

“Why is it that particles with half-integer spin are Fermi particles whereas particles with integer spin are Bose particles? An explanation has been worked out by Pauli from complicated arguments from quantum field theory and relativity. He has shown that the two must necessarily go together but we have not been able to reproduce his arguments on an elementary level. This probably means we do not have a complete understanding of the fundamental principle involved..”

*R. P. Feynman, R.B. Leighton and M. Sands,
Feynman Lectures on Physics, Volume 3, Chapter 4
Section 1, Addison-Wesley, Reading, MA (1963).*

Chapter 16

Ideal Fermi Gas

16.1 *Introduction*

Particles with $\frac{1}{2}$ -integer angular momentum obey the *Pauli Exclusion Principle* (PEP) – a restriction that no more than one such identical particle can occupy the same quantum state. Alternatively, PEP states that any single particle $\frac{1}{2}$ -integer spin quantum state *can have an occupation number of only 0 or 1*. This restriction was *announced* by W. Pauli in 1924, for which, in 1945, he received the Nobel Prize in Physics. The exclusion principle was later generalized – independently – by P. Dirac and E. Fermi who integrated it into quantum mechanics. As a consequence $\frac{1}{2}$ -integer spin particles are called Fermi-Dirac particles or fermions. PEP applies to electrons, protons, neutrons, neutrinos, quarks – and their antiparticles – as well as composite fermions, such as He^3 atoms. Thermodynamic properties of metals and semiconductors are dominated by electron behavior. Metals exhibit the uniquely fermionic low temperature heat capacity $C_V \propto T$ while some Fermi-Dirac systems undergo a transition to a distinctive superconducting state with signature properties of magnetic flux exclusion (no interior \mathcal{B} field) and zero electrical resistivity.

16.2 *Ideal Gas Eigen-energies*

The eigen-energies and eigenfunctions of any ideal (non-interacting) gas are constructed from the eigenvalue problem

$$\mathbf{h}_{op}|\bar{E}_s\rangle = E_s|\bar{E}_s\rangle \quad (16.1)$$

with E_s the single particle eigen-energies and $|\bar{E}_s\rangle$ the corresponding single particle eigenstates. \mathbf{h}_{op} is the single particle hamiltonian operator for the i^{th} particle¹

$$\mathbf{h}_{op}(i) = -\frac{\hbar^2}{2m}\nabla_i^2 + \mathcal{V}(i) . \quad (16.2)$$

We must first solve the single particle problem, i.e. Eqs.16.1 and 16.2. Generally, $\mathcal{V}(i)$ represents some charge neutralizing positive background potential, e.g. positive ions, so obtaining a solution may be computationally intensive. It is, however, common and useful to assume that the background ions are replaced by a uniform positive charge, which is called the free electron approximation. Presuming a solution is obtained, we can proceed to introduce Fermi-Dirac particle thermodynamics.

16.2.1 *Least Bias Postulate: Grand Partition Function*

A many particle (open system) thermodynamic hamiltonian is

$$\hat{\mathcal{H}}_{op} = \mathcal{H}_{op} - \mu\mathcal{N}_{op} \quad (16.3)$$

where the many-particle hamiltonian \mathcal{H}_{op} is a sum of single particle hamiltonians [see Eq.16.2]

$$\mathcal{H}_{op} = \sum_{i=1} \mathbf{h}_{op}(i) . \quad (16.4)$$

Using single particle eigen-energies from Eq.16.1 the many-particle eigen-energies of $\hat{\mathcal{H}}_{op}$ are

$$\hat{E}(n_1, n_2, \dots) = \sum_s (E_s - \mu) n_s \quad (16.5)$$

¹This initial step applies to particles of either integer or half-integer spin.

with $n_s = 0, 1, 2, \dots$ the number of particles occupying the single particle eigenstate $|\bar{E}_s\rangle$ and μ the system chemical potential. The sum over s is a sum over *all* states.²

However, temporarily ignoring spin, each (non-degenerate) single particle state is restricted by PEP to have occupation number *zero* (no particles) or *one* (one particle), i.e. $n_s = 0$ or 1 .

We continue by constructing the “Least Biased” Lagrangian \mathcal{L}_{F-D}

$$\begin{aligned} \mathcal{L}_{F-D} = & -k_B \sum_{n_1, n_2, \dots = 0, 1} \mathbf{P}_{F-D}(n_1, n_2, \dots) \ln \mathbf{P}_{F-D}(n_1, n_2, \dots) \\ & - \lambda_1 \sum_{n_1, n_2, \dots = 0, 1} \mathbf{P}_{F-D}(n_1, n_2, \dots) \sum_s (E_s - \mu) n_s \\ & - \lambda_0 \sum_{n_1, n_2, \dots = 0, 1} \mathbf{P}_{F-D}(n_1, n_2, \dots) \end{aligned} \quad (16.6)$$

where $\mathbf{P}_{F-D}(n_1, n_2, \dots)$ is the probability there are n_1 particles in the single particle state $|\bar{E}_1\rangle$, n_2 particles in the single particle state $|\bar{E}_2\rangle$, etc. and where the sums mean

$$\sum_{n_1, n_2, \dots = 0, 1} \Rightarrow \sum_{n_1=0, 1} \sum_{n_2=0, 1} \dots \quad (16.7)$$

Maximizing \mathcal{L}_{F-D} with respect to $\mathbf{P}_{F-D}(n_1, n_2, \dots)$ we find the “Least Biased” probabilities

$$\mathbf{P}_{F-D}(n_1, n_2, \dots) = \frac{e^{-\beta \sum_s (E_s - \mu) n_s}}{\sum_{n_1, n_2, \dots = 0, 1} e^{-\beta \sum_s (E_s - \mu) n_s}} \quad (16.8)$$

where, as usual, $\lambda_1 = \frac{1}{T}$ with $\beta = \frac{1}{k_B T}$.

The denominator in Eq.16.8 is identified as the *grand partition function*, \mathcal{Z}_{gr}^{F-D}

$$\mathcal{Z}_{gr}^{F-D} = \sum_{n_1, n_2, \dots = 0, 1} e^{-\beta \sum_s (E_s - \mu) n_s}. \quad (16.9)$$

²Here we picture each single particle state as an “open system” into and out of which particles “diffuse” so that in thermodynamic (diffusive) equilibrium all particles have the same chemical potential.

It is evaluated by rearranging the sum over states s in the exponential to give

$$\mathcal{Z}_{gr}^{F-D} = \prod_s \sum_{n_s=0,1} e^{-\beta(E_s-\mu)n_s} \quad (16.10)$$

and then first summing $n_s = 0, 1$

$$\mathcal{Z}_{gr}^{F-D} = \prod_s \left[1 + e^{-\beta(E_s-\mu)} \right]. \quad (16.11)$$

16.2.1.1 *Electron Spin*

Taking electron spin into account, the non-degenerate pair of single particle states $|\bar{E}_{s,\uparrow}\rangle, |\bar{E}_{s,\downarrow}\rangle$ emerges with corresponding eigen-energies $E_{s,\uparrow}$ and $E_{s,\downarrow}$ so that

$$\mathcal{Z}_{gr}^{F-D} = \prod_s \left[1 + e^{-\beta(E_{s,\uparrow}-\mu)} \right] \left[1 + e^{-\beta(E_{s,\downarrow}-\mu)} \right]. \quad (16.12)$$

With zero magnetic field $E_{s,\uparrow} = E_{s,\downarrow}$. As a result of this two-fold degeneracy

$$\mathcal{Z}_{gr}^{F-D} = \prod_s \left[1 + e^{-\beta(E_s-\mu)} \right]^2 \quad (16.13)$$

16.2.2 *Fermi-Dirac Thermodynamics*

Thermodynamic properties of Fermi-Dirac systems are determined from Ω_{gr}^{F-D} , the *grand potential*

$$\Omega_{gr}^{F-D} = -\frac{1}{\beta} \ln \mathcal{Z}_{gr}^{F-D}, \quad (16.14)$$

which, with Eq.16.13, is

$$\Omega_{gr}^{F-D} = -\frac{2}{\beta} \sum_s \ln \left[1 + e^{-\beta(E_s-\mu)} \right] \quad (16.15)$$

where the sum \sum_s is over *all states* $|\bar{E}_s\rangle$.

1. The **average particle number** $\langle \mathcal{N}_{op} \rangle$ follows from Eq.16.8, i.e.

$$\langle \mathcal{N}_{op} \rangle = \frac{2 \prod_s \left[\sum_{n_s=0,1} n_s e^{-\beta(E_s-\mu)n_s} \right]}{\prod_s \left[\sum_{n_s=0,1} e^{-\beta(E_s-\mu)n_s} \right]} \quad (16.16)$$

where the factor 2 accounts for spin degeneracy.

However, more transparent is the equivalent result, Eq.13.14,

$$\langle \mathcal{N}_{op} \rangle = - \left(\frac{\partial \Omega_{gr}^{F-D}}{\partial \mu} \right)_{T,V} \quad (16.17)$$

$$= 2 \sum_s \left[\frac{1}{e^{\beta(E_s-\mu)} + 1} \right] \quad (16.18)$$

$$= 2 \sum_s \langle n_s^{F-D} \rangle \quad (16.19)$$

where $\langle n_s^{F-D} \rangle$ is a *single particle average occupation number* for the state s ,

$$\boxed{\langle n_s^{F-D} \rangle = \left[\frac{1}{e^{\beta(E_s-\mu)} + 1} \right]}, \quad (16.20)$$

which is often called the “Fermi-Dirac” function.

Using Eq.16.18, the chemical potential can be calculated as a function of $\langle \mathcal{N}_{op} \rangle / V$ and temperature T . [See examples below.] In free electron models the average fermion (electron) density $\langle \mathcal{N}_{op} \rangle / V$ [see Table 16.1] is a parameter that distinguishes among different materials.³

2. The **F-D internal energy** \mathcal{U} is found from the general result of Eq.13.16

$$\mathcal{U} = \left[\frac{\partial}{\partial \beta} (\beta \Omega_{gr}^{F-D}) \right]_{V,\mu} + \mu \langle \mathcal{N}_{op} \rangle. \quad (16.21)$$

which, using Eq.16.15, yields the intuitively logical

$$\mathcal{U} = 2 \sum_s E_s \langle n_s^{F-D} \rangle. \quad (16.22)$$

³At $T = 0$ the electron chemical potential is referred to as the *Fermi energy*, ϵ_F : i.e., $\mu(T=0) \equiv \epsilon_F$.

Element	$\frac{\langle \mathcal{N}_{op} \rangle}{V}$
Cu	8.47
Ag	5.86
Au	5.90
Al	18.1
Be	24.7
Sn	14.8
Mg	8.61
Fe	17.0
Pb	13.2
Hg	8.65

Table 16.1: Average electron density $\langle \mathcal{N}_{op} \rangle / V$ of several elemental metals. Densities are in units of $10^{28} m^{-3}$. (N. W. Ashcroft and N. D. Mermin, *Solid State Physics*, Saunders, 1976.)

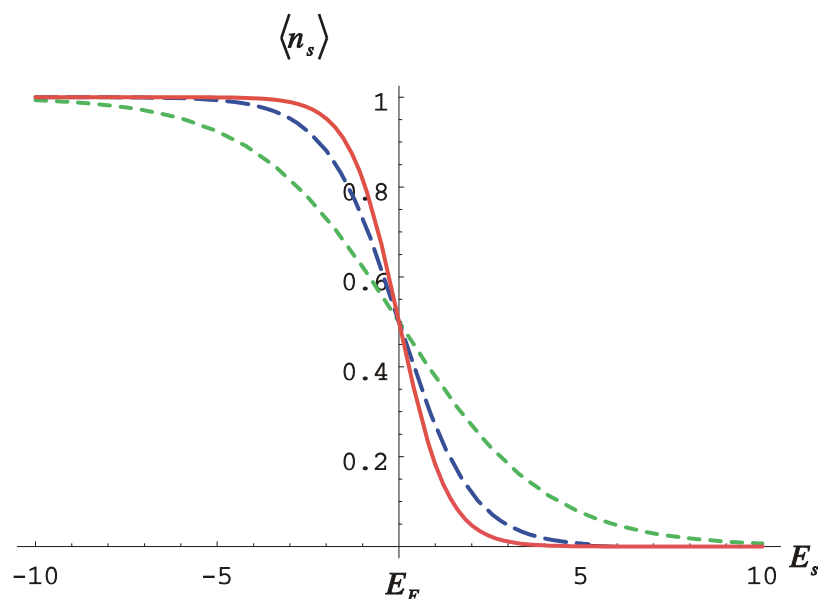


Figure 16.1: The Fermi-Dirac function in Eq.16.20. The solid line (red) represents $\langle n_s \rangle$ at low temperature. The short dashed curve (green) is representative at higher temperatures. With increasing temperature $\langle n_s \rangle \rightarrow \exp(-\beta E_s)$ – the semi-classical limit. At $T = 0$ $\langle n_s \rangle$ is a step function with *unit* jump at $E_s = E_F$.

3. The **equation-of-state** of the Fermi-Dirac gas is [see Eq.13.41]

$$pV = -\Omega_{gr}^{F-D} \quad (16.23)$$

$$= \frac{2}{\beta} \sum_s \ln(1 - \langle n_s^{F-D} \rangle) . \quad (16.24)$$

4. The **entropy** \mathcal{S} , as discussed in Chapter 5, is

$$\mathcal{S} = -k_B \sum_{n_1, n_2, \dots = 0, 1} \mathbf{P}_{F-D}(n_1, n_2, \dots) \ln \mathbf{P}_{F-D}(n_1, n_2, \dots) \quad (16.25)$$

where $\mathbf{P}_{F-D}(n_1, n_2, \dots)$ is as calculated in Eq.16.8. Equivalently, from the general result of Eq.13.22,

$$\mathcal{S} = k_B \beta^2 \left[\frac{\partial \Omega_{gr}^{F-D}}{\partial \beta} \right]_{\mu, V} \quad (16.26)$$

$$= -2k_B \sum_s \left[(1 - \langle n_s^{F-D} \rangle) \ln(1 - \langle n_s^{F-D} \rangle) + \langle n_s^{F-D} \rangle \ln \langle n_s^{F-D} \rangle \right] \quad (16.27)$$

16.2.3 Independent Fermion Model

The independent Fermion model for metals usually includes interactions with a static array of positive ions (band structure approximation) but neglects both electron-electron repulsion (correlations) and interactions with vibrating ions (electron-phonon interactions.)⁴

Independent fermion thermodynamics begins with Ω_{gr}^{F-D} [see Eq.16.15] where, using Dirac's δ -function⁵ property

$$\int_{-\infty}^{\infty} d\omega f(\omega) \delta(\omega - E_s) = f(E_s) . \quad (16.28)$$

Ω_{gr}^{F-D} may tactically be rewritten as

$$\Omega_{gr}^{F-D} = -\frac{2}{\beta} \int_0^{\infty} d\omega \sum_s \delta(\omega - E_s) \ln[1 + e^{-\beta(\omega - \mu)}] \quad (16.29)$$

⁴These effects can often be summarized by a few parameters, such as an effective mass m^* and an electron-phonon coupling constant λ which characterize individual systems. This "electron gas" model has been successful in describing simple metals and semiconductors.

⁵ $\delta(\omega - E_s)$ is the Dirac delta-function.

where the *density of states* $\mathcal{D}(\omega)$ [see Appendix E] is defined by

$$\mathcal{D}(\omega) = \sum_s \delta(\omega - E_s) \quad (16.30)$$

so that Ω_{gr}^{F-D} becomes

$$\Omega_{gr}^{F-D} = -\frac{2}{\beta} \int_0^\infty d\omega \mathcal{D}(\omega) \ln [1 + e^{-\beta(\omega-\mu)}] . \quad (16.31)$$

Replacing eigen-energies E_s with a *density of states* simplifies modeling more complex fermi systems, as will be demonstrated below.⁶

Similarly Eqs.16.18 and 16.22 become

$$\langle \mathcal{N}_{op} \rangle = 2 \int_0^\infty d\omega \frac{\mathcal{D}(\omega)}{e^{\beta(\omega-\mu)} + 1} \quad (16.32)$$

and

$$\mathcal{U} = 2 \int_0^\infty d\omega \omega \frac{\mathcal{D}(\omega)}{e^{\beta(\omega-\mu)} + 1} \quad (16.33)$$

Evaluating integrals Eqs.16.32, 16.33 and related types that appear in degenerate fermion models ($\beta\mu \gg 1$) often require specialized techniques. The simplest of these is Sommerfeld's asymptotic expansion, which is discussed in Appendix H.^{7,8}

16.2.3.1 The Chemical Potential

Thermodynamic properties of a degenerate fermi gas ($\beta\mu \gg 1$) can be expressed as expansions⁹ in T . In particular, the chemical potential is determined from the

⁶Approximate densities of states can be obtained from optical or tunneling experiments.

⁷Sommerfeld's approximation takes advantage of the property that for $T \approx 0$ the Fermi-Dirac function $\langle n_s^{F-D} \rangle$ approaches the unit step function at $\omega \approx \mu$ [see fig. 16.1].

⁸With increasing temperature, $\beta\mu \ll 1$, the steepness of the step declines over a width $\sim k_B T$ eventually taking the form of a smooth classical exponential $\frac{1}{e^{\beta(\omega-\mu)} + 1} \rightarrow e^{-\beta(\omega-\mu)}$.

⁹These expansions are not the usual convergent Taylor series type but belong to a special category called asymptotic expansions.

density $\frac{\langle \mathcal{N}_{op} \rangle}{V}$ by expanding the right hand side of Eq.16.32 about $\mu = \epsilon_F$ to *linear order*, where ϵ_F is the chemical potential at $T = 0$, to give

$$\langle \mathcal{N}_{op} \rangle = \int_0^\infty d\omega \frac{\mathcal{D}(\omega)}{e^{\beta(\omega - \epsilon_F)} + 1} + \frac{\beta(\mu - \epsilon_F)}{4} \int_0^\infty d\omega \mathcal{D}(\omega) \operatorname{sech}^2 \left[\frac{\beta}{2} (\omega - \epsilon_F) \right] \quad (16.34)$$

The first integral on the right-hand-side can be approximated by Sommerfeld's method [see Appendix H] to give

$$\int_0^\infty d\omega \frac{\mathcal{D}(\omega)}{e^{\beta(\omega - \epsilon_F)} + 1} \approx \int_0^{\epsilon_F} d\omega \mathcal{D}(\omega) + \frac{\pi^2}{6\beta^2} \mathcal{D}'(\epsilon_F) \quad (16.35)$$

while the second integral is approximated by

$$\frac{\beta}{4} \int_0^\infty d\omega \mathcal{D}(\omega) \operatorname{sech}^2 \left[\frac{\beta}{2} (\omega - \epsilon_F) \right] \approx \mathcal{D}(\epsilon_F) + \frac{\pi^2}{6\beta^2} \mathcal{D}''(\epsilon_F) \quad (16.36)$$

where $\mathcal{D}(\omega)$ has been similarly expanded about $\mu = \epsilon_F$. Combining Eqs.16.35 and 16.36 with Eq.16.34, while noting that

$$\int_0^{\epsilon_F} d\omega \mathcal{D}(\omega) = \langle \mathcal{N}_{op} \rangle \quad (16.37)$$

gives,

$$\mu = \epsilon_F - \frac{\frac{\pi^2}{6\beta^2} \mathcal{D}'(\epsilon_F)}{\mathcal{D}(\epsilon_F) + \frac{\pi^2}{6\beta^2} \mathcal{D}''(\epsilon_F)} \quad (16.38)$$

which to order T^2 is

$$\boxed{\mu = \epsilon_F - \frac{\pi^2}{6\beta^2} \frac{\mathcal{D}'(\epsilon_F)}{\mathcal{D}(\epsilon_F)}} \quad (16.39)$$

Note the important result that the chemical potential decreases from ϵ_F with increasing temperature. At very high temperature, $T \gg 0$, $\mu \rightarrow -\infty$.

16.2.3.2 Internal Energy

Similar to the argument above, the internal energy, [Eq.16.33], is first expanded about $\mu = \epsilon_F$

$$\int_0^\infty d\omega \frac{\omega \mathcal{D}(\omega)}{e^{\beta(\omega-\mu)} + 1} \approx \int_0^\infty d\omega \frac{\omega \mathcal{D}(\omega)}{e^{\beta(\omega-\epsilon_F)} + 1} + \frac{\beta(\mu - \epsilon_F)}{4} \int_0^\infty d\omega \omega \mathcal{D}(\omega) \operatorname{sech}^2 \left[\frac{\beta}{2} (\omega - \epsilon_F) \right]. \quad (16.40)$$

Applying Sommerfeld's approximation to the first term on the right-hand-side

$$\int_0^\infty d\omega \frac{\omega \mathcal{D}(\omega)}{e^{\beta(\omega-\epsilon_F)} + 1} \approx \int_0^{\epsilon_F} d\omega \omega \mathcal{D}(\omega) + \frac{\pi^2}{6\beta^2} \left\{ \frac{d}{d\omega'} [\omega' \mathcal{D}(\omega')] \right\}_{\omega'=\epsilon_F} \quad (16.41)$$

while the second term becomes

$$\int_0^\infty d\omega \omega \mathcal{D}(\omega) \operatorname{sech}^2 \left[\frac{\beta}{2} (\omega - \epsilon_F) \right] \approx \frac{4}{\beta} \epsilon_F \mathcal{D}(\epsilon_F). \quad (16.42)$$

Combining Eqs.16.40, 16.41, 16.42 with 16.39

$$\mathcal{U} = 2 \int_0^{\epsilon_F} d\omega \omega \mathcal{D}(\omega) + \frac{\pi^2}{3\beta^2} \mathcal{D}(\epsilon_F) \quad (16.43)$$

from which the constant volume heat capacity

$$\mathcal{C}_V = \left(\frac{\partial \mathcal{U}}{\partial T} \right)_V$$

is

$$\mathcal{C}_V = \frac{2\pi^2}{3} k_B^2 \mathcal{D}(\epsilon_F) T, \quad (16.44)$$

a result quite different from the constant valued classical result. This follows since PEP allows only the small fraction of particles near ϵ_F , i.e. $N_{eff} \sim N k_B T$, to participate in low temperature thermal processes. Interestingly, measured macroscopic heat capacity is proportional to the density of states $\mathcal{D}(\epsilon_F)$, a *microscopic quantum property*.

16.2.4 *Electron Gas Model*

The electron gas model is frequently used to describe the behavior of simple (monovalent) metals. It assumes a macroscopic collection of non-interacting *spin* $-\frac{1}{2}$ particles, neutralized only by a uniform positive background, with a single particle hamiltonian for the (i^{th}) particle

$$\mathbf{h}_{op}(i) = -\frac{\hbar^2}{2m} \nabla_i^2. \quad (16.45)$$

Choosing periodic boundary conditions [see Appendix G], the single particle eigen-energies are

$$\epsilon(\mathbf{k}) = \frac{\hbar^2 |\mathbf{k}|^2}{2m} \quad (16.46)$$

where k_j ($j = x, y, z$) are the eigenstate quantum numbers which assume the values

$$k_j = \frac{2\pi}{L} \nu_j \quad (16.47)$$

where

$$\nu_j = 0, \pm 1, \pm 2, \pm 3, \dots \quad (16.48)$$

Here L is the length of one side of a “notional” periodic cube with macroscopic volume $V = L^3$. From Appendix E the density of states, $\mathcal{D}(\omega)$, corresponding to this model [see Eq.16.46] is

$$\begin{aligned} \mathcal{D}(\omega) &= \sum_{\mathbf{k}} \delta[\omega - \epsilon(\mathbf{k})] \\ &= \frac{V}{4\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \sqrt{\omega}, \end{aligned} \quad (16.49)$$

which is the distinctive 3 – *dimensional* electron gas result.¹⁰

With Eq.16.49 the grand potential [see Eq.16.31] is

$$\Omega_{gr}^{F-D} = -\frac{V}{2\pi^2\beta} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_0^\infty d\omega \sqrt{\omega} \ln [1 + e^{-\beta(\omega-\mu)}] \quad (16.50)$$

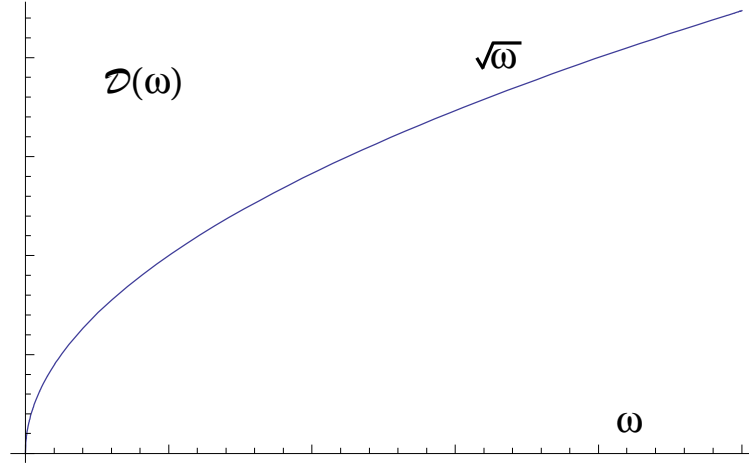


Figure 16.2: Density of states for 3 – D electron gas model. [See Eq.16.49].

which is integrated by parts to give

$$\Omega_{gr}^{F-D} = -\frac{V}{3\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_0^\infty d\omega \frac{\omega^{3/2}}{e^{\beta(\omega-\mu)} + 1}. \quad (16.51)$$

Since

$$p = - \left(\frac{\partial \Omega_{gr}}{\partial V} \right)_{T, \mu} \quad (16.52)$$

the fermion gas pressure is

$$p^{F-D} = \frac{1}{3\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_0^\infty d\omega \frac{\omega^{3/2}}{e^{\beta(\omega-\mu)} + 1}. \quad (16.53)$$

The *degenerate gas pressure*, ($\beta\mu \gg 1$), is expanded in powers of T by applying the

¹⁰The mass can be replaced by an effective mass, $m \rightarrow m^*$, where m^* approximates the effect of a crystal lattice.

steps in Sec. 16.2.3.2, starting with an expansion of the integrand about $\mu = \epsilon_F$,

$$\int_0^\infty d\omega \frac{\omega^{3/2}}{e^{\beta(\omega-\mu)} + 1} = \int_0^\infty d\omega \frac{\omega^{3/2}}{e^{\beta(\omega-\epsilon_F)} + 1} + \frac{\beta(\mu - \epsilon_F)}{4} \int_0^\infty d\omega \omega^{3/2} \operatorname{sech}^2 \left[\frac{\beta}{2} (\omega - \epsilon_F) \right]. \quad (16.54)$$

The first integral on the right-hand-side is approximated using Sommerfeld's method

$$\int_0^\infty d\omega \frac{\omega^{3/2}}{e^{\beta(\omega-\epsilon_F)} + 1} \approx \frac{2}{5} \epsilon_F^{5/2} + \frac{\pi^2}{4\beta^2} \epsilon_F^{1/2} \quad (16.55)$$

while the second integral is

$$\int_0^\infty d\omega \omega^{3/2} \operatorname{sech}^2 \left[\frac{\beta}{2} (\omega - \epsilon_F) \right] \approx \frac{4}{\beta} \epsilon_F^{3/2}. \quad (16.56)$$

Therefore Eq.16.54, together with Eqs.16.55, 16.56 and 16.39, gives the pressure of a degenerate ($\beta\mu \gg 1$) Femi-Dirac gas model

$$p^{F-D} = \frac{2}{15\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \epsilon_F^{5/2} \left(1 + \frac{5\pi^2}{12\beta^2 \epsilon_F^2} \right). \quad (16.57)$$

Unlike a classical ideal gas for which $p \approx T$, the pressure of a degenerate Fermi-Dirac gas is nearly independent of temperature.

Finally, in the Fermi gas model we have [see Eq.16.37]

$$\frac{\langle \mathcal{N}_{op} \rangle}{V} = \frac{1}{2\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \int_0^{\epsilon_F} d\omega \sqrt{\omega} \quad (16.58)$$

$$= \frac{1}{3\pi^2} \left(\frac{2m}{\hbar^2} \right)^{3/2} \epsilon_F^{3/2} \quad (16.59)$$

or

$$\boxed{\epsilon_F = \frac{\hbar^2}{2m} \left(3\pi^2 \frac{\langle \mathcal{N}_{op} \rangle}{V} \right)^{2/3}}. \quad (16.60)$$

Thermodynamic properties can now be expressed in terms of the fermion density [e.g. Table16.1].

16.2.5 *White Dwarf Stars*

The thermodynamics of stars is a study in self-gravitating systems – a titanic struggle between gravity, which compresses stellar matter, and outward gas pressure usually associated with high temperatures within stellar interiors.¹¹ As a result of nuclear and chemical evolution, equilibrium between these forces is in constant flux, resulting in continuous stellar transformations.

Stars that are not too massive, say on the order of our sun’s mass, eventually “burn” nearly all their core hydrogen resulting in a cooler stellar core richer in helium (and heavier nuclei.) Consequently gas pressure is reduced which, seemingly, throws the dynamical advantage to gravitational forces, suggesting the star’s continued collapse, perhaps without limit. Although ordinary gas pressure is no longer capable of resisting gravitational collapse, star stability can now be maintained by *quantum behavior of degenerate (but not necessarily relativistic) electrons*, whose source is the now fully ionized *He* atoms.

These stars can collapse to radii the order of the earth’s while maintaining masses on the order of the sun – a density of 10^6 times that of water. Such stars were discovered by British astronomer William Herschel (1783) who failed to realize their significance. These relatively nearby (on a galactic scale) but intrinsically dim stars are called White Dwarfs. They are resident corpses in the evolutionary stellar graveyard for stars of about a solar mass and less.

16.2.5.1 *White Dwarf Thermodynamics*

A white dwarf can be modeled as an aggregate of *He* nuclei densely packed at the star’s hot central region. The *He* nuclei intermingle with a degenerate ($\beta\mu \gg 1$) electron gas originating from the fully ionized *He* atoms.

The degenerate electron gas behavior is governed by the PEP, so using Eqs.16.57 and 16.60, the nearly temperature independent outward pressure of the degenerate electron gas component is

$$p^e = \frac{(3\pi^2)^{2/3}}{5} \left(\frac{\hbar^2}{m_e} \right) \left(\frac{\langle \mathcal{N}_{op}^e \rangle}{V} \right)^{5/3}, \quad (16.61)$$

¹¹High temperatures result from nuclear fusion at the star’s center in which hydrogen “burns” to helium and, eventually, heavier nuclei including carbon and oxygen.

where m_e is the electron mass and $\frac{\langle \mathcal{N}_{op}^e \rangle}{V}$ is the electron number density. The stellar mass, M , is approximately

$$M = 2\langle \mathcal{N}_{op}^e \rangle m_{He}. \quad (16.62)$$

where m_{He} is the mass of a He nucleus.

The star's dense He core exerts inward gravitational pressure

$$p^{core} = - \left(\frac{\partial \Omega_{gr}^{core}}{\partial V} \right)_{T,\mu} \quad (16.63)$$

where, for bosonic He nuclei,

$$\Omega_{gr}^{core} = - \frac{V}{3\pi^2} \left(\frac{2m_{He}}{\hbar^2} \right)^{3/2} \int_0^\infty d\omega \frac{\omega^{3/2}}{e^{\beta(\omega-\mu+W)} - 1} \quad (16.64)$$

with W the mean field gravitational potential energy per He nucleus [see Chapter 14]

$$W = m_{He} \langle \Phi(\mathbf{r}) \rangle. \quad (16.65)$$

$\Phi(\mathbf{r})$ is the gravitational potential determined by Poisson's equation

$$\nabla^2 \Phi(\mathbf{r}) = 4\pi G \rho(\mathbf{r}) \quad (16.66)$$

with a solution

$$\Phi(r) = - \frac{G}{r} \int_0^r \rho(\mathbf{r}') d\mathbf{r}' \quad (16.67)$$

where G is the universal gravitational constant and $\rho(\mathbf{r})$ is the He mass density.

In the extreme *non-degenerate* (classical) limit ($\beta\mu \ll 1$), where boson-fermion statistics are irrelevant, Eq.16.64 becomes

$$\Omega_{gr}^{core} = - \frac{V}{3\pi^2} \left(\frac{2m_{He}}{\hbar^2} \right)^{3/2} \int_0^\infty d\omega \omega^{3/2} e^{-\beta(\omega-\mu+W)} \quad (16.68)$$

$$= \frac{V}{(2\pi^3)^{1/2} \beta^{5/2}} \left(\frac{m_{He}}{\hbar^2} \right)^{3/2} e^{\beta(\mu+W)}. \quad (16.69)$$

Assuming a spherically distributed uniform mass density ρ

$$\rho = \frac{M}{V} \quad (16.70)$$

$$= \frac{3M}{4\pi R^3} \quad (16.71)$$

where R is the star radius, we find¹²

$$W = -m_{He} \frac{3GM}{5R}. \quad (16.72)$$

Applying Eq.16.63 the inward nuclear core pressure is

$$p^{core} = \frac{1}{(2\pi^3)^{1/2} \beta^{5/2}} \left(\frac{m_{He}}{\hbar^2} \right)^{3/2} e^{\beta(\mu-W)} - \frac{1}{5\sqrt{\pi}} \frac{GM^2}{V^{1/3}} \left(\frac{m_{He}}{\beta\hbar^2} \right)^{3/2} \left(\frac{\sqrt{2}}{3\pi^2} \right)^{1/3} e^{\beta(\mu-W)}. \quad (16.73)$$

Next, applying Eq.16.17 to Eq.16.69, the average number of He nuclei is

$$\langle \mathcal{N}_{op}^{He} \rangle = \frac{V}{\sqrt{2}} \left(\frac{m_{He}}{\pi\beta\hbar^2} \right)^{3/2} e^{\beta(\mu-W)}. \quad (16.74)$$

Therefore the nuclear core pressure becomes

$$p^{core} = \frac{\langle \mathcal{N}_{op}^{He} \rangle}{\beta V} - \frac{1}{5} \left(\frac{4\pi}{3} \right)^{1/3} \frac{GM^2}{V^{4/3}} \quad (16.75)$$

where the first term is the *classical ideal He gas* nuclei pressure, which is negligible for this cool star. The total pressure – i.e. the equation of state – for the white dwarf star, is therefore

$$p_{total} = \frac{(3\pi^2)^{2/3}}{5} \left(\frac{\hbar^2}{m_e} \right) \left(\frac{\langle \mathcal{N}_{op}^e \rangle}{V} \right)^{5/3} - \frac{1}{5} \left(\frac{4\pi}{3} \right)^{1/3} \frac{GM^2}{V^{4/3}}. \quad (16.76)$$

At mechanical equilibrium $p_{total} = 0$, giving

$$\frac{(3\pi^2)^{2/3}}{5} \left(\frac{\hbar^2}{m_e} \right) \left(\frac{\langle \mathcal{N}_{op}^e \rangle}{V} \right)^{5/3} = \frac{1}{5} \left(\frac{4\pi}{3} \right)^{1/3} \frac{GM^2}{V^{4/3}} \quad (16.77)$$

¹²The factor $3/5$ in the expression that follows is due to assumed spherical symmetry. For other mass distributions a different constant ≈ 1 will appear.

which with

$$\langle \mathcal{N}_{op}^e \rangle = 2 \langle \mathcal{N}_{op}^{He} \rangle \text{ (charge neutrality)} \quad (16.78)$$

$$= \frac{M}{2m_{He}} \quad (16.79)$$

yields the remarkable (non-relativistic) white dwarf mass-radius condition

$$\left[\frac{3^{4/3} \pi^{2/3} \hbar^2}{8G(m_{He})^{5/3} m_e} \right] = RM^{1/3}. \quad (16.80)$$