A Variant of Propositional Logic

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Idempotence, *associativity* and *commutativity* of conjunction and disjunction lead to the idea of applying conjunction and disjunction to sets of formulas (instead of applying them to two arguments). In this way, for instance, equivalence of two conjunctions is easier to decide.

Example 1.

$$A \vee B \vee C \equiv B \vee C \vee A$$

We have

$$A \vee B \vee C : \{A, B, C\}$$
$$B \vee C \vee A : \{B, C, A\}$$

Example 2.

$$(A \ \Rightarrow C \ ) \ \wedge \ (\neg C \ \Rightarrow \neg B \ ) \quad \equiv \quad (B \vee A) \ \Rightarrow \ C.$$

We have

$$(A \ \Rightarrow C \ ) \ \wedge \ (\neg C \ \Rightarrow \neg B \ ) \equiv (\neg A \ \vee C \ ) \ \wedge \ (C \ \vee \neg B \ ) \equiv \wedge \{\vee\{\neg A, C\} \ , \{C, \neg B\}\}$$
$$(B \vee A) \ \Rightarrow \ C \equiv \neg (B \vee A) \vee C \equiv (\neg B \wedge \neg A) \vee C \equiv (\neg B \vee C) \wedge (\neg A \vee C) \equiv \wedge \{\vee\{\neg B, C\} \ , \{\neg A, C\}\}$$

1 Syntax

Let $\mathcal{V}$ be an enumerable set, whose elements will be called propositional variables (e.g. P, Q, R, ...).

Definition 1. *The set $\mathcal{F}$ of propositional formulas is the smallest set having the properties:*

1. *$\mathcal{V} \subseteq \mathcal{F}$ (expressions consisting of one variable)*
2. *For all $A \in \mathcal{F}$, $\neg A \in \mathcal{F}$.*
3. *For all $\mathcal{A} \subset \mathcal{F}$, $\wedge \mathcal{A} \subset \mathcal{F}$.*

4. *For all* $\mathcal{A} \subset \mathcal{F}$, $\vee\mathcal{A} \subset \mathcal{F}$.

Formulas of type $P \in \mathcal{V}$ are called *atoms*, formulas of type P and $\neg P$ (also denoted by $\bar{P} \in \mathcal{V}$) are called *literals*. Note that $\mathcal{A}$ may be empty. Then we denote $\wedge\emptyset$ by $\mathbb{T}$ and $\vee\emptyset$ by $\mathbb{F}$.

2 Semantics

An interpretation I is a subset of $\mathcal{V}$: $I \subset \mathcal{V}$.
Given an interpretation I, the truth value of a formula A under the interpretation I ($\langle A\rangle_I$) is defined by:

1. For any $v \in \mathcal{V}:\ \langle v\rangle_I = \mathbb{T}$ if $v \in \mathcal{V}$ and $\mathbb{F}$ otherwise.
2. For all $A \in \mathcal{F}:\ \langle\neg A\rangle_I = \mathbb{T}$ if $\langle A\rangle_I = \mathbb{F}$ and $\mathbb{F}$ otherwise.
3. For all $\mathcal{A} \subset \mathcal{F} : \langle\wedge\mathcal{A}\rangle_I = \mathbb{T}$ if $\langle A\rangle_I = \mathbb{T}$ for all $A \in \mathcal{A}$ and $\mathbb{F}$ otherwise.
4. For all $\mathcal{A} \subset \mathcal{F} : \langle\vee\mathcal{A}\rangle_I = \mathbb{T}$ if $\langle A\rangle_I = \mathbb{T}$ for some $A \in \mathcal{A}$ and $\mathbb{F}$ otherwise.

Remark 1. $\wedge\{\varphi\} \equiv \varphi$, $\vee\{\varphi\} \equiv \varphi$.
Why *singletons are defined like this?*
$\langle\wedge\{\varphi\}\rangle_I = \mathbb{T}$ *iff* $\underset{\varphi\in\{\varphi\}}{\forall}\langle\varphi\rangle_I = \mathbb{T}$ *iff* $\langle\varphi\rangle_I = \mathbb{T}$

Remark 2. $\wedge\emptyset = \mathbb{T}$, $\vee\emptyset = \mathbb{F}$
Why *the truth constants are defined like this?*

$$\wedge\emptyset = \mathbb{T} \textit{ iff } \underset{\varphi\in\emptyset}{\forall}\langle\varphi\rangle_I = \mathbb{T} \textit{ iff } \underset{\varphi}{\forall}\underbrace{\left(\underbrace{\varphi\in\emptyset}_{\mathbb{F}} \Rightarrow \langle\varphi\rangle_I = \mathbb{T}\right)}_{\mathbb{T}}$$

3 Exercise

1. Prove $\neg\bigvee\Phi \equiv \bigwedge\bar{\Phi}$.

   Proof. We need to prove that for all I, $\langle\neg\bigvee\Phi\rangle_I = \left\langle\bigwedge\bar{\Phi}\right\rangle_I$. Let I be arbitrary.

   - $\langle\neg\bigvee\Phi\rangle_I = \mathbb{T}$ iff $\langle\bigvee\Phi\rangle_I = \mathbb{F}$ iff $\neg(\langle\bigvee\Phi\rangle_I = \mathbb{T})$ iff $\neg\left(\underset{\varphi\in\Phi}{\exists}\langle\varphi\rangle_I = \mathbb{T}\right)$ iff $\underset{\varphi\in\Phi}{\forall}\neg(\langle\varphi\rangle_I = \mathbb{T})$ iff $\underset{\varphi\in\Phi}{\forall}\langle\varphi\rangle_I = \mathbb{F}$
   - $\left\langle\bigwedge\bar{\Phi}\right\rangle_I = \mathbb{T}$ iff $\underset{\varphi\in\Phi}{\forall}\langle\bar{\varphi}\rangle_I = \mathbb{T}$ iff $\underset{\varphi\in\Phi}{\forall}\langle\varphi\rangle_I = \mathbb{F}$