Solutions (odd-numbered questions)

Q1

Let the joint PDF be

$$f(x, y) = 2e^{-x}e^{-2y}, \qquad x > 0,\ y > 0.$$

Note that this factors as $f_X(x)f_Y(y)$ with $X \sim \mathrm{Exp}(1)$ and $Y \sim \mathrm{Exp}(2)$, so X and Y are independent. Hence

$$\begin{aligned} P(X > 1,\ Y < 1) &= P(X > 1)\,P(Y < 1) \\ &= e^{-1}\big(1 - e^{-2}\big). \end{aligned}$$

Q3

Let $X \sim \mathrm{Bin}(n, p)$ with n large, p small and $\lambda = np$ fixed. The Poisson approximation gives for integer $k \geq 0$

$$P(X = k) \approx e^{-\lambda}\frac{\lambda^k}{k!}.$$

In particular,

$$P(X = 0) \approx e^{-\lambda}, \qquad P(X \geq 1) = 1 - P(X = 0) \approx 1 - e^{-\lambda}.$$

(Justification: $(1 - p)^n \approx e^{-np} = e^{-\lambda}$ and the binomial mass concentrates according to the Poisson law when $n \to \infty$, $p \to 0$, $np = \lambda$.)

Q5

For the set $\{1, 2, \ldots, 10\}$ let $n = 10$. The sample mean is

$$\bar{x} = \frac{1 + 10}{2} = 5.5.$$

The population variance of consecutive integers $1, \ldots, n$ is $\sigma^2_{\text{pop}} = (n^2 - 1)/12$. For $n = 10$,

$$\sigma^2_{\text{pop}} = \frac{10^2 - 1}{12} = \frac{99}{12} = 8.25.$$

The (Bessel-corrected) sample variance is

$$s^2 = \frac{n}{n-1}\sigma^2_{\text{pop}} = \frac{10}{9} \cdot 8.25 = \frac{82.5}{9} = 9.\overline{16} = \frac{55}{6}.$$

For the set $\{5, 6, 7, \ldots, 14\}$ this is just the first set shifted by $+4$. Variance is invariant under shifts, so the sample variance is the same:

$$s^2\big(\{5, \ldots, 14\}\big) = \frac{55}{6}.$$

Comment: Shifting all observations by a constant changes the mean but leaves the (sample) variance unchanged; hence both sets have identical sample variance.

Q7

The joint pmf table (rows $X = 0, 1, 2$, columns $Y = 0, 1, 2$) is

	$y = 0$	$y = 1$	$y = 2$
$x = 0$	0.1	0.1	0.1
$x = 1$	0.2	0.1	0.1
$x = 2$	0.1	0.1	0.1

(a) *Validity:* Sum of all entries

$$0.3 + 0.4 + 0.3 = 1,$$

so it is a valid joint distribution.

(b) *Marginals:*

$$p_X(0) = 0.1 + 0.1 + 0.1 = 0.3, \quad p_X(1) = 0.2 + 0.1 + 0.1 = 0.4, \quad p_X(2) = 0.1 + 0.1 + 0.1 = 0.3,$$
$$p_Y(0) = 0.1 + 0.2 + 0.1 = 0.4, \quad p_Y(1) = 0.1 + 0.1 + 0.1 = 0.3, \quad p_Y(2) = 0.1 + 0.1 + 0.1 = 0.3.$$

(c) $P(X = 1,\ Y \le 1) = P(X = 1, Y = 0) + P(X = 1, Y = 1) = 0.2 + 0.1 = 0.3.$

(d) Expectations:

$$E[X] = \sum_x x\, p_X(x) = 0 \cdot 0.3 + 1 \cdot 0.4 + 2 \cdot 0.3 = 1.0,$$
$$E[Y] = \sum_y y\, p_Y(y) = 0 \cdot 0.4 + 1 \cdot 0.3 + 2 \cdot 0.3 = 0.9.$$

(e) *Independence:* For independence we would require $p_{X,Y}(x, y) = p_X(x)p_Y(y)$ for all x, y. For example,

$$p_{X,Y}(1, 1) = 0.1 \quad \text{but} \quad p_X(1)p_Y(1) = 0.4 \times 0.3 = 0.12 \neq 0.1.$$

Therefore X and Y are not independent.