} = f_j$ with $f_j \in PB_2(I_1)$ ($\bar{q} = f_j$ with $f_j \in PB_2(I_t)$) will suffice. $\square$

Now let $(A, B)^{PE}$ be defined as the subgame of (A, B) in which all pure strategies f_j of player 2 are deleted for which it holds that

$$f_j \notin \bigcup_{k=1}^{t} PB_2(I_k) \cup \bigcup_{k=1}^{t-1} PB_2(p_k).$$

The game $(A, B)^{PE}$ plays a decisive role in determining the perfect equilibria of (A, B).

Theorem 15.10

Let (A, B) be a $2 \times n$ bimatrix game. Then

$$PE(A,B) = \begin{cases} E(A,B)^{PE} & \text{if both } J([1]) \neq \emptyset \text{ and } J([2]) \neq \emptyset \\ & \text{or both } J([1]) = \emptyset \text{ and } J([2]) = \emptyset, \\ E(A,B)^{PE} \cap \{(e_1, q) | q \in \Delta_n\} & \text{if } J([1]) \neq \emptyset \text{ and } J([2]) = \emptyset, \\ E(A,B)^{PE} \cap \{(e_2, q) | q \in \Delta_n\} & \text{if } J([1]) = \emptyset \text{ and } J([2]) \neq \emptyset. \end{cases}$$

Example 15.6

Reconsider the 2×6 bimatrix game (A, B) of Example 14.3 given by

$$(A,B) = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{cccccc} f_1 & f_2 & f_3 & f_4 & f_5 & f_6 \end{array} \\ \left[\begin{array}{cccccc} 2,0 & 0,6 & 0,8 & 0,6 & 3,8 & -1,3 \\ 0,8 & 0,6 & 1,0 & 2,6 & 1,-8 & 0,7 \end{array}\right] \end{array}.$$

From Figure 14.2 it was deduced that

$$E(A,B) = T_1 \cup T_2 \cup T_3$$

with

$$T_1 = \{\tfrac{1}{4}e_1 + \tfrac{3}{4}e_2\} \times \mathrm{Conv}\{f_2, \tfrac{1}{2}f_1 + \tfrac{1}{2}f_4, \tfrac{1}{3}f_1 + \tfrac{2}{3}f_6\}$$

$$T_2 = \mathrm{Conv}\{\tfrac{1}{4}e_1 + \tfrac{3}{4}e_2, \tfrac{3}{4}e_1 + \tfrac{1}{4}e_2\} \times \{f_2\}$$

and

$$T_3 = \{e_1\} \times \mathrm{Conv}\{f_5, \tfrac{2}{3}f_3 + \tfrac{1}{3}f_5\}$$

Deleting f_5 one obtains

$$(A,B)^{PE} = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{ccccc} f_1 & f_2 & f_3 & f_4 & f_6 \end{array} \\ \left[\begin{array}{ccccc} 2,0 & 0,6 & 0,8 & 0,6 & -1,3 \\ 0,8 & 0,6 & 1,0 & 2,6 & 0,7 \end{array}\right] \end{array}$$

and hence, since both $J([1]) \neq \emptyset$ and $J([2]) \neq \emptyset$, Theorem 15.10 implies that

$$PE(A,B) = E(A,B)^{PE} = T_1 \cup T_2. \qquad \diamond$$

It turns out that $PR(A,B) = PE(A,B)$ if either $J([1]) = \emptyset$ or $J([2]) = \emptyset$. If both $J([1]) \neq \emptyset$ and $J([2]) \neq \emptyset$ the decisive role to determine all proper equilibria will be played by the subgame $(A,B)^{PR}$ of (A,B) derived by deleting pure strategies of player 2 in the following way:

(1) First delete all $f_j \in J([12])$. The resulting subgame is denoted by $(A,B)_{-[12]}$.

(2) Next, define $(A,B)^{PR} = ((A,B)_{-[12]})^{PE}$.

Theorem 15.11
Let (A, B) be a $2 \times n$ bimatrix game. Then

$$PR(A,B) = PE(A,B)$$

if $J([1]) = \emptyset$ or $J([2]) = \emptyset$, and

$$PR(A,B) = \{(p,q) \in PE(A,B) \mid \exists_{\bar{q} \in \Delta_n} : (p,\bar{q}) \in E(A,B)^{PR}\}$$

if both $J([1]) \neq \emptyset$ and $J([2]) \neq \emptyset$.

Example 15.7
We further analyze the 2×6 bimatrix game (A, B) of Example 15.6. Note that both $J([1]) \neq \emptyset$ and $J([2]) \neq \emptyset$. Determining $(A, B)_{-[12]}$, we have to eliminate f_2 in step (1). Subsequently, f_5 is deleted in step (2). Hence,

$$(A,B)^{PR} = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{cccc} f_1 & f_3 & f_4 & f_6 \end{array} \\ \begin{bmatrix} 2,0 & 0,8 & 0,6 & -1,3 \\ 0,8 & 1,0 & 2,6 & 0,7 \end{bmatrix} \end{array}.$$

Since it readily follows that there is a $\bar{q} \in \Delta_n$ such that $(p, \bar{q}) \in E(A, B)^{PR}$ if and only if $p = \frac{1}{4}e_1 + \frac{3}{4}e_2$, Theorem 15.11 and the fact hat $PE(A, B) = T_1 \cup T_2$ imply that

$$PR(A,B) = T_1. \qquad \diamond$$

It turns out that also $SPE(A, B) = PE(A, B)$ if $J([1]) = \emptyset$ or $J([2]) = \emptyset$. If both $J([1]) \neq \emptyset$ and $J([2]) \neq \emptyset$, the decisive role to determine all strictly perfect equilibria will be played by the subgame $(A, B)^{SPE}$ of (A, B). This subgame is defined by $(A, B)^{SPE} = ((A, B)^{PE})_{-[12]}$. To obtain $(A, B)^{SPE}$ one in fact reverses the two steps (1) and (2) for determining $(A, B)^{PR}$. Moreover, it is clear that $(A, B)^{SPE}$ is a subgame of $(A, B)^{PR}$. $(A, B)^{SPE}$, however, could be a vacuous game (without Nash equilibria) in the sense that no pure strategies of player 2 are left over.

Theorem 15.12
Let (A, B) be a $2 \times n$ bimatrix game. Then

$$SPE(A,B) = PE(A,B)$$

if $J([1]) = \emptyset$ or $J([2]) = \emptyset$, and

$$SPE(A,B) = \{(p,q) \in PE(A,B) \mid \exists_{\tilde{q} \in \Delta_n} : (p,\tilde{q}) \in E(A,B) \cap E(A,B)^{SPE}\}$$

if both $J([1]) \neq \emptyset$ and $J([2]) \neq \emptyset$.

Since $(A,B)^{SPE}$ is a subgame of $(A,B)^{PR}$ and $(p,\tilde{q}) \in E(A,B) \cap E(A,B)^{SPE}$ implies that $(p,\tilde{q}) \in E(A,B)^{PR}$ it follows that $SPE(A,B) \subset PR(A,B)$ for every $2 \times n$ bimatrix game (A,B). For the 2×6 bimatrix game of Example 15.6 and Example 15.7 we find that $SPE(A,B) = PR(A,B)$.

Example 15.8
Consider the 2×3 bimatrix game (A,B) given by

$$(A,B) = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \begin{bmatrix} 2,0 & 0,6 & 0,8 \\ 0,8 & 0,6 & 1,0 \end{bmatrix} \end{array}$$

It is readily derived that

$$E(A,B) = \mathrm{Conv}\{\tfrac{1}{4}e_1 + \tfrac{3}{4}e_2, \tfrac{3}{4}e_1 + \tfrac{1}{4}e_2\} \times \{f_2\}.$$

Clearly both $J([1]) \neq \emptyset$ and $J([2]) \neq \emptyset$, $(A,B)^{PE} = (A,B)$ and hence $PE(A,B) = E(A,B)$. Moreover

$$(A,B)^{PR} = (A,B)^{SPE} = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{cc} f_1 & f_3 \end{array} \\ \begin{bmatrix} 2,0 & 0,8 \\ 0,8 & 1,0 \end{bmatrix} \end{array}$$

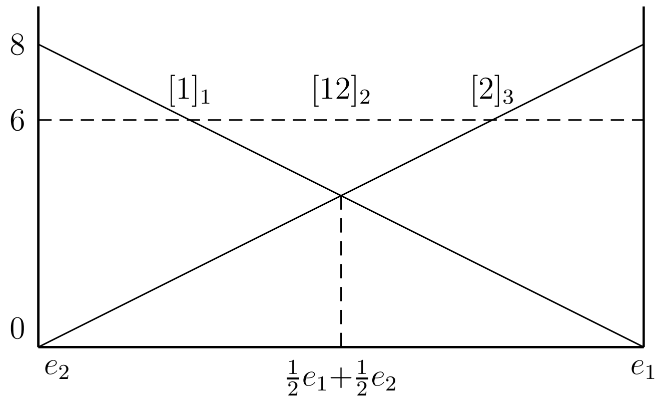


Figure 15.3: A representation of $(A,B)^{PR} = (A,B)^{SPE}$ in Example 15.8.

From Figure 15.3 we derive that there is a $\bar{q} \in \Delta_n$ with $(p,\bar{q}) \in E(A,B)^{PR}$ only if $p = \frac{1}{2}e_1 + \frac{1}{2}e_2$. Hence

$$PR(A,B) = \{(\tfrac{1}{2}e_1 + \tfrac{1}{2}e_2, f_2)\}$$

Note that the unique proper equilibrium does not correspond to a face of the unique Nash component. So Theorem 15.6 cannot be translated to proper equilibria. What can be shown, however, is that the set of proper equilibria of a bimatrix game is the finite union of polytopes.

Moreover, since $(\frac{1}{2}e_1 + \frac{1}{2}e_2, \tilde{q}) \in E(A, B)$ if and only if $\tilde{q} = f_2$ while, on the other hand, $(\frac{1}{2}e_1 + \frac{1}{2}e_2, \tilde{q}) \in E(A, B)^{SPE}$ if and only if $\tilde{q} = \frac{1}{3}f_1 + \frac{2}{3}f_3$ we have that $SPE(A, B) = \emptyset$. $\diamond$

Exercises

Exercise 15.1

Drawing restricted best reply correspondences:

(a) Determine all Nash equilibria and all perfect equilibria for the 2×2 matrix game A given by

$$A = \begin{array}{c c} & \begin{array}{cc} f_1 & f_2 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{bmatrix} 1000 & 0 \\ 0 & 0 \end{bmatrix} \end{array}$$

(b) Determine all Nash equilibria and all perfect equilibria for the 2×2 bimatrix game (A, B) given by

$$(A, B) = \begin{array}{c c} & \begin{array}{cc} f_1 & f_2 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{bmatrix} 1,1 & 3,0 \\ 0,-2 & 3,3 \end{bmatrix} \end{array}$$

Do you notice something remarkable?

(c) Determine all perfect equilibria for the 2×2 bimatrix game (A, B) given by

$$(A, B) = \begin{array}{c c} & \begin{array}{cc} f_1 & f_2 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{bmatrix} 1,1 & 0,0 \\ 0,0 & a,2 \end{bmatrix} \end{array}$$

with $a \in \mathbb{R}$.

Exercise 15.2

Let (A, B) be an $m \times n$ bimatrix game. Show that any Nash equilibrium $(p, q) \in E(A, B)$ with $p \in \mathring{\Delta}_m$ and $q \in \mathring{\Delta}_n$ is perfect.

Exercise 15.3

(a) Extend the definition of a perfect equilibrium to a trimatrix game (A, B, C). Also reformulate Lemma 15.1.

Consider the trimatrix game (A, B, C) given by

$$\begin{array}{cc} \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{cc} f_1 & \qquad\qquad f_2 \end{array} \\ \begin{bmatrix} 1,1,1 & \qquad\qquad 1,0,1 \\ 1,1,1 & \qquad\qquad 0,0,1 \end{bmatrix} \end{array} & \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{cc} f_1 & \qquad\qquad f_2 \end{array} \\ \begin{bmatrix} 1,1,0 & \qquad\qquad 0,0,0 \\ 0,1,0 & \qquad\qquad 1,0,0 \end{bmatrix} \end{array} \\ g_1 & g_2 \end{array}$$

(b) Show that (e_2, f_1, g_1) is a Nash equilibrium of (A, B, C) in undominated strategies.

(c) Show that (e_1, f_1, g_1) is the unique perfect equilibrium of (A, B, C).

Exercise 15.4

Consider the 3×4 bimatrix game (A, B) given by

$$(A, B) = \begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \end{array} \begin{array}{c} \begin{array}{cccc} f_1 & \quad f_2 & \quad f_3 & \quad f_4 \end{array} \\ \begin{bmatrix} 1,3 & 0,0 & 0,2 & 0,2 \\ 7,0 & 7,0 & -3,0 & 0,0 \\ 0,0 & 1,3 & 0,2 & 0,2 \end{bmatrix} \end{array}$$

Clearly $(e_2, f_4) \in E(A, B)$.

(a) Show that (e_2, f_4) is a strictly perfect equilibrium. To do so, let $\{(\varepsilon^1(k), \varepsilon^2(k))\}_{k \in \mathbb{N}}$ be a sequence of mistake vectors converging to 0.

If $\varepsilon_1^1(k) \geq \varepsilon_3^1(k)$, define

$$p^k = (\varepsilon_1^1(k), 1 - 3\varepsilon_1^1(k), 2\varepsilon_1^1(k))$$

and

$$q^k = (\varepsilon_1^2(k), \tfrac{7}{6}\varepsilon_1^2(k) + \varepsilon_2^2(k) + \varepsilon_3^2(k), \tfrac{14}{3}\varepsilon_1^2(k) + 2\varepsilon_2^2(k) + 2\varepsilon_3^3(k), 1 - \tfrac{41}{6}\varepsilon_1^2(k) - 3\varepsilon_2^2(k) - 3\varepsilon_3^2(k))$$

Otherwise, if $\varepsilon_1^1(k) < \varepsilon_3^1(k)$, reversing the roles of e_1 and e_3 and of f_1 and f_2, define

$$p^k = (2\varepsilon_3^1(k), 1 - 3\varepsilon_3^1(k), \varepsilon_3^1(k))$$

and

$$q^k = (\varepsilon_1^2(k) + \tfrac{7}{6}\varepsilon_2^2(k) + \varepsilon_3^2(k), \varepsilon_2^2(k), 2\varepsilon_1^2(k) + \tfrac{14}{3}\varepsilon_2^2(k) + 2\varepsilon_3^2(k), 1 - \tfrac{41}{6}\varepsilon_1^2(k) - 3\varepsilon_2^2(k) - 3\varepsilon_3^2(k))$$

Show that $(p^k, q^k) \in E(A, B; \varepsilon^1(k), \varepsilon^2(k))$ if k is large enough and that (p^k, q^k) tends to (e_2, f_4) if k tends to infinity.

(b) Show that (e_2, f_4) is not a proper equilibrium.

To do so, let $\{\varepsilon_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ be a sequence of positive real numbers converging to 0, and suppose that (p^k, q^k) is ε_k-proper for all $k \in \mathbb{N}$ such that $\{p^k, q^k)\}_{k\in\mathbb{N}}$ converges to (e_2, f_4).

Show consecutively that, for large k,

(i) $p^kBf_1 \le p^kBf_4$, $p^kBf_2 \le p^kBf_4$, $p^kBf_3 = p^kBf_4$

(ii) $e_1Aq^k = e_3Aq^k$, $q_1^k = q_2^k$, $p^kBf_1 = p^kBf_2$

(iii) $e_2Aq^k \ge e_3Aq^k$, $13q_2^k \ge 3q_3^k$

(iv) $p^kBf_2 \ge p^kBf_3$

(v) $p^kBf_2 = p^kBf_3$

(vi) $p^kBf_1 = p^kBf_2 = p^kBf_3 = p^kBf_4$, establishing a contradiction.

Exercise 15.5

Consider the 2×5 bimatrix game (A, B) given by

$$(A,B) = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{ccccc} f_1 & f_2 & f_3 & f_4 & f_5 \end{array} \\ \begin{bmatrix} 1,10 & 2,0 & 3,8 & 0,6 & 0,-8 \\ 1,10 & 0,8 & 0,0 & 1,6 & 2,8 \end{bmatrix} \end{array}$$

(a) Determine $E(A, B)$.

Note that both $J([1]) \neq \emptyset$ and $J([2]) \neq \emptyset$.

(b) Determine $(A, B)^{PE}$, $(A, B)^{PR}$ and $(A, B)^{SPE}$.

(c) Determine $PE(A, B)$, $PR(A, B)$, and $SPE(A, B)$.

Exercise 15.6

Determine the sets of perfect, proper and strictly perfect equilibria for each of the following bimatrix games.

(a) $$(A,B) = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{bmatrix} 0,3 & 0,0 & 1,1 & 2,2 \\ 0,0 & 2,2 & 0,2 & 0,2 \end{bmatrix} \end{array}.$$

(b) $$(A,B) = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{bmatrix} 0,3 & 0,0 & 1,1 & 0,2 \\ 1,0 & 2,2 & 0,2 & 0,2 \end{bmatrix} \end{array}.$$

(c) $$(A,B) = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{bmatrix} 0,3 & 0,0 & 0,1 & 0,2 \\ 1,0 & 2,2 & 1,2 & 0,2 \end{bmatrix} \end{array}.$$

(d) $$(A,B) = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \begin{bmatrix} -2,2 & -4,4 & 1,b \\ -2,2 & 0,0 & -4,4 \end{bmatrix} \end{array} \text{ with } b \in \mathbb{R}.$$

Exercise 15.7 (Robust equilibria)
Let (A,B) be an $m \times n$ bimatrix game. A strategy combination $(\hat{p},\hat{q}) \in \Delta_m \times \Delta_n$ is called *robust* for (A,B) if there exist open sets $U \subset \Delta_m$ and $V \subset \Delta_n$ such that $\hat{p} \in U$ and $\hat{q} \in V$ with

$$\hat{p} \in B_1(q) \text{ for all } q \in V$$

and

$$\hat{q} \in B_1(p) \text{ for all } p \in U.$$

Clearly, a robust strategy combination constitutes a Nash equilibrium.
Let $R(A,B)$ denote the set of all robust equilibria of (A,B).

(a) Determine $R(A,B)$ if

$$(A,B) = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{cc} f_1 & f_2 \end{array} \\ \begin{bmatrix} 2,1 & 0,0 \\ 0,0 & 1,2 \end{bmatrix} \end{array}$$

(b) Provide a 2×2 bimatrix game (A,B) such that $R(A,B) = \emptyset$.

(c) Show that each robust equilibrium is strictly perfect.

Exercise 15.8 (Informationally robust equilibria)
This exercise analyzes a refinement of Nash equilibria, called informationally robust equilibria or IRE. The thought experiment underlying the concept of IRE is based on the possibility of leakage of information. This is reflected in the definition of an informationally perturbed game. Consider an $m \times n$ bimatrix game (A, B). With small (but positive) probability ε_1 player 2 is informed about the action (which may be the outcome of a mixed strategy) of player 1. Thereafter, player 2 can base his decision on this information. Player 1 cannot distinguish between this case and the normal one, i.e., he does not know if is his action has been revealed or not. Similarly, there is small probability ε_2 that player 1 is informed about the action of player 1. With large probability $1 - \varepsilon_1 - \varepsilon_2$, no information leakage takes place. Formally, the informationally perturbed $m \times n$ bimatrix game $(A^{\smile}(\varepsilon^1, \varepsilon^2),\ B^{\smile}(\varepsilon^1, \varepsilon^2))$ is defined by

$$A^{\smile}_{ij}(\varepsilon_1, \varepsilon_2) = (1 - \varepsilon_1 - \varepsilon_2)A_{ij} + \varepsilon_1 \max_{s: f_s \in PB_2(e_i)} A_{is} + \varepsilon_2 \max_{r \in \{1,\ldots,m\}} A_{rj}$$

and

$$B^{\smile}_{ij}(\varepsilon_1, \varepsilon_2) = (1 - \varepsilon_1 - \varepsilon_2)B_{ij} + \varepsilon_2 \max_{r: e_r \in PB_1(f_j)} B_{rj} + \varepsilon_1 \max_{s \in \{1,\ldots,n\}} B_{is}$$

The second part of both formulas reflects an optimistic tie-breaking choice (hence the $\smile$ superscript). With probability ε_1, player 2 knows that player 1 is going to play e_i. Of course, player 2 will select a pure best reply f_s to e_i, but which one to select if $|PB_2(e_i)| \geq 2$? Player 2 is indifferent between all pure strategies in the set $PB_2(e_i)$ and in definition of $A^{\smile}$ we assume that he is benevolent to player 1 in selecting one with highest payoff to player 1.
A strategy combination $(p, q) \in \Delta_m \times \Delta_n$ is called *informationally robust* if there is a sequence $\{(\varepsilon_1(k), \varepsilon_2(k))\}_{k \in \mathbb{N}}$ of positive real numbers converging to 0 and a sequence $\{(p^k, q^k)\}_{k \in \mathbb{N}}$ of strategy combinations converging for (p, q) such that, for all $k \in \mathbb{N}$,

$$(p^k, q^k) \in E(A^{\smile}(\varepsilon_1(k), \varepsilon_2(k)), B^{\smile}(\varepsilon_1(k), \varepsilon_2(k))).$$

The set of all informationally robust strategy combinations for (A, B) is denoted by $IRE(A, B)$.

(a) Determine $IRE(A, B)$ for the 2×2 bimatrix game (A, B) given by

$$(A, B) = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{cc} f_1 & \quad f_2 \end{array} \\ \begin{bmatrix} 1,1 & 0,0 \\ 0,0 & 0,0 \end{bmatrix} \end{array}$$

(b) Show that $IRE(A, B) \neq \emptyset$ and that $IRE(A, B) \subset E(A, B)$ for every bimatrix game (A, B).

(c) Determine $IRE(A,B)$ for the 2×2 bimatrix game (A,B) given by

$$(A,B) = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{cc} f_1 & f_2 \end{array} \\ \begin{bmatrix} 1,1 & 0,0 \\ 1,0 & 0,2 \end{bmatrix} \end{array}$$

Try and motivate the outcome from the perspective of player 1. Also determine $PE(A,B)$, $PR(A,B)$ and $SPE(A,B)$.

(d) Determine $IRE(A,B)$ and $PE(A,B)$ for the 2×3 bimatrix game (A,B) given by

$$(A,B) = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{ccc} f_1 & f_2 & f_3 \end{array} \\ \begin{bmatrix} 1,0 & 1,1 & 2,0 \\ 0,1 & 1,0 & 2,1 \end{bmatrix} \end{array}$$

(e) Provide a definition for the pessimistic informationally perturbed game $(A^{\frown}(\varepsilon_1,\varepsilon_2), B^{\frown}(\varepsilon_1,\varepsilon_2))$ and the corresponding notion of IRE. Calculate this alternative type of IRE for the games in part (a), (c) and (d).

Exercise 15.9 (Evolutionary stable strategies)
Consider a symmetric $m \times m$ bimatrix game (A, A^T). A strategy $\hat{p} \in \Delta_m$ is called *evolutionary stable* for (A, A^T) if the following two conditions hold:

(i) $\hat{p} \in B_1(\hat{p})$

(ii) $p \in B_1(\hat{p}),\ p \neq \hat{p} \Rightarrow \hat{p}Ap > pAp$

The first condition just means that $(\hat{p},\hat{p}) \in E(A, A^T)$ and the second condition is an additional stability condition in the sense that if both players decide to move out of the equilibrium $(\hat{p},\hat{p})$ to a situation (p,p) by choosing a strategy $p \in \Delta_m$ which is as good as $\hat{p}$ against p, there is an incentive to go back to $\hat{p}$.

(a) Determine all evolutionary strategies for the 2×2 bimatrix games (A, A^T) given by

$$(A,A^T) = \begin{array}{c} \\ e_1 \\ e_2 \end{array} \begin{array}{c} \begin{array}{cc} e_1 & e_2 \end{array} \\ \begin{bmatrix} 1,1 & 4,0 \\ 0,4 & 2,2 \end{bmatrix} \end{array}$$

(b) Determine all evolutionary strategies for the 2×2 bimatrix game (A, A^T) given by

$$(A, A^T) = \begin{array}{c c} & \begin{array}{cc} e_1 & e_2 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{bmatrix} -1,-1 & 4,0 \\ 0,4 & 2,2 \end{bmatrix} \end{array}$$

(c) Show that there is no evoluationary stable strategy for the 2×2 bimatrix game (A, A^T) given by

$$(A, A^T) = \begin{array}{c c} & \begin{array}{cc} e_1 & e_2 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{bmatrix} 1,1 & 2,1 \\ 1,2 & 2,2 \end{bmatrix} \end{array}$$

Exercise 15.10 (Fall back equilibria)
Define the notion of a *perturbed fall back game* if in the underlying thought experiment (à la fall back proper) at most one action per player can be blocked.Also define the corresponding concept of *fall back equilibria*.
Determine all fall back proper equilibria and all fall back equilibria for the 2×4 bimatrix game given by

$$(A, B) = \begin{array}{c c} & \begin{array}{cccc} f_1 & f_2 & f_3 & f_4 \end{array} \\ \begin{array}{c} e_1 \\ e_2 \end{array} & \begin{bmatrix} 0,100 & 3,99 & 4,1 & 0,0 \\ 0,100 & 3,99 & 0,0 & 5,1 \end{bmatrix} \end{array}$$

Some relevant literature

Selten, R. (1975). Reexamination of the perfectness concept for equilibrium points in extensive games. *International Journal of Game Theory*, 4,25-55.

Damme, E. van (1983). Refinements of the Nash equilibrium concept. Springer, Berlin.

Okada, A. (1984). Strictly perfect equilibrium points of bimatrix games. *International Journal of Game Theory*, 13, 145-154.

Myerson, R. (1978). Refinements of the Nash equilibrium point concept. *International Journal of Game Theory*, 7, 73-80.

Borm, P. (1992). On perfectness concepts for bimatrix games, *OR Spektrum*, 14, 33-42.

Vermeulen, A. and M. Jansen (1996). Are strictly perfect equilibria proper? A counterexample. *Journal of Optimization Theory and Applications*, 90, 225-230.

Borm, P., Jansen, M., Potters, J. and S. Tijs (1993). On the structure of the set of perfect equilibria in bimatrix games. *OR Spektrum*, 15, 17-20.

Jansen, M. (1993). On the set of proper equilibria of a bimatrix game. *International Journal of Game Theory*, 22, 97-106.

Reijnierse, H., Borm, P. and M. Voorneveld (2007). On informationally robust equilibria for bimatrix games. *Economic Theory*, 30, 539-560.

Maynard Smith, J. (1974). The theory of games and the evolution of animal contests. *Journal of Theoretical Biology*, 47, 209-221.

Kleppe, J., Borm, P. and R. Hendrickx (2008). Fall back equilibrium. CentER Discussion Paper 2008-31. Tilburg University. The Netherlands.