Automated Theorem Proving, SS 2017. Homework 5 (due May 10, 2017)

1. Find the truth value of the formula $F \ :\Longleftrightarrow \ \underset{x}{\forall}\ (P[x] \Longrightarrow Q[f[x], a])$, where

$$I : \begin{cases} D = \{1, 2\} \\ a_I = 1 \\ f_I : D \to D & \begin{cases} f_I[1] = 1 \\ f_I[2] = 1 \end{cases} \\ P_I : D \to \{\mathbb{T}, \mathbb{F}\} & \begin{cases} P_I[1] = \mathbb{T} \\ P_I[2] = \mathbb{F} \end{cases} \\ Q_I : D^2 \to \{\mathbb{T}, \mathbb{F}\} & \begin{cases} Q_I[1,1] = \mathbb{T} & Q_I[1,2] = \mathbb{F} \\ Q_I[2,1] = \mathbb{F} & Q_I[2,2] = \mathbb{T} \end{cases} \end{cases}$$

2. For the interpretation $D = \{a, b\}$, $P[a, a] = \mathbb{T}$, $P[a, b] = \mathbb{F}$, $P[b, a] = \mathbb{F}$, $P[b, b] = \mathbb{T}$, determine the truth value of the following formulas:

   (a) $\underset{x}{\forall}\underset{y}{\exists} P[x, y]$

   (b) $\underset{x}{\exists}\underset{y}{\forall} P[x, y]$

   (c) $\underset{x,y}{\forall}\ (P[x, y] \implies P[y, x])$

   (a) $\underset{x,y}{\forall} P[x, y]$

   (b) $\underset{y}{\exists}\neg P[a, y]$

   (c) $\underset{x}{\forall} P[x, x]$

3. Transform the following formulas into prenex normal form:

   (a) $\underset{x}{\forall}\left(P[x] \implies \underset{y}{\exists} Q[x, y]\right)$

   (b) $\underset{x}{\exists}\left(\neg\left(\underset{y}{\exists} P[x, y]\right) \implies \left(\left(\underset{z}{\exists} Q[z]\right) \implies R[x]\right)\right)$

   (c) $\underset{x,y}{\forall}\left(\underset{z}{\exists} P[x, y, z] \ \wedge \ \left(\underset{u}{\exists} Q[x, u] \implies \underset{v}{\exists} Q[y, v]\right)\right)$

4. Transform the following formulas into Skolem normal form:

   (a) $\neg\left(\underset{x}{\forall} P[x] \implies \underset{y}{\exists}\underset{z}{\forall} Q[y, z]\right)$

   (b) $\neg\left(\underset{x}{\forall} P[x] \implies \underset{y}{\exists} P[y]\right)$

   (c) $\underset{x,y}{\forall}\underset{z}{\exists} P[x, y, z]$

   (d)

$$\left(\underset{x,y,z,u,v,w}{\forall}(P[x, y, u] \ \wedge \ P[y, z, v] \ \wedge \ P[u, z, w] \ \Rightarrow \ P[x, v, w])\right)$$
$$\wedge$$
$$\left(\underset{x,y,z,u,v,w}{\forall}(P[x, y, u] \ \wedge \ P[y, z, v] \ \wedge \ P[x, v, w] \ \Rightarrow \ P[u, z, w])\right)$$

(e) $\underset{x}{\forall} P[x, e, x] \wedge \underset{x}{\forall} P[e, x, x]$

(f) $\underset{x}{\forall} P[x, i[x], e] \wedge \underset{x}{\forall} P[i[x], x, e]$

(g) $\left(\underset{x}{\forall} P[x, x, e]\right) \Rightarrow \left(\underset{u,v,w}{\forall} (P[u, v, w] \Rightarrow P[v, u, w])\right)$