Formal Methods in Software Development

First-Order Logic

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Based on slides of the lecture Satisfiability Checking (Erika Ábrahám), RTWH Aachen

WS 2019/2020

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- 3 is fixed
- Fixing 1 and 2 gives different FO instances

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Constants:	$0, 1$
Variables:	$x, y, z, \ldots$
Function symbol	binary $+$

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Terms (theory expressions) are inductively defined by the following rules:

1. All constants and variables are terms.
2. If $t_1, \ldots, t_n$ $(n > 0)$ are terms and f an n-ary function symbol then $f(t_1, ..., t_n)$ is a term.

Only strings obtained by finitely many applications of these rules are terms.

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Example terms: 0, x, $+(0, 1)$, $+(x, 1)$, $+(x, +(y, 1))$

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Example terms: 0, x, $(0 + 1)$, $(x + 1)$, $(x + (y + 1))$

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(Theory) constraints are inductively defined by the following rule:

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Example constraints: $x < (x + 1)$, $\quad ((x + 1) + y) = ((x + y) + 1)$

Logical connectives and quantifiers, formulas

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1. If c is a constraint then c is a formula (called atomic formula).
2. If φ is a formula then $(\neg\varphi)$ is a formula.
3. If φ and ψ are formulas then $(\varphi \wedge \psi)$ is a formula.
4. Similar rules apply to other binary logical connectives.
5. If φ is a formula and x is a variable, then $(\forall x.\ \varphi)$ and $(\exists x.\ \varphi)$ are formulas.

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Example formulas:

- $x < (x + 1)$ (atomic formula)
- $(\neg\ x < 0)$
- $(x < (x + 1) \wedge ((x + 1) + y) = ((x + y) + 1))$
- $\forall x. \exists y.\ y = (x + 1)$

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Formalization:

1. $\forall x.\ isMen(x) \rightarrow isMortal(x)$
2. $isMen(Socrates)$
3. $isMortal(Socrates)$

Some remarks and notation

- Constants can also be seen as function symbols of arity 0.
- Sometimes equality ($=$) is included as a logical symbol.
- Note: the logical connectives negation ($\neg$) and conjunction ($\wedge$) and the existential quantifier ($\exists$) would be sufficient, the remaining syntax ($\vee, \rightarrow, \leftrightarrow, \ldots, \forall$) are syntactic sugar.

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We omit parentheses whenever we may restore them through operator precedence (with left-to-right binding for several occurrences of the same operator):

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Thus, we write:

$$\begin{array}{lll} \neg\neg a & \text{for} & (\neg(\neg a)), \\ \exists a.\ \exists b.\ (a \wedge b \rightarrow P(a,b)) & \text{for} & \exists a.\ \exists b.\ ((a \wedge b) \rightarrow P(a,b)) \end{array}$$

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- The same rule holds with $\forall$ in place of $\exists$.

Free and bound variables

Examples:

- In $P(z) \vee \forall x.\ \forall y.\ (P(x) \rightarrow Q(z))$, x and y are bound variables, z is a free variable, and w is neither bound nor free.
- In $Q(z) \vee \forall z.P(z)$, z is both bound and free.

Being free or bound is for specific occurrences of variables in a formula.

- In $Q(z) \vee \forall z.P(z)$, the first occurrence of z is free while the second is bound.

Signature Σ, Σ-formula, Σ-sentence

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The formulas

1. $\forall x.\ isMen(x) \rightarrow isMortal(x)$
2. $isMen(Socrates)$
3. $isMortal(Socrates)$

are Σ-sentences (the only variable x is bound).

Further examples

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- Examples of Σ-sentences:

$$\exists x.\ \forall y.\ x > y$$
$$\forall x.\ \exists y.\ x > y$$
$$\forall x.\ x + 1 > x$$
$$\forall x.\ \neg(x + 0 > x \vee x > x + 0)$$

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- $\Sigma = \{0, 1, +, *, <, \mathit{isPrime}\}$
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  - $0, 1$ constant symbols
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  - *isPrime* unary predicate symbol
- An example Σ-sentence:

  $$\forall n.\ (1 < n \rightarrow (\exists p.\ \mathit{isPrime}(p) \land n < p < 2 * n))$$

Example

- Let $\Sigma = \{0, 1, +, =\}$ where $0, 1$ are constants, $+$ is a binary function symbol and $=$ a binary predicate symbol.
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- Q: Is φ true in $\mathbb{N}_0$?
- A: Depends on the interpretation of '$+$' and '$=$'!

Structures, satisfiability, validity

- A Σ-structure is given by:
  - a domain D,
  - an interpretation I of the non-logical symbols in Σ that maps
    - each constant symbol to a domain element,
    - each function symbol of arity n to a function of type $D^n \to D$, and
    - each predicate symbol of arity n to a predicate of type $D^n \to \{0, 1\}$.
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  Notation: $S, \alpha \models \varphi$. For Σ-sentences we also write $S \models \varphi$.
- A Σ-formula φ is valid if it is satisfied by all Σ-structures and all assignments. Notation: $\models \varphi$.

Semantics

Semantics of terms and formulas under a structure $S = (D, I)$ and an assignment α:

constants:	$[\![c]\!]_{S,\alpha}$	$= I(c)$
variables:	$[\![x]\!]_{S,\alpha}$	$= \alpha(x)$
functions:	$[\![f(t_1, \ldots, t_n)]\!]_{S,\alpha}$	$= I(f)([\![t_1]\!]_{S,\alpha}, \ldots, [\![t_n]\!]_{S,\alpha})$
predicates:	$S, \alpha \models p(t_1, \ldots, t_n)$	iff $I(p)([\![t_1]\!]_{S,\alpha}, \ldots, [\![t_n]\!]_{S,\alpha})$

logical structure:

$S, \alpha \models \neg\varphi$	iff $S, \alpha \not\models \varphi$
$S, \alpha \models \varphi \wedge \psi$	iff $S, \alpha \models \varphi$ and $S, \alpha \models \psi$
$S, \alpha \models \exists x.\ \varphi$	iff there exists $v \in D$ such that $S, \sigma[x \mapsto v] \models \varphi$

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- Q: Is φ satisfiable?
- A: Yes. Consider the structure S:
  - Domain: $\mathbb{N}_0$
  - Interpretation:
    - 0 and 1 are mapped to 0 and 1 in $\mathbb{N}_0$
    - $+$ means addition
    - $=$ means equality

S satisfies φ. S is said to be a model of φ.

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- Q: Is φ valid?
- A: No. Consider the structure S':
  - Domain: $\mathbb{N}_0$
  - Interpretation:
    - 0 and 1 are mapped to 0 and 1 in $\mathbb{N}_0$
    - $+$ means multiplication
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  S' does not satisfy φ.

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- The number of sentences that are necessary for defining a theory may be large or infinite.
- Instead, it is common to define a theory through a set of axioms.
- The theory is defined by these axioms and everything that can be inferred from them by a sound inference system.

Examples

- $\Sigma = \{0, 1, +, =\}$
- $\varphi = \exists x.\ x + 0 = 1$ a Σ-formula.
- We now define the Σ-theory T by the following axioms:
  1. $\forall x.\ x = x$ $\quad$ //$=$ must be reflexive
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- A: Yes, S is a model.
- Q: Is φ T-valid?

Examples

- $\Sigma = \{0, 1, +, =\}$
- $\varphi = \exists x.\ x + 0 = 1$ a Σ-formula.
- We now define the Σ-theory T by the following axioms:
  1. $\forall x.\ x = x$ $\quad$ ($=$ is reflexive)
  2. $\forall x, y, z.\ ((x = y \wedge y = z) \to x = z)$ $\quad$ ($=$ is transitive)
  3. $\forall x.\ \forall y.\ x + y = y + x$ $\quad$ ($+$ is commutative)
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- Q: Is φ T-satisfiable?
- A: Yes, S is a model.
- Q: Is φ T-valid?
- A: Yes. (S' does not satisfy the fourth axiom.)

Example

- $\Sigma = \{=\}$
- $\varphi = (x = y \wedge y \neq z) \to x \neq z$ a Σ-formula
- We now define the Σ-theory T by the following axioms:
  1. $\forall x.\ x = x$ (reflexivity)
  2. $\forall x.\ \forall y.\ x = y \to y = x$ (symmetry)
  3. $\forall x.\ \forall y.\ \forall z.\ x = y \wedge y = z \to x = z$ (transitivity)

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- A: Yes.
- Q: Is φ T-valid?
- A: Yes. Every structure that satisfies T also satisfies φ.

Example

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- $\varphi : \forall x.\ \exists y.\ y < x$ a Σ-formula
- Consider the Σ-theory T defined by the axioms:
  1. $\forall x.\ \forall y.\ \forall z.\ x < y \land y < z \to x < z$ (transitivity)
  2. $\forall x.\ \forall y.\ x < y \to \neg(y < x)$ (anti-symmetry)

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- Q: Is φ T-valid?
- A: No. We construct a structure to the contrary:
  - Domain: $\mathbb{N}_0$
  - $<$ means "less than"

Logic fragments

- So far we only restricted the non-logical symbols by signatures and their interpretation by theories.
- Sometimes we want to restrict the grammar and the logical symbols that we can use as well.
- These are called logic fragments.
- Examples:
  - The quantifier-free fragment over $\Sigma = \{0, 1, +, =\}$
  - The conjunctive fragment over $\Sigma = \{0, 1, +, =\}$

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  Example: $x_1 \rightarrow (x_2 \vee x_3)$

  Thus, propositional logic is also a first-order theory.
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- Q: What if we allow quantifiers?
- A: We get the theory of quantified boolean formulas (QBF).
  Example:
  - $\forall x_1.\ \exists x_2.\ \forall x_3.\ x_1 \rightarrow (x_2 \vee x_3)$

Some famous theories

- Presburger arithmetic: $\Sigma = \{0, 1, +, >\}$ over integers
- Peano arithmetic: $\Sigma = \{0, 1, +, *, >\}$ over integers
- Linear real arithmetic: $\Sigma = \{0, 1, +, >\}$ over reals
- Real arithmetic: $\Sigma = \{0, 1, +, *, >\}$ over reals
- Theory of arrays
- Theory of pointers
- ...

The algorithmic point of view...

- It is also common to present theory fragments via an abstract grammar rather than restrictions on the generic first-order grammar.
- We assume that the interpretation of symbols is fixed to their common use.
  - Thus $+$ is plus, ...

The algorithmic point of view...

- Example: Equality logic
- Grammar:

$$\mathit{formula} ::= \quad \mathit{atom} \mid \mathit{formula} \land \mathit{formula} \mid \neg \mathit{formula}$$

$$\begin{aligned} \mathit{atom} ::= \quad & \mathit{Boolean\text{-}variable} \mid \\ & \mathit{variable} = \mathit{variable} \mid \\ & \mathit{variable} = \mathit{constant} \mid \\ & \mathit{constant} = \mathit{constant} \end{aligned}$$

- Interpretation: $=$ is equality.

Expressivity of a theory

- Each formula defines a language:
  The set of satisfying assignments (models) are the words accepted by this language.

- Consider the fragment '2-CNF':

$$\begin{array}{lcl} \textit{formula} & ::= & (\textit{literal} \vee \textit{literal}) \mid \textit{formula} \wedge \textit{formula} \\ \textit{literal} & ::= & \textit{Boolean-variable} \mid \neg \textit{Boolean-variable} \end{array}$$

- Example formula:

$$(x_1 \vee \neg x_2) \wedge (\neg x_3 \vee x_2)$$

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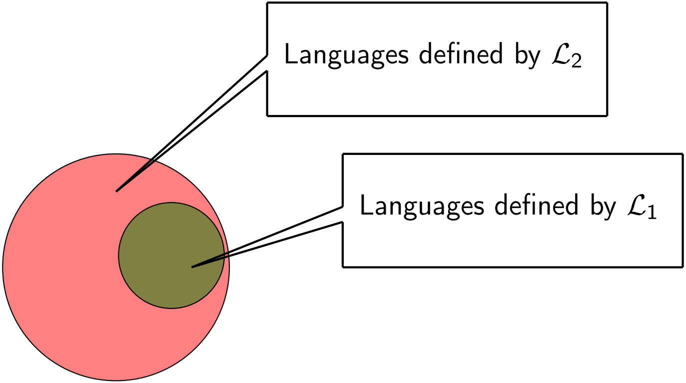
- Q: Can we express this language with 2-CNF?
- A: No.
- Proof:
  - The language accepted by φ has 7 words: all assignments other than $x_1 = x_2 = x_3 = 0$ (*false*).
  - A 2-CNF clause removes 2 assignments, which leaves us with 6 accepted words.
    E.g., $(x_1 \vee x_2)$ removes the assignments $x_1 = x_2 = x_3 = 0$ and $x_1 = x_2 = 0$, $x_3 = 1$.
  - Additional clauses only remove more assignments.

Examples



$\mathcal{L}_2$ is more expressive than $\mathcal{L}_1$. Notation: $\mathcal{L}_1 \prec \mathcal{L}_2$.

- Claim: 2-CNF $\prec$ propositional logic.

The tradeoff

- So we see that theories can have different expressive power.
- Q: Why would we want to restrict ourselves to a theory or a fragment? Why not take some 'maximal theory'?

The tradeoff

- So we see that theories can have different expressive power.
- Q: Why would we want to restrict ourselves to a theory or a fragment? Why not take some 'maximal theory'?
- A: Adding axioms to the theory may make it harder to decide or even undecidable.

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1. halts and answers *yes* iff $w \in L$
2. halts and answers *no* if $w \notin L$, or
3. does not halt if $w \notin L$.

Example: First-order Peano arithmetic

- $\Sigma = \{0, 1, +, *, =\}$
- Domain: Natural numbers
- Axioms ("semantics"):
  1. $\forall x.\ (x \neq x + 1)$
  2. $\forall x.\ \forall y.\ (x \neq y) \rightarrow (x + 1 \neq y + 1)$
  3. Induction
  4. $\forall x.\ x + 0 = x$
  5. $\forall x.\ \forall y : (x + y) + 1 = x + (y + 1)$
  6. $\forall x.\ x * 0 = 0$
  7. $\forall x.\ \forall y.\ x * (y + 1) = x * y + x$

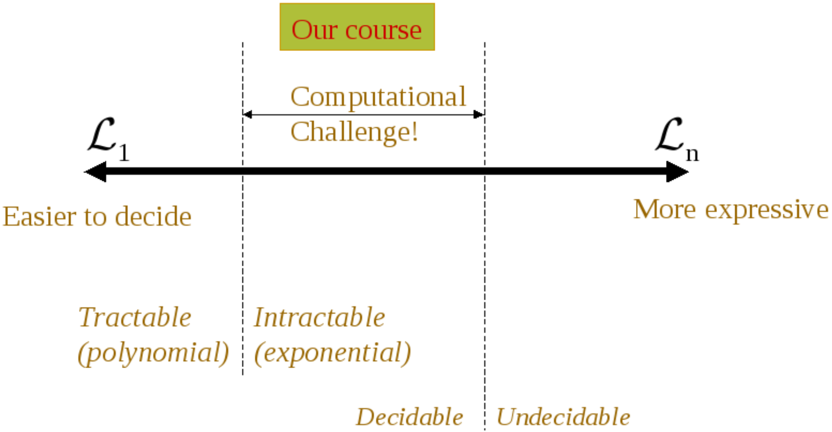
UNDECIDABLE!

Reduction: Peano arithmetic to Presburger arithmetic

- $\Sigma = \{0, 1, +, \cancel{*}, =\}$
- Domain: Natural numbers
- Axioms ("semantics"):
  1. $\forall x.\ (\neq x + 1)$
  2. $\forall x.\ \forall y.\ (x \neq y) \rightarrow (x + 1 \neq y + 1)$
  3. Induction
  4. $\forall x.\ x + 0 = x$
  5. $\forall x.\ \forall y.\ (x + y) + 1 = x + (y + 1)$
  6. $\cancel{\forall x.\ x * 0 = 0}$
  7. $\cancel{\forall x.\ \forall y.\ x * (y + 1) = x * y + x}$

DECIDABLE!

Tradeoff: Expressivity vs. computational hardness



When is a specific theory useful?

- Expressible enough to state something interesting.
- Decidable (or semi-decidable) and more efficiently solvable than richer theories.
- More expressible, or more natural for expressing some models in comparison to 'leaner' theories.