Formal Methods in Software Development

Propositional Logic on Examples

Mădălina Erașcu

West University of Timișoara
Faculty of Mathematics and Informatics
Department of Computer Science

Based on slides of the lecture Satisfiability Checking (Erika Ábrahám), RTWH Aachen

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Satisfiability with truth table

Example formula:

$$\phi := \neg(a \rightarrow (b \vee \neg c))$$

Satisfiability with truth table

Example formula:

$$\phi := \neg(a \to (b \vee \neg c))$$

a	b	c	$\neg(a \to (b \vee \neg c))$
0	0	0	
0	0	1	
0	1	0	
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Satisfiability with semantical algorithm

$$
\begin{array}{lcl}
\text{Eval}(\alpha, p) & = & \alpha(p) \\
\text{Eval}(\alpha, \neg A) & = & \neg\text{Eval}(\alpha, A) \\
\text{Eval}(\alpha, A \vee B) & = & \text{Eval}(\alpha, A) \vee \text{Eval}(\alpha, B) \\
\text{Eval}(\alpha, A \wedge B) & = & \text{Eval}(\alpha, A) \wedge \text{Eval}(\alpha, B) \\
\text{Eval}(\alpha, A \rightarrow B) & = & \text{Eval}(\alpha, \neg A) \vee \text{Eval}(\alpha, B) \\
\text{Eval}(\alpha, A \leftrightarrow B) & = & \text{Eval}(\alpha, A \rightarrow B) \wedge \text{Eval}(\alpha, A \leftarrow B)
\end{array}
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NNF conversion

Only operators $\neg, \vee, \wedge$, negation only in front of atomic propositions.

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= \quad & a \wedge \neg b \wedge c
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CNF conversion: The exponential way

CNF: $\wedge_{i=1,\dots,n} \vee_{j=1,\dots,m} l_{ij}$

Example formula:

$$\phi := (a \wedge b) \vee (\neg c \wedge (d \vee e))$$

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& (a \wedge b) \vee (\neg c \wedge (d \vee e)) \\
= \;& (a \vee (\neg c \wedge (d \vee e))) \wedge (b \vee (\neg c \wedge (d \vee e))) \\
= \;& (a \vee (\neg c \wedge d) \vee (\neg c \wedge e)) \wedge (b \vee (\neg c \wedge d) \vee (\neg c \wedge e)) \\
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$$

CNF conversion: The exponential way

CNF: $\bigwedge_{i=1,\ldots,n} \bigvee_{j=1,\ldots,m} l_{ij}$

Example formula:

$$\phi := (a \land b) \lor (\neg c \land (d \lor e))$$

$$\begin{aligned}
& (a \land b) \lor (\neg c \land (d \lor e)) \\
= \; & (a \lor (\neg c \land (d \lor e))) \land (b \lor (\neg c \land (d \lor e))) \\
= \; & (a \lor (\neg c \land d) \lor (\neg c \land e)) \land (b \lor (\neg c \land d) \lor (\neg c \land e)) \\
= \; & (a \lor \neg c \lor (\neg c \land e)) \land (a \lor d \lor (\neg c \land e)) \land \\
& (b \lor \neg c \lor (\neg c \land e)) \land (b \lor d \lor (\neg c \land e)) \\
= \; & (a \lor \neg c \lor \neg c) \land (a \lor \neg c \lor e) \land (a \lor d \lor \neg c) \land (a \lor d \lor e) \land \\
& (b \lor \neg c \lor \neg c) \land (b \lor \neg c \lor e) \land (b \lor d \lor \neg c) \land (b \lor d \lor e)
\end{aligned}$$

CNF conversion: Tseitin's encoding

Example formula:

$$\phi := (a \wedge b) \vee (\neg c \wedge (d \vee e))$$

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CNF conversion: Tseitin's encoding

Example formula:

$$\phi := (a \wedge b) \vee (\neg c \wedge (d \vee e))$$

- $a_1 \leftrightarrow (a_2 \vee a_3)$
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- $a_3 \leftrightarrow (\neg c \wedge a_4)$

CNF conversion: Tseitin's encoding

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- $a_3 \leftrightarrow (\neg c \wedge a_4)$
- $a_4 \leftrightarrow (d \vee e)$
- a_1

CNF conversion: Tseitin's encoding

In the previous formula, we have the following templates:

1. $h \leftrightarrow (p_1 \vee p_2)$

2. $h \leftrightarrow (p_1 \wedge p_2)$

CNF conversion: Tseitin's encoding

In the previous formula, we have the following templates:

1. $h \leftrightarrow (p_1 \vee p_2)$
   $= (h \rightarrow (p_1 \vee p_2)) \quad \wedge \quad (h \leftarrow (p_1 \vee p_2))$

2. $h \leftrightarrow (p_1 \wedge p_2)$

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&= (\neg h \vee p_1 \vee p_2) && \wedge \quad (h \vee (\neg p_1 \wedge \neg p_2)) \\
&= (\neg h \vee p_1 \vee p_2) && \wedge \quad (h \vee \neg p_1) \quad \wedge \quad (h \vee \neg p_2)
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&= (\neg h \vee p_1 \vee p_2) && \wedge \quad (h \vee (\neg p_1 \wedge \neg p_2)) \\
&= (\neg h \vee p_1 \vee p_2) && \wedge \quad (h \vee \neg p_1) \quad \wedge \quad (h \vee \neg p_2)
\end{aligned}
$$

2. $h \leftrightarrow (p_1 \wedge p_2)$
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&= (\neg h \vee p_1 \vee p_2) && \wedge \quad (h \vee (\neg p_1 \wedge \neg p_2)) \\
&= (\neg h \vee p_1 \vee p_2) && \wedge \quad (h \vee \neg p_1) \quad \wedge \quad (h \vee \neg p_2)
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&= (\neg h \vee p_1 \vee p_2) && \wedge \quad (h \vee (\neg p_1 \wedge \neg p_2)) \\
&= (\neg h \vee p_1 \vee p_2) && \wedge \quad (h \vee \neg p_1) \quad \wedge \quad (h \vee \neg p_2)
\end{aligned}
$$

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$$
\begin{aligned}
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&= (\neg h \vee (p_1 \wedge p_2)) && \wedge \quad (h \vee \neg(p_1 \wedge p_2)) \\
&= (\neg h \vee p_1) \quad \wedge \quad (\neg h \vee p_2) && \wedge \quad (h \vee \neg p_1 \vee \neg p_2)
\end{aligned}
$$

CNF conversion: Tseitin's encoding

$$\phi := (a \land b) \lor (\neg c \land (d \lor e))$$

$$\boxed{\begin{array}{l} h \leftrightarrow (p_1 \lor p_2) = (\neg h \lor p_1 \lor p_2) \land (h \lor \neg p_1) \land (h \lor \neg p_2) \\ h \leftrightarrow (p_1 \land p_2) = (\neg h \lor p_1) \land (\neg h \lor p_2) \land (h \lor \neg p_1 \lor \neg p_2) \end{array}}$$

$$\begin{array}{ll} a_1 \leftrightarrow (a_2 \lor a_3) & = \\ a_2 \leftrightarrow (a \land b) & = \\ a_3 \leftrightarrow (\neg c \land a_4) & = \\ a_4 \leftrightarrow (d \lor e) & = \end{array}$$

CNF conversion: Tseitin's encoding

$$\phi := (a \wedge b) \vee (\neg c \wedge (d \vee e))$$

$$\boxed{\begin{array}{l} h \leftrightarrow (p_1 \vee p_2) = (\neg h \vee p_1 \vee p_2) \wedge (h \vee \neg p_1) \wedge (h \vee \neg p_2) \\ h \leftrightarrow (p_1 \wedge p_2) = (\neg h \vee p_1) \wedge (\neg h \vee p_2) \wedge (h \vee \neg p_1 \vee \neg p_2) \end{array}}$$

$$\begin{array}{lll} a_1 \leftrightarrow (a_2 \vee a_3) & = & (\neg a_1 \vee a_2 \vee a_3) \quad \wedge (a_1 \vee \neg a_2) \quad \wedge (a_1 \vee \neg a_3) \\ a_2 \leftrightarrow (a \wedge b) & = & \\ a_3 \leftrightarrow (\neg c \wedge a_4) & = & \\ a_4 \leftrightarrow (d \vee e) & = & \end{array}$$

CNF conversion: Tseitin's encoding

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$$\begin{array}{llll} a_1 \leftrightarrow (a_2 \vee a_3) & = (\neg a_1 \vee a_2 \vee a_3) & \wedge (a_1 \vee \neg a_2) & \wedge (a_1 \vee \neg a_3) \\ a_2 \leftrightarrow (a \wedge b) & = (\neg a_2 \vee a) & \wedge (\neg a_2 \vee b) & \wedge (a_2 \vee \neg a \vee \neg b) \\ a_3 \leftrightarrow (\neg c \wedge a_4) & = & & \\ a_4 \leftrightarrow (d \vee e) & = & & \end{array}$$

CNF conversion: Tseitin's encoding

$$\phi := (a \land b) \lor (\neg c \land (d \lor e))$$

$$\boxed{\begin{array}{l} h \leftrightarrow (p_1 \lor p_2) = (\neg h \lor p_1 \lor p_2) \land (h \lor \neg p_1) \land (h \lor \neg p_2) \\ h \leftrightarrow (p_1 \land p_2) = (\neg h \lor p_1) \land (\neg h \lor p_2) \land (h \lor \neg p_1 \lor \neg p_2) \end{array}}$$

$$\begin{array}{llll}
a_1 \leftrightarrow (a_2 \lor a_3) & = (\neg a_1 \lor a_2 \lor a_3) & \land(a_1 \lor \neg a_2) & \land(a_1 \lor \neg a_3) \\
a_2 \leftrightarrow (a \land b) & = (\neg a_2 \lor a) & \land(\neg a_2 \lor b) & \land(a_2 \lor \neg a \lor \neg b) \\
a_3 \leftrightarrow (\neg c \land a_4) & = (\neg a_3 \lor \neg c) & \land(\neg a_3 \lor a_4) & \land(a_3 \lor c \lor \neg a_4) \\
a_4 \leftrightarrow (d \lor e) & = & &
\end{array}$$

CNF conversion: Tseitin's encoding

$$\phi := (a \wedge b) \vee (\neg c \wedge (d \vee e))$$

$$\boxed{\begin{array}{l} h \leftrightarrow (p_1 \vee p_2) = (\neg h \vee p_1 \vee p_2) \wedge (h \vee \neg p_1) \wedge (h \vee \neg p_2) \\ h \leftrightarrow (p_1 \wedge p_2) = (\neg h \vee p_1) \wedge (\neg h \vee p_2) \wedge (h \vee \neg p_1 \vee \neg p_2) \end{array}}$$

$$\begin{array}{llll}
a_1 \leftrightarrow (a_2 \vee a_3) & = (\neg a_1 \vee a_2 \vee a_3) & \wedge (a_1 \vee \neg a_2) & \wedge (a_1 \vee \neg a_3) \\
a_2 \leftrightarrow (a \wedge b) & = (\neg a_2 \vee a) & \wedge (\neg a_2 \vee b) & \wedge (a_2 \vee \neg a \vee \neg b) \\
a_3 \leftrightarrow (\neg c \wedge a_4) & = (\neg a_3 \vee \neg c) & \wedge (\neg a_3 \vee a_4) & \wedge (a_3 \vee c \vee \neg a_4) \\
a_4 \leftrightarrow (d \vee e) & = (\neg a_4 \vee d \vee e) & \wedge (a_4 \vee \neg d) & \wedge (a_4 \vee \neg e)
\end{array}$$

CNF conversion: Tseitin's encoding

$$\phi := (a \wedge b) \vee (\neg c \wedge (d \vee e))$$

$$
\begin{array}{l}
\mathrm{CNF}(\phi) = \\
\quad \begin{array}{llllll}
(\neg a_1 \vee a_2 \vee a_3) & \wedge & (a_1 \vee \neg a_2) & \wedge & (a_1 \vee \neg a_3) & \wedge \\
(\neg a_2 \vee a) & \wedge & (\neg a_2 \vee b) & \wedge & (a_2 \vee \neg a \vee \neg b) & \wedge \\
(\neg a_3 \vee \neg c) & \wedge & (\neg a_3 \vee a_4) & \wedge & (a_3 \vee c \vee \neg a_4) & \wedge \\
(\neg a_4 \vee d \vee e) & \wedge & (a_4 \vee \neg d) & \wedge & (a_4 \vee \neg e) & \wedge \\
a_1 & & & & &
\end{array}
\end{array}
$$

where: a_1, a_2, a_3, a_4 are newly introduced variables.