Formal Methods in Software Development

Branch and Bound

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Faculty of Mathematics and Informatics
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Based on slides of the lecture Satisfiability Checking (Erika Ábrahám), RTWH Aachen

WS 2019/2020

Integer linear systems

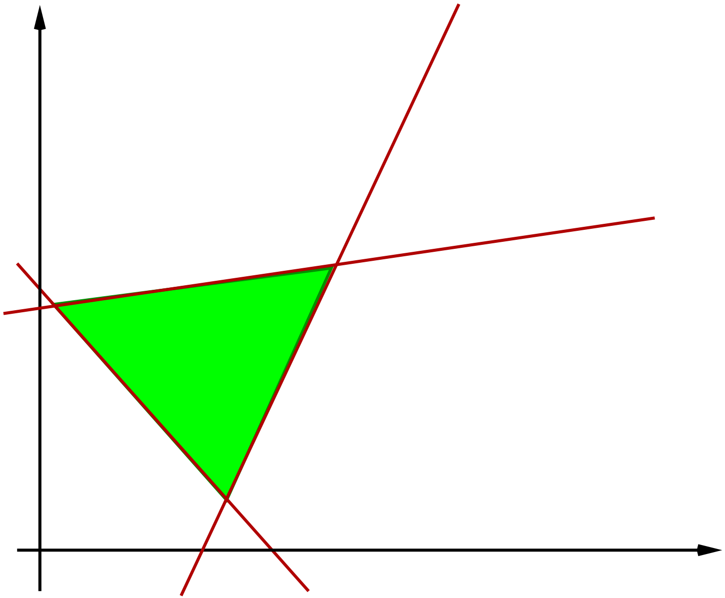
Definition

An integer linear system S is a linear system $Ax = 0$, $\bigwedge_{i=1}^{m} l_i \leq s_i \leq u_i$, with the additional integrality requirement that all variables are of type integer.

Definition (relaxed system)

Given an integer linear system S, its relaxation $\text{relaxed}(S)$ is S without the integrality requirement.

Geometric interpretation



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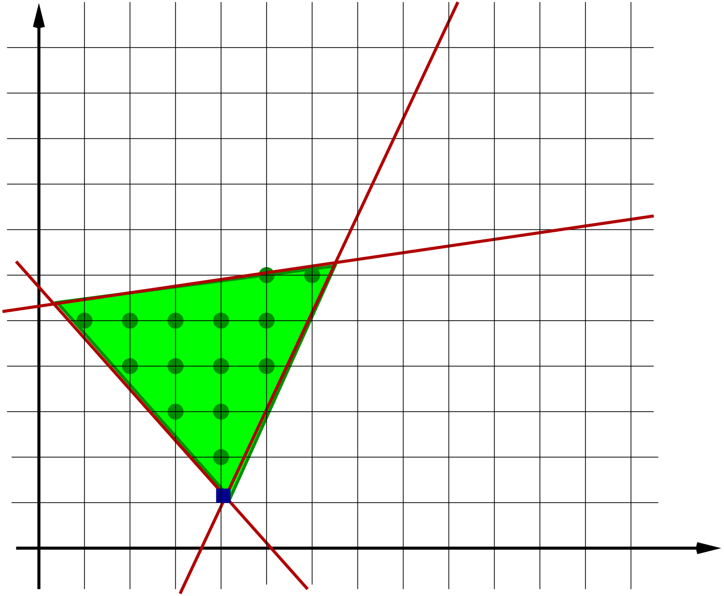

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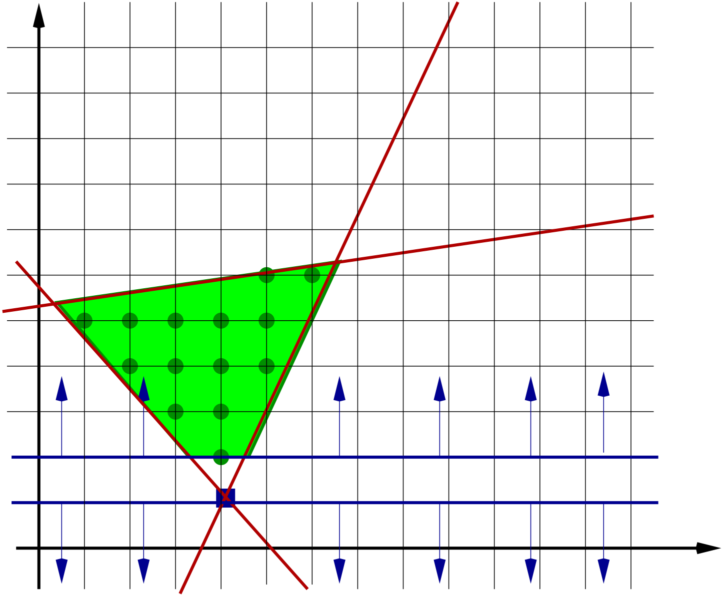

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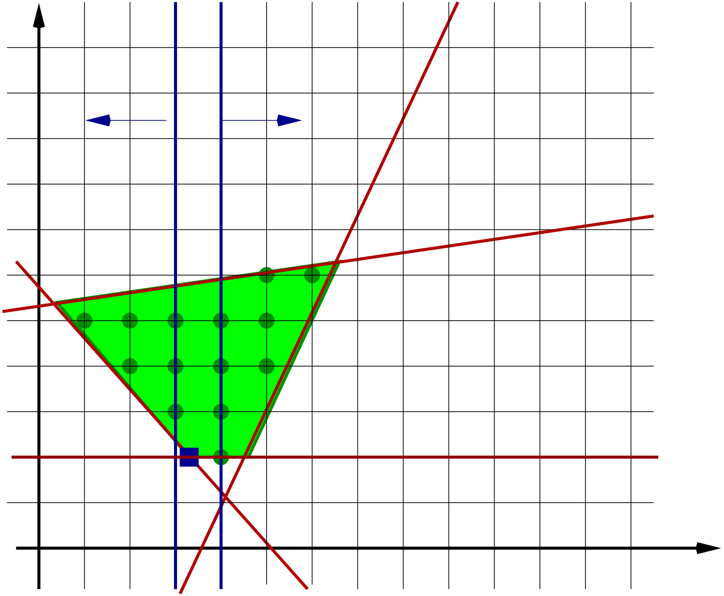

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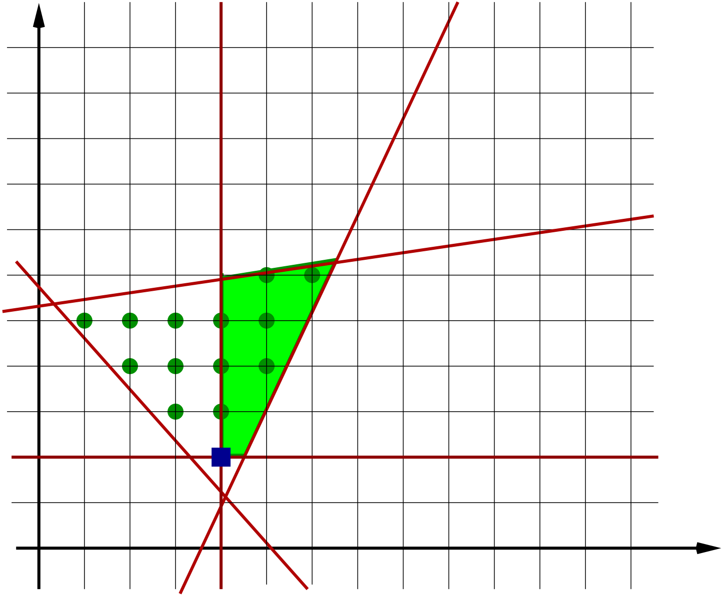

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- Let $x_1, ..., x_4$ be the variables of S. Assume that LP returns the solution $(1, 0.7, 2.5, 3)$;
- Branch-and-Bound algorithm chooses between x_2 and x_3; assume we choose x_2;
- S (the linear system solved at the current recursion level) is then augmented with the constraint $x_2 \leq 0$ and sent for solving at a deeper recursion level. If no solution is found in this branch, S is augmented instead with $x_2 \geq 1$ and, once again, is sent to a deeper recursion level. If both these calls do not return integral values, UNSAT is returned. If one call returns SAT, then the same is performed for x_3.

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- The Branch and Bound algorithm can be extended to mixed integer linear programming, where some of the variables are integer-valued while the others are real-valued.

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- Branch: Split the search space
- Bound: Exclude unsatisfiable sub-spaces
- We have seen: Depth-first search
- Also possible: Breadth-first search

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  From the first two constraints we get $x_1 \leq -1$
- Assume a constraint $\sum_i a_i x_i \leq b$ with $l_i \leq x_i \leq u_i$.
  If $a_k > 0$, we have $x_k \leq \frac{b - \sum_{i \neq k} a_i l_i)}{a_k}$.
  If $a_k < 0$, we have $x_k \geq \frac{b - \sum_{i \neq k} a_i u_i}{a_k}$.