Infographic of Adiabatic Theorem Proof
(from Griffiths 2nd Edition)

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Step 1: Start with the behavior and properties of time-independent Hamiltonians

$$H_n\psi_n = E_n\psi_n$$

$$\Psi_n(t) = \psi_n e^{-iE_nt/\hbar}$$

Eigenstates are stable for time independent Hamiltonians

Step 2: Make foundational observations and generalizations assuming time-dependent Hamiltonian

$$H_n(t)\psi_n(t) = E_n(t)\psi_n(t)$$

If the Hamiltonian changes with time, then the eigenfunctions and eigenvalues are time-dependent

$$\langle\psi_n(t)|\psi_m(t)\rangle = \delta_{nm}$$

But still an orthonormal and complete set

$$i\hbar\frac{\partial\Psi}{\partial t} = H\Psi$$

Time dependent Schrödinger

$$\Psi(t) = \sum_n c_n(t)\psi_n(t)e^{i\theta_n(t)}$$

Can still be represented as the sum. $c_n(t)$ is 'the amount of' $\psi_n(t)$ in $\Psi(t)$.

$$\theta_n(t) \equiv -\frac{1}{\hbar}\int_0^t E_n(t')\,dt'$$

Generalization of the phase factor for time depedent E

This is **not** the time-independent Schrödinger equation

Orthonormality and completeness properties are important throughout

Two important notes:
1. If E was a constant, would be the familiar phase factor:

$$\theta_n = -\frac{1}{\hbar}\int_0^t E_n\,dt' = -\frac{1}{\hbar}E_n\int_0^t dt' = -E_nt/\hbar$$

2. This relationship, which follows from above, will come in handy below.

$$-\hbar\dot{\theta}_n = E_n$$

Step 3: Use these properties to rewrite the coefficient derivatives

$$i\hbar\sum_n[\dot{c}_n\psi_n + c_n\dot{\psi}_n + ic_n\psi_n\dot{\theta}_n]e^{i\theta_n} = \sum_n c_n(H\psi_n)e^{i\theta_n}$$

Plug the expansion into the time-dependent Schrödinger equation

$$\sum_n \dot{c}_n\psi_n e^{i\theta_n} = -\sum_n c_n\dot{\psi}_n e^{i\theta_n}$$

Left and right hand side reduce

$$\sum_n \dot{c}_n\delta_{mn}e^{i\theta_n} = -\sum_n c_n\left\langle\psi_m\middle|\dot{\psi}_n\right\rangle e^{i\theta_n}$$

Use inner product strategy and orhonormality property

$$\dot{c}_m = -\sum_n c_n\left\langle\psi_m\middle|\dot{\psi}_n\right\rangle e^{i(\theta_n-\theta_m)}$$

Reaarange by multplying through with m state phase factor

Step-by-step:

$$i\hbar\sum_n[\dot{c}_n\psi_n + c_n\dot{\psi}_n + ic_n\psi_n\dot{\theta}_n]e^{i\theta_n} = \sum_n c_n(H\psi_n)e^{i\theta_n}$$
$$i\hbar\sum_n\dot{c}_n\psi_ne^{i\theta_n} + i\hbar\sum_n c_n\dot{\psi}_ne^{i\theta_n} + i^2\hbar\sum_n c_n\psi_n\dot{\theta}_ne^{i\theta_n} = \sum_n c_n(H\psi_n)e^{i\theta_n}$$
$$i\hbar\sum_n\dot{c}_n\psi_ne^{i\theta_n} + i\hbar\sum_n c_n\dot{\psi}_ne^{i\theta_n} - \hbar\sum_n c_n\psi_n\dot{\theta}_ne^{i\theta_n} = \sum_n c_n(E_n\psi_n)e^{i\theta_n}$$
$$i\hbar\sum_n\dot{c}_n\psi_ne^{i\theta_n} + i\hbar\sum_n c_n\dot{\psi}_ne^{i\theta_n} = \sum_n c_n(E_n\psi_n)e^{i\theta_n} + \hbar\sum_n c_n\psi_n\dot{\theta}_ne^{i\theta_n}$$
$$i\hbar\sum_n\dot{c}_n\psi_ne^{i\theta_n} + i\hbar\sum_n c_n\dot{\psi}_ne^{i\theta_n} = -\hbar\sum_n c_n\psi_n\dot{\theta}_ne^{i\theta_n} + \hbar\sum_n c_n\psi_n\dot{\theta}_ne^{i\theta_n}$$
$$i\hbar\sum_n\dot{c}_n\psi_ne^{i\theta_n} + i\hbar\sum_n c_n\dot{\psi}_ne^{i\theta_n} = 0$$
$$\sum_n\dot{c}_n\psi_ne^{i\theta_n} = -\sum_n c_n\dot{\psi}_ne^{i\theta_n}$$

Step 4: Revisit the time-independent Schrödinger equation and rewrite the on and off diagonal coefficient derivatives

$$\dot{H}_n\psi_n + H_n\dot{\psi}_n = \dot{E}_n\psi_n + E_n\dot{\psi}_n$$

Differentiate the products on both sides

$$\langle\psi_m|\dot{H}|\psi_n\rangle + \langle\psi_m|H|\dot{\psi}_n\rangle = \dot{E}_n\delta_{nm} + E_n\langle\psi_m|\dot{\psi}_n\rangle$$

$$\langle\psi_m|H|\dot{\psi}_n\rangle = E_m\langle\psi_m|\dot{\psi}_n\rangle$$

Hermitcity and eigenvalue equation

$$\langle\psi_m|\dot{\psi}_n\rangle = \frac{\langle\psi_m|\dot{H}|\psi_n\rangle}{(E_n - E_m)}$$

Plug the latter into former and rearrange.

$$\dot{c}_m = -\sum_n c_n\frac{\langle\psi_m|\dot{H}|\psi_n\rangle}{(E_n - E_m)}e^{i(\theta_n-\theta_m)}$$

Plugging the result from above into the result of Step 3

$$\underbrace{\dot{c}_m = -c_m\langle\psi_m|\dot{\psi}_m\rangle - \sum_{n\neq m} c_n\frac{\langle\psi_m|\dot{H}|\psi_n\rangle}{(E_n - E_m)}e^{-i/\hbar\int_0^t[E_n(t')-E_m(t')]dt'}}_{\text{This is the exact solution}}$$

Inner product strategy again!

Step-by-step:

$$\langle\psi_m|\dot{H}|\psi_n\rangle + \langle\psi_m|H|\dot{\psi}_n\rangle = \dot{E}_n\delta_{nm} + E_n\langle\psi_m|\dot{\psi}_n\rangle$$
$$\langle\psi_m|\dot{H}|\psi_n\rangle + E_m\langle\psi_m|\dot{\psi}_n\rangle = 0 + E_n\langle\psi_m|\dot{\psi}_n\rangle$$
$$\langle\psi_m|\dot{H}|\psi_n\rangle = E_n\langle\psi_m|\dot{\psi}_n\rangle - E_m\langle\psi_m|\dot{\psi}_n\rangle$$
$$\langle\psi_m|\dot{H}|\psi_n\rangle = (E_n - E_m)\langle\psi_m|\dot{\psi}_n\rangle$$

Commentary:

1. From the expected change in H in the numerator, you can see why we insist on small dH/dt for this to hold.

2. Both the denominator and the exponential term *hint* at the requirement of a '**mimimum gap**' between the ground eigenenergy and the 1st excited state. Is this the correct interpretation?

Step 5: Impose the approximation by assuming dH/dt is small and the off diagonal terms drop

$$\dot{c}_m(t) = -c_m\langle\psi_m|\dot{\psi}_m\rangle$$

This can be thought of a the first-order approximation in an adiabatic series.

$$c_m(t) = c_m(0)e^{i\gamma_m(t)}$$

The solution to the above differential equation.

$$\gamma_m(t) \equiv i\int_0^t\langle\psi_m(t')|\frac{\partial}{\partial t'}\psi_m(t')\rangle\,dt'$$

Berry or 'geometric phase'

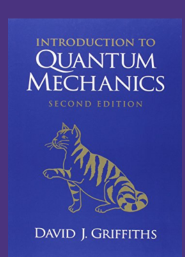
$$\Psi_n(t) \approx e^{i\theta_n(t)}e^{i\gamma_n(t)}\psi_n(t)$$

If the particle starts in the nth eigenstate and the process is adiabatic

$$\Psi_n(t) \approx (\text{dynamic phase})\,(\text{Berry phase})\,\psi_n(t)$$

Assume dH is small and the drop the diagonal.
See "Comment on the Adiabatic Condition"
Pinto et al 2000 Am. J. Phys.

Resources Used



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