Homework Assignment # 2

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1. Show that $(\sqrt{3}+i)^4 = -8 + i8\sqrt{3}$ in the two ways listed.

   (a) considering $z = x + iy, i^2 = -1$

$$(\sqrt{3}+i)^2(\sqrt{3}+i)^2$$
$$(2+2\sqrt{3}i)^2$$
$$(2+2\sqrt{3}i)(2+2\sqrt{3}i)$$
$$(4+4\sqrt{3}i+4\sqrt{3}i+12i^2)$$
$$-8+i8\sqrt{3}$$

   (b) consider, $r = |z| = \sqrt{x^2+y^2}, z^n = r^n r^{in\theta}, (cos(\theta)+isin(\theta))^n = cos(n\theta)+isin(n\theta)$

$$e^{i\theta} = cos(\pi/6) + isin(\pi/6)$$
$$e^{in\theta} = cos(n\pi/6) + isin(n\pi/6)$$
$$e^{in\theta} = cos(\frac{2\pi}{3}) + isin(\frac{2\pi}{3}), n = 4$$
$$e^{in\theta} = \frac{-1}{2} + \frac{\sqrt{3}}{2}i$$
$$r = \sqrt{\sqrt{3}^2 + 1^2} = 2$$
$$r^4 = 2^4 = 16$$
$$r^4 e^{i\frac{2\pi}{3}} = 16(\frac{-1}{2} + \frac{\sqrt{3}}{2}i) = -8 + 8\sqrt{3}i$$

2. Find Arg(z) for the following values of z:

   (a) $Arg(1-i) = -\frac{\pi}{4}$
   (b) $Arg((1-i)^3 = (-2-2i)) = -\frac{3\pi}{4}$
   (c) $Arg(-\sqrt{3}+i) = \frac{2\pi}{3}$
   (d) $Arg(\frac{2}{(i-1)}) = Arg(-1-i) = -\frac{3\pi}{4}$

3. Prove of the following properties of Arg for complex numbers z, z_1 and z_2.

   (a) Arg($z\overline{z}$) = 0
   Let z = $x + iy, \overline{z} = x - iy$
   Let w $z\overline{z} = (x^2+y^2) + i(-xy+xy)$, thus $Re(w) = x^2+y^2$ and $Im(w) = (-xy+xy) = 0$, since $Re(w) = x^2+y^2 > 0$, and $Im(w) = 0$ this means w is one dimensional extending indefinitely in the

real positive axis with no imaginary dimension which means $Arg(w) = 0$, also the angle between $< x^2 + y^2, 0 >= 0$

$$(a)cos(\theta) = \frac{z \cdot \overline{z}}{||z||||\overline{z}||}$$
$$cos(\theta) = \frac{x^2 + y^2}{\sqrt{(x^2 + y^2)^2}} = 1$$
$$\theta = arcos(1) = 0$$

(b) $\text{Arg}(z + \overline{z}) = 0$ when $\text{Re}(z) > 0$

Let $z = x + iy, \overline{z} = x - iy$ then let w $= z + \overline{z} = 2x$
consider $2(-x) = 2(-Re(z))$, since -2 is a point the real axis and the imaginary point is 0 then the $Arg(w) = 0, \pi$ if and only if $Re(z) > 0$ because $Re(z) = Re(\overline{z})$.

(c) $\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$ when $\text{Re}(z_1) > 0$ and $\text{Re}(z_2) > 0$
$Arg(z\overline{z}) = Arg((r_1 e^{i\theta_1})(r_2 e^{i\theta_1}))$
$Arg(r_1 r_2 e^{i(\theta_1+\theta_2)}) = \theta_1 + \theta_2$
$\theta_1 + \theta_2 = Arg(z_1) + Arg(z_2)$ iff $\frac{-\pi}{2} < Arg(z_1), Arg(z_2) < \frac{pi}{2}$
therefore $Arg(z_1) + Arg(z_2)$ must be grater than $-\pi$ and less than π consequently $Re(z_1)$ and $Re(z_2) > 0$ for this to be true. QED

4. Recall that arg(z) is the full set of possible values for θ. Explain (in complete sentences) why $\arg(1/z) = -\arg z$

In order to find the argument of z we find the angle made by real and imaginary coefficients in the xy-plane this angle can be any possible value so therefore it is added or subtracted to multiple periods. Recall that $z^n = r^n e^{in\theta}$ where n can be negative or positive which means the angle when n = -1 is negative thus we have $arg(z^{-1}) = -\theta + 2n_2\pi$ since n_2 can be plus or minus factoring the negative gives rise to $arg(z^{-1}) = -(\theta - 2n_2\pi)$ however $-2n_2\pi$ is equivalent to $2n_2\pi$ therefore $arg(z^{-1}) = -arg(z)$. In conclusion $arg(z^{-1}) = -arg(z)$ because the angle is negated and since it can be any angle it can be added or subtracted by a period multiple.
let $n = -1$, we know $n_2 = \pm n_2 = \mp n_2$

$$z^{-1} = \frac{1}{r} e^{-i\theta}$$
$$arg(\frac{1}{r} e^{-i\theta}) = -\theta \pm 2n_2\pi$$
$$(-1)arg(\frac{1}{r} e^{-i\theta}) = (-1)(-\theta \pm 2n_2\pi)$$
$$-arg(\frac{1}{r} e^{-i\theta}) = \theta \mp 2n_2\pi$$

$$-arg(1/z) = \theta \mp 2n_2\pi = arg(z)$$

Q.E.D

5. Let $P(z) = a_n z^n + a_{n-1} z^{(n-1)} \dots a_1 z^1 + a_0$ be a polynomial where each $a_i \in \mathbb{R}$ i.e., each a_i is a real number). Show that if z_r is a complex number which satisfies $P(z_r) = 0$, then $P(\overline{z_r}) = 0$.

   We know $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$, and $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ also since $a_i \in \mathbb{R}$ then $\overline{a_i} = a_i$
   Given: $P(z_r) = \sum_{i=o}^{n} a_r z_n^r = 0$
   Since $\overline{P(z_r)} = \overline{0} = 0$,
   so we prove $\overline{P(z_r)} = P(\overline{z_r})$
   Using properties known.
   $*\overline{z_i^n} = \overline{z_i}^n$, let $z = x + iy$ and $\overline{z} = x - iy$
   $*\overline{(x+iy)^n} = \overline{(x-iy)^n}$
   By de moivre's formula.
   $*\overline{r^n[cos(n\theta) + sin(n\theta)i]} = r^n[cos(n\theta) - sin(n\theta)i]$
   $r^n[cos(n\theta) - sin(n\theta)i] = r^n[cos(n\theta) - sin(n\theta)i]$
   $\overline{P(z_r)} = \overline{\sum_{i=0}^{n} a_r z_r^n} = \sum_{i=0}^{n} \overline{a_r}\,\overline{z_r^n} = \sum_{i=0}^{n} a_r \overline{z_r^n} = \sum_{i=0}^{n} a_r \overline{z_r}^n = P(\overline{z_r})$
   Q.E.D

   Note: This establishes that complex roots of a real valued polynomial come in complex conjugate pairs.

6. Find all the roots of the equation $z^4 - 4z^3 + 6z^2 - 4z + 5 = 0$ given that $z_1 = i$ is a root. (Hint: Use the previous problem!) We know i is a root so $-i$ is also a root therefore:

$$(z-i)(z+i) = z^2 + 1$$

$$LongDivision: (z^2+1)\sqrt{z^4 - 4z^3 + 6z^2 - 4z + 5} = z^2 - 4z + 5$$

$$Roots: z^2 - 4z + 5 = (2 \pm i)$$

   Answer: $i, -i, (2 \pm i)$

7. Let z be any complex number and let n be an integer. Prove that $z^n + (\overline{z})^n$ must be a real number.
   let $z = x + iy, \overline{z} = x - iy$
   $x = rcos(\theta), y = rsin(\theta)$
   $z^n + (\overline{z})$
   $r^n[(cos(\theta) + isin(\theta))^n + (cos(\theta) - sin(\theta))^n]$, we can use Demovier's formula.

   $r^n[cos(n\theta) + isin(n\theta) + (cos(-n\theta) + sin(-n\theta))]$, since $n = \pm n$
   $r^n(2cos(n\theta) = \in \mathbb{R}$, by properties of trig functions. Q.E.D

8. Below I have found and plotted the sixth roots of unity. That is, the six solutions to $z^6 = 1$. Find and plot the eight roots which solve solution to $z^8 = 1$. (Hint: If you are doing this in LaTeX, you can either include a figure as a PDF, or you can draw the figure itself in LaTeX using the tikz package like I did when making the Figure below.)
let $z^n = (re^{i\theta}) = 1e^{0+2\pi k}$

$$r^n = 1$$
$$n\theta = \frac{2\pi k}{n}$$

$n = 1, 2, 3, 4, 5, 6, 7, \theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{\pi}{1}, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}$

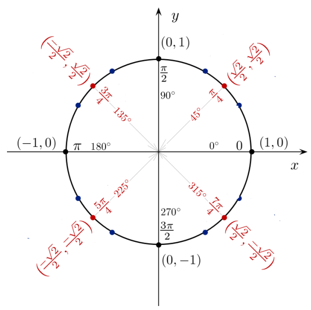

Figure 1: ShareLaTeX logo