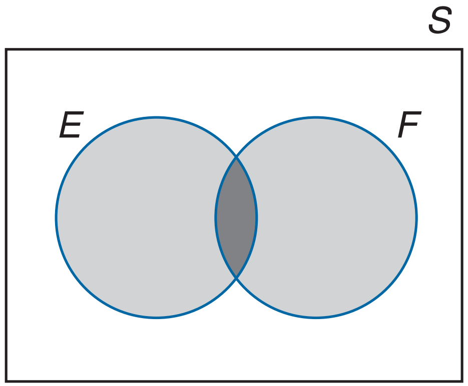
Probability

Sample space and Events

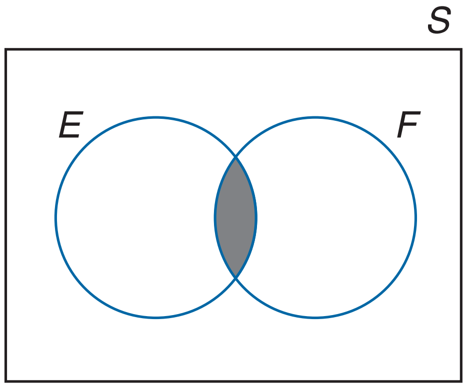
This set of all possible outcomes of an experiment is known as the **sample space** of the experiment and is denoted by S. Some examples are the following.

1. If the outcome of an experiment consists in the determination of the sex of a newborn child, then
$$S = g, b$$
where the outcome g means that the child is a girl and b that it is a boy.
2. If the experiment consists of the running of a race among the seven horses having post positions 1, 2, 3, 4, 5, 6, 7, then
$$S = \{\text{all orderings of } (1, 2, 3, 4, 5, 6, 7)\}$$
The outcome (2, 3, 1, 6, 5, 4, 7) means, for instance, that the number 2 horse is first, then the number 3 horse, then the number 1 horse, and so on.
3. Suppose we are interested in determining the amount of dosage that must be given to a patient until that patient reacts positively. One possible sample space for this experiment is to let S consist of all the positive numbers. That is, let
$$S = (0, \infty)$$
where the outcome would be x if the patient reacts to a dosage of value x but not to any smaller dosage.

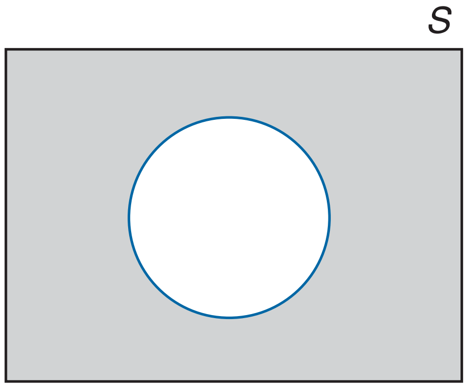
Venn Diagrams



(a) Shaded region: $E \cup F$

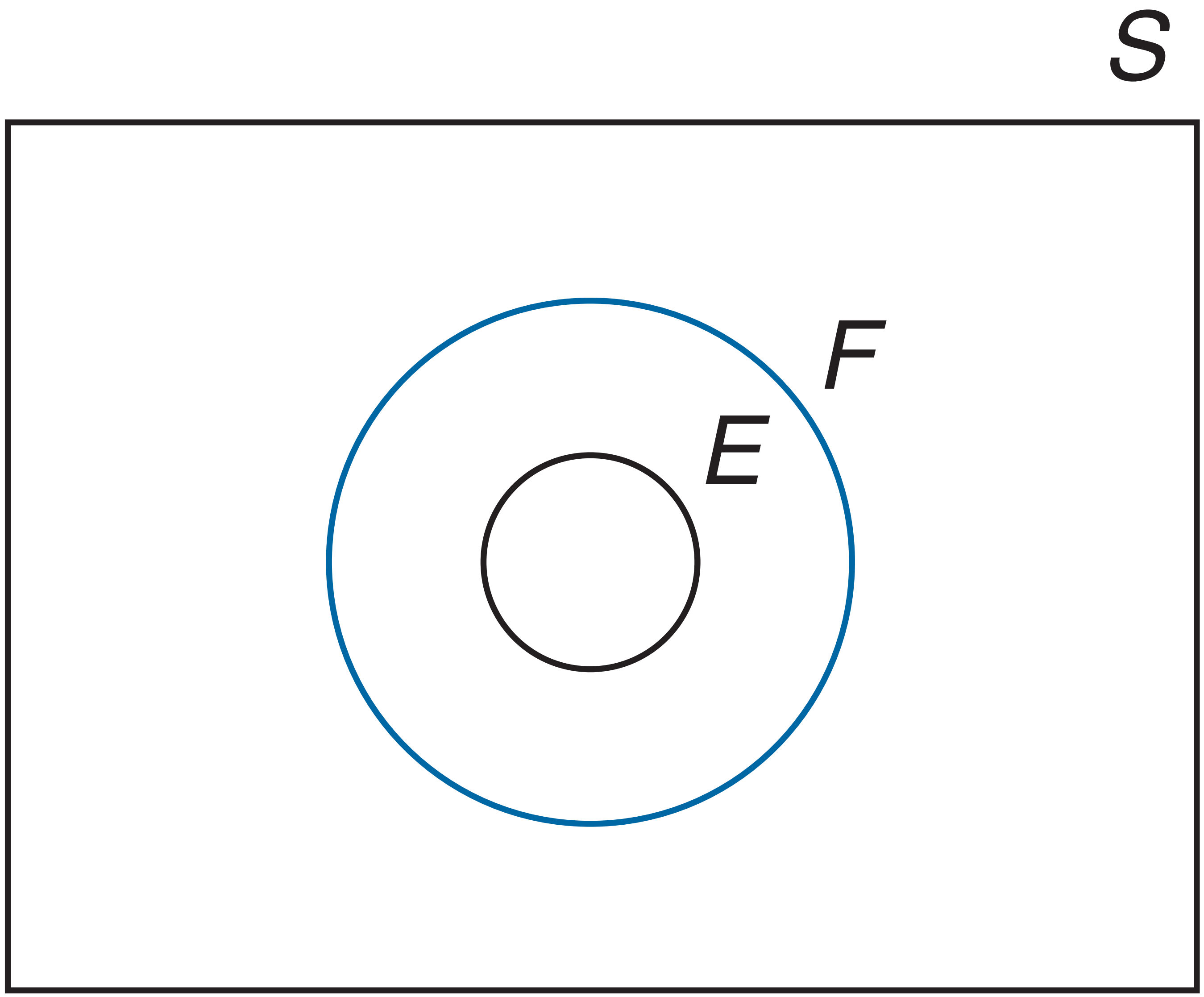


(b) Shaded region: EF



(c) Shaded region: E^c

Venn Diagrams



$E \subset F$

Algebra of Events

Commutative Law	$E \cup F = F \cup E$	$EF = FE$
Associative Law	$(E \cup F) \cup G = E \cup (F \cup G)$	$(EF)G = E(FG)$
Distributive Law	$(E \cup F)G = EG \cup FG$	$EF \cup G = (E \cup G)(F \cup G)$

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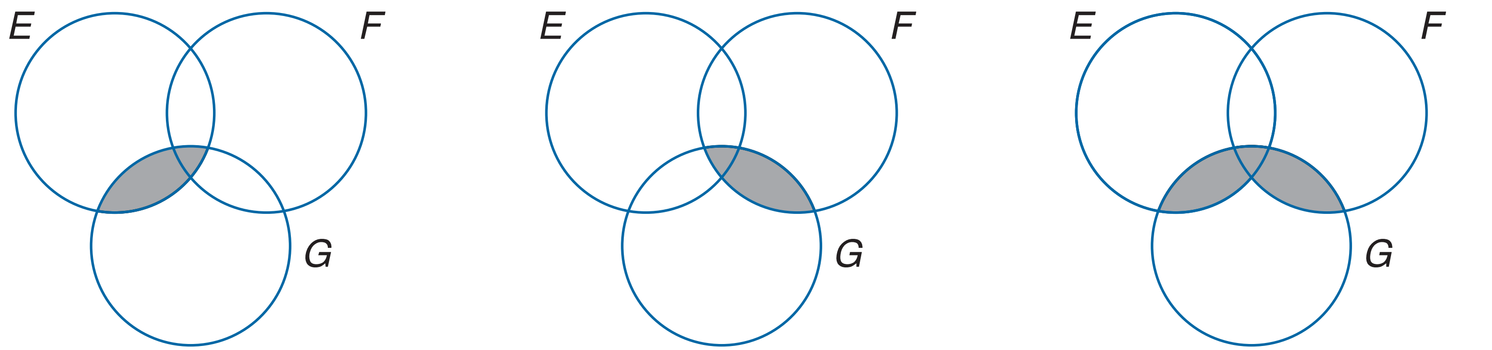
De Morgan's Laws

$$(E \cup F)^c = E^c F^c$$

$$(EF)^c = E^c \cup F^c$$

Proof of Distributive Law



(a) Shaded region: EG

(b) Shaded region: FG

(c) Shaded region: $(E \cup F)G$

$(E \cup F)G = EG \cup FG$

Axioms of Probability

Axiom 1

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Axiom 3 For any sequence of mutually exclusive events $E_1, E_2, \ldots$ (that is,events for which $E_i E_j = \emptyset$ when $i \neq j$),

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i), \qquad n = 1, 2, \ldots, \infty$$

We call $P(E)$ the probability of the event E.

Proposition 1

Since $E \cup E^c = S$

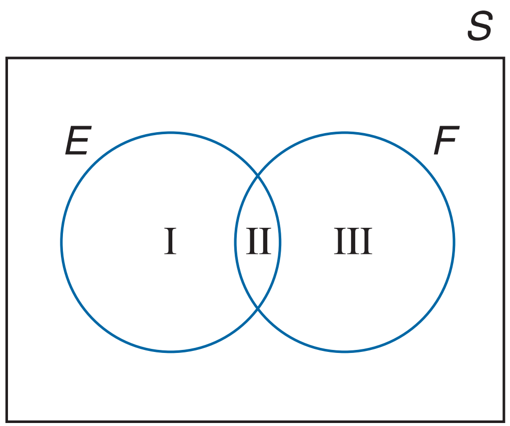
$$1 = P(S) = P(E \cup E^c) = P(E) + P(E^c)$$

Equivalently

$$P(E^c) = 1 - P(E)$$

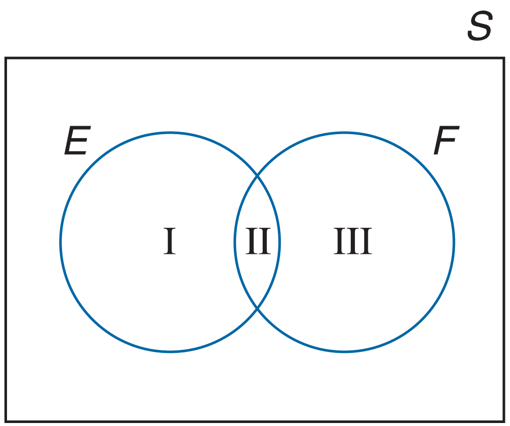
Proposition 2

$$P(E \cup F) = P(E) + P(F) - P(EF)$$



Proposition 2

$$P(E \cup F) = P(E) + P(F) - P(EF)$$



Proof

$$P(E \cup F) = P(I) + P(II) + P(III)$$
$$P(E) = P(I) + P(II)$$
$$P(F) = P(II) + P(III)$$

which shows that

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

Sample spaces with equally likely outcomes

For a large number of experiments, it is natural to assume that each point in the sample space is equally likely to occur. That is, for many experiments whose sample space S is a finite set, say $S = \{1, 2, \ldots, N\}$, it is often natural to assume that

$$P(\{1\}) = P(\{2\}) = \cdots = P(\{N\}) = p$$

Now it follows from Axioms 2 and 3 that

$$1 = P(S) = P(\{1\}) + P(\{2\}) = + \cdots + P(\{N\}) = Np$$

which shows that

$$P(\{i\}) = p = \frac{1}{N}$$

from Axiom 3 it follows that for any event E,

$$P(E) = \frac{\text{Number of points in} E}{N}$$

In words, if we assume that each outcome of an experiment is equally likely to occur, then the probability of any event E equals the proportion of points in the sample space that are contained in E.

Generalized Basic Principle of Counting

If r experiments that are to be performed are such that the first one may result in any of n_1 possible outcomes, and if for each of these n_1 possible outcomes there are n_2 possible outcomes of the second experiment, and if for each of the possible outcomes of the first two experiments there are n_3 possible outcomes of the third experiment, and if, . . . , then there are a total of $n_1 \cdot n_2 \cdots n_r$ possible outcomes of the r experiments.

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Notation

$$\begin{pmatrix} n \\ r \end{pmatrix} = \frac{n!}{(n-r)!r!}$$

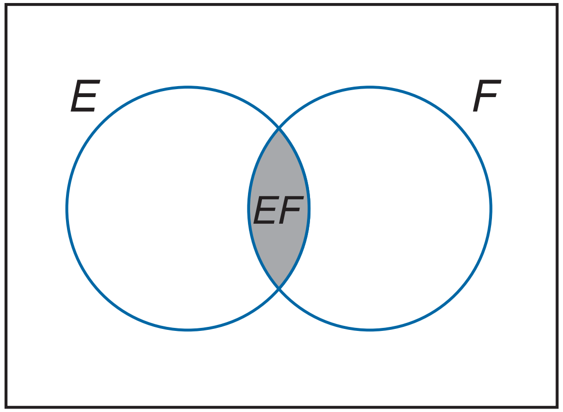
Also since $0! = 1$

$$\begin{pmatrix} n \\ 0 \end{pmatrix} = \begin{pmatrix} n \\ n \end{pmatrix} = 1$$

Conditional Probability

Conditional probability of E given that F has occurred, and is denoted by

$$P(E|F)$$



A general formula for $P(E|F)$ that is valid for all events E and F. If the event F occurs, then in order for E to occur it is necessary that the actual occurrence be a point in both E and F; that is, it must be in EF. Now, since we know that F has occurred, it follows that F becomes our new (reduced) sample space and hence the probability that the event EF occurs will equal the probability of EF relative to the probability of F. That is,

$$P(E|F) = \frac{P(EF)}{P(F)}$$

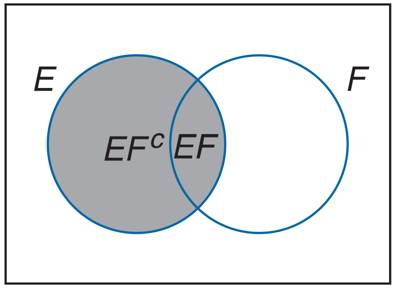
Bayes Formula

Let E and F be events. We may express E as

$$E = EF \cup EF^c$$

As EF and EF^c are clearly mutually exclusive, we have by Axiom 3 that

$$\begin{aligned} P(E) &= P(EF) + P(EF^c) \\ &= P(E|F)P(F) + P(E|F^c)P(F^c) \\ &= P(E|F)P(F) + P(E|F^c)[1 - P(F)] \end{aligned}$$



Equation states that the probability of the event E is a weighted average of the conditional probability of E given that F has occurred and the conditional probability of E given that F has not occurred: Each conditional probability is given as much weight as the event it is conditioned on has of occurring.