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Astro 507: Thermodynamics
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Homework 3

1. (a) The entropy for an ideal nonrelativistic Fermi gas is given as $S = (U + PV - N\mu)/T$. The pressure and number density are given in the notes, from which we can easily get PV and N:

$$PV = \frac{4(2s+1)VkT}{3\pi^{1/2}\lambda^3} F_{3/2}(z) \tag{1}$$
$$N = \frac{2(2s+1)V}{\pi^{1/2}\lambda^3} F_{1/2}(z) \tag{2}$$
$$\tag{3}$$

We know that $U = \frac{3}{2}P$ for an ideal gas, so:

$$U = \frac{2(2s+1)VkT}{\pi^{1/2}\lambda^3} F_{3/2}(z) \tag{4}$$

Putting these together, we have:

$$S = \left(\frac{2(2s+1)VkT}{\pi^{1/2}\lambda^3} F_{3/2}(z) + \frac{4(2s+1)VkT}{3\pi^{1/2}\lambda^3} F_{3/2}(z) - \frac{2(2s+1)V\mu}{\pi^{1/2}\lambda^3} F_{1/2}(z) \right) T^{-1} \tag{5}$$
$$= \frac{2(2s+1)V}{\pi^{1/2}\lambda^3 T} \left(\frac{5}{3} kT F_{3/2}(z) - \mu F_{1/2}(z) \right) \tag{6}$$

(b) Expanding P/nkT as a series in z is equivalent to Taylor expanding $F_{3/2}(z)$ about $z = 0$. The linear term is:

$$\frac{d}{dz} \left(\frac{w^{3/2}}{e^w/z + 1} \right) = \frac{e^w w^{3/2}}{(e^w + z)^2} \tag{7}$$

At $z = 0$, this is simply $w^{3/2}/e^w$. We then plug this back into the Fermi-Dirac integral and evaluate:

$$F_{3/2}(z) = z \int_0^\infty w^{3/2}/e^w dw \tag{8}$$
$$= \frac{3\sqrt{\pi}}{4} z \tag{9}$$

Plugging this back in to our expression for P/nkT, we have:

$$\frac{P}{nkT} = \frac{4(2s+1)}{3\pi^{1/2}\lambda^3 n} \frac{3\sqrt{\pi}}{4} z \tag{10}$$
$$= \frac{(2s+1)}{\lambda^3 n} z \tag{11}$$

This indicates that pressure will increase with fugacity, and fugacity increases with degeneracy.

2. (a) All calculations are in `https://github.com/meredith-durbin/ASTR507/blob/master/HW3/HW3.ipynb`.

   First, we need to find the electron number density. For 10% helium by number, $n_H = \rho_c/(m_H + 0.1m_{He})$. Assuming full ionization, we then have $n_e = 1.2n_H$. We find $F_{1/2}(z) = \sqrt{\pi}\lambda^3 n_e/2(2s+1) = 0.83$. We then numerically solve the Fermi-Dirac integral for z and find $z = 1.3$.

   (b) Using $P = \frac{4(2s+1)kT}{3\pi^{1/2}\lambda^3}F_{3/2}(z)$, we find the pressure at the center of the star to be 1.6×10^{17} dyn/cm^2.

   (c) The relativistic density is $\rho_{rel} = 2 \times 10^6(\mu_e/2m_p)$ g/cm^3, or about 1.4×10^6 g/cm^3. This is much higher than the given density, so the electrons are not relativistic.

   (d) Using $P = nkT$, we find a pressure of 1.15×10^{17} dyn/cm^2, which isn't that far from what we found with the other equation, so degeneracy is not important.

   (e) Solving for M and R, we find:

$$M = \frac{0.016\sqrt{\rho_c}}{\mu_e F_{1/2}(z)} \tag{12}$$

$$R = \frac{0.514}{(\mu_e F_{3/2}(z)\sqrt{\rho_c})^{1/3}} \tag{13}$$

   This gives us $M = 0.25M_\odot$ and $R = 0.19R_\odot$.