9th September, 2019

Fluid Kinematics > Flow Visualization Lines

Streamlines, Pathlines & Streaklines for a given unsteady, incompressible, two-dimensional velocity field

Link to Streamlines plot

https://bit.ly/FMstreamlines

Link to Pathlines plot

https://bit.ly/FMpathlines

Link to Streaklines plot

https://bit.ly/FMstreaklines

Link to this document:

https://bit.ly/FMvisual

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Velocity field:

Consider an unsteady, incompressible, two-dimensional velocity field given by

$\vec{V} = u\,\hat{\imath} \;+\; v\,\hat{\jmath}$

Where,

$$u = \frac{dx}{dt} = 0.5 + 0.8x \tag{1}$$

$$v = \frac{dy}{dt} = 1.5 + 2.5\sin(2\pi t\,) - 0.8y \tag{2}$$

Equation for Streamlines

By definition,

$$\left.\frac{dy}{dx}\right|_{along\ streamline} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{v}{u}$$

$$=> \frac{dy}{dx} = \frac{1.5 + 2.5\sin(2\pi t\,) - 0.8y}{0.5 + 0.8x}$$

$$=> \frac{dy}{1.5 + 2.5\sin(2\pi t\,) - 0.8y} = \frac{dx}{0.5 + 0.8x}$$

$$=> \int_{y_0}^{y} \frac{dy}{1.5 + 2.5\sin(2\pi t\,) - 0.8y} = \int_{x_0}^{x} \frac{dx}{0.5 + 0.8x}$$

$$=> \frac{\ln(1.5 + 2.5\sin(2\pi t\,) - 0.8y) - \ln\,(1.5 + 2.5\sin(2\pi t\,) - 0.8y_0)}{-0.8} = \frac{\ln(0.5 + 0.8x) - \ln\,(0.5 + 0.8x_0)}{0.8}$$

$$=> \ln\left(\frac{1.5 + 2.5\sin(2\pi t\,) - 0.8y_0}{1.5 + 2.5\sin(2\pi t\,) - 0.8y}\right) = \ln\left(\frac{0.5 + 0.8x}{0.5 + 0.8x_0}\right)$$

$$=> \frac{1.5 + 2.5\sin(2\pi t\,) - 0.8y_0}{1.5 + 2.5\sin(2\pi t\,) - 0.8y} = \frac{0.5 + 0.8x}{0.5 + 0.8x_0} \tag{a}$$

Plotting Streamlines

Streamline passing through any point (x_0, y_0), at any instant "$t = t_1$", can now be plotted using Equation(a).

See table-1 (or go to https://bit.ly/Streamlines) to see some plots of equation(a)

Above link plots Instantaneous Streamlines (at any instant "$t = t_1$") passing through three points (0.5,0.5), (0.5,2.5) & (0.5,4.5). Time "t_1" can be changed via slider from the left of the screen.

Table 1 Streamlines passing through (0.5,0.5), (0.5,2.5) & (0.5,4.5) at various instant of time "t"

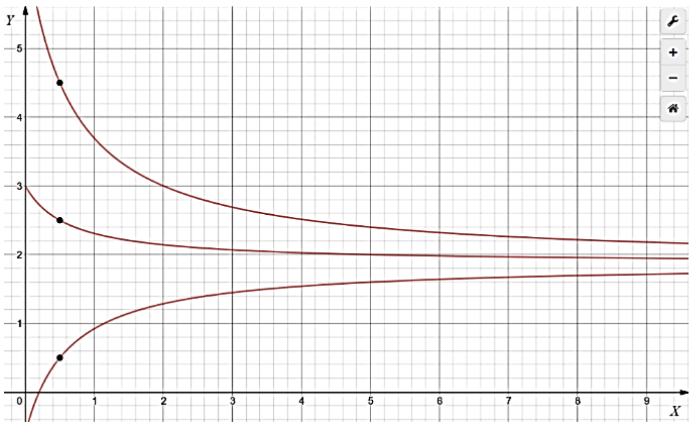

Figure 1   streamlines at t at any instant $t = t_1$ = 0 sec

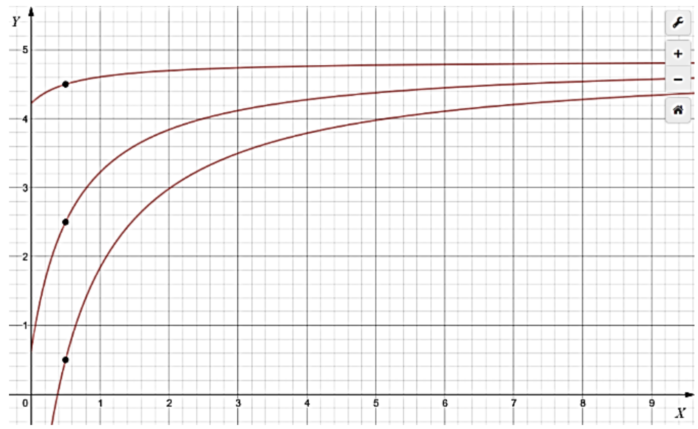

Figure 2   streamlines at $t = t_1$ = 0.2 sec

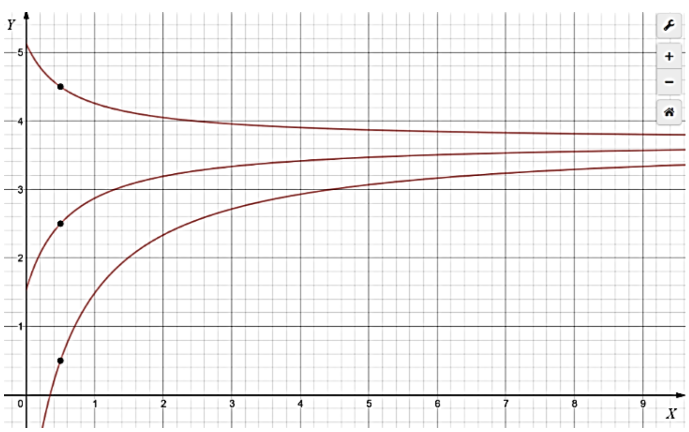

Figure 3   streamlines at $t = t_1$ = 0.4 sec

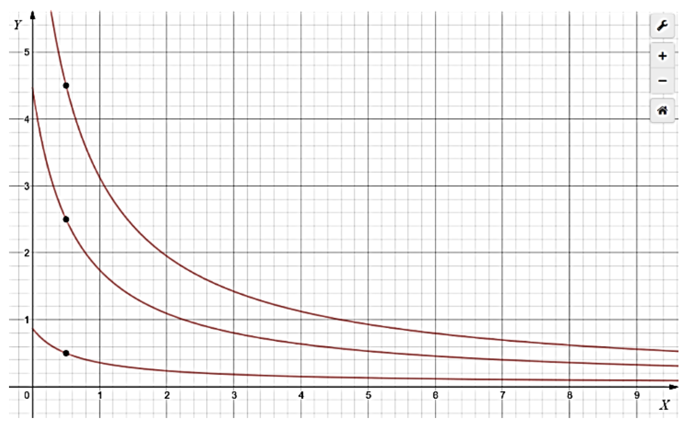

Figure 4   streamlines at $t = t_1$ = 0.6 sec

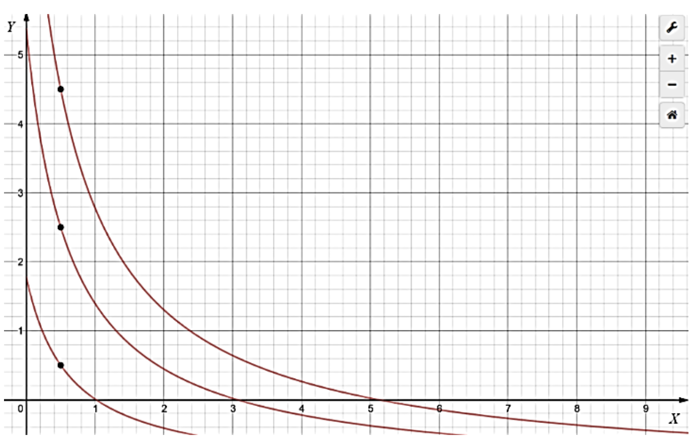

Figure 5   streamlines at $t = t_1$ = 0.8 sec

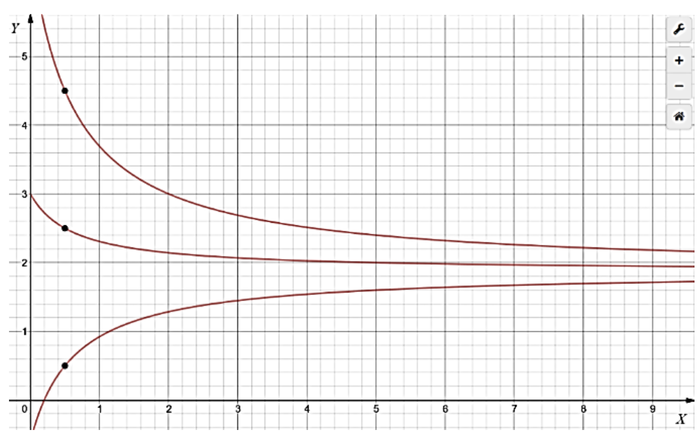

Figure 6   streamlines at $t = t_1$ = 1 sec

Equation for pathline

Equation for pathlines can be obtained by first solving the differential equations (1) & (2)

Consider equation (1),

$$u = \frac{dx}{dt} = 0.5 + 0.8x \tag{1}$$

Let

$$x(s) = x_0$$

$$\int_{x_0}^{x} \frac{dx}{0.5 + 0.8x} = \int_{s}^{t} dt$$

$$\frac{1}{0.8} \times \ln\left(\frac{0.5 + 0.8x}{0.5 + 0.8x_0}\right) = t - s$$

$$x = x(x_0, s, t) = \left(x_0 + \frac{5}{8}\right) e^{0.8(t-s)} - \frac{5}{8} \tag{b}$$

Now, Consider equation (2),

$$\frac{dy}{dt} = 1.5 + 2.5\sin(2\pi t\ ) - 0.8y \tag{2}$$

$$\frac{dy}{dt} + 0.8y = 1.5 + 2.5\sin(2\pi t\ )$$

Using the Bernoulli differential equation, multiplying both sides by $e^{0.8t}$

$$e^{0.8t}\frac{dy}{dt} + 0.8e^{0.8t}y = e^{0.8t}(1.5 + 2.5\sin(2\pi t\ ))$$

$$d(ye^{0.8t}) = e^{0.8t}(1.5 + 2.5\sin(2\pi t\ ))\,dt$$

Let

$$y(s) = y_0$$

$$\int_{y_0,s}^{y,t} d(ye^{0.8t}) = \int_{s}^{t} e^{0.8t}(1.5 + 2.5\sin(2\pi t\ ))\,dt$$

Integrating by parts

$$\left[ye^{\frac{4\,t}{5}}\right]_{y_0,s}^{y,t} = \left[\frac{15\,\mathrm{e}^{\frac{4\,t}{5}}}{8} + \frac{25\,\mathrm{e}^{\frac{4\,t}{5}}\,(2\ \sin(2\,\pi\,t) - 5\,\pi\ \cos(2\,\pi\,t))}{4\,(25\,\pi^2 + 4)}\right]_s^t$$

Substituting the condition and rearranging for y

$$\begin{gathered} y = y(y_0, s, t) = \ y_0\mathrm{e}^{-\frac{4}{5}(t-s)} + \left(\frac{15}{8} + \frac{25\,(2\ \sin(2\,\pi\,t) - 5\,\pi\ \cos(2\,\pi\,t))}{4\,(25\,\pi^2 + 4)}\right) \\ -\mathrm{e}^{-\frac{4}{5}(t-s)}\left(\frac{15}{8} + \frac{25\ (2\ \sin(2\,\pi\,s) - 5\,\pi\ \cos(2\,\pi\,s))}{4\,(25\,\pi^2 + 4)}\right) \end{gathered} \tag{c}$$

Plotting Pathlines (Using time "t" as the parameter)

Pathline over a period from t_0 to t_1 can be plotted for a fluid particle whose location (x_0, y_0) is known at an earlier point in time t_0 $(t_0 < t_1)$ using the locus given by

$$\left(x(x_0, s, t), y(y_0, s, t)\right)$$

Where

$$s = t_0 \tag{3}$$

$$t_0 \leq t \leq t_1$$

Plotting Pathlines (An alternate method: using dimensionless parameter "τ")

Note that in above method, the parameter "t" varies from t_0 to t_1. An alternate method of plotting pathlines involves introduction of another parameter such that the new parameter varies always from 0 to 1 irrespective of the values of t_0 and t_1.

Let's define another parameter "τ" which varies linearly from 0 to 1 as "t" varies from t_0 to t_1.

We can write,

$$\frac{\tau - 0}{1 - 0} = \frac{t - t_0}{t_1 - t_0}$$

$$\Rightarrow t = \tau(t_1 - t_0) + t_0 \tag{4}$$

Substituting (3) & (4) in (b) & (c) gives

$$x_{pathline}(x_0, t_0, t_1, \tau) = \left(x_0 + \frac{5}{8}\right) e^{0.8\tau(t_1 - t_0)} - \frac{5}{8} \tag{d}$$

$$\begin{aligned} y_{pathline}(y_0, t_0, t_1, \tau) = \; & y_0 e^{-\frac{4}{5}\tau(t_1 - t_0)} + \left(\frac{15}{8} + \frac{25\,(2\,sin(2\pi\tau(t_1 - t_0) + 2\pi t_0) - 5\pi\, cos(2\pi\tau(t_1 - t_0) + 2\pi t_0))}{4\,(25\pi^2 + 4)}\right) \\ & - e^{-\frac{4}{5}\tau(t_1 - t_0)} \left(\frac{15}{8} + \frac{25\,(2\,sin(2\,\pi\,t_0) - 5\,\pi\,cos(2\,\pi\,t_0))}{4\,(25\,\pi^2 + 4)}\right) \end{aligned} \tag{e}$$

The locus of point $\left(x_{pathline}(x_0, t_0, t_1, \tau), y_{pathline}(y_0, t_0, t_1, \tau)\right)$ for $0 \leq \tau \leq 1$ represents pathline over a period from t_0 to t_1.

Equations (d) and (e) can now be used to draw the pathline for the fluid particle whose location (x_0, y_0) is known at an earlier point in time t_0, over a period from t_0 to t_1. See Table-2 or go to https://bit.ly/FMpathlines[1] to see the plot of pathlines.

[1] Since the plotting tool (desmos.com) only recognizes "t" as the parameter for any parametric equation, the above link uses "t" instead of "τ" in writing equation (d) and (e) in the plotter's commands.

Table 2 Plot of pathlines for fluid particles that were initially at $(0.5,0.5)$, $(0.5,2.5)$ & $(0.5,4.5)$ at an earlier instant of time $t = t_0 = 0$ in the flow field, over a period from $t = t_0 = 0$ to $t = t_1$.

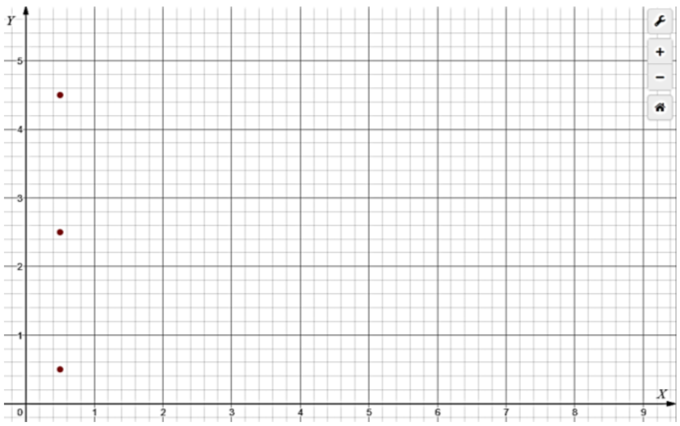

Figure 7 (Trivial) Pathlines from $t = t_0 = 0sec$ to $t = t_1 = 0sec$

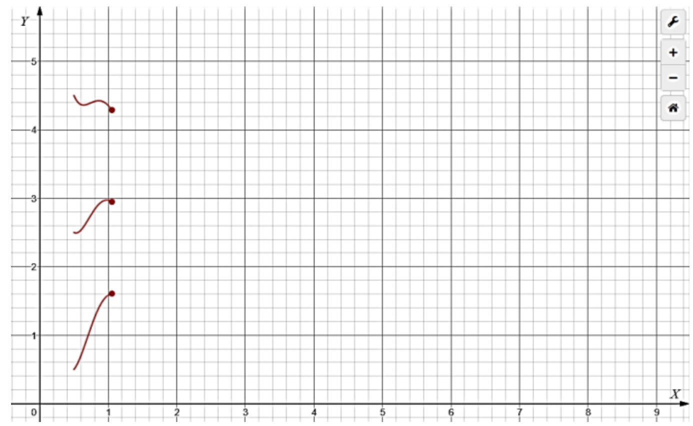

Figure 8 Pathlines from $t = t_0 = 0sec$ to $t = t_1 = 0.5sec$

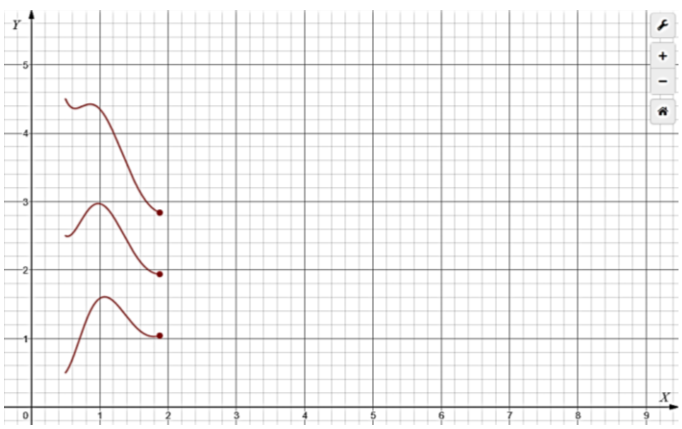

Figure 9 Pathlines from $t = t_0 = 0sec$ to $t = t_1 = 1sec$

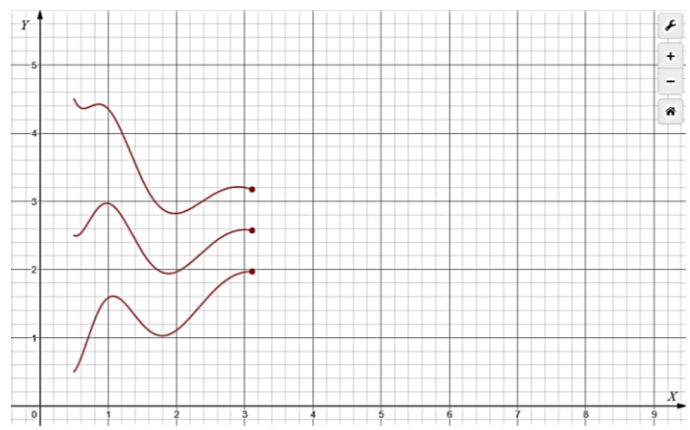

Figure 10 Pathlines from $t = t_0 = 0sec$ to $t = t_1 = 1.5sec$

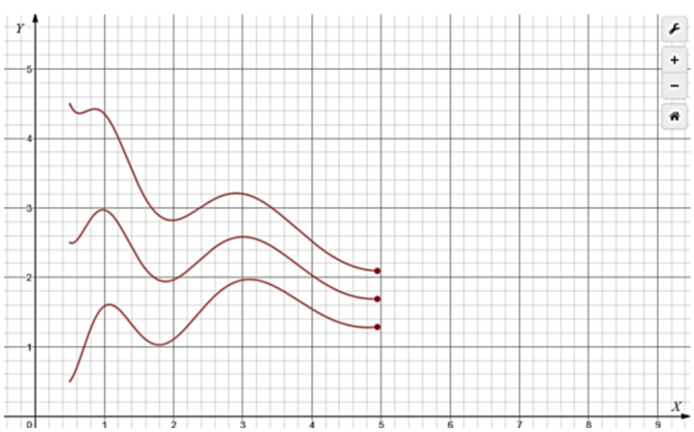

Figure 11 Pathlines from $t = t_0 = 0sec$ to $t = t_1 = 2sec$

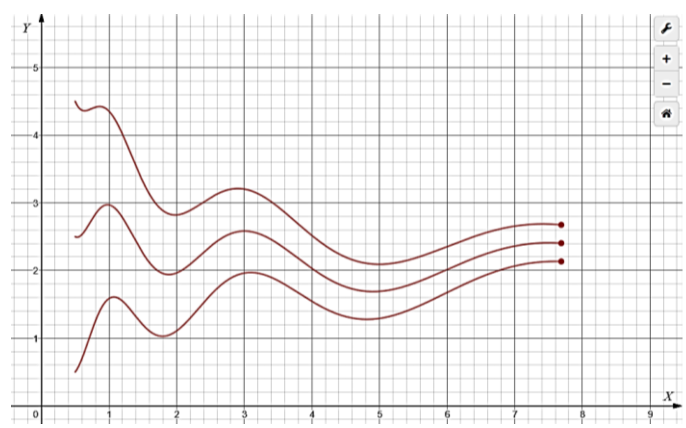

Figure 12 Pathlines from $t = t_0 = 0sec$ to $t = t_1 = 2.5sec$

Plotting Streaklines (Using time "s" as the parameter)

Instantaneous streaklines for all the fluid particles that have previously passed (x_0, y_0) over a period from t_0 to $t_1$can be drawn using the locus

$$\left(x(x_0, \boldsymbol{s}, t), y(y_0, \boldsymbol{s}, t)\right)$$

Where

$$t_0 < s < t_1$$
$$t = t_1 \tag{5}$$

Plotting Streaklines (An alternate method: using dimensionless parameter "σ")

Note that in above method, the parameter "s" varies from t_0 to t_1. An alternate method of plotting streaklines involves introduction of another parameter such that the new parameter varies always from 0 to 1 irrespective of the values of t_0 and t_1.

Let's define another parameter "σ" which varies linearly from 0 to 1 as "s" varies from t_0 to t_1.

We can write,

$$\frac{\sigma - 0}{1 - 0} = \frac{s - t_0}{t_1 - t_0}$$
$$\Rightarrow s = \sigma(t_1 - t_0) + t_0 \tag{6}$$

Substituting (5) & (6) in (b) & (c) gives

$$x_{streakline}(x_0, t_0, t_1, \sigma) = \left(x_0 + \frac{5}{8}\right) e^{\frac{4}{5}(1-\sigma)(t_1-t_0)} - \frac{5}{8} \tag{f}$$

$$y_{streakline}\left(y_0, t_0, t_1, \sigma\right) = y_0 e^{-\frac{4}{5}(1-\sigma)(t_1-t_0)} + \left(\frac{15}{8} + \frac{25\,(2\ sin(2\,\pi\,t_1) - 5\,\pi\ cos(2\,\pi\,t_1))}{4\,(25\,\pi^2 + 4)}\right)$$
$$-e^{-\frac{4}{5}(1-\sigma)(t_1-t_0)} \left(\frac{15}{8} + \frac{25\ (2\ sin(2\,\pi\sigma(t_1 - t_0) + 2\pi t_0) - 5\,\pi\ cos(2\,\pi\sigma(t_1 - t_0) + 2\pi t_0))}{4\,(25\,\pi^2 + 4)}\right) \tag{g}$$

The locus of point $\left(x_{streakline}(x_0, t_0, t_1, \sigma), y_{streakline}\left(y_0, t_0, t_1, \sigma\right)\right)$ for $0 \le \tau \le 1$ represents pathline over a period from t_0 to t_1. See Table-2 or go to https://bit.ly/FMstreaklines[2] to see the plot of streaklines.

[2] Since the plotting tool (desmos.com) only recognizes "t" as the parameter for any parametric equation, the above link uses "t" instead of "σ" in writing equation (d) and (e) in the plotter's commands.

Table 3 Instantaneous streaks for all the fluid particles that have previously passed (0.5,0.5), (0.5,2.5) & (0.5,4.5) at an earlier point in time $t = t_0 = 0\ sec$, over a period from t_0 to t_1

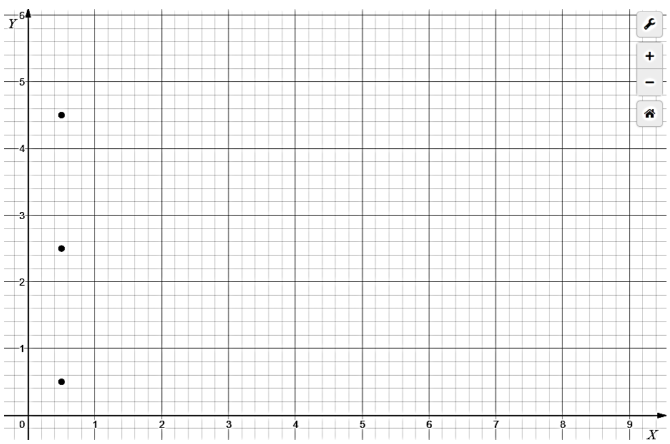

Figure 13 (Trivial) Streaklines from $t = t_0 = 0sec$ to $t = t_1 = 0sec$

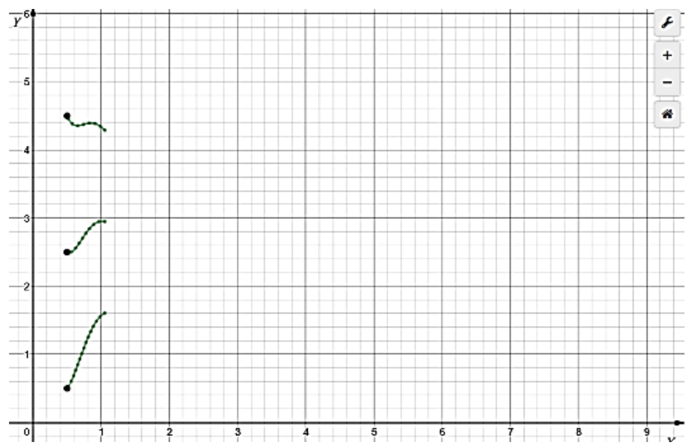

Figure 14 Streaklines from $t = t_0 = 0sec$ to $t = t_1 = 0.5sec$

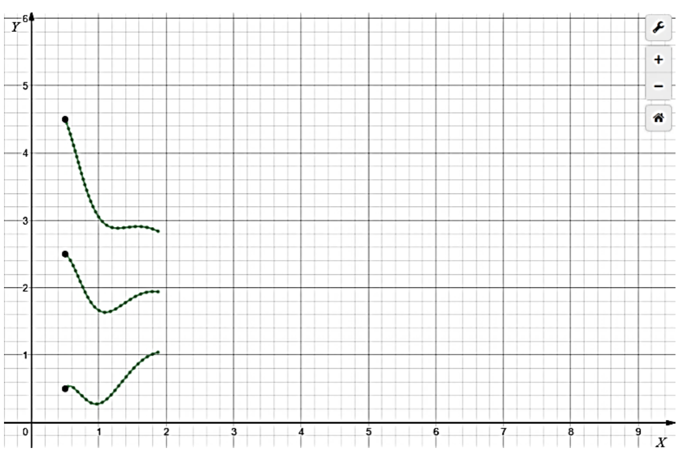

Figure 15 Streaklines from $t = t_0 = 0sec$ to $t = t_1 = 1sec$

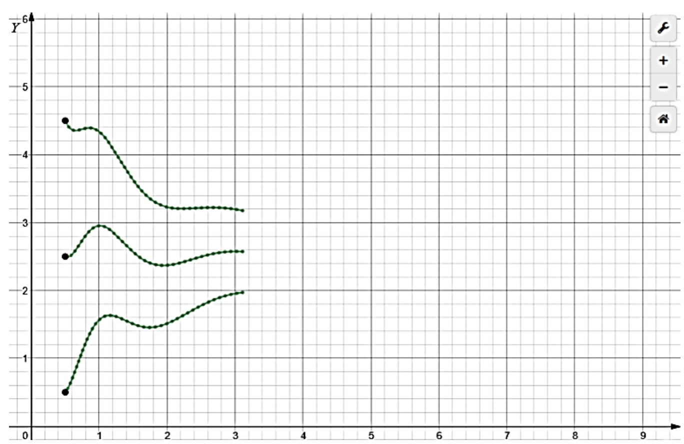

Figure 16 Streaklines from $t = t_0 = 0sec$ to $t = t_1 = 1.5sec$

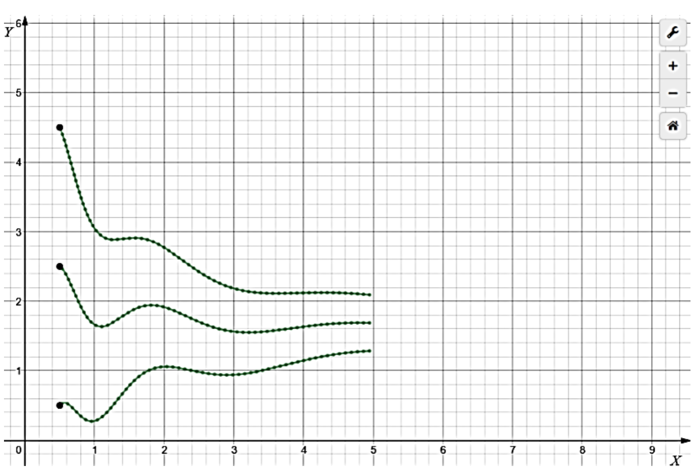

Figure 17 Streaklines from $t = t_0 = 0sec$ to $t = t_1 = 2sec$

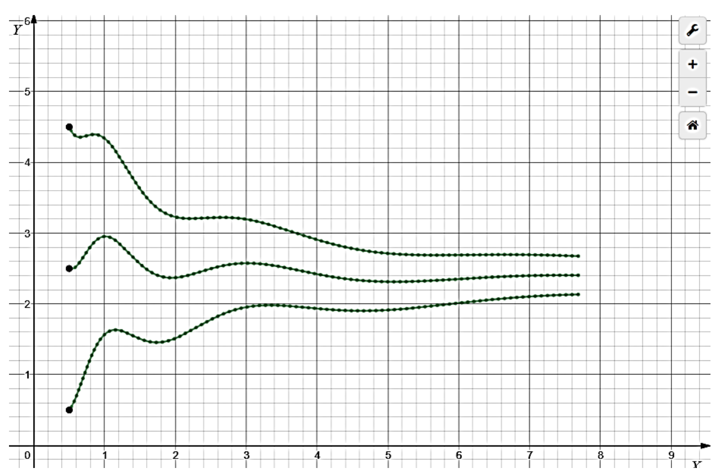

Figure 18 Streaklines from $t = t_0 = 0sec$ to $t = t_1 = 2.5sec$