Formal Methods in Software Developement

Propositional Logic - refresher

Mădălina Erașcu

West University of Timișoara
Faculty of Mathematics and Informatics

Based on slides of the lecture Satisfiability Checking (Erika Ábrahám), RTWH Aachen

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Propositional logic

The slides are partly taken from:

www.decision-procedures.org/slides/

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- Syntax of propositional logic
- Semantics of propositional logic
- Satisfiability and validity
- Normal forms
- Enumeration and deduction

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Formulae

- Examples of well-formed formulae:
  - $(\neg a)$
  - $(\neg(\neg a))$
  - $(a \wedge (b \wedge c))$
  - $(a \rightarrow (b \rightarrow c))$

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- Examples of well-formed formulae:
  - $(\neg a)$
  - $(\neg(\neg a))$
  - $(a \wedge (b \wedge c))$
  - $(a \to (b \to c))$
- We omit parentheses whenever we may restore them through operator precedence:

$$\xleftarrow{\text{binds stronger}}$$

$$\neg \quad \wedge \quad \vee \quad \to \quad \leftrightarrow$$

chaining the same operator: left binds stronger
e.g., $a \to b \to c$ means $((a \to b) \to c)$

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Semantics: Assignments

Structures for predicate logic:

- The domain is $\mathbb{B} = \{0, 1\}$.
- The interpretation assigns Boolean values to the variables:

$$\alpha : AP \to \{0, 1\}$$

We call these special interpretations assignments and use *Assign* to denote the set of all assignments.

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Example: $AP = \{a, b\}, \alpha(a) = 0, \alpha(b) = 1$

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p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$	$p \bigoplus q$
0	0	1	0	0	1	1	0
0	1	1	0	1	1	0	1
1	0	0	0	1	0	0	1
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Each possible assignment is covered by a line of the truth table.
α satisfies φ iff in the line for α and the column for φ the entry is 1.

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- Q: Does α satisfy φ?
- A1: Replace values of α in φ.

Semantics II: Satisfaction relation

Satisfaction relation: $\models \subseteq \mathit{Assign} \times \mathit{PropForm}$

Instead of $(\alpha, \varphi) \in \models$ we write $\alpha \models \varphi$ and say that

- α satisfies φ or
- φ holds for α or
- α is a model of φ.

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Note: More elegant but semantically equivalent to truth tables.

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A2: Compute with the satisfaction relation:

$$
\begin{array}{ll}
 & \alpha \models (a \vee (b \rightarrow c)) \\
\textit{iff} & \alpha \models a \textit{ or } \alpha \models (b \rightarrow c) \\
\textit{iff} & \alpha \models a \textit{ or } (\alpha \models b \textit{ implies } \alpha \models c) \\
\textit{iff} & 0 \textit{ or } (0 \textit{ implies } 1) \\
\textit{iff} & 0 \textit{ or } 1 \\
\textit{iff} & 1
\end{array}
$$

- Using the satisfaction relation we can define an algorithm for the problem to decide whether an assignment $\alpha : AP \to \{0, 1\}$ is a model of a propositional logic formula $\varphi \in \mathit{PropForm}$:

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\begin{array}{l}
Eval(\alpha, \varphi) \ \{ \\
\quad \textbf{if} \ \varphi \equiv a \ \textbf{return} \ \alpha(a); \\
\quad \textbf{if} \ \varphi \equiv (\neg\varphi_1) \ \textbf{return} \ \textbf{not} \ Eval(\alpha, \varphi_1); \\
\quad \textbf{if} \ \varphi \equiv (\varphi_1 \ \mathrm{op} \ \varphi_2) \\
\qquad\qquad \textbf{return} \ Eval(\alpha, \varphi_1) \ [\![\mathrm{op}]\!] \ Eval(\alpha, \varphi_2); \\
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\end{array}
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- Using the satisfaction relation we can define an algorithm for the problem to decide whether an assignment $\alpha : AP \to \{0, 1\}$ is a model of a propositional logic formula $\varphi \in PropForm$:

```
Eval(α, φ) {
   if φ ≡ a return α(a);
   if φ ≡ (¬φ1) return not Eval(α, φ1);
   if φ ≡ (φ1 op φ2)
        return Eval(α, φ1) [[op]] Eval(α, φ2);
}
```

- Equivalent to the $\models$ relation, but from the algorithmic view.

Semantics III: Example

- Recall our example
  - $\varphi = (a \vee (b \rightarrow c))$
  - $\alpha : \{a, b, c\} \rightarrow \{0, 1\}$ with $\alpha(a) = 0$, $\alpha(b) = 0$, and $\alpha(c) = 1$.

Semantics III: Example

- Recall our example
  - $\varphi = (a \vee (b \to c))$
  - $\alpha : \{a, b, c\} \to \{0, 1\}$ with $\alpha(a) = 0$, $\alpha(b) = 0$, and $\alpha(c) = 1$.

- $Eval(\alpha, \varphi) \ =$

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  - $\varphi = (a \vee (b \rightarrow c))$
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- $Eval(\alpha,\ \varphi) \ = \ Eval(\alpha,\ a) \ or \ Eval(\alpha,\ b \rightarrow c) =$

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- Hence, $\alpha \models \varphi$.

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- For $\varphi \in PropForm$ and $\alpha \in Assign$ it holds that

$$\alpha \models \varphi \qquad \textit{iff} \qquad \alpha \in sat(\varphi)$$

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Short summary for propositional logic

- Syntax of propositional formulae $\varphi \in \mathit{PropForm}$:

$$\varphi \;:=\; AP \mid (\neg\varphi) \mid (\varphi \wedge \varphi)$$

- Semantics:
  - Assignments $\alpha \in \mathit{Assign}$:

$$\begin{array}{l} \alpha : AP \to \{0,1\} \\ \alpha \in 2^{AP} \\ \alpha \in \{0,1\}^{AP} \end{array}$$

  - Satisfaction relation:

$$\begin{array}{llll} \models \subseteq \mathit{Assign} \times \mathit{PropForm} & , & (\text{e.g., } \alpha & \models \varphi\ ) \\ \models \subseteq 2^{\mathit{Assign}} \times \mathit{PropForm} & , & (\text{e.g., } \{\alpha_1, \ldots, \alpha_n\} & \models \varphi\ ) \\ \models \subseteq \mathit{PropForm} \times \mathit{PropForm}, & & (\text{e.g., } \varphi_1 & \models \varphi_2) \\ \mathit{sat} : \ \mathit{PropForm} \to 2^{\mathit{Assign}} & , & (\text{e.g., } \ \mathit{sat}(\varphi) & \ ) \end{array}$$

Propositional logic - Outline

- Syntax of propositional logic
- Semantics of propositional logic
- Satisfiability and validity
- Normal forms
- Enumeration and deduction

Semantic classification of formulae

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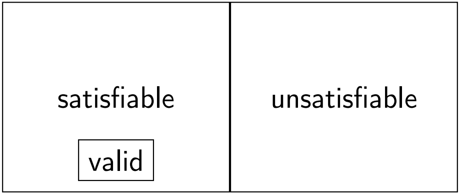

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- Some more (De Morgan rules):
  - $\models \neg(a \wedge b) \leftrightarrow (\neg a \vee \neg b)$
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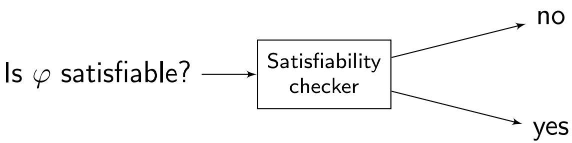

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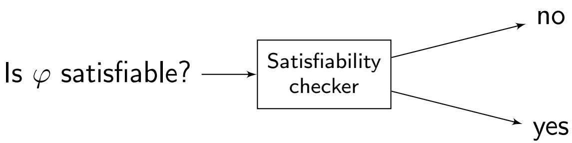

Note: A formula φ is valid iff $\neg\varphi$ is unsatisfiable.

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- $\varphi_2 = \neg a \wedge b$ is in NNF

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- DNF is a special case of NNF.

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\end{aligned}$$

- Now consider $\varphi_n = (a_1 \vee b_1) \wedge (a_2 \vee b_2) \wedge \ldots \wedge (a_n \vee b_n)$.
- Q: How many clauses will the DNF have?

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\varphi \quad &= (a \vee b) \wedge (\neg c \vee d) \\
&= ((a \vee b) \wedge (\neg c)) \vee ((a \vee b) \wedge d) \\
&= (a \wedge \neg c) \vee (b \wedge \neg c) \vee (a \wedge d) \vee (b \wedge d)
\end{aligned}
$$

- Now consider $\varphi_n = (a_1 \vee b_1) \wedge (a_2 \vee b_2) \wedge \ldots \wedge (a_n \vee b_n)$.
- Q: How many clauses will the DNF have?

  A: 2^n

Satisfiability of DNF

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A: Yes, because the term $a_2 \land a_1$ is satisfiable.

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- A: No, it would violate the NP-completeness of the problem.

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- Also CNF is a special case of NNF.

Converting to CNF

- Every formula can be converted to CNF in exponential time and space:
  1. Convert to NNF
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- Every formula can be converted to CNF in linear time and space if new variables are added.
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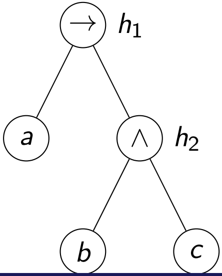

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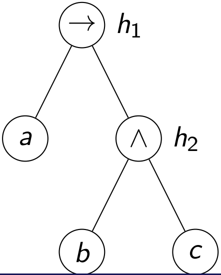

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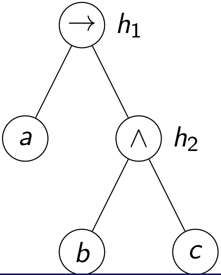
- Add constraints that define these new variables.

Parse tree:



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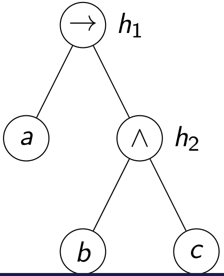
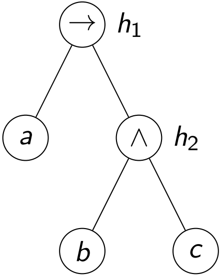
  $\varphi = (a \rightarrow (b \wedge c))$
- Associate a new auxiliary variable with each gate.
- Add constraints that define these new variables.
- Finally, enforce the root node.

Parse tree:



Converting to CNF: Tseitin's encoding

- Need to satisfy:
  $(h_1 \leftrightarrow (a \rightarrow h_2)) \wedge$
  $(h_2 \leftrightarrow (b \wedge c)) \wedge$
  (h_1)



- Each gate encoding has a CNF representation with 3 or 4 clauses.

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- Second: $(\neg h_2 \vee b) \wedge (\neg h_2 \vee c) \wedge (h_2 \vee \neg b \vee \neg c)$

Converting to CNF: Tseitin's encoding

- Let's go back to
  $\varphi_n = (x_1 \wedge y_1) \vee (x_2 \wedge y_2) \vee \cdots \vee (x_n \wedge y_n)$

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- With Tseitin's encoding we need:
  - n auxiliary variables $h_1, \ldots, h_n$.
  - Each adds 3 constraints.
  - Top clause: $(h_1 \vee \cdots \vee h_n)$

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  - n auxiliary variables $h_1, \ldots, h_n$.
  - Each adds 3 constraints.
  - Top clause: $(h_1 \vee \cdots \vee h_n)$

- Hence, we have
  - $3n + 1$ clauses, instead of 2^n.
  - $3n$ variables rather than $2n$.

Propositional logic - Outline

- Syntax of propositional logic
- Semantics of propositional logic
- Satisfiability and validity
- Normal forms