

The Art of Proving

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Outline

Proofs

Proof Rules

- Conjunction

- Disjunction

- Implication

- Equivalence

- Universal Quantification

- Existential Quantification

- Indirect Proofs

Example

Exercises

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Proofs

A **proof** is a structured argument that a formula is true.

- ▶ A tree whose nodes represent **proof situations (states)**.
- ▶ Each proof situation consists of **knowledge base** and a **goal**: $K_1, \dots, K_n \vdash G$
 - ▶ Knowledge K_1, \dots, K_n are assumed to be true
 - ▶ Goal G to be proved wrt knowledge
- ▶ The **root** of the tree is the initial proof situation.
 - ▶ Knowledge K_1, \dots, K_n – axioms of mathematical background theories
 - ▶ G – formula to be proved

Proof rules describes how a proof situation can be reduced to zero, one, or more “subsituations”.

$$\frac{\dots \vdash \dots \quad \dots \vdash \dots}{K_1, \dots, K_n \vdash G}$$

- ▶ Rule may or may not close the (sub)proof:
 - ▶ **Zero subproofs** (rule G has been proved, (sub)proof is closed).
 - ▶ **One or more subproofs** (rule G is proved, \forall subproofs are proved).
- ▶ **Top-down rules**: focus on G ; G is decomposed into simpler goals G_1, G_2, \dots
- ▶ **Bottom-up rules**: focus on K_1, \dots, K_n ; knowledge is extended to K_1, \dots, K_n, K_{n+1}

In each proof situation, we aim at showing that the goal is true with respect to the given knowledge.

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 - ▶ **Top-down rules:** G has been proved, (sub)proof is closed.
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- ▶ Rule may or may not close the (sub)proof:
 - ▶ **Top-down rules:** G has been proved, but proof is closed.
 - ▶ **Bottom-up rules:** G is proved, but not proof is proved.
- ▶ **Top-down rules:** focus on G ; G is decomposed into simpler goals G_1, G_2, \dots
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$$\frac{K \vdash G_1 \quad K \vdash G_2}{K \vdash G_1 \wedge G_2}$$

$$\frac{..., K_1 \wedge K_2, K_1, K_2 \vdash G}{..., K_1 \wedge K_2 \vdash G}$$

► Goal $G_1 \wedge G_2$

- Create two subsituations with goals G_1 and G_2 .

Subs. 1 has to prove G_1 in K_1 . Subs. 2 has to prove G_2 in K_2 .
The goal $G_1 \wedge G_2$ will prove G_1 and G_2 simultaneously with the goal G_1 .

► Knowledge $K_1 \wedge K_2$

- Create one subsituation with K_1 and K_2 in knowledge.

Sub. 1 uses K_1 in K_1 . The goal also uses K_1 and K_2 . Subs. 2 uses K_2 in K_2 .
With context K_1 and K_2 in the knowledge, the goal $G_1 \wedge G_2$ can prove G_1 and G_2 .

Conjunction

$$\frac{K \vdash G_1 \quad K \vdash G_2}{K \vdash G_1 \wedge G_2}$$

$$\frac{..., K_1 \wedge K_2, K_1, K_2 \vdash G}{..., K_1 \wedge K_2 \vdash G}$$

► Goal $G_1 \wedge G_2$

- Create two subsituations with goals G_1 and G_2 .

We have to show $G_1 \wedge G_2$. We show G_1 : ... (proof continues with the goal G_1). We show G_2 : ... (proof continues with the goal G_2).

► Knowledge $K_1 \wedge K_2$

- Create one subsituation with K_1 and K_2 in knowledge.

We have to show G under $K_1 \wedge K_2$. We show that under K_1 and K_2 (which combined with context Γ and additional knowledge K yield Σ)

Conjunction

$$\frac{K \vdash G_1 \quad K \vdash G_2}{K \vdash G_1 \wedge G_2}$$

$$\frac{..., K_1 \wedge K_2, K_1, K_2 \vdash G}{..., K_1 \wedge K_2 \vdash G}$$

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► Knowledge $K_1 \wedge K_2$

- Create one subsituation with K_1 and K_2 in knowledge.

We have to show $K_1 \wedge K_2, K_1, K_2 \vdash G$. We show K_1 : ... (proof continues with the goal K_1). We show K_2 : ... (proof continues with the goal K_2).

Conjunction

$$\frac{K \vdash G_1 \quad K \vdash G_2}{K \vdash G_1 \wedge G_2}$$

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We have to show $G_1 \wedge G_2$. We show G_1 : ... (proof continues with the goal G_1). We show G_2 : ... (proof continues with the goal G_2).

► Knowledge $K_1 \wedge K_2$

- Create one subsituation with K_1 and K_2 in knowledge.

Let's assume K_1 and K_2 are both true.
We have to show $G_1 \wedge G_2$.

Conjunction

$$\frac{K \vdash G_1 \quad K \vdash G_2}{K \vdash G_1 \wedge G_2}$$

$$\frac{..., K_1 \wedge K_2, K_1, K_2 \vdash G}{..., K_1 \wedge K_2 \vdash G}$$

► Goal $G_1 \wedge G_2$

- Create two subsituations with goals G_1 and G_2 .

We have to show $G_1 \wedge G_2$. We show G_1 : ... (proof continues with the goal G_1). We show G_2 : ... (proof continues with the goal G_2).

► Knowledge $K_1 \wedge K_2$

- Create one subsituation with K_1 and K_2 in knowledge.

We know $K_1 \wedge K_2$. We thus also know K_1 and K_2 . (proof continues with current goal and additional knowledge K_1 and K_2)

Conjunction

$$\frac{K \vdash G_1 \quad K \vdash G_2}{K \vdash G_1 \wedge G_2}$$

$$\frac{..., K_1 \wedge K_2, K_1, K_2 \vdash G}{..., K_1 \wedge K_2 \vdash G}$$

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► Goal $G_1 \wedge G_2$

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We have to show $G_1 \wedge G_2$. We show G_1 : ... (proof continues with the goal G_1). We show G_2 : ... (proof continues with the goal G_2).

► Knowledge $K_1 \wedge K_2$

- Create one subsituation with K_1 and K_2 in knowledge.

We know $K_1 \wedge K_2$. We thus also know K_1 and K_2 . (proof continues with current goal and additional knowledge K_1 and K_1)

Disjunction

$$\frac{K, \neg G_1 \vdash G_2}{K \vdash G_1 \vee G_2}$$

$$\frac{..., K_1 \vdash G \quad ..., K_2 \vdash G}{..., K_1 \vee K_2 \vdash G}$$

► Goal $G_1 \vee G_2$

- Create one subsituation where G_2 is proved under the assumption that G_1 does not hold (or vice versa):

We have to show $G_1 \vee G_2$. We proceed by cases. Case 1: $\neg G_1$.
Assume $\neg G_1$. Then we have to prove G_2 (using the goal, the additional knowledge $\neg G_1$).

► Knowledge $K_1 \vee K_2$

- Create two subsituations, one with K_1 and one with K_2 in knowledge.

We have $K_1 \vee K_2$. We have proved by the induction that K_1 is proved separately with the goal and with additional knowledge $\neg G_1$.
Case K_2 ... (goal completed with current goal and additional knowledge K_2).

Disjunction

$$\frac{K, \neg G_1 \vdash G_2}{K \vdash G_1 \vee G_2}$$

$$\frac{..., K_1 \vdash G \quad ..., K_2 \vdash G}{..., K_1 \vee K_2 \vdash G}$$

► Goal $G_1 \vee G_2$

- Create one subsituation where G_2 is proved under the assumption that G_1 does not hold (or vice versa):

We have to show $G_1 \vee G_2$. We assume $\neg G_1$ and show G_2 . (proof continues with goal G_2 and additional knowledge $\neg G_1$)

► Knowledge $K_1 \vee K_2$

- Create two subsituations, one with K_1 and one with K_2 in knowledge.

We have $K_1 \vee K_2$. We have proved by this rule that: Case K_1 is proved together with current goal and additional knowledge $\neg K_2$. Case K_2 ... (goal continues with current goal and additional knowledge $\neg K_1$)

Disjunction

$$\frac{K, \neg G_1 \vdash G_2}{K \vdash G_1 \vee G_2}$$

$$\frac{..., K_1 \vdash G \quad ..., K_2 \vdash G}{..., K_1 \vee K_2 \vdash G}$$

► Goal $G_1 \vee G_2$

- Create one subsituation where G_2 is proved under the assumption that G_1 does not hold (or vice versa):

We have to show $G_1 \vee G_2$. We assume $\neg G_1$ and show G_2 . (proof continues with goal G_2 and additional knowledge $\neg G_1$)

► Knowledge $K_1 \vee K_2$

- Create two subsituations, one with K_1 and one with K_2 in knowledge.

We have $K_1 \vee K_2$. We assume $\neg K_1$ and show K_2 .
We have $K_1 \vee K_2$. We assume $\neg K_2$ and show K_1 .
We have $K_1 \vee K_2$. We assume $\neg K_1$ and show K_2 .
We have $K_1 \vee K_2$. We assume $\neg K_2$ and show K_1 .

Disjunction

$$\frac{K, \neg G_1 \vdash G_2}{K \vdash G_1 \vee G_2}$$

$$\frac{..., K_1 \vdash G \quad ..., K_2 \vdash G}{..., K_1 \vee K_2 \vdash G}$$

► Goal $G_1 \vee G_2$

- Create one subsituation where G_2 is proved under the assumption that G_1 does not hold (or vice versa):

We have to show $G_1 \vee G_2$. We assume $\neg G_1$ and show G_2 . (proof continues with goal G_2 and additional knowledge $\neg G_1$)

► Knowledge $K_1 \vee K_2$

- Create two subsituations, one with K_1 and one with K_2 in knowledge.

We have to show $G_1 \vee G_2$ and assume $\neg G_1$.
We have to show G_2 and assume $\neg G_1$.
We have to show G_2 and assume $\neg G_1$.
We have to show G_2 and assume $\neg G_1$.
We have to show G_2 and assume $\neg G_1$.

Disjunction

$$\frac{K, \neg G_1 \vdash G_2}{K \vdash G_1 \vee G_2}$$

$$\frac{..., K_1 \vdash G \quad ..., K_2 \vdash G}{..., K_1 \vee K_2 \vdash G}$$

► **Goal $G_1 \vee G_2$**

- Create one subsituation where G_2 is proved under the assumption that G_1 does not hold (or vice versa):

We have to show $G_1 \vee G_2$. We assume $\neg G_1$ and show G_2 . (proof continues with goal G_2 and additional knowledge $\neg G_1$)

► **Knowledge $K_1 \vee K_2$**

- Create two subsituations, one with K_1 and one with K_2 in knowledge.

We know $K_1 \vee K_2$. We thus proceed by case distinction: Case K_1 : ... (proof continues with current goal and additional knowledge K_1). Case K_2 : ... (proof continues with current goal and additional knowledge K_2).

Disjunction

$$\frac{K, \neg G_1 \vdash G_2}{K \vdash G_1 \vee G_2}$$

$$\frac{..., K_1 \vdash G \quad ..., K_2 \vdash G}{..., K_1 \vee K_2 \vdash G}$$

► Goal $G_1 \vee G_2$

- Create one subsituation where G_2 is proved under the assumption that G_1 does not hold (or vice versa):

We have to show $G_1 \vee G_2$. We assume $\neg G_1$ and show G_2 . (proof continues with goal G_2 and additional knowledge $\neg G_1$)

► Knowledge $K_1 \vee K_2$

- Create two subsituations, one with K_1 and one with K_2 in knowledge.

We know $K_1 \vee K_2$. We thus proceed by case distinction: *Case K_1 :* ... (proof continues with current goal and additional knowledge K_1).
Case K_2 : ... (proof continues with current goal and additional knowledge K_2).

Disjunction

$$\frac{K, \neg G_1 \vdash G_2}{K \vdash G_1 \vee G_2}$$

$$\frac{..., K_1 \vdash G \quad ..., K_2 \vdash G}{..., K_1 \vee K_2 \vdash G}$$

► Goal $G_1 \vee G_2$

- Create one subsituation where G_2 is proved under the assumption that G_1 does not hold (or vice versa):

We have to show $G_1 \vee G_2$. We assume $\neg G_1$ and show G_2 . (proof continues with goal G_2 and additional knowledge $\neg G_1$)

► Knowledge $K_1 \vee K_2$

- Create two subsituations, one with K_1 and one with K_2 in knowledge.

We know $K_1 \vee K_2$. We thus proceed by case distinction: *Case K_1* : ... (proof continues with current goal and additional knowledge K_1).
Case K_2 : ... (proof continues with current goal and additional knowledge K_2).

Implication

$$\frac{K, G_1 \vdash G_2}{K \vdash G_1 \Rightarrow G_2}$$

$$\frac{\dots \vdash K_1 \quad \dots, K_2 \vdash G}{\dots, K_1 \Rightarrow K_2 \vdash G}$$

► Goal $G_1 \Rightarrow G_2$

- Create one subsituation where G_2 is proved under the assumption that G_1 holds:

Subsituation where G_1 holds: G_1 is added to the context. G_2 is proved under this context. The result is $G_1 \Rightarrow G_2$, and additional knowledge K_2 .

► Knowledge $K_1 \Rightarrow K_2$

- Create two subsituations, one with goal K_1 and one with knowledge K_2 .

Subsituation K_1 : K_1 is added to the context. K_2 is proved under this goal. The result is $K_1 \Rightarrow K_2$.
Subsituation K_2 : K_2 is added to the context. G is proved under this context. The result is $\dots, K_2 \vdash G$.

Implication

$$\frac{K, G_1 \vdash G_2}{K \vdash G_1 \Rightarrow G_2}$$

$$\frac{\dots \vdash K_1 \quad \dots, K_2 \vdash G}{\dots, K_1 \Rightarrow K_2 \vdash G}$$

► Goal $G_1 \Rightarrow G_2$

- Create one subsituation where G_2 is proved under the assumption that G_1 holds:

We have to show $G_1 \Rightarrow G_2$. We assume G_1 and show G_2 . (proof continues with goal G_2 and additional knowledge G_1)

► Knowledge $K_1 \Rightarrow K_2$

- Create two subsituations, one with goal K_1 and one with knowledge K_2 .

We have to show $K_1 \Rightarrow K_2$. We show K_1 and prove K_2 under the goal K_1 . We have K_2 as a goal statement with current goal and additional knowledge K_1 .

Implication

$$\frac{K, G_1 \vdash G_2}{K \vdash G_1 \Rightarrow G_2}$$

$$\frac{\dots \vdash K_1 \quad \dots, K_2 \vdash G}{\dots, K_1 \Rightarrow K_2 \vdash G}$$

- Goal $G_1 \Rightarrow G_2$

- Create one subsituation where G_2 is proved under the assumption that G_1 holds:

We have to show $G_1 \Rightarrow G_2$. We assume G_1 and show G_2 . (proof continues with goal G_2 and additional knowledge G_1)

- Knowledge $K_1 \Rightarrow K_2$

- Create two subsituations, one with goal K_1 and one with knowledge K_2 .

We have to show $K_1 \Rightarrow K_2$. We assume K_1 and show K_2 . (proof continues with goal K_2 and additional knowledge K_1)

Implication

$$\frac{K, G_1 \vdash G_2}{K \vdash G_1 \Rightarrow G_2}$$

$$\frac{\dots \vdash K_1 \quad \dots, K_2 \vdash G}{\dots, K_1 \Rightarrow K_2 \vdash G}$$

► Goal $G_1 \Rightarrow G_2$

- Create one subsituation where G_2 is proved under the assumption that G_1 holds:

We have to show $G_1 \Rightarrow G_2$. We assume G_1 and show G_2 . (proof continues with goal G_2 and additional knowledge G_1)

► Knowledge $K_1 \Rightarrow K_2$

- Create two subsituations, one with goal K_1 and one with knowledge K_2 .

We have to show $K_1 \Rightarrow K_2$. We assume K_1 and show K_2 . (proof continues with goal K_2 and additional knowledge K_1)

Implication

$$\frac{K, G_1 \vdash G_2}{K \vdash G_1 \Rightarrow G_2}$$

$$\frac{\dots \vdash K_1 \quad \dots, K_2 \vdash G}{\dots, K_1 \Rightarrow K_2 \vdash G}$$

► Goal $G_1 \Rightarrow G_2$

- Create one subsituation where G_2 is proved under the assumption that G_1 holds:

We have to show $G_1 \Rightarrow G_2$. We assume G_1 and show G_2 . (proof continues with goal G_2 and additional knowledge G_1)

► Knowledge $K_1 \Rightarrow K_2$

- Create two subsituations, one with goal K_1 and one with knowledge K_2 .

We know $K_1 \Rightarrow K_2$. We show K_1 : ... (proof continues with goal K_1). We know K_2 : ... (proof continues with current goal and additional knowledge K_2).

Implication

$$\frac{K, G_1 \vdash G_2}{K \vdash G_1 \Rightarrow G_2}$$

$$\frac{\dots \vdash K_1 \quad \dots, K_2 \vdash G}{\dots, K_1 \Rightarrow K_2 \vdash G}$$

► Goal $G_1 \Rightarrow G_2$

- Create one subsituation where G_2 is proved under the assumption that G_1 holds:

We have to show $G_1 \Rightarrow G_2$. We assume G_1 and show G_2 . (proof continues with goal G_2 and additional knowledge G_1)

► Knowledge $K_1 \Rightarrow K_2$

- Create two subsituations, one with goal K_1 and one with knowledge K_2 .

We know $K_1 \Rightarrow K_2$. We show K_1 : ... (proof continues with goal K_1). We know K_2 : ... (proof continues with current goal and additional knowledge K_2).

Implication

$$\frac{K, G_1 \vdash G_2}{K \vdash G_1 \Rightarrow G_2}$$

$$\frac{\dots \vdash K_1 \quad \dots, K_2 \vdash G}{\dots, K_1 \Rightarrow K_2 \vdash G}$$

► Goal $G_1 \Rightarrow G_2$

- Create one subsituation where G_2 is proved under the assumption that G_1 holds:

We have to show $G_1 \Rightarrow G_2$. We assume G_1 and show G_2 . (proof continues with goal G_2 and additional knowledge G_1)

► Knowledge $K_1 \Rightarrow K_2$

- Create two subsituations, one with goal K_1 and one with knowledge K_2 .

We know $K_1 \Rightarrow K_2$. We show K_1 : ... (proof continues with goal K_1). We know K_2 : ... (proof continues with current goal and additional knowledge K_2).

Equivalence

$$\frac{K \vdash G_1 \Rightarrow G_2 \quad K \vdash G_2 \Rightarrow G_1}{K \vdash G_1 \Leftrightarrow G_2}$$

$$\frac{\dots \vdash (\neg)K_1 \quad \dots, (\neg)K_2 \vdash G}{\dots, K_1 \Leftrightarrow K_2 \vdash G}$$

► Goal $G_1 \Leftrightarrow G_2$

- Create two subsituations with implications in both directions as goals:

We have to reach G_1 in the first situation, and reach G_2 in the second situation with goal G_1 in the first situation G_2 in the second situation with goal G_2 in the first situation.

► Knowledge $K_1 \Leftrightarrow K_2$

- Create two subsituations, one with goal $(\neg)K_1$ and one with knowledge $(\neg)K_2$.

We know $K_1 \Rightarrow K_2$. We show $(\neg)K_2 \Rightarrow (\neg)K_1$ (goal achieved with goal $(\neg)K_1$). We have $(\neg)K_2$ as given knowledge with current goal $(\neg)K_1$ and additional knowledge $(\neg)K_2$.

Equivalence

$$\frac{K \vdash G_1 \Rightarrow G_2 \quad K \vdash G_2 \Rightarrow G_1}{K \vdash G_1 \Leftrightarrow G_2}$$

$$\frac{\dots \vdash (\neg)K_1 \quad \dots, (\neg)K_2 \vdash G}{\dots, K_1 \Leftrightarrow K_2 \vdash G}$$

► Goal $G_1 \Leftrightarrow G_2$

- Create two subsituations with implications in both directions as goals:

We have to show $G_1 \Leftrightarrow G_2$. We show $G_1 \Rightarrow G_2$: ... (proof continues with goal $G_1 \Rightarrow G_2$). We show $G_2 \Rightarrow G_1$: ... (proof continues with goal $G_2 \Rightarrow G_1$)

► Knowledge $K_1 \Leftrightarrow K_2$

- Create two subsituations, one with goal $(\neg)K_1$ and one with knowledge $(\neg)K_2$.

We know $K_1 \Rightarrow K_2$. We show $(\neg)K_2 \Rightarrow \dots$ (proof continues with goal $(\neg)K_2$). We know $(\neg)K_2 \Rightarrow \dots$ (proof continues with current goal and additional knowledge $(\neg)K_1$).

Equivalence

$$\frac{K \vdash G_1 \Rightarrow G_2 \quad K \vdash G_2 \Rightarrow G_1}{K \vdash G_1 \Leftrightarrow G_2}$$

$$\frac{\dots \vdash (\neg)K_1 \quad \dots, (\neg)K_2 \vdash G}{\dots, K_1 \Leftrightarrow K_2 \vdash G}$$

► Goal $G_1 \Leftrightarrow G_2$

- Create two subsituations with implications in both directions as goals:

We have to show $G_1 \Leftrightarrow G_2$. We show $G_1 \Rightarrow G_2$: ... (proof continues with goal $G_1 \Rightarrow G_2$). We show $G_2 \Rightarrow G_1$: ... (proof continues with goal $G_2 \Rightarrow G_1$)

► Knowledge $K_1 \Leftrightarrow K_2$

- Create two subsituations, one with goal $(\neg)K_1$ and one with knowledge $(\neg)K_2$.

We know $K_1 \Leftrightarrow K_2$. We show $(\neg)K_1 \vdash (\neg)K_2$ (proof continues with goal $(\neg)K_1$). We know $(\neg)K_2 \vdash K_1$ (proof continues with goal $(\neg)K_2$). We know $(\neg)K_1 \vdash K_2$ (proof continues with goal $(\neg)K_1$). We know $K_2 \vdash (\neg)K_1$ (proof continues with goal $(\neg)K_2$).

Equivalence

$$\frac{K \vdash G_1 \Rightarrow G_2 \quad K \vdash G_2 \Rightarrow G_1}{K \vdash G_1 \Leftrightarrow G_2}$$

$$\frac{\dots \vdash (\neg)K_1 \quad \dots, (\neg)K_2 \vdash G}{\dots, K_1 \Leftrightarrow K_2 \vdash G}$$

► Goal $G_1 \Leftrightarrow G_2$

- Create two subsituations with implications in both directions as goals:

We have to show $G_1 \Leftrightarrow G_2$. We show $G_1 \Rightarrow G_2$: ... (proof continues with goal $G_1 \Rightarrow G_2$). We show $G_2 \Rightarrow G_1$: ... (proof continues with goal $G_2 \Rightarrow G_1$)

► Knowledge $K_1 \Leftrightarrow K_2$

- Create two subsituations, one with goal $(\neg)K_1$ and one with knowledge $(\neg)K_2$.

The knowledge base K is extended with $(\neg)K_1$ (goal $(\neg)K_1$) and $(\neg)K_2$ (knowledge $(\neg)K_2$). The goal $(\neg)K_1$ is proved by the knowledge base K extended with $(\neg)K_2$. The goal $(\neg)K_2$ is proved by the knowledge base K extended with $(\neg)K_1$.

Equivalence

$$\frac{K \vdash G_1 \Rightarrow G_2 \quad K \vdash G_2 \Rightarrow G_1}{K \vdash G_1 \Leftrightarrow G_2}$$

$$\frac{\dots \vdash (\neg)K_1 \quad \dots, (\neg)K_2 \vdash G}{\dots, K_1 \Leftrightarrow K_2 \vdash G}$$

► Goal $G_1 \Leftrightarrow G_2$

- Create two subsituations with implications in both directions as goals:

We have to show $G_1 \Leftrightarrow G_2$. We show $G_1 \Rightarrow G_2$: ... (proof continues with goal $G_1 \Rightarrow G_2$). We show $G_2 \Rightarrow G_1$: ... (proof continues with goal $G_2 \Rightarrow G_1$)

► Knowledge $K_1 \Leftrightarrow K_2$

- Create two subsituations, one with goal $(\neg)K_1$ and one with knowledge $(\neg)K_2$.

We know $K_1 \Leftrightarrow K_2$. We show $(\neg)K_1$: ... (proof continues with goal $(\neg)K_1$). We know $(\neg)K_2$: ... (proof continues with current goal and additional knowledge $(\neg)K_2$).

Equivalence

$$\frac{K \vdash G_1 \Rightarrow G_2 \quad K \vdash G_2 \Rightarrow G_1}{K \vdash G_1 \Leftrightarrow G_2}$$

$$\frac{\dots \vdash (\neg)K_1 \quad \dots, (\neg)K_2 \vdash G}{\dots, K_1 \Leftrightarrow K_2 \vdash G}$$

► Goal $G_1 \Leftrightarrow G_2$

- Create two subsituations with implications in both directions as goals:

We have to show $G_1 \Leftrightarrow G_2$. We show $G_1 \Rightarrow G_2$: ... (proof continues with goal $G_1 \Rightarrow G_2$). We show $G_2 \Rightarrow G_1$: ... (proof continues with goal $G_2 \Rightarrow G_1$)

► Knowledge $K_1 \Leftrightarrow K_2$

- Create two subsituations, one with goal $(\neg)K_1$ and one with knowledge $(\neg)K_2$.

We know $K_1 \Leftrightarrow K_2$. We show $(\neg)K_1$: ... (proof continues with goal $(\neg)K_1$). We know $(\neg)K_2$: ... (proof continues with current goal and additional knowledge $(\neg)K_2$).

Equivalence

$$\frac{K \vdash G_1 \Rightarrow G_2 \quad K \vdash G_2 \Rightarrow G_1}{K \vdash G_1 \Leftrightarrow G_2}$$

$$\frac{\dots \vdash (\neg)K_1 \quad \dots, (\neg)K_2 \vdash G}{\dots, K_1 \Leftrightarrow K_2 \vdash G}$$

► Goal $G_1 \Leftrightarrow G_2$

- Create two subsituations with implications in both directions as goals:

We have to show $G_1 \Leftrightarrow G_2$. We show $G_1 \Rightarrow G_2$: ... (proof continues with goal $G_1 \Rightarrow G_2$). We show $G_2 \Rightarrow G_1$: ... (proof continues with goal $G_2 \Rightarrow G_1$)

► Knowledge $K_1 \Leftrightarrow K_2$

- Create two subsituations, one with goal $(\neg)K_1$ and one with knowledge $(\neg)K_2$.

We know $K_1 \Leftrightarrow K_2$. We show $(\neg)K_1$: ... (proof continues with goal $(\neg)K_1$). We know $(\neg)K_2$: ... (proof continues with current goal and additional knowledge $(\neg)K_2$).

Universal Quantification

$$\frac{K \vdash G\{x \rightarrow x_0\}}{K \vdash \forall_x G} \quad (x_0 \text{ new for } K, G)$$

$$\frac{\dots, \forall_x K, K\{x \rightarrow T\} \vdash G}{\dots, \forall_x K \vdash G}$$

► Goal $\forall_x G$

- Introduce new (arbitrarily named) constant x_0 and create one subsituation with goal $G\{x \rightarrow x_0\}$.

We need to show $\forall_x G$. To do arbitrary x , we choose x_0 as a fresh constant with goal $G\{x \rightarrow x_0\}$.

► Knowledge $\forall_x K$

- Choose term T to create one subsituation with formula $K\{x \rightarrow T\}$ added to the knowledge.

We know $\forall_x K$ and thus also $K\{x \rightarrow T\}$ (ground instances w.r.t. x). We choose T as arbitrary term T .

Universal Quantification

$$\frac{K \vdash G\{x \rightarrow x_0\}}{K \vdash \forall_x G} \quad (x_0 \text{ new for } K, G)$$

$$\frac{\dots, \forall_x K, K\{x \rightarrow T\} \vdash G}{\dots, \forall_x K \vdash G}$$

► Goal $\forall_x G$

- Introduce new (arbitrarily named) constant x_0 and create one subsituation with goal $G[x \rightarrow x_0]$.

We have to show $\forall_x G$. Take arbitrary x_0 . We show $G\{x \rightarrow x_0\}$.
(proof continues with goal $G\{x \rightarrow x_0\}$)

► Knowledge $\forall_x K$

- Choose term T to create one subsituation with formula $K\{x \rightarrow T\}$ added to the knowledge.

We know $\forall_x K$ and thus also $K\{x \rightarrow T\}$ (closed universal quantifier).
(proof continues with knowledge $K\{x \rightarrow T\}$)

Universal Quantification

$$\frac{K \vdash G\{x \rightarrow x_0\}}{K \vdash \forall_x G}$$

(x_0 new for K, G)

$$\frac{\dots, \forall_x K, K\{x \rightarrow T\} \vdash G}{\dots, \forall_x K \vdash G}$$

► Goal $\forall_x G$

- Introduce new (arbitrarily named) constant x_0 and create one subsituation with goal $G[x \rightarrow x_0]$.

We have to show $\forall_x G$. Take arbitrary x_0 . We show $G\{x \rightarrow x_0\}$.
(proof continues with goal $G\{x \rightarrow x_0\}$)

► Knowledge $\forall_x K$

- Choose term T to create one subsituation with formula $K\{x \rightarrow T\}$ added to the knowledge.

We know $\forall_x K$ and thus also $K\{x \rightarrow T\}$ (closed instances of K).
The knowledge in this subsituation is $K \cup \{K\{x \rightarrow T\}\}$.

Universal Quantification

$$\frac{K \vdash G\{x \rightarrow x_0\}}{K \vdash \forall_x G}$$

(x_0 new for K, G)

$$\frac{\dots, \forall_x K, K\{x \rightarrow T\} \vdash G}{\dots, \forall_x K \vdash G}$$

► Goal $\forall_x G$

- Introduce new (arbitrarily named) constant x_0 and create one subsituation with goal $G\{x \rightarrow x_0\}$.

We have to show $\forall_x G$. Take arbitrary x_0 . We show $G\{x \rightarrow x_0\}$.
(proof continues with goal $G\{x \rightarrow x_0\}$)

► Knowledge $\forall_x K$

- Choose term T to create one subsituation with formula $K\{x \rightarrow T\}$ added to the knowledge.

The theory of universal quantification is a special case of the theory of substitution

Universal Quantification

$$\frac{K \vdash G\{x \rightarrow x_0\}}{K \vdash \forall_x G}$$

(x_0 new for K, G)

$$\frac{\dots, \forall_x K, K\{x \rightarrow T\} \vdash G}{\dots, \forall_x K \vdash G}$$

► Goal $\forall_x G$

- Introduce new (arbitrarily named) constant x_0 and create one subsituation with goal $G[x \rightarrow x_0]$.

We have to show $\forall_x G$. Take arbitrary x_0 . We show $G\{x \rightarrow x_0\}$.
(proof continues with goal $G\{x \rightarrow x_0\}$)

► Knowledge $\forall_x K$

- Choose term T to create one subsituation with formula $K\{x \rightarrow T\}$ added to the knowledge.

We know $\forall_x K$ and thus also $K\{x \rightarrow T\}$. (proof continues with current goal and additional knowledge $K\{x \rightarrow T\}$)

Universal Quantification

$$\frac{K \vdash G\{x \rightarrow x_0\}}{K \vdash \forall_x G} \quad (x_0 \text{ new for } K, G) \qquad \frac{..., \forall_x K, K\{x \rightarrow T\} \vdash G}{..., \forall_x K \vdash G}$$

► Goal $\forall_x G$

- Introduce new (arbitrarily named) constant x_0 and create one subsituation with goal $G[x \rightarrow x_0]$.

We have to show $\forall_x G$. Take arbitrary x_0 . We show $G\{x \rightarrow x_0\}$.
(proof continues with goal $G\{x \rightarrow x_0\}$)

► Knowledge $\forall_x K$

- Choose term T to create one subsituation with formula $K\{x \rightarrow T\}$ added to the knowledge.

We know $\forall_x K$ and thus also $K\{x \rightarrow T\}$. (proof continues with current goal and additional knowledge $K\{x \rightarrow T\}$)

Universal Quantification

$$\frac{K \vdash G\{x \rightarrow x_0\}}{K \vdash \forall_x G} \quad (x_0 \text{ new for } K, G) \qquad \frac{..., \forall_x K, K\{x \rightarrow T\} \vdash G}{..., \forall_x K \vdash G}$$

► Goal $\forall_x G$

- Introduce new (arbitrarily named) constant x_0 and create one subsituation with goal $G[x \rightarrow x_0]$.

We have to show $\forall_x G$. Take arbitrary x_0 . We show $G\{x \rightarrow x_0\}$.
(proof continues with goal $G\{x \rightarrow x_0\}$)

► Knowledge $\forall_x K$

- Choose term T to create one subsituation with formula $K\{x \rightarrow T\}$ added to the knowledge.

We know $\forall_x K$ and thus also $K\{x \rightarrow T\}$. (proof continues with current goal and additional knowledge $K\{x \rightarrow T\}$)

Existential Quantification

$$\frac{K \vdash G\{x \rightarrow T\}}{K \vdash \exists_x G}$$

$$\frac{\dots, K\{x \rightarrow x_0\}, \vdash G}{\dots, \exists_x K \vdash G}$$

(x_0) new for K, G

► Goal $\exists_x G$

- Choose term T to create one subsituation with goal $G\{x \rightarrow T\}$.

We have to show $\exists x. G$ because we chose $G\{x \rightarrow T\}$. (Goal introduced with goal $\exists(x \rightarrow T)$.)

► Knowledge $\exists_x K$

- Introduce new (arbitrarily named constant) x_0 and create one subsituation with additional knowledge $K\{x \rightarrow x_0\}$.

We know $\exists x. K$. Let x_0 be such that $\vdash K\{x \rightarrow x_0\}$. (Knowledge introduced with knowledge $\exists(x \rightarrow K)$.)

Existential Quantification

$$\frac{K \vdash G\{x \rightarrow T\}}{K \vdash \exists_x G}$$

$$\frac{\dots, K\{x \rightarrow x_0\}, \vdash G}{\dots, \exists_x K \vdash G}$$

(x_0) new for K, G

► Goal $\exists_x G$

- Choose term T to create one subsituation with goal $G\{x \rightarrow T\}$.

We have to show $\exists_x G$. It suffices to show $G\{x \rightarrow T\}$. (proof continues with goal $G\{x \rightarrow T\}$)

► Knowledge $\exists_x K$

- Introduce new (arbitrarily named constant) x_0 and create one subsituation with additional knowledge $K\{x \rightarrow x_0\}$.

We know $\exists_x K$. Let x_0 be such that $x \{x \rightarrow x_0\}$. (proof continues with additional knowledge $K\{x \rightarrow x_0\}$)

Existential Quantification

$$\frac{K \vdash G\{x \rightarrow T\}}{K \vdash \exists_x G}$$

$$\frac{..., K\{x \rightarrow x_0\}, \vdash G}{..., \exists_x K \vdash G}$$

(x_0) new for K, G

► Goal $\exists_x G$

- Choose term T to create one subsituation with goal $G\{x \rightarrow T\}$.

We have to show $\exists_x G$. It suffices to show $G\{x \rightarrow T\}$. (proof continues with goal $G\{x \rightarrow T\}$)

► Knowledge $\exists_x K$

- Introduce new (arbitrarily named constant) x_0 and create one subsituation with additional knowledge $K\{x \rightarrow x_0\}$.

The idea is to let x_0 be such that $x \rightarrow x_0$ is proved (because we know $\exists_x K$). Then we can show that $K\{x \rightarrow x_0\} \vdash G\{x \rightarrow x_0\}$.

Existential Quantification

$$\frac{K \vdash G\{x \rightarrow T\}}{K \vdash \exists_x G}$$

$$\frac{..., K\{x \rightarrow x_0\}, \vdash G}{..., \exists_x K \vdash G}$$

(x_0) new for K, G

► Goal $\exists_x G$

- Choose term T to create one subsituation with goal $G\{x \rightarrow T\}$.

We have to show $\exists_x G$. It suffices to show $G\{x \rightarrow T\}$. (proof continues with goal $G\{x \rightarrow T\}$)

► Knowledge $\exists_x K$

- Introduce new (arbitrarily named constant) x_0 and create one subsituation with additional knowledge $K\{x \rightarrow x_0\}$.

The idea is to show $\exists_x G$ by showing that G holds for x_0 . (proof continues with goal $G\{x \rightarrow x_0\}$)

Existential Quantification

$$\frac{K \vdash G\{x \rightarrow T\}}{K \vdash \exists_x G}$$

$$\frac{\dots, K\{x \rightarrow x_0\}, \vdash G}{\dots, \exists_x K \vdash G}$$

(x_0) new for K, G

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► Knowledge $\exists_x K$

- Introduce new (arbitrarily named constant) x_0 and create one subsituation with additional knowledge $K\{x \rightarrow x_0\}$.

We know $\exists_x K$. Let x_0 be such that $K\{x \rightarrow x_0\}$. (proof continues with current goal and additional knowledge $K\{x \rightarrow x_0\}$)

Existential Quantification

$$\frac{K \vdash G\{x \rightarrow T\}}{K \vdash \exists_x G}$$

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Example

Show that $\exists \forall_{x,y} P[x,y] \Rightarrow \forall \exists_{y,x} P[x,y]$. (a)

We assume $\exists \forall_{x,y} P[x,y]$ (1) and show $\forall \exists_{y,x} P[x,y]$ (b).

Take y_0 arbitrary. We show $\exists_x P[x,y]$. (c)

From (1) we know that for some x_0 , $\forall_y P[x_0,y]$. (2)

From (2) we know $P[x_0,y_0]$. (3)

From (3), we know (c).

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Exercises

Show that $\left(\exists_x P[x]\right) \wedge \left(\forall_x P[x] \Rightarrow \exists_y Q[x, y]\right) \Rightarrow \left(\exists_{x,y} Q[x, y]\right)$.

Indirect Proofs

$$\frac{K, \neg G \vdash \text{false}}{K \vdash G}$$

$$\frac{K, \neg G \vdash F \quad K, \neg G \vdash \neg F}{K \vdash G}$$

$$\frac{\dots, \neg G \vdash \neg K}{K \vdash G}$$

- Add $\neg G$ to the knowledge and show a contradiction.

Prove that *false* is *true*.

Prove that a formula F is *true* and also prove that it is *false*.

Prove that some knowledge K is *false*, i.e. that $\neg K$ is *true*.

(Switches goal G and knowledge K (negating both)).

Sometimes simpler than a direct proof.

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Outline

Proofs

Proof Rules

- Conjunction

- Disjunction

- Implication

- Equivalence

- Universal Quantification

- Existential Quantification

- Indirect Proofs

Example

Exercises

Example

Show that $\exists \forall_{x,y} P[x,y] \Rightarrow \forall \exists_{y,x} P[x,y]$. (a)

We assume $\exists \forall_{x,y} P[x,y]$ (1) and show $\forall \exists_{y,x} P[x,y]$ (b).

We assume $\neg \forall \exists_{y,x} P[x,y]$ (2) and show a contradiction.

From (2), we know $\exists \forall_{y,x} \neg P[x,y]$. (3)

Let y_0 be such that $\forall_x \neg P[x,y_0]$. (4)

From (1) we know for some x_0 , $\forall_y P[x_0,y]$. (5)

From (5), we know $P[x_0,y_0]$. (6)

From (4), we know $\neg P[x_0,y_0]$ (7)

From (6) and (7), we have a contradiction.

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Proof Rules

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Example

Exercises

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Prove that:

1. $((P \Rightarrow Q) \wedge (R \Rightarrow Q)) \Rightarrow ((P \vee R) \Rightarrow Q)$
2. $((P \Rightarrow Q) \vee (R \Rightarrow Q)) \iff ((P \wedge R) \Rightarrow Q)$