Deductive Verification of Programs

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Outline

Computer Systems and Correctness

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  Deductive verification of programs

A Small Example

Consider the following program:

```
int* allocateArray(int length)
{
  int i;
  int* array;
  array = malloc(sizeof(int)*length);

  for (i = 0;i <= length;i++)
    array[i] = 0;
  return array;
}
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Is this program correct?

Hardly: it writes into memory that has not been allocated.

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int* allocateArray(int length)
{
  int i;
  int* array;
  array = malloc(sizeof(int)*length); // may return 0!

  for (i = 0;i < length;i++)
    array[i] = 0;
  return array;
}
```

Is this program correct?

No: it may write to the null address.

A Small Example

Consider the following program:

```
int* allocateArray(int length)
{
  int i;
  int* array;
  array = malloc(sizeof(int)*length);
  if (!array) return 0;
  for (i = 0;i < length;i++)
    array[i] = 0;
  return array;
}
```

Is this program correct?

A Small Example

Consider the following program:

```
/* Returns a new array of integers of a given
 length initialised by a non-zero value */
int* allocateArray(int length)
{
  int i;
  int* array;
  array = malloc(sizeof(int)*length);
  if (!array) return 0;
  for (i = 0;i < length;i++)
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  return array;
}
```

Is this program correct?
No: it initialises the array by zeros

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We discussed correctness of a program without defining what it means.

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}
```

We discussed correctness of a program without defining what it means.
So what is correctness?

Notes

- We could spot the first two errors without knowing anything about the intended meaning of the program. But we had to understand the meaning of the program in general and some specific properties of programming in C-like languages (or other programming languages).

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- We could spot the first two errors without knowing anything about the intended meaning of the program. But we had to understand the meaning of the program in general and some specific properties of programming in C-like languages (or other programming languages).
- To understand the last “error” we had to know something about the intended behaviour of the program.

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- Consider the program/system as a mathematical object.
  Build a formal model of the system.

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  The semantics is normally based on notions is true, is false, satisfies.

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Deductive verification, model checking

Deductive Verification of Programs

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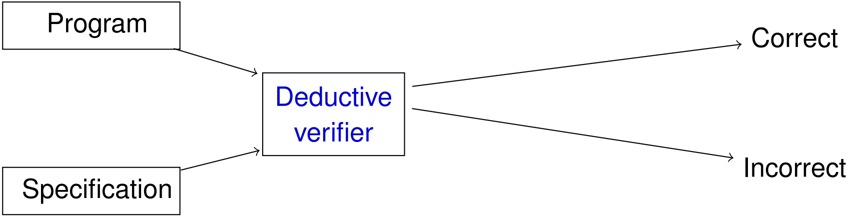
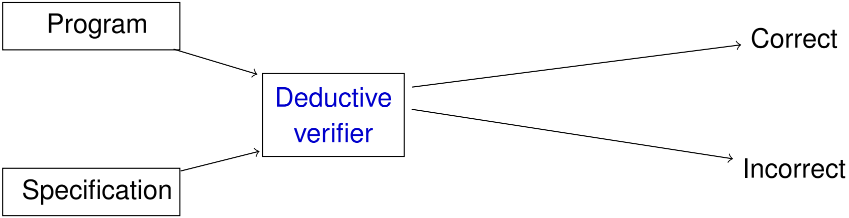
Deductive Verification of Programs

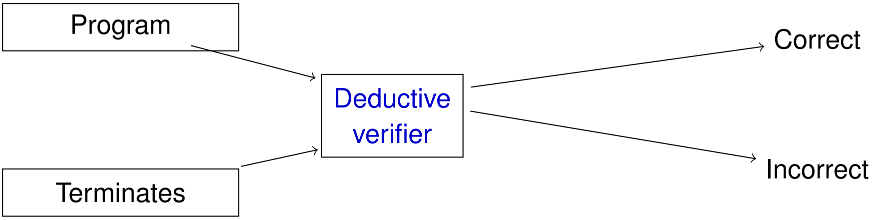


Deductive Verification of Programs



- Statically reason about program behavior without executing the program: consider all possible program executions under all inputs
- Examples of specifications: absence of arithmetic overflow, array is sorted, program terminates . . .

Deductive Verification of Programs



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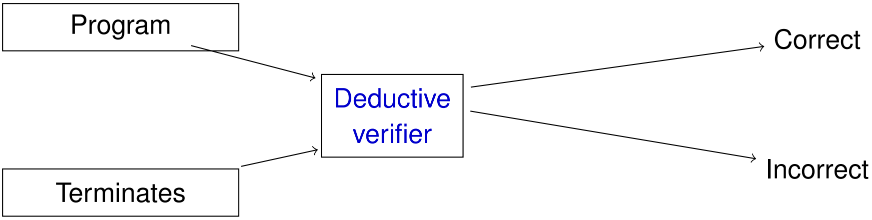
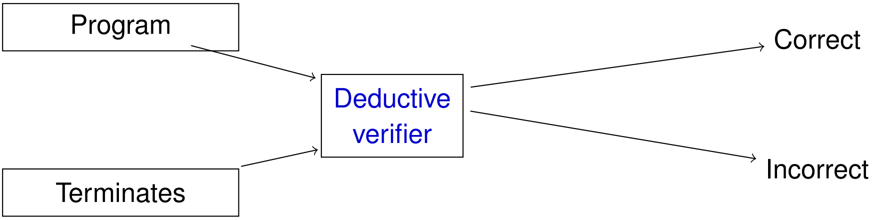


The Halting Problem

Deductive Verification of Programs



The Halting Problem: Undecidable

- A problem is undecidable if there does not exist a Turing machine that can solve it

Alan Turing, 1936

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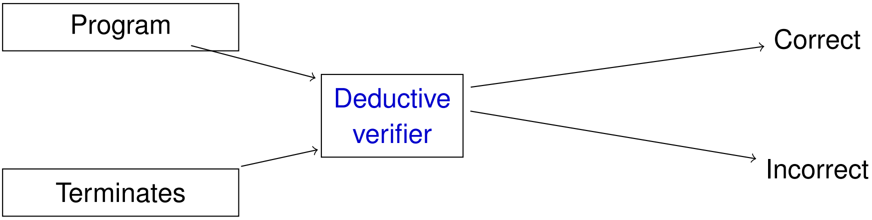

The Halting Problem: Undecidable

- A problem is undecidable if there does not exist a C/Java program that can solve it

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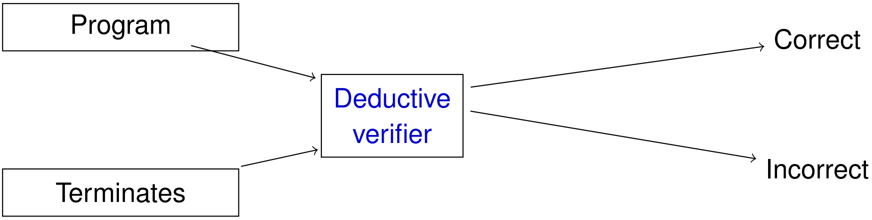

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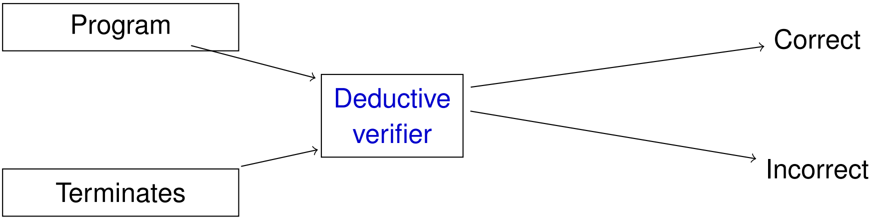

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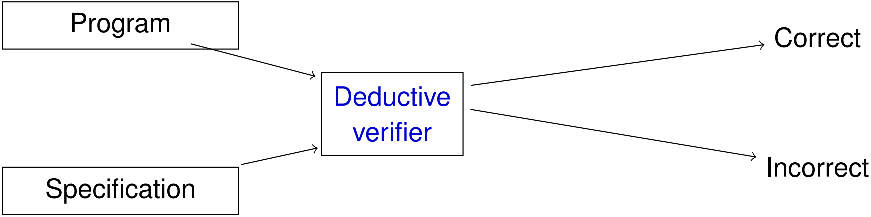

Undecidable: Verifier cannot exist

- There is no computer program that can always decide whether a computer program satisfies a non-trivial specification (Rice Theorem)

Alan Turing, 1936

Deductive Verification – Living with Undecidability

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  - finite-state, loop-free, bounded arithmetic, ...

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  - **Testing:** analyze program executions on a fixed input set

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```
if (x-100<=0)
 if (y-100<=0)
   if (x+y-200==0)
     crash();
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- $1/2^{64}$

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Testing is hard!

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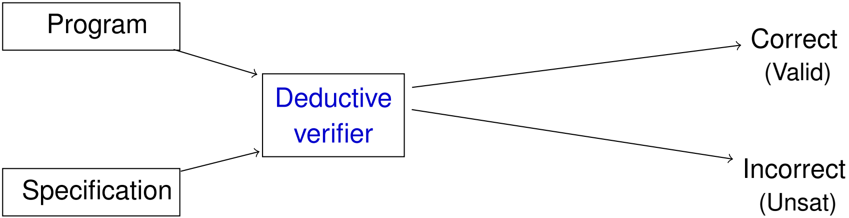
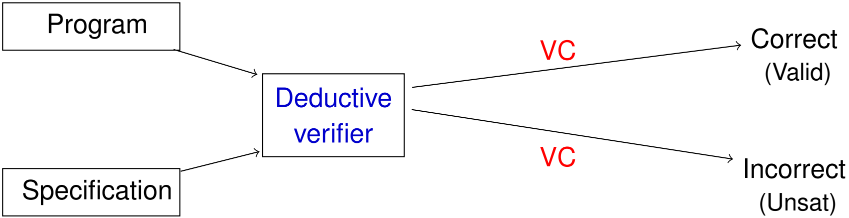
    Algorithmic approach to deductive verification: **(my research area)**
    - automate the generation of annotations,
    - fully automatic, push-button verification for classes of programs

Deductive Verification of Programs

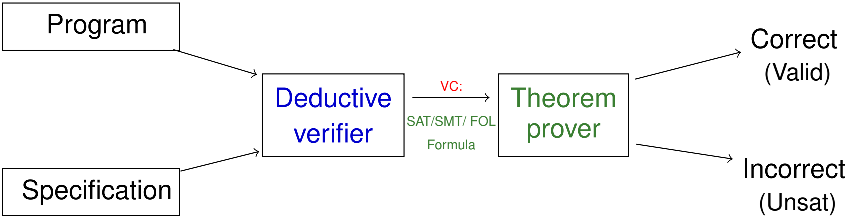


Deductive Verification of Programs in Practice



- Verification condition (VC): A logical formula ϕ s.t. the program is correct iff ϕ is valid

Deductive Verification of Programs in Practice



- Verification condition (VC): SAT/SMT/FOL formula ϕ s.t. the program is correct iff ϕ is valid
- Available tools: Dafny (Microsoft), Why3, KeY, ESC/Java, ...

Deductive Verification of Programs in Dafny

Revisiting our `allocateArray` example

Deductive Verification of Programs in Dafny

```
method allocateArray(a:array<int>, length:int)



{
  var i:int;
  i:=0;
  while (i<length)

  {
    a[i]:=1; i:=i+1;
  }
}
```

Deductive Verification of Programs in Dafny

```
method allocateArray(a:array<int>, length:int)
requires 0<=length<a.Length
modifies a

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Deductive Verification of Programs in Dafny

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method allocateArray(a:array<int>, length:int)
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ensures forall k::0<=k<length==>a[k]!=0
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ensures forall k::0<=k<length==>a[k]!=0
{
  var i:int;
  i:=0;
  while (i<length)
  invariant (forall k::0<=k<i&& k<length==>a[k]!=0);
  {
    a[i]:=1; i:=i+1;
  }
}
```

Deductive Verification of Programs

- Living with undecidability;
- Algorithmic approach, using Hoare logic;
- Automated approach, thanks to SAT/SMT solving and theorem proving.

Deductive Verification of Programs

Tentative list of topics to be addressed:

- Small imperative language IMP
  - Syntax and semantics
- Hoare logic: basis of all deductive verification techniques
- Deductive verification of IMP
  - Axiomatic Semantics
  - Predicate transformers - for algorithmic verification
- Inferring loop invariants automatically

Outline

Small Imperative Language IMP – Syntax

We consider a small imperative language IMP.

Syntactic Sets of IMP

•	Int	positive and negative (integer) numerals	$n, m, \ldots$ (e.g. $-5, 0, 10$)
•	Loc	locations	$x, y, z, \ldots$
•	AExp	arithmetic expressions	a
•	BExp	boolean expressions	b
•	P	programs	p

Small Imperative Language IMP – Syntax

Abstract Syntax of IMP: Arithmetic Expressions AExp

$$
\begin{array}{rcll}
a & ::= & n & \text{for } n \in \mathrm{Int} \\
& \mid & x & \text{for } x \in \mathrm{Loc} \\
& \mid & a_1 + a_2 & \text{for } a_1, a_2 \in \mathrm{AExp} \\
& \mid & a_1 - a_2 & \text{for } a_1, a_2 \in \mathrm{AExp} \\
& \mid & a_1 * a_2 & \text{for } a_1, a_2 \in \mathrm{AExp} \\
& \mid & a_1 / a_2 & \text{for } a_1, a_2 \in \mathrm{AExp}
\end{array}
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\end{array}
$$

- In the abstract synthax, we omit parantheses.

$$2 + 3 * 4 - 5$$

can be parsed as $2 + ((3 * 4) - 5$ or as $(2 + 3) * (4 - 5)$, depending on the *concrete syntax* (e.g. using operator precedence).

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  $$2 + 3 * 4 - 5$$

  can be parsed as $2 + ((3 * 4) - 5$ or as $(2 + 3) * (4 - 5)$, depending on the *concrete syntax* (e.g. using operator precedence).
- $3 + 5$ is not syntactically identical to $5 + 3$.
  But, $(3 + 5)$ is syntactically identical to $3 + 5$.

Small Imperative Language IMP – Syntax

Abstract Syntax of IMP: Boolean Expressions BExp

$$
\begin{array}{rcll}
b & ::= & \mathrm{true} & \\
& | & \mathrm{false} & \\
& | & a_1 \ \mathcal{AOP} \ a_2 & \text{for } a_1, a_2 \in \mathrm{AExp} \text{ and } \mathcal{AOP} \in \{=, <, >, \leq, \geq\} \\
& | & \neg b & \text{for } b \in \mathrm{BExp} \\
& | & b_1 \ \mathcal{BOP} \ b_2 & \text{for } b_1, b_2 \in \mathrm{BExp} \text{ and } \mathcal{BOP} \in \{\wedge, \vee\}
\end{array}
$$

Small Imperative Language IMP – Syntax

Abstract Syntax of IMP: Programs P

$$
\begin{array}{rcll}
p & ::= & \text{skip} & \\
 & | & \text{abort} & \\
 & | & x := a & \text{for } x \in \text{Loc and } a \in \text{AExp} \\
 & | & p_1 ; p_2 & \text{for } p_1, p_2 \in \text{P} \\
 & | & \text{if } b \text{ then } p_1 \text{ else } p_2 & \text{for } b \in \text{BExp and } p_1, p_2 \in \text{P} \\
 & | & \text{while } b \text{ do } p \text{ od} & \text{for } b \in \text{BExp and } p \in \text{P}
\end{array}
$$

Small Imperative Language IMP – Syntax

Example of an IMP Program

p

$$\Longrightarrow^* \quad x := 10; y := 0; \text{ while } x > 0 \text{ do } y := y + x; x := x - 1 \text{ od}$$

Small Imperative Language IMP – Syntax

Example of an IMP Program

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\begin{aligned}
p \quad &\Longrightarrow \quad p_1 ; p_2 \\
&\Longrightarrow \quad p_1 ; p_3 ; p_4 \\
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Small Imperative Language IMP – Syntax

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&\Longrightarrow^* x := 10; y := 0;\ \text{while } a_1 > a_2 \text{ do } p_6; p_7 \text{ od}\\
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\end{aligned}
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- Meaning of IMP programs and expressions depends on the values of variables in locations.

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  - $\sigma(x)$ is the value/content of locations x in state σ

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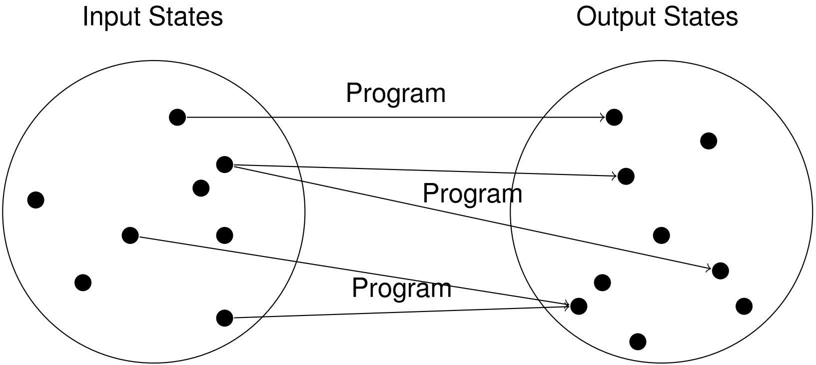
- A state σ is a function from Loc to Int.
  - $\sigma(x)$ is the value/content of locations x in state σ
  - the set of states $\Sigma = \{\sigma | \sigma : \mathrm{Loc} \to \mathrm{Int}\}$

Programs as State Transformers



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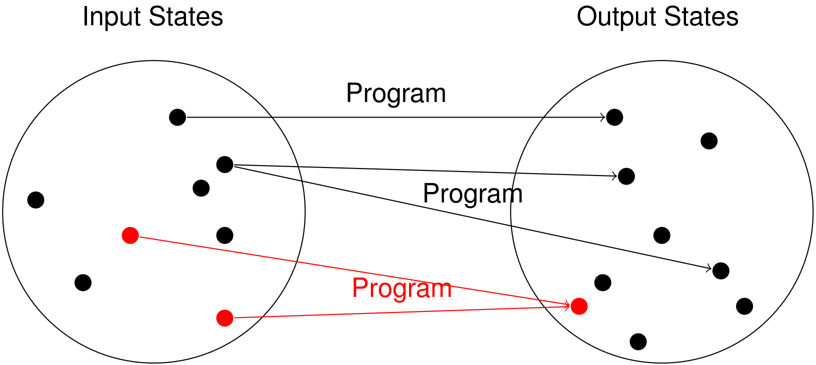
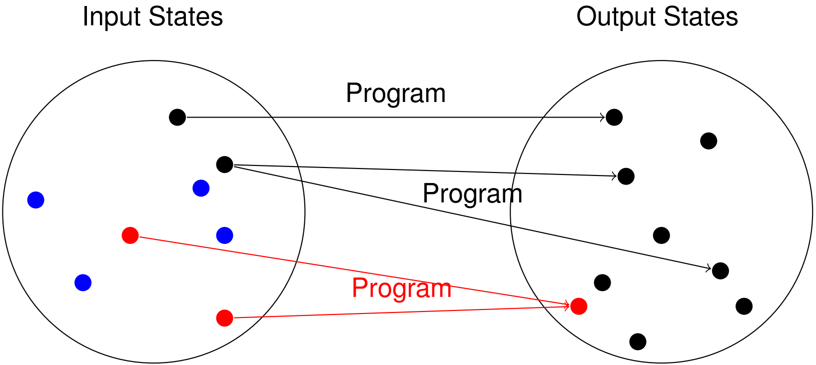


- Several inputs may be mapped to the same output.

Programs as State Transformers



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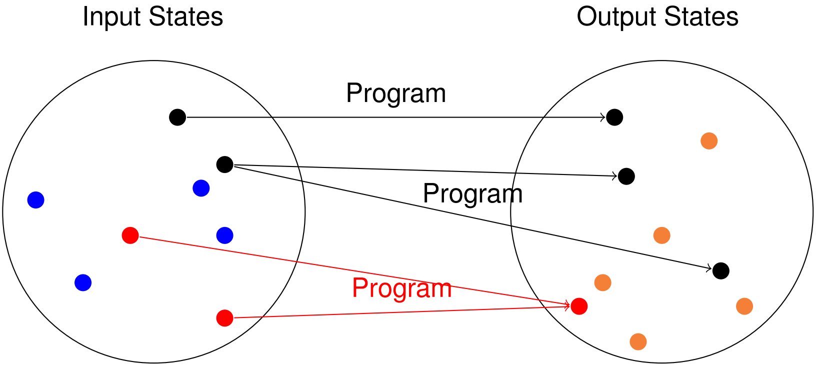

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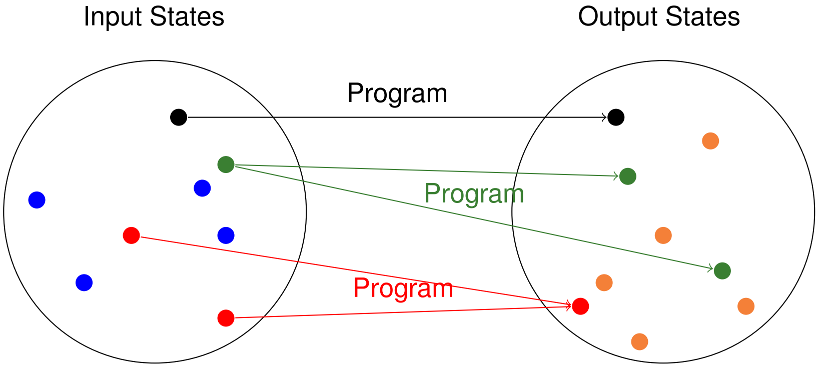

- Several inputs may be mapped to the same output.
- Some inputs are not mapped to any output.
  (Program aborts or loops.)
- Some outputs are not reached from any input.
- One input may be mapped to several outputs (non-determinism).

Small Imperative Language IMP – Semantics

- Evaluation of IMP expressions:
  - Ternary relation on an $\mathrm{AExp}/\mathrm{BExp}$ expression, a state σ and a value v
  - Denoted as $\langle exp, \sigma \rangle \to v$, where $exp \in \mathrm{AExp}$ or $exp \in \mathrm{BExp}$:

    "Expression exp in state σ evaluates to value v".

Small Imperative Language IMP – Operational Semantics

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  Evaluation of IMP expressions has no side-effects (the state remains unchanged)
- Can evaluations of expressions be considered as a function of 2 arguments exp and σ?

  Only if there is a unique value v for $\langle exp, \sigma\rangle \rightarrow v$

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    Operational semantics: "mapping of expressions to values"

- The pair $\langle exp, \sigma\rangle$ is:
  - an arithmetic expression configuration, for $exp \in \mathrm{AExp}$;
  - a boolean expression configuration, for $exp \in \mathrm{BExp}$;

- Evaluation of IMP programs:
  - Ternary relation on a program p, a state σ and a new state σ'
  - Denoted as $\langle p, \sigma \rangle \to \sigma'$:

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Small Imperative Language IMP – Operational Semantics

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Small Imperative Language IMP – Operational Semantics

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- Execution of a program has effect
  - but no direct value
  - "result" of a program is a new state σ'
- Evaluation of a program:
  - may terminate in a final state
  - may diverge and never yield a final state

Small Imperative Language IMP – Operational Semantics

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Small Imperative Language IMP – Operational Semantics

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- Can evaluations of programs be considered as a function of 2 arguments p and σ?

  Only if there is a unique successor state

Small Imperative Language IMP – Operational Semantics

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- Set of program configurations $\mathcal{C} = (P \times \Sigma) \cup \Sigma$
  - Transition relation $==>$ of IMP programs is a relation over $\mathcal{C} \times \mathcal{C}$
    - $\langle p, \sigma \rangle ==> \langle p', \sigma' \rangle$ intermediate step of program execution
    - $\langle p, \sigma \rangle ==> \sigma'$ last step of program execution

Evaluation Rule Notation:

$$\frac{F_1 \quad \ldots \quad F_k}{G} \ ,$$

to denote “if $F_1, \ldots, F_k$, then G”.

When $k = 0$, then G is an axiom:

$$\frac{}{G} \ .$$

Small Imperative Language IMP – Operational Semantics

Note: In $\langle a, \sigma \rangle \to v$, the value v is an integer value.

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Evaluation Rules for AExp

$$\frac{}{\langle n, \sigma \rangle \to \mathbf{n}} \; ,$$

where $\mathbf{n} \in \mathbf{Z}$ represents the integer number corresponding to $n \in \mathrm{Int}$

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$$\frac{}{\langle x, \sigma \rangle \to \sigma(x)}$$

Small Imperative Language IMP – Operational Semantics

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$$\frac{}{\langle x, \sigma \rangle \rightarrow \sigma(x)}$$

$$\frac{\langle a_1, \sigma \rangle \rightarrow n_1 \quad \langle a_2, \sigma \rangle \rightarrow n_2}{\langle a_1 + a_2, \sigma \rangle \rightarrow n_1 +_{\mathbf{Z}} n_2} \ ,$$

where $+_{\mathbf{Z}}$ is the integer sum operator

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$$\frac{\langle a_1, \sigma \rangle \rightarrow n_1 \quad \langle a_2, \sigma \rangle \rightarrow n_2}{\langle a_1 - a_2, \sigma \rangle \rightarrow n_1 -_{\mathbf{Z}} n_2} \;,$$

where $-_{\mathbf{Z}}$ is the integer substraction operator

$$\frac{\langle a_1, \sigma \rangle \rightarrow n_1 \quad \langle a_2, \sigma \rangle \rightarrow n_2}{\langle a_1 * a_2, \sigma \rangle \rightarrow n_1 *_{\mathbf{Z}} n_2} \;,$$

where $*_{\mathbf{Z}}$ is the integer product operator

$$\frac{\langle a_1, \sigma \rangle \rightarrow n_1 \quad \langle a_2, \sigma \rangle \rightarrow n_2}{\langle a_1 / a_2, \sigma \rangle \rightarrow n_1 /_{\mathbf{Z}} n_2} \;,$$

where $/_{\mathbf{Z}}$ is the of integer division operator

An Example of Evaluating $a \in \mathrm{AExp}$

Let a be $(x + 5) + (8 + 10)$.
Consider the state σ_0 with $\sigma_0(x) = 0$.

$$\frac{}{\langle (x + 5) + (8 + 10), \sigma_0 \rangle \to}$$

An Example of Evaluating $a \in \mathrm{AExp}$

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$$\dfrac{\dfrac{}{\langle x + 5, \sigma_0 \rangle \to} \qquad \dfrac{}{\langle 8 + 10, \sigma_0 \rangle \to}}{\langle (x + 5) + (8 + 10), \sigma_0 \rangle \to} \; \mathit{sum}$$

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Small Imperative Language IMP – Operational Semantics

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An Example of Evaluating $a \in \mathrm{AExp}$

Let a be $(x+5)+(8+10)$.
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$$
\dfrac{
  \dfrac{
    \dfrac{}{\langle x, \sigma_0\rangle \to 0}\,\textit{locations}
    \quad
    \dfrac{}{\langle 5, \sigma_0\rangle \to 5}\,\textit{numbers}
  }{\langle x+5, \sigma_0\rangle \to 5}\,\textit{sum}
  \qquad
  \dfrac{
    \dfrac{}{\langle 8, \sigma_0\rangle \to 8}\,\textit{numbers}
    \quad
    \dfrac{}{\langle 10, \sigma_0\rangle \to 10}\,\textit{numbers}
  }{\langle 8+10, \sigma_0\rangle \to 18}\,\textit{sum}
}{\langle (x+5)+(8+10), \sigma_0\rangle \to 23}\,\textit{sum}
$$

An Example of Evaluating $a \in \mathrm{AExp}$

Let a be $(x + 5) + (8 + 10)$.
Consider the state σ_0 with $\sigma_0(x) = 0$.

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The above structure of evaluation rules illustrates the derivation tree, or simply derivation of $\langle a, \sigma_0 \rangle \to 23$.

Small Imperative Language IMP – Operational Semantics

Note: In $\langle b, \sigma \rangle \to v$, the value v is a boolean value/truth value.

Small Imperative Language IMP – Operational Semantics

Evaluation Rules for BExp

$$\frac{}{\langle \textit{true}, \sigma \rangle \rightarrow \textbf{true}} \,,$$

where **true** represents the boolean value 1 (**true**)

$$\frac{}{\langle \textit{false}, \sigma \rangle \rightarrow \textbf{false}} \,,$$

where **false** represents the boolean value 0 (**false**)

Small Imperative Language IMP – Operational Semantics

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where **false** represents the boolean value 0 (**false**)

$$\frac{\langle a_1, \sigma \rangle \to n_1 \quad \langle a_2, \sigma \rangle \to n_2}{\langle a_1 = a_2, \sigma \rangle \to \mathbf{true}} \,,$$

if n_1 and n_2 are equal

$$\frac{\langle a_1, \sigma \rangle \to n_1 \quad \langle a_2, \sigma \rangle \to n_2}{\langle a_1 = a_2, \sigma \rangle \to \mathbf{false}} \,,$$

if n_1 and n_2 are not equal

Small Imperative Language IMP – Operational Semantics

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if n_1 and n_2 are not equal

$$\frac{\langle a_1, \sigma\rangle \to n_1 \quad \langle a_2, \sigma\rangle \to n_2}{\langle a_1 \leq a_2, \sigma\rangle \to \mathbf{true}} \;,$$

if n_1 is less than or equal to n_2

$$\frac{\langle a_1, \sigma\rangle \to n_1 \quad \langle a_2, \sigma\rangle \to n_2}{\langle a_1 \leq a_2, \sigma\rangle \to \mathbf{false}} \;,$$

if n_1 is not less than or equal to n_2

Small Imperative Language IMP – Operational Semantics

Evaluation Rules for BExp (contd)

$$\frac{\langle a_1, \sigma\rangle \to n_1 \quad \langle a_2, \sigma\rangle \to n_2}{\langle a_1 \geq a_2, \sigma\rangle \to \mathbf{true}}\ ,$$

if n_1 is greater than or equal to n_2

$$\frac{\langle a_1, \sigma\rangle \to n_1 \quad \langle a_2, \sigma\rangle \to n_2}{\langle a_1 \geq a_2, \sigma\rangle \to \mathbf{false}}\ ,$$

if n_1 is not greater than or equal to n_2

$$\frac{\langle a_1, \sigma\rangle \to n_1 \quad \langle a_2, \sigma\rangle \to n_2}{\langle a_1 < a_2, \sigma\rangle \to \mathbf{true}}\ ,$$

if n_1 is less than n_2

$$\frac{\langle a_1, \sigma\rangle \to n_1 \quad \langle a_2, \sigma\rangle \to n_2}{\langle a_1 < a_2, \sigma\rangle \to \mathbf{false}}\ ,$$

if n_1 is not less than n_2

$$\frac{\langle a_1, \sigma\rangle \to n_1 \quad \langle a_2, \sigma\rangle \to n_2}{\langle a_1 > a_2, \sigma\rangle \to \mathbf{true}}\ ,$$

if n_1 is greater than n_2

$$\frac{\langle a_1, \sigma\rangle \to n_1 \quad \langle a_2, \sigma\rangle \to n_2}{\langle a_1 > a_2, \sigma\rangle \to \mathbf{false}}\ ,$$

if n_1 is not greater than n_2

Evaluation Rules for BExp (contd)

$$\frac{\langle b, \sigma \rangle \to \textbf{true}}{\langle \neg b, \sigma \rangle \to \textbf{false}}\ \textit{neg} \qquad \frac{\langle b, \sigma \rangle \to \textbf{false}}{\langle \neg b, \sigma \rangle \to \textbf{true}}\ \textit{neg}$$

Small Imperative Language IMP – Operational Semantics

Evaluation Rules for BExp (contd)

$$\frac{\langle b, \sigma \rangle \to \textbf{true}}{\langle \neg b, \sigma \rangle \to \textbf{false}}\ \textit{neg} \qquad \frac{\langle b, \sigma \rangle \to \textbf{false}}{\langle \neg b, \sigma \rangle \to \textbf{true}}\ \textit{neg}$$

$$\frac{\langle b_1, \sigma \rangle \to t_1 \quad \langle b_2, \sigma \rangle \to t_2}{\langle b_1 \wedge b_2, \sigma \rangle \to t}\ ,$$

where t is **true** if both t_1 and t_2 are **true**, and is **false** otherwise.

Small Imperative Language IMP – Operational Semantics

Evaluation Rules for BExp (contd)

$$\frac{\langle b, \sigma \rangle \to \textbf{true}}{\langle \neg b, \sigma \rangle \to \textbf{false}} \; neg \qquad \frac{\langle b, \sigma \rangle \to \textbf{false}}{\langle \neg b, \sigma \rangle \to \textbf{true}} \; neg$$

$$\frac{\langle b_1, \sigma \rangle \to t_1 \quad \langle b_2, \sigma \rangle \to t_2}{\langle b_1 \wedge b_2, \sigma \rangle \to t} \; ,$$

where t is **true** if both t_1 and t_2 are **true**, and is **false** otherwise.

$$\frac{\langle b_1, \sigma \rangle \to t_1 \quad \langle b_2, \sigma \rangle \to t_2}{\langle b_1 \vee b_2, \sigma \rangle \to t} \; ,$$

where t is **false** if both t_1 and t_2 are **false**, and is **true** otherwise.

Evaluation Rules for P

$$\frac{}{\langle \textbf{skip}, \sigma \rangle \to} \qquad \frac{}{\langle \textbf{abort}, \sigma \rangle \to}$$

Small Imperative Language IMP – Operational Semantics

Evaluation Rules for P

$$\frac{}{\langle \textbf{skip}, \sigma\rangle \to \sigma} \qquad \frac{}{\langle \textbf{abort}, \sigma\rangle \not\to \qquad \text{undefined}}$$

Evaluation of IMP programs is a partial relation!

Small Imperative Language IMP – Operational Semantics

Evaluation Rules for P

$$\frac{}{\langle \mathbf{skip}, \sigma \rangle \to \sigma}$$

$$\frac{}{\langle \mathbf{abort}, \sigma \rangle \not\to} \quad \text{undefined}$$

$$\frac{}{\langle x := a, \sigma \rangle \to}$$

Evaluation Rules for P

$$\frac{}{\langle \textbf{skip}, \sigma\rangle \to \sigma} \qquad \frac{}{\langle \textbf{abort}, \sigma\rangle \not\to} \ \text{undefined}$$

$$\frac{\langle a, \sigma\rangle \to v}{\langle x := a, \sigma\rangle \to \sigma[x/v]}$$

Notation: We write $\sigma[x/v]$ to denote the state obtained from σ by replacing x by v. That is:

$$\sigma[x/v](y) = \begin{cases} v, & \text{if } y = x, \\ \sigma(y), & \text{otherwise} \end{cases}$$

Small Imperative Language IMP – Operational Semantics

Evaluation Rules for P

$$\frac{}{\langle \textbf{skip}, \sigma \rangle \to \sigma}$$

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$$\frac{\langle a, \sigma \rangle \to v}{\langle x := a, \sigma \rangle \to \sigma[x/v]}$$

$$\frac{}{\langle p_1 ; p_2, \sigma \rangle \to}$$

Small Imperative Language IMP – Operational Semantics

Evaluation Rules for P

$$\frac{}{\langle \textbf{skip}, \sigma\rangle \rightarrow \sigma} \qquad \frac{}{\langle \textbf{abort}, \sigma\rangle \not\rightarrow \qquad \text{undefined}}$$

$$\frac{\langle a, \sigma\rangle \rightarrow v}{\langle x := a, \sigma\rangle \rightarrow \sigma[x/v]}$$

$$\frac{\langle p_1, \sigma\rangle \rightarrow \sigma' \quad \langle p_2, \sigma'\rangle \rightarrow \sigma''}{\langle p_1 ; p_2, \sigma\rangle \rightarrow \sigma''}$$

Small Imperative Language IMP – Operational Semantics

Evaluation Rules for P

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$$\frac{\langle a, \sigma \rangle \to v}{\langle x := a, \sigma \rangle \to \sigma[x/v]}$$

$$\frac{\langle p_1, \sigma \rangle \to \sigma' \quad \langle p_2, \sigma' \rangle \to \sigma''}{\langle p_1 ; p_2, \sigma \rangle \to}$$

$$\frac{}{\langle \textbf{if } b \textbf{ then } p_1 \textbf{ else } p_2, \sigma \rangle \to}$$

Small Imperative Language IMP – Operational Semantics

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$$\frac{}{\langle \textbf{skip}, \sigma\rangle \to \sigma}$$

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$$\frac{\langle a, \sigma\rangle \to v}{\langle x := a, \sigma\rangle \to \sigma[x/v]}$$

$$\frac{\langle p_1, \sigma\rangle \to \sigma' \quad \langle p_2, \sigma'\rangle \to \sigma''}{\langle p_1 ; p_2, \sigma\rangle \to}$$

$$\frac{\langle b, \sigma\rangle \to \textbf{true} \quad \langle p_1, \sigma\rangle \to \sigma'}{\langle \textbf{if } b \textbf{ then } p_1 \textbf{ else } p_2, \sigma\rangle \to \sigma'}$$

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$$\frac{\langle a, \sigma\rangle \to v}{\langle x := a, \sigma\rangle \to \sigma[x/v]}$$

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$$\frac{\langle b, \sigma\rangle \to \textbf{true} \quad \langle p_1, \sigma\rangle \to \sigma'}{\langle \textbf{if } b \textbf{ then } p_1 \textbf{ else } p_2, \sigma\rangle \to \sigma'} \qquad \frac{\langle b, \sigma\rangle \to \textbf{false} \quad \langle p_2, \sigma\rangle \to \sigma'}{\langle \textbf{if } b \textbf{ then } p_1 \textbf{ else } p_2, \sigma\rangle \to \sigma'}$$

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$$\frac{\langle p_1, \sigma\rangle \to \sigma' \quad \langle p_2, \sigma'\rangle \to \sigma''}{\langle p_1 ; p_2, \sigma\rangle \to}$$

$$\frac{\langle b, \sigma\rangle \to \textbf{true} \quad \langle p_1, \sigma\rangle \to \sigma'}{\langle \textbf{if } b \textbf{ then } p_1 \textbf{ else } p_2, \sigma\rangle \to \sigma'}$$

$$\frac{\langle b, \sigma\rangle \to \textbf{false} \quad \langle p_2, \sigma\rangle \to \sigma'}{\langle \textbf{if } b \textbf{ then } p_1 \textbf{ else } p_2, \sigma\rangle \to \sigma'}$$

$$\frac{}{\langle \textbf{while } b \textbf{ do } p \textbf{ od }, \sigma\rangle \to}$$

Small Imperative Language IMP – Operational Semantics

Evaluation Rules for P

$$\frac{}{\langle \textbf{skip}, \sigma\rangle \to \sigma} \qquad \frac{}{\langle \textbf{abort}, \sigma\rangle \not\to \quad \text{undefined}}$$

$$\frac{\langle a, \sigma\rangle \to v}{\langle x := a, \sigma\rangle \to \sigma[x/v]}$$

$$\frac{\langle p_1, \sigma\rangle \to \sigma' \quad \langle p_2, \sigma'\rangle \to \sigma''}{\langle p_1 ; p_2, \sigma\rangle \to}$$

$$\frac{\langle b, \sigma\rangle \to \textbf{true} \quad \langle p_1, \sigma\rangle \to \sigma'}{\langle \textbf{if } b \textbf{ then } p_1 \textbf{ else } p_2, \sigma\rangle \to \sigma'} \qquad \frac{\langle b, \sigma\rangle \to \textbf{false} \quad \langle p_2, \sigma\rangle \to \sigma'}{\langle \textbf{if } b \textbf{ then } p_1 \textbf{ else } p_2, \sigma\rangle \to \sigma'}$$

$$\frac{}{\langle \textbf{while } b \textbf{ do } p \textbf{ od }, \sigma\rangle \to \sigma}$$

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$$\frac{\langle b, \sigma \rangle \to \mathbf{true} \quad \langle p, \sigma \rangle \to \sigma' \quad \langle \mathbf{while}\ b\ \mathbf{do}\ p\ \mathbf{od}\ , \sigma' \rangle \to \sigma''}{\langle \mathbf{while}\ b\ \mathbf{do}\ p\ \mathbf{od}\ , \sigma \rangle \to \sigma''}$$

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Summary:

The operational semantics of IMP is given by:

- Evaluation rules for AExp expressions,
- Evaluation rules for BExp expressions,
- Evaluation rules for P programs expressions,

by using (partial) evaluation relations.

Learning Objectives

- Understanding the task of deductive program verification
- Limitations and trends in deductive program verification
- Familiarize with the Dafny program verifier
- Syntax of the small imperative language IMP
- Operational semantics of IMP