ME 102: Lecture 9

S. Gopalakrishnan

September 3, 2016

Normal Random Variable

A random variable is said to be normally distributed with parameters μ and σ^2, and we write $X \sim \mathcal{N}(\mu, \sigma^2)$, if its density is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad -\infty < x < \infty$$

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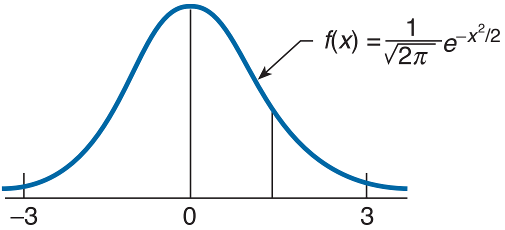

Normal distribution with $\mu = 0$ and $\sigma = 1$.

The moment generating function of a normal random variable with parameters μ and σ^2 is derived as

$$
\begin{aligned}
\phi(t) &= E[e^{tX}] \\
&= \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{\infty} e^{tx} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx \\
&= \frac{1}{\sqrt{2\pi}} e^{\mu t}\int_{-\infty}^{\infty} e^{t\sigma y} e^{\frac{-y^2}{2}}\,dy \qquad \text{by letting} y = \frac{x-\mu}{\sigma} \\
&= \frac{e^{\mu t}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} exp\left\{-\left[\frac{y^2 - 2t\sigma y}{2}\right]\right\} dy \\
&= \frac{e^{\mu t}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} exp\left\{-\frac{(y-t\sigma)^2}{2} + \frac{t^2\sigma^2}{2}\right\} dy \\
&= exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\} \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(y-t\sigma)^2}{2}}\,dy \\
&= exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}
\end{aligned}
$$

since,

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(y-t\sigma)^2}{2}}\,dy = 1$$

Differentiating the moment generating function

$$\phi'(t) = (\mu + t\sigma^2) exp \left\{ \mu t + \frac{\sigma^2 t^2}{2} \right\}$$

$$\phi''(t) = \sigma^2 exp \left\{ \mu t + \frac{\sigma^2 t^2}{2} \right\} + (\mu + t\sigma^2)^2 exp \left\{ \mu t + \frac{\sigma^2 t^2}{2} \right\}$$

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Hence

$$E[X] = \phi'(0) = \mu$$

$$E[X^2] = \phi''(0) = \sigma^2 + \mu^2$$

and

$$Var(X) = E[X^2] - (E[X])^2 = \sigma^2$$

If X is normal random variable with mean μ and variance σ^2, then $Y = \alpha X + \beta$ is a normal random variable with mean $\alpha\mu + \beta$and variance $\alpha^2\sigma^2$. By Using the moment generating function.

$$\begin{aligned} E[e^{t(\alpha X+\beta)}] &= e^{t\beta}E[e^{\alpha tX}] \\ &= e^{t\beta} exp\left\{\mu\alpha t + \frac{\sigma^2(\alpha t)^2}{2}\right\} \\ &= exp\left\{(\beta + \mu\alpha)t + \frac{\alpha^2\sigma^2 t^2}{2}\right\} \end{aligned}$$

The final equation is the moment generating function of the normal random variable with mean $\mu\alpha + \beta$ and variance $\alpha^2\sigma^2$

If $X \sim \mathcal{N}(\mu, \sigma^2)$ then

$$Z = \frac{X - \mu}{\sigma}$$

is a normal random variable with mean 0 and variance 1. Such a random variable Z is said to have a standard, or unit, normal distribution. Let $\Phi(\cdot)$ denote its distribution function

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy \qquad -\infty < x < \infty$$

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This enables us to write all probability statements about X in terms of probabilities for Z

$$P\{X < b\} = P\left\{\frac{X - \mu}{\sigma} < \frac{b - \mu}{\sigma}\right\}$$
$$= \Phi\left(\frac{b - \mu}{\sigma}\right)$$

Similarly for any $a < b$

$$
\begin{aligned}
P\{a < X < b\} &= P\left\{\frac{a-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right\} \\
&= P\left\{\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right\} \\
&= P\left\{Z < \frac{b-\mu}{\sigma}\right\} - P\left\{Z < \frac{a-\mu}{\sigma}\right\} \\
&= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)
\end{aligned}
$$

Tabulated values of $\Phi(x)$ now can be used for calculations based on the normal distribution. Only positive values of x are generally tabulated, negative values can be found by using

$$
\begin{aligned}
\Phi(-x) &= P\{Z < -x\} \\
&= P\{Z > x\} \qquad \text{by symmetry} \\
&= 1 - \Phi(x)
\end{aligned}
$$

Table

TABLE A1 *Standard Normal Distribution Function:* $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy$

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Example

If X is a normal random variable with mean $\mu = 3$and variance $\sigma^2 = 16$,find

1. $P\{X < 11\}$
2. $P\{X > -1\}$
3. $P\{2 < X < 7\}$

Example

Suppose that a binary message – either 0 or 1 – must be transmitted by wire from location A to location B. However, the data sent over the wire are subject to a channel noise disturbance and so to reduce the possibility of error, the value -2 is sent over the wire when the message is 1 and the value -2 is sent when the message is 0. If $x, x = \pm 2$, is the value sent at location A then R, the value received at location B, is given by $R = x + N$, where N is the channel noise disturbance. When the message is received at location B, the receiver decodes it according to the following rule:

$$\text{if } R \geq .5, \text{ then 1 is concluded}$$
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There are two types of errors that can occur: One is that the message 1 can be incorrectly concluded to be 0 and the other that 0 is incorrectly concluded to be 1. The first type of error will occur if the message is 1 and $2 + N < .5$, whereas the second will occur if the message is 0 and $-2 + N \geq .5$. Hence

$$P\{\text{error} \mid \text{message is "1"}\} = P\{N < 1.5\} = 1 - \Phi(1.5) = 0.0668$$

$$P\{\text{error} \mid \text{message is "0"}\} = P\{N \geq 1.5\} = 1 - \Phi(2.5) = 0.0062$$

Example

The power W dissipated in a resistor is proportional to the square of the voltage V. That is,

$$W = rV^2$$

where r is a constant. If $r = 3$, and V can be assumed (to a very good approximation) to be a normal random variable with mean 6 and standard deviation 1, find

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Solution

1.

$$\begin{aligned} E[W] &= E[3V^2] = 3E[V^2] \\ &= 3(Var[V] + (E[V])^2) == 3(1 + 36) = 111 \end{aligned}$$

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2.
$$\begin{aligned} P\{W > 120\} &= P\{3V^2 > 120\} = P\{V > \sqrt{40}\} \\ &= P\{V - 6 > \sqrt{40} - 6\} = P\{Z > 0.3246\} \\ &= 1 - \Phi(0.3246) = 0.3237 \end{aligned}$$

The sum of independent normal random variables is also a normal random variable. To see this, suppose that $X_i, i = 1, \ldots, n$, are independent, with X_i being normal with mean μ_i and variance σ_i^2. The moment generating function of $\sum_{i=1}^{n} X_i$ is

$$
\begin{aligned}
E\left[exp\left\{t\sum_{i=1}^{n} X_i\right\}\right] &= E[e^{tX_1}e^{tX_2}\ldots e^{tX_n}] \\
&= \prod_{i=1}^{n} E[e^{tX_i}] \\
&= \prod_{i=1}^{n} e^{\mu_i t + \frac{\sigma_i^2 t^2}{2}} \\
&= e^{\mu t + \frac{\sigma^2 t^2}{2}}
\end{aligned}
$$

Where

$$\mu = \sum_{i=1}^{n} \mu_i \qquad \sigma^2 = \sum_{i=1}^{n} \sigma_i^2$$

Example

Data from the National Oceanic and Atmospheric Administration indicate that the yearly precipitation in Los Angeles is a normal random variable with a mean of 12.08 inches and a standard deviation of 3.1 inches.

1. Find the probability that the total precipitation during the next 2 years will exceed 25 inches.
2. Find the probability that next years precipitation will exceed that of the following year by more than 3 inches.

Assume that the precipitation totals for the next 2 years are independent.

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2. Find the probability that next years precipitation will exceed that of the following year by more than 3 inches.

Assume that the precipitation totals for the next 2 years are independent. **Solution:** Let X_1 and X_2 be the precipitation totals for the next 2 years

1. Since $X_1 + X_2$ is normal with mean 24.16and variance $2(3.1)^2 = 19.22$, it follows that

$$
\begin{aligned}
P\{X_1 + X_2 > 25\} &= P\left\{\frac{X_1 + X_2 - 24.16}{\sqrt{19.22}} > \frac{25 - 24.16}{\sqrt{19.22}}\right\} \\
&= P\{Z > 0.1916\} \approx 0.4240
\end{aligned}
$$

2. Since $-X_2$ is a normal random variable with mean -12.08 and variance $(-1)2(3.1)^2$, it follows that $X_1 - X_2$ is normal with mean 0 and variance 19.22. Hence,

$$\begin{aligned} P\{X_1 > X_2 + 3\} &= P\{X_1 - X_2 > 3\} \\ &= P\left\{\frac{X_1 - X_2}{\sqrt{19.22}} > \frac{3}{\sqrt{19.22}}\right\} \\ &= P\{Z > 0.6843\} \approx 0.2469 \end{aligned}$$

Thus there is a 42.4 percent chance that the total precipitation in Los Angeles during the next 2 years will exceed 25 inches, and there is a 24.69 percent chance that next years precipitation will exceed that of the following year by more than 3 inches.