Formal Methods in Software Development

Gauß and Fourier-Motzkin Variable Elimination for Linear Real Arithmetic

Mădălina Erașcu

West University of Timișoara
Faculty of Mathematics and Informatics
Department of Computer Science

Based on slides of the lecture Satisfiability Checking (Erika Ábrahám), RTWH Aachen

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The Xmas problem

There are three types of Xmas presents Santa Claus can make.

- Santa Claus wants to reduce the overhead by making only two types.
- He needs at least 100 presents.
- He needs at least 5 of either type 1 or type 2.
- He needs at least 10 of the third type.
- Each present of type 1, 2, and 3 need 1, 2, resp. 5 minutes to make.
- Santa Claus is late, and he has only 3 hours left.
- Each present of type 1, 2, and 3 costs 3, 2, resp. 1 EUR.
- He has 300 EUR for presents in total.

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$$(p_1 = 0 \lor p_2 = 0 \lor p_3 = 0) \land p_1 + p_2 + p_3 \geq 100 \land$$
$$(p_1 \geq 5 \lor p_2 \geq 5) \land p_3 \geq 10 \land p_1 + 2p_2 + 5p_3 \leq 180 \land$$
$$3p_1 + 2p_2 + p_3 \leq 300$$

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Terms:	t	$::=$	0	$\mid 1$	$\mid x$	$\mid t + t$
Constraints:	c	$::=$	$t < t$			
Formulas:	φ	$::=$	c	$\mid \neg\varphi$	$\mid \varphi \land \varphi$	$\mid \exists x.\ \varphi$

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- Linear real arithmetic is also called linear real algebra.
- We consider the satisfiability problem for the quantifier-free fragment QFLRA (or equivalently the existential fragment, i.e., no universal quantifiers and no negation of expressions containing existential quantifiers).

Gauß variable elimination (based on equations)

Reminder: In an SMT solver for QFLRA, the theory solver needs to check the satisfiability of sets of constraints $\sum_{k=1}^{N} a_{ik} \cdot x_k \sim_i b_i$, where $a_{i,k}$ and b_i are integer (or rational) constants, x_k are real-valued variables, and $\sim_i \in \{=, \leq, <\}$ for $k=1, \ldots, N$ and $i=1, \ldots, M$. (Note: $t > b$ is equivalent to $-t < -b$ and similarly for $\geq$.)

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- Assume that the ith constraint is an equation with $a_{ij} \neq 0$ for some $1 \leq j \leq N$:

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\Rightarrow \quad a_{ij} \cdot x_j &= b_i - \sum_{k \in \{1,\ldots,j-1,j+1,\ldots,N\}} a_{ik} \cdot x_k \\
\Rightarrow \quad x_j &= \frac{b_i}{a_{ij}} - \sum_{k \in \{1,\ldots,j-1,j+1,\ldots,N\}} \frac{a_{ik}}{a_{ij}} \cdot x_k := \beta_j
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- Replace x_j by β_j in all constraints (and multiply the involved constraints by a_{ij} if integer coefficients are wanted).
- After removing tautologies, this substitutiton leads to an equisatisfiable problem with (at most) $M-1$ constraints in (at most) $N-1$ variables (at least the ith constraint and x_j are eliminated).

What remains to be solved

- Let us assume first that after applying Gauß variable elimination as long as possible, m non-strict inequalities in n variables are left (i.e., there are no strict inequalities):

$$\bigwedge_{1 \leq i \leq m} \sum_{1 \leq j \leq n} a_{ij} x_j \leq b_i$$

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- Input in matrix form: $A\overline{x} \leq \overline{b}$

$$\begin{array}{cc} m \text{ constraints} & \begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ a_{m1} & a_{22} & \cdots & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} \leq \begin{pmatrix} b_1 \\ \vdots \\ \vdots \\ b_m \end{pmatrix} \\ & n \text{ variables} \end{array}$$

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  - Continue until all variables are eliminated.
- Fourier-Motzkin: Collect requirements on the lower an upper bounds on the variable we want to eliminate.

Variable bounds

- For a variable x_n, we can partition the constraints according to the coefficients of x_n:
  - $a_{in} = 0$: constraint i puts no bound on x_n
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$$(a) \quad \overset{a_{in}>0}{\Rightarrow} \quad x_n \ \leq \ \frac{b_i}{a_{in}} - \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \cdot x_j \qquad \textcolor{red}{\text{upper bound}}$$

$$(b) \quad \overset{a_{in}<0}{\Rightarrow} \quad x_n \ \geq \ \frac{b_i}{a_{in}} - \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \cdot x_j \qquad \textcolor{red}{\text{lower bound}}$$

Example for upper and lower bounds

Category for x_1?

$$
\begin{array}{ll}
(1) & x_1 - x_2 \leq 0 \\
(2) & x_1 - x_3 \leq 0 \\
(3) & -x_1 + x_2 + 2x_3 \leq 0 \\
(4) & -x_3 \leq -1
\end{array}
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Eliminating unbounded variables

- Iteratively remove variables that are not bounded in both ways (and all the constraints that use them).
- The new problem has a solution iff the old problem has one!

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\begin{aligned}
-8x + 7y &\leq 0 \\
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- For each such pair, add the constraint

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Fourier-Motzkin: Example

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eliminate x_2

Fourier-Motzkin: Example

Category for x_1?

~~(1) $x_1 - x_2 \leq 0$~~

~~(2) $x_1 - x_3 \leq 0$~~

~~(3) $-x_1 + x_2 + 2x_3 \leq 0$~~

(4) $-x_3 \leq -1$ Lower bound

---

eliminate x_1

(5) $2x_3 \leq 0$ (from 1,3) Upper bound

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$(4) \quad -x_3 \leq -1$ — Lower bound

eliminate x_1

$(5) \quad 2x_3 \leq 0$ (from 1,3) — Upper bound

$\cancel{(6) \quad x_2 + x_3 \leq 0}$ (from 2,3) — Upper bound

eliminate x_2

$(7) \quad 1 \leq 0$ (from 4,5)

$\rightarrow$ Contradiction (the system is UNSAT)

Strict inequalities

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Strict inequalities

The approach works also if we have both non-strict and strict inequalities. All we need to change is that

- we distinguiosh between strict and non-strict lower and upper bounds (defined by strict respectively non-strict inequalities), and
- for each pair of lower and upper bounds, if any of them is strict then we add the constraint

  $$\beta_l \quad < \quad \beta_u$$

  instead of

  $$\beta_l \quad \leq \quad \beta_u \; .$$

Linear integer arithmetic, non-linear real arithmetic

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- Reason: The integer domain is not dense.

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- Question: Does this method work also for non-linear real arithmetic, i.e., if the variables range over the reals but also multiplication is allowed?

Linear integer arithmetic, non-linear real arithmetic

- Question: Does this method work also for linear integer arithmetic, i.e., if the variables range over the integers (instead of the reals)?
- Answer: No.
- Reason: The integer domain is not dense.

- Question: Does this method work also for non-linear real arithmetic, i.e., if the variables range over the reals but also multiplication is allowed?
- Answer: No.

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- Question: Does this method work also for non-linear real arithmetic, i.e., if the variables range over the reals but also multiplication is allowed?
- Answer: No.
- Reason: in general it is not possible to transform constraints containing non-linear polynomial expressions such that we have a single variable on the left-hand-side and a real-arithmetic expression on the right-hand side (we would need complicated case distinctions, fractions and roots).

Complexity

- Worst-case complexity:

$$m \quad \rightarrow$$

Complexity

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&\to \dots
\end{aligned}
$$

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&\rightarrow \ldots \\
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- Heavy!

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\end{aligned}
$$

- Heavy!
- The bottleneck: case-splitting

Requirements on theory solver in the SMT context

1. Incrementality?
2. Minimal infeasible subsets?
3. Backtracking?

Next Lecture: Linear Integer Arithmetic