ME 102: Lecture 9

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Bernoulli Random Variable

A random variable X is said to be a Bernoulli random variable (after the Swiss mathematician James Bernoulli) if its probability mass function is given by

$$P\{X = 0\} = 1 - p$$

$$P\{X = 1\} = p$$

for some $p \in (0, 1)$.

The expected value of a Bernoulli Variable is

$$E[X] = 1 \cdot P\{X = 1\} + 0 \cdot P\{X = 0\} = p$$

Binomial Random Variable

Suppose now that n independent trials, each of which results in a success with probability p and in a failure with probability $1 - p$, are to be performed. If X represents the number of successes that occur in the n trials, then X is said to be a binomial random variable with parameters (n, p).

The probability mass function of a binomial random variable with parameters n and p is given by

$$P\{X = i\} = \binom{n}{i} p^i (1-p)^{n-i}, \qquad i = 0, 1, 2, \ldots, n$$

Where

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

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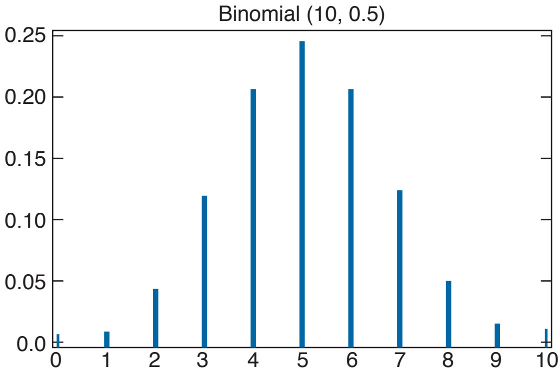
Where

$$\left(\begin{array}{c} n \\ i \end{array} \right) = \frac{n!}{i!(n-i)!}$$

By the binomial theorem, the probabilities sum to 1, that is,

$$\sum_{i=0}^{\infty} p(i) = \sum_{i=0}^{n} \left(\begin{array}{c} n \\ i \end{array} \right) p^i (1-p)^{n-i} = [p + (1-p)]^n = 1$$


Binomial (10, 0.5)
0.25
0.20
0.15
0.10
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1
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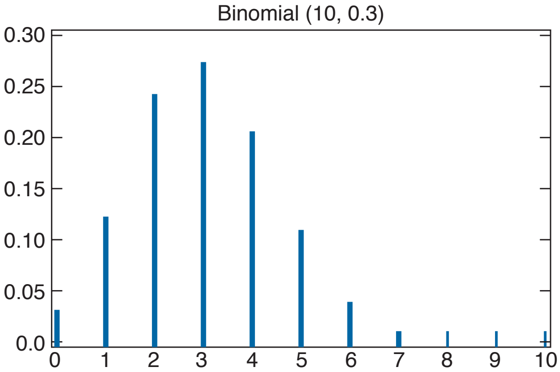
Binomial (10, 0.3)
0.30
0.25
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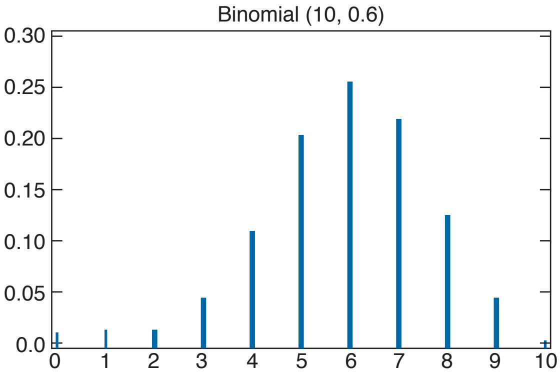
Binomial (10, 0.6)
0.30
0.25
0.20
0.15
0.10
0.05
0.0
0
1
2
3
4
5
6
7
8
9
10

Mean and Variance of Binomial Distribution function

Mean

$$E[X] = \sum_{i=1}^{n} E[X_i]$$
$$= np$$

Variance

$$Var(X) = \sum_{i=1}^{n} Var(X_i) \quad \text{since the } X_i \text{ are independent}$$
$$= np(1-p)$$

Binomial Distribution Function

$$P\{X \le i\} = \sum_{k=0}^{i} \binom{n}{k} p^k (1-p)^{n-k}, \quad i = 0, 1, 2, \ldots, n$$

Example Problem

It is known that disks produced by a certain company will be defective with probability .01 independently of each other. The company sells the disks in packages of 10 and offers a money-back guarantee that at most 1 of the 10 disks is defective. What proportion of packages is returned? If someone buys three packages, what is the probability that exactly one of them will be returned?

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The probability that a package will have to be replaced is

$$\begin{aligned} P\{X > 1\} &= 1 - P\{X = 0\} - P\{X = 1\} \\ &= 1 - \binom{10}{0}(0.01)^0(0.99)^{10} - \binom{10}{1}(0.01)^1(0.99)^9 \approx 0.005 \end{aligned}$$

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It follows from the foregoing that the number of packages that the person will have to return is a binomial random variable with parameters $n = 3$ and $p = 0.005$. Therefore, the probability that exactly one of the three packages will be returned is

$$\binom{3}{1}(0.005)(0.995)^2 = 0.015$$

Example Problem

The color of ones eyes is determined by a single pair of genes, with the gene for brown eyes being dominant over the one for blue eyes. This means that an individual having two blue-eyed genes will have blue eyes, while one having either two brown-eyed genes or one brown-eyed and one blue-eyed gene will have brown eyes. When two people mate, the resulting offspring receives one randomly chosen gene from each of its parents gene pair. If the eldest child of a pair of brown-eyed parents has blue eyes, what is the probability that exactly two of the four other children (none of whom is a twin) of this couple also have blue eyes?

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The probability that exactly two of remaining 4 children have blue eyes is

$$\binom{4}{2} (0.25)^2 (0.75)^2 = 0.2109$$

Example Problem

A communications system consists of n components, each of which will, independently, function with probability p. The total system will be able to operate effectively if at least one-half of its components function. For what values of p is a 5-component system more likely to operate effectively than a 3-component system?

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SolutionThe probability that a 5-component system will be effective is

$$\begin{pmatrix} 5 \\ 3 \end{pmatrix} p^3(1-p)^2 + \begin{pmatrix} 5 \\ 4 \end{pmatrix} p^4(1-p) + p^5$$

The probability that a 3-component system will be effective is

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} p^2(1-p) + p^3$$

Hence, the 5-component system is better if

$$10p^3(1-p)^2 + 5p^4(1-p) + p^5 \geq 3p^2(1-p) + p^3$$

which is

$$p \geq \frac{1}{2}$$

Poisson Random Variable

A random variable X, taking on one of the values $0, 1, 2, \ldots,$ is said to be a Poisson random variable with parameter $\lambda, \lambda > 0$ if its probability mass function is given by

$$P\{X = i\} = e^{-\lambda} \frac{\lambda^i}{i!}, \qquad i = 0, 1, 2, \ldots$$

Since above equation defines the probability mass function

$$\sum_{i=0}^{\infty} p(i) = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{-\lambda} e^{\lambda} = 1$$

The Poisson probability distribution was introduced by S. D. Poisson in a book he wrote dealing with the application of probability theory to lawsuits, criminal trials, and the like. This book, published in 1837, was entitled *Recherches sur la probabilité des jugements en matière criminelle et en matière civile*.

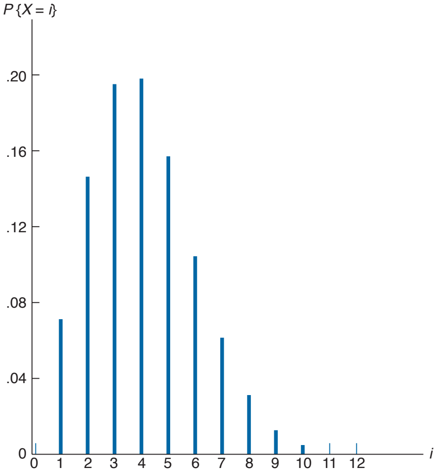

Figure: The Poisson probability mass function with $\lambda = 4$.

Moment generating function of Poisson random variable

$$
\begin{aligned}
\phi(t) &= E[e^{tX}] \\
&= \sum_{i=0}^{\infty} e^{ti} e^{-\lambda} \frac{\lambda^i}{i!} \\
&= e^{-\lambda} \sum_{i=0}^{\infty} \frac{(\lambda e^t)^i}{i!} \\
&= e^{-\lambda} e^{\lambda} e^{t} \\
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Differentiating

$$\phi'(t) = \lambda e^t exp[\lambda(e^t - 1)]$$

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Evaluating at $t = 0$ gives

$$E[X] = \phi'(0) = \lambda$$

$$\begin{aligned}Var(X) &= \phi''(0) - (E[X])^2 \\ &= \lambda^2 + \lambda - \lambda = \lambda\end{aligned}$$

The Poisson random variable may be used as an approximation for a binomial random variable with parameters (n, p) when n is large and p is small.
Let $\lambda = np$

$$
\begin{aligned}
P\{X = i\} &= \frac{n!}{(n-i)!i!} p^i (1-p)^{n-i} \\
&= \frac{n!}{(n-i)!i!} \left(\frac{\lambda}{n}\right)^i \left(1 - \frac{\lambda}{n}\right)^{n-i} \\
&= \frac{n(n-1)\ldots(n-i+1)}{n^i} \frac{\lambda^i}{i!} \frac{\left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^i}
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\end{aligned}
$$

Now, for n large and p small

$$
\left(1 - \frac{\lambda}{n}\right)^n \approx e^{-\lambda} \qquad \frac{n(n-1)\ldots(n-i+1)}{n^i} \approx 1 \qquad \left(1 - \frac{\lambda}{n}\right)^i \approx 1
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$$

Hence for large n and small p,

$$
P\{X = i\} \approx e^{-\lambda} \frac{\lambda^i}{i!}
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Example Problem

Suppose that the average number of accidents occurring weekly on a particular stretch of a highway equals 3. Calculate the probability that there is at least one accident this week.

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question during this week.

$$\begin{aligned} P\{X \geq 1\} &= 1 - P\{X = 0\} \\ &= 1 - e^{-3}\frac{3^0}{0!} \\ &= 1 - e^{-3} \\ &\approx 0.95 \end{aligned}$$

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Suppose the probability that an item produced by a certain machine will be defective is .1. Find the probability that a sample of 10 items will contain at most one defective item. Assume that the quality of successive items is independent.

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Considering a binomial random variable, the probability is

$$\begin{pmatrix} 10 \\ 0 \end{pmatrix} (0.1)^0 (0.9)^{10} + \begin{pmatrix} 10 \\ 1 \end{pmatrix} (0.1)^1 (0.9)^9 = 0.7361$$

Considering Poisson approxximation

$$e^{-1} \frac{1^0}{0!} + e^{-1} \frac{1^1}{1!} = 2e^{-1} \approx 0.7358$$

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Suppose that the number of claims handled daily, call it X, is a Poisson random variable. Now $E[X] = 5$
The probability that there will be fewer than 3 claims on any given day is

$$\begin{aligned} P\{X \leq 3\} &= P\{X = 0\} + P\{X = 1\} + P\{X = 2\} \\ &= e^{-5} + e^{-5}\frac{5^1}{1!} + e^{-5}\frac{5^2}{2!} \\ &= \frac{37}{2}e^{-1} \\ &\approx 0.1247 \end{aligned}$$

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The probability that there will be 4 claims in exactly 3 of the next 5 days is computed as

$$P\{X = 4\} = e^{-5}\frac{5^4}{4!} \approx 0.1755$$

The number of claims per day is independent , hence probability is

$$\binom{5}{3}(0.1755)^3(0.8245)^2 \approx 0.0367$$