ME 102: Lecture 5

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Empirical rule of normal datasets

If a data set is approximately normal with sample mean $\bar{x}$ and sample standard deviation s, then the following statements are true.

1. Approximately 68 percent of the observations lie within
$$\bar{x} \pm s$$
2. Approximately 95 percent of the observations lie within
$$\bar{x} \pm 2s$$
3. Approximately 99.7 percent of the observations lie within
$$\bar{x} \pm 3s$$

Example Problem

The following stem and leaf plot gives the scores on a statistics exam taken by industrial engineering students.

9	0,1,4
8	3,5,5,7,8
7	2,4,4,5,7,7,8
6	0,2,3,4,6,6
5	2,5,5,6,8
4	3,6

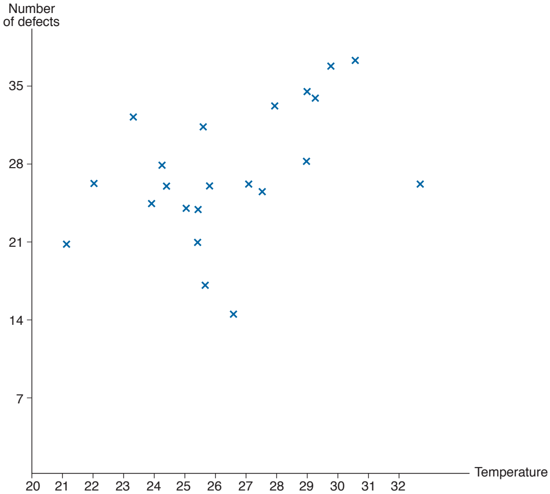
Use it to assess the empirical rule.

Paired data sets: Example Problem

TABLE 2.8 *Temperature and Defect Data*

Day	Temperature	Number of Defects
1	24.2	25
2	22.7	31
3	30.5	36
4	28.6	33
5	25.5	19
6	32.0	24
7	28.6	27
8	26.5	25
9	25.3	16
10	26.0	14
11	24.4	22
12	24.8	23
13	20.6	20
14	25.1	25
15	21.4	25
16	23.7	23
17	23.9	27
18	25.2	30
19	27.4	33
20	28.3	32
21	28.8	35
22	26.6	24

Example Problem



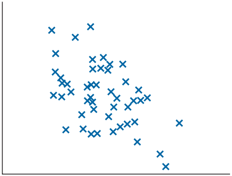
Paired data sets and correlations

Let s_x and s_y denote, respectively, the sample standard deviations of the x values and the y values. The sample correlation coefficient, call it r , of the data pairs $(x_i, y_i), i = 1, \ldots, n$ is defined by

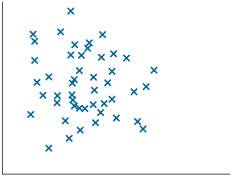
$$r = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{(n-1)s_x s_y}$$
$$= \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2 \sum_{i=1}^{n}(y_i - \bar{y})^2}}$$

When $r > 0$ we say that the sample data pairs are positively correlated, and when $r < 0$ we say that they are negatively correlated.

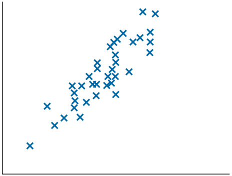
Different correlations



$r = -.50$



$r = 0$



$r = -.90$

Properties of r

1. $-1 \leq r \leq 1$
2. If for constants a and b,with $b > 0$,

$$y_i = a + bx_i, \quad i = 1, \ldots, n$$

   then $r = 1$.
3. If for constants a and b,with $b < 0$,

$$y_i = a + bx_i, \quad i = 1, \ldots, n$$

   then $r = -1$.
4. If r is sample correlation coefficient for the data pairs $x_i, y_1, i = 1, \ldots, n$ then it is also the sample correlation coefficient of the data pairs.

$$a + bx_i, \quad c + dy_i, \quad i = 1, \ldots, n$$

   provided that b and d are both positive or both negative.

Example Problem

The following data gives the resting pulse rates (in beats per minute) and the years of schooling of 10 individuals.

Person	1	2	3	4	5	6	7	8	9	10
Years of school	12	16	13	18	19	12	18	19	12	14
Pulse rate	73	67	74	63	7384	60	62	76	71	

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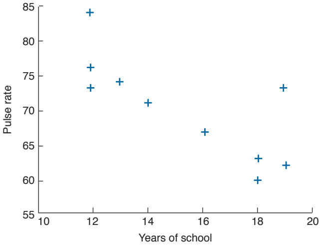
A scatter diagram of these data is presented in the figure. The sample correlation coefficient for these data is $r = 0.7638$.



We will now prove the first three properties of the sample correlation coefficient r . That is, we will prove that $|r| \leq 1$ with equality when the data lie on a straight line. To begin, note that,

$$\sum \left(\frac{x_i - \bar{x}}{s_x} - \frac{y_i - \bar{y}}{s_y} \right)^2 \geq 0$$

$$\sum \frac{(x_i - \bar{x})^2}{s_x^2} + \sum \frac{(y_i - \bar{y})^2}{s_y^2} - 2 \sum \frac{(x_i - \bar{x})(y_i - \bar{y})}{s_x s_y} \geq 0$$

$$n - 1 + n - 1 - 2(n-1)r \geq 0$$

showing that

$$r \leq 0$$

Note also that $r = 1$ if and only if there is equality in first equation on the previous slide. That is, $r = 1$ if and only if for all i

$$\frac{\bar{y} - y_i}{s_y} = \frac{\bar{x} - x_i}{s_x}$$

$$y_i = \bar{y} - \frac{s_y}{s_x}\bar{x} + \frac{s_y}{s_x}x_i$$

That is, $r = 1$ if and only if the data values (x_i, y_i) lie on a straight line having a positive slope.

To show that $r \geq -1$, with equality if and only if the data values (x_i, y_i) lie on a straight line having a negative slope, start with

$$\sum \left(\frac{x_i - \bar{x}}{s_x} + \frac{y_i - \bar{y}}{s_y} \right)^2 \geq 0$$

and use an argument analogous to the one just given.

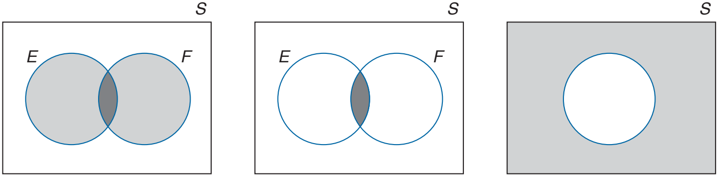
Probability

Sample space and Events

This set of all possible outcomes of an experiment is known as the **sample space** of the experiment and is denoted by S. Some examples are the following.

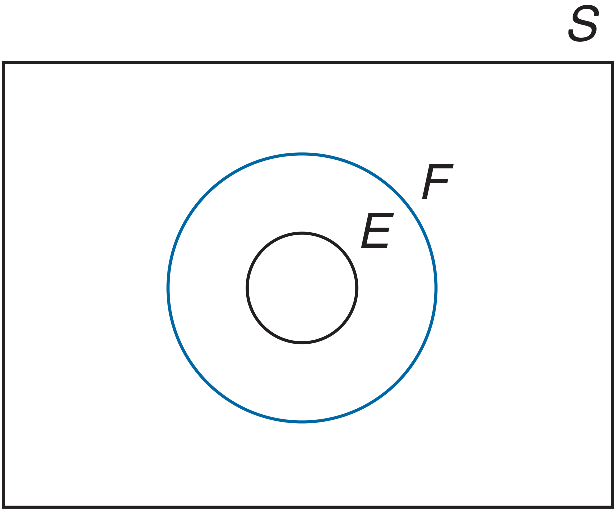
1. If the outcome of an experiment consists in the determination of the sex of a newborn child, then
$$S = g, b$$
where the outcome g means that the child is a girl and b that it is a boy.
2. If the experiment consists of the running of a race among the seven horses having post positions 1, 2, 3, 4, 5, 6, 7, then
$$S = \{\text{all orderings of } (1, 2, 3, 4, 5, 6, 7)\}$$
The outcome (2, 3, 1, 6, 5, 4, 7) means, for instance, that the number 2 horse is first, then the number 3 horse, then the number 1 horse, and so on.
3. Suppose we are interested in determining the amount of dosage that must be given to a patient until that patient reacts positively. One possible sample space for this experiment is to let S consist of all the positive numbers. That is, let
$$S = (0, \infty)$$
where the outcome would be x if the patient reacts to a dosage of value x but not to any smaller dosage.

Venn Diagrams



(a) Shaded region: $E \cup F$ (b) Shaded region: EF (c) Shaded region: E^c

Venn Diagrams



$E \subset F$

Algebra of Events

Commutative Law	$E \cup F = F \cup E$	$EF = FE$
Associative Law	$(E \cup F) \cup G = E \cup (F \cup G)$	$(EF)G = E(FG)$
Distributive Law	$(E \cup F)G = EG \cup FG$	$EF \cup G = (E \cup G)(F \cup G)$

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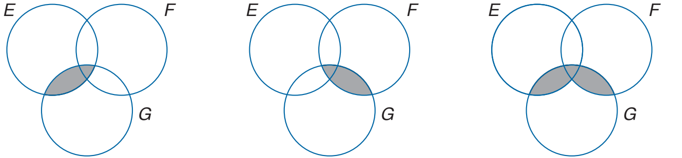
De Morgan's Laws

$$(E \cup F)^c = E^c F^c$$

$$(EF)^c = E^c \cup F^c$$

Proof of Distributive Law



(a) Shaded region: EG (b) Shaded region: FG (c) Shaded region: $(E \cup F)G$

$(E \cup F)G = EG \cup FG$

Axioms of Probability

Axiom 1

$$0 \leq P(E) \leq 1$$

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Axiom 3 For any sequence of mutually exclusive events $E_1, E_2, \ldots$ (that is,events for which $E_i E_j = \emptyset$ when $i \neq j$),

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i), \qquad n = 1, 2, \ldots, \infty$$

We call $P(E)$ the probability of the event E.

Proposition 1

Since $E \cup E^c = S$

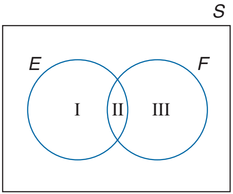
$$1 = P(S) = P(E \cup E^c) = P(E) + P(E^c)$$

Equivalently

$$P(E^c) = 1 - P(E)$$

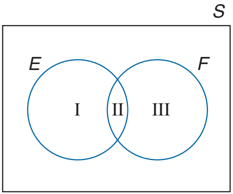
Proposition 2

$$P(E \cup F) = P(E) + P(F) - P(EF)$$



Proposition 2

$$P(E \cup F) = P(E) + P(F) - P(EF)$$



Proof

$$P(E \cup F) = P(I) + P(II) + P(III)$$
$$P(E) = P(I) + P(II)$$
$$P(F) = P(II) + P(III)$$

which shows that

$$P(E \cup F) = P(E) + P(F) - P(EF)$$

Sample spaces with equally likely outcomes

For a large number of experiments, it is natural to assume that each point in the sample space is equally likely to occur. That is, for many experiments whose sample space S is a finite set, say $S = \{1, 2, \ldots, N\}$, it is often natural to assume that

$$P(\{1\}) = P(\{2\}) = \cdots = P(\{N\}) = p$$

Now it follows from Axioms 2 and 3 that

$$1 = P(S) = P(\{1\}) + P(\{2\}) = + \cdots + P(\{N\}) = Np$$

which shows that

$$P(\{i\}) = p = \frac{1}{N}$$

from Axiom 3 it follows that for any event E,

$$P(E) = \frac{\text{Number of points in} E}{N}$$

In words, if we assume that each outcome of an experiment is equally likely to occur, then the probability of any event E equals the proportion of points in the sample space that are contained in E.

Generalized Basic Principle of Counting

If r experiments that are to be performed are such that the first one may result in any of n_1 possible outcomes, and if for each of these n_1 possible outcomes there are n_2 possible outcomes of the second experiment, and if for each of the possible outcomes of the first two experiments there are n_3 possible outcomes of the third experiment, and if, . . . , then there are a total of $n_1 \cdot n_2 \cdots n_r$ possible outcomes of the r experiments.

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Notation

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}$$

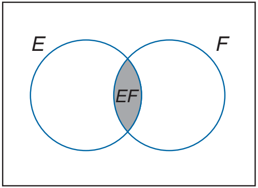
Also since $0! = 1$

$$\binom{n}{0} = \binom{n}{n} = 1$$

Conditional Probability

Conditional probability of E given that F has occurred, and is denoted by

$$P(E|F)$$



A general formula for $P(E|F)$ that is valid for all events E and F. If the event F occurs, then in order for E to occur it is necessary that the actual occurrence be a point in both E and F; that is, it must be in EF. Now, since we know that F has occurred, it follows that F becomes our new (reduced) sample space and hence the probability that the event EF occurs will equal the probability of EF relative to the probability of F. That is,

$$P(E|F) = \frac{P(EF)}{P(F)}$$

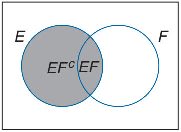
Bayes Formula

Let E and F be events. We may express E as

$$E = EF \cup EF^c$$

As EF and EF^c are clearly mutually exclusive, we have by Axiom 3 that

$$\begin{aligned} P(E) &= P(EF) + P(EF^c) \\ &= P(E|F)P(F) + P(E|F^c)P(F^c) \\ &= P(E|F)P(F) + P(E|F^c)[1 - P(F)] \end{aligned}$$



Equation states that the probability of the event E is a weighted average of the conditional probability of E given that F has occurred and the conditional probability of E given that F has not occurred: Each conditional probability is given as much weight as the event it is conditioned on has of occurring.