Spatial Data Analysis

Week 8: Areal Data III

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Areal Data III

- Simultaneous Autoregressive (SAR) Models
- Conditional Autoregressive (CAR) Models
- Mixed Effects Models
- Geographically Weighted Regression

Areal Data

- We have examined neighbourhood relationships and defined weights applied to areal units.
- Weights connect our areal units in a particular way.
- We have examined evidence of global and local spatial (auto)correlation in the areal units through Moran's I, Geary's c, Local Moran's I and the Getis-Ord G* statistics.
- We now move more formally to fitting spatial regression models for areal data, most commonly called autoregressive models.
- Simultaneous and Conditional Auto-Regressive models (SAR and CAR).
- Both SAR and CAR models are known as network-based or graphical models.
- The SAR model can be considered as a spatial extension of the autoregressive model often used in time series.

Areal Data

- Recall from geostatistics $E[Z(s)] = \mu$, so we can define the spatial process as a Gaussian process, $Z(s) \sim N(\mu, \Sigma(\theta))$.
- $\Sigma(\theta) = \tau^2 I + \sigma^2 C(\theta; h)$
- We can let the mean depend on a set of covariates:

$$Z(s) \sim N(X\beta, \tau^2 I + \sigma^2 C(\theta; h))$$

- In terms of areal data, we typically use $Y(s)$ to represent the response variable of interest as multivariate normal with mean $\mu = X\beta$ and variance-covariance matrix Σ.

Autoregressive Models

- Similar to universal kriging or regression kriging.
- Use regression on values from neighbouring areal units to account for spatial dependence.
- Autocorrelation reflects self regression where you use observations of the outcome at other locations as additional covariates in the model.
- $Y(s) \sim MVN(X\beta, \Sigma)$
- In universal kriging we modeled Σ as a parametric function of distance.
- In areal modeling, we restrict distances to those between our areal units.
- In R we use the `spatialreg` package to fit autoregressive models.

Simultaneous Autoregressive (SAR) Models

There are several SAR model types:

- SAR lag model: Spatial dependence is in the dependent variable(s) only (errors iid).
- SAR error model: Spatial dependence is in the error terms only (like ordinary kriging).
- SARAR or spatial autocorrelation model: spatial dependence is in th dependent variable(s) and error terms (like modified universal kriging).

SAR Model Types

- SAR models explicitly encode dependence via a weights matrix W.
- Three common specifications:
  - **Lag**: $Y = \rho WY + X\beta + \varepsilon$, $\varepsilon \sim N(0, \sigma^2 I)$
  - **Error**: $Y = X\beta + u, \quad u = \lambda Wu + \varepsilon$
  - **SARAR**: both lag and error components

SAR Model Types

R `spatialreg` functions:

- SAR lag model: `lagsarlm`
- SAR error model: `errorsarlm`
- SARAR or spatial autocorrelation model: `sacsarlm`

SAR Lag Model - Introduction

- **Purpose:** model spatial dependence in the *outcome* by adding a spatially lagged Y.
- **Model idea:** nearby units' outcomes influence each other through a weights matrix W.
- **What is lagged?** The *dependent variable*: WY enters the mean. Covariates X are *not* lagged in SAR–lag.
- **R function:** `spatialreg::lagsarlm(...)`.
- **Pieces:** Y (outcome), X (covariates), β (coefficients), W (spatial weights), ρ (lag strength), ε (iid error).

SAR Lag Model - Details

- SAR lag models are represented as $Y = X\beta + \rho WY + \varepsilon$
- Y is the dependent variable available in each block (areal unit), W is the spatial weights matrix, ρ is the spatial autoregressive parameter, X is the matrix of covariates, β is a vector of regression coefficients, and ε is the error term.
- $\rho WY + \varepsilon$ represents the spatial lag of the dependent variable (influence of neighboring values of Y on the outcome).
- $X\beta$ is the usual linear regression component (not lagged).
- ε is assumed IID for the lag only model.

SAR Lag Model - Details

$$
\begin{aligned}
Y &= \rho WY + X\beta + \varepsilon, \qquad \varepsilon \sim N(0, \sigma^2 I) \\
(I - \rho W)Y &= X\beta + \varepsilon \\
Y &= \underbrace{(I - \rho W)^{-1}}_{S(\rho)} X\beta \, + \, (I - \rho W)^{-1}\varepsilon
\end{aligned}
$$

- W: $n \times n$ spatial weights (often row-standardized, diagonal 0).
- $S(\rho) = (I - \rho W)^{-1}$ is the *network multiplier*: it propagates effects through neighbors and higher-order neighbors.
- **Existence / range:** need $I - \rho W$ invertible. With row-standardized W, a practical rule is $\rho \in (-1, 1)$ (more generally bounded by reciprocals of W's extreme eigenvalues).

SAR Lag Model - Likelihood

Let $r = (I - \rho W)Y - X\beta$. The ML log-likelihood is

$$\ell(\rho, \beta, \sigma^2) = -\frac{n}{2}\log(2\pi\sigma^2) + \log|I - \rho W| - \frac{1}{2\sigma^2} r'r.$$

- Estimate (ρ, β, σ^2) by ML (e.g., `lagsarlm`); $\log|I - \rho W|$ is computed via W's eigenvalues/approximations.
- Compare models using AIC/logLik; check residual Moran's I (should be small if dependence captured).
- After fitting, compute *impacts* (direct/indirect/total) for substantive interpretation.

SAR Lag Model - Existence and Parameter Range

- Need $I - \rho W$ invertible $\Rightarrow$ restrict ρ using eigenvalues of W.
- If $\omega_{\min}$ and $\omega_{\max}$ are the min/max eigenvalues of W, an admissible interval is:

$$\rho \in \left(\frac{1}{\omega_{\min}}, \frac{1}{\omega_{\max}} \right) \quad \text{(endpoints excluded).}$$

- For **row-standardized** W (common in areal data), typically $\omega_{\max} = 1$ and $\omega_{\min} \geq -1$, so a practical rule is $\rho \in (-1, 1)$.
- Numerical optimizers enforce this range to keep $|I - \rho W| \neq 0$.

Eigenvalue Mini-Demo (Valid Row-Standardized W)

$$W = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \Rightarrow \quad \det(W - \omega I) = \begin{vmatrix} -\omega & 1 \\ 1 & -\omega \end{vmatrix} = \omega^2 - 1 = 0$$

$$\Rightarrow \ \omega \in \{1, -1\} \quad \Longrightarrow \quad \rho \in \left(\frac{1}{-1}, \frac{1}{1} \right) = (-1, 1)$$

- This matches the usual admissible range for row-standardized W.
- For other W (e.g., non-standardized or distance-based), recompute bounds via its eigenvalues.

SAR Lag Model - Interpretation of the Effects

$$Y = (I - \rho W)^{-1} X\beta \, + \, (I - \rho W)^{-1} \varepsilon$$

$$E[Y] \, = \, S(\rho) X\beta, \qquad S(\rho) = (I - \rho W)^{-1}.$$

Consider a $+\mathbf{1}$ change in regressor x_j at one unit (holding others fixed).

- "Effects" are the marginal change in the expected outcome Y when you change a regressor x_j at one areal unit (direct) and how that change propagates to other units through the spatial network (indirect/spillovers).
- **Direct effect (own-unit):** average diagonal of $S(\rho)\beta_j$.
- **Indirect effect (spillovers):** average row-sum of off-diagonals of $S(\rho)\beta_j$ (effects on other units).
- **Total effect:** direct + indirect.

Key points

- β is *not* the marginal effect because $S(\rho)$ induces feedback across the network.
- With row-standardized W, a quick check is average total effect $\approx \beta/(1-\rho)$).

SAR Lag Model - Interpretation of the Effects

Decomposing the change:

- **Direct effect at k (own-unit):** $\Delta E[Y_k] = \beta_j S_{kk}(\rho)$
- **Indirect (spillover) effects on others $i \neq k$:** $\Delta E[Y_i] = \beta_j S_{ik}(\rho)$

Network view: $S(\rho) = I + \rho W + \rho^2 W^2 + \cdots$. Direct effects include feedback loops back to k; indirect effects collect neighbors, neighbors-of-neighbors, etc.

SAR Lag Model - Interpretation of the Effects

What we report (averaged over all starting units k):

- **Average Direct Effect (ADE) = Direct**
  Own-unit effect (after feedback), averaged across k:

$$\mathrm{ADE}_j = \frac{1}{n}\,\mathrm{tr}\big(S(\rho)\big)\,\beta_j.$$

- **Average Indirect Effect (AIE) = Indirect**
  Spillover to all *other* units, averaged across k:

$$\mathrm{AIE}_j = \left[\frac{1}{n}\mathbf{1}^\top S(\rho)\mathbf{1} - \frac{1}{n}\,\mathrm{tr}\big(S(\rho)\big)\right]\beta_j = \mathrm{ATE}_j - \mathrm{ADE}_j.$$

- **Average Total Effect (ATE) = Total**
  Own + spillovers, averaged across k:

$$\mathrm{ATE}_j = \frac{1}{n}\mathbf{1}^\top S(\rho)\mathbf{1}\,\beta_j.$$

SAR Lag Model - Interpretation of the Effects

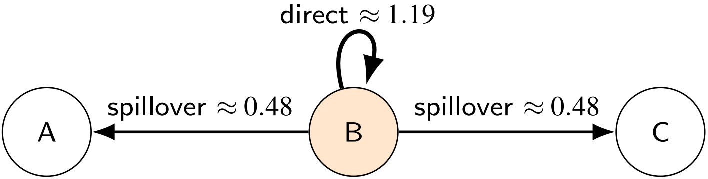
- Pick one areal unit, **B**. Increase one predictor at B by **+1**.
- Because regions are linked (spatial lag), B's change spreads to neighbors; neighbors nudge B back a bit; this continues.
- After this back-and-forth finishes, we look at the **final** changes:
  - **Direct effect** = final change in B itself (includes feedback).
  - **Indirect effect** = average final change in other regions (spillovers).
  - **Total effect** = sum of final changes over all regions.
- In SAR–lag, report these effects (not just the raw coefficient).

Example 3 regions in a line; row-standardized W, $\rho = 0.4$, coefficient $= 1$):

Region	Final change	Interpretation
A (neighbor of B)	0.48	indirect (spillover)
B (changed here)	**1.19**	**direct (own)**
C (neighbor of B)	0.48	indirect (spillover)

Totals: direct $= 1.19$, indirect (avg) $= 0.48$, total (A+B+C) $= 2.14$.

SAR Lag Model — Diagram of Effects



Example with $\rho = 0.4$ and coefficient $\beta_j = 1$ (row-standardized W): bump x_j at **B** by $+1$. The curved arrow is B's *own* final change (direct); straight arrows are *neighbor* changes (indirect). Total = sum over all units.

SAR Error Model — Introduction

- **Goal:** Capture spatial structure in the residuals after explaining Y with X.
- **Mean (trend):** $E[Y] = X\beta$ (no WY term in the mean).
- **Residual pattern:** neighboring units tend to have related departures from the fitted trend.
- **Model idea:** let the error term follow a spatial autoregression.

$$Y = X\beta + u, \qquad u = \lambda Wu + \varepsilon, \quad \varepsilon \sim N(0, \sigma^2 I).$$

- **Interpretation of λ:**
  - $\lambda > 0$: residuals are *locally clustered* (similar residuals across neighbors).
  - $\lambda < 0$: residuals *counterbalance* locally (peaks beside troughs).

SAR Error Model — Model formulation

$$Y = X\beta + u, \qquad u = \lambda W u + \varepsilon, \quad \varepsilon \sim N(0, \sigma^2 I)$$

$$(I - \lambda W)u = \varepsilon \quad \Rightarrow \quad u = (I - \lambda W)^{-1}\varepsilon.$$

Implications

- **Mean:** $E[Y] = X\beta$ (regression coefficients keep their usual partial-regression meaning).
- **Covariance:**
$$\mathrm{Var}(Y) = \mathrm{Var}(u) = \sigma^2\left[(I - \lambda W)^\top (I - \lambda W)\right]^{-1}.$$
- λ**:** choose λ so $I - \lambda W$ is invertible (for row-standardized W, a practical rule is $\lambda \in (-1, 1)$; more generally bounded by reciprocals of W's extreme eigenvalues).

SAR Error Model — Likelihood and estimation

Define the transformed residual

$$e = (I - \lambda W)(Y - X\beta).$$

The (concentrated) log-likelihood is

$$\ell(\lambda, \beta, \sigma^2) = -\frac{n}{2}\log(2\pi\sigma^2) + \log|I - \lambda W| - \frac{1}{2\sigma^2} e'e.$$

Fitting (ML)

- Maximize w.r.t. $(\lambda, \beta, \sigma^2)$; the Jacobian term $\log|I - \lambda W|$ is evaluated from W's eigenvalues (or approximations).
- In R: `spatialreg::errorsarlm(...)` or `spdep::spautolm(...)`.
- Compare AIC/logLik with OLS and SAR–lag; residual Moran's I should be small if the error structure is well captured.

SAR Error Model — Interpretation

What to read off the output

- **Regression effects $\hat{\beta}$:** marginal effects on the mean. A one-unit increase in x_j changes $E[Y]$ by β_j (no spatial effects decomposition here).
- **Residual spatial parameter $\hat{\lambda}$:** strength/sign of remaining local spatial pattern after controlling for X.
  - $\hat{\lambda} > 0$ (significant): residuals still cluster locally.
  - $\hat{\lambda} < 0$ (significant): residuals alternate locally (local suppression).
- **Diagnostics:** residual Moran's I should be non-significant; if not, revisit W or model form.
- **Model comparison:** report AIC/logLik vs OLS and SAR–lag; pick the lowest AIC and verify residual spatial autocorrelation is addressed.

$\hat{\beta}$ describes the mean trend; $\hat{\lambda}$ summarizes the remaining local spatial pattern in the residuals; residual Moran's I checks that spatial pattern has been accounted for.

SAR Error Model and Covariance

$$Y = X\beta + u, \qquad u = \lambda W u + \varepsilon, \qquad \varepsilon \sim N(0, \sigma^2 I)$$
$$(I - \lambda W)u = \varepsilon \;\Rightarrow\; Y \sim N\Big(X\beta,\, \sigma^2 \big[(I - \lambda W)^\top (I - \lambda W)\big]^{-1}\Big).$$

- Spatial dependence is in the *errors*; β retains the usual partial-regression meaning.
- λ is restricted analogously to ρ so that $I - \lambda W$ is invertible.

SAR Lag–Error - Introduction

- **Motivation:** spatial dependence may appear in both the *outcome* (neighbors influence each other) and the *residual pattern* (nearby units share unmodeled variation).
- **Model idea:** combine a spatial lag in the mean with a spatial autoregression in the errors.
$$y = \rho\, Wy + X\beta + u, \qquad u = \lambda\, Wu + \varepsilon, \quad \varepsilon \sim N(0, \sigma^2 I).$$
- **Roles:**
  - ρ captures *diffusion/feedback* in outcomes (broad spatial trend).
  - λ captures *residual* local spatial pattern after controlling for X and ρWy.
- **R function:** `spatialreg::sacsarlm(...)`.

SAR Lag-Error — Model formulation

Mean (trend):

$$(I-\rho W)y = X\beta + u \quad \Rightarrow \quad E[y] = (I-\rho W)^{-1}X\beta.$$

Error process: $u = \lambda Wu + \varepsilon \Rightarrow (I-\lambda W)u = \varepsilon \Rightarrow u = (I-\lambda W)^{-1}\varepsilon.$

Implications

- **Network multiplier on the mean:** $S(\rho) = (I-\rho W)^{-1}$ propagates $X\beta$ through neighbors (as in SAR–lag).
- **Covariance of y:**

$$\mathrm{Var}(y) = \sigma^2 (I-\rho W)^{-1} \left[(I-\lambda W)^{\top}(I-\lambda W)\right]^{-1} (I-\rho W)^{-\top}.$$

- Choose (ρ,λ) so $I-\rho W$ and $I-\lambda W$ are invertible (for row-standardized W, a practical rule is $\rho,\lambda \in (-1,1)$; more precise bounds use W's extreme eigenvalues).

SAR Lag-Error - Likelihood and estimation

Define the transformed residual

$$e = (I - \lambda W)\big[(I - \rho W)y - X\beta\big].$$

The ML log-likelihood is

$$\ell(\rho, \lambda, \beta, \sigma^2) = -\frac{n}{2}\log(2\pi\sigma^2) \; + \; \log|I - \rho W| \; + \; \log|I - \lambda W| \; - \; \frac{1}{2\sigma^2}\, e'e.$$

Fitting (ML)

- Maximize w.r.t. $(\rho, \lambda, \beta, \sigma^2)$ (`sacsarlm`); Jacobians $\log|I - \rho W|, \log|I - \lambda W|$ computed from W's spectrum/approximations.
- Compare AIC/logLik vs OLS, SAR–lag, SAR–error; residual Moran's I on SAC residuals should be small if dependence is captured.

SAC — How to interpret ρ, λ, and β

Parameters

- ρ **(lag on y):** strength of outcome diffusion/feedback. If $\rho > 0$, nearby outcomes tend to be similar (positive clustering in the mean).
- λ **(spatial error):** strength/sign of *residual* local pattern after accounting for ρWy.
  - $\lambda > 0$: residuals still *cluster* locally.
  - $\lambda < 0$: residual residuals *counterbalance* locally (local suppression).
- β **(regression coefficients):** relate X to the mean, but *marginal effects are not* β because $E[y] = S(\rho)X\beta$.

Report impacts for each x_j (as in SAR–lag):

$$\text{Direct (ADE)}, \ \text{Indirect (AIE)}, \ \text{Total (ATE)} \quad \text{from } S(\rho) = (I - \rho W)^{-1}.$$

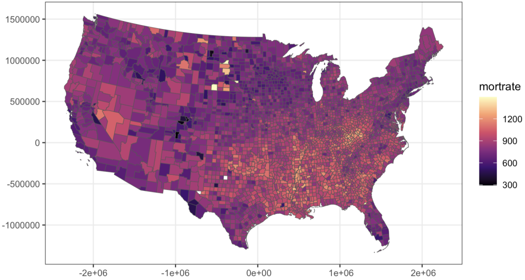
- λ affects *uncertainty* (SEs) of impacts through the covariance, but not their mean values.
- In R: `impacts(sacsarlm_fit, listw = W, R = ...)` returns ADE/AIE/ATE (with simulation-based SEs).

Example SAR Lag-Error vs OLS

- We can look at the spatial parameter estimates of SAR lag-error vs OLS regression in terms of testing for $\mathrm{H}_0 : \rho = 0$ vs $\mathrm{H}_1 : \rho \neq 0$ and $\mathrm{H}_0 : \lambda = 0$ vs $\mathrm{H}_1 : \lambda \neq 0$.
- Furthermore, we can test for residual spatial autocorrelation by using Moran's I (good practice for all of these models).

OLS vs SAR

- Let's look at mortality rates in all 3067 counties in the U.S. (data are available from https://www.nhgis.org/)
- Use mortality rate as the dependent variable Y and X independent variables include proportion of people in poverty (ppersonspo), proportion aged 65+ (p65plus), proportion White (pwhite), proportion Black (pblack_1), proportion Hispanic (phisp), and proportion unemployed (punemp_1).
- Note the coordinates are projected.



OLS output

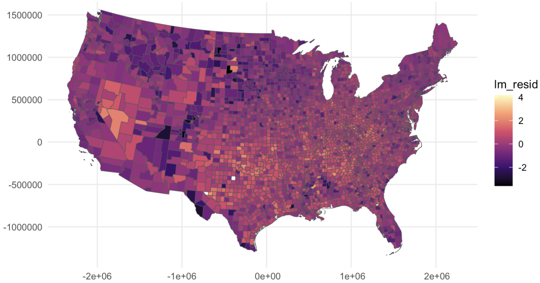
```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  -1.3411     0.2122  -6.321 2.97e-10 ***
ppersonspo    7.8135     0.3237  24.136  < 2e-16 ***
p65plus      -2.1017     0.3679  -5.713 1.22e-08 ***
pwhite        0.4330     0.2103   2.059   0.0396 *
pblack_1      1.2972     0.2081   6.232 5.23e-10 ***
phisp        -2.1099     0.1405 -15.021  < 2e-16 ***
punemp_1      3.4119     0.7665   4.451 8.85e-06 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 0.7626 on 3060 degrees of freedom
Multiple R-squared:  0.4196,	Adjusted R-squared:  0.4185
F-statistic: 368.7 on 6 and 3060 DF,  p-value: < 2.2e-16
```

OLS Moran's I Residuals

To test the residual spatial autocorrelation, we look at the Moran's I of the residuals:

```
Moran I statistic standard deviate = 25.538, p-value < 2.2e-16
alternative hypothesis:  greater
sample estimates:
Moran I statistic Expectation Variance
0.4281131935 -0.0003261579 0.0002814596
```

The null hypothesis of no spatial autocorrelation is rejected.



SAR Lag-Error

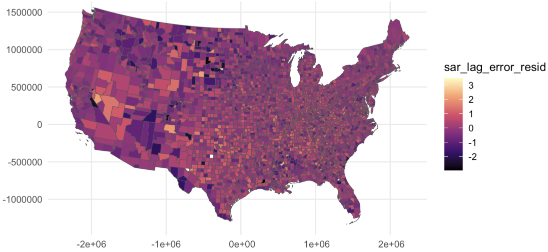
```
Type: sac
Coefficients: (asymptotic standard errors)
             Estimate Std. Error  z value  Pr(>|z|)
(Intercept) -0.136275   0.129906  -1.0490  0.294166
ppersonspo   3.107012   0.228828  13.5780 < 2.2e-16
p65plus     -0.526626   0.226140  -2.3288  0.019872
pwhite      -0.328792   0.127311  -2.5826  0.009806
pblack_1    -0.068370   0.127117  -0.5379  0.590680
phisp       -1.010330   0.087409 -11.5587 < 2.2e-16
punemp_1     2.146500   0.483957   4.4353 9.194e-06

Rho: 0.67872
Asymptotic standard error: 0.015331
    z-value: 44.272, p-value: < 2.22e-16
Lambda: -0.41814
Asymptotic standard error: 0.024425
    z-value: -17.119, p-value: < 2.22e-16

LR test value: 829.33, p-value: < 2.22e-16

Log likelihood: -3102.42 for sac model
ML residual variance (sigma squared): 0.3372, (sigma: 0.58069)
Number of observations: 3067
Number of parameters estimated: 10
AIC: 6224.8, (AIC for lm: 7050.2)
```

SAR Lag-Error Moran's I Residuals

To test the residual spatial autocorrelation, we look at the Moran's I of the residuals:

```
Moran I statistic standard deviate = 0.011781, p-value =0.4953
alternative hypothesis:  greater
sample estimates:
Moran I statistic Expectation Variance
-0.0001285362 -0.0003261579 0.0002813893
```

We fail to reject the null hypothesis of no spatial autocorrelation.



Conditional Autoregressive (CAR) Models — Introduction

- **CAR vs SAR:**
  - **SAR** is specified via the *joint* distribution (and its covariance).
  - **CAR** is specified via *full conditionals*: each Y_i given the others Y_{-i}.
- **Markov property:** Y_i depends on Y_j only if $j \in \mathcal{N}_i$ (neighbors).
- **Use cases:** Widely used in Bayesian disease mapping and hierarchical models to encode local smoothing over areal units.
- **Key object:** neighborhood/weights matrix W (e.g., Queen/rook adjacency).

CAR — Conditional specification (Gaussian)

Assume Gaussian full conditionals:

$$Y_i \mid Y_{-i} \sim N\Big(\mu_i + \sum_{j \in \mathcal{N}_i} c_{ij} (Y_j - \mu_j) \,,\ \sigma_i^2 \Big), \quad \mu_i = x_i^\top \beta .$$

Notes

- $c_{ii} = 0$; $c_{ij} \neq 0$ only for $j \in \mathcal{N}_i$.
- $x_i^\top \beta$ is the mean (trend) at area i; the c_{ij} encode local dependence.
- **Compatibility (to ensure a valid joint Gaussian):** require the symmetry constraint

$$\boxed{\sigma_i^2 c_{ij} \;=\; \sigma_j^2 c_{ji}} \quad \text{for all } i \neq j.$$

This guarantees the set of conditionals corresponds to a unique joint $N(X\beta, Q^{-1})$.

CAR — Joint distribution via precision matrix

Under the compatibility constraints, the joint distribution is

$$Y \sim N\big(X\beta,\ Q^{-1}\big), \qquad Q \text{ is sparse and encodes neighbors (a GMRF).}$$

Common parameterizations (with $D = \mathrm{diag}(w_i)$, $w_i = \sum_j W_{ij}$):

- **Proper CAR (pCAR / Besag–York–Mollié):**

$$\boxed{Q \;=\; \tau\left(D - \rho W\right)}, \quad |\rho| < \frac{1}{\omega_{\max}\left(D^{-1/2} W D^{-1/2}\right)}.$$

- **Intrinsic CAR (ICAR):** $\rho = 1$ in the above $\Rightarrow Q = \tau(D - W)$ is singular (improper prior); use constraints (e.g., sum-to-zero) in Bayesian fitting.
- **Leroux CAR:**

$$Q \;=\; \tau\Big[(1-\rho)I \;+\; \rho(D - W)\Big], \qquad \rho \in [0,1).$$

Interpretation of Q: off-diagonal $Q_{ij} \neq 0$ only for neighbors; the partial correlation

$$\mathrm{Corr}(Y_i, Y_j \mid Y_{-(i,j)}) \;=\; -\frac{Q_{ij}}{\sqrt{Q_{ii} Q_{jj}}}.$$

CAR — Estimation (Gaussian case)

Two common routes

- **Bayesian (most common):** place priors on (β, τ, ρ) and sample/posterior-mode:
  - Packages: `CARBayes`, `R-INLA`, `spBayes`, `brms` (via BYM2/Leroux).
  - ICAR used as a structured random effect with a sum-to-zero constraint.
- **Frequentist ML/REML (proper CAR):** maximize the Gaussian likelihood

$$\ell(\beta, \tau, \rho) = -\tfrac{1}{2}\log|Q| - \tfrac{1}{2}(Y - X\beta)^\top Q\,(Y - X\beta) + \text{const.}$$

  - Sparse Cholesky for Q makes evaluation efficient.
  - In spdep/spatialreg, `spautolm(..., family="CAR")` fits a CAR error structure; in GLMM frameworks, specify Q as a GMRF precision.

Tuning/constraints

- Ensure Q is positive definite (proper CAR) by restricting ρ to the admissible range.
- For ICAR, enforce an identifiability constraint (e.g., $\sum_i u_i = 0$) and interpret u up to a constant.

CAR Models — What can W be?

Key requirements (so the joint $N(X\beta, Q^{-1})$ exists cleanly):

- **Undirected, symmetric:** $W_{ij} = W_{ji} \geq 0$, and $W_{ii} = 0$.
- **Sparsity:** $W_{ij} > 0$ only if $j \in \mathcal{N}_i$ (neighbors).
- **Precision form (examples):** proper CAR $Q = \tau(D - \rho W)$; ICAR $Q = \tau(D - W)$; Leroux $Q = \tau[(1-\rho)I + \rho(D - W)]$, with $D = \operatorname{diag}(w_i)$ and $w_i = \sum_j W_{ij}$.

Common, safe choices

- **Binary adjacency (default in many CAR papers):** $W_{ij} = 1$ if areas share a border (rook) or any boundary point (queen); 0 otherwise.
- **Weighted adjacency (also fine):** use nonnegative, symmetric weights, e.g. *shared boundary length*, *inverse distance within a cutoff*, or *average traffic/flow* — then set $D = \operatorname{diag}(\sum_j W_{ij})$ from those weights.
- *Generally avoid* row-standardizing for CAR, because $Q = \tau(D - \rho W)$ relies on symmetry.
- If you want scale invariance, use **Leroux/BYM2** parameterization or a *scaled ICAR* (software often handles this).

CAR Models — What can W be?

Checklist

- Keep W **symmetric, nonnegative, zero-diagonal**.
- Handle **islands** (no neighbors): connect with kNN or treat separately (ICAR needs constraints).
- If you start with asymmetric weights, **symmetrize**: $W \leftarrow (W + W^\top)/2$.

CAR — How to interpret parameters

- β **(fixed effects):** standard partial-regression effects on $E[Y] = X\beta$ (no lag multiplier on the mean).
- ρ **(spatial strength):** controls how strongly neighbors are tied in the precision Q:
  - Larger $\rho \Rightarrow$ stronger smoothing/borrowing across neighbors.
  - $\rho = 0$ reduces to independent errors with precision τD (in pCAR form).
- τ **(precision / 1-variance scale):** overall strength of the spatial random field (larger τ shrinks toward the trend $X\beta$ more tightly).
- **Local dependence (partial correlations):** sign and magnitude of Q_{ij} describe conditional association between neighboring areas after accounting for all others.
- **Contrast with SAR–lag:** CAR does *not* create "impacts" (no $(I - \rho W)^{-1}$ on the mean). It encodes *covariance/partial-correlation* structure for spatial smoothing.

From conditionals to joint – Brook's lemma & MRF

- A collection of compatible full conditionals $\{p(Y_i \mid Y_{-i})\}$ defines a unique joint density $p(Y)$ up to a constant. For Gaussian CAR, the compatibility constraints yield a joint $N(X\beta, Q^{-1})$ with sparse Q.
- This joint is a **Gaussian Markov Random Field (GMRF)**: $Q_{ij} = 0$ for non-neighbors $\Rightarrow$ conditional independence.
- Practical takeaway: you can work with either
  - conditionals (easier to specify in hierarchical Bayes), or
  - the joint precision Q (easier for computation/penalization).

CAR

- In SAR, the joint pdf $p[Y(s_1), Y(s_2), ..., Y(s_n)]$ determines the full conditional distribution $p[Y(s_i)|Y(s_j), j \neq i]$
- However, when does $p[Y(s_i)|Y(s_j), j \neq i]$ uniquely determine $p[Y(s_1), Y(s_2), ..., Y(s_n)]$?
- Through Brook's lemma we can recover the joint distribution of $Y(s_i)$ from the full conditionals
- Let $y_0 = (y_{10}, ..., y_{n0})$

$$p(y_1, ..., y_n) = \frac{p(y_1|y_2, ..., y_n)}{p(y_{10}|y_2, ..., y_n)} \frac{p(y_2|y_{10}, y_3, ..., y_n}{p(y_{20}|y_{10}, y_3, ... y_n)} \cdots$$
$$\frac{p(y_n|y_{10}, ..., y_{n-1,0})}{p(y_{n0}|y_{10}, ..., y_{n-1,0})} p(y_{10}, ..., y_{n0})$$

- The conditions needed for a set of conditional distributions to define a valid joint distribution are less straightforward than for joint distributions to define conditional distributions (in the Gaussian case).
- A set of conditional distributions defined over spatial neighborhoods and meeting the conditions defines a Markov Random Field, where each observation given the other observations depends only on values at neighboring locations.

Which Autoregressive Model?

- Is there a particular reason why we would choose one over the other?
- The conditional autoregressive model (CAR) is often appropriate for situations with first order dependency or relatively local spatial autocorrelation, and simultaneous autoregressive model (SAR) is more suitable where there are second order dependency or a more global spatial autocorrelation.
- The CAR model obeys the properties of a Markov random field, namely it assumes that the state of a particular area is influenced only by its neighbors and not neighbors of neighbors, etc. (i.e. it is spatially "memoryless"), whereas SAR does not assume such.
- This is due to the different ways in which the variance-covariance matrices are specified.
- So, when the Markov random field property holds, CAR provides a simpler way to model autocorrelated areal data.

Autoregressive vs Mixed Effects Models

- While mixed effects models are seen in a different context than spatial models, they are central to multi-level models and small area estimation, which can be used in the analysis of spatial data.
- The errors in the SAR and CAR models are used to account for between-area variation, following a specified correlation structure. This is typically known as random effects (random effects can change from area to area).
- Mixed effects models can be formulated as:

$$Y = x\beta + Zb + \varepsilon$$

  Where e represents the random effects which are $N(0, \Sigma_e)$and Z accounts for the structure in the random effects. We can set Z to reproduce SAR or CAR specification, but it also must depend on ρ, the spatial parameter.
- In R, we can use the `nlme` package.

Mixed Effects Models — Why in spatial areal data?

- **Goal:** separate *global* effects (fixed) from *area-level* heterogeneity (random).
- **Model skeleton:** $Y = X\beta + Zb + \varepsilon$
  $b \sim N(0, \Sigma_b)$ (area random effects), $\varepsilon \sim N(0, \sigma^2 I)$.
- **Spatial twist:** impose a *structured* Σ_b that encodes neighbor ties:
  - *CAR random effect:* $b \sim N\big(0,\ \tau^{-1} Q^{-1}\big)$ with sparse precision Q from adjacency ($Q = \tau(D - \rho W)$ or Leroux/BYM2).
  - *(Less common) SAR-error as RE:* $u \sim N\big(0,\ \sigma^2 [(I - \lambda W)^\top (I - \lambda W)]^{-1}\big)$.
- **Payoff:** regularized, coherent smoothing at the areal level; easy to extend to *random slopes* (spatially varying coefficients).

Mixed Effects as a unifying view of CAR/SAR

CAR as a random effect (GMRF):

$Y = X\beta + b + \varepsilon, \quad b \sim N\big(0,\ \tau^{-1}Q^{-1}\big),\ Q = \tau(D - \rho W)$ (proper CAR), or Leroux: $Q = \tau[(1-\rho)I + \rho(D - W)]$.

Spatially varying coefficients (SVC):

$$Y = X\beta + \sum_{j=1}^{p} x_j \odot b_j + \varepsilon, \quad b_j \sim N\big(0,\ \tau_j^{-1}Q^{-1}\big),$$

so coefficient j varies by area with CAR smoothing.

Remarks

- Z is often the identity for random intercepts; for random slopes, Z stacks the covariates per area.
- *Estimation:* ML/REML (proper CAR); Bayesian/INLA for ICAR/Leroux/BYM2 or GLM/Poisson outcomes.
- *Contrast to SAR–lag:* CAR/SVC live in the *random effect/variance* part, SAR–lag modifies the *mean* with $(I - \rho W)^{-1}$.

Mixed Effects — Estimation options and choices

Gaussian responses

- **Frequentist:** REML with sparse Q (proper CAR); efficient via sparse Cholesky.
- **Bayesian:** ICAR/Leroux/BYM2 with priors on (β, τ, ρ); works for non-Gaussian too.

Tips

- Prefer **proper CAR/Leroux** for ML/REML; use **ICAR** as a prior with constraints (sum-to-zero) in Bayesian fits.
- Random slopes (SVC) substantially improve flexibility but need shrinkage (penalization via τ_j) to avoid overfitting.
- Compare by AIC/WAIC/LOO and check residual spatial pattern (Moran's I) for adequacy.

Mixed Effects — Interpretation & When to Use

Interpretation:

- β: global fixed effects (partial regression effects).
- b (CAR intercept): area deviations; map $\hat{b}$ to visualize smoothed residual risk.
- SVC b_j: spatial departures from global slope β_j; map $\beta_j + b_{j,i}$.

When to use:

- **Areal smoothing / disease mapping**, small-area estimation, partial pooling across neighbors.
- **Non-stationary relationships:** prefer SVC (random slopes) over geographic weighted regression (next slides) when you want full uncertainty quantification and shrinkage.
- **If diffusion/spillovers (impacts) are the target**, use SAR–lag/SAC instead.

Geographically Weighted Regression (GWR) — Idea

- **Goal:** allow regression coefficients to *vary by location* using local weighted fits.
- **Local model at site i:** $y = X\beta(\mathbf{s}_i) + \varepsilon, \quad \hat{\beta}(\mathbf{s}_i) = \left(X^\top W_i X\right)^{-1} X^\top W_i y.$
- **Weights:** $W_i = \mathrm{diag}\{K(d_{ij}/h)\}$, kernel $K(\cdot)$ decays with distance from site i, bandwidth h controls locality.
- **Output:** a surface for each coefficient $\beta_j(\mathbf{s})$ and local $R^2(\mathbf{s})$.
- The distance in the weighting function d_{ij} is the Euclidean distance between the location of observation i and j, and h is the bandwidth. The bandwidth may be user defined or by minimization of the root mean square prediction error.

GWR — Kernels and bandwidths

Kernels (examples)

$$\text{Gaussian: } K(t) = \exp\big(-\tfrac{1}{2}t^2\big), \quad \text{Exponential: } K(t) = \exp(-t), \quad \text{Bi-square: } K(t) = (1-t^2)^2 \mathbf{1}\{t<1\}.$$

Bandwidth h

- **Fixed:** same distance h everywhere (good for uniform sampling).
- **Adaptive:** include k nearest neighbors at each site (good for uneven densities).

Selecting h

- **CV (LOO)**: minimize $\sum_i (y_i - \hat{y}_i^{(-i)}(h))^2$.
- **AICc**: penalizes complexity; common in practice.

GWR — Diagnostics and mapping

Diagnostics:

- **Local** $R^2(\mathbf{s})$ and **residual maps** to spot where the model fits well/poorly.
- **Local SEs** for $\hat{\beta}(\mathbf{s})$ and **Monte Carlo tests** for spatial variability (is $\beta_j(\mathbf{s})$ non-constant?).
- **Collinearity check:** local condition numbers / VIFs to guard against unstable $\hat{\beta}(\mathbf{s})$.

Mapping:

- Standardize covariates for comparability before mapping coefficient surfaces.
- Use common color scales across maps; mask areas with very large local condition numbers.

GWR — Caveats and alternatives

Caveats

- **Multiple comparisons:** many local tests inflate false positives.
- **Edge effects & bandwidth sensitivity:** results can change with h and kernel.
- **Residual dependence:** GWR does not guarantee spatially independent residuals.
- **Interpretation:** local coefficients are descriptive; avoid causal claims.

Alternatives / complements

- **Spatially varying coefficients (SVC) via mixed/CAR models:** $\beta_j(\text{area}) = \beta_j + b_j$, $b_j \sim \text{CAR}$; includes shrinkage and full uncertainty.
- **Multiscale GWR (MGWR):** allow a different bandwidth per predictor to reflect different spatial scales.
- **SAR/SAC with interactions or regional dummies** if diffusion is the primary mechanism.

When to choose which? (CAR/SAR/SVC/GWR)

- **CAR (random effect smoothing):** areal data; interest in residual risk maps; partial pooling across neighbors.
- **SAR–lag/SAC (impacts):** interest in diffusion/spillovers; report direct/indirect/total impacts.
- **Mixed (SVC):** need coefficient non-stationarity with principled shrinkage and uncertainty.
- **GWR/MGWR:** exploratory mapping of local relationships; quick diagnostics of non-stationarity; follow up with SVC for inference.

The Modifiable Areal Unit Problem (MAUP)

Areal data require aggregation. **MAUP** is the sensitivity of results to how we define areal units.

- **Two components:**
  - **Scale effect** — changing the *level* of aggregation (tract $\rightarrow$ county $\rightarrow$ state) alters statistics (means stable, but variances, correlations, regressions can change in magnitude/sign).
  - **Zoning effect** — at a fixed scale, different *boundary schemes* (contiguity patterns) yield different results.
- Classic empirical observations (Gehlke & Biehl; Openshaw & Taylor): correlations and model coefficients can *increase or decrease* with coarser units; alternative boundary layouts can reverse findings.
- **Implications:** effect sizes, p-values, and even model selection can depend on arbitrary cartography.

Practice: report the *target support* used; perform *sensitivity analyses* over plausible scales/zonings; prefer theory-driven or administrative units when interpretation requires them.

The Ecological Fallacy

Drawing individual-level conclusions from aggregate data is hazardous.

- **Definition:** ecological (aggregate) associations generally *differ* from individual-level associations $\Rightarrow$ individual inference from grouped data can be biased (*ecological bias*).
- **Why it happens:** within-area heterogeneity, confounding that varies across areas, and MAUP (scale/zone choices) distort relationships.
- **Not always wrong, but...** valid individual inference needs *additional assumptions/models* (e.g., multilevel/partial pooling, ecological inference methods). Absent that, treat aggregate results as *group-level* only.

Practice: where possible, use *multilevel models* (individual + area covariates), include *area random effects* (CAR/SVC) to capture contextual variation, and avoid individual claims from purely areal regressions.

Change of Support & Spatial Misalignment

Often, variables live on different *supports* (points vs. polygons; differing polygons). Analyses then require **change of support (CoS)**.

- **Support** = the spatial footprint a value represents (size, shape, orientation).
- **Common tasks:**
  - **Point → Areal (P→A):** *block averaging/kriging* to polygon means.
  - **Areal → Areal (A→A):** *areal interpolation* between incompatible polygons (e.g., tract→health region).
  - **Areal → Point (A→P):** *downscaling* (e.g., dasymetric, area-to-point kriging) to create fine grids/points.
- **Consequences:** changing support changes variance, correlation, and spatial smoothness; uncertainty must be propagated.

Practice: declare the *analysis support* up front; when aggregating or disaggregating, use methods that are *mass-preserving* (totals/means conserved) and quantify CoS uncertainty.

Solutions for Spatial Misalignment

Deterministic / data-driven

- **Areal interpolation (A→A):** simple areal weighting; *dasymetric mapping* using ancillary rasters (land use, pop); *pycnophylactic* (smooth, mass-preserving); regression-based (ancillary covariates).
- **Raster resampling:** nearest/bilinear/cubic (document target resolution; preserve totals when appropriate).
- **Interpolation (P→A):** *block kriging* (Gaussian), *Poisson/area kriging* for counts/rates with offsets.

Model-based

- **Hierarchical Bayesian models:** latent spatial fields (ICAR/Leroux/BYM2 or SPDE) with *change-of-support operators* (integrate latent field over polygons); naturally fuses multiple supports.
- **Data fusion / covariate alignment:** joint models that link variables at different supports via shared latent processes or measurement models.

Checklist

- Be explicit about *source* and *target* supports; ensure *mass conservation* when needed.
- Assess sensitivity to bandwidths/ancillary choices; propagate CoS uncertainty into final intervals.
- Remember MAUP: repeat key analyses at multiple scales/zonings where feasible.