Formal Methods in Software Development

Propositional Logic - refresher

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Based on slides of the lecture Satisfiability Checking (Erika Ábrahám), RTWH Aachen

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Propositional logic

The slides are partly taken from:

www.decision-procedures.org/slides/

Propositional logic - Outline

- Syntax of propositional logic
- Semantics of propositional logic
- Satisfiability and validity
- Normal forms
- Enumeration and deduction

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Formulae

- Examples of well-formed formulae:
  - $(\neg a)$
  - $(\neg(\neg a))$
  - $(a \wedge (b \wedge c))$
  - $(a \rightarrow (b \rightarrow c))$

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- Examples of well-formed formulae:
  - $(\neg a)$
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  - $(a \wedge (b \wedge c))$
  - $(a \rightarrow (b \rightarrow c))$
- We omit parentheses whenever we may restore them through operator precedence:

binds stronger

$$\longleftarrow$$

$$\neg \quad \wedge \quad \vee \quad \rightarrow \quad \leftrightarrow$$

chaining the same operator: left binds stronger
e.g., $a \rightarrow b \rightarrow c$ means $((a \rightarrow b) \rightarrow c)$

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Semantics: Assignments

Structures for predicate logic:

- The domain is $\mathbb{B} = \{0, 1\}$.
- The interpretation assigns Boolean values to the variables:

$$\alpha : AP \to \{0, 1\}$$

We call these special interpretations assignments and use *Assign* to denote the set of all assignments.

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Example: $AP = \{a, b\}, \alpha(a) = 0, \alpha(b) = 1$

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p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$	$p \bigoplus q$
0	0	1	0	0	1	1	0
0	1	1	0	1	1	0	1
1	0	0	0	1	0	0	1
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1	1	0	1	1	1	1	0

Each possible assignment is covered by a line of the truth table.
α satisfies φ iff in the line for α and the column for φ the entry is 1.

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- Q: Does α satisfy φ?
- A1: Replace values of α in φ.

Semantics II: Satisfaction relation

Satisfaction relation: $\models \subseteq$ *Assign* $\times$ *PropForm*
Instead of $(\alpha, \varphi) \in \models$ we write $\alpha \models \varphi$ and say that

- α satisfies φ or
- φ holds for α or
- α is a model of φ.

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Note: More elegant but semantically equivalent to truth tables.

Semantics II: Example

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A2: Compute with the satisfaction relation:

$$
\begin{array}{ll}
 & \alpha \models (a \vee (b \rightarrow c)) \\
\textit{iff} & \alpha \models a \textit{ or } \alpha \models (b \rightarrow c) \\
\textit{iff} & \alpha \models a \textit{ or } (\alpha \models b \textit{ implies } \alpha \models c) \\
\textit{iff} & 0 \textit{ or } (0 \textit{ implies } 1) \\
\textit{iff} & 0 \textit{ or } 1 \\
\textit{iff} & 1
\end{array}
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Semantics III: The algorithmic view

- Using the satisfaction relation we can define an algorithm for the problem to decide whether an assignment $\alpha : AP \to \{0, 1\}$ is a model of a propositional logic formula $\varphi \in PropForm$:

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```
Eval(α, φ) {
   if φ ≡ a return α(a);
   if φ ≡ (¬φ1) return not Eval(α, φ1);
   if φ ≡ (φ1 op φ2)
        return Eval(α, φ1) ⟦op⟧ Eval(α, φ2);
}
```

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\begin{array}{l}
\mathit{Eval}(\alpha, \varphi)\ \{ \\
\quad \textbf{if}\ \ \varphi \equiv a\ \ \textbf{return}\ \ \alpha(a); \\
\quad \textbf{if}\ \ \varphi \equiv (\neg\varphi_1)\ \ \textbf{return}\ \ \textbf{not}\ \ \mathit{Eval}(\alpha, \varphi_1); \\
\quad \textbf{if}\ \ \varphi \equiv (\varphi_1\ \mathrm{op}\ \varphi_2) \\
\qquad\qquad \textbf{return}\ \ \mathit{Eval}(\alpha, \varphi_1)\ [\![\mathrm{op}]\!]\ \mathit{Eval}(\alpha, \varphi_2); \\
\}
\end{array}
$$

- Equivalent to the $\models$ relation, but from the algorithmic view.

Semantics III: Example

- Recall our example
  - $\varphi = (a \vee (b \rightarrow c))$
  - $\alpha : \{a, b, c\} \rightarrow \{0, 1\}$ with $\alpha(a) = 0$, $\alpha(b) = 0$, and $\alpha(c) = 1$.

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- Recall our example
  - $\varphi = (a \vee (b \rightarrow c))$
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- $Eval(\alpha, \varphi) \; =$

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- Hence, $\alpha \models \varphi$.

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- For $\varphi \in \mathit{PropForm}$ and $\alpha \in \mathit{Assign}$ it holds that

$$\alpha \models \varphi \qquad \mathit{iff} \qquad \alpha \in \mathit{sat}(\varphi)$$

Satisfying assignments: Example

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Short summary for propositional logic

- Syntax of propositional formulae $\varphi \in \mathit{PropForm}$:

$$\varphi \ := \ AP \mid (\neg\varphi) \mid (\varphi \wedge \varphi)$$

- Semantics:
  - Assignments $\alpha \in \mathit{Assign}$:

$$\begin{array}{l} \alpha : AP \to \{0,1\} \\ \alpha \in 2^{AP} \\ \alpha \in \{0,1\}^{AP} \end{array}$$

  - Satisfaction relation:

$$\begin{array}{lll} \models \subseteq \mathit{Assign} \times \mathit{PropForm} & , & (\text{e.g., } \alpha \models \varphi\ ) \\ \models \subseteq 2^{\mathit{Assign}} \times \mathit{PropForm} & , & (\text{e.g., } \{\alpha_1, \ldots, \alpha_n\} \models \varphi\ ) \\ \models \subseteq \mathit{PropForm} \times \mathit{PropForm}, & & (\text{e.g., } \varphi_1 \models \varphi_2) \\ \mathit{sat} : \ \mathit{PropForm} \to 2^{\mathit{Assign}} & , & (\text{e.g., } \ \mathit{sat}(\varphi)\ ) \end{array}$$

Propositional logic - Outline

- Syntax of propositional logic
- Semantics of propositional logic
- Satisfiability and validity
- Normal forms
- Enumeration and deduction

Semantic classification of formulae

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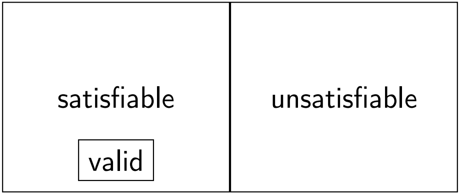

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- Some more (De Morgan rules):
  - $\models \neg(a \wedge b) \leftrightarrow (\neg a \vee \neg b)$
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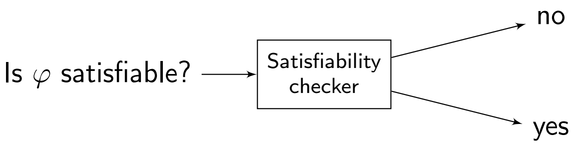

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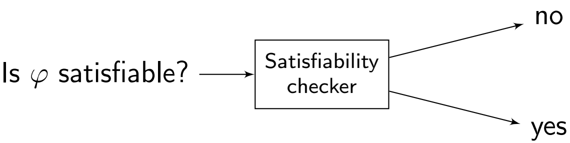

Note: A formula φ is valid iff $\neg\varphi$ is unsatisfiable.

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- $\varphi_2 = \neg a \wedge b$ is in NNF

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- Every formula can be converted to NNF in linear time:
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- DNF is a special case of NNF.

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- A: No, it would violate the NP-completeness of the problem.

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- Also CNF is a special case of NNF.

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- Every formula can be converted to CNF in linear time and space if new variables are added.
- The original and the converted formulae are not equivalent but equisatisfiable.

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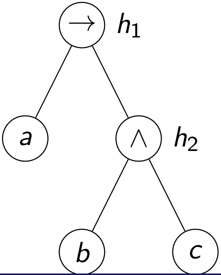

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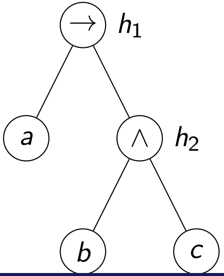

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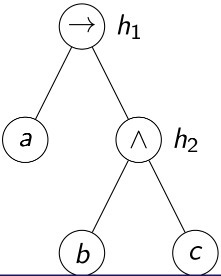

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- Consider the formula
  $\varphi = (a \to (b \wedge c))$
- Associate a new auxiliary variable with each gate.
- Add constraints that define these new variables.
- Finally, enforce the root node.

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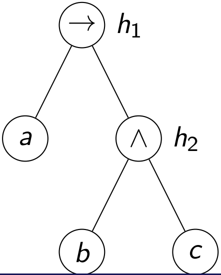
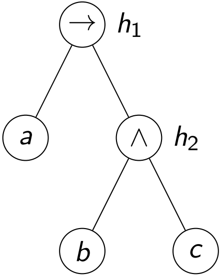


Converting to CNF: Tseitin's encoding

- Need to satisfy:
  $(h_1 \leftrightarrow (a \ \rightarrow \ h_2)) \wedge$
  $(h_2 \leftrightarrow (b \wedge c)) \wedge$
  (h_1)



- Each gate encoding has a CNF representation with 3 or 4 clauses.

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- Let's go back to

  $\varphi_n = (x_1 \wedge y_1) \vee (x_2 \wedge y_2) \vee \cdots \vee (x_n \wedge y_n)$

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  - n auxiliary variables $h_1, \ldots, h_n$.
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  - Top clause: $(h_1 \vee \cdots \vee h_n)$

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  - n auxiliary variables $h_1, \ldots, h_n$.
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- Hence, we have
  - $3n + 1$ clauses, instead of 2^n.
  - $3n$ variables rather than $2n$.

Propositional logic - Outline

- Syntax of propositional logic
- Semantics of propositional logic
- Satisfiability and validity
- Normal forms
- Enumeration and deduction

Two classes of algorithms for validity

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- More generally (beyond propositional logic):
  - Enumeration is possible only in some logics.
  - Deduction cannot necessarily be fully automated.

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Use substitution to eliminate all variables one by one:

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- Q: What is the difference?
  A: Branching on complete vs. partial assignments.

Deduction requires axioms and inference rules

- Inference rules:

$$\frac{\mathit{Antecedents}}{\mathit{Consequents}} \qquad (\mathit{rule\ name})$$

Meaning: If all antecedents hold then at least one of the consequents can be derived.

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- Examples:

$$\frac{a \rightarrow b \qquad b \rightarrow c}{a \rightarrow c} \qquad (\textit{Trans})$$

$$\frac{a \rightarrow b \qquad a}{b} \qquad (\textit{M.P.})$$

Axioms

- Axioms are inference rules with no antecedents, e.g.,

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$$\frac{}{a \to (b \to a)} \quad (H1)$$

- A proof system consists of a set of axioms and inference rules.

Proofs

- Let $\mathcal{H}$ be a proof system.
- $\Gamma \vdash_{\mathcal{H}} \varphi$ means: There is a proof of φ in system $\mathcal{H}$ whose premises are included in Γ
- $\vdash_{\mathcal{H}}$ is called the provability (derivability) relation.

Example

- Let $\mathcal{H}$ be the proof system comprised of the rules Trans and M.P. that we saw earlier:

$$\frac{a \rightarrow b \quad b \rightarrow c}{a \rightarrow c} \qquad (Trans)$$

$$\frac{a \rightarrow b \quad a}{b} \qquad (M.P.)$$

- Does the following relation hold?

$$a \rightarrow b,\ b \rightarrow c,\ c \rightarrow d,\ d \rightarrow e,\ a \quad \vdash_{\mathcal{H}} \quad e$$

Deductive proof: Example

$$\frac{a \to b \quad b \to c}{a \to c} \quad (Trans) \qquad \frac{a \to b \quad a}{b} \quad (M.P.)$$

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Deductive proof: Example

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7. $a \to e$ 3, 6, *Trans*

Deductive proof: Example

$$\frac{a \to b \quad b \to c}{a \to c} \quad (Trans) \qquad \frac{a \to b \quad a}{b} \quad (M.P.)$$

$$a \to b,\ b \to c,\ c \to d,\ d \to e,\ a \quad \vdash_{\mathcal{H}} \quad e$$

1. $a \to b$ *premise*
2. $b \to c$ *premise*
3. $a \to c$ 1, 2, *Trans*
4. $c \to d$ *premise*
5. $d \to e$ *premise*
6. $c \to e$ 4, 5, *Trans*
7. $a \to e$ 3, 6, *Trans*
8. a

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$$\frac{a \to b \quad b \to c}{a \to c} \quad (Trans) \qquad \frac{a \to b \quad a}{b} \quad (M.P.)$$

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2.	$b \to c$	*premise*
3.	$a \to c$	1, 2, *Trans*
4.	$c \to d$	*premise*
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2.	$b \to c$	*premise*
3.	$a \to c$	1, 2, *Trans*
4.	$c \to d$	*premise*
5.	$d \to e$	*premise*
6.	$c \to e$	4, 5, *Trans*
7.	$a \to e$	3, 6, *Trans*
8.	a	*premise*
9.	e	

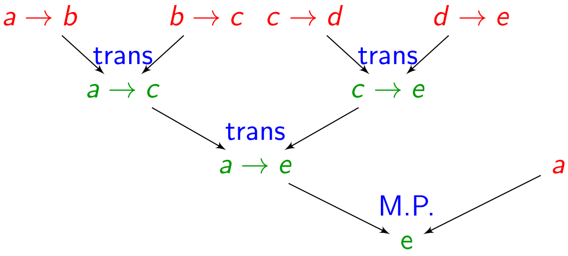
Deductive proof: Example

$$\frac{a \rightarrow b \quad b \rightarrow c}{a \rightarrow c} \quad (Trans) \qquad \frac{a \rightarrow b \quad a}{b} \quad (M.P.)$$

$$a \rightarrow b,\ b \rightarrow c,\ c \rightarrow d,\ d \rightarrow e,\ a \quad \vdash_{\mathcal{H}} \quad e$$

1.	$a \rightarrow b$	*premise*
2.	$b \rightarrow c$	*premise*
3.	$a \rightarrow c$	1, 2, *Trans*
4.	$c \rightarrow d$	*premise*
5.	$d \rightarrow e$	*premise*
6.	$c \rightarrow e$	4, 5, *Trans*
7.	$a \rightarrow e$	3, 6, *Trans*
8.	a	*premise*
9.	e	7, 8, *M.P.*

Proof graph



$a \rightarrow b$
$b \rightarrow c$
$c \rightarrow d$
$d \rightarrow e$
trans
trans
$a \rightarrow c$
$c \rightarrow e$
trans
$a \rightarrow e$
a
M.P.
e

Soundness and completeness

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  With respect to the semantic definition of the logic. In the case of propositional logic truth tables give us this.

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Soundness of $\mathcal{H}$:	*if*	$\vdash_{\mathcal{H}} \varphi$	*then*	$\models \varphi$
Completeness of $\mathcal{H}$:	*if*	$\models \varphi$	*then*	$\vdash_{\mathcal{H}} \varphi$

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$$\begin{array}{llllll} \textit{Soundness of } \mathcal{H}: & \textit{if} & \vdash_{\mathcal{H}} & \varphi & \textit{then} & \models \ \varphi \\ \textit{Completeness of } \mathcal{H}: & \textit{if} & \models & \varphi & \textit{then} & \vdash_{\mathcal{H}} \ \varphi \end{array}$$

- How to prove soundness and completeness?