Formal Methods in Software Developement

SAT Solving

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Based on slides of the lecture Satisfiability Checking (Erika Ábrahám), RTWH Aachen

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Given:

- Propositional logic formula φ in CNF.

Question:

- Is φ satisfiable?
  (Is there a model for φ?)

SAT-solving: Components

- Decision (enumeration)
- Boolean constraint propagation (BCP) – discussed in the previous lecture
- Conflict resolution and backtracking – discussed in the previous lecture

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- Decision (enumeration)
- Boolean constraint propagation (BCP)
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Enumeration algorithm

- Naive approach yields 2^n candidate models to check
- Solution: decision heuristics

Example CNF: Decision heuristics

$$\underbrace{(\neg x \lor y \lor z)}_{c_1} \land \underbrace{(y \lor \neg z)}_{c_2} \land \underbrace{(\neg x \lor \neg y)}_{c_3}$$

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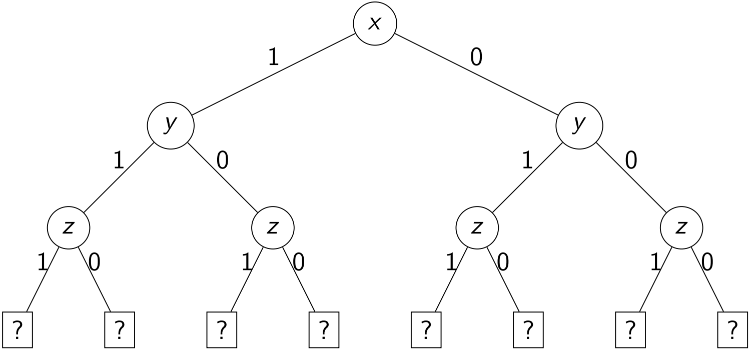

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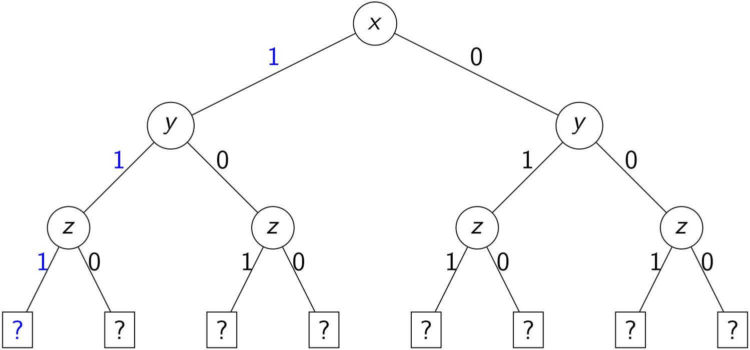

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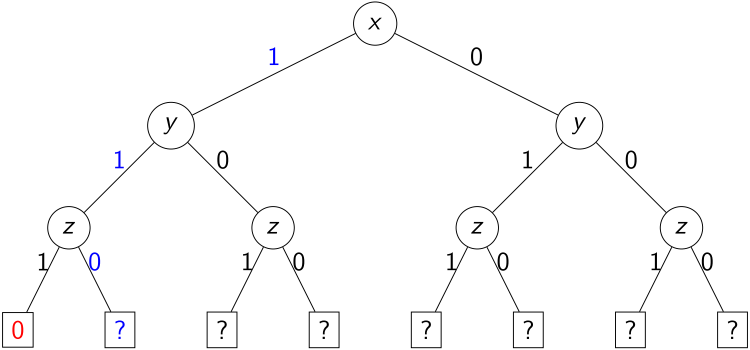

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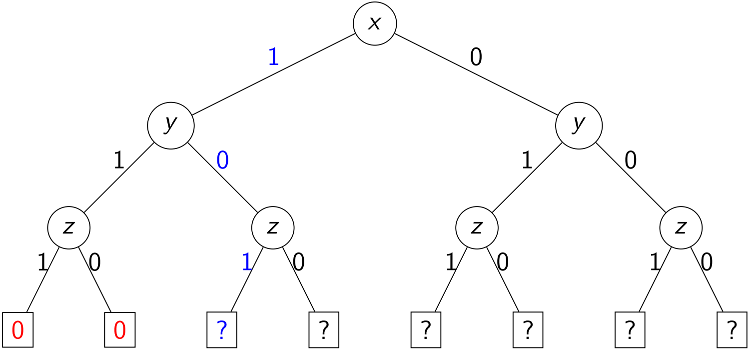

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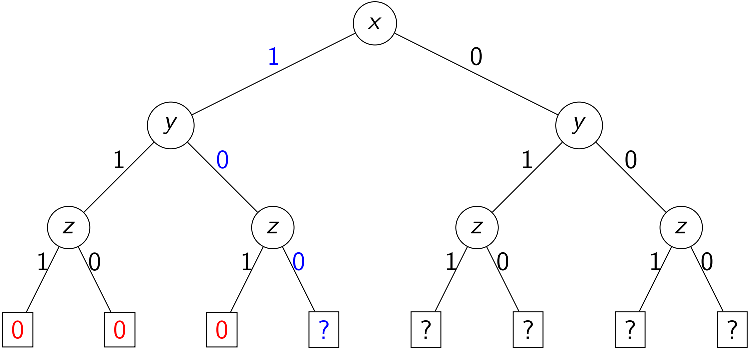

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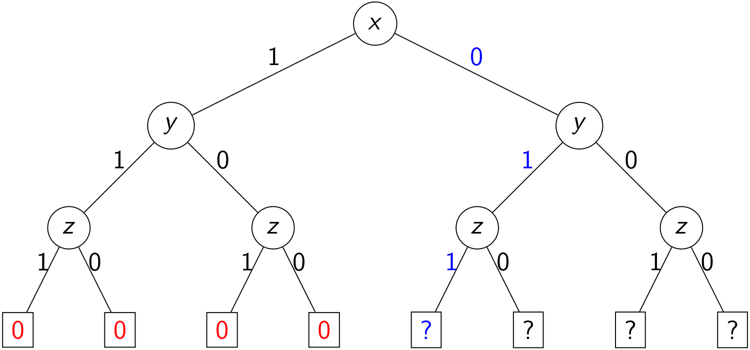

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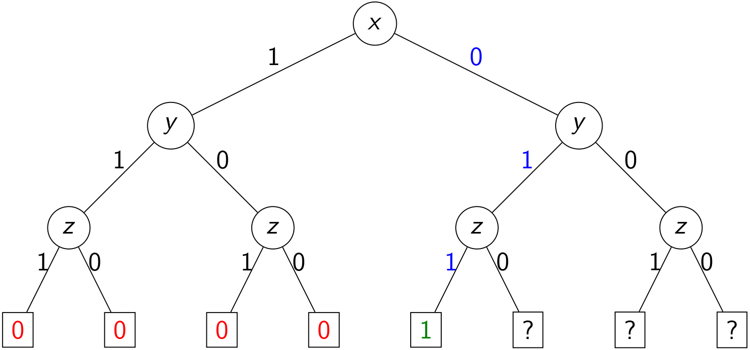

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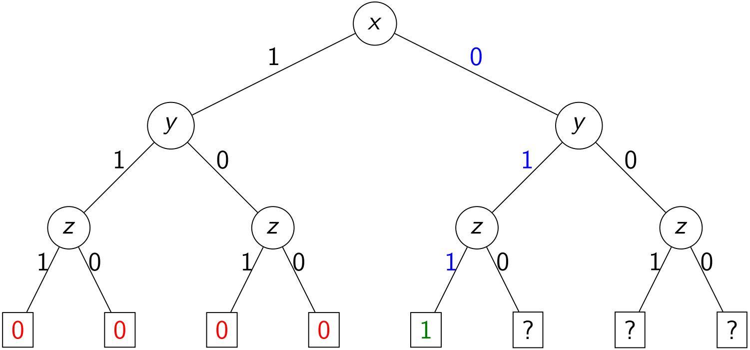

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For satisfiable problems, variable and sign ordering might strongly influence the running time.

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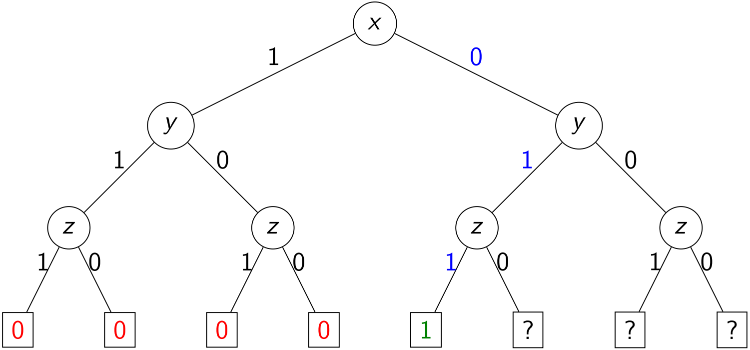

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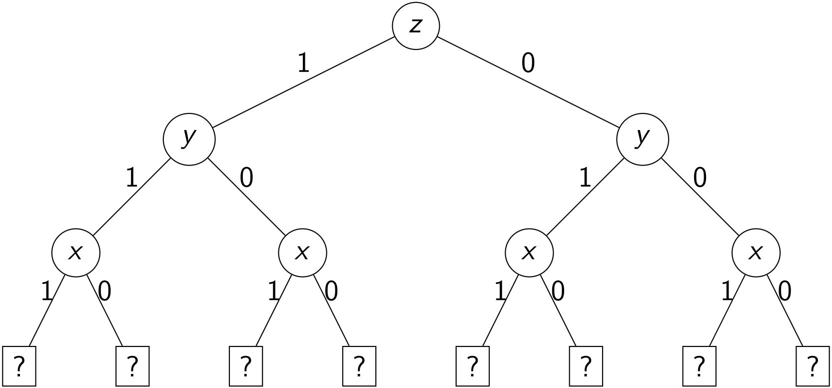

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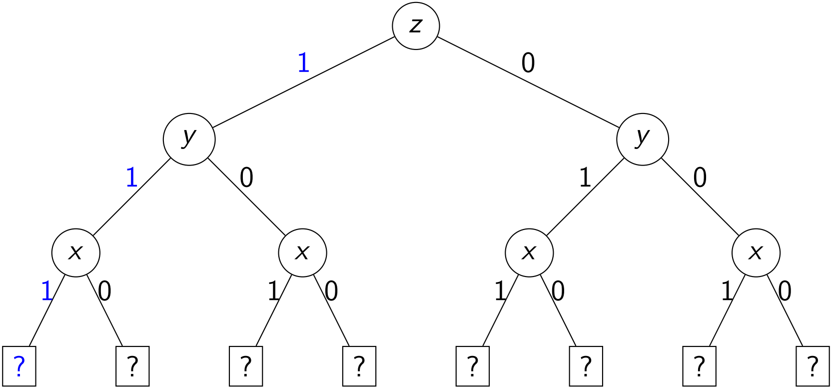

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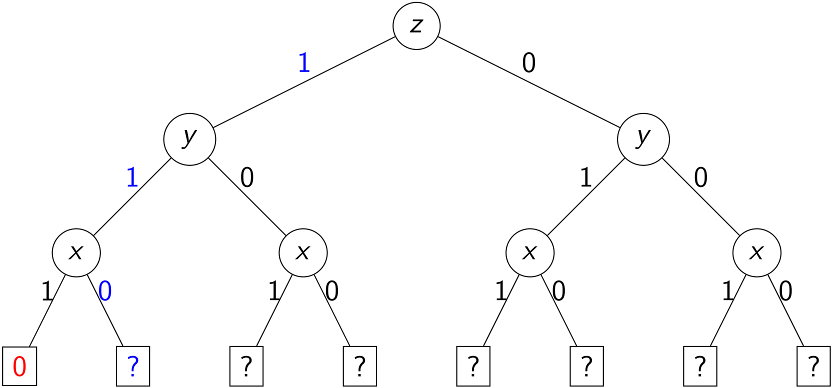

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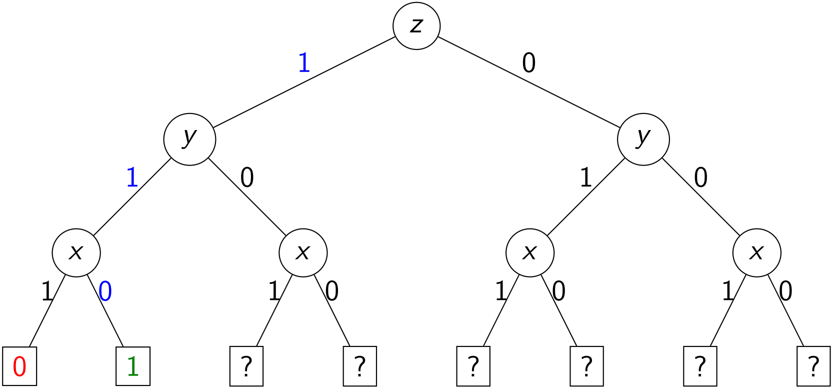

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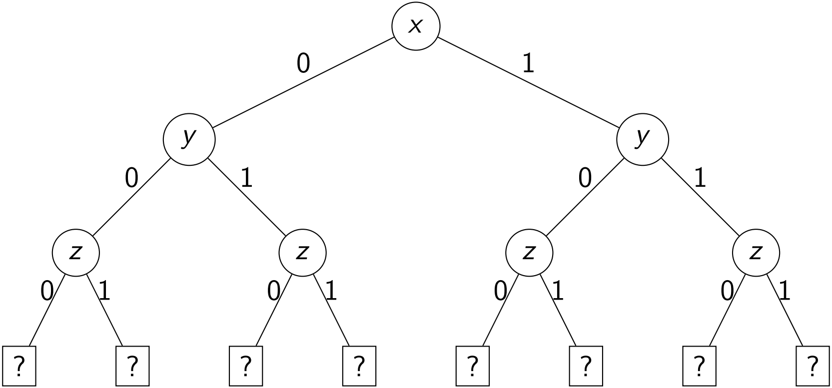

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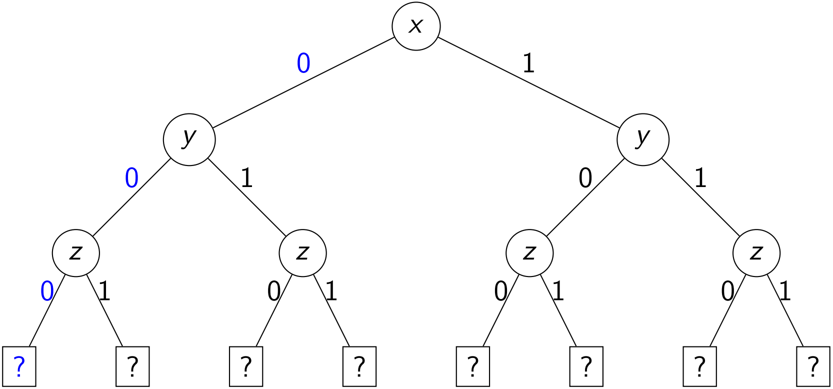

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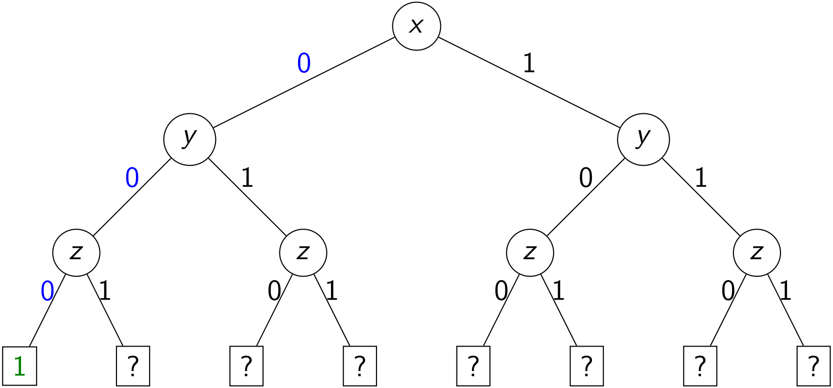

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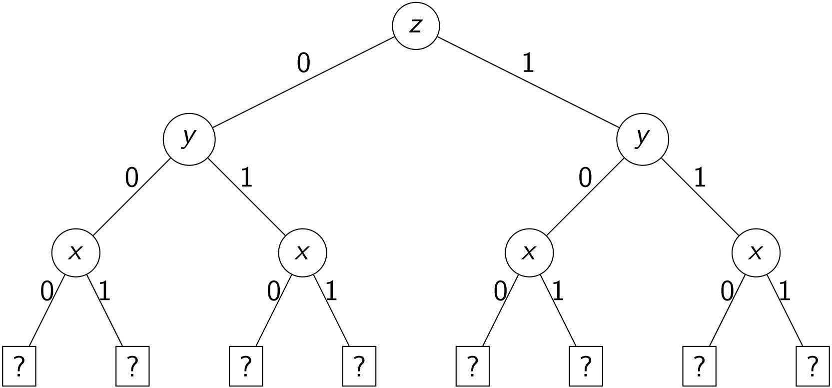

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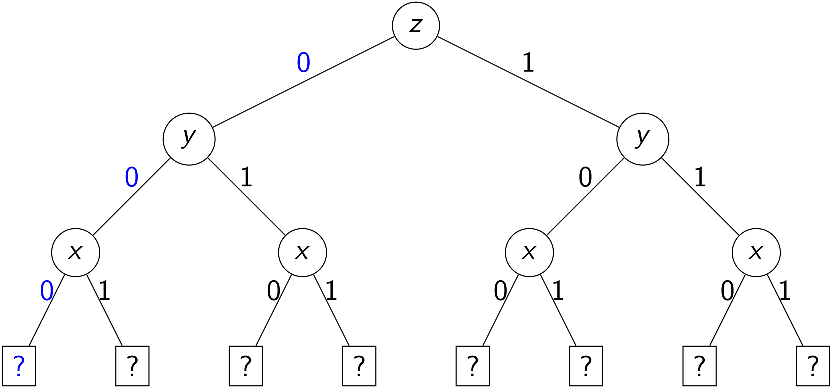

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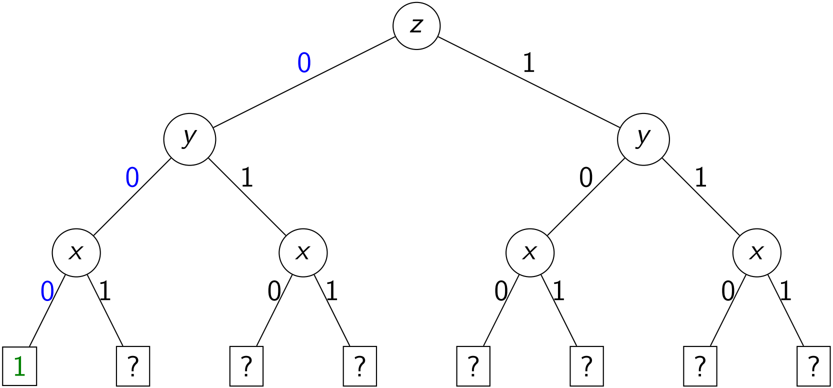

Decision heuristics

Dynamic Largest Individual Sum (DLIS): Choose an assignment that increases the most the number of satisfied clauses

- For each variable x, let C_x be the number of unresolved clauses in which x appears positively.
- For each variable x, let $C_{\neg x}$ be the number unresolved clauses in which x appears negatively.
- Let x be a variable for which C_x is maximal ($C_x \geq C_z$ for all variables z).
- Let y be a variable for which $C_{\neg y}$ is maximal ($C_{\neg y} \geq C_{\neg z}$ for all variables z).
- If $C_x > C_{\neg y}$ choose x and assign it TRUE.
- Otherwise choose y and assign it FALSE.

- Requires $\mathcal{O}(\#\mathit{literals})$ queries for each decision.

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$$\begin{array}{lll} C_x = 0 & C_y = 2 & C_z = 1 \\ C_{\neg x} = 2 & C_{\neg y} = 1 & C_{\neg z} = 1 \end{array}$$

Dynamic Largest Individual Sum (DLIS) literal order

Fallback literal order (in case of equal values: $\neg x$ and y): $\neg x < x < \neg z < z < \neg y < y$

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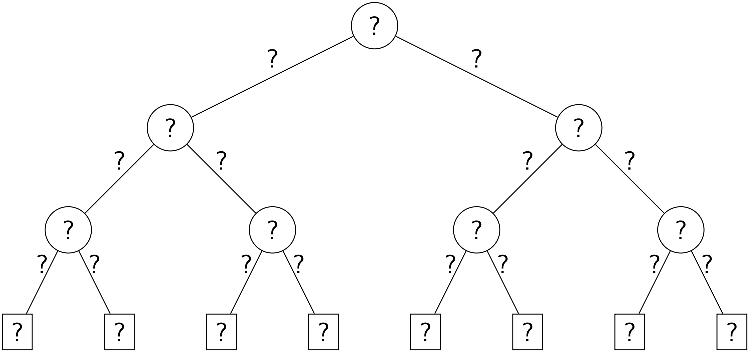

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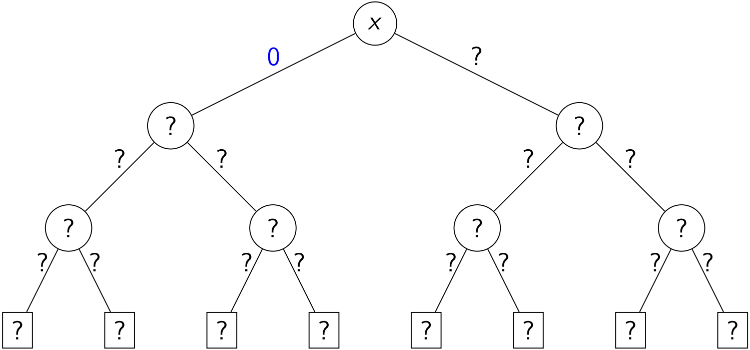

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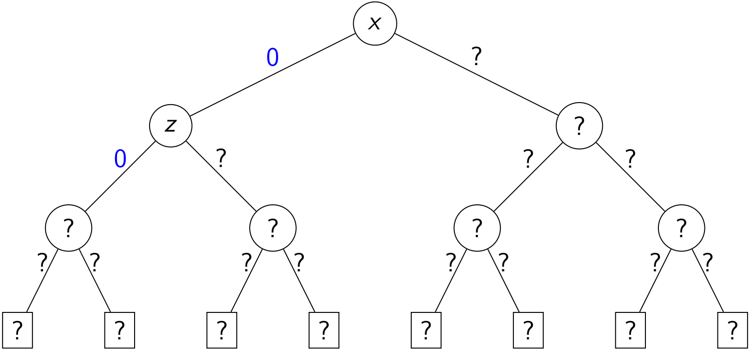

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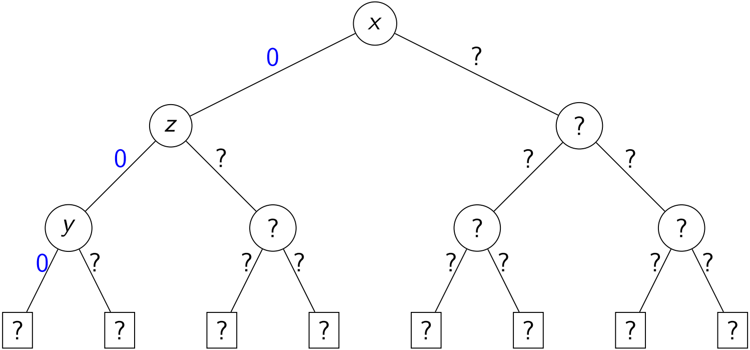

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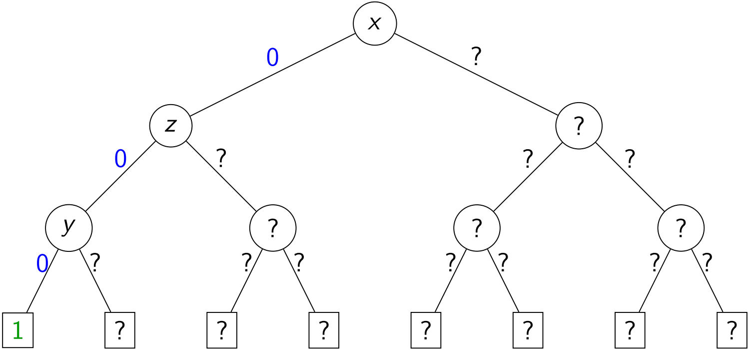

Decision heuristics

Jersolow-Wang method

Compute for every literal l the following static value:

$$J(l) : \sum_{l \in c, c \in \phi} 2^{-|c|}$$

c – clause, ϕ – formula

- Choose a literal l that maximizes $J(l)$.
- This gives an exponentially higher weight to literals in shorter clauses

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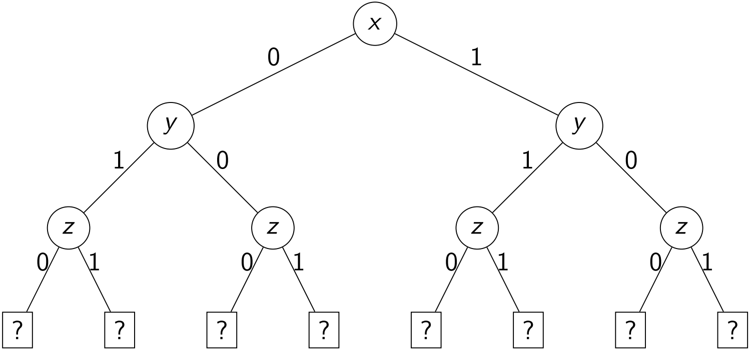

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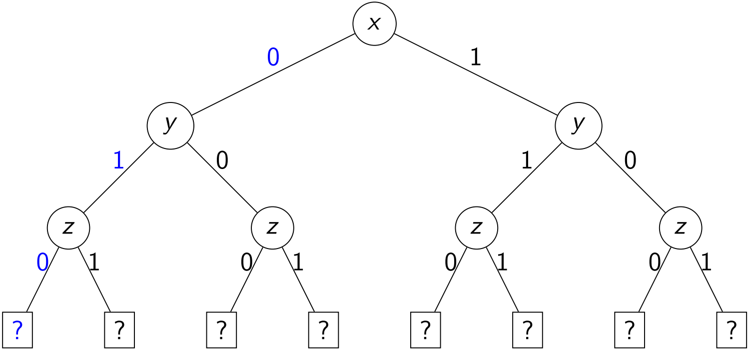

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