Controllability of Complex Networks

Yang-Yu Liu

Center of Complex Network Research
Northeastern University

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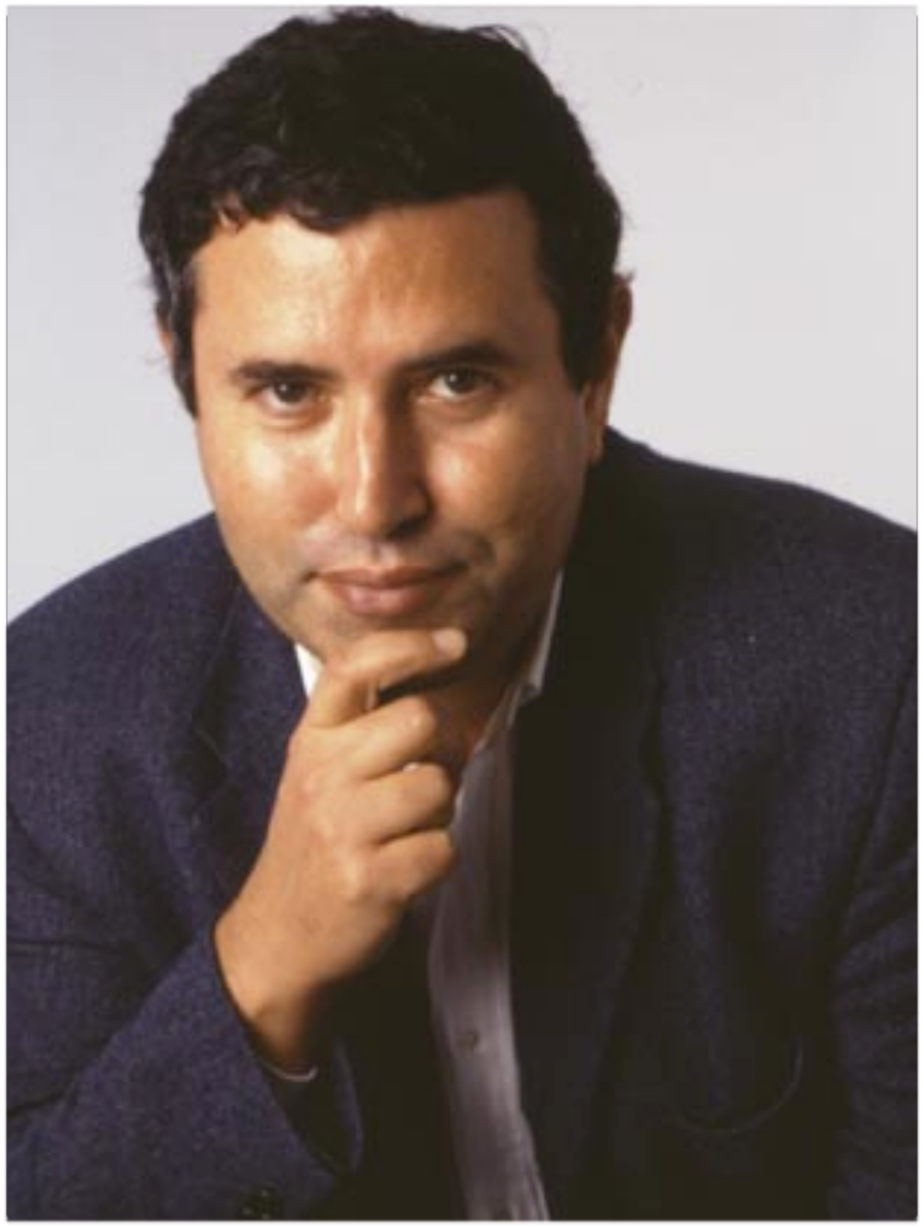
Collaborators



Jean-Jacques Slotine
MIT

Albert-László Barabási
NEU/HMS

ARTICLE

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Controllability of complex networks

Yang-Yu Liu[1,2], Jean-Jacques Slotine[3,4] & Albert-László Barabási[1,2,5]

The ultimate proof of our understanding of natural or technological systems is reflected in our ability to control them. Although control theory offers mathematical tools for steering engineered and natural systems towards a desired state, a framework to control complex self-organized systems is lacking. Here we develop analytical tools to study the controllability of an arbitrary complex directed network, identifying the set of driver nodes with time-dependent control that can guide the system's entire dynamics. We apply these tools to several real networks, finding that the number of driver nodes is determined mainly by the network's degree distribution. We show that sparse inhomogeneous networks, which emerge in many real complex systems, are the most difficult to control, but that dense and homogeneous networks can be controlled using a few driver nodes. Counterintuitively, we find that in both model and real systems the driver nodes tend to avoid the high-degree nodes.

According to control theory, a dynamical system is controllable if, with a suitable choice of inputs, it can be driven from any initial state to any desired final state within finite time[1–3]. This definition agrees with our intuitive notion of control, capturing an ability to guide a system's behaviour towards a desired state through the appropriate manipulation of a few input variables, like a driver prompting a car to move with the desired speed and in the desired direction by manipulating the pedals and the steering wheel. Although control theory is a mathematically highly developed branch of engineering with applications to electric circuits, manufacturing processes, communication systems[4–6], aircraft, spacecraft and robots[2,3], fundamental questions pertaining to the controllability of complex systems emerging in nature and engineering have resisted advances. The difficulty is rooted in the fact that two independent factors contribute to controllability, each with its own layer of unknown: (1) the system's architecture, represented by the network encapsulating which components interact with each other; and (2) the dynamical rules that capture the time-dependent interactions between the components. Thus, progress has been possible only in systems where both layers are well mapped, such as the control of synchronized networks[7–10], small biological circuits[11] and rate control for communication networks[4–6]. Recent advances towards quantifying the topological characteristics of complex networks[12–16] have shed light on factor (1), prompting us to wonder whether some networks are easier to control than others and how network topology affects a system's controllability. Despite some pioneering conceptual work[17–23] (Supplementary Information, section II), we continue to lack general answers to these questions for large weighted and directed networks, which most commonly emerge in complex systems.

Network controllability

Most real systems are driven by nonlinear processes, but the controllability of nonlinear systems is in many aspects structurally similar to that of linear systems[3], prompting us to start our study using the canonical linear, time-invariant dynamics

$$\frac{d\mathbf{x}(t)}{dt} = A\mathbf{x}(t) + B\mathbf{u}(t) \tag{1}$$

where the vector $\mathbf{x}(t) = (x_1(t), \ldots, x_N(t))^T$ captures the state of a system of N nodes at time t. For example, $x_i(t)$ can denote the amount of traffic that passes through a node i in a communication network[24] or transcription factor concentration in a gene regulatory network[25]. The $N \times N$ matrix A describes the system's wiring diagram and the interaction strength between the components, for example the traffic on individual communication links or the strength of a regulatory interaction. Finally, B is the $N \times M$ input matrix ($M \leq N$) that identifies the nodes controlled by an outside controller. The system is controlled using the time-dependent input vector $\mathbf{u}(t) = (u_1(t), \ldots, u_M(t))^T$ imposed by the controller (Fig. 1a), where in general the same signal $u_i(t)$ can drive multiple nodes. If we wish to control a system, we first need to identify the set of nodes that, if driven by different signals, can offer full control over the network. We will call these 'driver nodes'. We are particularly interested in identifying the minimum number of driver nodes, denoted by N_D, whose control is sufficient to fully control the system's dynamics.

The system described by equation (1) is said to be controllable if it can be driven from any initial state to any desired final state in finite time, which is possible if and only if the $N \times NM$ controllability matrix

$$C = (B, AB, A^2B, \ldots, A^{N-1}B) \tag{2}$$

has full rank, that is

$$\text{rank}(C) = N \tag{3}$$

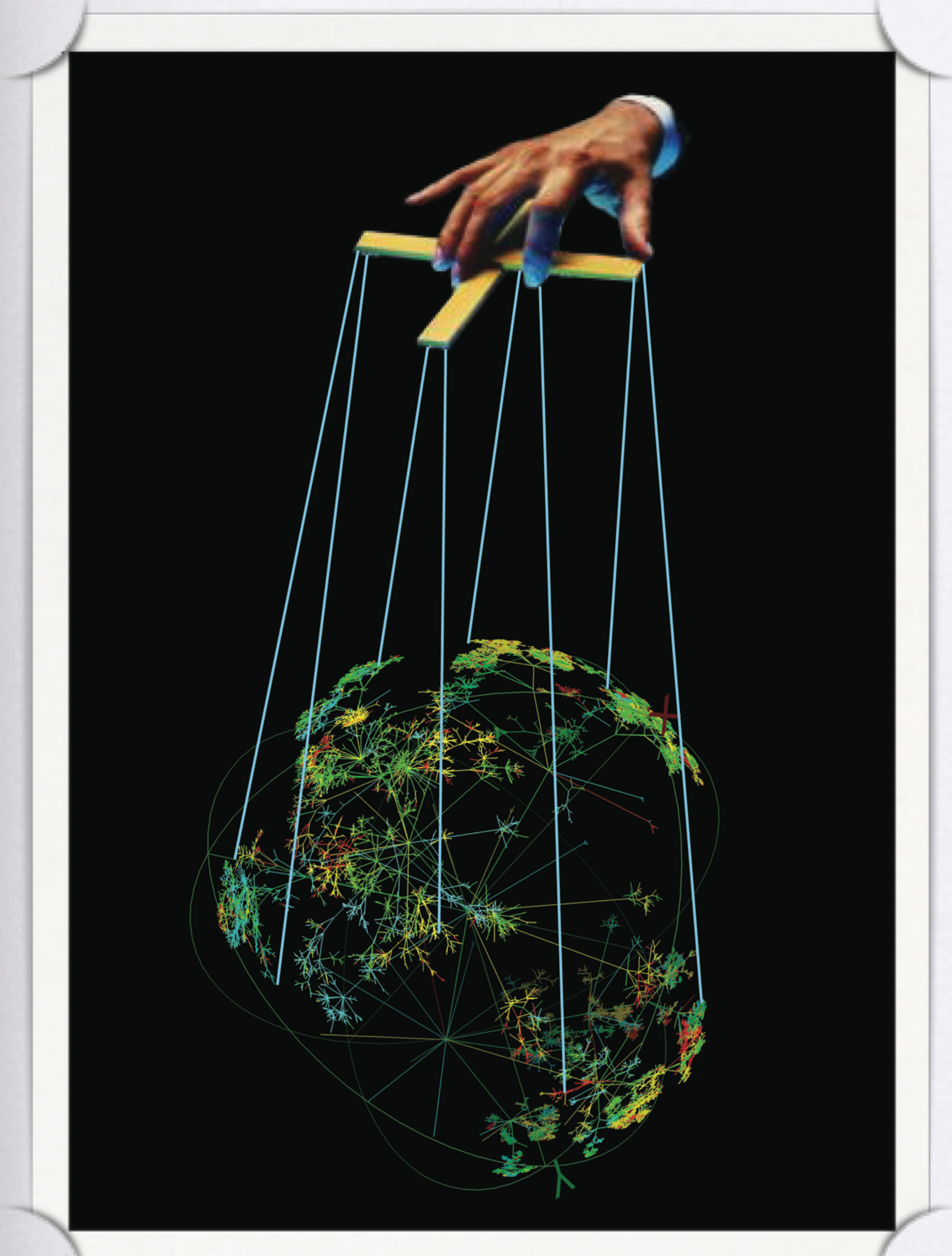
This represents the mathematical condition for controllability, and is called Kalman's controllability rank condition[1,2] (Fig. 1a). In practical terms, controllability can be also posed as follows. Identify the minimum number of driver nodes such that equation (3) is satisfied. For example, equation (3) predicts that controlling node x_1 in Fig. 1b with the input signal u_1 offers full control over the system, as the states of nodes x_1, x_2, x_3 and x_4 are uniquely determined by the signal $u_1(t)$ (Fig. 1c). In contrast, controlling the top node in Fig. 1e is not sufficient for full control, as the difference $a_{31}x_2(t) - a_{21}x_3(t)$ (where a_{ij} are the elements of A) is not uniquely determined by $u_1(t)$ (see Fig. 1f and Supplementary Information section III.A). To gain full control, we must simultaneously control node x_1 and any two nodes among $\{x_2, x_3, x_4\}$ (see Fig. 1h, i for a more complex example).

To apply equations (2) and (3) to an arbitrary network, we need to know the weight of each link (that is, the a_{ij}), which for most real

[1]Center for Complex Network Research and Departments of Physics, Computer Science and Biology, Northeastern University, Boston, Massachusetts 02115, USA. [2]Center for Cancer Systems Biology, Dana-Farber Cancer Institute, Boston, Massachusetts 02115, USA. [3]Nonlinear Systems Laboratory, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA. [4]Department of Mechanical Engineering and Department of Brain and Cognitive Sciences, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA. [5]Department of Medicine, Brigham and Women's Hospital, Harvard Medical School, Boston, Massachusetts 02115, USA.

Motivation

Motivation



- How to control a network with minimum number of nodes?

Motivation

- How to control a network with minimum number of nodes?
- How does the network topology affect its controllability?

Rudolf E. Kalman

J.S.I.A.M. Control
Ser. A, Vol. 1, No. 2
Printed in U.S.A., 1963

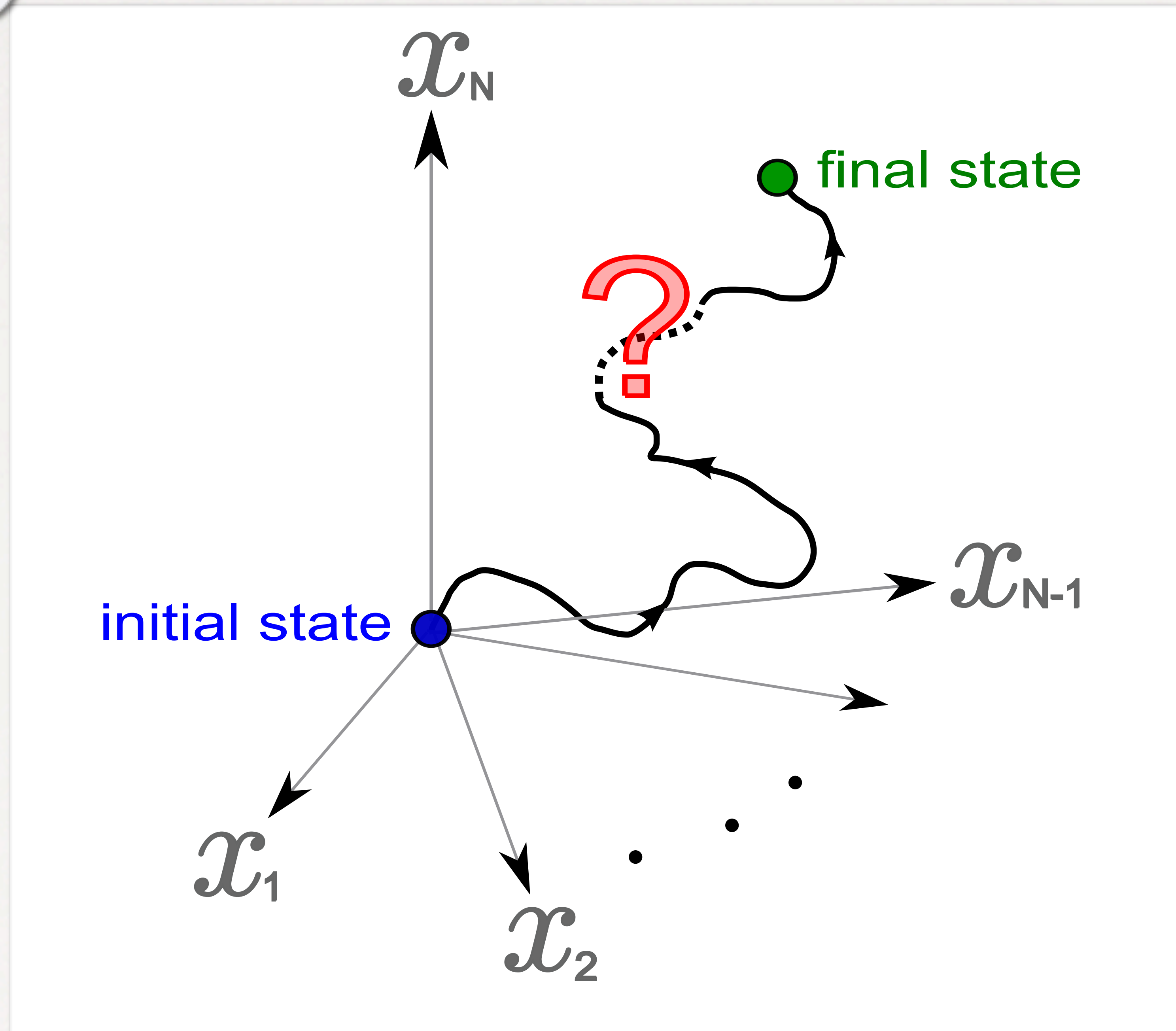
MATHEMATICAL DESCRIPTION OF LINEAR DYNAMICAL SYSTEMS*

R. E. KALMAN†

Abstract. There are two different ways of describing dynamical systems: (i) by means of state variables and (ii) by input/output relations. The first method may be regarded as an axiomatization of Newton's laws of mechanics and is taken to be the basic definition of a system.

It is then shown (in the linear case) that the input/output relations determine only one part of a system, that which is completely observable and completely controllable. Using the theory of controllability and observability, methods are given for calculating irreducible realizations of a given impulse-response matrix. In particular, an explicit procedure is given to determine the minimal number of state variables necessary to realize a given transfer-function matrix. Difficulties arising from the use of reducible realizations are discussed briefly.

Kalman, *J.S.I.A.M. Control* (1963)



Controllable: the system can be driven from any initial state to any desired final state in finite time.

Linear System

Kalman, *J.S.I.A.M. Control* (1963)

Linear System

- **Linear Time-Invariant System**

$$\dot{\mathbf{x}}(t) = \mathbf{A}\,\mathbf{x}(t) + \mathbf{B}\,\mathbf{u}(t)$$

$\mathbf{x}(t) \in \mathbb{R}^{N\times 1}$: state vector.

$\mathbf{u}(t) \in \mathbb{R}^{M\times 1}$: input signals ($M \leq N$).

$\mathbf{A} \in \mathbb{R}^{N\times N}$: state matrix

(weighted wiring diagram).

$\mathbf{B} \in \mathbb{R}^{N\times M}$: input matrix

($\Rightarrow$ control configuration).

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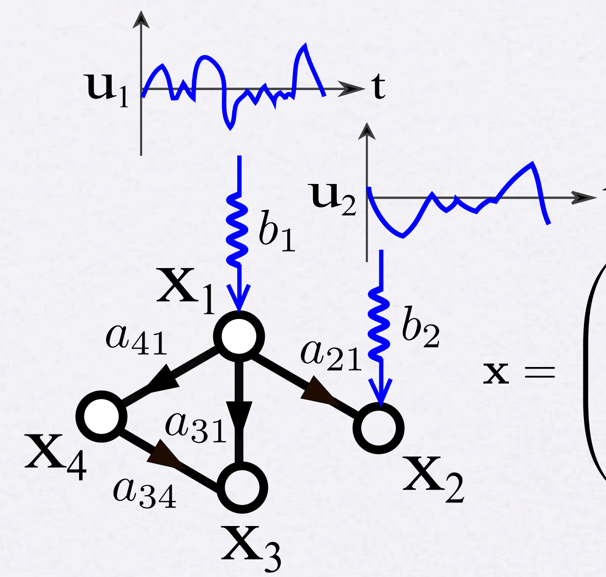

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix};\ \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_{21} & 0 & 0 & 0 \\ a_{31} & 0 & 0 & a_{34} \\ a_{41} & 0 & 0 & 0 \end{pmatrix};\ \mathbf{B} = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix};$$

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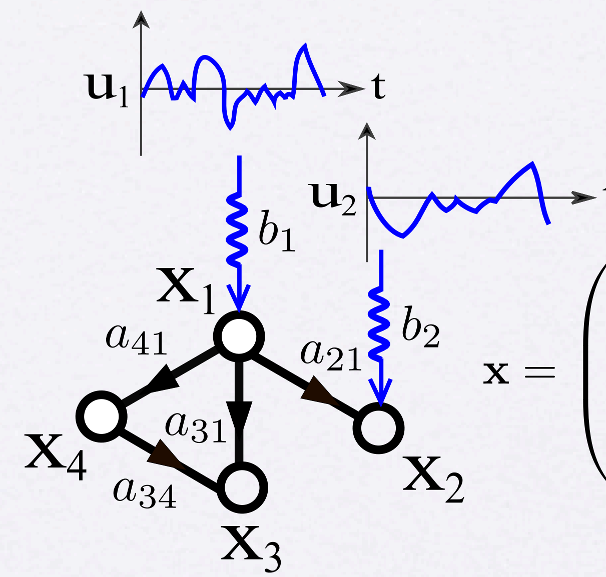

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- **Kalman's Rank Condition**:

The LTI system is controllable iff its controllability matrix has full rank.

$$\boxed{\operatorname{rank}\mathbf{C} = N}$$

$$\mathbf{C} = [\mathbf{B}, \mathbf{A}\,\mathbf{B}, \mathbf{A}^2\,\mathbf{B}, \cdots, \mathbf{A}^{N-1}\,\mathbf{B}]$$

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Example 1: Controllable

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Consider a single-input discrete-time system ($N = 3$)

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t)$$

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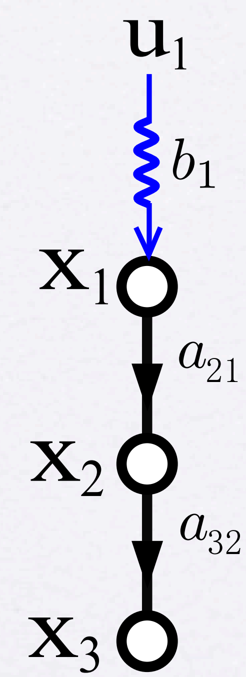

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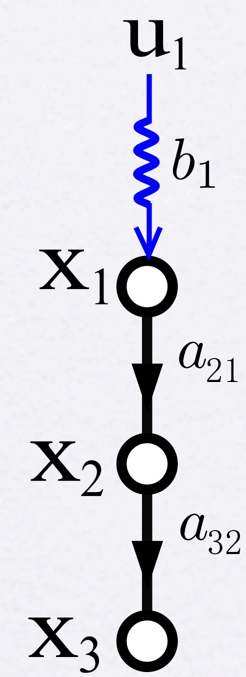

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Assume $\mathbf{x}(t = 0) = \mathbf{0}$, then the state sequence is given by

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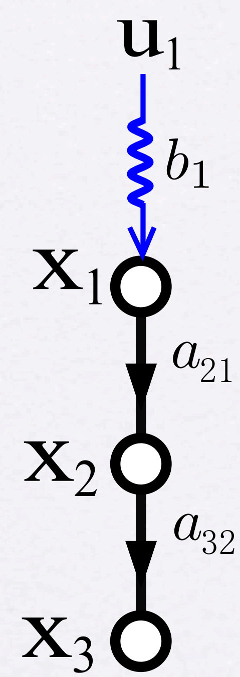

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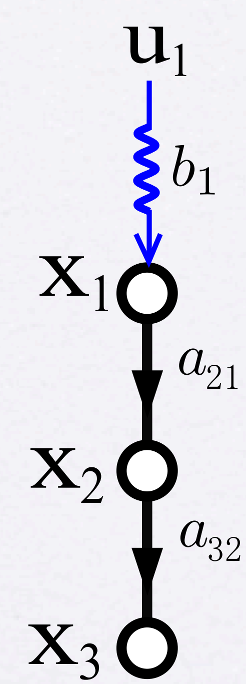

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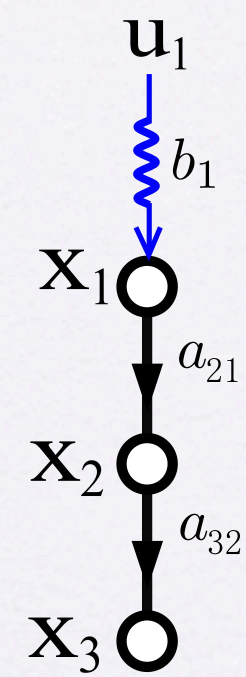

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Controllability Matrix: $\mathbf{C} = [\mathbf{b}, \mathbf{Ab}, \mathbf{A^2b}]$

Since C has full rank, with suitable choice of input signals, we can explore the whole state space.

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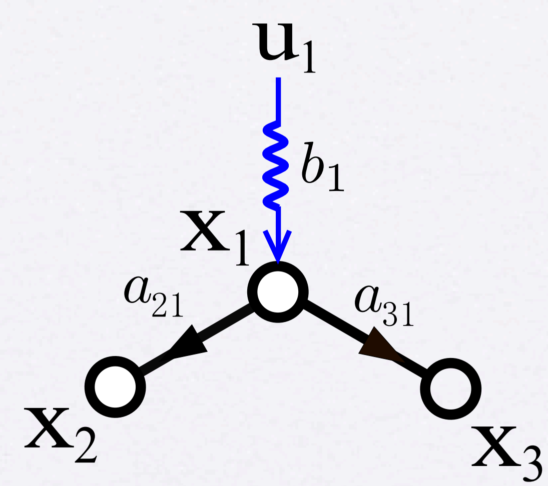

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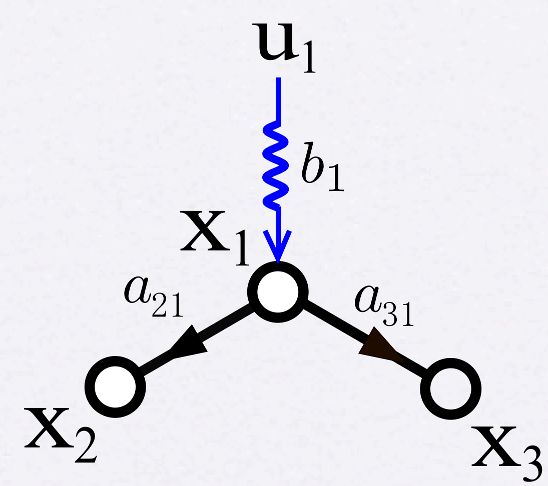

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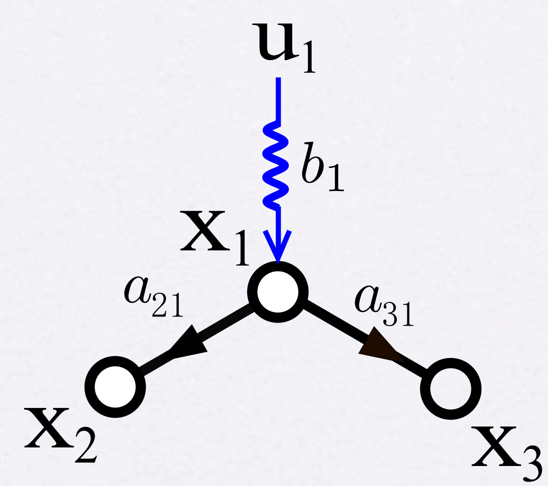

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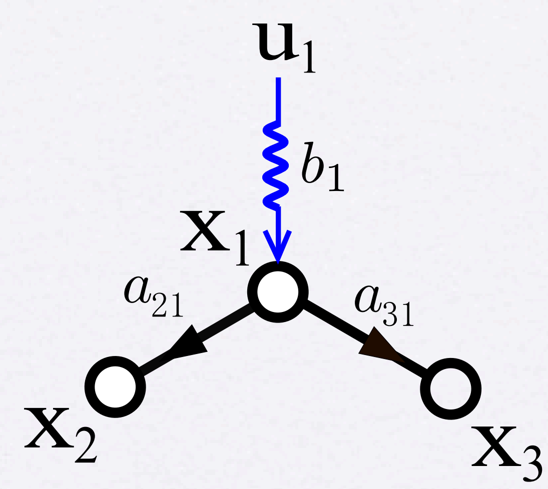

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$$\text{Controllability Matrix: } \mathbf{C} = [\mathbf{b}, \mathbf{A}\mathbf{b}, \mathbf{A}^2\mathbf{b}]$$

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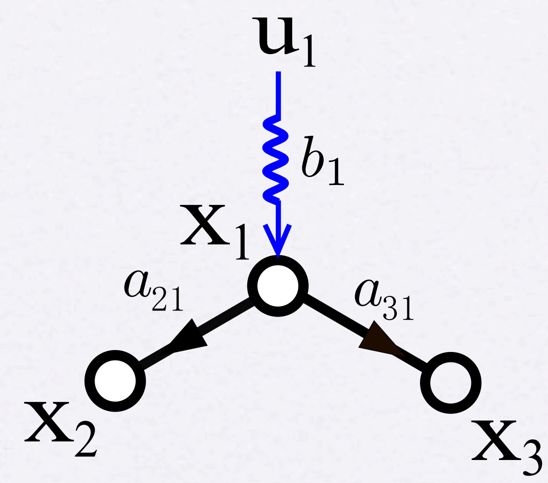

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Since C is rank deficient, no matter how we tune input signals, we can never explore the whole state space.

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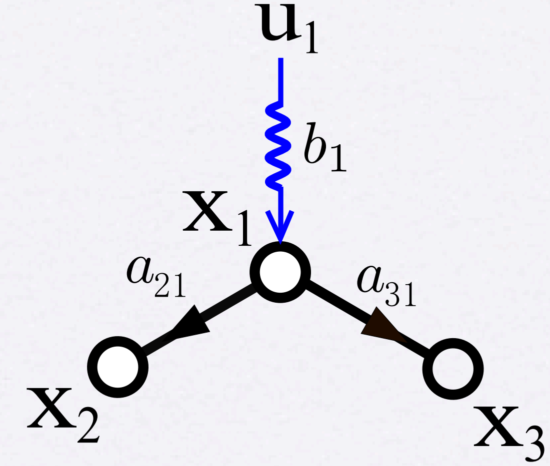
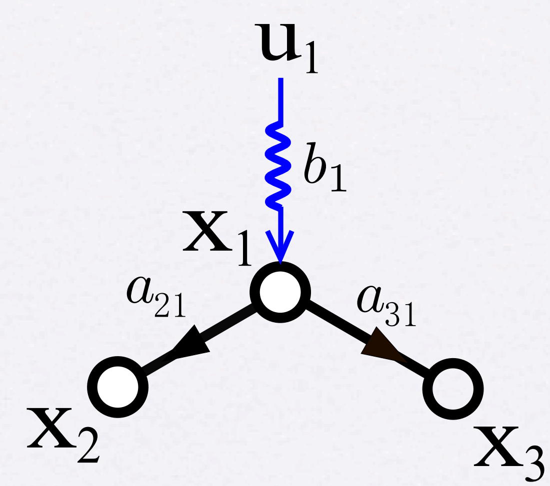


Example 2: Uncontrollable

$$\mathbf{x}(t+1) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t)$$

$$\Leftrightarrow \quad \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} u(t).$$



$$\Leftrightarrow \quad \begin{cases} x_1(t+1) &= \quad bu(t) \\ x_2(t+1) &= \quad a_{21}x_1(t) \\ x_3(t+1) &= \quad a_{31}x_1(t) \end{cases}$$

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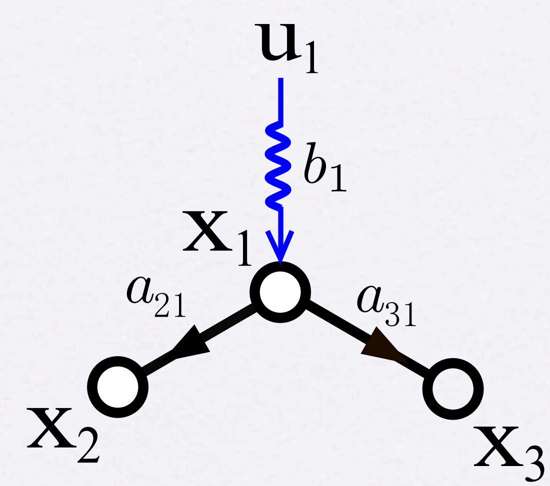
$$\Leftrightarrow \quad \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ x_3(t+1) \end{bmatrix} = \begin{bmatrix} 0 & 0 & \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} b \\ 0 \\ 0 \end{bmatrix} u(t).$$



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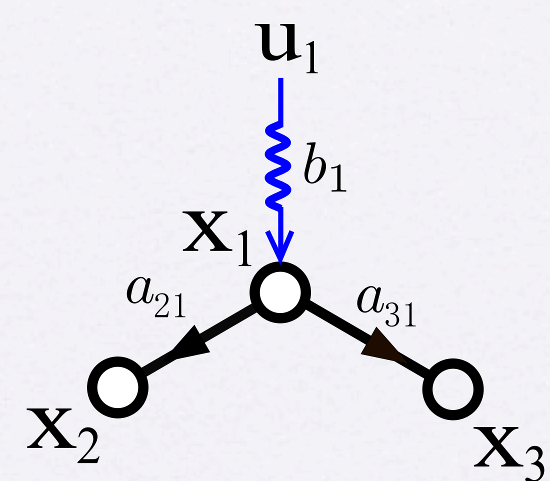

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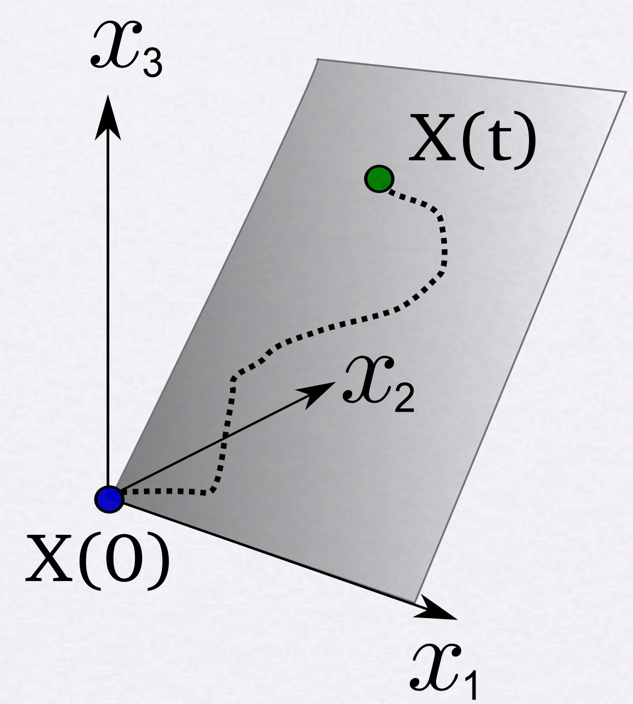

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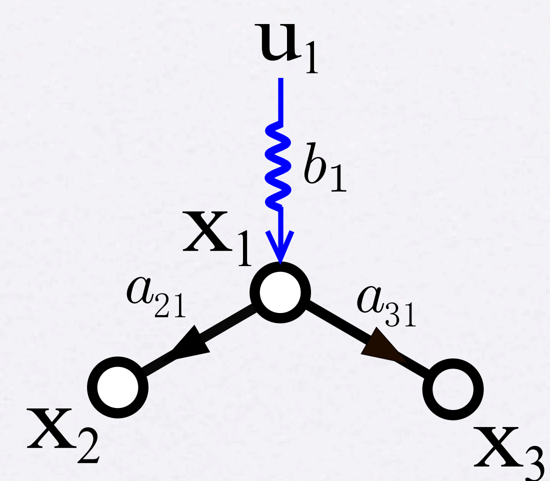

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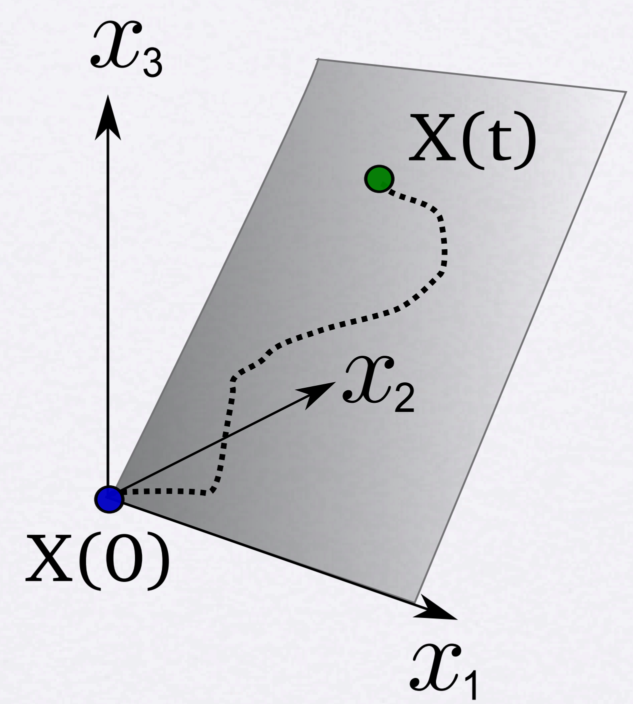

The system is stuck in a plane in the state space.

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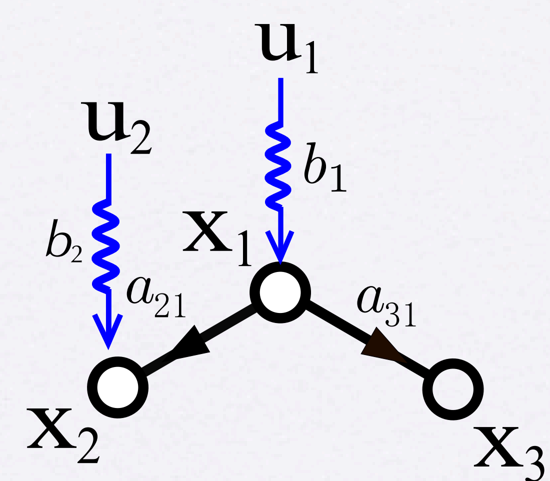

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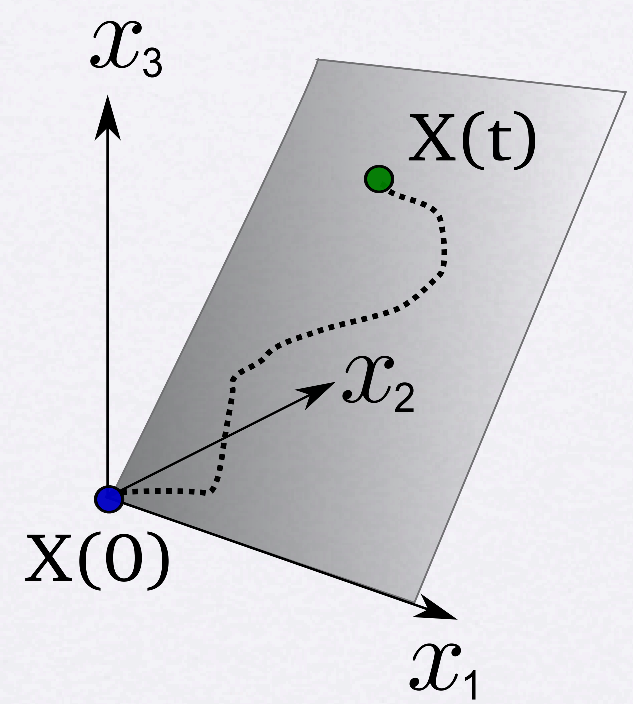

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Questions

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For an arbitrary complex network:

- What's the minimum number of driver nodes (N_D) ?
- How to efficiently identify them?
- Which network characteristics determine N_D ?

Difficulties

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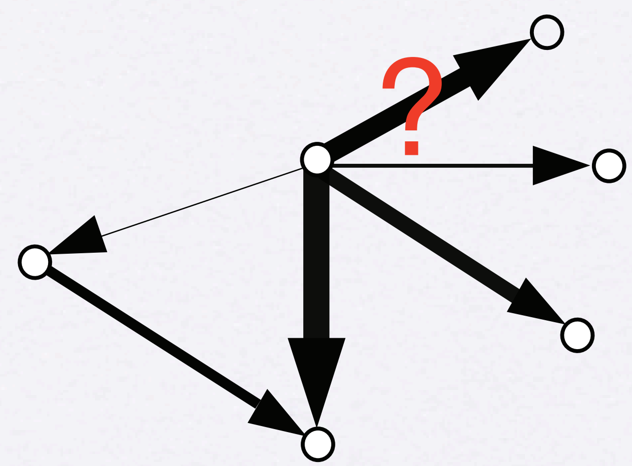
- Parameters (link weights): usually unknown.



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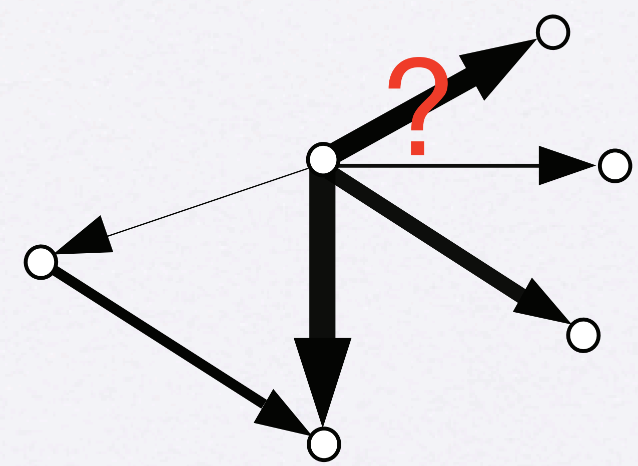

- Kalman's rank condition: hard to check for large system (C is of dimension *N*x*NM*).

$$\boxed{\operatorname{rank} \mathbf{C} = N}$$

$$\mathbf{C} = [\mathbf{B}, \mathbf{A}\,\mathbf{B}, \mathbf{A}^2\,\mathbf{B}, \cdots, \mathbf{A}^{N-1}\,\mathbf{B}]$$

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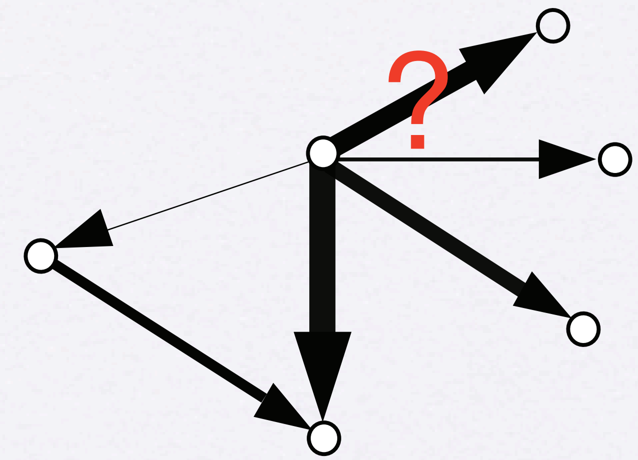

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- **If brute-force search: (2^N-1) combinations.**

$$\binom{N}{1} + \binom{N}{2} + \cdots + \binom{N}{N} = 2^N - 1$$

Solution

Solution

- **Structural Controllability**
- **Maximum Matching**

Liu *et al.*, *Nature* (in press)

IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. AC-19, NO. 3, JUNE 1974

Structural Controllability

CHING-TAI LIN, MEMBER, IEEE

Abstract—The new concepts of "structure" and "structural controllability" for a linear time-invariant control system (described by a pair (A,b)) are defined and studied. The physical justification of these concepts and examples are also given.

The graph of a pair (A,b) is also defined. This gives another way of describing the structure of this pair. The property of structural controllability is reduced to a property of the graph of the pair (A,b). To do this, the basic concept of a "cactus" and the related concept of a "precactus" are introduced. The main result of this paper states that the pair (A,b) is structurally controllable if an only if the graph of (A,b) is "spanned by a cactus." The result is also expressed in a more conventional way, in terms of some properties of the pair (A,b).

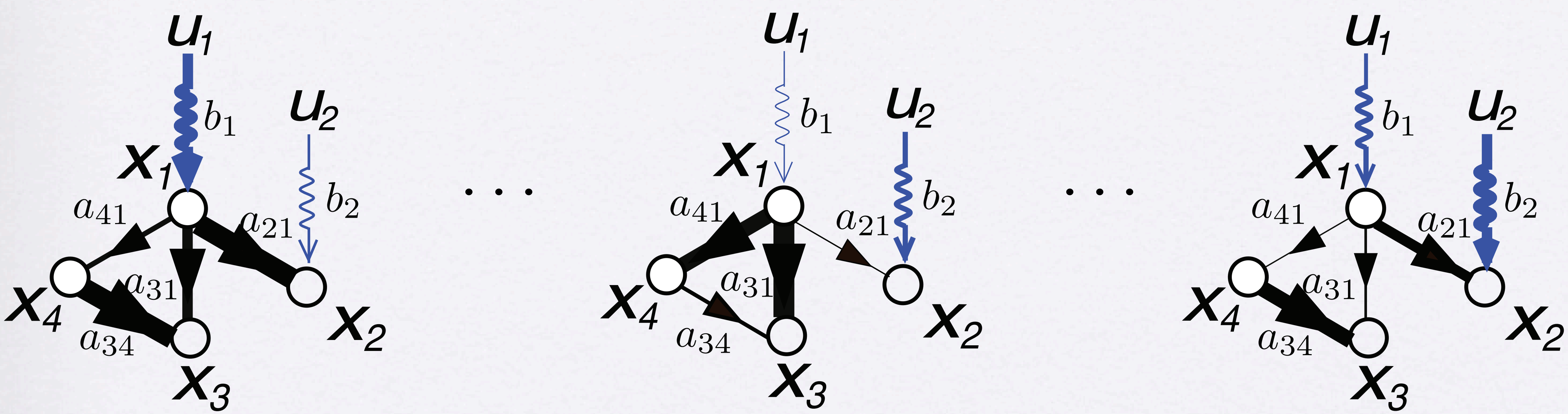
ing entry of (Ab) is also fixed (zero). Then one defines the pair (A_0,b_0) to be *structurally controllable* if and only if there exists a completely controllable pair (A,b) which has the same structure as (A_0,b_0).

The concept of "structural controllability" of a pair (A_0,b_0) makes the meaning of controllability (in the usual sense) more complete from the physical point of view. In fact, it is preferred whenever (A_0,b_0) represents an actual physical system (that involves parameters only approximately determined). Actually, the completely controllable pair (A,b) can be considered as "physically

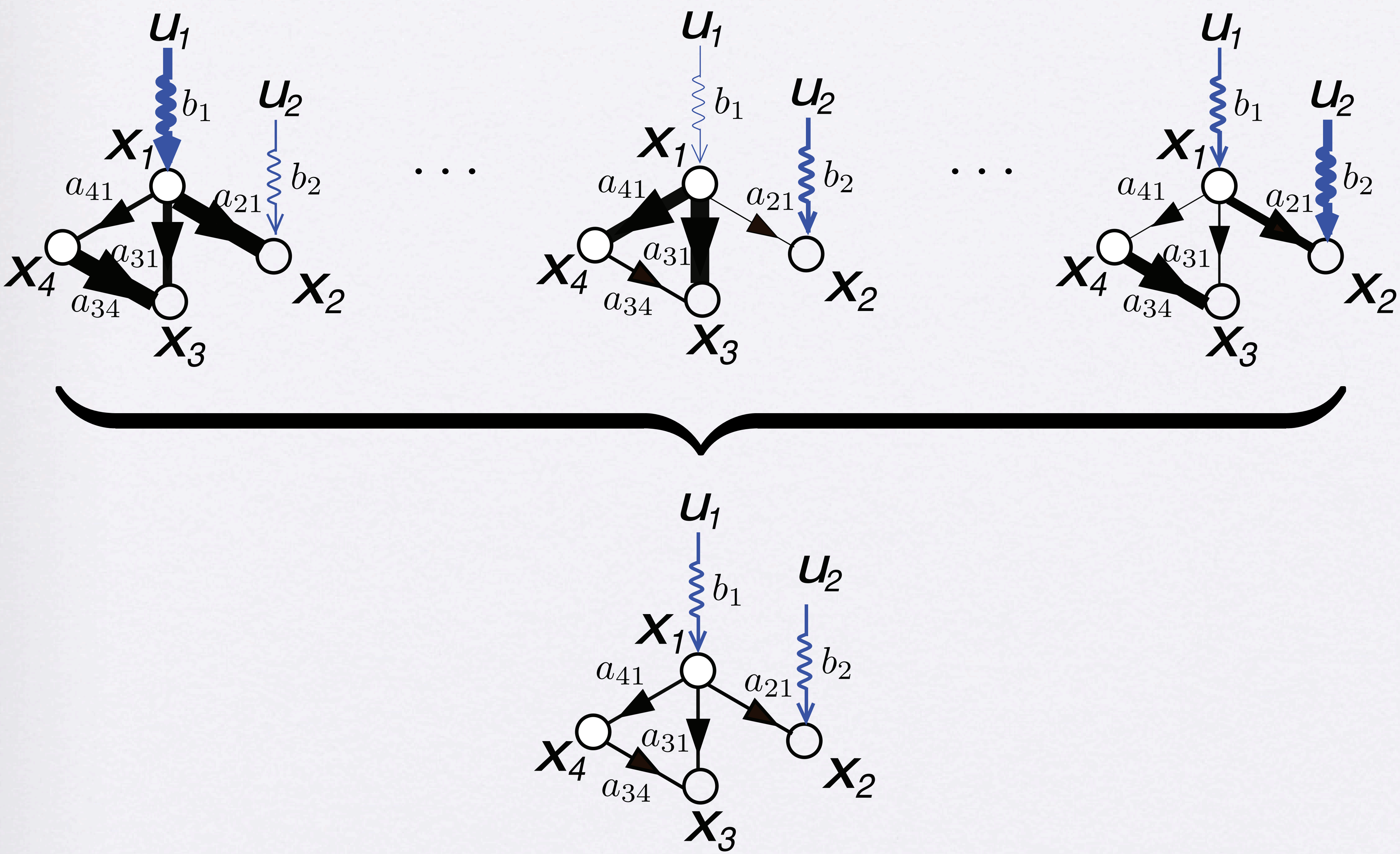
Lin, *IEEE Trans. Auto. Contr.* (1974)

Structural Controllability

Structural Controllability



Structural Controllability



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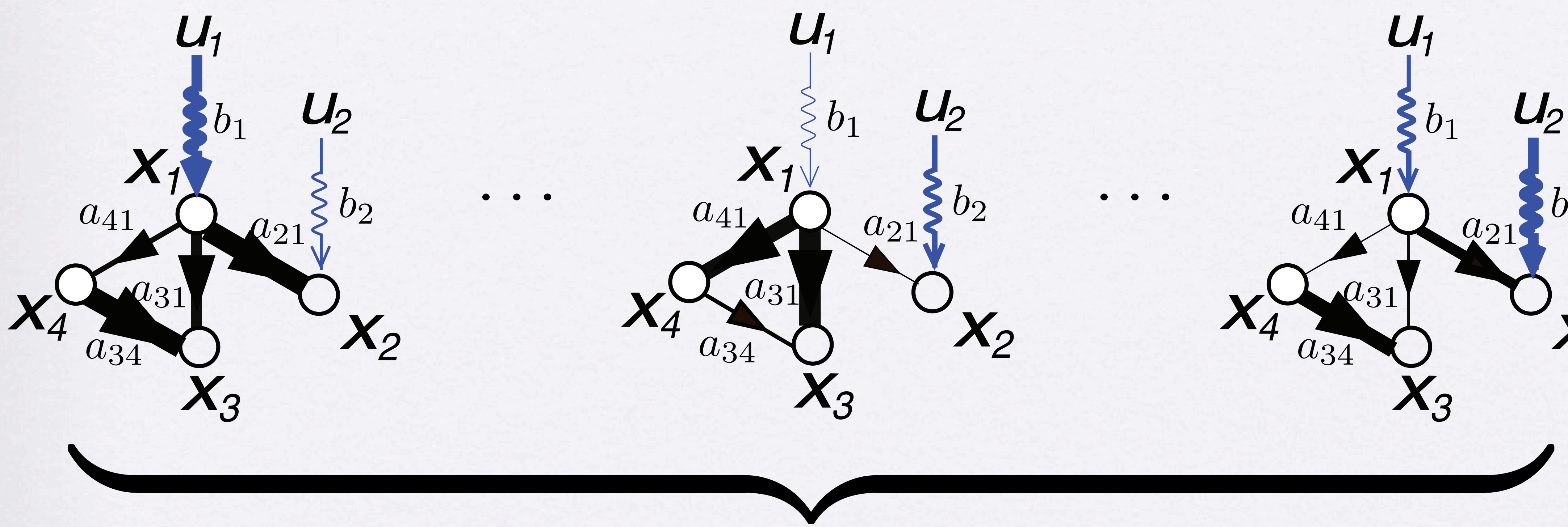
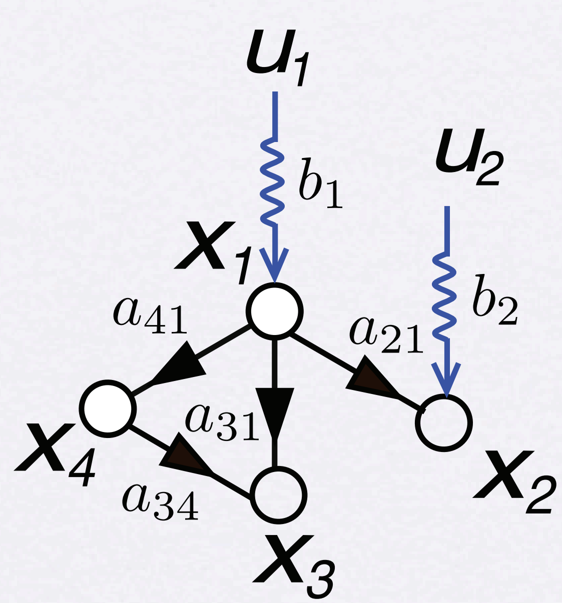
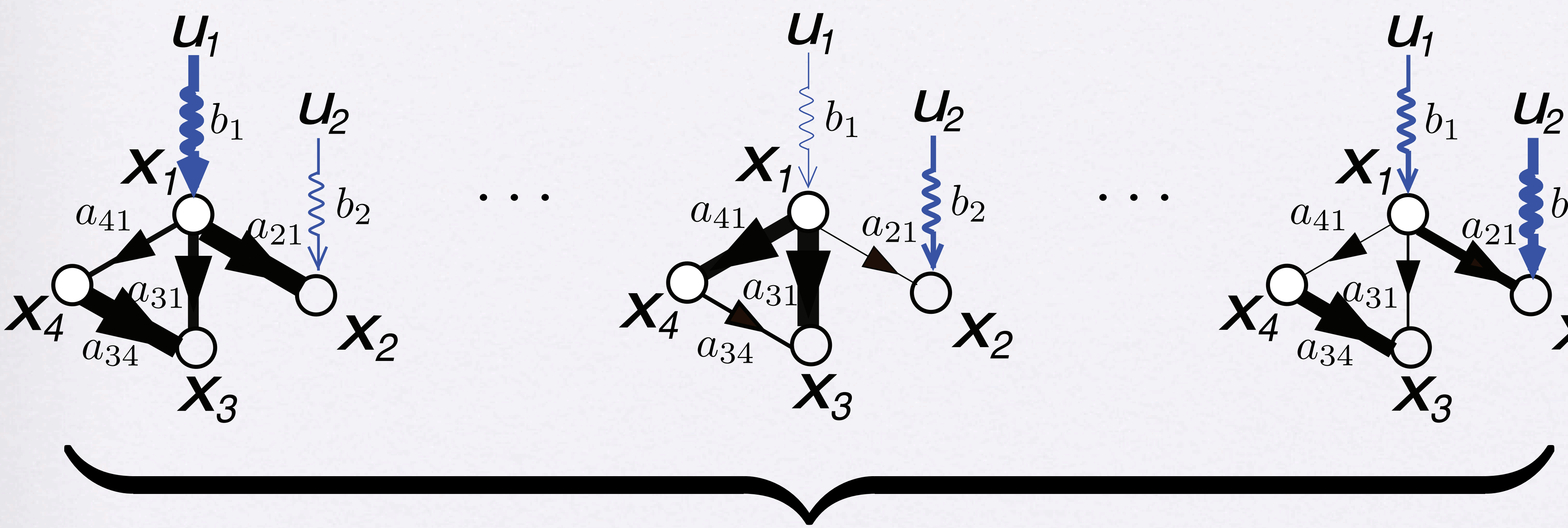


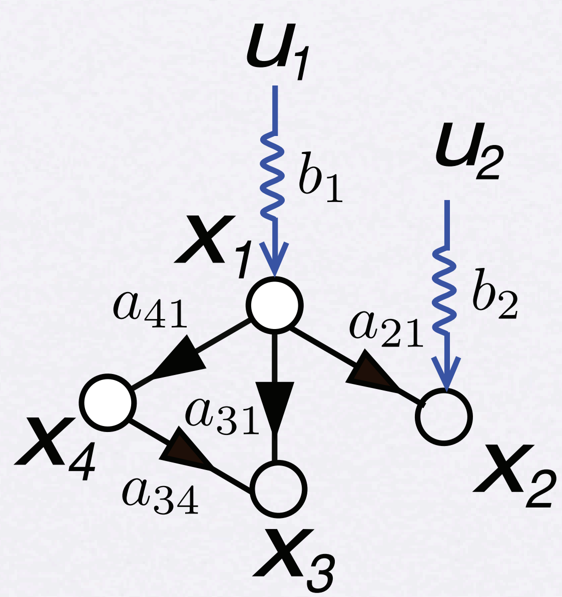


$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a_{21} & 0 & 0 & 0 \\ a_{31} & 0 & 0 & a_{34} \\ a_{41} & 0 & 0 & 0 \end{pmatrix}; \quad \mathbf{B} = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix};$$

Structured Matrix: Its elements are either **fixed zeros** or **independent free** parameters.

Structural Controllability





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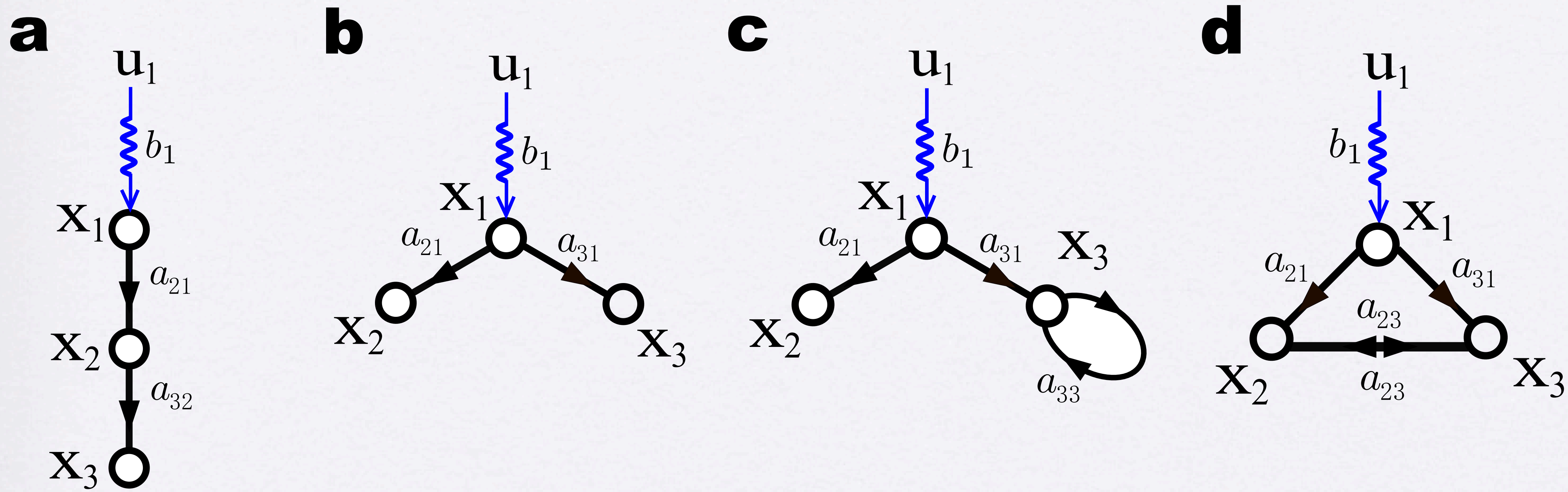
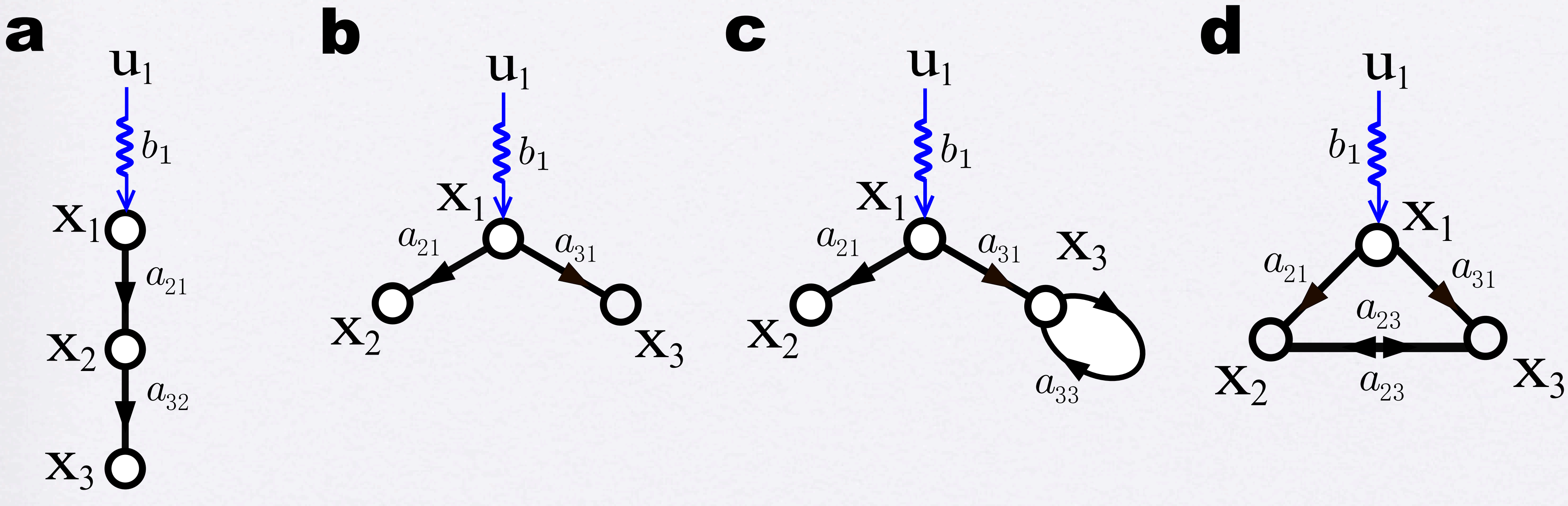
Only the structure matters!

Examples



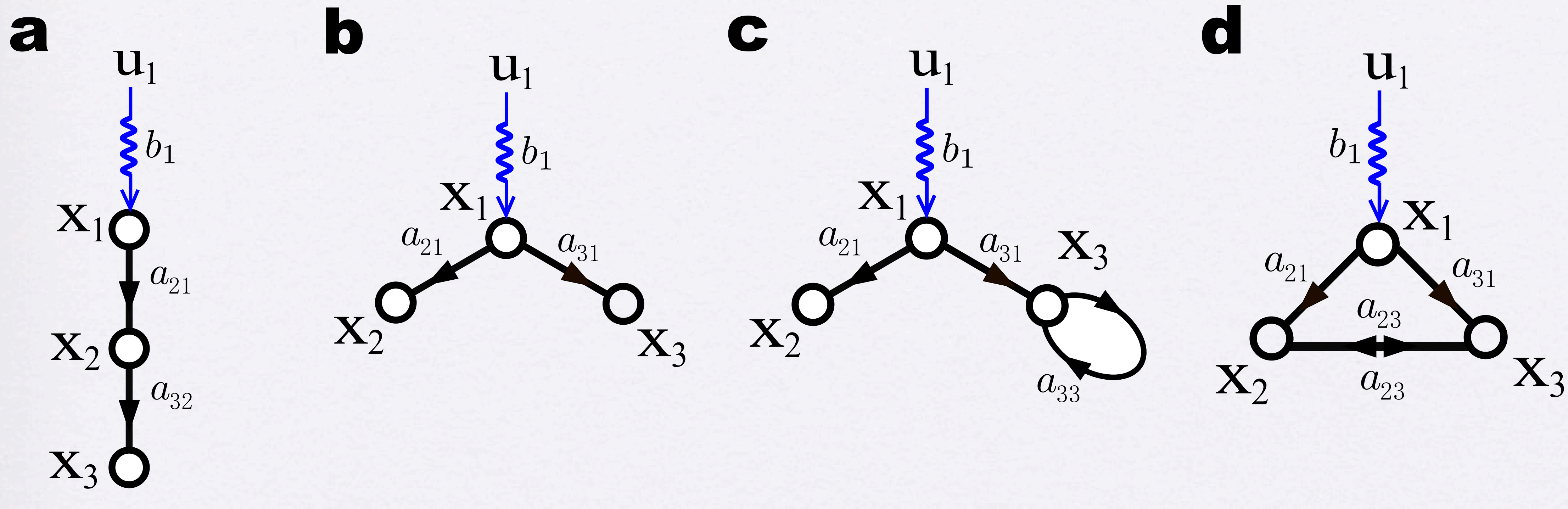
a
u_1
b_1
x_1
a_{21}
x_2
a_{32}
x_3
b
u_1
b_1
x_1
a_{21}
a_{31}
x_2
x_3
c
u_1
b_1
x_1
a_{21}
a_{31}
x_3
x_2
a_{33}
d
u_1
b_1
x_1
a_{21}
a_{31}
a_{23}
x_2
a_{23}
x_3

Examples



$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & 0 \\ a_{31} & 0 & a_{33} \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{bmatrix}$$

Examples



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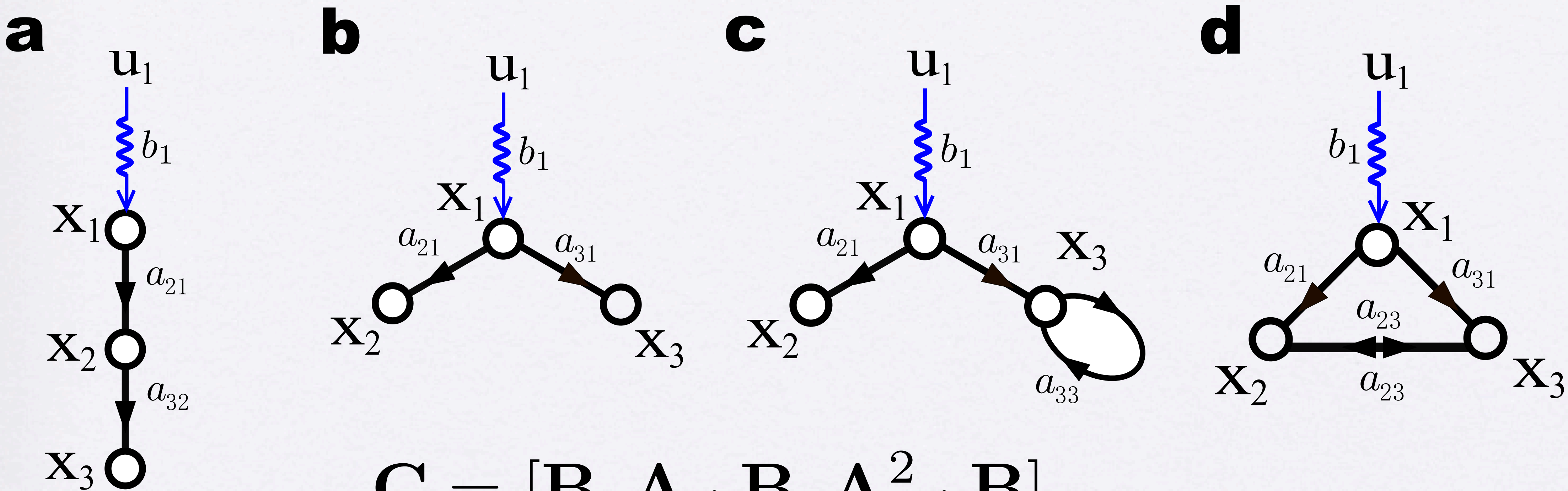
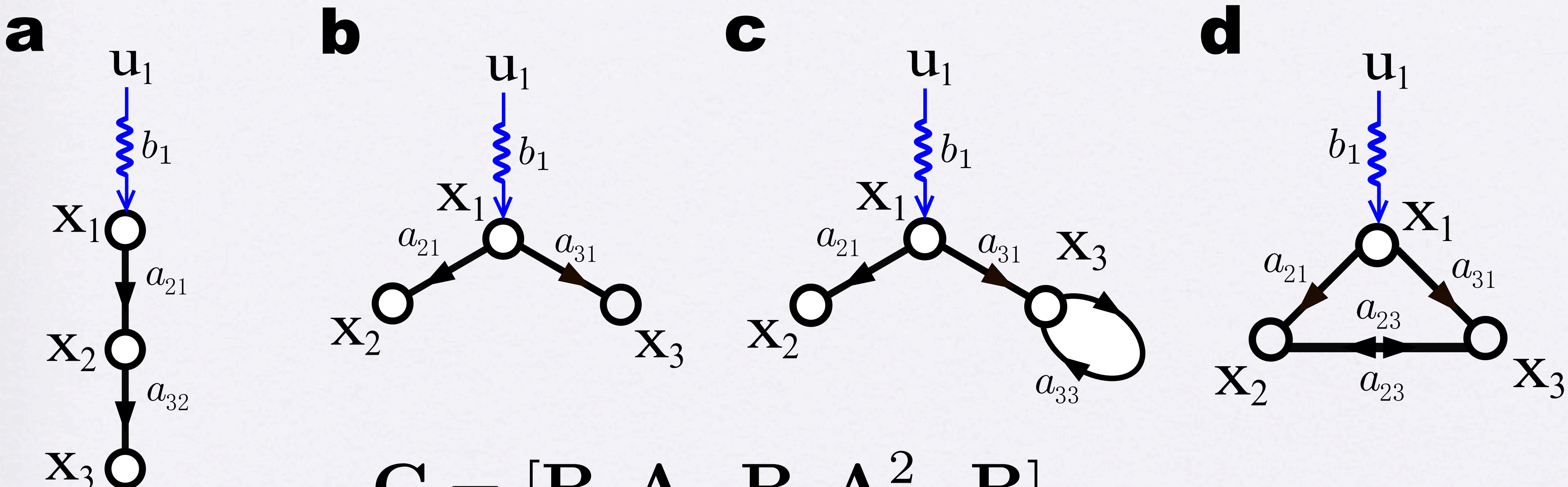
Examples



$$\mathbf{C} = [\mathbf{B}, \mathbf{A} \cdot \mathbf{B}, \mathbf{A}^2 \cdot \mathbf{B}]$$

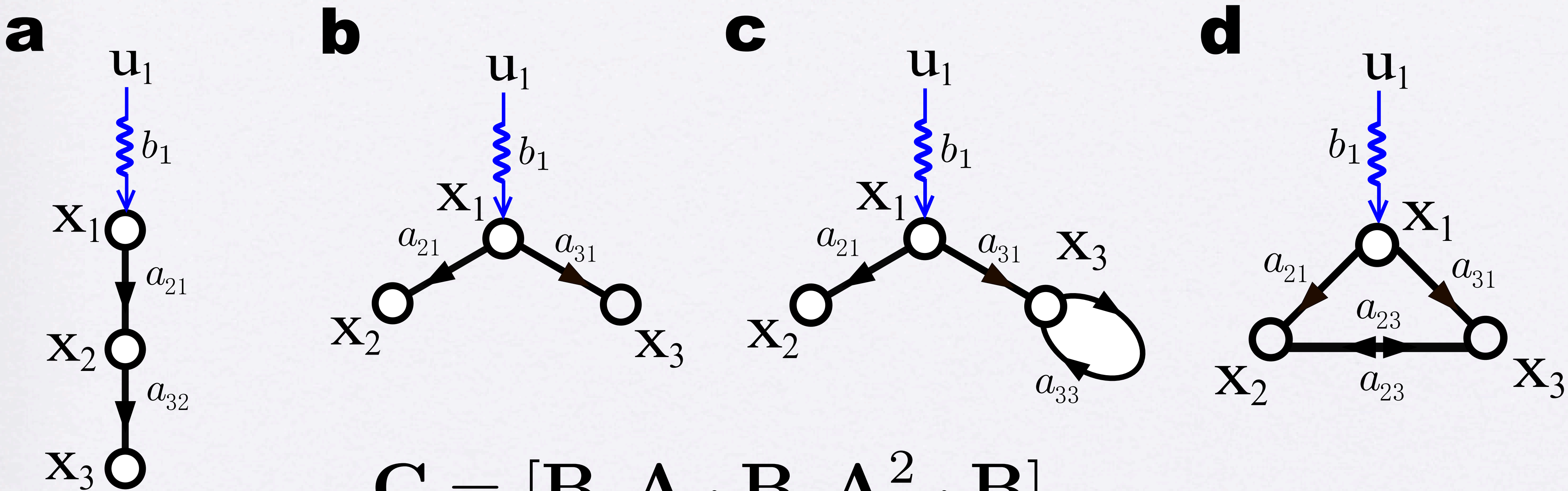
Examples



$$\mathbf{C} = [\mathbf{B}, \mathbf{A} \cdot \mathbf{B}, \mathbf{A}^2 \cdot \mathbf{B}]$$

$$b_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{21} & 0 \\ 0 & 0 & a_{32}a_{21} \end{bmatrix}, \quad b_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{21} & 0 \\ 0 & a_{31} & 0 \end{bmatrix}, \quad b_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{21} & 0 \\ 0 & a_{31} & a_{33}a_{31} \end{bmatrix}, \quad b_1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{21} & a_{23}a_{31} \\ 0 & a_{31} & a_{32}a_{21} \end{bmatrix}$$

Examples



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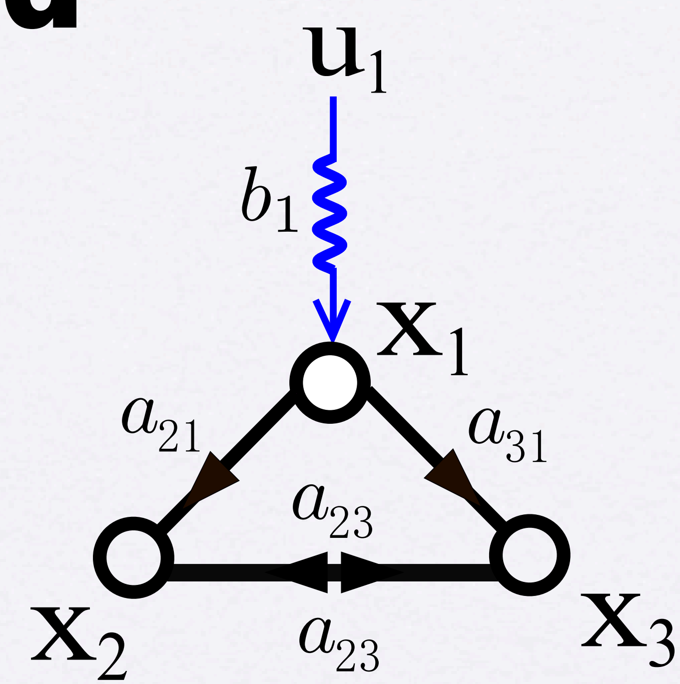
rank **C** = 3 = N
controllable

rank **C** = 2< N =3
uncontrollable

rank **C** = 3 = N
controllable

rank **C** = ?
controllable ?

d



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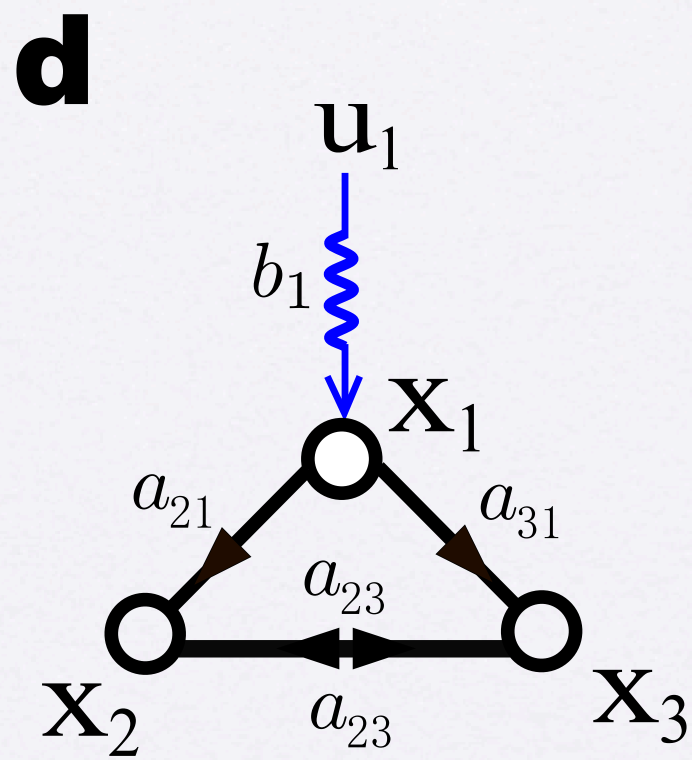

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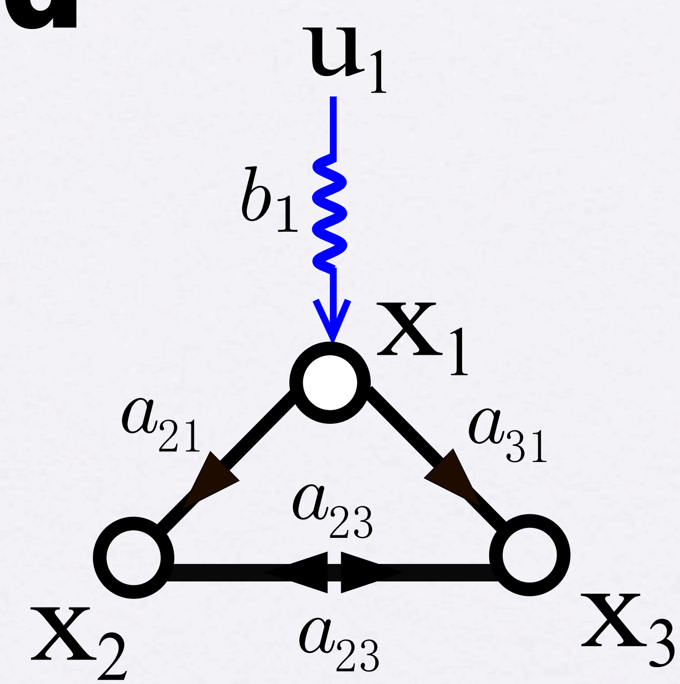
then $\operatorname{rank}(\mathbf{C}) = 2 < N \Rightarrow$ uncontrollable! However, this case is pathological. In most cases, the system is controllable.

d



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structurally controllable

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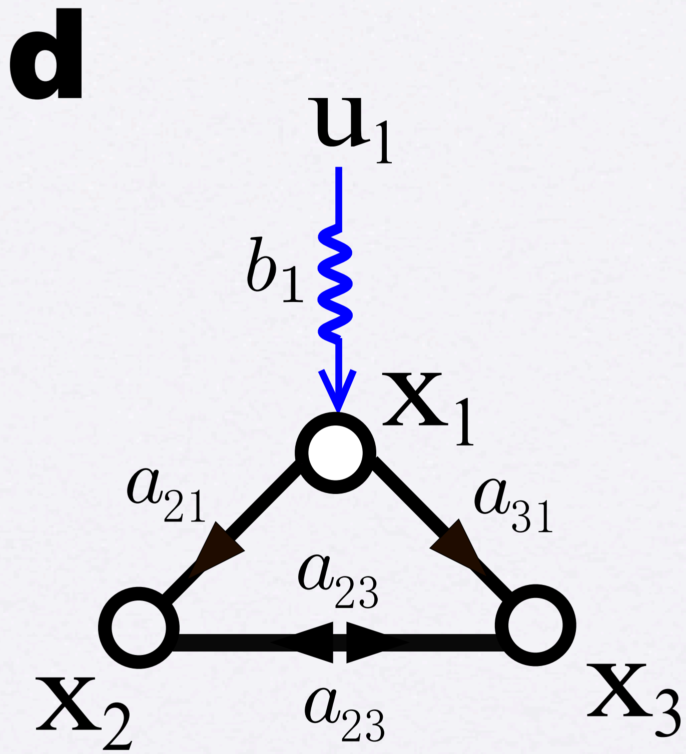
Structurally Controllable ≈ Controllable



$$
\begin{array}{cc}
 & \\
b_1 & \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_{21} & a_{23}a_{31} \\ 0 & a_{31} & a_{32}a_{21} \end{bmatrix}
\end{array}
$$

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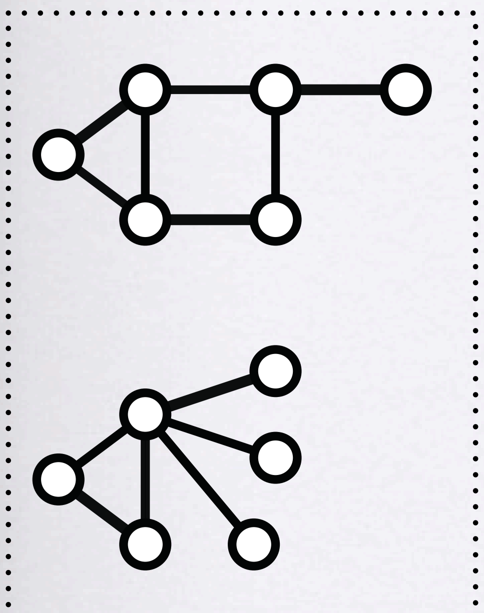
Matching

Lovász & Plummer, *Matching Theory*

Matching

Undirected network



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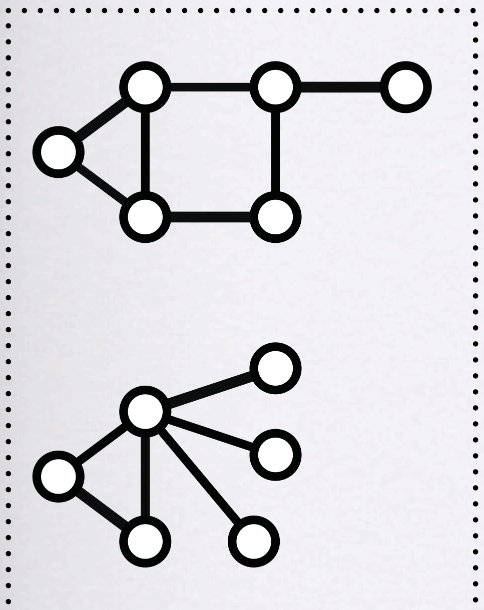
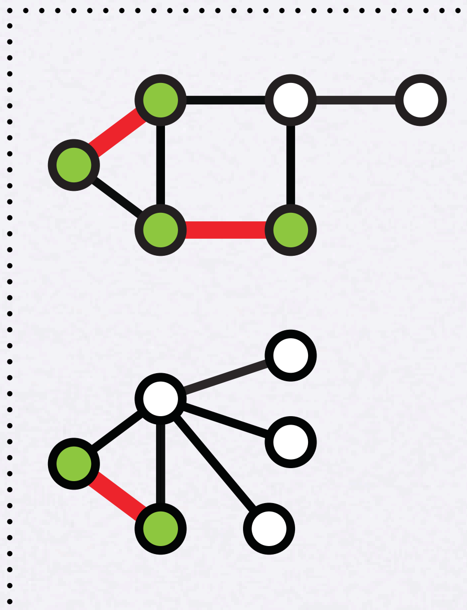


Matching
a set of edges without common vertices



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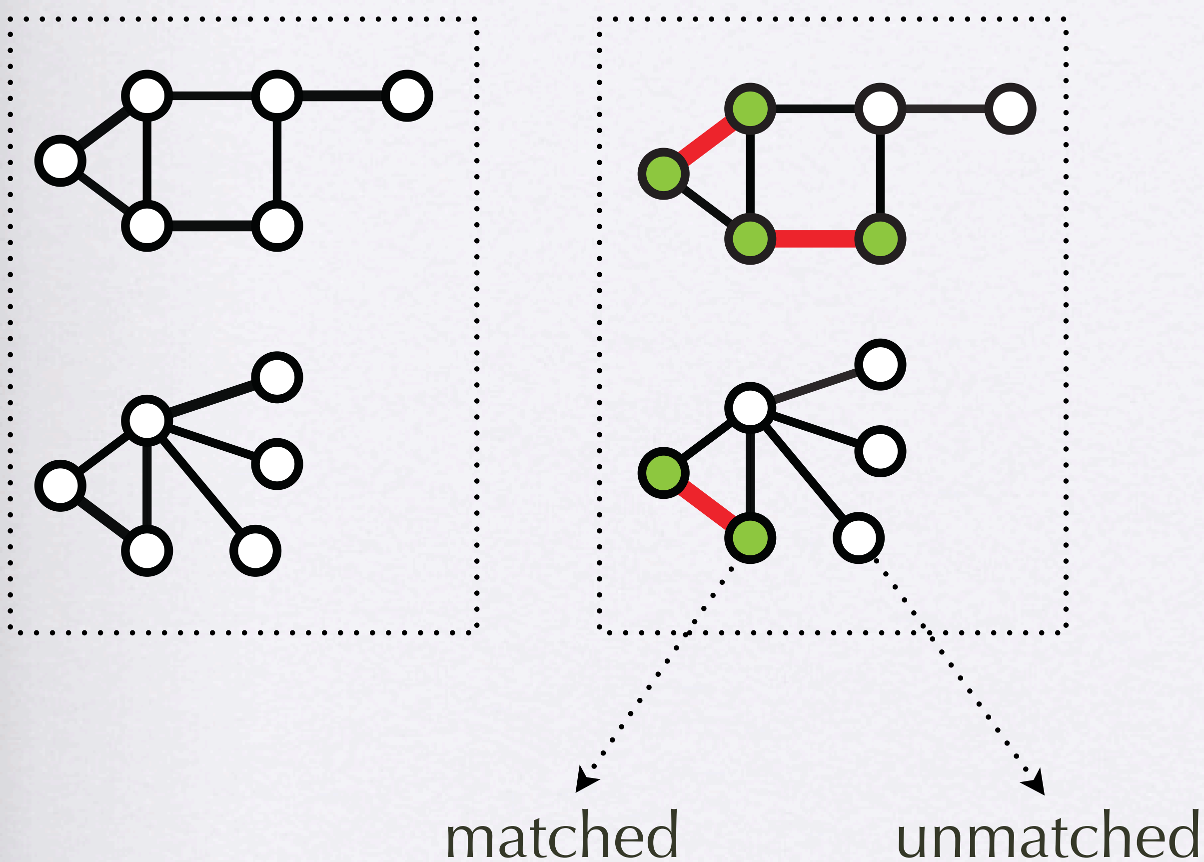

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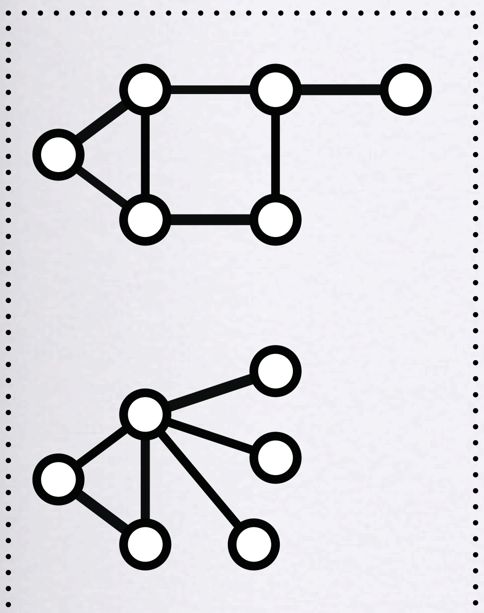

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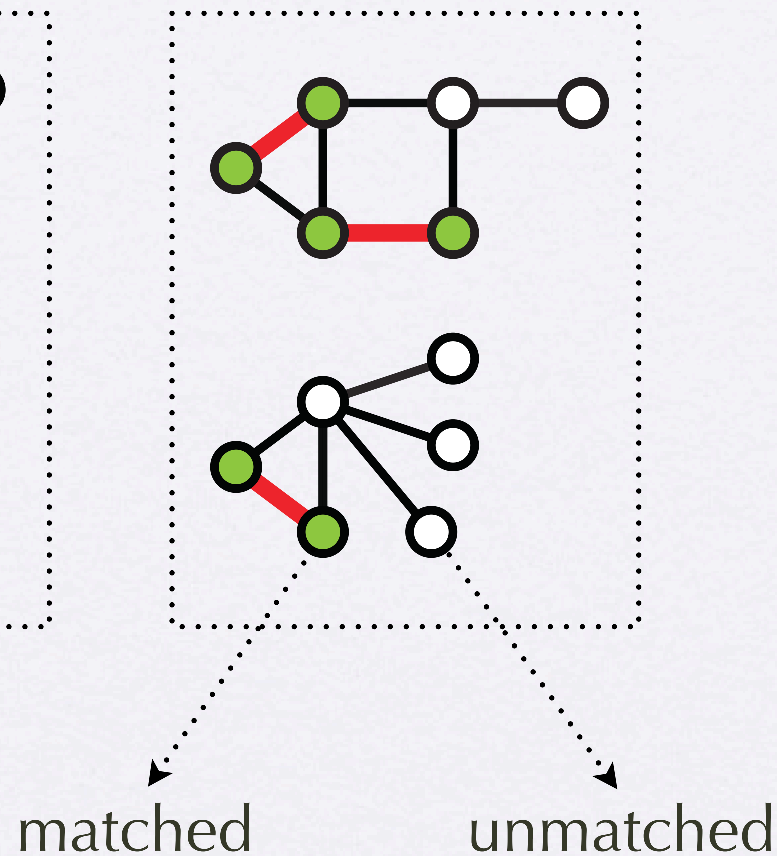
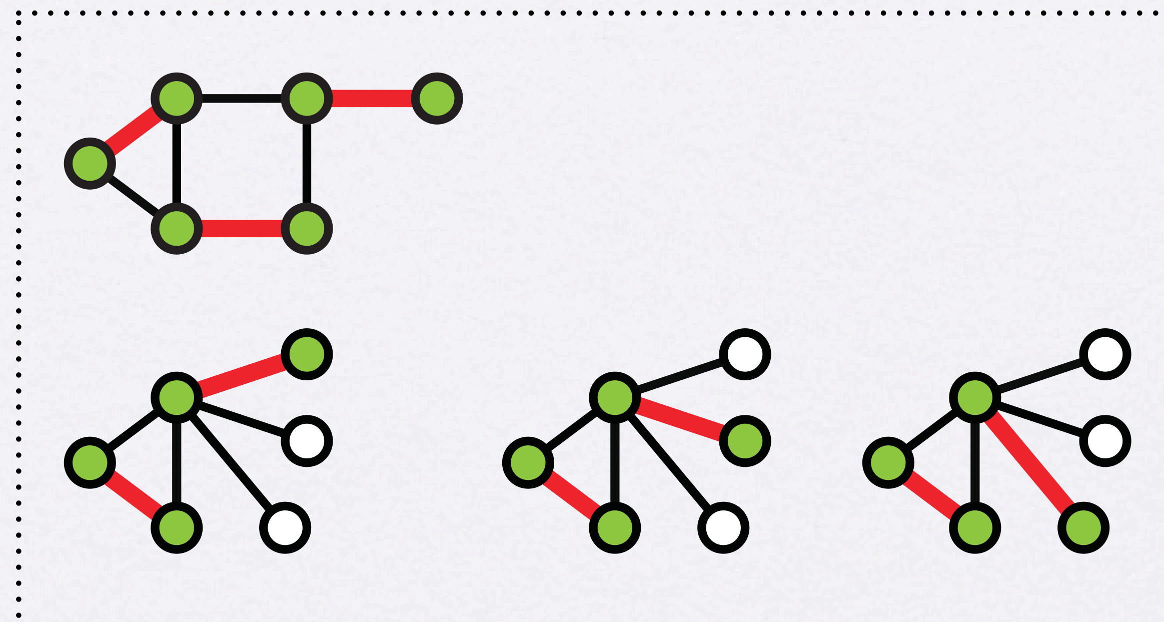


Maximum Matching
a matching of the largest size



Lovász & Plummer, *Matching Theory*

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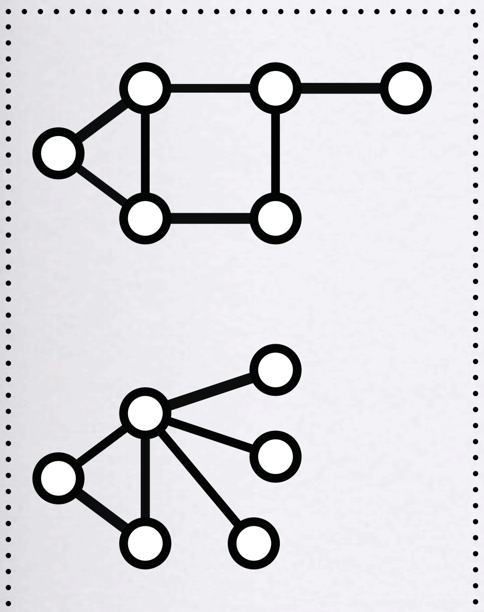

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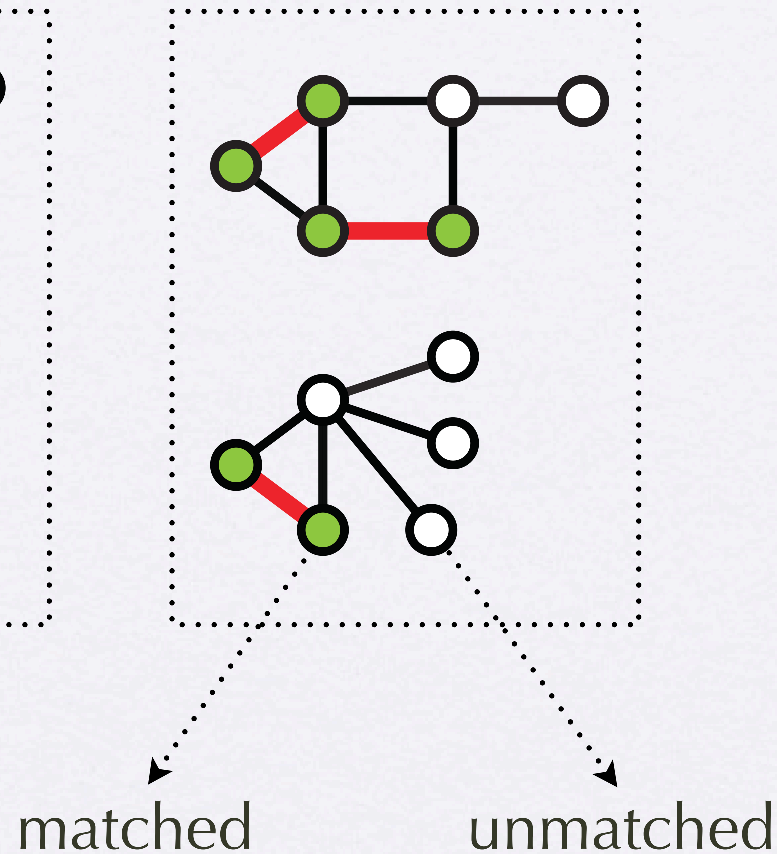

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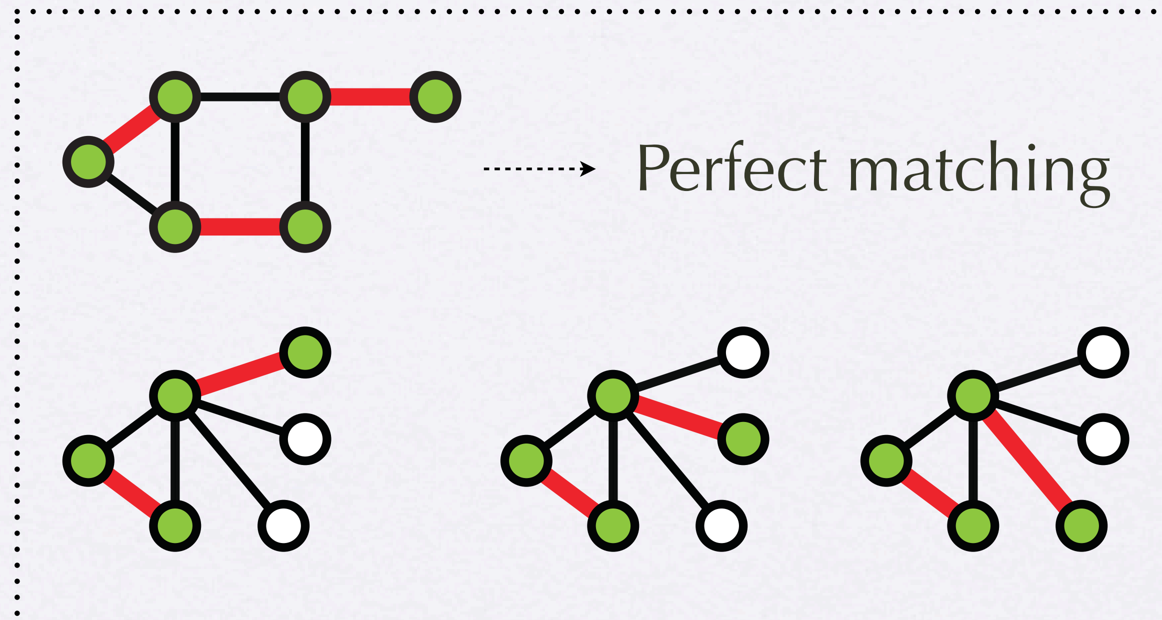

Lovász & Plummer, *Matching Theory*

Matching in Directed Networks

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Matching : a set of directed edges without common heads or tails.

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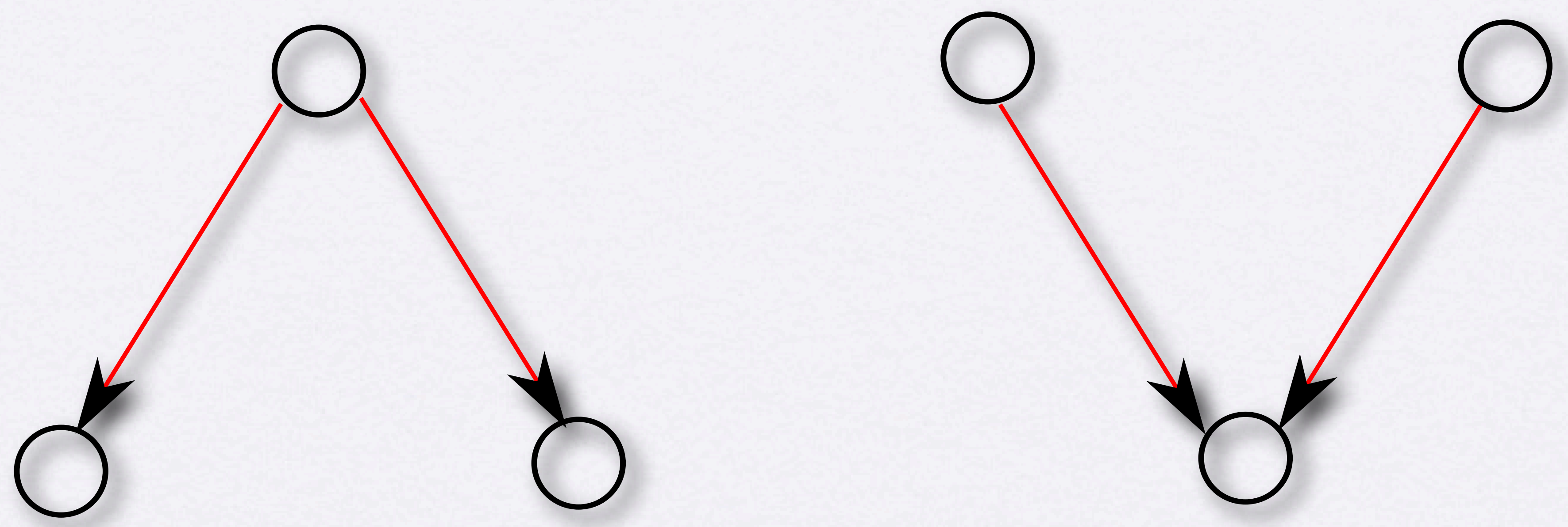
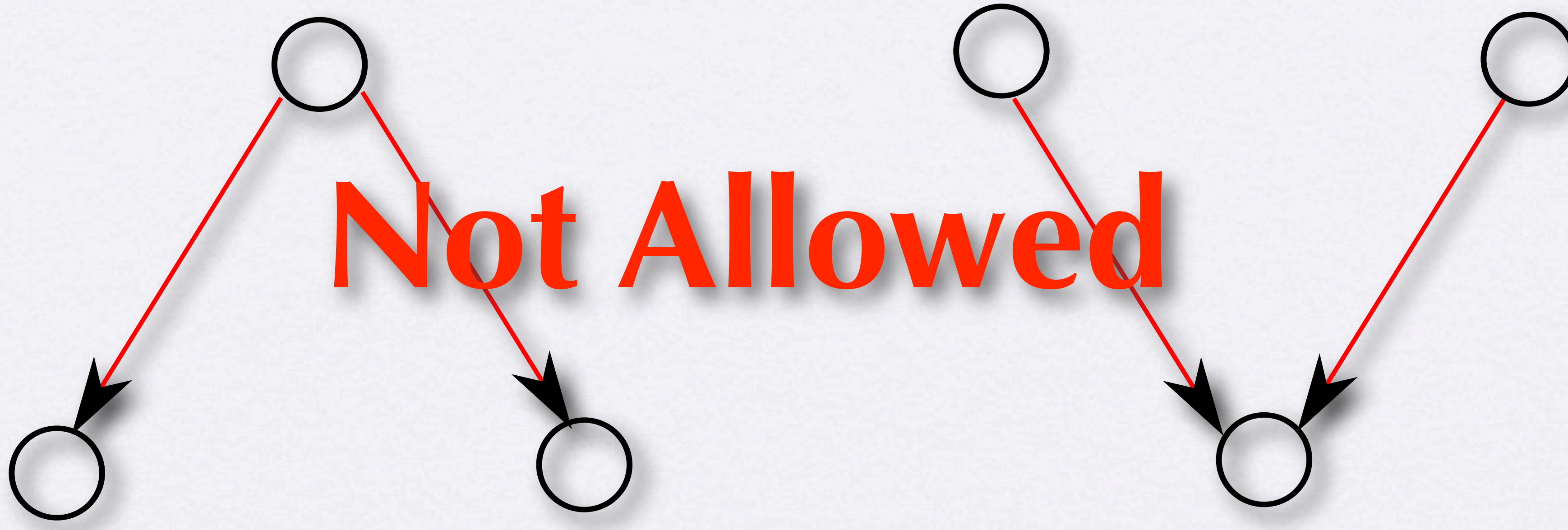


Matching in Directed Networks

Matching : a set of directed edges without common heads or tails.



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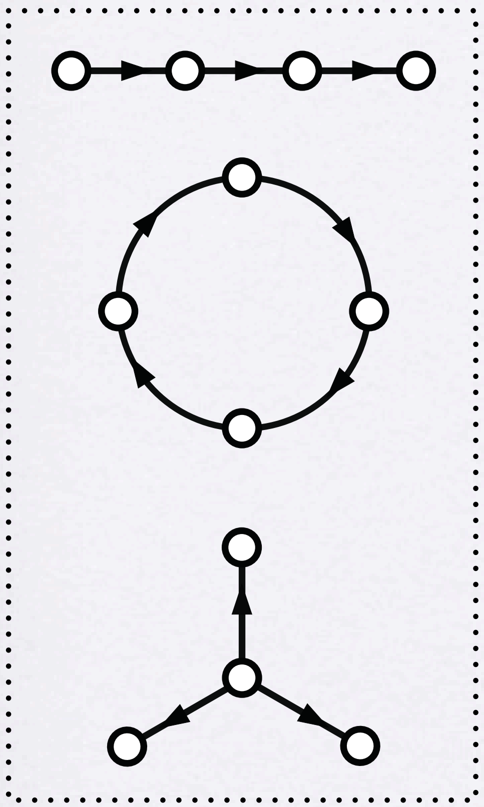
Directed network



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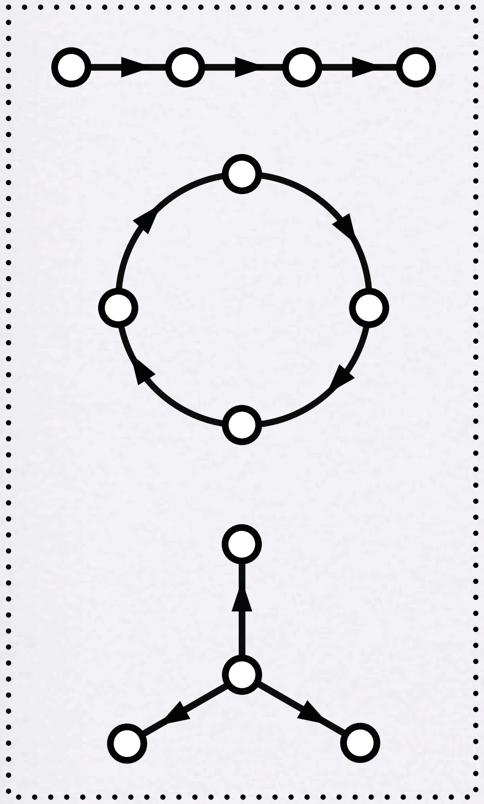
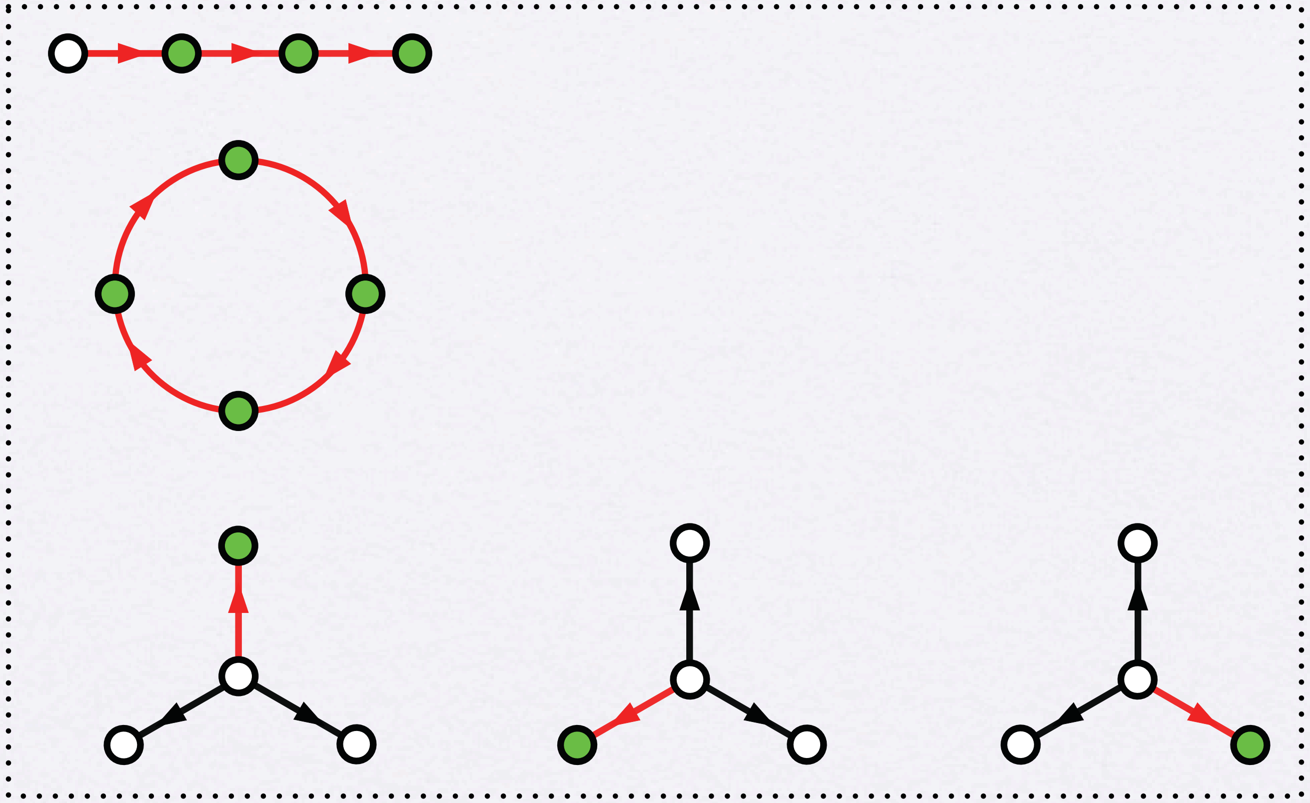


Maximum matching



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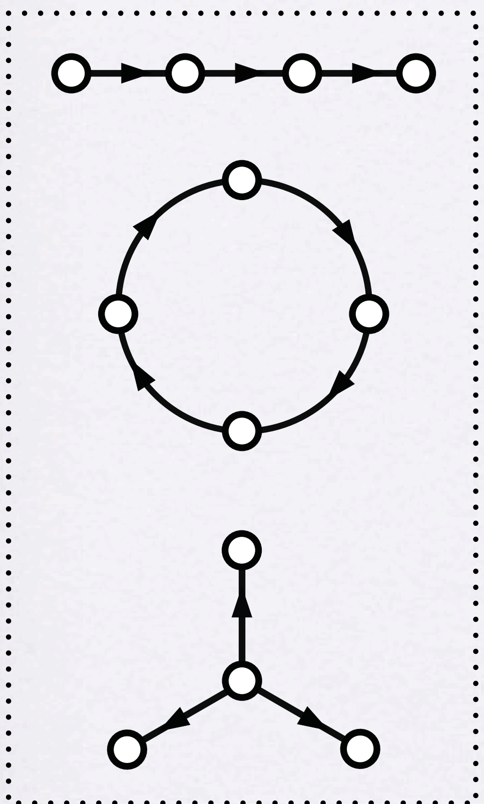

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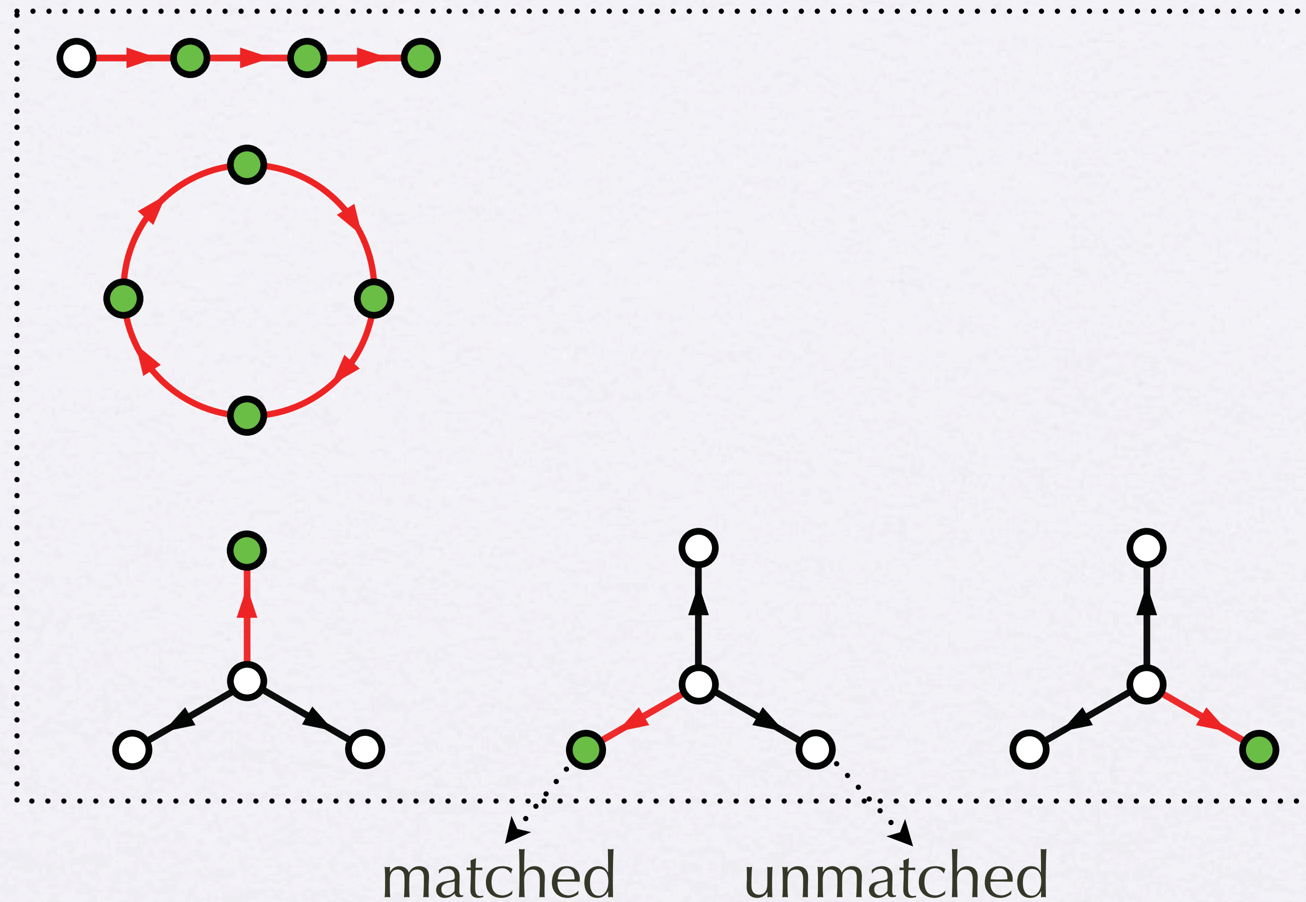

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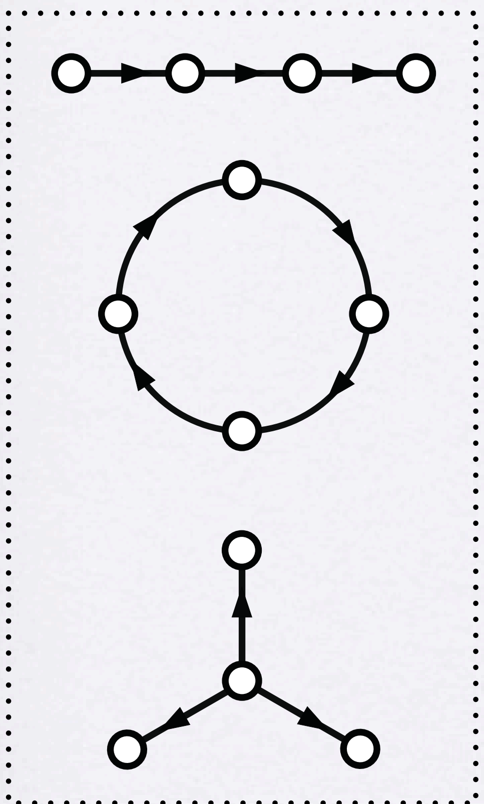

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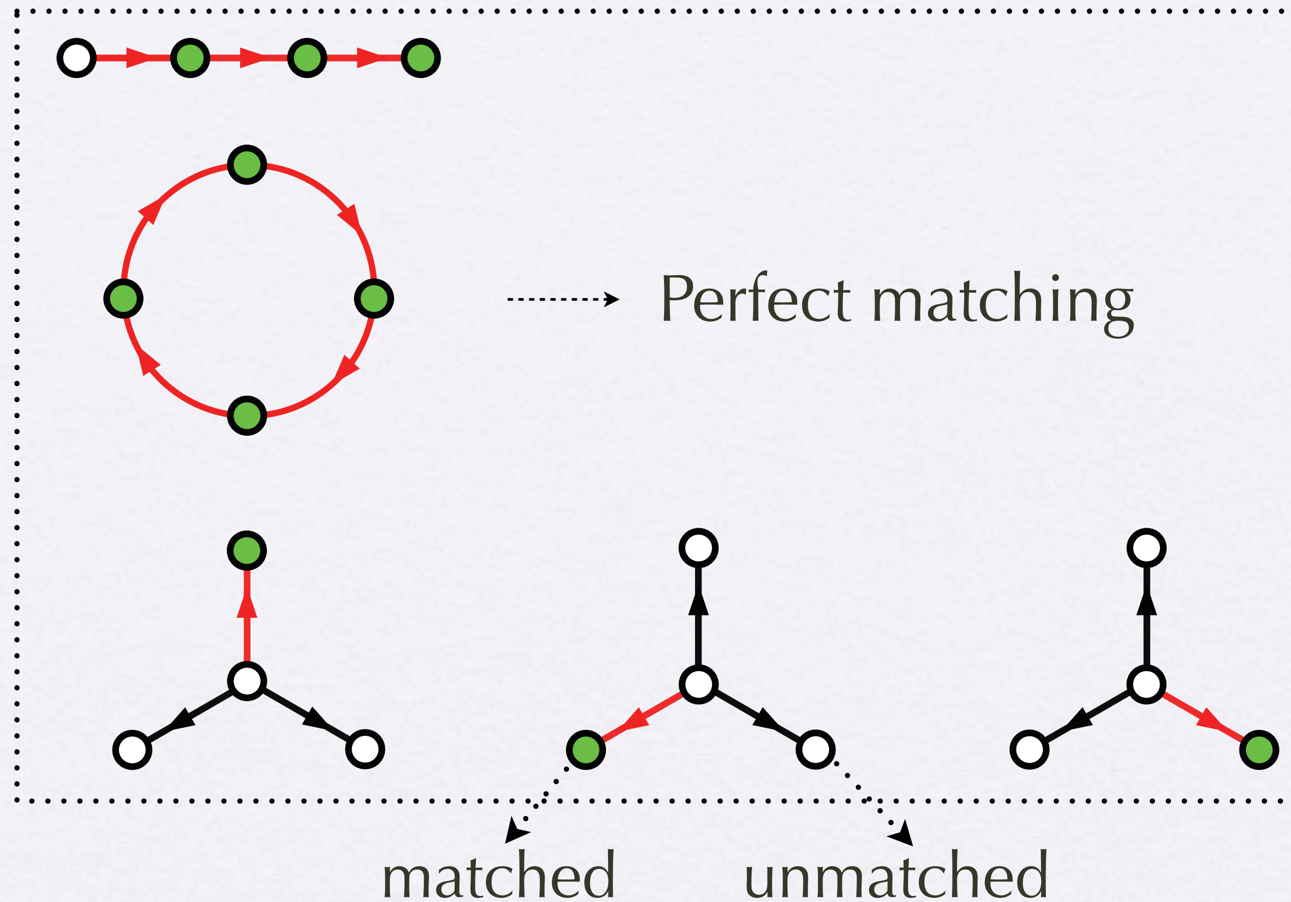

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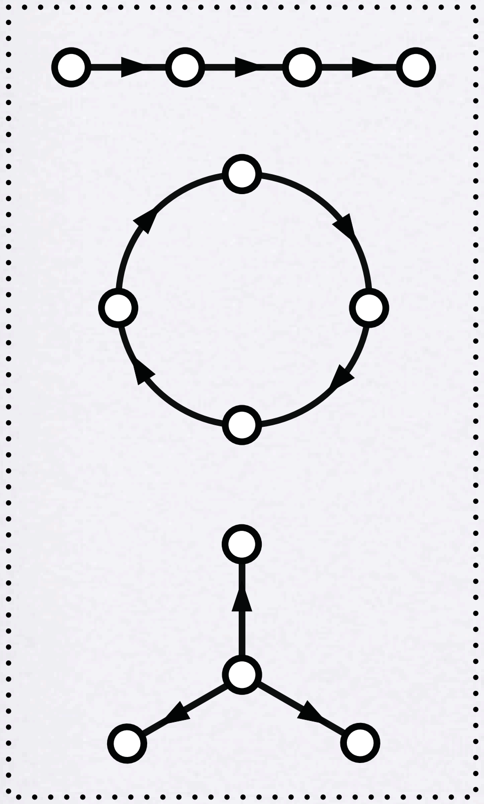

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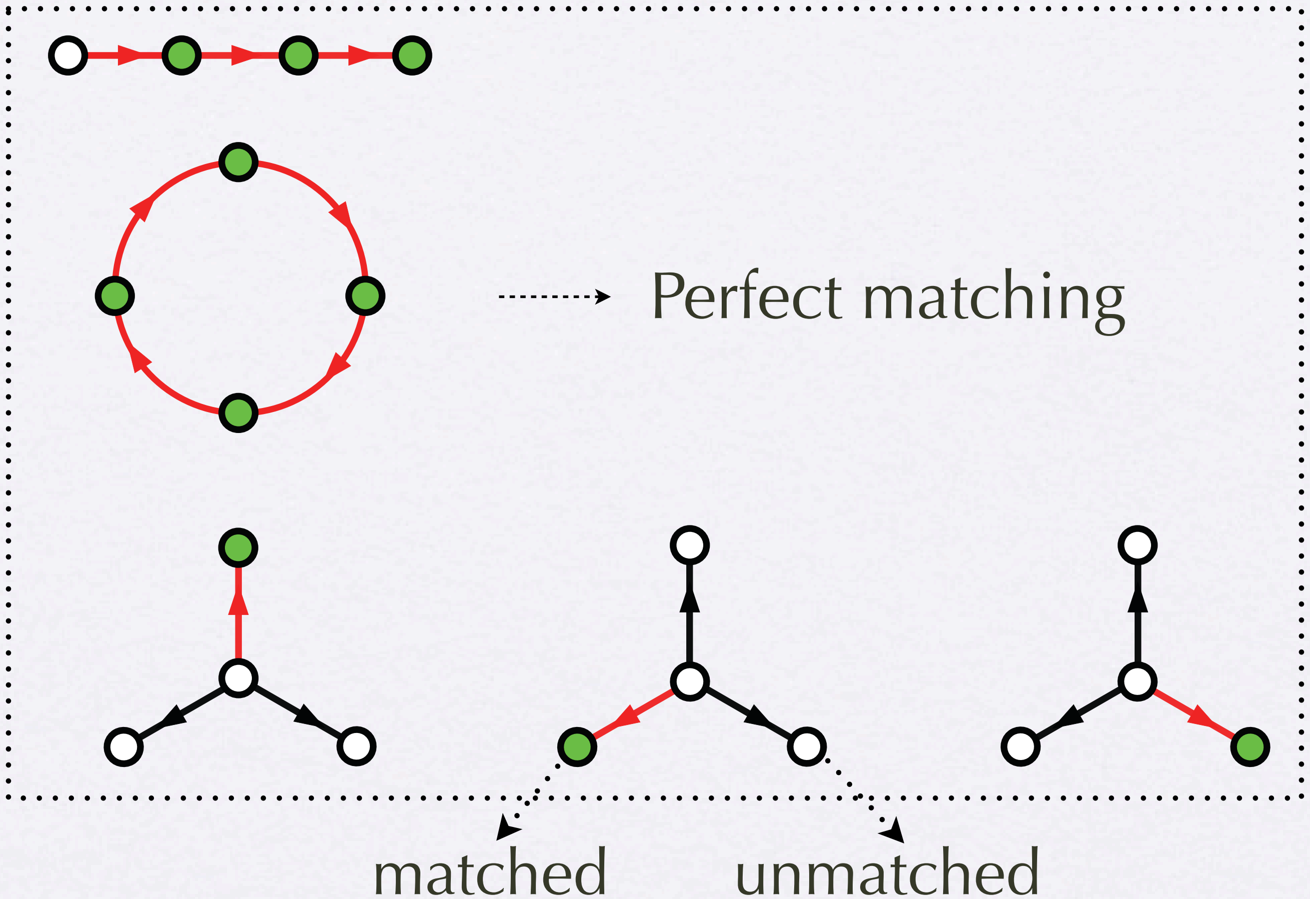


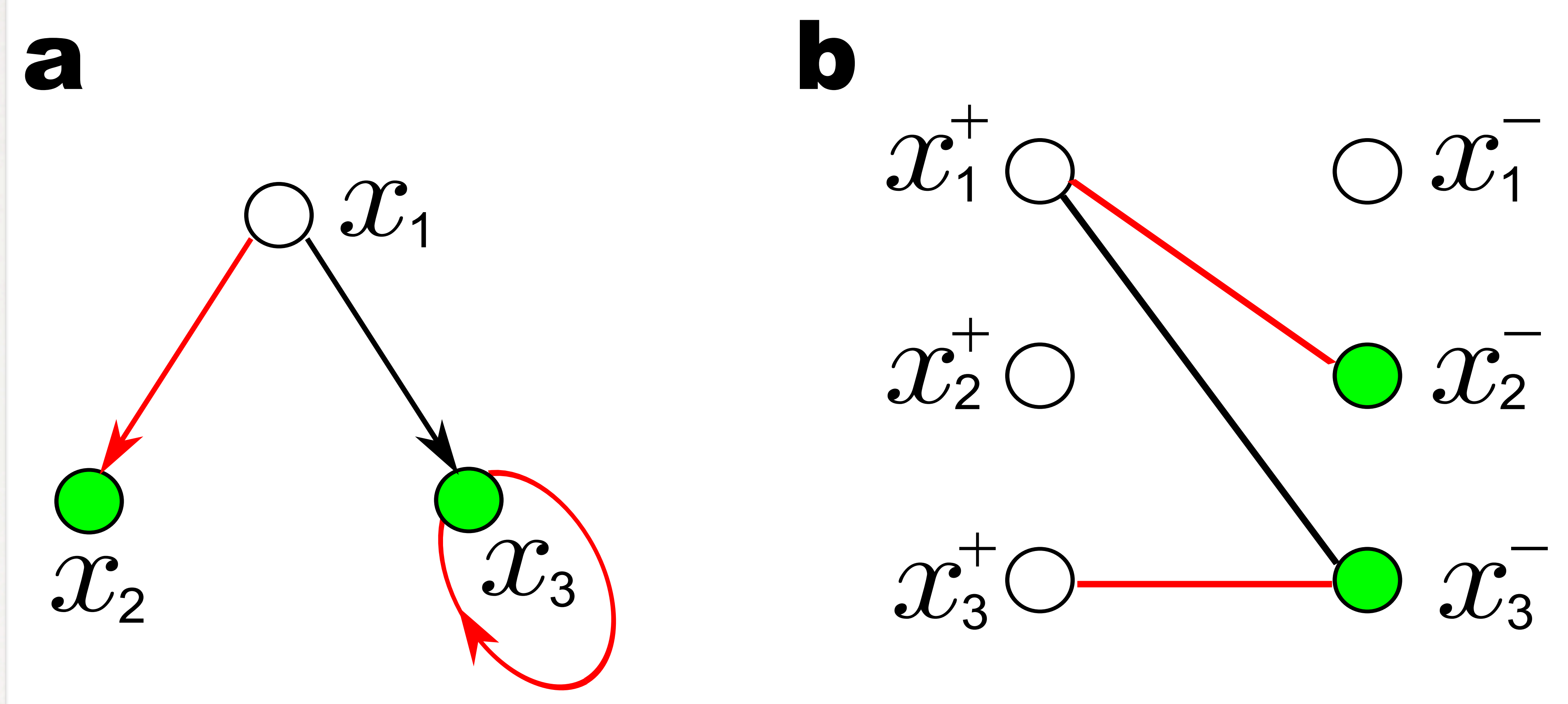
Minimum Inputs Theorem

Driver nodes = unmatched nodes

$N_D = \max\{1, N_{unmatched}\}$

Liu et al., *Nature* (in press)

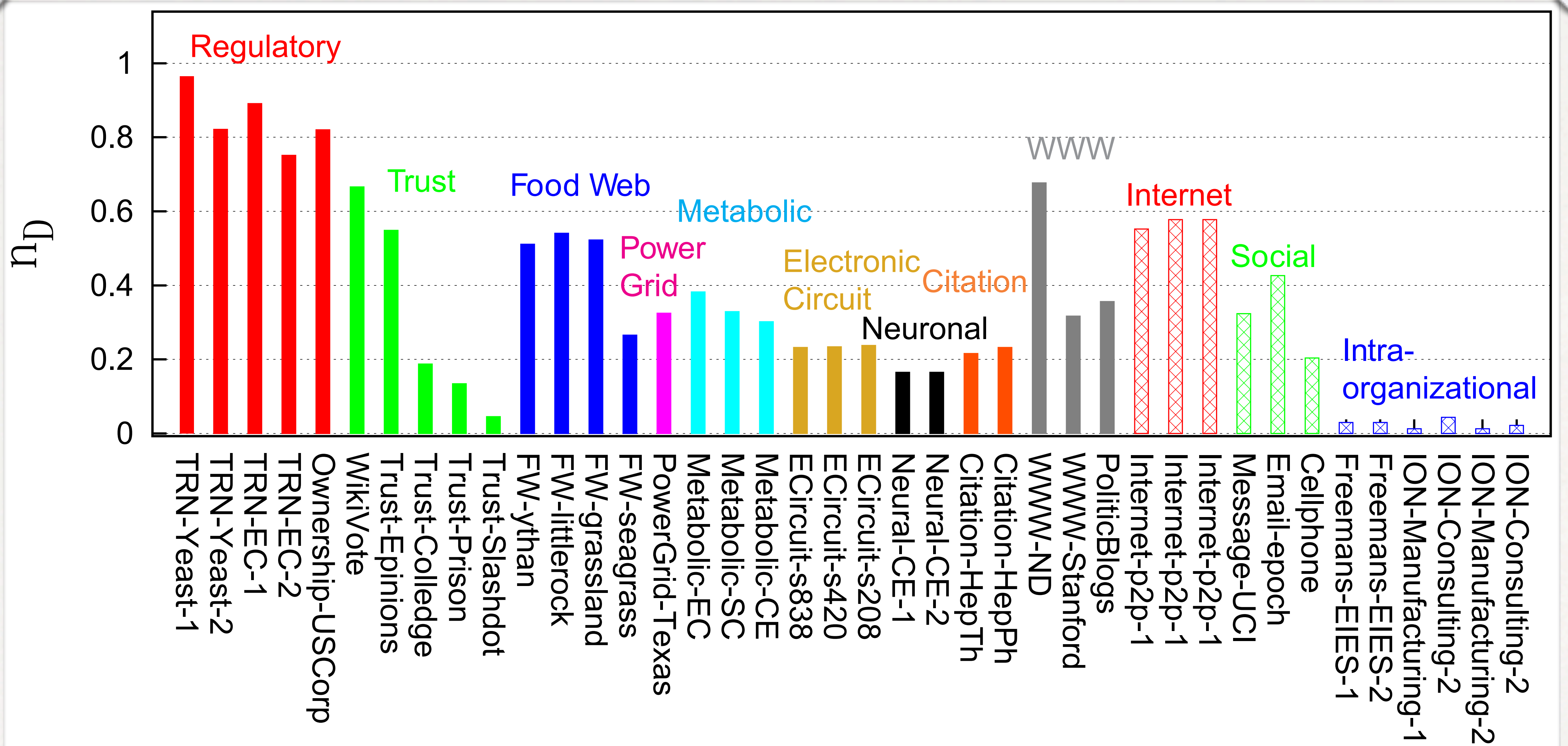
How to calculate maximum matchings in Digraph?



A maximum matching of a digraph can be easily found in its bipartite representation, using the Hopcroft-Karp algorithm ($\mathcal{O}(\sqrt{N}E)$).

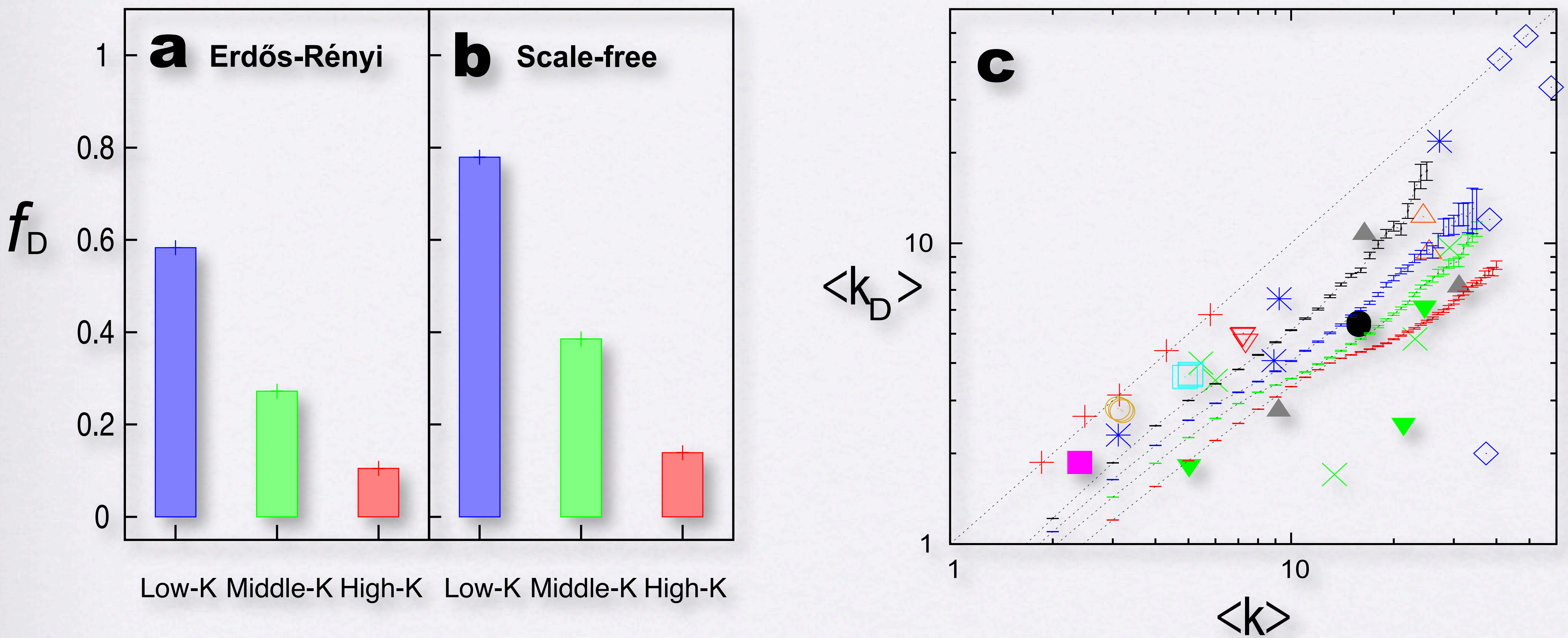
Results

N_D of real networks



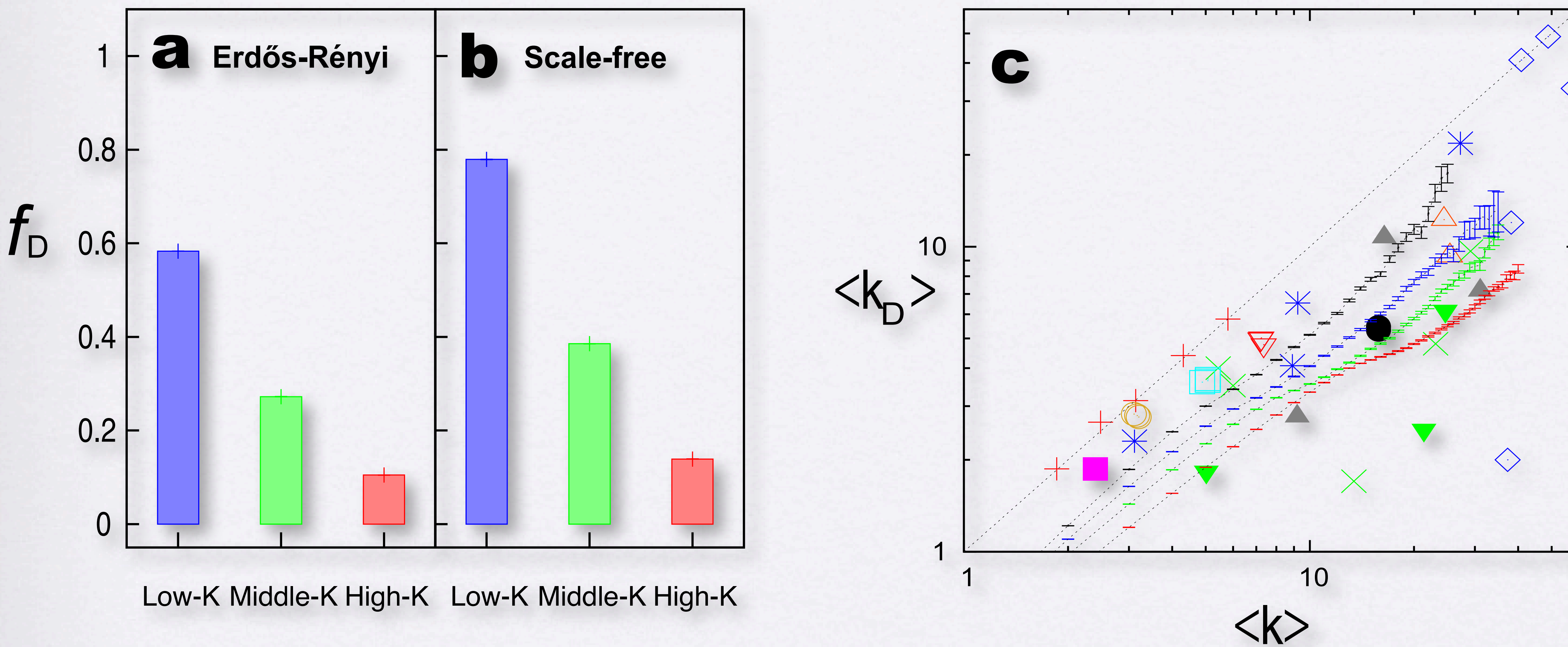
1. As a group, regulatory networks display very high $n_D \approx 0.8$.
2. A few social networks display the smallest observed n_D values.
3. Overall we see no obvious trend in n_D (or N_D) across those networks.

Role of hubs



1. The fraction of driver nodes is significantly higher among low degree nodes than among the hubs.
2. Mean degree of driver nodes <k_D> is either significantly smaller or comparable to <k>.

Role of hubs



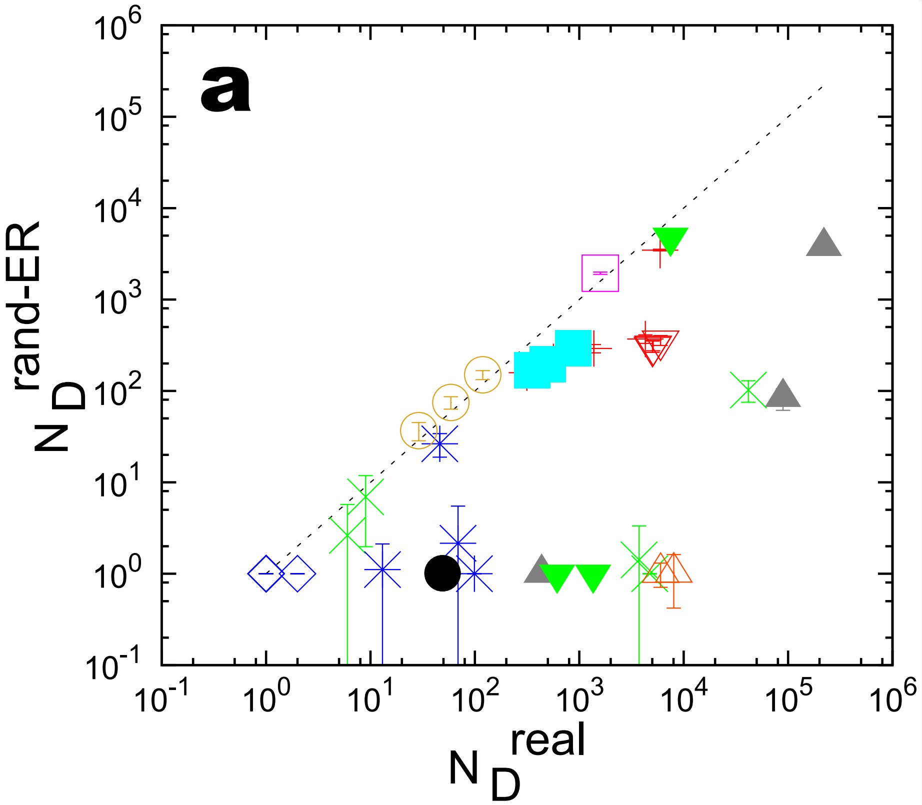
1. The fraction of driver nodes is significantly higher among low degree nodes than among the hubs.
2. Mean degree of driver nodes $\langle k_D \rangle$ is either significantly smaller or comparable to $\langle k \rangle$.

Driver nodes tend to avoid hubs.

N_D^{real} vs. N_D^{rand}

N_D^{real} vs. N_D^{rand}

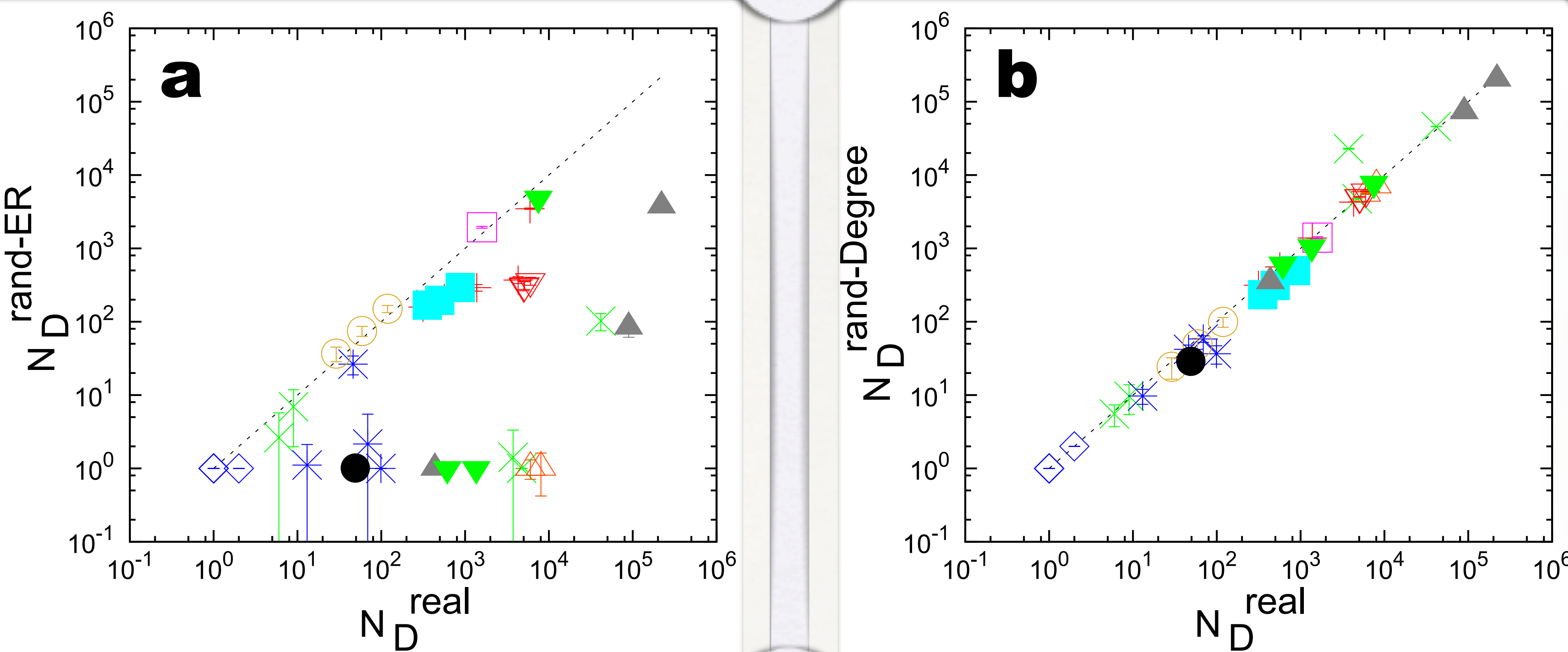
Complete randomization



N_D^{real} vs. N_D^{rand}

Complete randomization

Degree-preserving randomization



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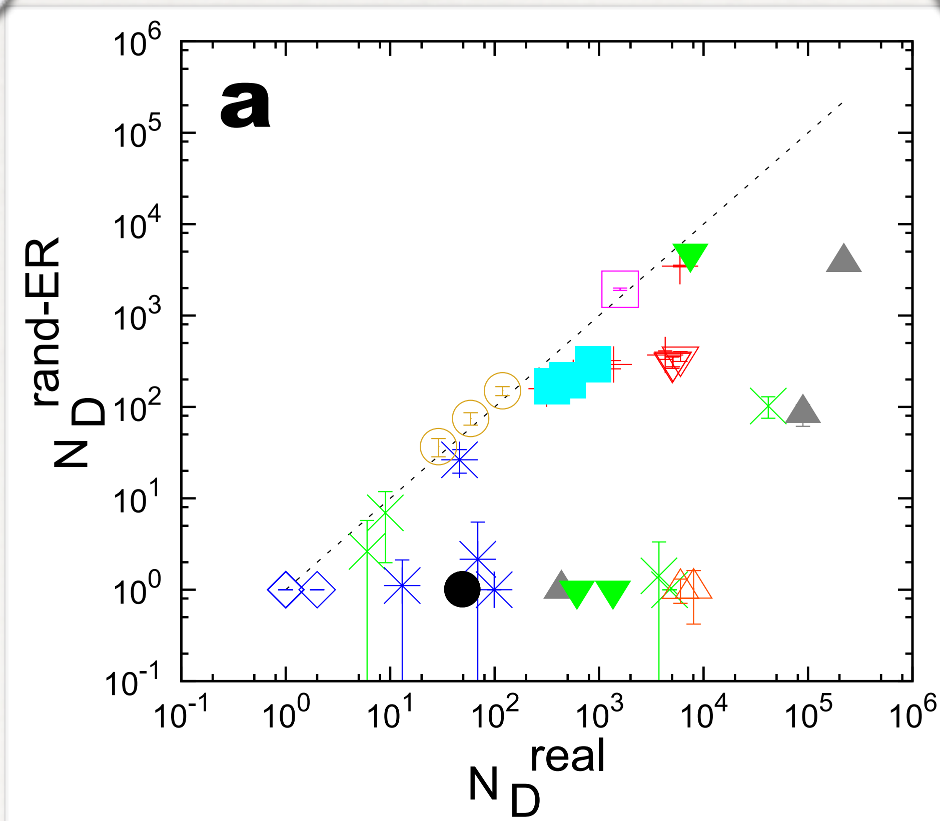

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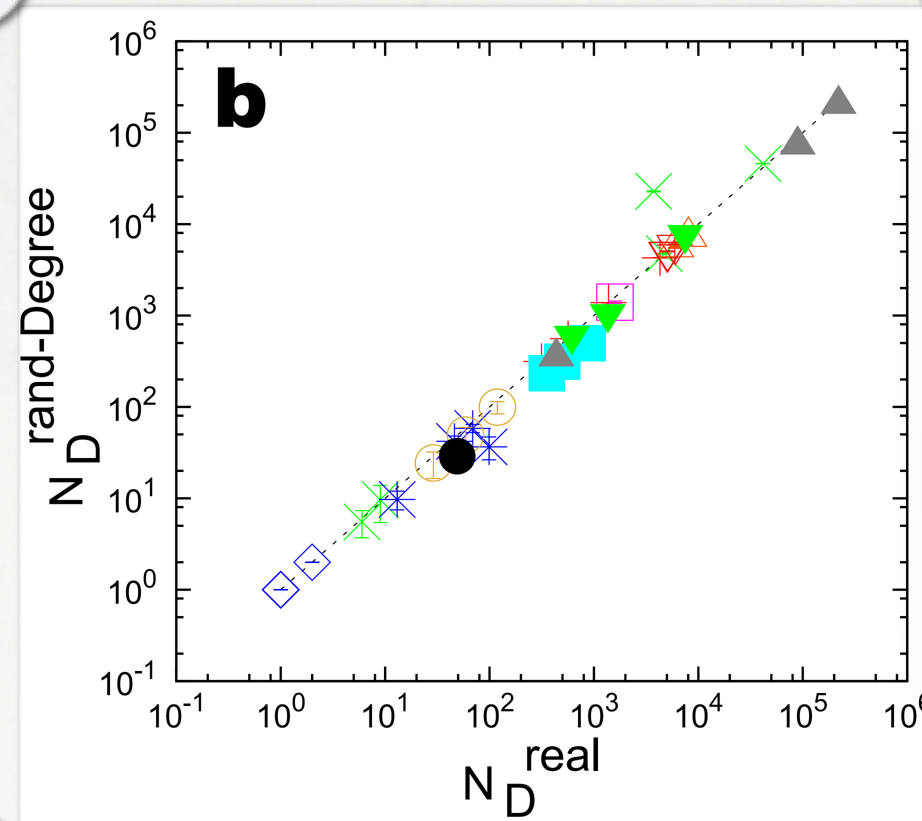

N_D is mainly determined by degree distribution.

Key Result

N_D is primarily determined by degree distribution.

$$\boxed{P(k_{\mathrm{in}}, k_{\mathrm{out}}) \rightarrow N_{\mathrm{D}}}$$

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$$\boxed{P(k_{\mathrm{in}}, k_{\mathrm{out}}) \to N_{\mathrm{D}}}$$

1. **Unexpected and counter-intuitive**:
   The specific wiring diagram does not matter!

Key Result

N_D is primarily determined by degree distribution.

$$\boxed{P(k_{\mathrm{in}}, k_{\mathrm{out}}) \to N_{\mathrm{D}}}$$

1. **Unexpected and counter-intuitive**:
   The specific wiring diagram does not matter!

2. Allows us to analytically calculate the average N_D over all network realizations compatible with $P(k_{in}, k_{out})$, using the **cavity method** in statistical physics.

Analytic Result

Define $Q(k_{\mathrm{out}}) := \frac{k_{\mathrm{out}} P(k_{\mathrm{out}})}{\langle k_{\mathrm{out}} \rangle}$ and $\widehat{Q}(k_{\mathrm{in}}) := \frac{k_{\mathrm{in}} \widehat{P}(k_{\mathrm{in}})}{\langle k_{\mathrm{in}} \rangle}$.
Generating functions:

$$\begin{cases} G(x) & := & \sum_{k_{\mathrm{out}}=0}^{\infty} P(k_{\mathrm{out}}) x^{k_{\mathrm{out}}} \\ \widehat{G}(x) & := & \sum_{k_{\mathrm{in}}=0}^{\infty} \widehat{P}(k_{\mathrm{in}}) x^{k_{\mathrm{in}}} \\ H(x) & := & \sum_{k_{\mathrm{out}}=0}^{\infty} Q(k_{\mathrm{out}}+1) x^{k_{\mathrm{out}}} \\ \widehat{H}(x) & := & \sum_{k_{\mathrm{in}}=0}^{\infty} \widehat{Q}(k_{\mathrm{in}}+1) x^{k_{\mathrm{in}}} \end{cases}$$

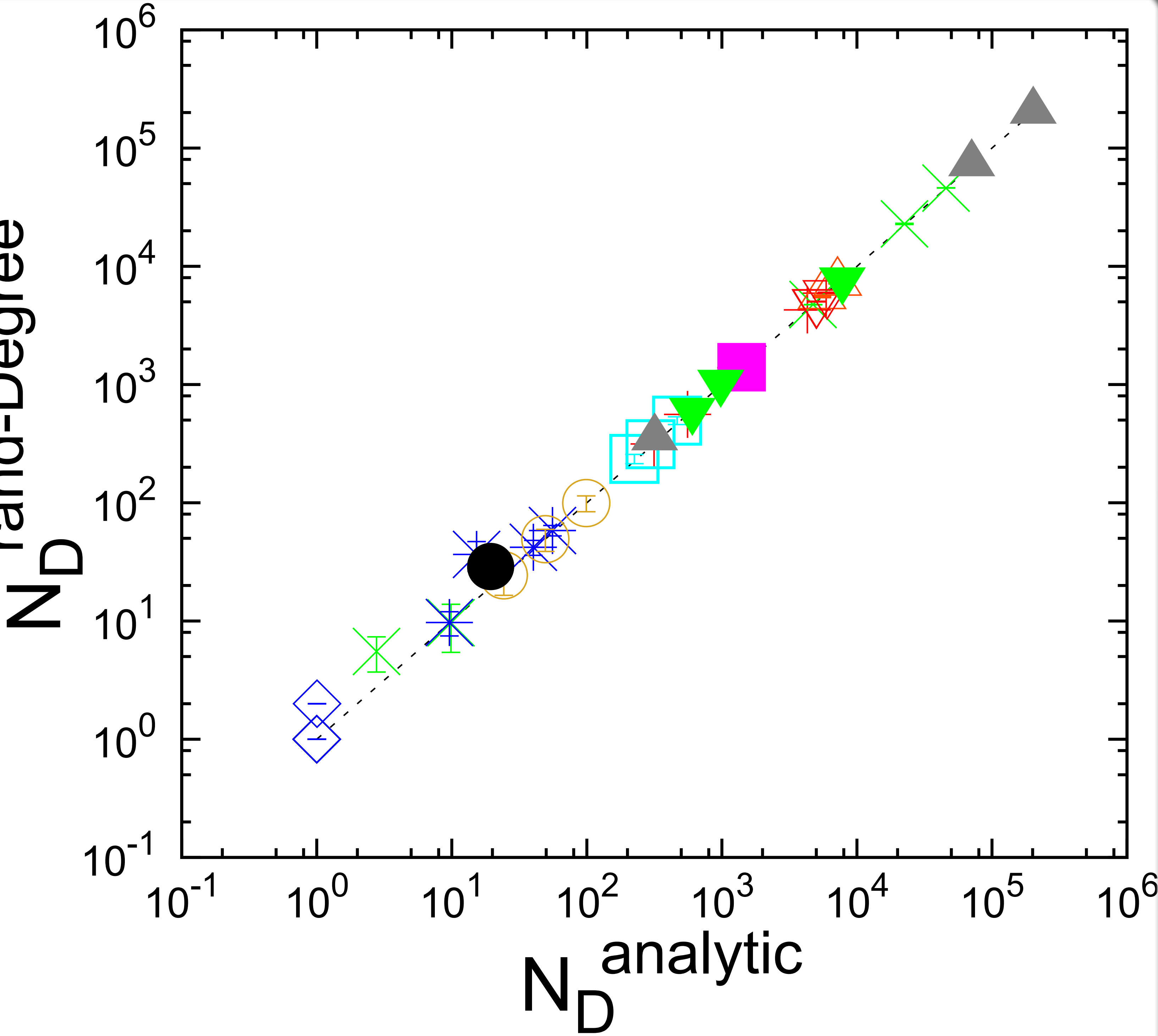
Self-consistent equations:

$$\begin{cases} w_1 & = & H(\widehat{w}_2) \\ w_2 & = & 1 - H(1-\widehat{w}_1) \\ \widehat{w}_1 & = & \widehat{H}(w_2) \\ \widehat{w}_2 & = & 1 - \widehat{H}(1-w_1) \end{cases}$$

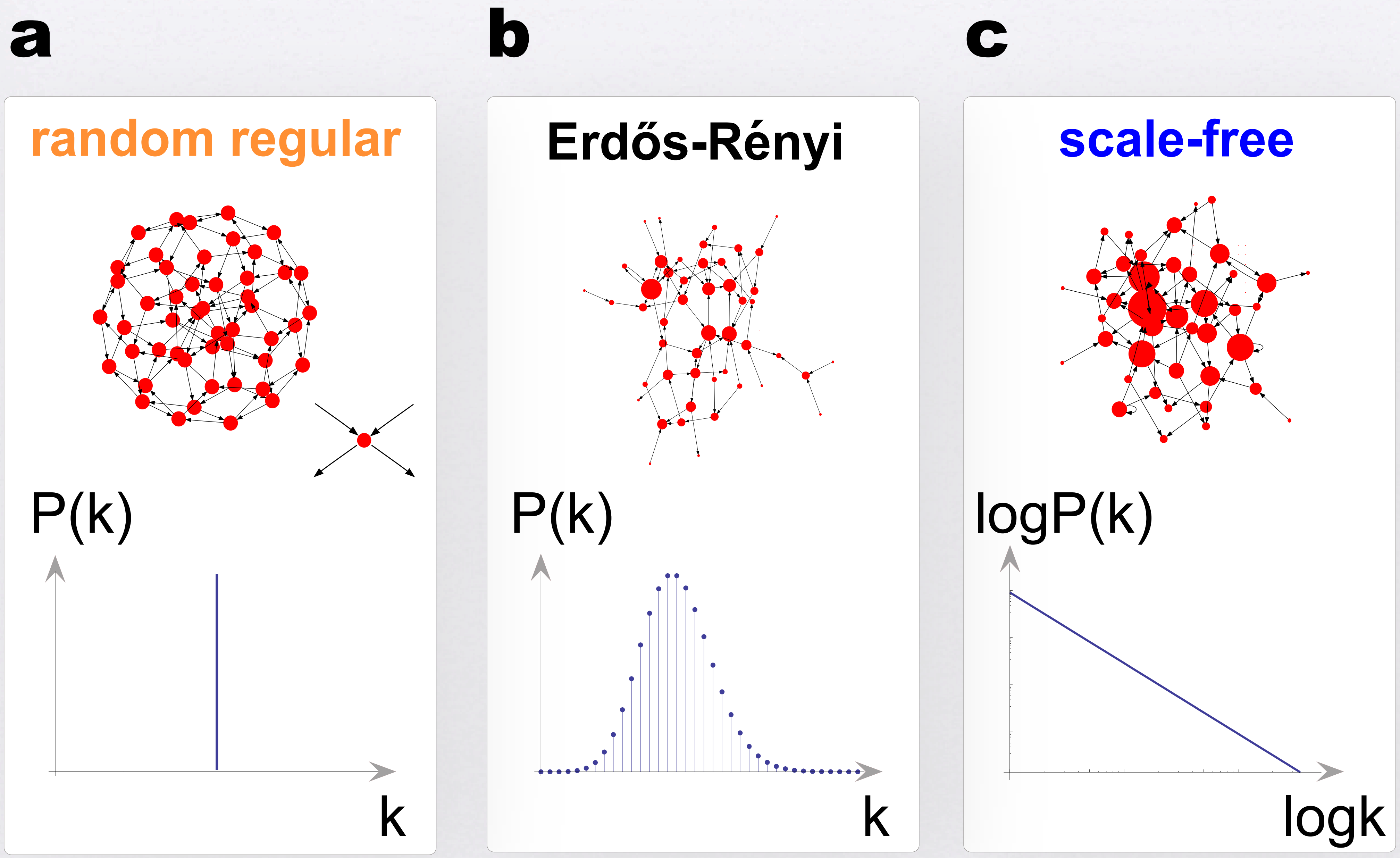
Average N_{D} over all network realizations compatible with $P(k_{\mathrm{in}}, k_{\mathrm{out}})$

$$\boxed{\begin{aligned} \mathbb{E}(N_{\mathrm{D}}) \quad = \quad & \tfrac{N}{2} \Big\{ \big[G(\widehat{w}_2) + G(1-\widehat{w}_1) - 1 \big] + \big[\widehat{G}(w_2) + \widehat{G}(1-w_1) - 1 \big] \\ & + \tfrac{z}{2} \big[\widehat{w}_1 (1-w_2) + w_1 (1-\widehat{w}_2) \big] \Big\} \end{aligned}}$$

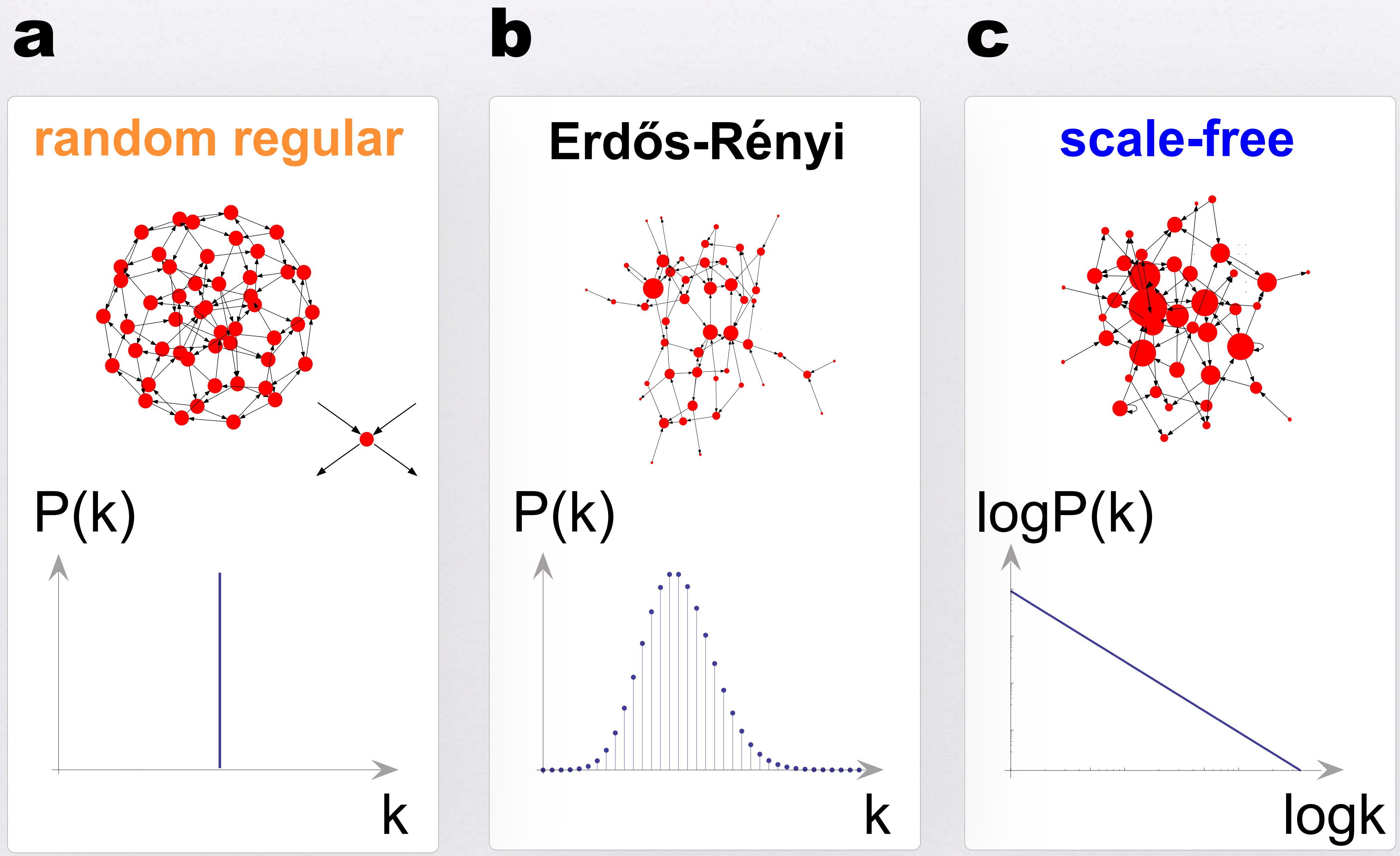
Analytical vs. Numerical



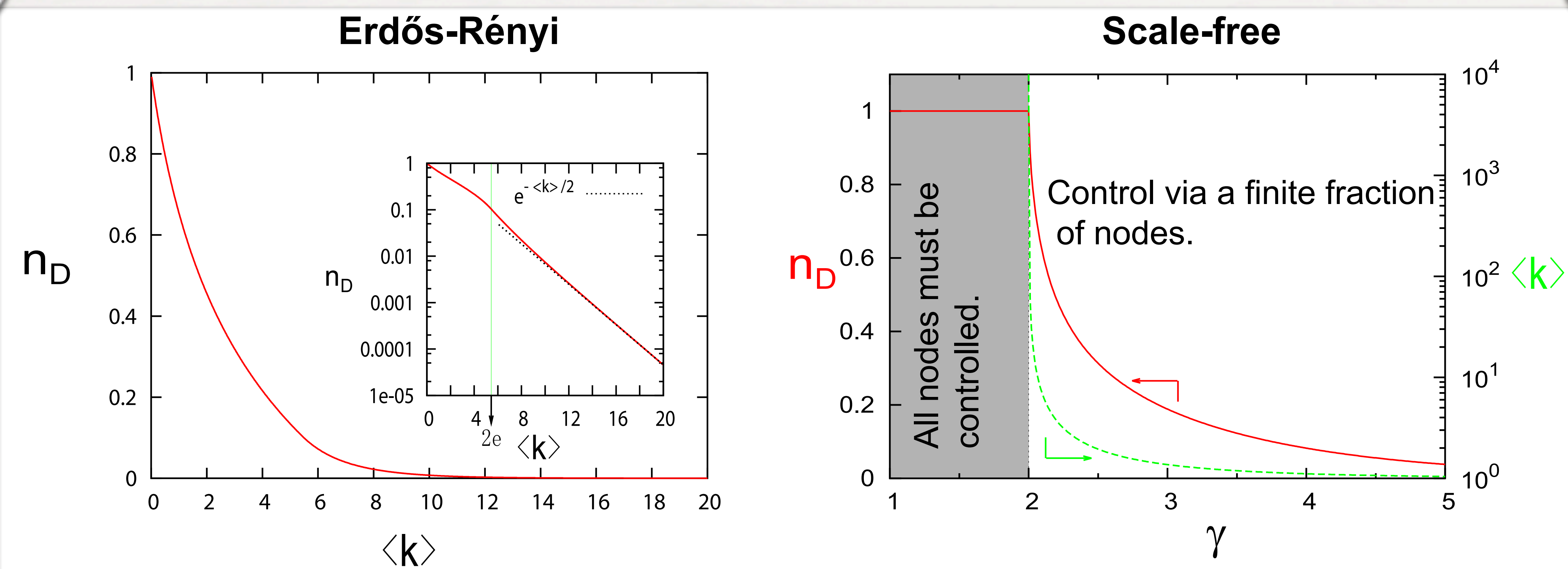
Which properties in $P(k)$ determine N_{D}?


a
b
c
random regular
Erdős-Rényi
scale-free
P(k)
k
P(k)
k
logP(k)
logk

Most homogeneous → Most heterogeneous



Analytical

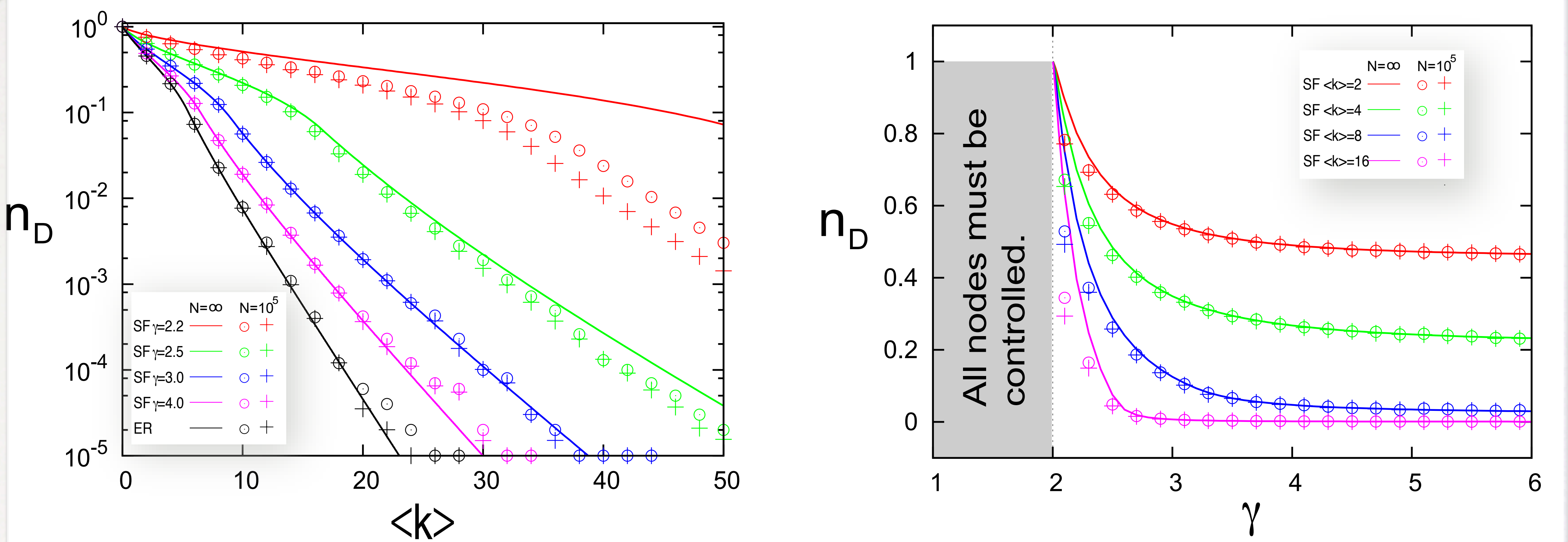


For infinite system $(N = \infty)$

1. ER : $n_{\mathrm{D}}(\langle k \rangle) \sim e^{-\langle k \rangle/2}$ as $\langle k \rangle \gg 1$

2. SF : $n_{\mathrm{D}}(\gamma) \to 1$ as $\gamma \to \gamma_{\mathrm{c}} = 2$

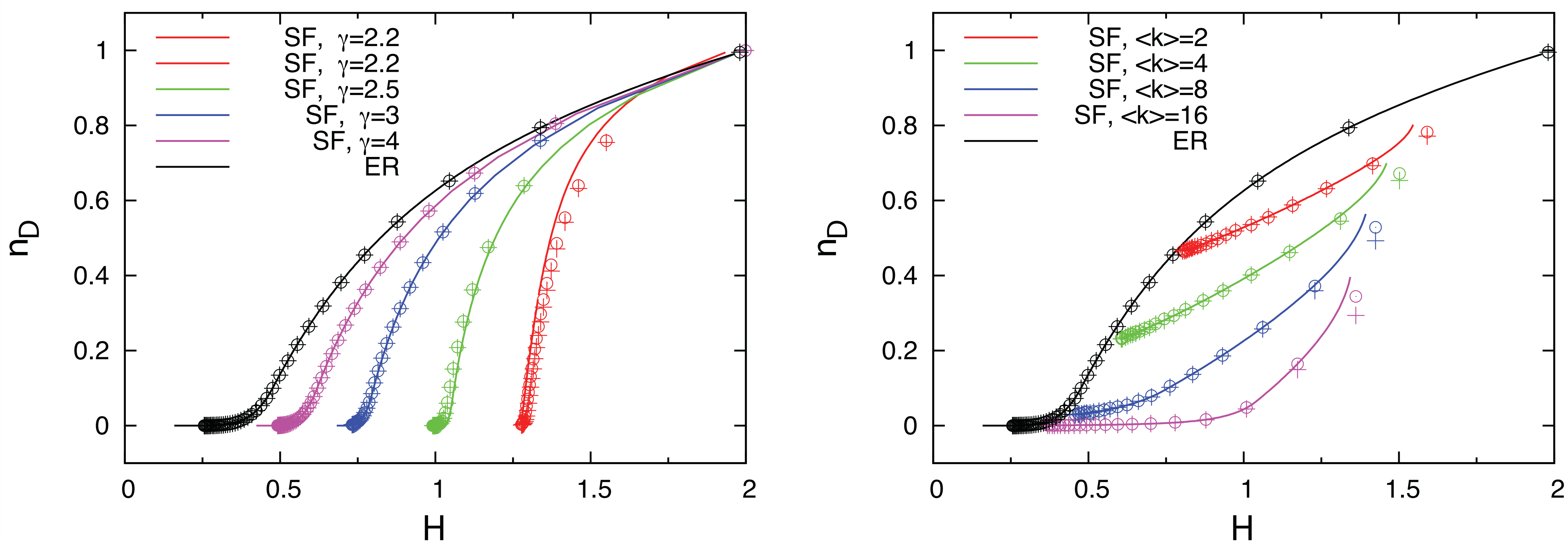
Analytical vs. Numerical



Finite system, asymptotically ($\langle k \rangle \gg 1$)

1. ER: $n_{\mathrm{D}}(\langle k \rangle) \sim e^{-\langle k \rangle/2}$.
2. SF: $n_{\mathrm{D}}(\langle k \rangle, \gamma) \sim e^{-f(\gamma)\langle k \rangle/2}$ with $f(\gamma) = 1 - \frac{1}{\gamma - 1}$ (consistent with $\gamma_{\mathrm{c}} = 2$).

Degree Heterogeneity



Degree heterogeneity H = 2 x Gini coefficient

$$H = \frac{\Delta}{\langle k \rangle} = \frac{\sum_i \sum_j |i-j| P(i) P(j)}{\langle k \rangle}$$

Summary

1. **The minimum set of driver nodes can be efficiently identified, with**

$$\boxed{N_{\mathrm{D}} = \max\{1, N_{\mathrm{unmatched}}\}.}$$

2. N_{D} **is mainly determined by the degree distribution.**

$$\boxed{P(k_{\mathrm{in}}, k_{\mathrm{out}}) \to N_{\mathrm{D}}.}$$

* Driver nodes tend to avoid hubs.
* Mean degree $\langle k \rangle$ and degree inhomogeneity H are two main factors.
* Sparse and heterogeneous networks are most difficult to control.

Courtesy of Mauro Martino

Many Thanks.

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yangliu61@gmail.com

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