Formal Methods in Software Developement

The basic ideas of the virtual substitution and the cylindrical algebraic decomposition for solving real arithmetic problems

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Based on slides of the lecture Satisfiability Checking (Erika Ábrahám), RTWH Aachen

December 24, 2018

Domain	$+$	$+ , \cdot$
Reals $\mathbb{R}$	linear real arithmetic decidable **Fourier-Motzkin** Simplex	non-linear real arithmetic decidable Interval constraint propagation Virtual substitution Cylindrical algebraic decomposition
Integers $\mathbb{Z}$	linear integer arithmetic decidable **Branch-and-bound** Omega test	non-linear integer arithmetic undecidable Bit-blasting Branch-and-bound

Reminder: Non-linear real arithmetic (NRA)

Real arithmetic: first-order theory $(\mathbb{R}, +, \cdot, 0, 1, <)$ over the reals with addition and multiplication.

Syntax of real arithmetic

Polynomials:	t	$::=$	0 |	1 |	x |	$t + t$ |	$t \cdot t$
Constraints:	c	$::=$	$t < t$				
Formulas:	φ	$::=$	c |	$\neg\varphi$ |	$\varphi \wedge \varphi$ |	$\exists x.\ \varphi$	

where x is a variable.

- Syntactic sugar for constraints: $t_1 \leq t_2$, $t_1 = t_2$, $t_1 \neq t_2$.
- Notation: $D[x_1, \ldots, x_n]$ $(n \geq 1)$ is the set of all polynomials in variables $x_1, \ldots, x_n$ with coefficients from some domain D.
- E.g., $\mathbb{Z}[x_1, \ldots, x_n]$ is the set of all polynomials over variables $x_1, \ldots, x_n$ with coefficients from $\mathbb{Z}$.
- Naming in math: real algebra (instead of real arithmetic).

Quantifier-free NRA (QFNRA)

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$$\exists x.\ \exists y.\ x + 2y > 10 \wedge x \geq y \wedge (x < 0 \vee 2y > x)$$

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- QFLRA (quantifier-free linear real arithmetic) example:

$$\exists x.\ \exists y.\ x + 2y > 10 \wedge x \geq y \wedge (x < 0 \vee 2y > x)$$

- QFNRA (quantifier-free non-linear real arithmetic) example:

$$\exists x.\ \exists y.\ (\ x^2 - 4x^3y^2 > 0 \wedge x - y = 1\ )$$

Notations

- Assume: variables $x_1, \dots, x_n$, coefficient domain $D = \mathbb{Z}$
- Monomial: product of variables (the empty product represents the constant 1).
  Examples: xy^2, u^3vz^2
- Term: product of an integer coefficient and a monomial.
  Examples: $2xy^2$, $3u^3vz^2$
- Polynomial $p \in \mathbb{Z}[x_1, \dots, x_n]$: sum of terms
  Example: $2xy^2 + 3u^3vz^2$
- Polynomial constraint in canonical form: $p \sim 0$, $\sim \in \{<, \leq, =, \geq, >\}$.
  Example: $2xy^2 + 3u^3vz^2 - 5 < 0$
- A polynomial in one variable is called univariate.

  A polynomial in more than one variable is called multivariate.

  Multivariate polynomials can be seen as univariate polynomials with polynomial coefficients (notation: $p \in \mathbb{Z}[x_1, \dots, x_{n-1}][x_n]$).

Real arithmetic: On the border of decidability

Theorem (Alfred Tarski 1948)

The FO *theory of* $(\mathbb{R}, +, \cdot, 0, 1, <)$ *is decidable.*

- Tarski's proof was constructive, i.e., it defined a decision procedure.
- However, its time-complexity in the number of variables was non-elementary ("greater than all finite towers of powers of 2").

Real arithmetic: Some historical facts

1637 Descartes' rule of signs

1835 Jaques Charles François Sturm's theorem

1948 Alfred Tarski's "A decision method for elementary algebra and geometry"

1975 Cylindrical algebraic decomposition (CAD) method by George E. Collins

1979–80 First implementation of the CAD method by Dennis S. Arnon

1988 Virtual substitution by Volker Weispfenning

1990 First implementation of virtual substitution (Klaus-Dieter Burhenne)

1993 Gröbner bases approach by P. Pedersen, M.-F. Roy, A. Szpirglas, later extended by V. Weispfenning

1994 Implementation of the Gröbner bases approach (Andreas Dolzmann)

Real arithmetic: Implementations

Interval arithmetic

- `Ariadne`, `iSAT`, `SMT-RAT`, ...

Virtual substitution (VS)

- Computer algebra system `Redlog`, `SMT-RAT`, ...

Cylindrical algebraic decomposition (CAD)

- `QEPCAD`, `Redlog`, `SMT-RAT`, ...

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The virtual substitution (VS) and the cylindrical algebraic decomposition (CAD) are quantifier elimination methods.

The idea of quantifier elimination

Given: FO sentence φ over $(\mathbb{R}, +, \cdot, 0, 1, <)$ containing n quantifiers

1. Transform φ into prenex normal form:

$$\varphi \quad \equiv \quad Q_1 x_1 \ldots Q_n x_n \varphi_n(x_1, \ldots, x_n)$$

   where φ_n is a quantifier-free NRA formula with variables $x_1, \ldots, x_n$.

2. Eliminate iteratively the quantifiers $Q_n \ldots Q_1$ and thus the quantified variables:

$$\begin{aligned}
\varphi \quad &\equiv \quad Q_1 x_1 \ldots Q_{n-1} x_{n-1}\ Q_n x_n \quad \varphi_n(x_1, \ldots, x_n) \\
&\equiv \quad Q_1 x_1 \ldots Q_{n-1} x_{n-1} \quad \varphi_{n-1}(x_1, \ldots, x_{n-1}) \\
&\ldots \\
&\equiv \quad Q_1 x_1 \quad \varphi_1(x_1) \\
&\equiv \quad \varphi_0()
\end{aligned}$$

Removing universal quantification

Is it sufficient to eliminate existential quantifiers?

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\begin{aligned}
& \exists x_1.\, \exists x_2.\, \forall x_3.\, \exists x_4.\, \forall x_5.\, \forall x_6.\, \exists x_7.\, \exists x_8.\, \varphi' \\
\equiv\; & \exists x_1.\, \exists x_2.\, \neg(\exists x_3.\, \neg(\exists x_4.\, \neg(\exists x_5.\, \neg(\neg(\exists x_6.\, \neg(\exists x_7.\, \exists x_8.\, \varphi')))))) \\
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Is it sufficient to handle equations?

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\begin{array}{lcl}
p \geq 0 & \equiv & \exists \epsilon.\ p - \epsilon^2 = 0 \\
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Quantifier elimination with VS and CAD: Finite abstraction

- The degree of a polynomial is the highest degree of its monomials, when expressed in canonical form. The degree of a monomial is the sum of the exponents of the variables that appear in it. The word degree is now standard, but in some older books, the word order may be used instead.
- Each univariate polynomial $p(x)$ of degree d has d complex roots.
- Each univariate polynomial $p(x)$ of degree d has at most d real roots.

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- The sign of p is invariant between each two successive real roots. This implies that, if we know all roots, we can partition $\mathbb{R}$ into at most $2d + 1$ sign invariant regions for p.

- Similar facts hold also for formulas: for each QFNRA formula there is a finite partitioning of the state space such that the formula's truth value is invariant in each partition.

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- Idea: Find a finite set $T \subset \mathbb{R}$ with

$$\exists x_1. \ldots \exists x_n. \varphi_n \quad \Leftrightarrow \quad \exists x_1. \ldots \exists x_{n-1}. \bigvee_{t \in T} \varphi_n[t/x_n]$$

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- What remains: Determine the real roots of polynomials.

Real roots of univariate polynomials

What are the degrees and the real roots of these polynomials?

Polynomial	Degree	Real roots
x		
$2x - 5$		
x^2		
$x^2 - 1$		
$x^2 + 1$		
$x^2 - 2$		
$2x^6 - 5x^4 + 3x^2 - 6$		

Real roots of univariate polynomials

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Polynomial	Degree	Real roots
x	1	
$2x - 5$	1	
x^2	2	
$x^2 - 1$	2	
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\begin{array}{ll}
\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} & \text{if } a \neq 0,\ b^2 - 4ac \geq 0 \\
x = -\frac{c}{b} & \text{if } a = 0,\ b \neq 0 \\
\mathbb{R} & \text{if } a = 0,\ b = 0,\ c = 0 \\
\text{none} & \text{else.}
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Problem: expressions not in QFNRA. Solution: virtual substitution.

CAD: Real root isolation

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- For polynomials of degree 5 or higher, no solution equations exist.
- Instead of computing the zeros, we will isolate them: for each zero we define an interval in which this single zero is included.
- This is the so-called interval representation of zeros: (p, I) with univariate polynomial p and real interval I, such that p has exactly one real root in I.
- We need to be able to compute with this representation, e.g., substitute such a real root for a variable in a univariate polynomial constraint and check its validity.