Meredith Durbin
Tom Quinn
Astro 507: Thermodynamics
January 30, 2018

Homework 2

1. Starting with $C_V = dU/dT = \frac{3}{2}Nk_B$ and $C_P = dU/dT + PdV/dT = \frac{5}{2}Nk_B$, we have:

$$C_P/C_V = \frac{\frac{5}{2}Nk_B}{\frac{3}{2}Nk_B} \tag{1}$$
$$= \frac{5}{3} \tag{2}$$
$$= \gamma \tag{3}$$
$$C_P - C_V = \frac{5}{2}Nk_B - \frac{3}{2}Nk_B \tag{4}$$
$$= Nk_B \tag{5}$$

2. To find the speed at which the fastest electron should be moving, we want to set nV times the integral of the Maxwell-Boltzmann distribution equal to one and solve for the velocity. Given that $\int_x^\infty y^2 e^{-y^2} dy \approx xe^{-x^2}/2$ for $x \gg 1$, we set $x = \sqrt{mv^2/2k_BT}$ and find:

$$1 = nV\frac{1}{2}xe^{-x^2} \tag{6}$$
$$= \frac{nV}{2}\sqrt{\frac{mv^2}{2k_BT}}e^{-\frac{mv^2}{2k_BT}} \tag{7}$$

Solving this numerically using `scipy.optimize.fsolve`, we find that the maximum expected velocity is $0.224c$, which is much less than the cosmic ray velocity. Therefore cosmic rays cannot be of thermal origin.

Numerical solving code: `https://github.com/meredith-durbin/ASTR507/blob/master/HW2/HW2.ipynb`

3. (a) For a particle cross section of σ, the number density at which the mean free path equals the scale height is:

$$\frac{1}{\sigma n_{\text{esc}}} = \frac{k_BT}{mg} \tag{8}$$
$$n_{\text{esc}} = \frac{mg}{k_BT\sigma} \tag{9}$$

(b) The upwards flux of particles is:

$$\phi(v)dv = \frac{f(v)dv}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi/2} v\cos\theta\sin\theta d\theta \tag{10}$$
$$= \frac{f(v)dv}{2}\int_0^{\pi/2} v\cos\theta\sin\theta d\theta \tag{11}$$
$$= \frac{vf(v)dv}{4} \tag{12}$$

(c) The total flux of escaping particles is:

$$
\begin{aligned}
\phi &= \int_{v_{\rm esc}}^{\infty} \phi(v) dv &(13)\\
&= \frac{1}{4} n_{\rm esc} \int_{v_{\rm esc}}^{\infty} v f(v) dv &(14)\\
&= \frac{1}{4} n_{\rm esc} \int_{v_{\rm esc}}^{\infty} 4\pi v^3 \left(\frac{m}{2\pi k_B T}\right)^{3/2} e^{-\frac{mv^2}{2k_B T}} dv &(15)\\
&= \pi n_{\rm esc} \left(\frac{m}{2\pi k_B T}\right)^{3/2} \int_{v_{\rm esc}}^{\infty} v^3 e^{-\frac{mv^2}{2k_B T}} dv &(16)\\
&= \pi n_{\rm esc} \left(\frac{m}{2\pi k_B T}\right)^{3/2} \frac{k_B T}{m^2} e^{\frac{-mv_{\rm esc}^2}{2k_B T}} \left(m v_{\rm esc}^2 + 2k_B T\right) &(17)\\
&= \pi n_{\rm esc} \left(\frac{m}{2\pi k_B T}\right)^{3/2} e^{\frac{-mv_{\rm esc}^2}{2k_B T}} \left(\frac{k_B T}{m} v_{\rm esc}^2 + \frac{2(k_B T)^2}{m^2}\right) &(18)
\end{aligned}
$$

Substituting in $v_s = \sqrt{2k_B T/m}$ and $\lambda_{\rm esc} = (v_{\rm esc}/v_s)^2 = m v_{\rm esc}^2/2k_B T$, we have:

$$
\begin{aligned}
\phi &= \pi n_{\rm esc} \left(\frac{m}{2\pi k_B T}\right)^{3/2} e^{\frac{-mv_{\rm esc}^2}{2k_B T}} \left(\frac{k_B T}{m} v_{\rm esc}^2 + \frac{2(k_B T)^2}{m^2}\right) &(19)\\
&= \frac{n_{\rm esc}}{2\sqrt{\pi}} v_s^{-3} e^{-\lambda_{\rm esc}} \left(v_s^2 v_{\rm esc}^2 + v_s^4\right) &(20)\\
&= \frac{v_s n_{\rm esc}}{2\sqrt{\pi}} e^{-\lambda_{\rm esc}} \left(\lambda_{\rm esc} + 1\right) &(21)
\end{aligned}
$$

(d) I find that the rate of H2 loss over the entire exosphere is 1.33×10^{26} particles per second, which means that over 1 Gyr 4.20×10^{42} particles will escape. This is about half of the current hydrogen content of Earth's atmosphere. (Code in notebook linked above.)

(e) For oxygen, I find that the loss rate is on the order of 10^{-70} particles per second, which comes out to much less than a single particle being lost over 1 Gyr. This implies a stable oxygen abundance over time.

(f) For deuterium, I find a loss rate of 9.6×10^{36} particles per second, and a total loss over 1 Gyr of 3×10^{36} particles. This is a slower loss rate than H2, but not slow enough to be stable.