Formal Methods in Software Developement

Equality and Fourier-Motzkin Variable Elimination for Linear Real Arithmetic

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Based on slides of the lecture Satisfiability Checking (Erika Ábrahám), RTWH Aachen

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The Xmas problem

There are three types of Xmas presents Santa Claus can make.

- Santa Claus wants to reduce the overhead by making only two types.
- He needs at least 100 presents.
- He needs at least 5 of either type 1 or type 2.
- He needs at least 10 of the third type.
- Each present of type 1, 2, and 3 need 1, 2, resp. 5 minutes to make.
- Santa Claus is late, and he has only 3 hours left.
- Each present of type 1, 2, and 3 costs 3, 2, resp. 1 EUR.
- He has 300 EUR for presents in total.

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\begin{aligned}
(p_1 = 0 \vee p_2 = 0 \vee p_3 = 0) \wedge p_1 + p_2 + p_3 \geq 100 \wedge \\
(p_1 \geq 5 \vee p_2 \geq 5) \wedge p_3 \geq 10 \wedge p_1 + 2p_2 + 5p_3 \leq 180 \wedge \\
3p_1 + 2p_2 + p_3 \leq 300
\end{aligned}
$$

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Syntax of linear real arithmetic

Terms:	t	$::=$	0	$\mid$	1	$\mid$	x	$\mid$	$t + t$
Constraints:	c	$::=$	$t < t$						
Formulas:	φ	$::=$	c	$\mid$	$\neg\varphi$	$\mid$	$\varphi \wedge \varphi$	$\mid$	$\exists x.\ \varphi$

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- Linear real arithmetic is also called linear real algebra.
- We consider the satisfiability problem for the quantifier-free fragment QFLRA (or equivalently the existential fragment, i.e., no universal quantifiers and no negation of expressions containing existential quantifiers).

Equality elimination

In an SMT solver for QFLRA, the theory solver needs to check the satisfiability of sets of constraints $\sum_{k=1}^{N} a_{ik} \cdot x_k \sim_i b_i$, where $a_{i,k}$ and b_i are integer (or rational) constants, x_k are real-valued variables, and $\sim_i \in \{=, \leq, <\}$ for $k=1, \ldots, N$ and $i=1, \ldots, M$. (Note: $t > b$ is equivalent to $-t < -b$ and similarly for $\geq$.)

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- Assume that the ith constraint is an equation with $a_{ij} \neq 0$ for some $1 \leq j \leq N$:
$$\sum_{k=1}^{N} a_{ik} \cdot x_k = b_i \quad (a_{i,k}, b_i\text{: } \textit{integer/rational constants, } x_k\text{: } \textit{variables})$$

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- Replace x_j by β_j in all constraints (and multiply the involved constraints by a_{ij} if integer coefficients are wanted).
- After removing tautologies, this substitutiton leads to an equisatisfiable problem with (at most) $M-1$ constraints in (at most) $N-1$ variables (at least the ith constraint and x_j are eliminated).

Eliminate $\leq$

- Let us assume first that after eliminating all equalities, m non-strict inequalities in n variables are left (i.e., there are no strict inequalities):

$$\bigwedge_{1 \leq i \leq m} \sum_{1 \leq j \leq n} a_{ij} x_j \leq b_i$$

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- Input in matrix form: $A\overline{x} \leq \overline{b}$

$$m \text{ constraints} \quad \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ a_{m1} & a_{22} & \cdots & \cdots & a_{mn} \end{pmatrix}}_{n \text{ variables}} \begin{pmatrix} x_1 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} \leq \begin{pmatrix} b_1 \\ \vdots \\ \vdots \\ b_m \end{pmatrix}$$

Fourier-Motzkin variable elimination

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- Fourier-Motzkin: Collect requirements on the lower an upper bounds on the variable we want to eliminate.

Variable bounds

- For a variable x_n, we can partition the constraints according to the coefficients of x_n:
  - $a_{in} = 0$: constraint i puts no bound on x_n
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$$(a) \quad \stackrel{a_{in}>0}{\Rightarrow} \quad x_n \;\leq\; \frac{b_i}{a_{in}} - \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \cdot x_j \qquad \text{upper bound}$$

$$(b) \quad \stackrel{a_{in}<0}{\Rightarrow} \quad x_n \;\geq\; \frac{b_i}{a_{in}} - \sum_{j=1}^{n-1} \frac{a_{ij}}{a_{in}} \cdot x_j \qquad \text{lower bound}$$

Example for upper and lower bounds

Category for x_1?

$$\begin{array}{ll} (1) & x_1 - x_2 \leq 0 \\ (2) & x_1 - x_3 \leq 0 \\ (3) & -x_1 + x_2 + 2x_3 \leq 0 \\ (4) & -x_3 \leq -1 \end{array}$$

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Eliminating unbounded variables

- Iteratively remove variables that are not bounded in both ways (and all the constraints that use them).
- The new problem has a solution iff the old problem has one!

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\begin{aligned}
-8x + 7y &\leq 0 \\
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- For each such pair, add the constraint

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(7) $1 \leq 0$ (from 4,5)

$\rightarrow$ Contradiction (the system is UNSAT)

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- we distinguiosh between strict and non-strict lower and upper bounds (defined by strict respectively non-strict inequalities), and
- for each pair of lower and upper bounds, if any of them is strict then we add the constraint

  $$\beta_l \quad < \quad \beta_u$$

  instead of

  $$\beta_l \quad \leq \quad \beta_u \ .$$

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- Answer: No.
- Reason: in general it is not possible to transform constraints containing non-linear polynomial expressions such that we have a single variable on the left-hand-side and a real-arithmetic expression on the right-hand side (we would need complicated case distinctions, fractions and roots).

Complexity

- Worst-case complexity (Recall: m constraints, n variables):

$$m \quad \xrightarrow[\frac{m}{2} \text{ lower bounds on } x]{\frac{m}{2} \text{ upper bounds on } x}$$

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$$m \quad \xrightarrow[\frac{m}{2} \text{ lower bounds on } x]{\frac{m}{2} \text{ upper bounds on } x} \quad \left(\frac{m}{2}\right)^2 = \frac{m^2}{4}$$

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- The bottleneck: case-splitting

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More efficient method: Simplex (not in this lecture).