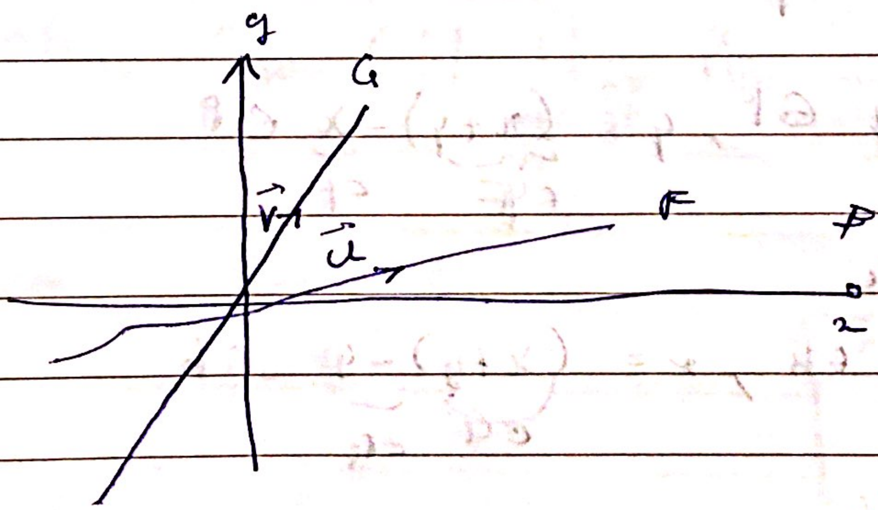
Ex1

E a $\mathbb{R}$-vs
F and G are 2 linear subspaces of E

Is $F \cup G$ a linear subspace ?

No example
$F \cup G$ linear subspace would ~~be a lin~~ mean
$* \ 0_E \in F \cup G$
$* \ \forall (u, v) \in (F \cup G)^2, (\lambda u + v) \in F \cup G$
$\forall \lambda \in \mathbb{R}$

$E = \mathbb{R}^2$



$u \in F \cup G$
$v \in F \cup G$
$\lambda u + v \notin F \cup G$

2) $F \cup G$ linear $\Leftrightarrow$ $\underbrace{F \subset G \text{ or } G \subset F}_{P_2}$

$\Leftarrow$: case 1 $F \subset G \Rightarrow F \cup G = G$ is linear

case 2 $G \subset F \rightarrow F \cup G = F$ ___

$\Rightarrow$ Suppose $F \cup G$ is a linear subspace
let's prove P_2

Contradic:

Suppose not ($F \subset G$ or $G \subset F$)

$\rightarrow$ So we suppose $\exists x \in F$, $x \notin G$ and $\exists y \in G$, $y \notin F$

Then $x + y \notin F \cup G$. Indeed:

. If $x + y \in F$, $y = \underbrace{(x+y)}_{\in F} - \underbrace{x}_{\in F} \in F$

Impossible.

. If $x + y \in G$, $x = \underbrace{(x+y)}_{\in G} - \underbrace{y}_{\in G} \in G$

Impossible

Hence, $x \in F \cup G$, $y \in F \cup G$, $x + y \notin F \cup G$, contradic

Ex2

$\mathcal{H}$

1) Let E be an $\mathbb{R}$-vs

Let F and G be two linear sey E
Prove $Span(F \cup G) = F + G$

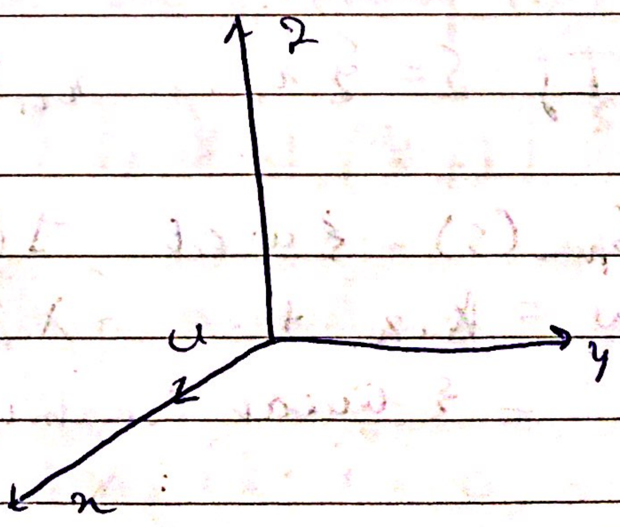
* $F + G = \{ w \in E, \exists u \in F, \exists v \in G, w = u + v \}$
It is a linear subspace of E

* Let S be a subset of E
Span (S) is the intersection of all the subspaces H st $S \subset H$

If $\mathcal{H}$ is the set of all subspaces of H st $S \subset H$

then $Span(S) = \bigcap_{H \in \mathcal{H}} H$

Ex : $S = \{u\}$
$E = \mathbb{R}^3$



$H_1 =$ plane $(xOz) \in \mathcal{H}$
$H_2 =$ plane $(xOy) \in \mathcal{H}$
the intersec' is the Ox axis

$\ast$ $F+G \in \mathcal{H}$ bec $F \cup G \subset F+G$

So $\bigcap_{H \in \mathcal{H}} H \subset F+G$

$\ast$ $\forall H \in \mathcal{H}$, let's prove $F+G \subset H$
$\forall u \in F, \forall v \in G$, then $\left| \begin{array}{l} u \in F \subset H \Rightarrow u \in H \\ v \in G \subset H \Rightarrow v \in H \end{array} \right.$

So $u+v \in H$
So $F+G \subset H$

Now, $\forall H \in \mathcal{H}$ $F+G \subset H \Rightarrow F+G \subset \bigcap_{H \in \mathcal{H}} H$

Finally $\underbrace{\bigcap_{H \in \mathcal{H}} H}_{\mathrm{Span}(F \cup G)} = F+G$

1) If $S = \{x_1, \dots x_n\}$ is a set of vects of E, then

$\mathrm{Span}(S) = \{w \in E,\ \exists (\lambda_1, \dots \lambda_n) \in \mathbb{R}^n$
$w = \lambda_1 x_1 + \dots + \lambda_n x_n$
$= \{$ linear combina' of $x_1 \dots x_n$

$\vdots$

Rk

If we def

$F = \{ \lambda_1 x_1 + \cdots + \lambda_n x_n \text{ with } \lambda_1 = \lambda_2 \}$

then $F \neq \mathrm{Span}(\{x_1, \ldots, x_n\})$

But $F = \{ \lambda_1 (x_1 + x_2) + \lambda_3 x_3 + \cdots + \lambda_n x_n \}$

$F = \mathrm{Span}(x_1 + x_2, x_3, \ldots, x_n \})$

Reminder : $S = \{ x_1, \ldots x_n \}$

$S = \{ x_1 \ldots x_n \}$ is lin ind

if $\forall (\lambda_1 \ldots \lambda_n) \in \mathbb{R}^n$

$\lambda_1 x_1 + \cdots + \lambda_n x_n \Rightarrow \lambda_1 = \cdots = \lambda_n = 0$

Ex 3

Are the following families linearly ind

1) $E = \mathbb{R}[X]$ $S = \{ 1, X-1, (X+1)^2 \}$

$\forall (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$,

$\lambda_1 x^1 + \lambda_2 (X-1) + \lambda_3 (X+1)^2 = 0_E$ (the only

$\Rightarrow \lambda_1 + \lambda_2 X - \lambda_2 + \lambda_3 (X^2 + 2X + 1) = 0$

$\Rightarrow \lambda_3 X^2 + (\lambda_2 + 2\lambda_3) X + (\lambda_1 - \lambda_2 + \lambda_3) = 0$

$\Rightarrow \begin{cases} \lambda_3 = 0 \\ \lambda_2 + 2\lambda_3 = 0 \\ \lambda_1 - \lambda_2 - \lambda_3 = 0 \end{cases} \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = 0$

2) $S = \{ f(x) = e^{2x}, g(x) = x^2, h(x) = x \}$

$\lambda_1 e^{2x} + \lambda_2 x^2 + \lambda_3 x = 0$

FALSE

λ_1 necessarily is 0 ~~but if x = 0 λ_2 and λ_3 can be anything~~

~~So S is linearly dependent~~

$\lambda_2 x^2 + \lambda_3 x = 0$

$x(\lambda_2 x + \lambda_3) = 0$

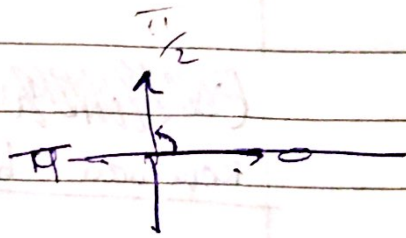
Correc:

Let $(\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3$

$\lambda_1 f + \lambda_2 g + \lambda_3 h = 0_E$

$\Leftrightarrow \forall x \in E, \lambda_1 e^{2x} + \lambda_2 x^2 + \lambda_3 x = 0$

$$\begin{cases} x=0 & \lambda_1 e^0 + \lambda_2 \times 0 + \lambda_3 = 0 \\ x=1 & \lambda_1 e^2 + \lambda_2 \times 1 + \lambda_3 \times 1 = 0 \\ x=-1 & \lambda_1 e^{-2} + \lambda_2 - \lambda_3 = 0 \end{cases}$$



$$\begin{cases} \lambda_1 e = 0 \Rightarrow \lambda_1 = 0 \\ \lambda_2 + \lambda_3 = 0 \\ +\lambda_2 - \lambda_3 = 0 \end{cases} \Bigg\} \Rightarrow \lambda_2 = \lambda_3 = 0$$

3) $S = \{ f(x) = e^x, g(x) = e^{x+1}, h(x) = e^{x+2} \}$

$$\lambda_1 e^x + \lambda_2 e^{x+1} + \lambda_3 e^{x+2} = 0$$

$$e^x (\lambda_1 + \lambda e + \lambda_3 e^2) = 0$$

if $\lambda_1 = 0$, $\lambda_2 = -e$, $\lambda_3 = 1$, it is true

LD

4) $S = \{ f(x) = \sin x, g(x) = \cos(x), h(x) = x)$

$$\lambda_1 \sin x + \lambda_2 \cos(x) + \lambda_3 x = 0$$

$$\begin{cases} x=0 & \lambda_1 \times 0 + \lambda_2 \times 1 + \lambda_3 \times 0 = 0 \\ x=\pi & \lambda_1 \times 0 - \lambda_2 + \lambda_3 \pi = 0 \\ x=\pi/2 & \lambda_1 \times 1 + \lambda_2 \times 0 + \lambda_3 \frac{\pi}{2} \end{cases} \Leftrightarrow \begin{cases} \lambda_2 = 0 \\ -\lambda_2 + \lambda_3 \pi = 0 \\ \lambda_1 + \lambda_3 \pi/2 = 0 \end{cases}$$

So $\lambda_1 = \lambda_2 = \lambda_3 = 0$

~~Exercise~~

Reminder : basis

A family $(e_1, e_2, \dots, e_n)$ is a basis of E if
- Span $(\{e_1, \dots e_n\}) = E$
- $(e_1 \dots e_n)$ is linearly indep

If E has a basis of n vectors, then all the other bases have n vectors : $\dim(E) = n$

Ex 4

Let E be a finite dimensional $\mathbb{R}$-VS
Let F be a linear subspace of E.
Using the incomplete basis theorem, prove that F admits a supplementary in E

* Let $n = \dim E$
* Incomplete basis th
  If $(v_1 \dots v_n)$ is a spanning family of E
  If $(e_1, \dots, e_k)$ is a linearly indep family

Then we can pick $n-k$ vectors in $(v_1 \dots v_n)$ such that

$(e_1 \dots e_k, v_{k+1}, v_{k+2}, \dots, v_n)$ is a basis of E

Ex: $E = \mathbb{R}_2[X]$ $\dim(E) = 3$

Spanning family : $(1, X, X^2)$

$e_1 = X+1$ $(X+1, X-1)$ is independant
$e_2 = X-1$

then $(X+1, X-1, X^2)$ is a basis

* Supplementary :
F and G are supplementary in E if
. $F \cap G = \{0_E\}$
. $F + G = E$

Back to Ex 4

Let $(e_1 \dots e_n)$ be a basis of E
Let F be a subspace. We suppose $F \neq E$
$\dim(F) < \dim(E)$
Let $p = \dim(E)$
Let $(u_1 \dots u_p)$ be a basis of F

From incomplete basis theorem we can pick $(e_{p+1} \dots e_n)$ in the basis such that $(v_1, \dots v_p, e_{p+1}, \dots e_n)$ is a basis of E

$F = \mathrm{Span}(\{v_1 \dots v_p\})$
Let $G = \mathrm{Span}(e_{p+1} \dots e_n)$

Let's prove

$F \oplus G = E$

<u>Hyp</u> : $(v_1 \dots v_p ; e_{p+1}, \dots, e_n)$ is linearly indep

$\lambda_1 v_1 + \dots + \lambda_p v_p + \lambda_{p+1} e_{p+1} + \dots + \lambda_n e_n = 0_E$

$\Rightarrow \lambda_1 = \dots = \lambda_n = 0$

It is a spanning family.

$\forall w \in E, \exists (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$

$W = \lambda_1 v_1 + \dots + \lambda_p v_p + \lambda_{p+1} e_{p+1} + \dots + e_n$

<u>To prove</u> :

. $F \cap G = \{0_E\}$

$\supset$: evident

$\subset$: $u \in F \cap G \Rightarrow u = 0_E$

. $F + G = E$

$\subset$: evident

$\supset$: $\forall w \in E, \exists u \in F, \exists v \in G, w = u + v$

Let $w \in E$

we know $\exists (\lambda_1 \dots \lambda_n) \in \mathbb{R}^n$

$$w = \underbrace{\lambda_1 v_1 + \dots + \lambda_p v_p}_{\in \text{Span}(v_1 \dots v_p) = F} + \underbrace{\lambda_{p+1} e_{p+1} + \dots + \lambda_n e_n}_{\in \text{Span}(e_{p+1} \dots e_n) = G}$$

$\in F + G$

Let be

Thus $E \subset F + G$

Let $u \in F \cap G$

$u \in F \Rightarrow \exists (\lambda_1 \dots \lambda_p) \in \mathbb{R}^n, u_1 = \lambda_1 v_1 + \dots + \lambda_p v_p$

$u \in G \Rightarrow \exists (\lambda_{p+1} \dots \lambda_n) \in \mathbb{R}^{n-p}, v = \lambda_{p+1} e_{p+1} + \dots + \lambda_n e_n$

Let's subtract:

$\lambda_1 v_1 + \dots + \lambda_p v_p - \lambda_{p+1} e_{p+1} - \dots - \lambda_n e_n = 0$

$\Rightarrow \lambda_1 = \dots = \lambda_p = (-\lambda_{p+1}) = \dots = -\lambda_n = 0$

$\Rightarrow u = 0_E$

24/10/2013

Ex 9

$E = \mathbb{R}^{\mathbb{R}} = \{\text{function } \mathbb{R} \to \mathbb{R}\}$

$\phi = E \longrightarrow E$

$f = F : \forall x \in \mathbb{R}, F(x) = f(x) + f(-x)$

1) Prove that ϕ is an endomorphism of E

To prove:

- $\forall f \in E, \ \phi(f) \in E$
- ϕ is linear

* Let $f \in E$, f is a fct $\mathbb{R} \to \mathbb{R}$

Let $F = \phi(f)$

F is a fct $\mathbb{R} \to \mathbb{R}$
$x \to f(x) + f(-x)$

So $\phi(f) \in E$

* ϕ is linear

Let $(f, g) \in E^2$, $\lambda \in \mathbb{R}$

We must prove that

$\phi(\lambda f + g) = \lambda \phi(f) + \phi(g)$

$\phi(f) = F : \mathbb{R} \to \mathbb{R}$

$x \to f(x) + f(-x)$

$\phi(g) = G : \mathbb{R} \to \mathbb{R}$

$x \to g(x) + g(-x)$

Let $h = \lambda f + g$ and $H = \phi(h)$

$\forall x \in \mathbb{R}$, $H(x) = h(x) + h(-x)$

$= \lambda f(x) + g(x) + \lambda f(-x) + g(-x)$

$= \lambda(f(x) + f(-x)) + g(x) + g(-x)$

$= \lambda F(x) + G(x)$

So $H = \lambda F + G$

$\phi(\lambda f + g) = \lambda \phi f + \phi g$

2) What is $\ker(\phi)$? Is ϕ injective?

$F(x) = 0_{\mathbb{R}}$ $\forall x \in \mathbb{R}$, $f(x) + f(-x) = 0$

$f(x) = -f(x)$

f is odd

$f \in \ker(\phi) \Leftrightarrow F = 0_E$

$\Leftrightarrow \forall x \in \mathbb{R}, F(x) = 0$

$\Leftrightarrow \forall x \in \mathbb{R}, f(x) + f(-x) = 0$

$\Leftrightarrow \forall x \in \mathbb{R}, f(-x) = -f(x)$

$\Rightarrow f$ is odd

$Ker(\phi) = \{ \text{odd fcts} \}$

$Ker(\phi) \neq \{0_E\} \Rightarrow \phi$ not injective

3) Prove $Im(\phi) = \{ \text{even fcts} \}$

$\subset$: let $g \in Im(\phi)$

$\exists f \in E,\ g = \phi(f) = F$ $\quad \cancel{\phi(-f) = -f - f(-x)}$

then $\forall x \in \mathbb{R},\ g(x) = f(x) + f(-x)$ $\quad \cancel{\phi(f) + f(-x) = -f}$

$g(-x) = f(-x) + f(-(-x))$

$= f(-x) + f(x)$ $\quad \cancel{f(-x) = \phi(-f) = f}$

So, $\forall x \in \mathbb{R},\ g(-x) = g(x)$

So g is even

$\supset$: Let f be an even fct $(\forall x \in \mathbb{R},\ f(-x) = f(x))$

$f(x) = -f(-x) - \phi(-f)(x)$

$\phi(-f) = -2f$

$f = \phi\left(\frac{f}{2}\right)$

Let $g = f/2$, $\forall x \in \mathbb{R}$, $g(x) = \frac{1}{2} f(x)$

Let $G = \phi(g)$

$\forall x \in \mathbb{R}$, $G(x) = g(x) + g(-x)$

$= \frac{1}{2} f(x) + \frac{1}{2} f(-x)$

$= f(x)$

So $G = f$

$\phi(g) = f$ and $f \in \operatorname{Im}(\phi)$

Ex 10

1) $f : \mathbb{R}^3 \to \mathbb{R}$

$(x, y, z) \to x - y + 2z$

Determine $\operatorname{Ker}(f)$ (Give a basis) and $\operatorname{Im}(f)$

. $\operatorname{Ker}(f)$: $\forall (x, y, z) \in \mathbb{R}^3$,

$(x, y, z) \in \operatorname{Ker}(f) \Leftrightarrow x - y + 2z = 0$

$\Leftrightarrow x = \underbrace{y - 2z}_{\text{free. pells}}$

unknowns

$$\Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y - 2z \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} y \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} -2z \\ 0 \\ z \end{pmatrix}$$

$$\operatorname{Ker}(f) = \operatorname{Span}\left(\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} ; \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\} \right)$$

So $\left(\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}-2\\0\\1\end{pmatrix}\right)$ is a spanning family of $\ker(f)$. Furthermore, it is linearly indep so it is a basis of $\ker(f)$

Rk: We can also do

$$\begin{pmatrix}x\\y\\z\end{pmatrix} \in \ker(f) \iff x - y + 2z = 0$$

$$\iff \underbrace{y}_{\text{imposed}} = \underbrace{x + 2z}_{\text{free variables}}$$

$$\iff \begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x\\x+2z\\z\end{pmatrix}$$

$$= x\begin{pmatrix}1\\1\\0\end{pmatrix} + z\begin{pmatrix}0\\2\\1\end{pmatrix}$$

So $\ker(f) = \mathrm{Span}\left(\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\2\\1\end{pmatrix}\right\}\right)$

- What is $\mathrm{Im}(f)$

Rk th: $\dim(f) = \dim(\ker f) + \dim(\mathrm{Im}(f))$

So $\dim(\mathrm{Im}(f)) = 1$

$$\left.\begin{array}{l}\mathrm{Im}(f) \subset \mathbb{R}\\ \dim(f) = \dim(\mathbb{R})\end{array}\right\} \Rightarrow \mathrm{Im}(f) = \mathbb{R}$$

$\dim(\mathrm{Im}(f))$

2) $g : \mathbb{R}^3 \to \mathbb{R}^2$

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} x - y + 2z \\ 2x - z \end{pmatrix}$

$\underline{\ker(g)}$, $\mathrm{Im}(g)$

find a basis

$\ker(g) = \{ (x,y,z) ,\ \begin{cases} x - y + 2z = 0 \\ 2x - z = 0 \end{cases}$

$\begin{cases} y = x + 2z \\ z = 2x \end{cases}$

$\begin{cases} y = x + 4x \\ z = 2x \end{cases}$

$\begin{cases} y = 5x \\ z = 2x \end{cases}$

$$\boxed{\ker(g) = \mathrm{Span}\left(\left\{ \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} \right\}\right)}$$

$$\ker(g) = \mathrm{Span}\left(\left\{ \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix} \right\}\right)$$

Rk th : $\dim(f) = \dim(\ker(f)) + \dim(\mathrm{im}(f))$

$\dim(f) = \dim(\mathbb{R}^3) = 3$

$\dim \ker = 1$

$\dim(\mathrm{Im}(f)) = 2$

$$\left.\begin{array}{l}\mathrm{Im}(f) \subset \mathbb{R}^2 \\ \dim(\mathrm{Im}(f)) = \dim(\mathbb{R})\end{array}\right\} \Rightarrow \mathrm{Im}(f) = \mathbb{R}^2$$

5) Additionale case

if $\dim(\mathrm{Im}(f)) < \dim(\text{output space})$

If $(e_1, \ldots, e_n)$ is a basis of E ← inp space

Then $(f(e_1), \ldots f(e_n))$ is a spanning family of $\mathrm{Im}(f)$ (see Ex 7.1)

$h : \mathbb{R}^3 \to \mathbb{R}^3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x - y + z \\ x + 2y - z \\ 3x + 3y - z \end{pmatrix}$$

$\mathrm{Ker}(h)$ $\mathrm{Im}(h)$

$$\mathrm{Ker}(h) = \left\{\begin{array}{l} y = x + z \\ z = x + 2y \\ z = 3x + 3y \end{array}\right. \Rightarrow \left\{\begin{array}{l} y = x + x + 2y \\ x + 2y = 3x + 3y \\ z = x + 2y \end{array}\right.$$

$$\Rightarrow \left\{\begin{array}{l} -2x = 2y \Leftrightarrow y = -x \\ 0 = 2x + y \\ z = x + 2y \end{array}\right.$$

$$\Rightarrow \left\{\begin{array}{l} y = -x \\ y = -2x \\ z \end{array}\right.$$

$$\mathrm{Ker}(h) = \left|\begin{array}{l} \cancel{y = x + z \quad z = y - x} \\ \cancel{z = x + 2y} \\ z = 3x + 3y \end{array}\right.$$

$$= \left|\begin{array}{l} y = x + z \\ 0 = x + 2x + 2z - z \\ z = 3x + 3y \end{array}\right.$$

$$\Rightarrow \left|\begin{array}{l} y = x + z \\ 3x + z = 0 \\ z = 3x + 3y \end{array}\right.$$

$$\Rightarrow \left|\begin{array}{l} y = x - 3x = -2x \\ z = -3x \\ 3x = -3x - 3(-2x) \end{array}\right.$$

$$3x = 3x$$

$$\mathrm{Ker}(h) = \mathrm{Span}\left(\left\{\begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}\right\}\right) \Rightarrow$$ ich habe, basis of Ker

$$\dim(\cancel{h}) \cancel{= \dim(\mathrm{Ker})} + \dim(\mathrm{Im})$$

$$\dim(\mathrm{Im}(h)) = 2$$

$$\cancel{\mathrm{Im}(h)} = \left|\begin{array}{l} 1 - (-2) + (-3) \\ 1 + (-2) \times 2 - (-3) \\ 3 + 3 \times 2 - (-3) \end{array}\right.$$

$\dim(\mathbb{R}^3) = \dim(\text{Ker}) + \dim(\text{Im})$

$\dim(\text{Im}(h)) = 2$

Let $e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $e_2 \begin{pmatrix}0\\1\\0\end{pmatrix}$, $e_3 \begin{pmatrix}0\\0\\1\end{pmatrix}$

Then $(h(e_1), h(e_2), h(e_3))$ ~~basis of~~ spanning family of $\text{Im}(f)$

It is $\left(\underbrace{\begin{pmatrix}1\\1\\3\end{pmatrix}}_{u_1}, \underbrace{\begin{pmatrix}-1\\2\\3\end{pmatrix}}_{u_2}, \underbrace{\begin{pmatrix}1\\-1\\-1\end{pmatrix}}_{u_3}\right)$

$u_1 - 2u_2 - 3u_3 = 0$

We can remove any vect whose coeff is not 0 in the rela

Basis of $\text{Im}(h)$ is $\begin{pmatrix}1\\1\\3\end{pmatrix} \begin{pmatrix}-1\\2\\3\end{pmatrix}$

Reminder : Polynomial of endomorphism

Let E be a $\mathbb{R}$-V_S

Let $f \in \mathcal{L}(E)$

$f^n = \underbrace{f \circ f \circ f \ldots \circ f}_{n \text{ times}}$

$f^0 = id$ ← identity map : $\forall x \in E, id(x) = x$

1) $P(X) = (X-2)(X-1) = X^2 - 3X + 2$

$P(f) = f^2 - 3f + 2id = (f-2id)\circ(f-id)$
$= (f-id)\circ(f-2id)$

But $(f-2id)\circ(f-id) = 0 \not\Rightarrow f = 2id$ or $f = id$

Generally $f\circ g = 0 \not\Rightarrow f = 0$ or $g = 0$

Ex 11

E a $\mathbb{R}$ vs, $f \in \mathcal{L}(E)$ $g \in \mathcal{L}(E)$

1) Prove $g\circ f = 0 \Leftrightarrow \mathrm{Im}(f) \subset \mathrm{Ker}(g)$

$\Leftarrow$: Suppose $\mathrm{Im}(f) \subset \mathrm{Ker}(g)$

To prove: $\forall u \in E$, $g(f(u)) = 0_E$

Let $u \in E$, $f(u) \in \mathrm{Im}(f) \Rightarrow f(u) \in \mathrm{Ker}(g)$
$\Rightarrow g(f(u)) = 0_E$

$\Rightarrow$ Suppose $g\circ f = 0$
Then $\forall u \in E$, $g(f(u)) = 0_E$
To prove: $\mathrm{Im}(f) \subset \mathrm{Ker}(g)$
Let $v \in \mathrm{Im}(f)$, then $\exists u \in E$, $v = f(u)$
so $g(v) = g(f(u)) = 0_E \Rightarrow v \in \mathrm{Ker}(g)$

2) Let $f \in \mathcal{L}(E)$ st

$$f^2 + f - 2id = 0$$

a) Prove $Im(f - id) \subset Ker(f + 2id)$

and $Im(f + 2id) \subset Ker(f - id)$

* $Im(f - id) \subset Ker(f + 2id)$

According to 1) , we need to prove that $(f + 2id) \circ (f - id) = 0$

$(f + 2id) \circ (f - id) = f^2 - f + 2f - 2id$

$$= f^2 + f - 2id$$
$$= 0$$

* $Im(f + 2id) \subset Ker(f - id)$

$(f - id) \circ (f + 2id) = f^2 + 2f - f - 2id$
$$= f^2 + f - 2id = 0$$

b) Prove $E = Ker(f + 2id) \oplus Ker(f - id)$

To prove:

* $Ker(f + 2id) \cap Ker(f - id) = \{0_E\}$
* $E = Ker(f + 2id) + Ker(f - id)$

$\forall w \in E, \exists u \in Ker(f + 2id)$, $w = u + v$
$\exists v \in Ker(f - id)$

Hint : $(f + 2id) - (f - id) = 3id$

So $(f + 2id)(w) - (f - id)(w) = 3w$

$\ker(f + 2id) \cap \ker(f - id) = \{0_E\}$

$\supset$: evident bc both kernels are linear subsp

$\subset$: let $u \in \ker(f + 2id) \cap \ker(f - id)$

$(f + 2id)(u) = 0_E \iff f(u) + 2u = 0_E$

$(f - id)(u) = 0_E \iff f(u) - u = 0_E$

$Eq_1 - Eq_2 = 3u = 0_E$, $u = 0_E$

27/10/2017

Ex11

E a $\mathbb{R}$-vs

1) $g \circ f = 0 \iff \mathrm{Im}(f) \subset \mathrm{Ker}(g)$

2) f such that $f^2 + f - 2id = 0$

$P(X) = X^2 + X - 2 = (X-1)(X+2)$

$P(f) = 0 \iff f^2 + f - 2id = \begin{cases} (f-id) \circ (f+2id) = 0 \\ (f+2id) \circ (f-id) = 0 \end{cases}$

as $\mathrm{Im}(f+2id) \subset \mathrm{Ker}(f-id)$

$\mathrm{Im}(f-id) \subset \mathrm{Ker}(f+2id)$

b) Prove $E = \mathrm{Ker}(f-id) \oplus \mathrm{Ker}(f+2id)$

* $\mathrm{Ker}(f-id) \cap \mathrm{Ker}(f+2id) = \{0_E\} \to$ done
* $E = \mathrm{Ker}(f-id) + \mathrm{Ker}(f+2id)$

$\supset$: evident

$\subset$: let $w \in E$

We're looking for $u \in \mathrm{Ker}(f-id)$

$v \in \mathrm{Ker}(f+2id)$

$w = u + v$

$(f+2id)(w) \in \mathrm{Im}(f+2id) \subset \mathrm{Ker}(f-id)$

$= f(w + 2w)$

$f(w) - w \in \mathrm{Ker}(f+2id)$

$(f(w) + 2w) - (f(w) - w) = 3w$

$\longrightarrow w = \underbrace{\left(f\left(\frac{w}{3}\right) + 2\frac{w}{3}\right)}_{\in \mathrm{Im}(f+2id) \subset \mathrm{Ker}(f-id)} - \underbrace{\left(f\left(\frac{w}{3}\right) - \frac{w}{3}\right)}_{\in \mathrm{Im}(f-id) \text{ so } \in \mathrm{Ker}(f+2id)}$

Ccl : $(f - id) \circ (f + 2id) = 0$

$\cancel{\Rightarrow}$ $f = id$ or $f = -2id$

$\Longrightarrow$ $E = \underbrace{F}_{\ker(f - id)} \oplus \underbrace{G}_{\ker(f + 2id)}$

$f = id$ $\quad$ $f = -2id$

Example

$\dim(E) = n$

Let $p = \dim(\ker(f - id))$

$n - p = \dim(\ker(f + 2id))$

Let $(e_1 \dots e_p)$ be a basis of $\ker(f - id)$

— $(e_{p+1} \dots e_n)$ — $(f + 2id)$

Then $(e_1 \dots e_n)$ is a basis of E

$$\mathrm{Mat}(f) = \begin{array}{c} \left(\begin{array}{cccc} 1 & 0 & 0 & \\ 0 & \vdots & \vdots & 0 \\ 1 & & & \\ 0 & 0 & 2 \\ & & \vdots & \\ & & 0 & 2 \end{array} \right) \\ \begin{array}{cccc} f(e_1) \dots & f(e_p) & f(e_{p+1}) \dots & f(e_n) \end{array} \end{array} \begin{array}{l} \leftarrow \text{coord } e_1 \\ \leftarrow \text{coord } e^2 \\ \vdots \\ \leftarrow \text{coord } e_n \end{array}$$

$f(e_1) - id(e_1) = 0_E \Rightarrow f(e_1) = e_1$

$\vdots$

$f(e_p) - id(e_p) = 0_E \Rightarrow f(e_p) = e_p$

$f(e_{p+1}) + 2id(e_{p+1}) = 0_E \Rightarrow f(e_{p+1}) = -2e_{p+1}$

$\vdots$

$f(e_n) + 2id(e_n) = 0_E \Rightarrow f(e_n) = -2e_n$

Ex 12

E a $\mathbb{R}$-vs

$p \in \mathcal{L}(E)$ $p^2 = p \iff p^2 - p = 0$

$\iff \begin{cases} p \circ (p - id) = 0 \\ (p - id) \circ (p) = 0 \end{cases}$

$S \in \mathcal{L}(E)$ $s^2 = id \iff S^2 - id = 0$

1) Prove $E = \ker(p) \oplus \mathrm{Im}(p)$

$u \in \ker(p) \iff u \in \{u, f(u) = 0\}$

$v \in \mathrm{Im}(p) \iff k \in \{a, f(a) = k\}$

* $u \in \ker(p) \cap \mathrm{Im}(p) \iff \forall x \in E, f(x) = u$ and $p(u) = 0$

$\Rightarrow p =$ null fct

$\Rightarrow$ ~~ft~~ $\ker(p) \cap \mathrm{Im}(p) =$ null fct

* let $w = u + v$

Correc°

* We must prove that

$\ker(p) \cap \mathrm{Im}(p) = \{0_E\}$

$\supset$: evident

$\subset$: let $u \in \ker(p) \cap \mathrm{Im}(p)$

- $p(u) = 0_E$
- $\exists x \in E, u = p(x)$

then $0_E = p(u) = p(p(x)) = p(x) = u$ $(p^2 = p)$

therefore $u = 0_E$

7

$\star$ $E = \mathrm{Ker}(p) + \mathrm{Im}(p)$

$\supset$ $\subseteq$ evident

$\subseteq$: let $\{w \in E\}$

$u \in \mathrm{Ker}\, p$

$v = p(x)$

$w = u + v = u + p(x)$

$p(w) = \underbrace{p(u)}_{=0} + (p(p(x)) = p(x) = v$

So $v = p(w)$ and $u = w - v = w - p(w)$

$w = \underbrace{(w - p(w))}_{\mathrm{Ker}(p)} + \underbrace{p(w)}_{\in \mathrm{Im}(p)}$

$w - p(w) \in \mathrm{Ker}(p)$ because

$p(w - p(w)) = p(w) - p(p(w))$

$= 0_E$ since $p \circ p = p$

Remark: $\forall v \in \mathrm{Im}(p)$, $p(v) = v$

Indeed, $v \in \mathrm{Im}(p) \Leftrightarrow \exists x \in E, v = p(x)$

Then $p(v) = p(p(x)) = p(x) = v$

2) Prove $\mathrm{Im}(s+id) \subset \mathrm{Ker}(s-id)$
$\mathrm{Im}(s-id) \subset \mathrm{Ker}(s+id)$ $\quad s^2 - id = 0$

a) $\Leftrightarrow \forall u$, $u \in \mathrm{Im}(s+id)$, $u \in \mathrm{Ker}(s-id)$

$u \in \mathrm{Im}(s+id) \Leftrightarrow \forall x \in E, \exists u \in E, s(x) = u + x$

$u = (u + x) - x$

$u = s(x) + x$

$s(u) = s^2(x) + s(x)$

$= x + s(x)$

$s(u) - u = \cancel{x} + \cancel{s(x)} - (s(x) - x)$

$= 0$

$u \in \ker(s - id)$

$v = s(x) - x$

$s(v) = s^2(x) - s(x)$

$s(v) = id\, x - s(x)$

$s(v) + v = \cancel{x} - \cancel{s(x)} + \cancel{s(x)} - \cancel{x}$

$s(v) + v = 0$

$v \in \ker s + id$

Correc°

$* \; s^2 - id = 0 \iff (s - id) \circ (s + id) = 0$

$\iff \operatorname{Im}(s + id) \subset \ker(s - id)$ (according to e

$*$ Let $v \in \operatorname{Im}(s + id)$

then $\exists u \in E$, $v = (s + id)(u)$

$= s(u) + u$

So $(s - id)(v) = s(v) - v$

$= s(s(u) + u) - (s(u) + u)$

$= u + (s(u) - s(u)) - u$

$= 0_E$

In the same way, $\forall v \in \operatorname{Im}(s - id)$

$\exists u \in E$, $v = s(u) - u$

then $(s + id)(v) = s(v) + v = s^2(u) - su + su - u$

$= 0_E$

3) $E = \mathrm{Ker}(s - id) \oplus \mathrm{Ker}(s + id)$

* $\mathrm{Ker}(s-id) \cap \mathrm{Ker}(s+id) = 0_E$

$\supset$ evident

$\subset$: Let $u \in \mathrm{Ker}(s-id) \cap \mathrm{Ker}(s+id)$

$$\left.\begin{array}{l} (s-id)(u) = 0_E \iff s(u) = u \\ (s+id)(u) = 0_E \iff s(u) = -u \end{array}\right\} \begin{array}{l} \Rightarrow u = -u \\ \Rightarrow 2u = 0_E \\ \Rightarrow u = 0_E \end{array}$$

* $E = \mathrm{Ker}(s-id) + \mathrm{Ker}(s+id)$

Let $w \in E$

$s(w) + w \in \mathrm{Ker}(s-id)$

$s(w) - w \in \mathrm{Ker}(s+id)$

$(s(w)+w) - (s(w)-w) = 2w$

Then $w = \underbrace{\left(s\left(\frac{w}{2}\right) + \left(\frac{w}{2}\right)\right)}_{\in \mathrm{Ker}(s-id)} - \underbrace{\left(s\left(\frac{w}{2}\right) - \frac{w}{2}\right)}_{\in \mathrm{Ker}(s+id)}$

If $\dim(E) = n$

$\mathrm{Ker}(s-id) =$ basis $(e_1, \dots, e_p)$

$\mathrm{Ker}(s+id) =$ basis $(e_{p+1}, \dots, e_n)$

Let $B = (e_1, \dots, e_n)$ basis of E

$$\mathrm{Mat}_B(s) = \begin{pmatrix} 1 & & & & & & 0 \\ & \ddots & & & & & \\ & & 1 & & & & \\ & & & -1 & & & \\ & & & & -1 & & \\ & & & & & \ddots & \\ 0 & & & & & & -1 \end{pmatrix}$$

Ex 13

Let E be a $\mathbb{R}$-VS

$u \in \mathcal{L}(E)$
$v \in \mathcal{L}(E)$ | $u \circ v = v \circ u$

Prove that

$E = \ker(u) \oplus \ker(v) \Rightarrow$ | $\mathrm{Im}(v) \subset \ker(u)$
| $\mathrm{Im}(u) \subset \ker(v)$

Remark : then $u(v(x)) = 0_E$, since $v(x) \in \mathrm{Im}(v) \subset \ker(u)$

Hyp: $\begin{cases} u \circ v = v \circ u \\ E = \ker(u) + \ker(v) \end{cases}$

To prove: $\mathrm{Im}(v) \subset \ker(u)$, $\forall x \in E$, $x \in \mathrm{Im}(v) \Leftrightarrow u(x) = 0_E$

$\mathrm{Im}(u) \subset \ker(v)$

* Let $x \in \mathrm{Im}(v)$. Let's prove $u(x) = 0_E$

$\exists z \in E$, $x = v(z)$

$\exists z_1 \in \ker(u)$, $\exists z_2 \in \ker(v)$, $z = z_1 + z_2$

then $x = (v(z))$

$= v(z_1) + \underbrace{v(z_2)}_{=0}$

$= v(z_1)$

$u(x) = u \circ v(z_1)$

$= v \circ u(z_1) = v(0) = \{0\}$

So $x \in \mathrm{Im}(v) \Rightarrow x \in \ker(u)$

We can deduce

* $u \circ v = 0 \Rightarrow v \circ u = 0$

$\Rightarrow \mathrm{Im}(u) \in \mathrm{Ker}(v)$

Ex 11.1 $v \circ u = 0 \Leftrightarrow \mathrm{Im}(u) \in \mathrm{Ker}(v)$

Reminder :

finite dim space. If $\dim(E) = n$

* for all F & G linear sub of E $F \subset G$, $\dim(F) = \dim(G)$ $\Rightarrow F = G$

* $\dim(F+G) + \dim(F \cap G) = \dim(F) + \dim(G)$

If $F \cap G = \{0_E\}$, $\dim(F \oplus G) = \dim(F) + \dim(G)$

* Rank th : $f \in \mathcal{L}(E, F)$, $\dim(E) = \dim(\mathrm{Ker}(f)) + \dim(\mathrm{Im}(f))$

Ex 14

Let E be a finite dim

Let $f \in \mathcal{L}(E)$

Prove $\underbrace{Im(f) = Im(f^2)}$ $\Leftrightarrow$ $E = Ker(f) \oplus Im(f)$

strong property $Im(f) \in Im(f^2)$

$\Leftarrow$ Suppose $E = Ker(f) \oplus Im(f)$

prove $Im f = Im f^2$

$\supset$ Evident

$\forall v \in Im(f^2)$, $\exists u \in E$, $v = f^2(u) = f(f(u)) \in Im(f)$

$\subset$: Let $v \in Im(f)$

Let's prove $v \in Im(f^2)$

$\exists u_1 \in Ker(f)$ $\quad u = u_1 + u_2$

$\exists_2 \in Im(f)$

then $v = f(u_1) + f(u_2) = f(u_2)$

$u_2 \in Im(f) \Rightarrow \exists x \in E, u_2 = f(x)$

then $v = f(f(x)) \in Im(f^2)$

$\Rightarrow$: Hyp: $Im(f) = Im(f^2)$

To prove: $E = Ker f \oplus Im(f)$

$\times$ Let's prove $E = Ker(f) + Im(f)$

$\supset$: evident (because $Ker(f) \in E$, $Im(f) \in E$

$\subset$: Let $w \in E$

Let $u \in Ker f$, $v \in Im(f)$ exist

$w = u + v$

$f(w) = fu + fv = f(v) = f(f(\ldots))$

$\cancel{\text{so}}$ $f(w) \in \mathrm{Im}(f) \Rightarrow f(w) \in \mathrm{Im}(f^2)$

$\Rightarrow \exists x \in E, f(w) = f(f(x))$

Let $v = f(x)$

$u = w - v = w - f(x) \Rightarrow f(u) = f(w - f(x)) = 0_E$

Then $w = \underbrace{w - f(x)}_{0 \in \ker(f)} + \underbrace{f(x)}_{\in \mathrm{Im} f}$

* Let's prove $\ker(f) \cap \mathrm{Im}(f) = \{0_E\}$

$\underbrace{\dim(\ker(f) + \mathrm{Im}(f))}_{\dim(E) \text{ since } E = \ker(f) + \mathrm{Im}(f)} + \dim(\ker(f) \cap \mathrm{Im}(f)) = \dim(\mathrm{Im}(f)) + \dim(\ker(f))$

$- \dim(E)$ from rank thm

So $\dim(\ker(f) \cap \mathrm{Im}(f)) = 0 \Rightarrow \ker(f) \cap \mathrm{Im}(f) = 0$

Matrices

If $\dim(E) = n$ $\quad f \in \mathcal{L}(E, F)$

$\dim(F) = p$

B a basis of E: $B = (e_1, \dots, e_n)$

B' a basis of F: $B' = (\varepsilon_1, \dots, \varepsilon_p)$

$$\mathrm{Mat}\, f = \begin{array}{c} \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \begin{array}{l} \leftarrow \text{coord } \varepsilon_1 \\ \vdots \\ \leftarrow \text{coord } \varepsilon_p \end{array} \\ \begin{array}{ccc} \uparrow & & \uparrow \\ f(e_1) & \dots & f(e_n) \end{array} \end{array}$$

size $p \times n$

If $u \in E$ has coordinates $X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$f(u) \in F$ has ______ $Y = \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix}$

Matrix mult

$\underbrace{A}_{p.n} = (a_{ij})$, $1 \leq i \leq p$, $1 \leq j \leq n$

$\underbrace{B}_{n.q} = (b_{ij})$, $1 \leq i \leq n$, $1 \leq j \leq q$

then $AB = (c_{ij})$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} + b_{kj}$$

= line i of A x column j of B

Transpose:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}, \quad {}^tA = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}$$

trace: if A is square,

$A = (a_{ij})$ $1 \leq i \leq n$

$1 \leq j \leq n$

$tr(A) = a_{11} + a_{22} + \dots + a_{nn}$

= sum of the diagonal terms

Ex-15

Let $(A, B, C) \in [\underbrace{M_n(\mathbb{R})}_{\text{mats of size } (n\times n)}]^3$

1) $A(BC) = (AB)C$

2) ${}^t(AB) = {}^tB\,{}^tA$

3) $tr(AB) = tr(BA)$

(Rk, $AB \neq BA$ in gen)

2) $A = (a_{ij})$

$B = (b_{ij})$ $\quad 1 \leqslant i \leqslant n$

$C = (c_{ij})$ $\quad 1 \leqslant j \leqslant n$

Let $D = AB = d(ij)$

$$d_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Let ${}^tD = (d'_{ij})$

$d'_{ij} = d_{ji} = \sum_{k=1}^{n} a_{jk} b_{ki} \rightarrow$ coef of ${}^t(AB)$

Let ${}^tB = (b'_{ij}) = b'_{ij} = b_{ji}$

${}^tA = (a'_{ij} = a'_{ij} = a_{ji}$

$E = {}^tB\,{}^tA = (e_{ij})$

$$e_{ij} = \sum_{k=1}^{n} b'_{ik} a'_{kj}$$

$$= \sum_{k=1}^{n} b_{ki} a_{jk}$$

$= d'_{ij}$

So $C = D'$

${}^tB\,{}^tA = {}^t(AB)$

3) * $tr(AB)$ – $AB = (d_{ij})$

$d_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

So $tr(AB) = \sum_{i=1}^{n} d_{ii}$

$= \sum_{i=1}^{n} \left(\sum_{k=1}^{n} a_{ik} b_{ki} \right)$

* $tr(BA)$

$BA = (\alpha_{ij})$

$\alpha_{ij} = \sum_{k=1}^{n} b_{ik} a_{kj}$

$tr(BA) = \sum_{i=1}^{n} \alpha_{ii} = \sum_{i=1}^{n} \left(\sum_{k=1}^{n} b_{ik} a_{ki} \right)$

$= \sum_{k=1}^{n} \left(\sum_{i=1}^{n} b_{ik} a_{ki} \right)$

$= \sum_{i=1}^{n} \left(\sum_{k=1}^{n} a_{ik} b_{ki} \right) = tr(AB)$

Ex 16

Let E be a $\mathbb{R}$-vs

$\dim(E) = n$

$u \in \mathcal{L}(E)$

$v \in \mathcal{L}(E)$

Purpose of the exercise :

Prove that $u \circ v = id \Rightarrow v \circ u = id$

$\Rightarrow v = u^{-1}$

(true only if output space = input space and finite dim)

Reminder $v = u^{-1} \Leftrightarrow \begin{cases} u \circ v = id \\ v \circ u = id \end{cases}$

a) Using ~~theorem~~ rk th, prove that

u injective $\Rightarrow$ u bijective

u surjective $\Rightarrow$ u bijective

Rank th : $\dim E = \dim \ker(u) + \dim(\operatorname{Im} f)$

* If u injective $\Rightarrow \dim(\ker(u)) = 0$

$\Rightarrow \dim(\operatorname{Im}(u)) = \dim(E)$

$\left.\begin{array}{l} \operatorname{Im}(u) \subset E \\ \dim(\operatorname{Im}(u)) = \dim(E) \end{array}\right| \begin{array}{l} \Rightarrow \operatorname{Im}(u) = E \\ \Rightarrow u \text{ surjective} \end{array}$

* u surjective $\Rightarrow \dim(\operatorname{Im}(u)) = \dim(E)$

$\Rightarrow \dim(\ker(u)) = 0$

$\left.\begin{array}{l} \{0_E\} \subset \ker(u) \\ \dim(\{0_E\}) = \dim(\ker(u)) \end{array}\right| \begin{array}{l} \Rightarrow \{0_E\} = \ker(u) \\ \Rightarrow u \text{ injective} \end{array}$

2) Prove $u \circ v = id \Rightarrow u$ surjective

$\forall x \in E, \; u(v(x)) = x$

$\Rightarrow \forall x \in E, \; x = u(v(x)) \in Im(u)$

$\Rightarrow E \subset Im(u)$

$\Rightarrow u$ surjective

3) Prove $v \circ u = id \Rightarrow u$ injective

Let's prove $Ker(u) = \{0_E\}$

$\supset$: evident

$\subset$: Let $x \in Ker(u)$: $u(x) = 0_E$

$v(u(x)) = x \Rightarrow v(0_E) = x$

$\Rightarrow 0_E = x$

4) Let $(A, B) \in \mathcal{M}_n(\mathbb{R})$ such that $AB = I_n$

$E = \mathbb{R}^n$

$u = \mathbb{R}^n \to \mathbb{R}^n$

$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \to AX$

$v = \mathbb{R}^n \to \mathbb{R}^n$

$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \to BX$

Thus $u \circ v = id \Rightarrow u$ surjective (Q2)

$\Rightarrow u$ bijective (Q1)

So u^{-1} exists

$u^{-1} \circ (u \circ v) = u^{-1} \circ id \Rightarrow v = u^{-1}$

$\Rightarrow v \circ u = id$

So $BA = I_n$ and $BA = I_n$, $B = A^{-1}$

Rk: This works only if output set = input space (or at least same dimension) and finite dim

Determinants

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(A) = ad - bc$

If $A = \begin{pmatrix} a^{+} & b^{-} & c^{+} \\ d^{-} & e^{+} & f^{-} \\ g^{+} & h^{-} & i^{+} \end{pmatrix}$

We choose a line or a column

For example line 1

$$\det(A) = +a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

We have the same result if we choose another line (column to expand it)

$$\det(A) = -b \begin{pmatrix} d & f \\ g & i \end{pmatrix} \cancel{+ e \begin{pmatrix} e & f \\ h & i \end{pmatrix}} \cancel{+ c \begin{pmatrix} d & e \\ g & h \end{pmatrix}}$$

$$+ e \begin{pmatrix} a & c \\ g & i \end{pmatrix} - h \begin{vmatrix} a & c \\ d & f \end{vmatrix}$$

If $A = \begin{pmatrix} 1^{+} & 2^{-} & 3^{+} \\ 0^{-} & 4^{+} & 0^{-} \\ 3^{+} & 2^{-} & 1^{+} \end{pmatrix}$

$$\det(A) = -0 \begin{vmatrix} & \\ & \end{vmatrix} + 4 \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} - 0$$

$$= 4(1 - \cancel{3}\,9) = -32$$

We don't change $\det(A)$, if we do a transforma'

$$L_i \longleftarrow L_i + \sum_{k \neq i} \alpha_k L_k$$

or $$C_i \longleftarrow C_i + \sum_{k \neq i} \alpha_k C_k$$

Theorem: if A is square matrix, A bijective $\iff \det(A) \neq 0$

17/11/2017

Ex 1t

$\det(A - XI)$ $\quad A = \begin{pmatrix} 0 & 1 & -1 \\ -3 & 4 & -3 \\ -1 & 1 & 0 \end{pmatrix}$

$$\det(A - XI) = \begin{vmatrix} -X & 1 & -1 \\ -3 & 4-X & -3 \\ -1 & 1 & -X \end{vmatrix}$$

$$= \begin{vmatrix} -X+1 & 1 & -1 \\ 1-X & 4-X & -3 \\ 0 & 1 & -X \end{vmatrix} \quad C_1 \leftarrow C_1 + C_2$$

| | : det

( ) : mat

$$= \begin{vmatrix} -X+1 & 1 & -1 \\ 0 & 5-X & -4 \\ 0 & 1 & -X \end{vmatrix} \quad \underbrace{L_2 \leftarrow L_2 + L_1}_{\text{mistake}}$$

$$= (-X+1) \begin{vmatrix} 5-X & -4 \\ 1 & -X \end{vmatrix}$$

$$= -(X-1)\left[X(X-5)+4\right]$$

$$= -(X-1)^2(X-4)$$

1 multiplicity 2
4 multiplicity 1 $\Rightarrow 1+1+4 = tr(A)$

FALSE

$\longrightarrow$ SO MISTAKE

$$= \begin{vmatrix} 1-X & 1 & -1 \\ 0 & 3-X & -2 \\ 0 & 1 & -X \end{vmatrix} \quad L_2 \leftarrow L_2 - L_1$$

$$= (1-X) \begin{vmatrix} 3-X & -2 \\ 1 & -X \end{vmatrix}$$

$$= (1-X)\,[\underbrace{X^2 - 3X + 2}_{(X-1)(X-2)}] = -(X-1)^2(X-2)$$

test of trace : $1+1+2=4$ $= tr(A)$

$$\det(B - XI) = \begin{vmatrix} -X & 1 & 1 \\ 1 & -X & 1 \\ 1 & 1 & -X \end{vmatrix}$$

$$\cancel{= \begin{vmatrix} -X-1 & 1 & 1 \\ 1+X & -X & 1 \\ 0 & 1 & -X \end{vmatrix} \quad C_1 \leftarrow C_1 - C_2}$$

$$= \begin{vmatrix} -X-1 & 1 & 1 \\ 1-X & -X & 1 \\ 0 & 1 & -X \end{vmatrix} \quad C_1 \leftarrow C_1 - C_2$$

$$= \begin{vmatrix} \cancel{2X} - 2X_2 & 1 & 1 \\ -X\,x & -X & 1 \\ -X & 1 & X \end{vmatrix}$$

$$x_1 = \frac{-b-\sqrt{\Delta}}{2a} = \frac{1-3}{2} =$$

$$\det(B - X I) = \begin{vmatrix} -X & 1 & 1 \\ 1 & -X & 1 \\ 1 & 1 & -X \end{vmatrix} = \begin{vmatrix} -X+1 & -X+1 & 2 \\ 1 & -X & 1 \\ 1 & 1 & -X \end{vmatrix} = \begin{vmatrix} 0 & -X+1 & 2 \\ 1+X & -X & 1 \\ 0 & 1 & -X \end{vmatrix} \quad x_2 = \frac{-1+3}{2}$$

$$= (\cancel{X-1})\, (1+X) \begin{vmatrix} -X+1 & 2 \\ 1 & -X \end{vmatrix}$$

$$= (X-1)\, \cancel{(X^2 - X + 2)}\, (X^2 - X - 2)$$

$$\cancel{(X-1)^2}$$

$$\cancel{= (X-1)^2 (X+1)^2 (X-1)}$$

$$\cancel{= (X+1)^2}$$

$$\cancel{= -(X+1)^2 (X-1)}$$

$$= (-X-1) \begin{vmatrix} -X+1 & 2 \\ 1 & -X \end{vmatrix}$$

$$= (-X-1)(+X^2 - X - 2)$$

$$= (X+1)(-X^2 + X + 2)$$

$$= (X+1)(X+1)(X-2)$$

$$= -(X+1)^2 (X-2$$

In a diagonal matrix (or triangular matrix) A,

$\det(A)$ = Product of diagonal terms

Ex |

$$V(x_1 \dots x_n) = \begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-1} \\ 1 & \vdots & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-1} \end{vmatrix}$$

$$V(x_1 \dots x_n) = \begin{vmatrix} 1 & 0 & \dots & \dots & 0 \\ 1 & x_2 - x_1 & \dots & \dots & x_2^{n-1} - x_1 x_2^{n-2} \\ \vdots & \vdots & & & \vdots \\ 1 & x_n - x_1 & & & x_n^{n-1} - x_1 x_n^{n-2} \end{vmatrix} \quad C_n = C_n - x_1 C_{n-1}$$

$$V(x_1 \dots x_n) = (x_2 - x_1)(x_3 - x_1) \dots (x_n - x_1)\, V(x_2 \dots x_n)$$

We use the property: $\det[C_1\ \lambda C_2\ C_3 \dots C_n] = \lambda \det[C_1 \dots C_n]$

$$\det(\lambda C_1\ \lambda C_2 \dots \lambda C_n) = \lambda^n \det(C_1 \dots C_n)$$

$$\det \begin{pmatrix} \lambda L_1 \\ L_2 \\ \vdots \\ L_n \end{pmatrix} = \lambda \det \begin{pmatrix} L_1 \\ \vdots \\ L_n \end{pmatrix}$$

$$\det(\lambda A + B) \neq \lambda \det(A) + \det(B)$$

But $\forall i$, $\det(C_1 C_2 (\lambda C_i + C_i') C_{i+1} \dots C_n) = \lambda \det(C_1 C_2 \dots C_i C_{i+1} \dots C_n) + \det(C_1 C_2 \dots C_i' C_{i+1} \dots C_n)$

$$\det \begin{pmatrix} L_1 \\ \lambda L_2 + L_2' \\ L_3 \\ \vdots \\ L_n \end{pmatrix} = \lambda \det \begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_n \end{pmatrix} + \det \begin{pmatrix} L_1 \\ L_2 \\ \vdots \\ L_n \end{pmatrix}$$

Explanation of the theorem.

In a right basis, the matrix A will be $\begin{pmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{pmatrix}$

$$P_A(X) = \begin{vmatrix} x_1 - X & 0 & 0 \\ 0 & x_2 - X & 0 \\ 0 & 0 & x_3 - X \end{vmatrix}$$

$$= (x_1 - X)(x_2 - X)(x_3 - X)$$

Ex-18

$$\Delta = \begin{vmatrix} a_1 & a_1 & \cdots & a_1 \\ a_1 & a_2 & a_2 \cdots & a_2 \\ a_1 & a_2 & \ddots & \vdots \\ \vdots & \vdots & & \vdots \\ a_1 & a_2 & \cdots & a_n \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & \cdots & & a_1 \\ a_1 & a_2 & \cdots & a_2 \\ \vdots & \vdots & & \vdots \\ a_1 & a_2 & & \\ 0 & 0 & 0 \; 0 \; 0 \cdots & a_n - a_{n-1} \end{vmatrix} \quad L_n \leftarrow L_n - L_{n-1}$$

$$= (a_n - a_{n-1}) \begin{vmatrix} a_1 & \cdots & & a_1 \\ a_1 & a_2 & \cdots & a_2 \\ \vdots & \vdots & a_3 \cdots & a_3 \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_{n-1} \end{vmatrix}$$

$$= (a_n - a_{n-1}) \Delta(a_1, \ldots, a_{n-1})$$

$$= (a_n - a_{n-1})(a_{n-1} - a_{n-2}) \Delta(a_1, \ldots a_{n-2})$$

$$= (a_n - a_{n-1}) \cdots (a_2 - a_1) a_1$$