Formal Methods in Software Development

SAT Solving

Mădălina Erașcu

West University of Timișoara
Faculty of Mathematics and Informatics
Department of Computer Science

Based on slides of the lecture Satisfiability Checking (Erika Ábrahám), RTWH Aachen

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Given:

- Propositional logic formula φ in CNF.

Question:

- Is φ satisfiable?
  (Is there a model for φ?)

SAT-solving: Components

- Decision (enumeration)
- Boolean constraint propagation (BCP) – discussed in the previous lecture
- Conflict resolution and backtracking – discussed in the previous lecture

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- Decision (enumeration)
- Boolean constraint propagation (BCP)
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Enumeration algorithm

- Naive approach yields 2^n candidate models to check
- Solution: decision heuristics

Example CNF: Decision heuristics

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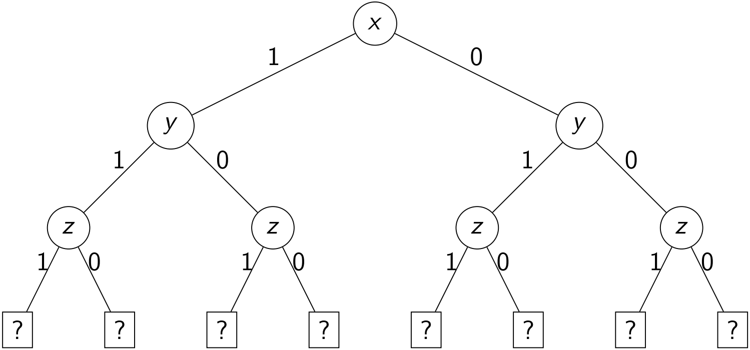

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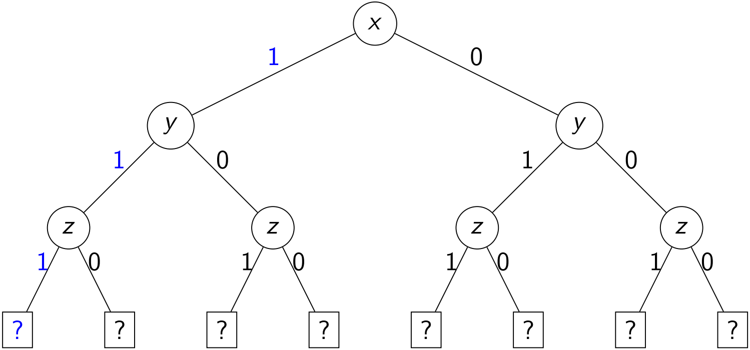

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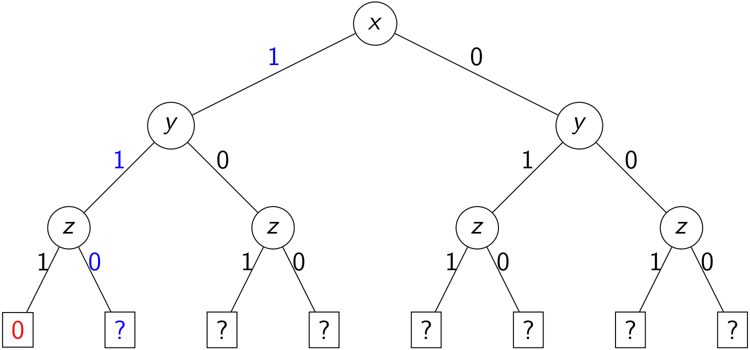

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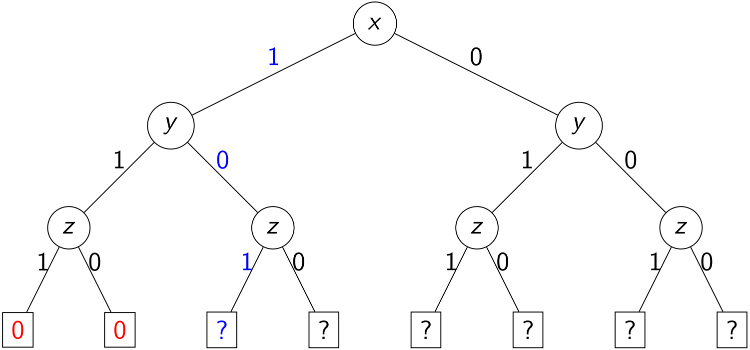

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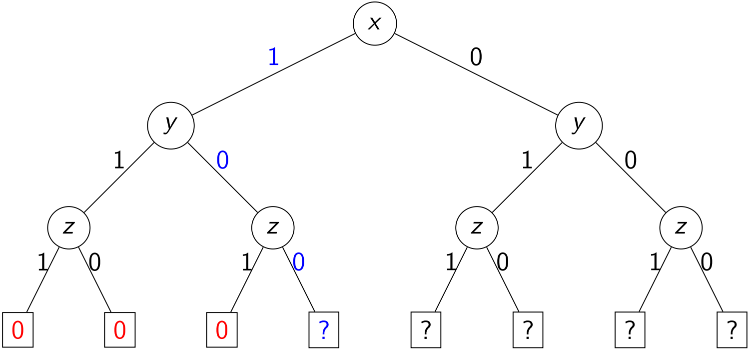

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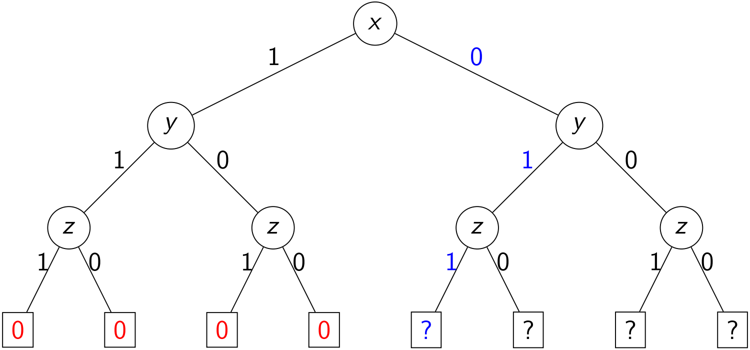

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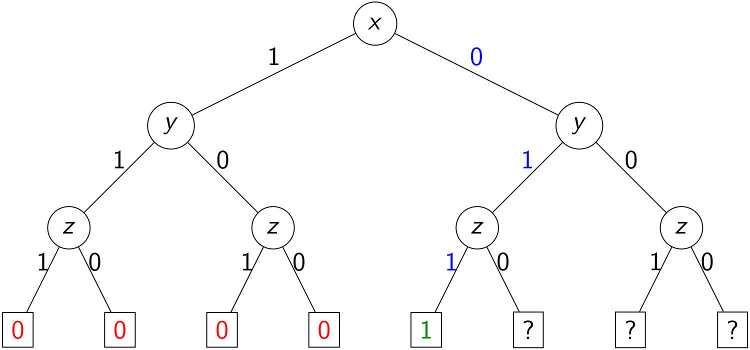

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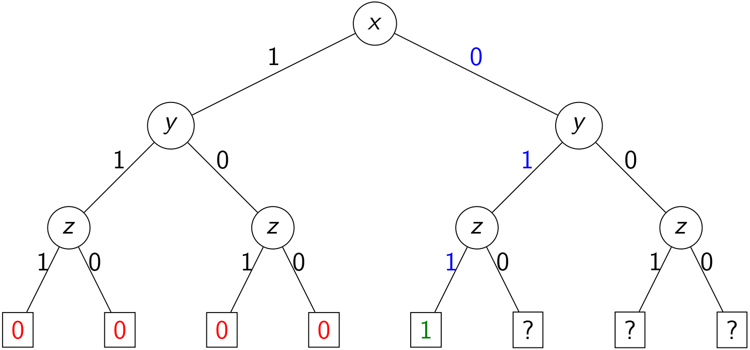

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For satisfiable problems, variable and sign ordering might strongly influence the running time.

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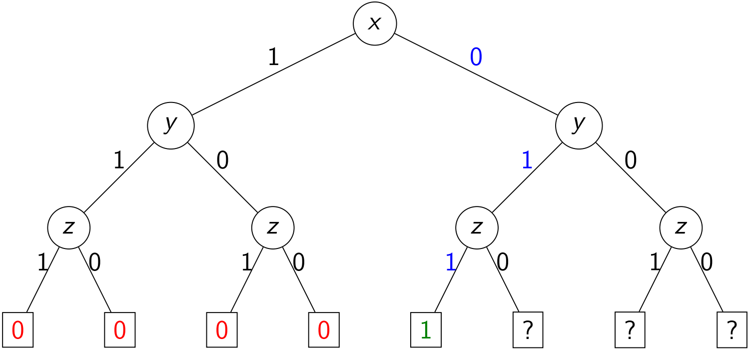
Static variable order $x < y < z$, sign: try positive first



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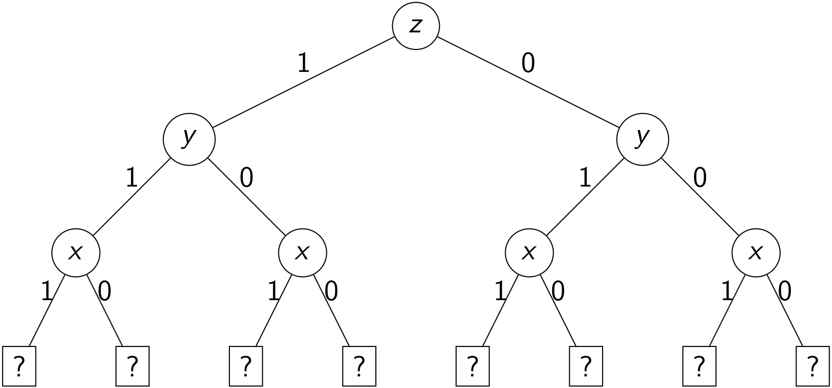

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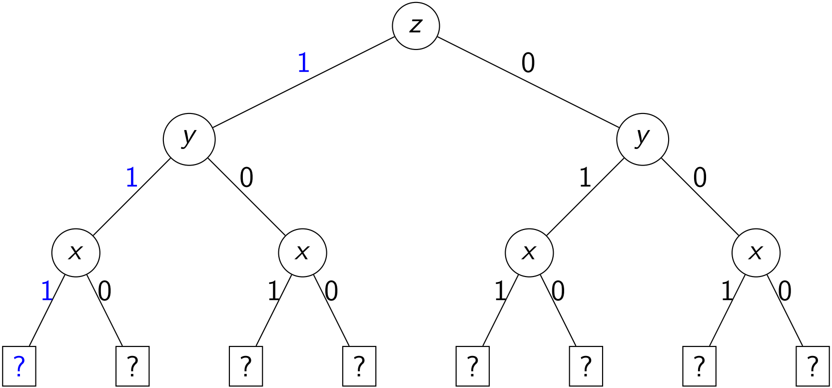

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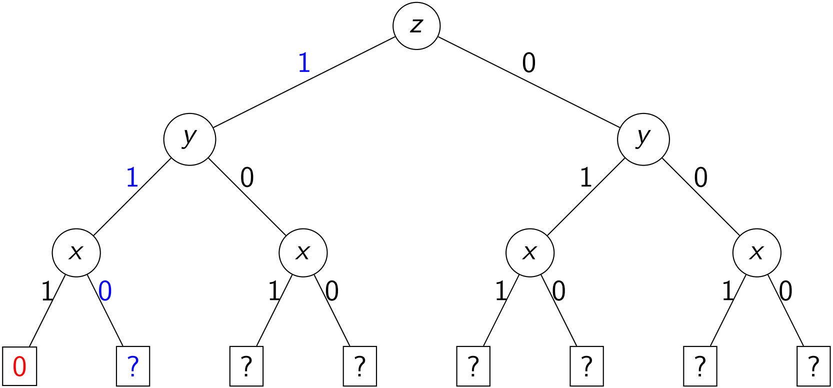

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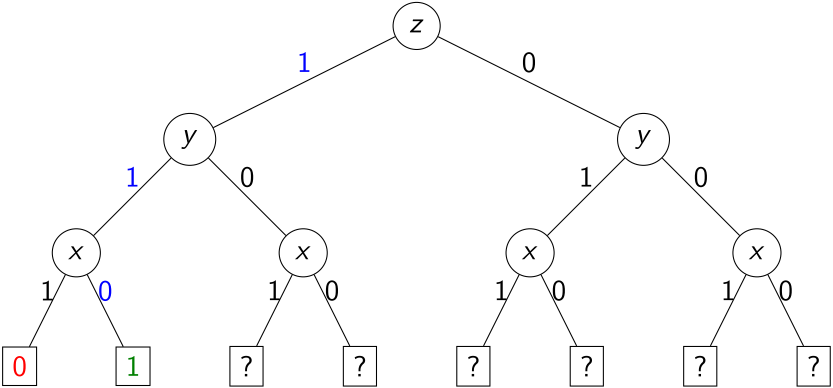

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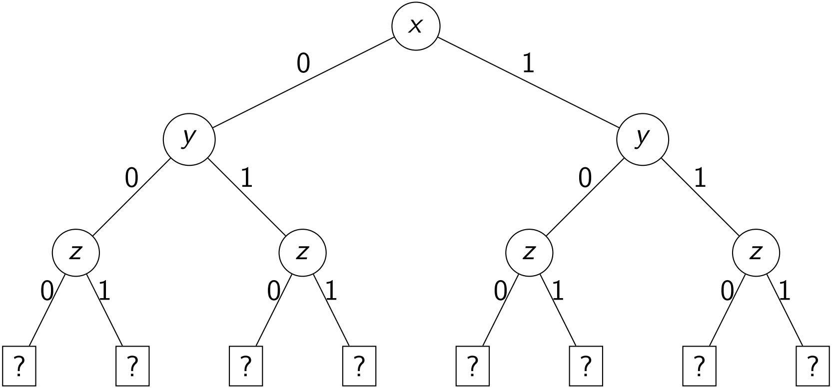

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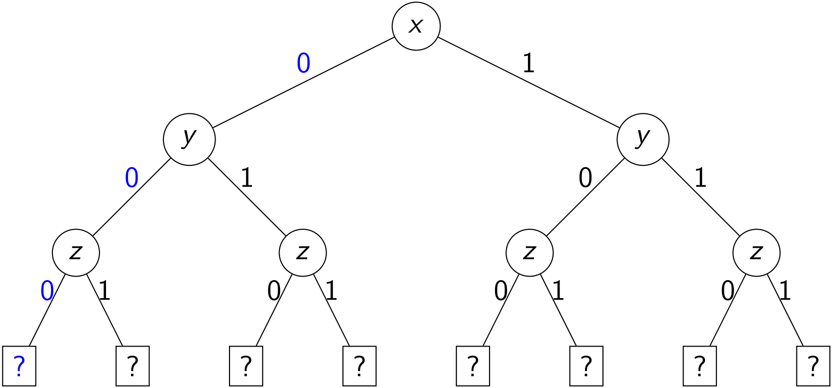

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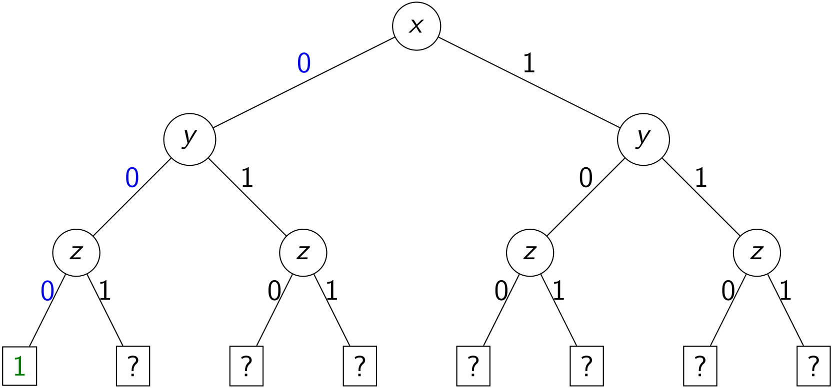

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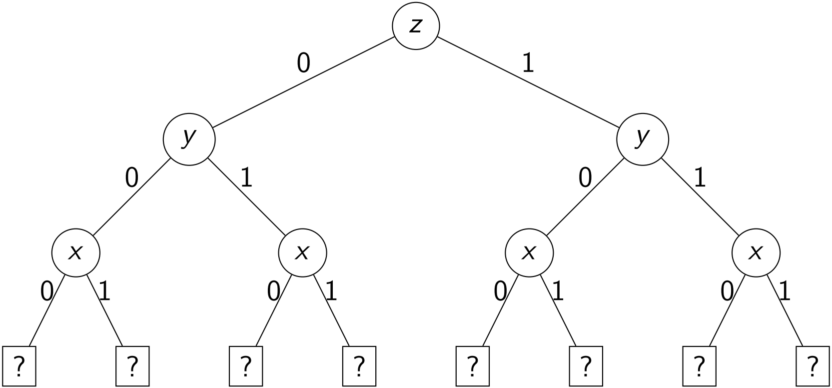

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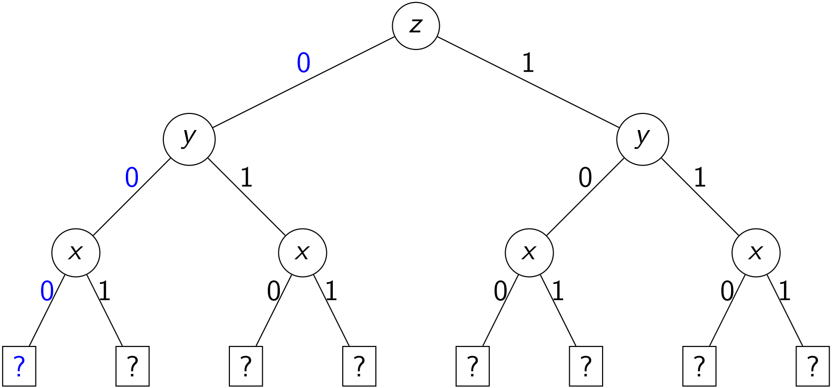

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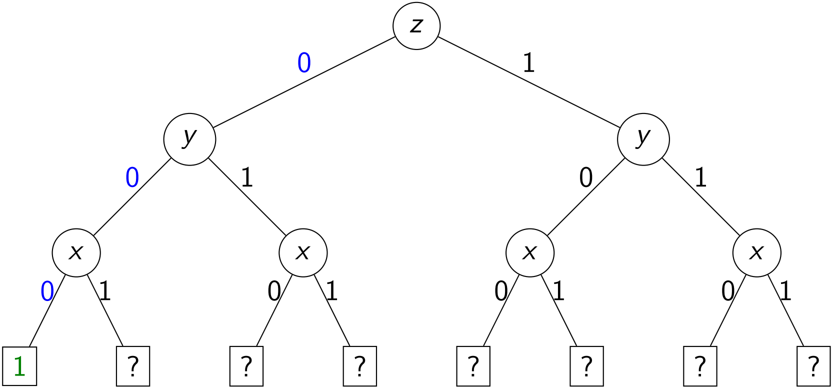

Decision heuristics

Dynamic Largest Individual Sum (DLIS): Choose an assignment that increases the most the number of satisfied clauses

- For each variable x, let C_x be the number of unresolved clauses in which x appears positively.
- For each variable x, let $C_{\neg x}$ be the number unresolved clauses in which x appears negatively.
- Let x be a variable for which C_x is maximal ($C_x \geq C_z$ for all variables z).
- Let y be a variable for which $C_{\neg y}$ is maximal ($C_{\neg y} \geq C_{\neg z}$ for all variables z).
- If $C_x > C_{\neg y}$ choose x and assign it TRUE.
- Otherwise choose y and assign it FALSE.

- Requires $\mathcal{O}(\#\mathit{literals})$ queries for each decision.

Example CNF: Decision heuristics DLIS

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$$\begin{array}{lll} C_x = 0 & C_y = 2 & C_z = 1 \\ C_{\neg x} = 2 & C_{\neg y} = 1 & C_{\neg z} = 1 \end{array}$$

Dynamic Largest Individual Sum (DLIS) literal order

Fallback literal order (in case of equal values: $\neg x$ and y): $\neg x < x < \neg z < z < \neg y < y$

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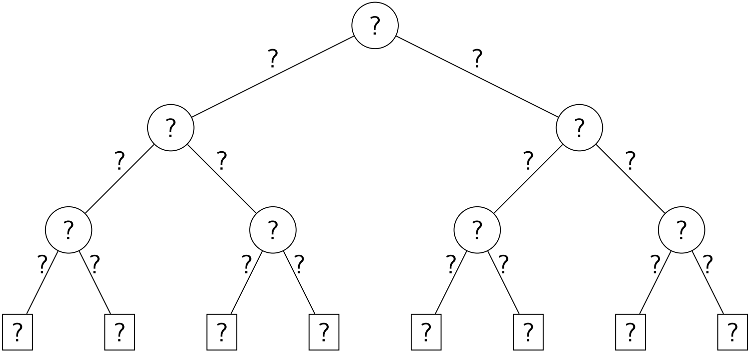

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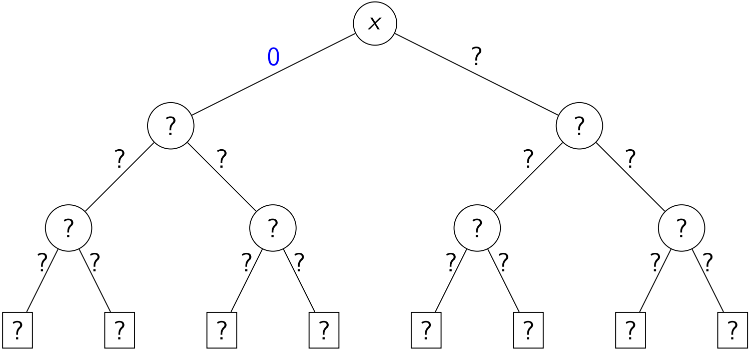

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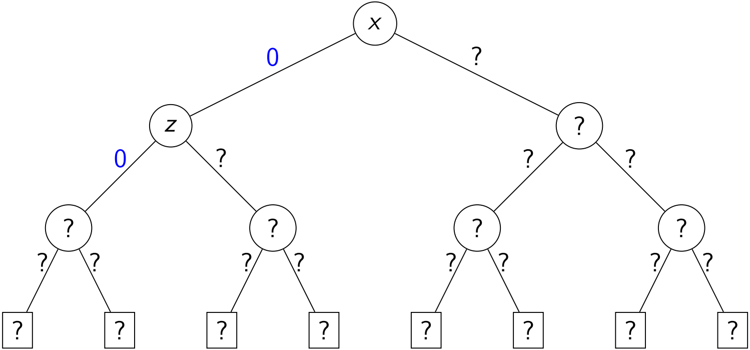

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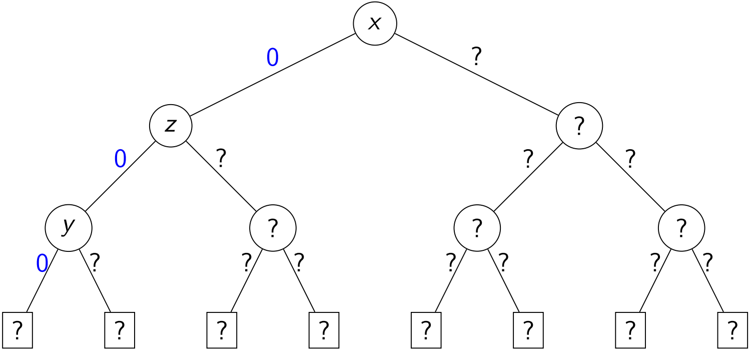

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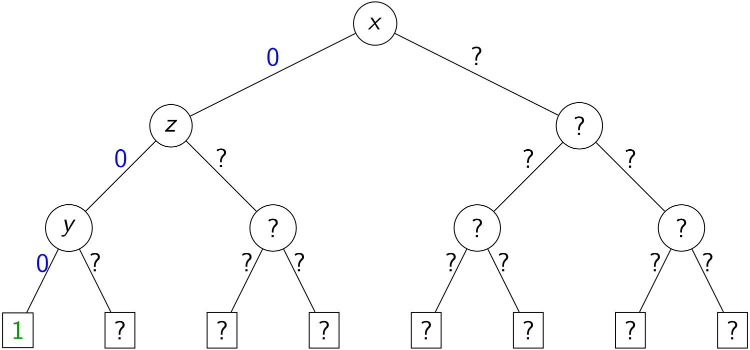

Decision heuristics

Jersolow-Wang method

Compute for every literal l the following static value:

$$J(l) : \sum_{l \in c, c \in \phi} 2^{-|c|}$$

c – clause, ϕ – formula

- Choose a literal l that maximizes $J(l)$.
- This gives an exponentially higher weight to literals in shorter clauses

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Static Jersolow-Wang method

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Static Jersolow-Wang method

$J(x) = 0,\ J(\neg x) = \frac{1}{8} + \frac{1}{4},\ J(y) = \frac{1}{8} + \frac{1}{4},\ J(\neg y) = \frac{1}{4},\ J(z) = \frac{1}{8},\ J(\neg z) = \frac{1}{4}$

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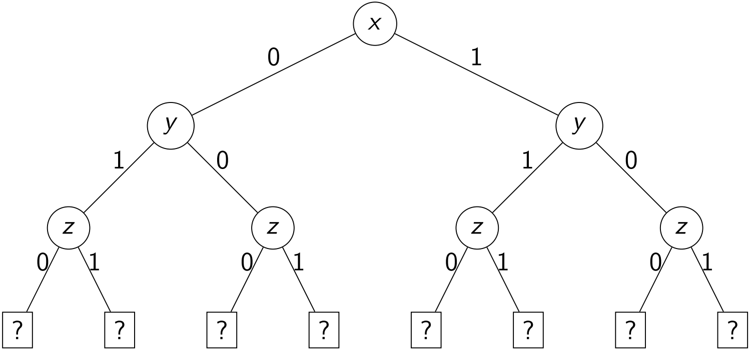

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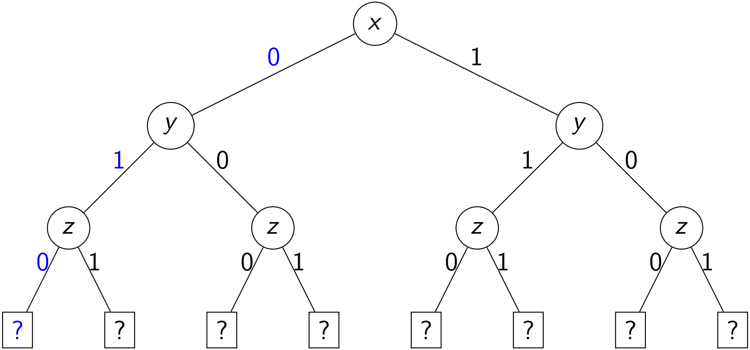

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