Propositional Logic

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1 Syntax

Definition 1 (Syntax) *A* proposition *is a declarative sentence that is either true ($\mathbb{T}$) or false ($\mathbb{F}$), but not both.*

Can you give some examples?
Symbols like P, Q, R, etc. used for denoting propositions are called *atomic formulas* or *atoms*.
Complex propositions are built using logical connectives: $\neg$, $\wedge$, $\vee$, , $\Rightarrow$ (implication), $\iff$ (equivalence).

Definition 2 Well-formed formulas (formulas) *in propositional logic are defined recursively as follows:*

1. *An* atom *is a formula.*
2. *If G is a formula, then $\neg G$ is a formula.*
3. *If G and H are formulas, then $G \wedge H$, $G \vee H$, $G \Rightarrow H$, and $G \iff H$ are formulas.*
4. *All formulas are generated by applying the rules above.*

The *alphabet* of propositional logic expressions

$$\{(,)\} \ \cup \ \{\neg, \wedge, \vee, \Rightarrow, \iff\} \cup \{\mathbb{F}, \mathbb{T}\} \cup \Theta,$$

where Θ is the set of all propositional variables.
What is the meaning of 4.? Can you give some examples/counterexamples of formulas?

2 Semantics

Definition 3 (Semantics) *The semantics of a formula G, is a function $f_G : \mathcal{I} \rightarrow \{\mathbb{T}, \mathbb{F}\}$ with $\mathcal{I} = \{I : Vars(G) \rightarrow \{\mathbb{T}, \mathbb{F}\}\}$.*

We introduce the notation $\langle G \rangle_I$ instead of $f_G(I)$ meaning *the truth evaluation of the formula G in the interpretation I.*

Definition 4 (Interpretation) *Given a propositional formula G, let A_1, ..., A_n be the atoms occurring in the formula G. Then an* interpretation *of G is an assignment of truth values to A_1, ..., A_n in which every A_i is assigned either $\mathbb{T}$ or $\mathbb{F}$, but not both.*

Example 1 *Evaluate the truth value of $(A \wedge (A \Rightarrow B)) \Rightarrow B$.*

To evaluate it we need to know an interpretation I as well as the semantics of the logical connectives.

$\mathcal{B}_\neg$	
$\mathbb{T}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{T}$

$\mathcal{B}_\vee$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{T}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{F}$

$\mathcal{B}_\Leftrightarrow$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{T}$

$\mathcal{B}_\wedge$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{F}$	$\mathbb{F}$

$\mathcal{B}_\Rightarrow$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{T}$	$\mathbb{T}$	$\mathbb{F}$
$\mathbb{F}$	$\mathbb{T}$	$\mathbb{T}$

Then we have

$$
\begin{aligned}
& \langle (A \wedge (A \Rightarrow B)) \Rightarrow B \rangle_I \\
= {} & \mathcal{B}_\Rightarrow (\langle A \wedge (A \Rightarrow B) \rangle_I,\ \langle B \rangle_I) \\
= {} & \ldots
\end{aligned}
$$

Definition 5 (Validity/Invalidity) *A formula F is said to be valid iff it is true under all its interpretations (For any $I \in \mathcal{I} : \langle F \rangle_I = \mathbb{T}$). A formula is said to be invalid iff it is not valid.*

Definition 6 (Inconsistent (unsatisfiable) / Consistent (satisfiable)) *A formula is said to be inconsistent (unsatisfiable) iff it is false under all its interpretations (For any $I \in \mathcal{I}$: $\langle F \rangle_I = \mathbb{F}$). A formula is said to be consistent (satisfiable) iff it is not inconsistent.*

3 Normal Forms in the Propositional Logic

Definition 7 (Equivalent Formulas) *Let F, G be two formulas. Then $F = G$ iff for any $I \in \mathcal{I} : \langle F \rangle_I = \langle G \rangle_I$.*

Proving the equivalence of two formulas:

1. By examining the truth tables of them.
2. By rewriting
3. By bringing the two formulas in the normal form

4 Equivalent transformations

Let $\square$ be the formula which is always false, $\blacksquare$ the formula which is always true.
We have the followings:

$F \iff G \;=\; (F \Rightarrow G) \wedge (G \Rightarrow F)$		
$F \Rightarrow G \;=\; \neg F \vee G$		
$F \vee G \;=\; G \vee F$	$F \wedge G \;=\; G \wedge F$	(comutativity)
$F \vee (G \vee H) \;=\; (F \vee G) \vee H$	$F \wedge (G \wedge H) \;=\; (F \wedge G) \wedge H$	(associativity)
$F \vee (G \wedge H) \;=\; (F \vee G) \wedge (F \vee H)$	$F \wedge (G \vee H) \;=\; (F \wedge G) \vee (F \wedge H)$	(distributivity)
$F \vee \square \;=\; F$	$F \wedge \blacksquare \;=\; F$	
$F \vee \blacksquare \;=\; \blacksquare$	$F \wedge \square \;=\; \square$	
$F \vee \neg F \;=\; \blacksquare$	$F \wedge \neg F \;=\; \square$	
$\neg(\neg F) \;=\; F$		
$\neg(F \vee G) \;=\; \neg F \wedge \neg G$	$\neg(F \wedge G) \;=\; \neg F \vee \neg G$	(de Morgan)

Definition 8 (Literal) *A* literal *is an atom or the negation of an atom.*

Definition 9 (Conjunctive Normal Form) *A formula F is in* conjunctive normal form (CNF) *iff F is in the form $F_1 \wedge ... \wedge F_n$, $n \geq 1$, where each F_i is a disjunction of literals.*

Definition 10 (Disjunctive Normal Form) *A formula F is in* disjunctive normal form (DNF) *iff F is in the form $F_1 \vee ... \vee F_n$, $n \geq 1$, where each F_i is a conjunction of literals.*

A formula can be brought into a normal form by following the next steps.

Step 1 Use the laws

1. $F \iff G \;=\; (F \Rightarrow G) \wedge (G \Rightarrow F)$
2. $F \Rightarrow G \;=\; \neg F \vee G$

to eliminate $\iff$ and $\Rightarrow$.

Step 2 Repeatedly use the laws

1. $\neg(\neg F) \;=\; F$

and de Morgan's laws

1. $\neg (F \vee G) \quad = \quad \neg F \wedge \neg G$
2. $\neg (F \wedge G) \quad = \quad \neg F \vee \neg G$

to bring the negation signs immediately before atoms.

Step 3 Repeatedly use the distributive laws

1. $F \vee (G \wedge H) \quad = \quad (F \vee G) \wedge (F \vee H)$
2. $F \wedge (G \vee H) \quad = \quad (F \wedge G) \vee (F \wedge H)$

and the other laws to obtain a normal form.