The Art of Proving

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Based on W. Schreiner lecture notes, Formal Methods on Software Development, $https://www.risc.jku.at/$

Outline

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Proofs

A proof is a structured argument that a formula is true.

- A tree whose nodes represent proof situations (states).
- Each proof situation consists of knowledge base and a goal: $K_1, \ldots, K_n \vdash G$
  - Knowledge $K_1, \ldots, K_n$ are assumed to be true
  - Goal G to be proved wrt knowledge
- The root of the tree is the initial proof situation.
  - Knowledge $K_1, \ldots, K_n$ – axioms of mathematical background theories
  - G – formula to be proved

Proof rules describes how a proof situation can be reduced to zero, one, or more "subsituations".

$$\frac{\ldots \vdash \ldots \qquad \ldots \vdash \ldots}{K_1, \ldots, K_n \vdash G}$$

- Rule may or may not close the (sub)proof:
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  - One or more subsituations: G is proved if all subgoals are proved.
- Top-down rules: focus on G; G is decomposed into simpler goals G_1, G_2, ...
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In each proof situation, we aim at showing that the goal is true with respect to the given knowledge.

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- Goal $G_1 \wedge G_2$
  - Create two subsituations with goals G_1 and G_2.

    We have to show $G_1 \wedge G_2$. We show G_1: … (proof continues with the goal G_1). We show G_2: … (proof continues with the goal G_2).

- Knowledge $K_1 \wedge K_2$
  - Create one subsituation with K_1 and K_2 in knowledge.

    We know $K_1 \wedge K_2$. We thus also know K_1 and K_2. (proof continues with current goal and additional knowledge K_1 and K_2).

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Disjunction

$$\frac{K,\ \neg G_1 \vdash G_2}{K \vdash G_1 \vee G_2} \qquad\qquad \frac{\ldots,\ K_1 \vdash G \qquad \ldots,\ K_2 \vdash G}{\ldots,\ K_1 \vee K_2 \vdash G}$$

- Goal $G_1 \vee G_2$
  - Create one subsituation where G_2 is proved under the assumtion that G_1 does not hold (or vice versa):

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- Knowledge $K_1 \vee K_2$
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    We know $K_1 \vee K_2$. We thus proceed by case distinction: Case K_1: ... (proof continues with current goal and additional knowledge K_1). Case K_2: ... (proof continues with current goal and additional knowledge K_2).

Disjunction

$$\frac{K,\ \neg G_1 \vdash G_2}{K \vdash G_1 \vee G_2} \qquad \frac{\ldots,\ K_1 \vdash G \qquad \ldots,\ K_2 \vdash G}{\ldots,\ K_1 \vee K_2 \vdash G}$$

- Goal $G_1 \vee G_2$
  - Create one subsituation where G_2 is proved under the assumtion that G_1 does not hold (or vice versa):

    We have to show $G_1 \vee G_2$. We assume $\neg G_1$ and show G_2. (proof continues with goal G_2 and additional knowledge $\neg G_1$)
- Knowledge $K_1 \vee K_2$
  - Create two subsituations, one with K_1 and one with K_2 in knowledge.

    We know $K_1 \vee K_2$. We thus proceed by case distinction: *Case K_1:* ... (proof continues with current goal and additional knowledge K_1). *Case K_2:* ... (proof continues with current goal and additional knowledge K_2).

Disjunction

$$\frac{K,\ \neg G_1 \vdash G_2}{K \vdash G_1 \vee G_2} \qquad \frac{\ldots,\ K_1 \vdash G \qquad \ldots,\ K_2 \vdash G}{\ldots,\ K_1 \vee K_2 \vdash G}$$

- Goal $G_1 \vee G_2$
  - Create one subsituation where G_2 is proved under the assumtion that G_1 does not hold (or vice versa):

    We have to show $G_1 \vee G_2$. We assume $\neg G_1$ and show G_2. (proof continues with goal G_2 and additional knowledge $\neg G_1$)
- Knowledge $K_1 \vee K_2$
  - Create two subsituations, one with K_1 and one with K_2 in knowledge.

    We know $K_1 \vee K_2$. We thus proceed by case distinction: *Case K_1:* ... (proof continues with current goal and additional knowledge K_1). *Case K_2:* ... (proof continues with current goal and additional knowledge K_2).

Disjunction

$$\frac{K,\ \neg G_1 \vdash G_2}{K \vdash G_1 \vee G_2} \qquad \frac{\ldots,\ K_1 \vdash G \qquad \ldots,\ K_2 \vdash G}{\ldots,\ K_1 \vee K_2 \vdash G}$$

- Goal $G_1 \vee G_2$
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- Knowledge $K_1 \vee K_2$
  - Create two subsituations, one with K_1 and one with K_2 in knowledge.

    We know $K_1 \vee K_2$. We thus proceed by case distinction: *Case K_1:* ... (proof continues with current goal and additional knowledge K_1). *Case K_2:* ... (proof continues with current goal and additional knowledge K_2).

Implication

$$\frac{K,\ G_1 \vdash G_2}{K \vdash G_1 \Rightarrow G_2} \qquad\qquad \frac{\ldots \vdash K_1 \qquad \ldots,\ K_2 \vdash G}{\ldots,\ K_1 \Rightarrow K_2 \vdash G}$$

- Goal $G_1 \Rightarrow G_2$
  - Create one subsituation where G_2 is proved under the assumption that G_1 holds:

    We have to show $G_1 \Rightarrow G_2$. We assume G_1 and show G_2. (proof continues with goal G_2 and additional knowledge G_1)
- Knowledge $K_1 \Rightarrow K_2$
  - Create two subsituations, one with goal K_1 and one with knowledge K_2.

    We know $K_1 \Rightarrow K_2$. We show K_1 ... (proof continues with goal K_1). We know K_2 ... (proof continues with current goal and additional knowledge K_2).

Implication

$$\frac{K,\ G_1 \vdash G_2}{K \vdash G_1 \Rightarrow G_2} \qquad \frac{\ldots \vdash K_1 \qquad \ldots,\ K_2 \vdash G}{\ldots,\ K_1 \Rightarrow K_2 \vdash G}$$

- Goal $G_1 \Rightarrow G_2$
  - Create one subsituation where G_2 is proved under the assumption that G_1 holds:

    We have to show $G_1 \Rightarrow G_2$. We assume G_1 and show G_2. (proof continues with goal G_2 and additional knowledge G_1)
- Knowledge $K_1 \Rightarrow K_2$
  - Create two subsituations, one with goal K_1 and one with knowledge K_2.

    We know $K_1 \Rightarrow K_2$. We show K_1: … (proof continues with goal K_1). We know K_2: … (proof continues with current goal and additional knowledge K_2).

Implication

$$\frac{K,\ G_1 \vdash G_2}{K \vdash G_1 \Rightarrow G_2} \qquad\qquad \frac{\ldots \vdash K_1 \qquad \ldots,\ K_2 \vdash G}{\ldots,\ K_1 \Rightarrow K_2 \vdash G}$$

- Goal $G_1 \Rightarrow G_2$
  - Create one subsituation where G_2 is proved under the assumption that G_1 holds:

    We have to show $G_1 \Rightarrow G_2$. We assume G_1 and show G_2. (proof continues with goal G_2 and additional knowledge G_1)
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  - Create two subsituations, one with goal K_1 and one with knowledge K_2.

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Implication

$$\frac{K,\ G_1 \vdash G_2}{K \vdash G_1 \Rightarrow G_2} \qquad\qquad \frac{\ldots \vdash K_1 \qquad \ldots,\ K_2 \vdash G}{\ldots,\ K_1 \Rightarrow K_2 \vdash G}$$

- Goal $G_1 \Rightarrow G_2$
  - Create one subsituation where G_2 is proved under the assumption that G_1 holds:

    We have to show $G_1 \Rightarrow G_2$. We assume G_1 and show G_2. (proof continues with goal G_2 and additional knowledge G_1)
- Knowledge $K_1 \Rightarrow K_2$
  - Create two subsituations, one with goal K_1 and one with knowledge K_2.

Implication

$$\frac{K,\ G_1 \vdash G_2}{K \vdash G_1 \Rightarrow G_2} \qquad \frac{\ldots \vdash K_1 \qquad \ldots,\ K_2 \vdash G}{\ldots,\ K_1 \Rightarrow K_2 \vdash G}$$

- Goal $G_1 \Rightarrow G_2$
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    We have to show $G_1 \Rightarrow G_2$. We assume G_1 and show G_2. (proof continues with goal G_2 and additional knowledge G_1)
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  - Create two subsituations, one with goal K_1 and one with knowledge K_2.

    We know $K_1 \Rightarrow K_2$. We show K_1: ... (proof continues with goal K_1). We know K_2: ... (proof continues with current goal and additional knowledge K_2).

Implication

$$\frac{K,\ G_1 \vdash G_2}{K \vdash G_1 \Rightarrow G_2} \qquad\qquad \frac{\ldots \vdash K_1 \qquad \ldots,\ K_2 \vdash G}{\ldots,\ K_1 \Rightarrow K_2 \vdash G}$$

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  - Create one subsituation where G_2 is proved under the assumption that G_1 holds:

    We have to show $G_1 \Rightarrow G_2$. We assume G_1 and show G_2. (proof continues with goal G_2 and additional knowledge G_1)
- Knowledge $K_1 \Rightarrow K_2$
  - Create two subsituations, one with goal K_1 and one with knowledge K_2.

    We know $K_1 \Rightarrow K_2$. We show K_1: ... (proof continues with goal K_1). We know K_2: ... (proof continues with current goal and additional knowledge K_2).

Implication

$$\frac{K,\ G_1 \vdash G_2}{K \vdash G_1 \Rightarrow G_2} \qquad\qquad \frac{\ldots \vdash K_1 \qquad \ldots,\ K_2 \vdash G}{\ldots,\ K_1 \Rightarrow K_2 \vdash G}$$

- Goal $G_1 \Rightarrow G_2$
  - Create one subsituation where G_2 is proved under the assumption that G_1 holds:

    We have to show $G_1 \Rightarrow G_2$. We assume G_1 and show G_2. (proof continues with goal G_2 and additional knowledge G_1)
- Knowledge $K_1 \Rightarrow K_2$
  - Create two subsituations, one with goal K_1 and one with knowledge K_2.

    We know $K_1 \Rightarrow K_2$. We show K_1: ... (proof continues with goal K_1). We know K_2: ... (proof continues with current goal and additional knowledge K_2).

Equivalence

$$\frac{K \vdash G_1 \Rightarrow G_2 \qquad K \vdash G_2 \Rightarrow G_1}{K \vdash G_1 \Leftrightarrow G_2} \qquad\qquad \frac{\ldots \vdash (\neg)K_1 \qquad \ldots,\ (\neg)K_2 \vdash G}{\ldots,\ K_1 \Leftrightarrow K_2 \vdash G}$$

- Goal $G_1 \Leftrightarrow G_2$
  - Create two subsituations with implications in both directions as goals:

    We have to show $G_1 \Leftrightarrow G_2$. We show $G_1 \Rightarrow G_2$. ... (proof continues with goal $G_1 \Rightarrow G_2$). We show $G_2 \Rightarrow G_1$. ... (proof continues with goal $G_2 \Rightarrow G_1$).
- Knowledge $K_1 \Leftrightarrow K_2$
  - Create two subsituations, one with goal $(\neg)K_1$ and one with knowledge $(\neg)K_2$.

    We know $K_1 \Leftrightarrow K_2$. We show $(\neg)K_1$. ... (proof continues with goal $(\neg)K_1$). We know $(\neg)K_2$. ... (proof continues with current goal and additional knowledge $(\neg)K_2$).

Equivalence

$$\frac{K \vdash G_1 \Rightarrow G_2 \qquad K \vdash G_2 \Rightarrow G_1}{K \vdash G_1 \Leftrightarrow G_2} \qquad\qquad \frac{\ldots \vdash (\neg)K_1 \qquad \ldots,\ (\neg)K_2 \vdash G}{\ldots,\ K_1 \Leftrightarrow K_2 \vdash G}$$

- Goal $G_1 \Leftrightarrow G_2$
  - Create two subsituations with implications in both directions as goals:

    We have to show $G_1 \Leftrightarrow G_2$. We show $G_1 \Rightarrow G_2$: ... (proof continues with goal $G_1 \Rightarrow G_2$). We show $G_2 \Rightarrow G_1$: ... (proof continues with goal $G_2 \Rightarrow G_1$)
- Knowledge $K_1 \Leftrightarrow K_2$
  - Create two subsituations, one with goal $(\neg)K_1$ and one with knowledge $(\neg)K_2$.

    We know $K_1 \Leftrightarrow K_2$. We show $(\neg)K_1$: ... (proof continues with goal $(\neg)K_1$). We know $(\neg)K_2$: ... (proof continues with current goal and additional knowledge $(\neg)K_2$).

Equivalence

$$\frac{K \vdash G_1 \Rightarrow G_2 \qquad K \vdash G_2 \Rightarrow G_1}{K \vdash G_1 \Leftrightarrow G_2} \qquad\qquad \frac{\ldots \vdash (\neg)K_1 \qquad \ldots,\ (\neg)K_2 \vdash G}{\ldots,\ K_1 \Leftrightarrow K_2 \vdash G}$$

- Goal $G_1 \Leftrightarrow G_2$
  - Create two subsituations with implications in both directions as goals:

    We have to show $G_1 \Leftrightarrow G_2$. We show $G_1 \Rightarrow G_2$: ... (proof continues with goal $G_1 \Rightarrow G_2$). We show $G_2 \Rightarrow G_1$: ... (proof continues with goal $G_2 \Rightarrow G_1$)
- Knowledge $K_1 \Leftrightarrow K_2$
  - Create two subsituations, one with goal $(\neg)K_1$ and one with knowledge $(\neg)K_2$.

Equivalence

$$\frac{K \vdash G_1 \Rightarrow G_2 \qquad K \vdash G_2 \Rightarrow G_1}{K \vdash G_1 \Leftrightarrow G_2} \qquad\qquad \frac{\ldots \vdash (\neg)K_1 \qquad \ldots,\ (\neg)K_2 \vdash G}{\ldots,\ K_1 \Leftrightarrow K_2 \vdash G}$$

- Goal $G_1 \Leftrightarrow G_2$
  - Create two subsituations with implications in both directions as goals:

    We have to show $G_1 \Leftrightarrow G_2$. We show $G_1 \Rightarrow G_2$: ... (proof continues with goal $G_1 \Rightarrow G_2$). We show $G_2 \Rightarrow G_1$: ... (proof continues with goal $G_2 \Rightarrow G_1$)
- Knowledge $K_1 \Leftrightarrow K_2$
  - Create two subsituations, one with goal $(\neg)K_1$ and one with knowledge $(\neg)K_2$.

    We know $K_1 \Leftrightarrow K_2$. We show $(\neg)K_1$: ... (proof continues with goal $(\neg)K_1$). We know $(\neg)K_2$: ... (proof continues with current goal and additional knowledge $(\neg)K_2$)

Equivalence

$$\frac{K \vdash G_1 \Rightarrow G_2 \qquad K \vdash G_2 \Rightarrow G_1}{K \vdash G_1 \Leftrightarrow G_2} \qquad\qquad \frac{\ldots \vdash (\neg)K_1 \qquad \ldots,\ (\neg)K_2 \vdash G}{\ldots,\ K_1 \Leftrightarrow K_2 \vdash G}$$

- Goal $G_1 \Leftrightarrow G_2$
  - Create two subsituations with implications in both directions as goals:

    We have to show $G_1 \Leftrightarrow G_2$. We show $G_1 \Rightarrow G_2$: ... (proof continues with goal $G_1 \Rightarrow G_2$). We show $G_2 \Rightarrow G_1$: ... (proof continues with goal $G_2 \Rightarrow G_1$)
- Knowledge $K_1 \Leftrightarrow K_2$
  - Create two subsituations, one with goal $(\neg)K_1$ and one with knowledge $(\neg)K_2$.

    We know $K_1 \Leftrightarrow K_2$. We show $(\neg)K_1$: ... (proof continues with goal $(\neg)K_1$). We know $(\neg)K_2$: ... (proof continues with current goal and additional knowledge $(\neg)K_2$).

Equivalence

$$\frac{K \vdash G_1 \Rightarrow G_2 \qquad K \vdash G_2 \Rightarrow G_1}{K \vdash G_1 \Leftrightarrow G_2} \qquad\qquad \frac{\ldots \vdash (\neg)K_1 \qquad \ldots,\ (\neg)K_2 \vdash G}{\ldots,\ K_1 \Leftrightarrow K_2 \vdash G}$$

- Goal $G_1 \Leftrightarrow G_2$
  - Create two subsituations with implications in both directions as goals:

    We have to show $G_1 \Leftrightarrow G_2$. We show $G_1 \Rightarrow G_2$: ... (proof continues with goal $G_1 \Rightarrow G_2$). We show $G_2 \Rightarrow G_1$: ... (proof continues with goal $G_2 \Rightarrow G_1$)
- Knowledge $K_1 \Leftrightarrow K_2$
  - Create two subsituations, one with goal $(\neg)K_1$ and one with knowledge $(\neg)K_2$.

    We know $K_1 \Leftrightarrow K_2$. We show $(\neg)K_1$: ... (proof continues with goal $(\neg)K_1$). We know $(\neg)K_2$: ... (proof continues with current goal and additional knowledge $(\neg)K_2$).

Equivalence

$$\frac{K \vdash G_1 \Rightarrow G_2 \qquad K \vdash G_2 \Rightarrow G_1}{K \vdash G_1 \Leftrightarrow G_2} \qquad\qquad \frac{\ldots \vdash (\neg)K_1 \qquad \ldots,\ (\neg)K_2 \vdash G}{\ldots,\ K_1 \Leftrightarrow K_2 \vdash G}$$

- Goal $G_1 \Leftrightarrow G_2$
  - Create two subsituations with implications in both directions as goals:

    We have to show $G_1 \Leftrightarrow G_2$. We show $G_1 \Rightarrow G_2$: ... (proof continues with goal $G_1 \Rightarrow G_2$). We show $G_2 \Rightarrow G_1$: ... (proof continues with goal $G_2 \Rightarrow G_1$)
- Knowledge $K_1 \Leftrightarrow K_2$
  - Create two subsituations, one with goal $(\neg)K_1$ and one with knowledge $(\neg)K_2$.

    We know $K_1 \Leftrightarrow K_2$. We show $(\neg)K_1$: ... (proof continues with goal $(\neg)K_1$). We know $(\neg)K_2$: ... (proof continues with current goal and additional knowledge $(\neg)K_2$).

Universal Quantification

$$\frac{K \vdash G\{x \to x_0\}}{K \vdash \underset{x}{\forall}\, G} \qquad (x_0 \text{ new for } K, G)$$

$$\frac{\ldots,\ \underset{x}{\forall}\, K,\ K\{x \to T\} \vdash G}{\ldots,\ \underset{x}{\forall}\, K \vdash G}$$

- Goal $\underset{x}{\forall} G$
  - Introduce new (arbitrarily named) constant x_0 and create one subsituation with goal $G[x \to x_0]$.

    We have to show $\underset{x}{\forall}\, G$. Take arbitrary x_0. We show $G\{x \to x_0\}$.

    (proof continues with goal $G\{x \to x_0\}$)

- Knowledge $\underset{x}{\forall}\, K$
  - Choose term T to create one subsituation with formula $K\{x \to T\}$ added to the knowledge.

    We know $\underset{x}{\forall}\, K$ and thus also $K\{x \to T\}$ (proof continues with current goal and additional knowledge $K\{x \to T\}$)

Universal Quantification

$$\frac{K \vdash G\{x \to x_0\}}{K \vdash \underset{x}{\forall}\, G} \quad (x_0 \text{ new for } K, G) \qquad\qquad \frac{\ldots,\ \underset{x}{\forall}\, K,\ K\{x \to T\} \vdash G}{\ldots,\ \underset{x}{\forall}\, K \vdash G}$$

- Goal $\underset{x}{\forall} G$
  - Introduce new (arbitrarily named) constant x_0 and create one subsituation with goal $G[x \to x_0]$.

    We have to show $\underset{x}{\forall}\, G$. Take arbitrary x_0. We show $G\{x \to x_0\}$.
    (proof continues with goal $G\{x \to x_0\}$)

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    We know $\underset{x}{\forall}\, K$ and thus also $K\{x \to T\}$ (proof continues with current goal and additional knowledge $K\{x \to T\}$).

Universal Quantification

$$\frac{K \vdash G\{x \to x_0\}}{K \vdash \underset{x}{\forall}\ G} \quad (x_0 \text{ new for } K, G) \qquad\qquad \frac{\ldots,\ \underset{x}{\forall}\ K,\ K\{x \to T\} \vdash G}{\ldots,\ \underset{x}{\forall}\ K \vdash G}$$

- Goal $\underset{x}{\forall} G$
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    We know $\underset{x}{\forall}\ K$ and thus also $K\{x \to T\}$ (proof continues with current goal and additional knowledge $K\{x \to T\}$)

Universal Quantification

$$\frac{K \vdash G\{x \to x_0\}}{K \vdash \underset{x}{\forall}\ G} \qquad (x_0 \text{ new for } K, G) \qquad\qquad \frac{\ldots,\ \underset{x}{\forall}\ K,\ K\{x \to T\} \vdash G}{\ldots,\ \underset{x}{\forall}\ K \vdash G}$$

- Goal $\underset{x}{\forall} G$
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    We know $\underset{x}{\forall}\ K$ and thus also $K\{x \to T\}$ (proof continues with current goal and additional knowledge $K\{x \to T\}$)

Universal Quantification

$$\frac{K \vdash G\{x \to x_0\}}{K \vdash \underset{x}{\forall}\ G} \qquad (x_0 \text{ new for } K, G) \qquad\qquad \frac{\ldots,\ \underset{x}{\forall}\ K,\ K\{x \to T\} \vdash G}{\ldots,\ \underset{x}{\forall}\ K \vdash G}$$

- Goal $\underset{x}{\forall} G$
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    We have to show $\underset{x}{\forall}\ G$. Take arbitrary x_0. We show $G\{x \to x_0\}$. (proof continues with goal $G\{x \to x_0\}$)
- Knowledge $\underset{x}{\forall}\ K$
  - Choose term T to create one subsituation with formula $K\{x \to T\}$ added to the knowledge.

    We know $\underset{x}{\forall}\ K$ and thus also $K\{x \to T\}$.(proof continues with current goal and additional knowledge $K\{x \to T\}$)

Universal Quantification

$$\frac{K \vdash G\{x \to x_0\}}{K \vdash \underset{x}{\forall}\ G} \qquad (x_0 \text{ new for } K, G) \qquad\qquad \frac{\ldots,\ \underset{x}{\forall}\ K,\ K\{x \to T\} \vdash G}{\ldots,\ \underset{x}{\forall}\ K \vdash G}$$

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    We know $\underset{x}{\forall}\ K$ and thus also $K\{x \to T\}$.(proof continues with current goal and additional knowledge $K\{x \to T\}$)

Universal Quantification

$$\frac{K \vdash G\{x \to x_0\}}{K \vdash \underset{x}{\forall}\ G} \qquad (x_0 \text{ new for } K, G) \qquad \frac{\ldots,\ \underset{x}{\forall}\ K,\ K\{x \to T\} \vdash G}{\ldots,\ \underset{x}{\forall}\ K \vdash G}$$

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  - Choose term T to create one subsituation with formula $K\{x \to T\}$ added to the knowledge.

    We know $\underset{x}{\forall}\ K$ and thus also $K\{x \to T\}$.(proof continues with current goal and additional knowledge $K\{x \to T\}$)

Existential Quantification

$$\frac{K \vdash G\{x \to T\}}{K \vdash \underset{x}{\exists}\ G} \qquad\qquad \frac{\ldots,\ K\{x \to x_0\}, \vdash G}{\ldots,\ \underset{x}{\exists}\ K \vdash G} \quad (x_0) \text{ new for } K, G$$

- Goal $\underset{x}{\exists} G$
  - Choose term T to create one subsituation with goal $G\{x \to T\}$.

    We have to show $\underset{x}{\exists}\ G$. It suffices to show $G\{x \to T\}$. (proof continues with goal $G\{x \to T\}$)
- Knowledge $\underset{x}{\exists}\ K$
  - Introduce new (arbitrarily named constant) x_0 and create one subsituation with additional knowledge $K\{x \to x_0\}$.

    We know $\underset{x}{\exists}\ K$. Let x_0 be such that $K\{x \to x_0\}$. (proof continues with current goal and additional knowledge $K\{x \to x_0\}$)

Existential Quantification

$$\frac{K \vdash G\{x \to T\}}{K \vdash \underset{x}{\exists}\ G} \qquad \frac{\ldots,\ K\{x \to x_0\}, \vdash G}{\ldots,\ \underset{x}{\exists}\ K \vdash G} \quad (x_0) \text{ new for } K, G$$

- Goal $\underset{x}{\exists} G$
  - Choose term T to create one subsituation with goal $G\{x \to T\}$.

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- Knowledge $\underset{x}{\exists}\ K$
  - Introduce new (arbitrarily named constant) x_0 and create one subsituation with additional knowledge $K\{x \to x_0\}$.

    We know $\underset{x}{\exists}\ K$. Let x_0 be such that $K\{x \to x_0\}$. (proof continues with current goal and additional knowledge $K\{x \to x_0\}$)

Existential Quantification

$$\frac{K \vdash G\{x \to T\}}{K \vdash \underset{x}{\exists}\ G} \qquad\qquad \frac{\ldots,\ K\{x \to x_0\}, \vdash G}{\ldots,\ \underset{x}{\exists}\ K \vdash G} \quad (x_0) \text{ new for } K, G$$

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  - Introduce new (arbitrarily named constant) x_0 and create one subsituation with additional knowledge $K\{x \to x_0\}$.

    We know $\underset{x}{\exists}\ K$. Let x_0 be such that $K\{x \to x_0\}$. (proof continues with current goal and additional knowledge $K\{x \to x_0\}$)

Existential Quantification

$$\frac{K \vdash G\{x \to T\}}{K \vdash \underset{x}{\exists}\ G} \qquad\qquad \frac{\ldots,\ K\{x \to x_0\}, \vdash G}{\ldots,\ \underset{x}{\exists}\ K \vdash G} \quad (x_0) \text{ new for } K, G$$

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    We know $\underset{x}{\exists}\ K$. Let x_0 be such that $K\{x \to x_0\}$. (proof continues with current goal and additional knowledge $K\{x \to x_0\}$)

Existential Quantification

$$\frac{K \vdash G\{x \to T\}}{K \vdash \underset{x}{\exists}\ G} \qquad\qquad \frac{\ldots,\ K\{x \to x_0\}, \vdash G}{\ldots,\ \underset{x}{\exists}\ K \vdash G} \quad (x_0) \text{ new for } K, G$$

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    We know $\underset{x}{\exists}\ K$. Let x_0 be such that $K\{x \to x_0\}$. (proof continues with current goal and additional knowledge $K\{x \to x_0\}$)

Existential Quantification

$$\frac{K \vdash G\{x \to T\}}{K \vdash \underset{x}{\exists}\ G} \qquad\qquad \frac{\ldots,\ K\{x \to x_0\}, \vdash G}{\ldots,\ \underset{x}{\exists}\ K \vdash G} \quad (x_0) \text{ new for } K, G$$

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  - Choose term T to create one subsituation with goal $G\{x \to T\}$.

    We have to show $\underset{x}{\exists}\ G$. It suffices to show $G\{x \to T\}$. (proof continues with goal $G\{x \to T\}$)
- Knowledge $\underset{x}{\exists}\ K$
  - Introduce new (arbitrarily named constant) x_0 and create one subsituation with additional knowledge $K\{x \to x_0\}$.

    We know $\underset{x}{\exists}\ K$. Let x_0 be such that $K\{x \to x_0\}$. (proof continues with current goal and additional knowledge $K\{x \to x_0\}$)

Existential Quantification

$$\frac{K \vdash G\{x \to T\}}{K \vdash \underset{x}{\exists}\, G} \qquad \frac{\ldots,\ K\{x \to x_0\}, \vdash G}{\ldots,\ \underset{x}{\exists}\, K \vdash G} \quad (x_0) \text{ new for } K, G$$

- Goal $\underset{x}{\exists} G$
  - Choose term T to create one subsituation with goal $G\{x \to T\}$.

    We have to show $\underset{x}{\exists}\, G$. It suffices to show $G\{x \to T\}$. (proof continues with goal $G\{x \to T\}$)
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    We know $\underset{x}{\exists}\, K$. Let x_0 be such that $K\{x \to x_0\}$. (proof continues with current goal and additional knowledge $K\{x \to x_0\}$)

Example

Show that $\underset{x}{\exists}\underset{y}{\forall}\, P[x,y] \;\Rightarrow\; \underset{y}{\forall}\underset{x}{\exists} P[x,y]$. (a)

We assume $\underset{x}{\exists}\underset{y}{\forall}\, P[x,y]$ (1) and show $\underset{y}{\forall}\underset{x}{\exists} P[x,y]$ (b).

Take y_0 arbitrary. We show $\underset{x}{\exists} P[x,y]$. (c)

From (1) we know that for some x_0, $\underset{y}{\forall}\, P[x_0,y]$. (2)

From (2) we know $P[x_0,y_0]$. (3)

From (3), we know (c).

Example

Show that $\underset{x}{\exists}\underset{y}{\forall} P[x,y] \quad \Rightarrow \quad \underset{y}{\forall}\underset{x}{\exists} P[x,y]$. (a)

We assume $\underset{x}{\exists}\underset{y}{\forall} P[x,y]$ (1) and show $\underset{y}{\forall}\underset{x}{\exists} P[x,y]$ (b).

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Example

Show that $\exists_x \forall_y P[x, y] \Rightarrow \forall_y \exists_x P[x, y]$. (a)

We assume $\exists_x \forall_y P[x, y]$ (1) and show $\forall_y \exists_x P[x, y]$ (b).

Take y_0 arbitrary. We show $\exists_x P[x, y]$. (c)

From (1) we know that for some x_0, $\forall_y P[x_0, y]$. (2)

From (2) we know $P[x_0, y_0]$. (3)

From (3), we know (c).

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Show that $\underset{x}{\exists}\underset{y}{\forall}\, P[x, y] \quad \Rightarrow \quad \underset{y}{\forall}\underset{x}{\exists} P[x, y]$. (a)

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From (3), we know (c).

Exercises

Show that $\left(\underset{x}{\exists}\ P[x]\right) \wedge \left(\underset{x}{\forall} P[x] \Rightarrow \underset{y}{\exists} Q[x, y]\right) \Rightarrow \left(\underset{x,y}{\exists}\ Q[x, y]\right)$.

Indirect Proofs

$$\frac{K, \neg G \vdash \mathit{false}}{K \vdash G} \qquad \frac{K, \neg G \vdash F \qquad K, \neg G \vdash \neg F}{K \vdash G} \qquad \frac{\ldots, \neg G \vdash \neg K}{K \vdash G}$$

- Add $\neg G$ to the knowledge and show a contradiction.

  Prove that *false* is *true*.
  Prove that a formula F is *true* and also prove that it is *false*.
  Prove that some knowledge K is *false*, i.e. that $\neg K$ is *true*.
  (Switches goal G and knowledge K (negating both)).

Sometimes simpler than a direct proof.

Indirect Proofs

$$\frac{K, \neg G \vdash \mathit{false}}{K \vdash G} \qquad \frac{K, \neg G \vdash F \qquad K, \neg G \vdash \neg F}{K \vdash G} \qquad \frac{..., \neg G \vdash \neg K}{K \vdash G}$$

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Outline

Example

Show that $\underset{x}{\exists}\underset{y}{\forall} P[x, y] \Rightarrow \underset{y}{\forall}\underset{x}{\exists}P[x, y]$. (a)

We assume $\underset{x}{\exists}\underset{y}{\forall} P[x, y]$ (1) and show $\underset{y}{\forall}\underset{x}{\exists}P[x, y]$ (b).

We assume $\neg\underset{y}{\forall}\underset{x}{\exists}P[x, y]$ (2) and show a contradiction.

From (2), we know $\underset{y}{\exists}\underset{x}{\forall}\neg P[x, y]$. (3)

Let y_0 be such that $\underset{x}{\forall}\neg P[x, y]$. (4)

From (1) we know for some x_0, $\underset{y}{\forall} P[x_0, y]$. (5)

From (5), we know $P[x_0, y_0]$. (6)

From (4), we know $\neg P[x_0, y_0]$ (7)

From (6) and (7), we have a contradiction.

Example

Show that $\exists_x \forall_y \, P[x, y] \;\; \Rightarrow \;\; \forall_y \exists_x P[x, y]$. (a)

We assume $\exists_x \forall_y \, P[x, y]$ (1) and show $\forall_y \exists_x P[x, y]$ (b).

We assume $\neg \forall_y \exists_x P[x, y]$ (2) and show a contradiction.

From (2), we know $\exists_y \forall_x \neg P[x, y]$. (3)

Let y_0 be such that $\forall_x \neg P[x, y]$. (4)

From (1) we know for some x_0, $\forall_y \, P[x_0, y]$. (5)

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From (6) and (7), we have a contradiction.

Outline

Exercises

Prove that:

1. $((P \Rightarrow Q) \wedge (R \Rightarrow Q)) \Rightarrow ((P \vee R) \Rightarrow Q)$
2. $((P \Rightarrow Q) \vee (R \Rightarrow Q)) \iff ((P \wedge R) \Rightarrow Q)$