Complex Analysis Homework 1

Ramiro González

August 2016

1 Proving properties of complex numbers

1.1 (a)Proof: $Re(z) = \frac{z+\overline{z}}{2}$

$x, y \in \mathbb{R}$
let $z = x + iy$,
$\overline{z} = x - iy$
Then $Re(z) = x$, $Im(z) = y$

$$\frac{z+\overline{z}}{2} = \frac{(x+iy)+(x-iy)}{2} = \frac{(x+x)+(iy-iy)}{2} = x$$

Since $x = Re(z)$ **therefore** $\frac{z+\overline{z}}{2} = x = Re(z)$,
$Q.E.D$

1.2 (b) Proof: $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$ if $z_2 \neq 0$

$x, y \in \mathbb{R}$
Given $z_1 = x_1 + iy_1$ **,** $z_2 = x_2 + iy_2$
, $\overline{z_1} = x_1 - iy_1$ **,** $\overline{z_2} = x_2 - iy_2$

Let $w = (\frac{z_1}{z_2})$

$$\frac{(z_1)(\overline{z_2})}{(z_2)(\overline{z_2})} = \frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)} = \frac{(x_1x_2+y_1y_2)+i(-x_1y_2+x_2y_2)}{(x_2^2+y_2^2)} = \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2} + \frac{i(-x_1y_2+x_2y_2)}{x_2^2+y_2^2}$$

Therefore $\overline{w} = \frac{x_1x_2+y_1y_2}{x_2^2+y_2^2} - \frac{i(-x_1y_2+x_2y_2)}{x_2^2+y_2^2}$ **which simplifies to:** $\frac{x_1x_2+y_1y_2+ix_1y_2-ix_2y_2}{x_2^2+y_2^2} = \frac{(x_1-iy_1)(x_2+iy_2)}{(x_2+iy_2)(x_2-iy_2)}$
hence $\frac{(x_1-iy_2)}{(x_2-iy_2)}$ **considering that** $\overline{z_1} = x_1 - iy_1$ **,** $\overline{z_2} = x_2 - iy_2$ **implies that** $\frac{(x_1-iy_2)}{(x_2-iy_2)} = \frac{\overline{z_1}}{\overline{z_2}}$

1.3 (c) Proof: $im(iz) = Re(z)$

Given: $z = x + iy$ **and** $Re(z) = x$

$$let(t) = iz = i(x+iy) = ix - y$$

from this we can conclude that $Im(t) = x$ **and from the given we know that Re(z) = x therefore** $Im(t) = Im(iz) = Re(z)$

2 Prove or disprove the following:

2.1 **(a)Proof:**$Re(z_1 + z_2) = Re(z_1) + Re(z_2)$
Given: $z_1 = (x_1 + iy_1), z_2 = (x_2 + iy_2)$ **and thus** $Re(z_1) = x_1, Re(z_2) = x_2$

let $u = z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$
which implies that $Re(u) = (x_1 + x_2)$
from the given we know $Re(z_1) = x_1, Re(z_2) = x_2$ **which proves that** $Re(u) = Re(z_1 + z_2) = Re(z_1) + Re(z_2)$ **Q.E.D**

2.2 **(b) Proof** $Im(z_1 + z_2) = Im(z_1) + Im(z_2)$
Given: $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$ **, implies** $Im(z_1) = y_1, Im(z_2) = y_2$

let u = $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

so $Im(u) = (y_1 + y_z)$ **since we know** $Im(z_1) = y_1, Im(z_2) = y_2$ **then** $Im(u) = Im(z_1) + Im(z_2)$ **where** $u = z_1 + z_2$ **thus** $Im(z_1 + z_2) = Im(z_1) + Im(z_2)$ **Q.E.D**

2.3 **(c) Proof:** $Re(z_1 z_2) = Re(z_1)Re(z_2)$
Given: $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$ **which implies** $Re(z_1) = x_1, Re(z_2) = x_2$

$$Letu = z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1$$

so $Re(u) = Re(z_1 z_2) = x_1 x_2 - y_1 y_2$ **since** $Re(z_1 z_2) = Re(z_1)Re(z_2) - Im(z_1)Im(z_2)$ **is not equal to** $Re(z_1)Re(z_2)$ **therefore this only holds if and only if** $Im(z_1)Im(z_2) = 0$ **which is not possible, since z**$_1 and z_2 will no longer be imaginary$

2.4 **(d) Proof:** $Im(z_1 z_2) = Im(z_1)Im(z_2)$
Given: $z_1 = x_1 + iy_1, z_2 = x_2 + iy_, Im(2_1) = y_1, Im(z_2) = y_2, Re(z_1) = x_1, Re(z_2) = x_2$
let u = $z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$ **thus** $Im(u) = x_1 y_2 + x_2 y_1 = Re(z_1)Im(z_1) + Re(z_2)Im(z_1)$ **is not equal to** $Im(z_1)Im(z_2)$ **therefore the proof is not valid.**

3 Which of the following points lie inside the circle |z-i| = 1

Given $z = x + iy$ by substitution $|(x + iy) - i| = 1$ simplifies to $x^2 + (y - 1)^2 = 1$ this is the formula of a cirle therefore radius is 1, this implies in order to be inside it must be less than 1, if it is more it is outside, if it is equal to 1 it is on the circle.
must satisfy $\sqrt{x^2 + (y - 1)^2} < 1$

3.1 (a) $\frac{1}{2}+i$ by substitution $(\sqrt{\frac{1}{2})^2+(1-1)}=\frac{1}{2}<1$ inside the circle

3.2 (b) $1+\frac{i}{2}$ by substitution we find $|1+\frac{i}{2}-i|=\frac{5}{4}>1$ therefore it is outside

3.3 (c) $\frac{1}{2}+i\frac{\sqrt{2}}{2}$ by substitution $(\sqrt{(\frac{1}{2})^2+(\frac{\sqrt(2)}{2}-1)^2}=\frac{1+(\sqrt{2}-2)^2}{4}<1$ inside the circle

4 Show

Let $z_1, z_2 \in \mathbb{C}$. Show that z_1 is perpendicular to z_2 if and only if $\Re(z_1\overline{z_2})=0$

Vector Product

$$<x_1, y_1><x_2, y_2>=x_1x_2+y_1y_2$$

since $Re(z_1\overline{z_2})=<x_1, y_1>\cdot<x_2, y_2>$ and dot product must be zero for two vectors to be perpendicular. Moreover $Re(z_1\overline{z_2})=0$ which uniquely corresponds to $<x_1, y_1>\cdot<x_2, y_2>$ therefore $<x_1, y_1>\cdot<x_2, y_2>=0$ iff $Re(z_1\overline{z_2})=0$, Q.E.D]$letz_1=x+iy, \overline{z_2}=x_2-iy_2$

$$letu=z_1z_2=x_1x_2+y_1y_2+i(x_1y_2+x_2y_1), thusRe(u)=Re(z_1z_2)=x_1x_2+y_1y_2$$

Vector Product

$$<x_1, y_1><x_2, y_2>=x_1x_2+y_1y_2$$

since $Re(z_1\overline{z_2})=<x_1, y_1>\cdot<x_2, y_2>$ and dot product must be zero for two vectors to be perpendicular. Moreover $Re(z_1\overline{z_2})=0$ which uniquely corresponds to $<x_1, y_1>\cdot<x_2, y_2>$ therefore $<x_1, y_1>\cdot<x_2, y_2>=0$ iff $Re(z_1\overline{z_2})=0$, Q.E.D

5 Show that $|z^n|=|z|^n$ for complex numbers z and integers n

5.1 Given: $z=x+iy$, $r^2=x^2+y^2, |z|=\sqrt{x^2+y^2}, x=rcos(\theta), y=rsin(\theta)$, de Moivre's formula $(cos(\theta)+isin(\theta))^n=cos(n\theta)+isin(\theta)$

$$|(x+iy)^n|=(\sqrt{x^2+y^2})^n$$

$$|r^n(cos(\theta)+isin(\theta))^n|=r^n|(cos(n\theta)+isin(n\theta))^n|$$

$$r^n|cos(n\theta)+isin(n\theta)|$$

$$r^n(cos^2(n\theta)+isin^2(n\theta))$$

$$r^n=(\sqrt{x^2+y^2})^n$$

$$r^n=(r)^n$$

since

$$r=\sqrt{x^2+y^2}$$

6 Use mathematical induction to prove the following for n complex numbers $z_1, z_2, \ldots, z_n$

$$\left|\sum_{k=1}^{n} z_k\right| \leq \sum_{k=1}^{n} |z_k|$$

] Base case: n =1

$$\left|\sum_{k=1}^{1} z_k\right| \leq \sum_{k=1}^{1} |z_k|$$
$$|z_k| \leq |z_k|$$

Inductive step: Assume: $\left|\sum_{k=1}^{v} z_k\right| \leq \sum_{k=1}^{v} |z_k|$ is true for n = v
now we show its true for n = v + 1

$$\left|\sum_{k=1}^{v+1} z_k\right| \leq \sum_{k=1}^{v+1} |z_k|$$

$$\left|\sum_{k=1}^{v+1} z_k\right| = |\sum_{k=1}^{v} z_k + z_(v+1)|$$

By the triangle inequality:

$$\left|\sum_{k=1}^{v+1} z_k\right| = |\sum_{k=1}^{v} z_k| + |z_(v+1)|$$

Therefore:

$$\left|\sum_{k=1}^{v+1} z_k\right| \leq |\sum_{k=1}^{v} z_k| + |z_(v+1)|$$

$$\left||z_n + z_(n+1)\right| \leq |\sum_{k=1}^{v} z_k| + |z_(v+1)|$$

QED.

7 Let z be an arbitrary complex number. Prove that:

$$\sqrt{2}|z| \geq |\Re(z)| + |\Im(z)|.$$

7.1 Given $|z| = \sqrt{x^2 + y^2}$,

$$\sqrt{2x^2 + 2y^2}$$
$$\sqrt{2(rcos(\theta)^2 + 2(rsin(\theta))^2}$$
$$\sqrt{2r^2(cos^2(\theta) + sin^2(\theta))}$$

$$\sqrt{2r^2}$$