Exponential Random Variables

A continuous random variable whose probability density function is given, for some $\lambda > 0$, by

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$$

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The cumulative distribution function $F(x)$ of an exponential random variable is given by

$$\begin{aligned} F(x) &= P\{X \leq x\} \\ &= \int_0^x \lambda e^{-\lambda y} dy \\ &= 1 - e^{-\lambda x}, \qquad x \geq 0 \end{aligned}$$

The moment generating function of the exponential is given by

$$\begin{aligned}\phi(t) &= E[e^{tX}] \\ &= \int_0^\infty e^{tx}\lambda e^{-\lambda x}dx \\ &= \lambda \int_0^\infty e^{-(\lambda - t)x}dx \\ &= \frac{\lambda}{\lambda - t}, \qquad t < \lambda\end{aligned}$$

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Differentiating,

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\begin{aligned}
\phi'(t) &= \frac{\lambda}{(\lambda - t)^2} \\
\phi''(t) &= \frac{2\lambda}{(\lambda - t)^3}
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And,

$$\begin{aligned}E[X] = \phi'(0) &= \frac{1}{\lambda} \\ Var(X) = \phi''(0) - (E[X])^2 \\ &= \frac{1}{\lambda^2}\end{aligned}$$

Suppose that a number of miles that a car can run before its battery wears out is exponentially distributed with an average value of 10,000 miles. If a person desires to take a 5,000-mile trip, what is the probability that she will be able to complete her trip without having to replace her car battery?

Suppose that a number of miles that a car can run before its battery wears out is exponentially distributed with an average value of 10,000 miles. If a person desires to take a 5,000-mile trip, what is the probability that she will be able to complete her trip without having to replace her car battery?

Solution:
The remaining lifetime (in thousands of miles) of the battery is exponential with parameter $\lambda = 1/10$. Hence the desired probability is

$$\begin{aligned} P\{\text{remaining lifetime } > 5\} &= 1 - F(5) \\ &= e^{-5\lambda} \\ &= e^{-0.5} \approx 0.604 \end{aligned}$$

Statement: If $X_1, X_2, \ldots, X_n$ are independent exponential random variables having respective parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ then $min(X_1, X_2, \ldots, X_n)$ is exponential with parameter $\sum_{t=1}^{n} \lambda_i$.

Statement: If $X_1, X_2, \ldots, X_n$ are independent exponential random variables having respective parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ then $min(X_1, X_2, \ldots, X_n)$ is exponential with parameter $\sum_{t=1}^{n} \lambda_i$.

Proof: Since the smallest value of a set of numbers is greater than x if and only if all values are greater than x, we have

$$\begin{aligned}
P\{min(X_1, X_2, \ldots, X_n) > x\} &= P\{X_1 > x, X_2 > x, \ldots, X_n > x\} \\
&= \prod_{i=1}^{n} P\{X_i > x\} \qquad \text{by independence} \\
&= \prod_{i=1}^{n} e^{-\lambda_i x} \\
&= e^{-\sum_{i=1}^{n} \lambda_i x}
\end{aligned}$$

A series system is one that needs all of its components to function in order for the system itself to be functional. For an n-component series system in which the component lifetimes are independent exponential random variables with respective parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ what is the probability that the system survives for a time t?

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Solution: Since the system life is equal to the minimal component life

$$P\{\text{system life exceeds } t\} = e^{-\sum_i \lambda_i t}$$

The Gamma Distribution

A random variable is said to have a gamma distribution with parameters $(\alpha, \lambda), \lambda > 0, \alpha > 0$, if its density function is given by

$$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

where

$$\begin{aligned} \Gamma(\alpha) &= \int_0^\infty \lambda e^{-\lambda x}(\lambda x)^{\alpha-1} dx \\ &= \int_0^\infty e^{-y} y^{\alpha-1} dy \qquad \text{by letting } y = \lambda x \end{aligned}$$

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Using integration by parts

$$\begin{aligned} \int_0^\infty e^{-y} y^{\alpha-1} dy &= -e^{-y} y^{\alpha-1} \Big|_{y=0}^{y=\infty} + \int_0^\infty e^{-y}(\alpha-1) y^{\alpha-2} dy \\ &= (\alpha-1) \int_0^\infty e^{-y} y^{\alpha-2} dy \end{aligned}$$

Hence

$$\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$$

When α is an integer n

$$\begin{aligned}\Gamma(n) &= (n-1)\Gamma(n-1)\\ &= (n-1)(n-2)\Gamma(n-2)\\ &\vdots\\ &= (n-1)!\Gamma(1)\end{aligned}$$

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Since

$$\Gamma(1) = \int_0^\infty e^{-y}\, dy = 1$$

It is shown

$$\Gamma(n) = (n-1)!$$

The function $\Gamma(\alpha)$ is called the **gamma** function.

The moment generating function of a gamma random variable X with parameters (α, λ) is obtained as follows:

$$\begin{aligned}
\phi(t) &= E[e^{tX}] \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{tx} e^{-\lambda x} x^{\alpha-1} dx \\
&= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty e^{-(\lambda-t)x} x^{\alpha-1} dx \\
&= \left(\frac{\lambda}{\lambda - t}\right)^\alpha \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-y} y^{\alpha-1} dy \qquad \text{by letting } y = (\lambda - t)x \\
&= \left(\frac{\lambda}{\lambda - t}\right)^\alpha
\end{aligned}$$

Differentiating

$$\phi'(t) = \frac{\alpha\lambda^\alpha}{(\lambda - t)^{\alpha+1}}$$

$$\phi''(t) = \frac{\alpha(\alpha+1)\lambda^\alpha}{(\lambda - t)^{\alpha+2}}$$

Computing Mean and Variance

$$E[X] = \phi'(0) = \frac{\alpha}{\lambda}$$

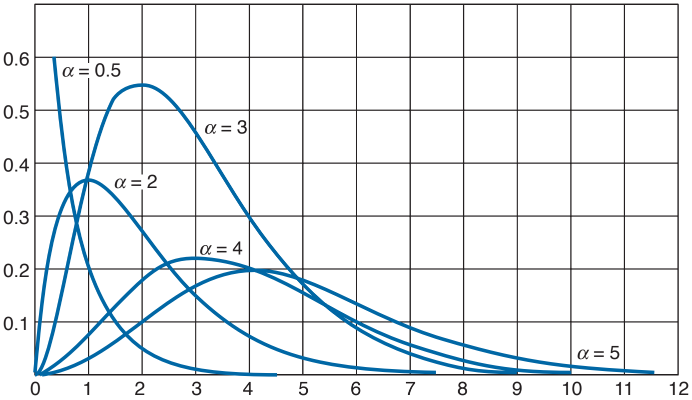
$$\begin{aligned} Var(X) &= E[X^2] - (E[X])^2 \\ &= \phi''(0) - \left(\frac{\alpha}{\lambda}\right)^2 \\ &= \frac{\alpha}{\lambda^2} \end{aligned}$$

If X_1 and X_2 are independent gamma random variables having respective parameters (α_1, λ) and (α_2, λ), then $X_1 + X_2$ is a gamma random variable with parameters $(\alpha_1 + \alpha_2, \lambda)$.

$$\begin{aligned}
\phi_{X_1+X_2}(t) &= E[e^{t(X_1+X_2)}] \\
&= \phi_{X_1}(t)\phi_{X_2}(t) \\
&= \left(\frac{\lambda}{\lambda - t}\right)_1^{\alpha} \left(\frac{\lambda}{\lambda - t}\right)_2^{\alpha} \\
&= \left(\frac{\lambda}{\lambda - t}\right)^{\alpha_1+\alpha_2}
\end{aligned}$$

which is seen to be the moment generating function of a gamma $(\alpha_1 + \alpha_2, \lambda)$ random variable.
This can be generalised to the following statement.

If $X_i, i = 1, \ldots, n$ are independent gamma random variables with respective parameters (α_i, λ), then $\sum_{i=1}^{n} X_i$ is gamma with parameters $\sum_{i=1}^{n} \alpha_i, \lambda$.


0.6
0.5
0.4
0.3
0.2
0.1
0
1
2
3
4
5
6
7
8
9
10
11
12
α = 0.5
α = 3
α = 2
α = 4
α = 5

The Chi–Square Distribution

If $Z_1, Z_2, \ldots, Z_n$ are independent standard normal random variables, then X , defined by

$$X = Z_1^2 + Z_2^2 + \cdots + Z_n^2$$

is said to have a chi(χ)-square distribution with n degrees of freedom.
The notation used to signify that X has a $chi - square$ distribution with n degrees of freedom.

$$X \sim \chi_n^2$$

The chi-square distribution has the additive property that if X_1 and X_2 are independent chi-square random variables with n_1 and n_2 degrees of freedom, respectively, then $X_1 + X_2$ is chi-square with $n_1 + n_2$ degrees of freedom.

If X is a chi-square random variable with n degrees of freedom, then for any $\alpha \in (0, 1)$, the quantity $\chi^2_{\alpha,n}$ is defined to be such that

$$P\{X \geq \chi^2_{\alpha,n}\} = \alpha$$

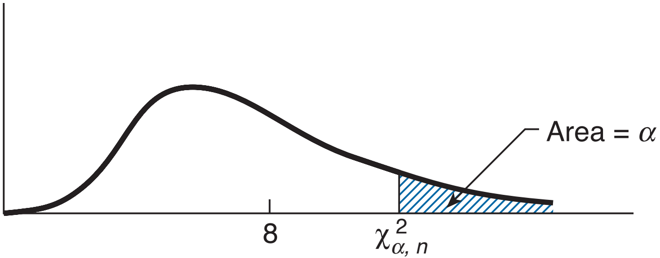


TABLE A2 *Values of* $x^2_{\alpha,n}$

n	$\alpha = .995$	$\alpha = .99$	$\alpha = .975$	$\alpha = .95$	$\alpha = .05$	$\alpha = .025$	$\alpha = .01$	$\alpha = .005$
1	.0000393	.000157	.000982	.00393	3.841	5.024	6.635	7.879
2	.0100	.0201	.0506	.103	5.991	7.378	9.210	10.597
3	.0717	.115	.216	.352	7.815	9.348	11.345	12.838
4	.207	.297	.484	.711	9.488	11.143	13.277	14.860
5	.412	.554	.831	1.145	11.070	12.832	13.086	16.750
6	.676	.872	1.237	1.635	12.592	14.449	16.812	18.548
7	.989	1.239	1.690	2.167	14.067	16.013	18.475	20.278
8	1.344	1.646	2.180	2.733	15.507	17.535	20.090	21.955
9	1.735	2.088	2.700	3.325	16.919	19.023	21.666	23.589
10	2.156	2.558	3.247	3.940	18.307	20.483	23.209	25.188
11	2.603	3.053	3.816	4.575	19.675	21.920	24.725	26.757
12	3.074	3.571	4.404	5.226	21.026	23.337	26.217	28.300
13	3.565	4.107	5.009	5.892	22.362	24.736	27.688	29.819
14	4.075	4.660	5.629	6.571	23.685	26.119	29.141	31.319
15	4.601	5.229	6.262	7.261	24.996	27.488	30.578	32.801
16	5.142	5.812	6.908	7.962	26.296	28.845	32.000	34.267
17	5.697	6.408	7.564	8.672	27.587	30.191	33.409	35.718
18	6.265	7.015	8.231	9.390	28.869	31.526	34.805	37.156
19	6.844	7.633	8.907	10.117	30.144	32.852	36.191	38.582
20	7.434	8.260	9.591	10.851	31.410	34.170	37.566	39.997
21	8.034	8.897	10.283	11.591	32.671	35.479	38.932	41.401
22	8.643	9.542	10.982	12.338	33.924	36.781	40.289	42.796
23	9.260	10.196	11.689	13.091	35.172	38.076	41.638	44.181
24	9.886	10.856	12.401	13.484	36.415	39.364	42.980	45.558
25	10.520	11.524	13.120	14.611	37.652	40.646	44.314	46.928
26	11.160	12.198	13.844	15.379	38.885	41.923	45.642	48.290
27	11.808	12.879	14.573	16.151	40.113	43.194	46.963	49.645
28	12.461	13.565	15.308	16.928	41.337	44.461	48.278	50.993
29	13.121	14.256	16.047	17.708	42.557	45.772	49.588	52.336
30	13.787	14.953	16.791	18.493	43.773	46.979	50.892	53.672

Other Chi-Square Probabilities:

$x^2_{.9,9} = 4.2 \quad P\{x^2_{16} < 14.3\} = .425 \quad P\{x^2_{11} < 17.1875\} = .8976.$

Suppose that we are attempting to locate a target in three–dimensional space, and that the three coordinate errors (in meters) of the point chosen are independent normal random variables with mean 0 and standard deviation 2. Find the probability that the distance between the point chosen and the target exceeds 3 meters.

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Solution: If D is the distance, then

$$D^2 = X_1^2 + X_2^2 + X_3^2$$

where X_i is the error in the ith coordinate. Since $Z_i = X_i/2, i = 1, 2, 3$, are all standard normal random variables, it follows that

$$\begin{aligned} P\{D^2 > 9\} &= P\{Z_1^2 + Z_2^2 + Z_3^2 > 9/4\} \\ &= P\{\chi_3^2 > 9/4\} \\ &\approx 0.5222 \end{aligned}$$

The Relation between Chi-Square and Gamma Random Variables

Let us compute the moment generating function of a chi-square random variable with n degrees of freedom. To begin, we have, when $n = 1$, that

$$
\begin{aligned}
E[e^{tX}] &= E[e^{tZ^2}] \quad \text{where } Z \sim \mathcal{N}(0,1) \\
&= \int_{-\infty}^{\infty} e^{tx^2} f_Z(x)dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx^2} e^{-\frac{x^2}{2}} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2 \frac{(1-2t)}{2}} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\bar{\sigma}^2}} dx \qquad \text{where } \bar{\sigma}^2 = (1-2t)^{-1} \\
&= (1-2t)^{-\frac{1}{2}} \frac{1}{\sqrt{2\pi}\bar{\sigma}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2\bar{\sigma}^2}} dx \\
&= (1-2t)^{-\frac{1}{2}}
\end{aligned}
$$

where the last equality follows since the integral of the normal $(0, \bar{\sigma}^2)$ density equals 1.

In the general case of n degrees of freedom

$$\begin{aligned} E[e^{tX} &= E\left[e^{t\sum_{i=1}^{n} Z_i^2}\right] \\ &= E\left[\prod_{i=1}^{n} e^{tZ_i^2}\right] \\ &= \prod_{i=1}^{n} E[e^{tZ_i^2}] \qquad \text{since } Z_i \text{ is independent} \\ &= (1-2t)^{-\frac{n}{2}} \end{aligned}$$

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&= \prod_{i=1}^{n} E[e^{tZ_i^2}] \qquad \text{since } Z_i \text{ is independent} \\
&= (1-2t)^{-\frac{n}{2}}
\end{aligned}
$$

However, we recognize $[1/(1?2t)]^{n/2}$ as being the moment generating function of a gamma random variable with parameters $(n/2, 1/2)$. Hence chi-square with n degrees of freedom and gamma with parameters $n/2$ and $1/2$ are identical.
Thus the density of X is given by

$$
f(x) = \frac{\frac{1}{2} e^{-\frac{x}{2}} \left(\frac{x}{2}\right)^{\frac{n}{2}-1}}{\Gamma\left(\frac{n}{2}\right)}
$$

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Solution: If D is the distance, then

$$D^2 = X_1^2 + X_2^2$$

Since $Z_i = X_i/2, i = 1, 2$ are all standard normal random variables, it follows that

$$\begin{aligned} P\{D^2 > 9\} &= P\{Z_1^2 + Z_2^2 > 9/4\} \\ &= P\{\chi_2^2 > 9/4\} \\ &= e^{-9/8} \\ &\approx 0.3247 \end{aligned}$$

where the preceding calculation used the fact that the chi-square distribution with 2 degrees of freedom is the same as the exponential distribution with parameter $1/2$.