Automated Theorem Proving
(Demonstrarea Automata a Teoremelor)
First-Order Logic

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Example 1 (Semantics) Let

$$F_1 \quad :\Longleftrightarrow \quad \underset{x}{\forall}\,\underset{y}{\forall}\, x \leq y$$

$$I : \begin{cases} D = \{0,1\} \\ \leq_I \,\to\, \leq_{\mathbb{Z}} \end{cases}$$

Solution. $\langle F_1 \rangle_I = \mathbb{T}$ iff for each $d \in D$:

$$\left\langle \underset{y}{\forall}\, x \leq y \right\rangle^I_{\sigma \uplus \{x \to d\}} = \mathbb{T}$$

- Case $x \to 0$. We have

$$\left\langle \underset{y}{\forall}\, x \leq y \right\rangle^I_{\{x \to 0\}} = \mathbb{T} \text{ iff for each } d \in D : \quad \langle x \leq y \rangle^I_{\{x\to 0\} \uplus \{y \to d\}} = \mathbb{T}$$

  - Case $y \to 0$. We have

$$\begin{aligned} & \langle x \leq y \rangle^I_{\{x\to 0, y\to 0\}} \\ \rightsquigarrow \quad & \leq_{\mathbb{Z}} \left[\langle x \rangle^I_{\{x\to 0, y\to 0\}}, \langle y \rangle^I_{\{x\to 0, y\to 0\}} \right] \\ \rightsquigarrow \quad & \leq_{\mathbb{Z}} [0,0] \\ \rightsquigarrow \quad & \mathbb{T} \end{aligned}$$

  - Case $y \to 1$. We have

$$\begin{aligned} & \langle x \leq y \rangle^I_{\{x\to 0, y\to 1\}} \\ \rightsquigarrow \quad & \leq_{\mathbb{Z}} \left[\langle x \rangle^I_{\{x\to 0, y\to 1\}}, \langle y \rangle^I_{\{x\to 0, y\to 1\}} \right] \\ \rightsquigarrow \quad & \leq_{\mathbb{Z}} [0,1] \\ \rightsquigarrow \quad & \mathbb{T} \end{aligned}$$

- Case $x \to 1$. We have

$$\left\langle \underset{y}{\forall}\, x \leq y \right\rangle^{I}_{\{x \to 1\}} = \mathbb{T} \text{ iff for each } d \in D: \quad \langle x \leq y \rangle^{I}_{\{x\to 1\} \uplus \{y \to d\}} = \mathbb{T}$$

  - Case $y \to 0$. We have

$$\begin{aligned} & \langle x \leq y \rangle^{I}_{\{x\to 1, y\to 0\}} \\ \rightsquigarrow\ & \leq_{\mathbb{Z}} \left[\langle x \rangle^{I}_{\{x\to 1, y\to 0\}}, \langle y \rangle^{I}_{\{x\to 1, y\to 0\}} \right] \\ \rightsquigarrow\ & \leq_{\mathbb{Z}} [1, 0] \\ \rightsquigarrow\ & \mathbb{F} \end{aligned}$$

Hence $\langle F_1 \rangle_I = \mathbb{F}$. ◀

Example 2 (Semantics) Let

$$F_2 \quad :\Longleftrightarrow \quad \underset{x}{\forall}\, \underset{y}{\exists}\, x + y > c$$

$$I : \begin{cases} D = \{0, 1\} \\ c_I = 0 \\ +_I \to +_{\mathbb{Z}} \\ >_I \to >_{\mathbb{Z}} \end{cases}$$

Solution. $\langle F_2 \rangle_I = \mathbb{T}$ iff for each $d \in D$:

$$\left\langle \underset{y}{\exists}\, x + y > c \right\rangle^{I}_{\sigma \uplus \{x \to d\}} = \mathbb{T}$$

- Case $x \to 0$. We have

$$\left\langle \underset{y}{\exists}\, x + y > c \right\rangle^{I}_{\{x \to 0\}} = \mathbb{T} \text{ iff for some } d \in D: \quad \langle x + y > c \rangle^{I}_{\{x\to 0\} \uplus \{y \to d\}} = \mathbb{T}$$

  - Case $y \to 0$. We have

$$\begin{aligned} & \langle x + y > c \rangle^{I}_{\{x\to 0, y\to 0\}} \\ \rightsquigarrow\ & >_{\mathbb{Z}} \left[\langle x + y \rangle^{I}_{\{x\to 0, y\to 0\}}, \langle c \rangle^{I}_{\{x\to 0, y\to 0\}} \right] \\ \rightsquigarrow\ & >_{\mathbb{Z}} \left[+_{\mathbb{Z}} \left[\langle x \rangle^{I}_{\{x\to 0, y\to 0\}}, \langle y \rangle^{I}_{\{x\to 0, y\to 0\}} \right], 0 \right] \\ \rightsquigarrow\ & >_{\mathbb{Z}} \left[+_{\mathbb{Z}} [0, 0], 0 \right] \\ \rightsquigarrow\ & >_{\mathbb{Z}} [0, 0] \\ \rightsquigarrow\ & \mathbb{F} \end{aligned}$$

  - Case $y \to 1$. We have

$$\begin{aligned} & \langle x + y > c \rangle^{I}_{\{x\to 0, y\to 1\}} \\ \rightsquigarrow\ & >_{\mathbb{Z}} \left[\langle x + y \rangle^{I}_{\{x\to 0, y\to 1\}}, \langle c \rangle^{I}_{\{x\to 0, y\to 1\}} \right] \\ \rightsquigarrow\ & >_{\mathbb{Z}} \left[+_{\mathbb{Z}} \left[\langle x \rangle^{I}_{\{x\to 0, y\to 1\}}, \langle y \rangle^{I}_{\{x\to 0, y\to 1\}} \right], 0 \right] \\ \rightsquigarrow\ & >_{\mathbb{Z}} \left[+_{\mathbb{Z}} [0, 1], 0 \right] \\ \rightsquigarrow\ & >_{\mathbb{Z}} [1, 0] \\ \rightsquigarrow\ & \mathbb{T} \end{aligned}$$

- Case $x \to 1$. We have

$$\left\langle \underset{y}{\exists}\, x + y > c \right\rangle^{I}_{\{x\to 1\}} = \mathbb{T} \text{ iff for some } d \in D: \quad \langle x + y > c \rangle^{I}_{\{x\to 1\} \uplus \{y \to d\}} = \mathbb{T}$$

  - Case $y \to 0$. We have

$$\begin{aligned}
& \langle x + y > c \rangle^{I}_{\{x\to 1, y\to 0\}} \\
\rightsquigarrow \quad & >_{\mathbb{Z}} \left[\langle x + y \rangle^{I}_{\{x\to 1, y\to 0\}}, \langle c \rangle^{I}_{\{x\to 1, y\to 0\}} \right] \\
\rightsquigarrow \quad & >_{\mathbb{Z}} \left[+_{\mathbb{Z}} \left[\langle x \rangle^{I}_{\{x\to 1, y\to 0\}}, \langle y \rangle^{I}_{\{x\to 1, y\to 0\}} \right], 0 \right] \\
\rightsquigarrow \quad & >_{\mathbb{Z}} \left[+_{\mathbb{Z}} [1, 0], 0 \right] \\
\rightsquigarrow \quad & >_{\mathbb{Z}} [1, 0] \\
\rightsquigarrow \quad & \mathbb{T}
\end{aligned}$$

Hence $\langle F_2 \rangle_I = \mathbb{T}$. ◀

Example 3 (CNF) Prove the following by bringing the formulas into CNF

$$\left(\underset{x}{\forall} P[x] \right) \Rightarrow Q \quad \equiv \quad \underset{x}{\exists} (P[x] \Rightarrow Q) \,.$$

Solution. We have

$$\left(\underset{x}{\forall} P[x] \right) \Rightarrow Q \quad \equiv \quad \neg \left(\underset{x}{\forall} P[x] \right) \vee Q \quad \equiv \quad \left(\underset{x}{\exists} \neg P[x] \right) \vee Q \quad \equiv \quad \underset{x}{\exists} (\neg P[x] \vee Q)$$

Further we have

$$\underset{x}{\exists} (P[x] \Rightarrow Q) \quad \equiv \quad \underset{x}{\exists} (\neg P[x] \vee Q)$$

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Example 4 (Skolem Standard Form) Bring the following formula into Skolem Standard Form

$$\underset{x}{\forall} \underset{y,z}{\exists} ((\neg P[x, y] \ \wedge \ Q[x, z]) \ \vee \ R[x, y, z])$$

Solution.

$$\begin{aligned}
& \underset{x}{\forall} \underset{y,z}{\exists} ((\neg P[x, y] \ \wedge \ Q[x, z]) \ \vee \ R[x, y, z]) \\
\Longleftrightarrow \quad & \underset{x}{\forall} \underset{y,z}{\exists} ((\neg P[x, y] \ \vee \ R[x, y, z]) \ \wedge \ (Q[x, z] \ \vee \ R[x, y, z])) \\
\rightsquigarrow \quad & \underset{x}{\forall} ((\neg P[x, f[x]] \ \vee \ R[x, f[x], g[x]]) \ \wedge \ (Q(x, g[x]) \ \vee \ R[x, f[x], g[x]]))
\end{aligned}$$

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Example 5 (Skolem Standard Form) Bring the following formula into Skolem Standard Form

$$\underset{x,y}{\forall} \left(\underset{z}{\exists} (P[x, z] \wedge P[y, z]) \ \Rightarrow \ \underset{u}{\exists} Q[x, y, u] \right)$$

Solution.

$$
\begin{aligned}
& \underset{x,y}{\forall}\left(\underset{z}{\exists}\,(P[x,z] \wedge P[y,z]) \;\Rightarrow\; \underset{u}{\exists} Q[x,y,u]\right) \\
\Longleftrightarrow\quad & \underset{x,y}{\forall}\left(\neg\left(\underset{z}{\exists}\,(P[x,z] \wedge P[y,z])\right) \;\vee\; \underset{u}{\exists} Q[x,y,u]\right) \\
\Longleftrightarrow\quad & \underset{x,y}{\forall}\left(\underset{z}{\forall}\,(\neg P[x,z] \vee \neg P[y,z]) \;\vee\; \underset{u}{\exists} Q[x,y,u]\right) \\
\Longleftrightarrow\quad & \underset{x,y,z}{\forall}\left(\neg P[x,z] \vee \neg P[y,z] \;\vee\; \underset{u}{\exists} Q[x,y,u]\right) \\
\Longleftrightarrow\quad & \underset{x,y,z}{\forall}\ \underset{u}{\exists}\ (\neg P[x,z] \vee \neg P[y,z] \;\vee\; Q[x,y,u]) \\
\rightsquigarrow\quad & \underset{x,y,z}{\forall} \neg P[x,z] \vee \neg P[y,z] \;\vee\; Q[x,y,f[x,y,z]]
\end{aligned}
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