Formal Methods in Software Developement

Propositional Logic - refresher

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Based on slides of the lecture Satisfiability Checking (Erika Ábrahám), RTWH Aachen

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Propositional logic

The slides are partly taken from:

www.decision-procedures.org/slides/

Propositional logic - Outline

- Syntax of propositional logic
- Semantics of propositional logic
- Satisfiability and validity
- Normal forms
- Enumeration and deduction

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$$\begin{array}{rcccl} & & \perp & & := (a \wedge \neg a) \\ & & \top & & := (a \vee \neg a) \\ (& \varphi_1 & \vee & \varphi_2 &) := \neg((\neg\varphi_1) \wedge (\neg\varphi_2)) \\ (& \varphi_1 & \rightarrow & \varphi_2 &) := \end{array}$$

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$$\begin{array}{ccccc}
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(\ \varphi_1 & \lor & \varphi_2 \ ) & & := \neg((\neg\varphi_1) \land (\neg\varphi_2)) \\
(\ \varphi_1 & \rightarrow & \varphi_2 \ ) & & := ((\neg\varphi_1) \lor \varphi_2) \\
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\end{array}$$

Formulae

- Examples of well-formed formulae:
  - $(\neg a)$
  - $(\neg(\neg a))$
  - $(a \wedge (b \wedge c))$
  - $(a \rightarrow (b \rightarrow c))$

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- Examples of well-formed formulae:
  - $(\neg a)$
  - $(\neg(\neg a))$
  - $(a \wedge (b \wedge c))$
  - $(a \rightarrow (b \rightarrow c))$
- We omit parentheses whenever we may restore them through operator precedence:

  binds stronger

  $\longleftarrow$

  $\neg \quad \wedge \quad \vee \quad \rightarrow \quad \leftrightarrow$

  chaining the same operator: left binds stronger
  e.g., $a \rightarrow b \rightarrow c$ means $((a \rightarrow b) \rightarrow c)$

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Structures for predicate logic:

- The domain is $\mathbb{B} = \{0, 1\}$.
- The interpretation assigns Boolean values to the variables:

$$\alpha : AP \to \{0, 1\}$$

We call these special interpretations assignments and use *Assign* to denote the set of all assignments.

Semantics: Assignments

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We call these special interpretations assignments and use *Assign* to denote the set of all assignments.

Example: $AP = \{a, b\}, \alpha(a) = 0, \alpha(b) = 1$

Semantics I: Truth tables

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p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$	$p \bigoplus q$
0	0	1	0	0	1	1	0
0	1	1	0	1	1	0	1
1	0	0	0	1	0	0	1
1	1	0	1	1	1	1	0

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1	1	0	1	1	1	1	0

Each possible assignment is covered by a line of the truth table.
α satisfies φ iff in the line for α and the column for φ the entry is 1.

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- Q: Does α satisfy φ?
- A1: Replace values of α in φ.

Semantics II: Satisfaction relation

Satisfaction relation: $\models \subseteq \mathit{Assign} \times \mathit{PropForm}$

Instead of $(\alpha, \varphi) \in \models$ we write $\alpha \models \varphi$ and say that

- α satisfies φ or
- φ holds for α or
- α is a model of φ.

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\alpha \models \varphi_1 \lor \varphi_2 & \textit{iff} & \alpha \models \varphi_1 \textit{ or } \alpha \models \varphi_2 \\
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\alpha \models \varphi_1 \rightarrow \varphi_2 & \mathit{iff} & \alpha \models \varphi_1 \ \mathit{implies}\ \alpha \models \varphi_2
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Note: More elegant but semantically equivalent to truth tables.

Semantics II: Example

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A2: Compute with the satisfaction relation:

$$
\begin{array}{ll}
 & \alpha \models (a \vee (b \rightarrow c)) \\
\textit{iff} & \alpha \models a \textit{ or } \alpha \models (b \rightarrow c) \\
\textit{iff} & \alpha \models a \textit{ or } (\alpha \models b \textit{ implies } \alpha \models c) \\
\textit{iff} & 0 \textit{ or } (0 \textit{ implies } 1) \\
\textit{iff} & 0 \textit{ or } 1 \\
\textit{iff} & 1
\end{array}
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Semantics III: The algorithmic view

- Using the satisfaction relation we can define an algorithm for the problem to decide whether an assignment $\alpha : AP \rightarrow \{0,1\}$ is a model of a propositional logic formula $\varphi \in \mathit{PropForm}$:

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Eval(α, φ) {
   if φ ≡ a return α(a);
   if φ ≡ (¬φ1) return not Eval(α, φ1);
   if φ ≡ (φ1 op φ2)
         return Eval(α, φ1) ⟦op⟧ Eval(α, φ2);
}
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}
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- Equivalent to the $\models$ relation, but from the algorithmic view.

Semantics III: Example

- Recall our example
  - $\varphi = (a \vee (b \rightarrow c))$
  - $\alpha : \{a, b, c\} \rightarrow \{0, 1\}$ with $\alpha(a) = 0$, $\alpha(b) = 0$, and $\alpha(c) = 1$.

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- $Eval(\alpha,\ \varphi)\ =\ Eval(\alpha,\ a)\ or\ Eval(\alpha,\ b \rightarrow c) =$

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- Hence, $\alpha \models \varphi$.

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- For $\varphi \in PropForm$ and $\alpha \in Assign$ it holds that

$$\alpha \models \varphi \qquad \textit{iff} \qquad \alpha \in sat(\varphi)$$

Satisfying assignments: Example

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Short summary for propositional logic

- Syntax of propositional formulae $\varphi \in \mathit{PropForm}$:

$$\varphi \ := \ AP \mid (\neg\varphi) \mid (\varphi \wedge \varphi)$$

- Semantics:
  - Assignments $\alpha \in \mathit{Assign}$:

$$\begin{array}{l} \alpha : AP \rightarrow \{0,1\} \\ \alpha \in 2^{AP} \\ \alpha \in \{0,1\}^{AP} \end{array}$$

  - Satisfaction relation:

$$\begin{array}{llll} \models \ \subseteq \mathit{Assign} \times \mathit{PropForm} & , & (\text{e.g., } \alpha & \models \varphi \ ) \\ \models \ \subseteq 2^{\mathit{Assign}} \times \mathit{PropForm} & , & (\text{e.g., } \{\alpha_1, \ldots, \alpha_n\} & \models \varphi \ ) \\ \models \ \subseteq \mathit{PropForm} \times \mathit{PropForm}, & & (\text{e.g., } \varphi_1 & \models \varphi_2) \\ \mathit{sat} : \ \mathit{PropForm} \rightarrow 2^{\mathit{Assign}} & , & (\text{e.g., } \ \mathit{sat}(\varphi) & \ ) \end{array}$$

Propositional logic - Outline

- Syntax of propositional logic
- Semantics of propositional logic
- Satisfiability and validity
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Semantic classification of formulae

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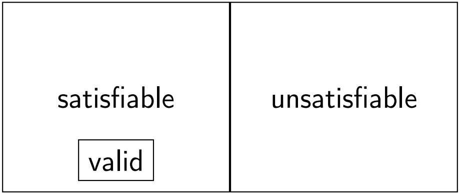

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- Some more (De Morgan rules):
  - $\models \neg(a \wedge b) \leftrightarrow (\neg a \vee \neg b)$
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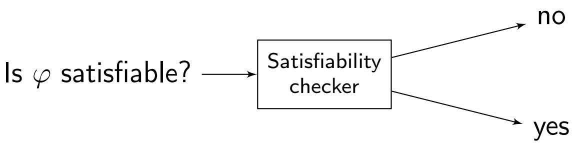

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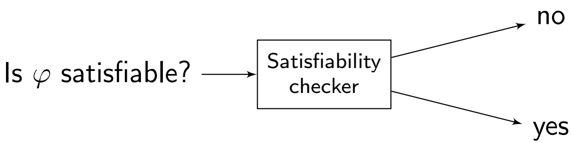

Note: A formula φ is valid iff $\neg\varphi$ is unsatisfiable.

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- $\varphi_2 = \neg a \wedge b$ is in NNF

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- Every formula can be converted to NNF in linear time:
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- DNF is a special case of NNF.

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$$\begin{aligned}\varphi &= (a \vee b) \wedge (\neg c \vee d) \\ &= ((a \vee b) \wedge (\neg c)) \vee ((a \vee b) \wedge d) \\ &= (a \wedge \neg c) \vee (b \wedge \neg c) \vee (a \wedge d) \vee (b \wedge d)\end{aligned}$$

- Now consider $\varphi_n = (a_1 \vee b_1) \wedge (a_2 \vee b_2) \wedge \ldots \wedge (a_n \vee b_n)$.
- Q: How many clauses will the DNF have?

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$$\begin{aligned}\varphi &= (a \vee b) \wedge (\neg c \vee d) \\ &= ((a \vee b) \wedge (\neg c)) \vee ((a \vee b) \wedge d) \\ &= (a \wedge \neg c) \vee (b \wedge \neg c) \vee (a \wedge d) \vee (b \wedge d)\end{aligned}$$

- Now consider $\varphi_n = (a_1 \vee b_1) \wedge (a_2 \vee b_2) \wedge \ldots \wedge (a_n \vee b_n)$.
- Q: How many clauses will the DNF have?

  A: 2^n

Satisfiability of DNF

- Q: Is the following DNF formula satisfiable?

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  A: Yes, because the term $a_2 \wedge a_1$ is satisfiable.

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- Q: Is the following DNF formula satisfiable?

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- A: No, it would violate the NP-completeness of the problem.

Conjunctive Normal Form (CNF)

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- Also CNF is a special case of NNF.

Converting to CNF

- Every formula can be converted to CNF in exponential time and space:
  1. Convert to NNF
  2. Distribute disjunctions following the rule:
     $\models \varphi_1 \vee (\varphi_2 \wedge \varphi_3) \leftrightarrow (\varphi_1 \vee \varphi_2) \wedge (\varphi_1 \vee \varphi_3)$

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- Every formula can be converted to CNF in linear time and space if new variables are added.
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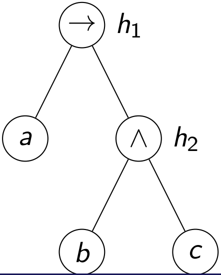

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- Associate a new auxiliary variable with each gate.

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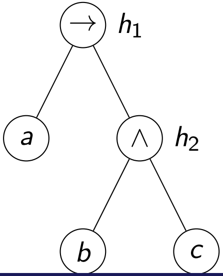

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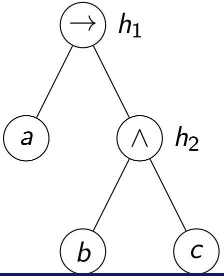

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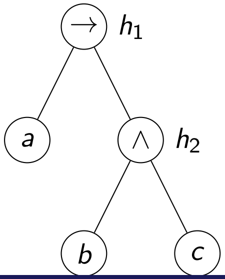
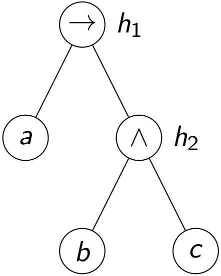
- Consider the formula
  $\varphi = (a \to (b \wedge c))$
- Associate a new auxiliary variable with each gate.
- Add constraints that define these new variables.
- Finally, enforce the root node.

Parse tree:



Converting to CNF: Tseitin's encoding

- Need to satisfy:
  $(h_1 \leftrightarrow (a \rightarrow h_2))\wedge$
  $(h_2 \leftrightarrow (b \wedge c))\wedge$
  (h_1)



- Each gate encoding has a CNF representation with 3 or 4 clauses.

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- Second: $(\neg h_2 \vee b) \wedge (\neg h_2 \vee c) \wedge (h_2 \vee \neg b \vee \neg c)$

Converting to CNF: Tseitin's encoding

- Let's go back to

  $\varphi_n = (x_1 \wedge y_1) \vee (x_2 \wedge y_2) \vee \cdots \vee (x_n \wedge y_n)$

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- With Tseitin's encoding we need:
  - n auxiliary variables $h_1, \ldots, h_n$.
  - Each adds 3 constraints.
  - Top clause: $(h_1 \vee \cdots \vee h_n)$

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- With Tseitin's encoding we need:
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- Hence, we have
  - $3n + 1$ clauses, instead of 2^n.
  - $3n$ variables rather than $2n$.

Propositional logic - Outline

- Syntax of propositional logic
- Semantics of propositional logic
- Satisfiability and validity
- Normal forms

Two classes of algorithms for validity

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- More generally (beyond propositional logic):
  - Enumeration is possible only in some logics.
  - Deduction cannot necessarily be fully automated.

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- Q: What is the difference?
  A: Branching on complete vs. partial assignments.

Deduction requires axioms and inference rules

- Inference rules:

$$\frac{\mathit{Antecedents}}{\mathit{Consequents}} \qquad (\mathit{rule\ name})$$

Meaning: If all antecedents hold then at least one of the consequents can be derived.

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- Examples:

$$\frac{a \to b \qquad b \to c}{a \to c} \qquad (Trans)$$

$$\frac{a \to b \qquad a}{b} \qquad (M.P.)$$

Axioms

- Axioms are inference rules with no antecedents, e.g.,

$$\frac{}{a \rightarrow (b \rightarrow a)} \quad (H1)$$

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$$\frac{}{a \to (b \to a)} \quad (H1)$$

- A proof system consists of a set of axioms and inference rules.

Proofs

- Let $\mathcal{H}$ be a proof system.
- $\Gamma \vdash_{\mathcal{H}} \varphi$ means: There is a proof of φ in system $\mathcal{H}$ whose premises are included in Γ
- $\vdash_{\mathcal{H}}$ is called the provability (derivability) relation.

Example

- Let $\mathcal{H}$ be the proof system comprised of the rules Trans and M.P. that we saw earlier:

$$\frac{a \to b \quad b \to c}{a \to c} \qquad (Trans)$$

$$\frac{a \to b \quad a}{b} \qquad (M.P.)$$

- Does the following relation hold?

$$a \to b,\ b \to c,\ c \to d,\ d \to e,\ a \quad \vdash_{\mathcal{H}} \quad e$$

Deductive proof: Example

$$\frac{a \to b \quad b \to c}{a \to c} \quad (Trans) \qquad \frac{a \to b \quad a}{b} \quad (M.P.)$$

$$a \to b,\; b \to c,\; c \to d,\; d \to e,\; a \quad \vdash_{\mathcal{H}} \quad e$$

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$$a \to b,\ b \to c,\ c \to d,\ d \to e,\ a \quad \vdash_{\mathcal{H}} \quad e$$

1. $a \to b$ *premise*

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3. $a \to c$ 1, 2, *Trans*

Deductive proof: Example

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2. $b \to c$ *premise*
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4. $c \to d$

Deductive proof: Example

$$\frac{a \rightarrow b \quad b \rightarrow c}{a \rightarrow c} \quad (Trans) \qquad \frac{a \rightarrow b \quad a}{b} \quad (M.P.)$$

$$a \rightarrow b,\ b \rightarrow c,\ c \rightarrow d,\ d \rightarrow e,\ a \quad \vdash_{\mathcal{H}} \quad e$$

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3. $a \rightarrow c$ 1, 2, *Trans*
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5. $d \rightarrow e$ *premise*
6. $c \rightarrow e$ 4, 5, *Trans*

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6. $c \to e$ 4, 5, *Trans*
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4. $c \to d$ *premise*
5. $d \to e$ *premise*
6. $c \to e$ 4, 5, *Trans*
7. $a \to e$ 3, 6, *Trans*

Deductive proof: Example

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7. $a \to e$ 3, 6, *Trans*
8. a

Deductive proof: Example

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$$a \to b,\ b \to c,\ c \to d,\ d \to e,\ a \quad \vdash_{\mathcal{H}} \quad e$$

1. $a \to b$ *premise*
2. $b \to c$ *premise*
3. $a \to c$ 1, 2, *Trans*
4. $c \to d$ *premise*
5. $d \to e$ *premise*
6. $c \to e$ 4, 5, *Trans*
7. $a \to e$ 3, 6, *Trans*
8. a *premise*

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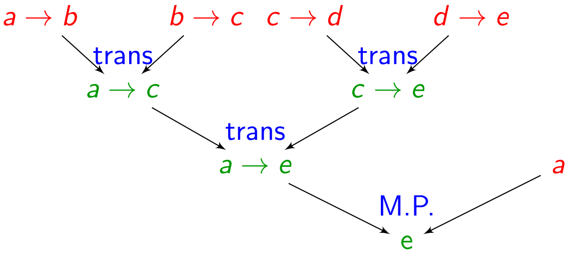
9.	e	7, 8, *M.P.*

Proof graph



$a \rightarrow b$
$b \rightarrow c$
$c \rightarrow d$
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trans
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  With respect to the semantic definition of the logic. In the case of propositional logic truth tables give us this.

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- How to prove soundness and completeness?

Example: Hilbert axiom system (H)

- Let H be (M.P.) together with the following axiom schemes:

$$\frac{}{a \rightarrow (b \rightarrow a)} \quad (H1)$$

$$\frac{}{((a \rightarrow (b \rightarrow c)) \rightarrow ((a \rightarrow b) \rightarrow (a \rightarrow c)))} \quad (H2)$$

$$\frac{}{(\neg b \rightarrow \neg a) \rightarrow (a \rightarrow b)} \quad (H3)$$

- H is sound and complete for propositional logic.

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a	b	$a \to (b \to a)$
0	0	1
0	1	1
1	0	1
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- Completeness: harder, but possible.

The resolution proof system

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- The resolution inference rule for CNF:

$$\frac{(l \lor l_1 \lor l_2 \lor \ldots \lor l_n) \quad (\neg l \lor l'_1 \lor \ldots \lor l'_m)}{(l_1 \lor \ldots \lor l_n \lor l'_1 \lor \ldots \lor l'_m)}\ \textit{Resolution}$$

- Example:

$$\frac{(a \lor b) \quad (\neg a \lor c)}{(b \lor c)}$$

- We first see some example proofs, before proving soundness and completeness.

Proof by resolution

- Let $\varphi = (a_1 \vee a_3) \wedge (\neg a_1 \vee a_2 \vee a_5) \wedge (\neg a_1 \vee a_4) \wedge (\neg a_1 \vee \neg a_4)$
- We want to prove $\varphi \ \rightarrow \ (a_3)$

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$$
\begin{tikzcd}
 & (\neg a_1 \vee a_4) \arrow[dr] & & (\neg a_1 \vee \neg a_4) \arrow[dl] \\
(a_1 \vee a_3) \arrow[dr] & & (\neg a_1) \arrow[dl] & \\
 & (a_3) & &
\end{tikzcd}
$$

Resolution

- Resolution is a sound and complete proof system for CNF.
- If the input formula is unsatisfiable, there exists a proof of the empty clause.

Example

Let $\varphi = (a_1 \vee a_3) \wedge (\neg a_1 \vee a_2) \wedge (\neg a_1 \vee a_4) \wedge (\neg a_1 \vee \neg a_4) \wedge (\neg a_3)$.

$$
\begin{tikzcd}
(\neg a_1 \vee a_4) \arrow[dr] & & (\neg a_1 \vee \neg a_4) \arrow[dl] & & \\
& (\neg a_1) \arrow[dr] & & (a_1 \vee a_3) \arrow[dl] & \\
& & a_3 \arrow[dr] & & \neg a_3 \arrow[dl] \\
& & & () &
\end{tikzcd}
$$

Soundness and completeness of resolution

- Soundness

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- Soundness is straightforward. Just prove by truth table that
  $\models ((\varphi_1 \vee a) \wedge (\varphi_2 \vee \neg a)) \to (\varphi_1 \vee \varphi_2)$.

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- Completeness is a bit more involved.

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- Soundness is straightforward. Just prove by truth table that

$$\models ((\varphi_1 \vee a) \wedge (\varphi_2 \vee \neg a)) \rightarrow (\varphi_1 \vee \varphi_2).$$

- Completeness is a bit more involved.
  Basic idea: Use resolution for variable elimination.

$$\begin{gathered}(a \vee \varphi_1) \wedge \ldots \wedge (a \vee \varphi_n) \wedge \\ (\neg a \vee \psi_1) \wedge \ldots (\neg a \vee \psi_m) \wedge \\ R \\ \Leftrightarrow \\ (\varphi_1 \vee \psi_1) \wedge \ldots \wedge (\varphi_1 \vee \psi_m) \wedge \\ \ldots \\ (\varphi_n \vee \psi_1) \wedge \ldots (\varphi_n \vee \psi_m) \wedge \\ R\end{gathered}$$

  where φ_i $(i = 1, \ldots, n)$, ψ_j $(j = 1, \ldots, m)$, and R contains neither a nor $\neg a$.