ity of the desired transfer-function matrix $T_d(s)$ is that the matrix equation (9) is consistent for $rm > 3m$ (i.e., for the case where there are more equations than unknowns). However, as it is seen below, this condition is not sufficient.

Suppose that it is required to test the admissibility of a desired transfer-function matrix $T_d(s)$. Also consider another transfer-function matrix $T_a(s)$, for which

$$T_d(a_i) = T_a(a_i), \qquad i = 1, 2, \cdots, p. \tag{11}$$

$T_d(s)$ and $T_a(s)$ give rise to the same values for the matrices J_p and K_p. Now, if these matrices make (9) consistent, only one of the transfer-function matrices need be admissible. And thus the condition that (9) is consistent for $r = p$ is not sufficient for admissibility of $T_d(s)$.

However, since the elements of $T_d(s)$ are ratios of finite-order polynomials in s, there is an upper limit on the value of "p" for which the above equality (11) holds for distinct $T_d(s)$ and $T_a(s)$. The closed-loop system with the PID controller is of order $(n+m)$(see [10]). The numerator of the elements of $T_d(s)$ are polynomials of maximal order $(n+m-1)$ and denominators are polynomials of maximal order $(n+m)$. Hence, taking the gains into account, (11) can hold for distinct $T_d(s)$ and $T_a(s)$ if $p \leqslant \{2(n+m)-1\}$. For any value of $p > \{2(n+m)-1\}$, the equality (11) will hold only if $T_d(s)$ and $T_a(s)$ are identical. The following result is thus reached.

Lemma: A desired transfer-function matrix $T_d(s)$ is admissible if and only if (9) is consistent for $r = 2(n+m)$.

From the lemma the problem of the admissibility test is reduced to a test for consistency of (9).

The necessary and sufficient condition for the admissibility of (9) is given by (see, e.g., [11])

$$J_r J_r^{(1)} K_r = K_r, \qquad r = 2(n+m) \tag{12}$$

where $J_r^{(1)}$ is a $\{1\}$-inverse of matrix J_r. (For a detailed treatment of and methods for the computation of $\{1\}$-inverses see, e.g., [11].)

IV. Evaluation of the Controller

Once admissibility has been established through (12), the controller parameters can be determined from (9), i.e.

$$\begin{bmatrix} Q' \\ P' \\ R' \end{bmatrix} = J_r^{(1)} K_r + \{ Y - J_r^{(1)} J_r Y \}, \qquad r \geqslant 3$$

where Y is an arbitrary $(3m \times m)$ matrix. Note that if the matrix $\{ Y - J_r^{(1)} J_r Y \}$ is the null matrix then the controller parameters are determined uniquely. Notice also that since the results are independent of numbers a_i chosen from the set Z, these numbers are, naturally, chosen to be real to facilitate computation.

V. Conclusion

A numerical method has been developed for testing the realizability of a given transfer-function matrix with the use of multivariable PID controllers. The admissibility matrix $[J_r J_r^{(1)} K_r - K_r]$ can be used, by a suitable definition of a norm, to generate admissible transfer-function matrices if the desired one is not admissible. This is the subject of future investigation. Apart from the aforementioned point, no guideline has been given in the report for the choice of the desired transfer matrix.

References

[1] W. A. Wolovich, "The determination of state-space representations for linear multivariable systems," *Automatica*, vol. 9, pp. 97–106, 1973.

[2] G. D. Forney, "Minimal bases of rational vector spaces, with application to multivariable linear systems," *SIAM J. Contr.*, vol. 13, May 1975.

[3] M. K. Sain, "A free-modular algorithm for minimal design of linear multivariable systems," in *Proc. I.F.A.C. World Congress*, 1975.

[4] S. Wang and E. J. Davison, "A minimisation algorithm for the design of linear multivariable systems," *IEEE Trans. Automat. Contr.*, vol. AC-18, pp. 220–225, June 1973.

[5] B. C. Moore and L. M. Silverman, "Model matching by state feedback and dynamic compensation," *IEEE Trans. Automat. Contr.*, vol. AC-17, pp. 491–497, Aug. 1972.

[6] H. H. Rosenbrock, *State-space and Multivariable Theory*. Camden, NJ: Nelson, 1970.

[7] E. J. Davison and A. Goldenberg, "Robust control of a general servomechanism problem," *Automatica*, vol. 11, pp. 461–471, 1975.

[8] E. J. Davison, "The robust control of a servomechanism problem for linear time-invariant multivariable systems," *IEEE Trans. Automat. Contr.*, vol. AC-21, pp. 25–34, Feb. 1976.

[9] G. Bengtsson, "Output regulation and internal models—a frequency domain approach," *Automatica*, vol. 13, pp. 333–345, 1977.

[10] H. Seraji and M. Tarokh, "Design of PID controllers for multivariable systems," *Int. J. Contr.*, vol. 26, July 1977.

[11] A. Ben-Israel and T. N. E. Greville, *Generalised Inverses*. New York: Wiley, 1973.

Guaranteed Margins for LQG Regulators

JOHN C. DOYLE

Abstract—**There are none.**

Introduction

Considerable attention has been given lately to the issue of robustness of linear–quadratic (LQ) regulators. The recent work by Safonov and Athans [1] has extended to the multivariable case the now well-known guarantee of 60° phase and 6 dB gain margin for such controllers. However, for even the single-input, single-output case there has remained the question of whether there exist any guaranteed margins for the full LQG (Kalman filter in the loop) regulator. By counterexample, this note answers that question; there are none.

A standard two-state single-input single-output LQG control problem is posed for which the resulting closed-loop regulator has arbitrarily small gain margin.

Example

Consider the following:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 1 \\ 1 \end{bmatrix} w$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + v$$

where (x_1, x_2), u, and y denote the usual states, control input, and measured output, and where w and v are Gaussian white noises with intensities $\sigma > 0$ and 1, respectively.

Let performance integral have weights

$$Q = qCC^T = q \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}, \qquad q > 0$$

Manuscript received September 23, 1977. This work was performed for Honeywell Systems and Research Center, Minneapolis, MN, 55413 with support from the Office of Naval Research under Contract N00014-75-C-0144.

The author is with Honeywell, Inc., Minneapolis, MN 55413.

0018-9286/78/0800-0756$00.75 ©1978 IEEE

and

$$R=1.$$

Note that the estimation and control problems have identical (dual) solution matrices.

It can be shown analytically that the optimal gain vector g in $u=-g'x$ may be written as a function of q as

$$g=\begin{bmatrix}1\\1\end{bmatrix}(2+\sqrt{4+q}).$$

A similar relation holds between the optimal filter gain k and σ. For simplicity, let

$$g=f\begin{bmatrix}1\\1\end{bmatrix}$$

and

$$k=d\begin{bmatrix}1\\1\end{bmatrix}$$

where $f=2+\sqrt{4+q}$ and $d=2+\sqrt{4+\sigma}$.

Suppose that the resulting closed-loop controller has a scalar gain m (nominally unity) associated with the input matrix. Only the nominal value of this gain is known to the filter. The full system matrix then becomes

$$\begin{bmatrix}1 & 1 & 0 & 0\\ 0 & 1 & -mf & -mf\\ d & 0 & 1-d & 1\\ d & 0 & -d-f & 1-f\end{bmatrix}.$$

Evaluation of the characteristic polynomial is rather tedious, but reveals that only the last two terms are functions of m. The linear term is

$$d+f-4+2(m-1)df$$

and the constant term is

$$1+(1-m)df.$$

A necessary condition for stability is that both terms be positive. It is easy to see that for sufficiently large d and f (or q and σ), the system is unstable for arbitrarily small perturbations in m in either direction. Thus, by choice of q and σ the gain margins may be made arbitrarily small.

It is interesting to note that the margins deteriorate as control weight gets small and/or system driving noise gets large. In modern control folklore, these have often been considered ad hoc means of improving sensitivity.

It is also important to recognize that vanishing margins are not only associated with open-loop unstable systems. It is easy to construct minimum phase, open-loop stable counterexamples for which the margins are arbitrarily small.

The point of these examples is that LQG solutions, unlike LQ solutions, provide no global system-independent guaranteed robustness properties. Like their more classical colleagues, modern LQG designers are obliged to test their margins for each specific design.

It may, however, be possible to improve the robustness of a given design by relaxing the optimality of the filter with respect to error properties. A promising approach appears to be the introduction of certain fictitious system noises in the filter design procedure. This approach will be the topic of future papers.

References

[1] M. G. Safonov and M. Athans, "Gain and phase margins for multiloop LQG regulators," *IEEE Trans. Automat. Contr.*, Apr. 1977.

A Note on the Characteristic Frequency Loci of Multivariable Linear Optimal Regulators

I. POSTLETHWAITE

***Abstract*—This note outlines a procedure for determining the asymptotic behavior of the optimal closed-loop poles of a multivariable time-invariant linear regulator, as the weight on the input in the performance criterion approaches zero. It is based on an association of the optimal characteristic frequency loci with the branches of an algebraic function.**

I. Introduction

In a recent paper by Kwakernaak [1], the asymptotic loci of the optimal closed-loop poles of the multivariable time-invariant linear regulator were considered, as the weight on the input in the performance criterion approached zero. Kwakernaak showed that the poles going to infinity group into several Butterworth configurations of different orders, and also gave a method for determining them. In this note an equivalent procedure is presented which determines the Butterworth patterns using a well-established technique in algebraic function theory, that is, the "Newton diagram" approach for finding the series expansions of an algebraic function $q(v)$ in the neighborhood of a point v_o, (see [2] or [3]). Although this is an equivalent method to that given by Kwakernaak, the essential simplicity of the approach is emphasized in the setting of algebraic function theory. The method also complements recent research, (for example [4], [5] and [6]), in which algebraic function theory has been used to develop complex variable methods in the analysis and design of linear multivariable feedback systems.

The method is founded on an association of the optimal characteristic frequency loci with the branches of an appropriate algebraic function. The Newton diagram technique is then used to find the first terms in the series expansions for the branches of the algebraic function which are sufficient to determine the asymptotic behavior of the optimal closed-loop poles. A fuller exposition of the use of the Newton diagram is found in [6], where the asymptotic behavior of the closed-loop poles for a time-invariant linear multivariable feedback system is determined.

The definition of an algebraic function is given in Section II, and in Section III the optimal characteristic frequency loci are shown to be branches of an appropriate algebraic function. In Section IV it is shown how the Newton diagram can be used to determine the asymptotic behavior of the optimal closed-loop poles. Finally, in Section V the procedure is demonstrated by an example.

II. Definition of an Algebraic Function

Let $\Lambda(q,v)$ be a polynomial in q of the form

$$\Lambda(q,v)=f_0(v)q^m+f_1(v)q^{m-1}+\cdots+f_m(v) \qquad (2.1)$$

where each coefficient $\{f_i(v)\colon i=1,2,\cdots,m\}$ is itself a polynomial in v with coefficients in the domain of complex numbers. Then an algebraic function is a function $q(v)$ defined for values of v in the complex v-plane by an equation of the form

$$\Lambda(q,v)=0. \qquad (2.2)$$

The polynomial $\Lambda(q,v)$ can be rewritten as a polynomial in v with coefficients which are themselves polynomials in q, and when considered in this way, (2.2) defines an algebraic function $v(q)$.

For a fixed value of v, v_0 say, (2.2) has m solutions which are called branches of $q(v)$, and in the neighborhood of v_0 the branches are representable by power series expansions [3].

It is assumed in the above definition that $\Lambda(q,v)$ is an irreducible polynomial in (q,v), that is, that $\Lambda(q,v)$ is not the product of two or

Manuscript received December 2, 1977.

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