Spatial Statistics

PM599, Fall 2013

Lecture 2: September 9, 2013

Data that varies continuously over space, but is measured only at discrete locations

Examples:

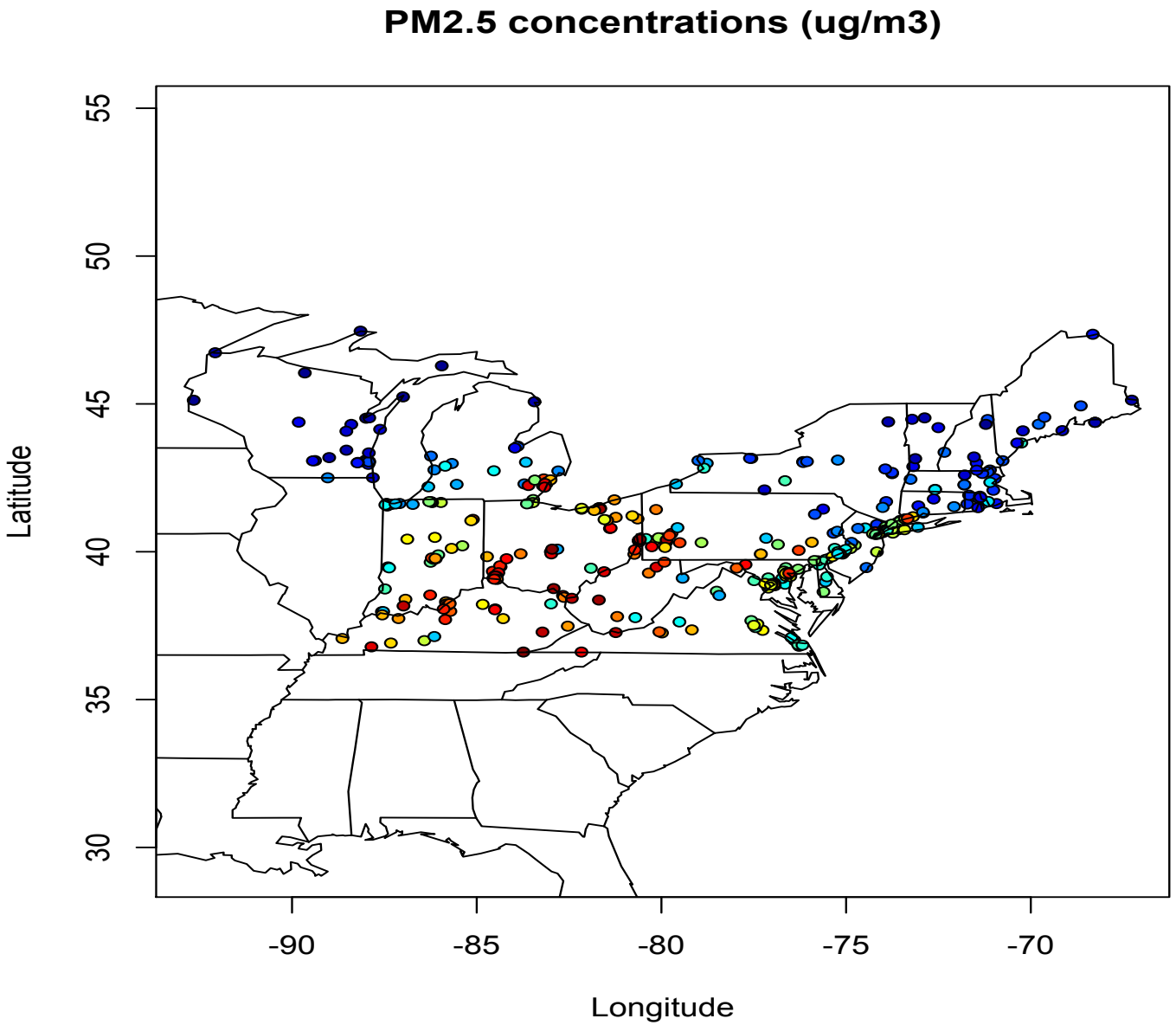
- field observations such as soil samples, air pollution measurements (environmental exposures)
- meteorological and climate data
- housing prices in a metropolitan area

The common thread that links the data is a random process (also called stochastic process or random field)

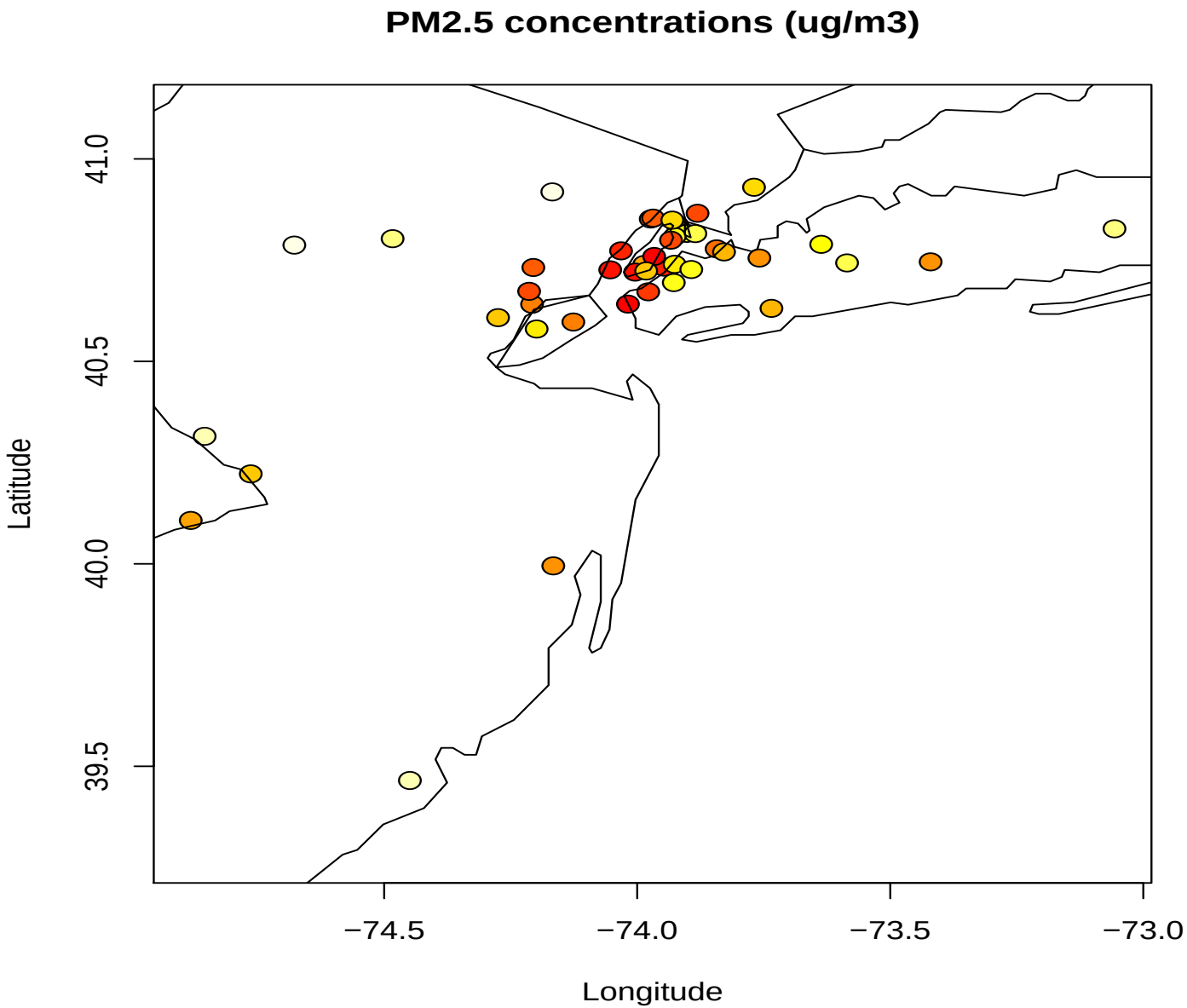
$$Z(\mathbf{s}) : \mathbf{s} \in D$$

where D is a domain in $\Re^d$ (d typically 2)

Geostatistical Data



Geostatistical Data



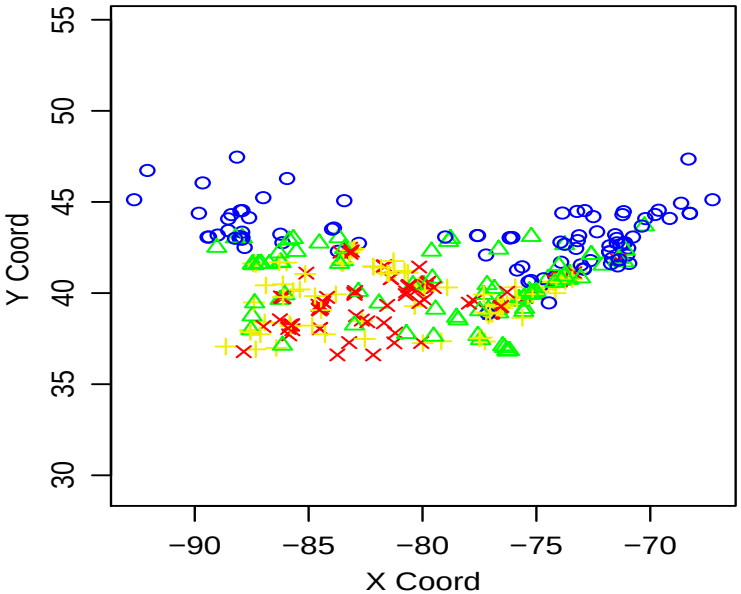
Goals of spatial statistics applied to point referenced data

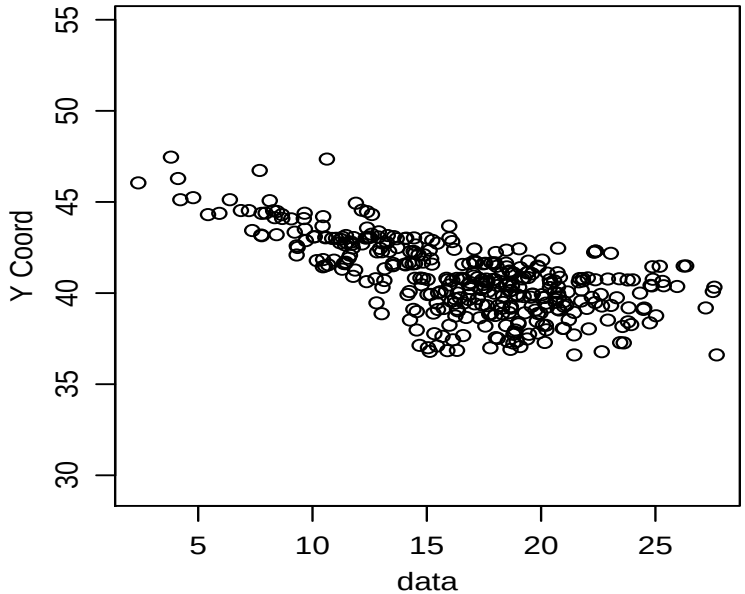
- Determining if there is a spatial pattern in the observations. (Often called spatial "structure")
- Testing null hypothesis of no spatial structure.
- Modeling the spatial correlation/covariance in the observations.
- Making predictions at unobserved locations: interpolation, smoothing.
- Accounting for spatial structure in regression models.
- Determining where new data could be collected.

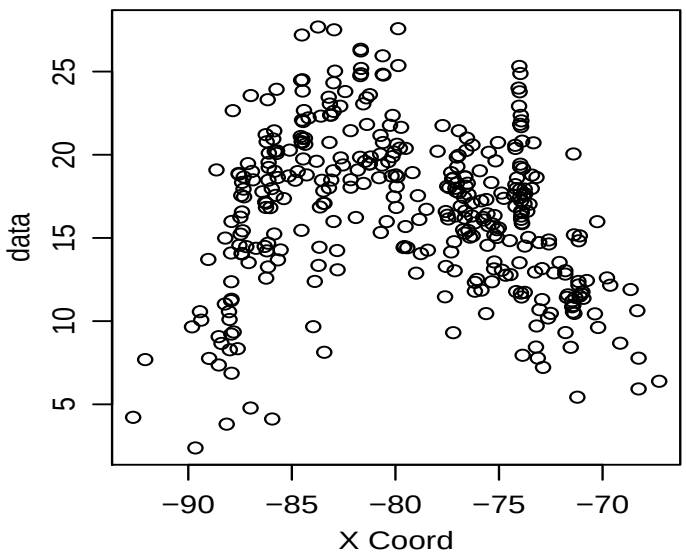
Exploratory Data Analysis

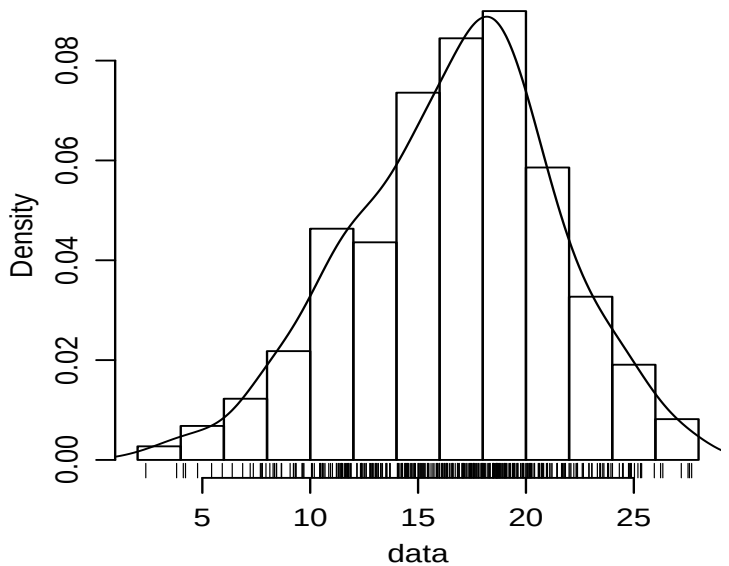
- Exploratory analysis is critial in applied statistics. We want to know what are the spread and distribution of the data, outliers, and in spatial statistics outlying locations
- Might need to transform since many methods rely on normality/symmetry of the data.
- As in regression, assumptions are generally based on residuals and not on original data.
- Log and square-root transformations are most common, primarily used to deal with skewed and non-negative data. However, there are sometimes issues of interpretation in moving back to the original scale if this is required for the analysis. One effect can be underestimation of peaks after back-transformation.

Geostatistical Data









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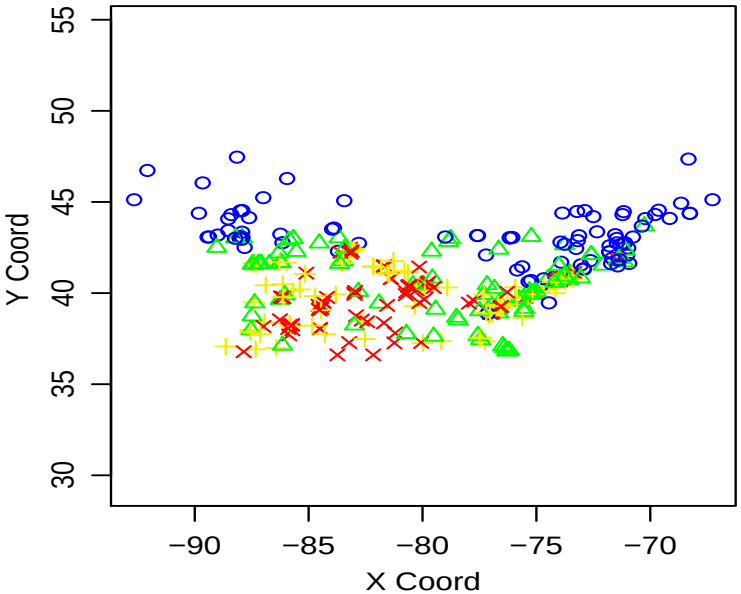
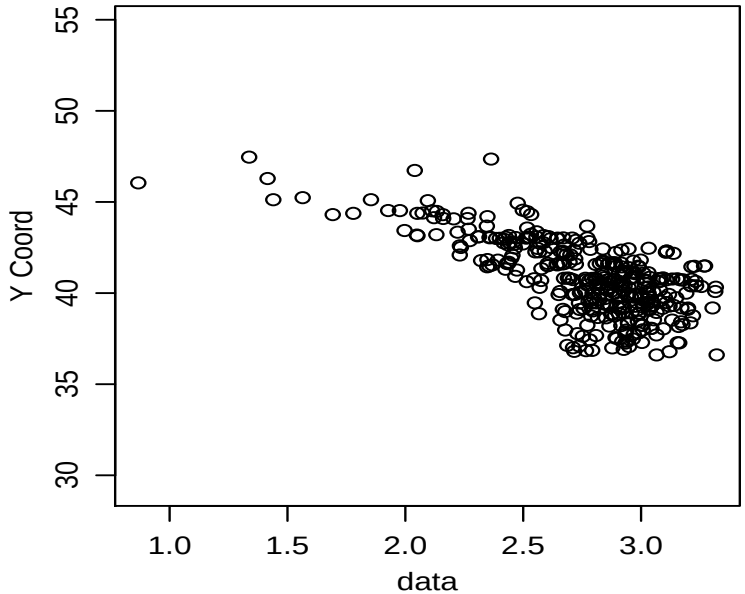
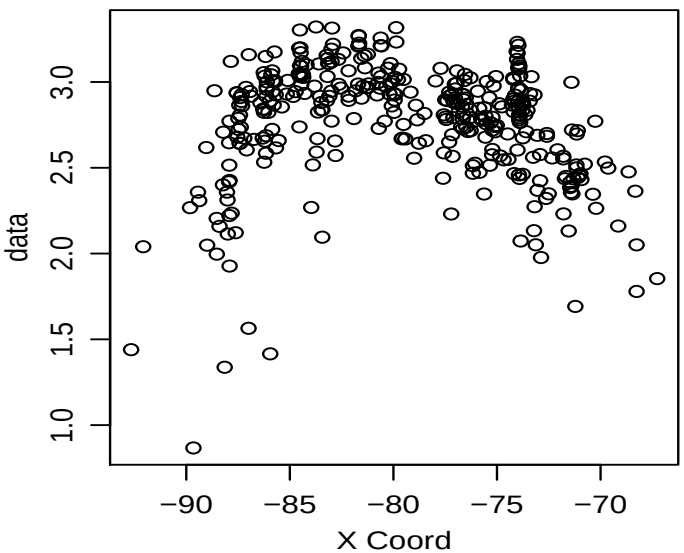
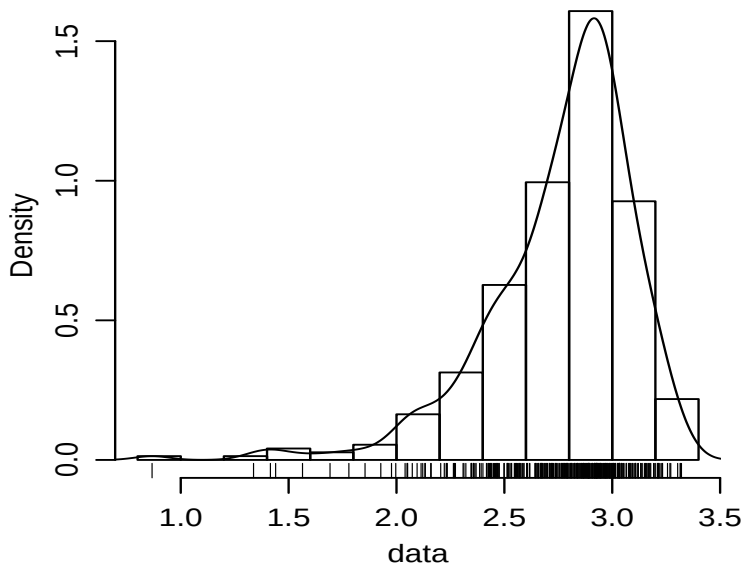








Statistical formulation

- Spatial pattern as a random process: $Z(\mathbf{s}) : \mathbf{s} \in D$ where the spatial domain D is fixed (e.g. Northeast US) and $\mathbf{s}$ are the spatial locations $s_1, s_2, ..., s_n$ in D (e.g. GPS locations of $PM_{2.5}$ monitor). The process is the collection of random variables $Z(\mathbf{s})$ (e.g. $Z(s_1) = PM_{2.5}$ concentration at location s_1)
- Since an infinite number of measurements could have been taken over the domain, D we think of the spatial locations we have measured $Z(\mathbf{s})$ as a realization of the random process

Geostatistical Data

- Understanding spatial structure based on covariance/correlation is more difficult than understanding temporal structure (time series)
- How is distance defined?
- Irregular spacing leads to few (usually one) pair of points for a given distance
- In time, there is only one dimension, but in space there are two, so direction can matter as well as distance

Geostatistical Data

- Covariance tells us whether knowing one observation gives us any information about another observation.
- Covariance allows borrowing of strength for local prediction and estimation, usually increasing efficiency (reduced uncertainty).

$$Cov(X, Y) = E[(X - E(X))(Y - E(Y))]$$
$$Corr(X, Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}}$$

- In spatial setting, covariance is a function of distance

Geostatistical Data

Covariance and correlation for random processes are often called autocovariance and autocorrelation

$$C(h) = E[Z(s) - E(Z(s)) \cdot (Z(s+h) - E(Z(s+h))]$$

$$Cov(Z(s_i), Z(s_j)) = C(s_i - s_j) = C(h)$$

The covariance depends only on the difference between locations s_i and s_j, not on the locations themselves. We will revisit this assumption later.

$$\rho(h) = \frac{C(h)}{C(0)} = \frac{C(h)}{Var(Z(s))}$$

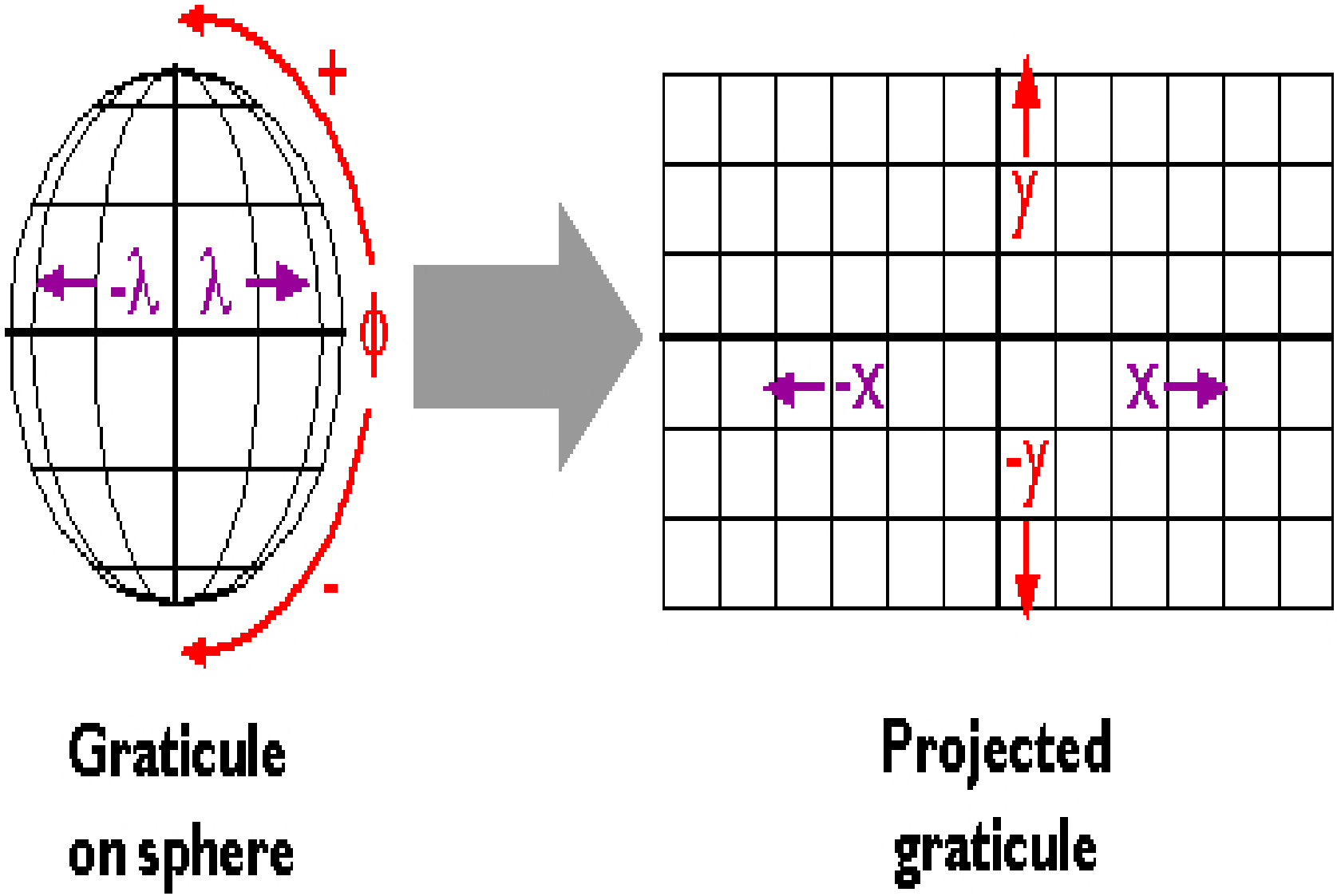
Geostatistical Data

- How is distance, h, defined?
- Several ways to define spatial distances but they must satisfy several technical conditions
  1. symmetry ($d(s_i, s_j) = d(s_j, s_i)$)
  2. the distance between a point and itself is zero
  3. the triangle inequality ($d(s_i, s_j) \leq d(s_i) + d(s_i)$)
- Euclidean distance
  $d(s_i, s_j) = \sqrt{(s_{ix} - s_{jx})^2 + (s_{iy} - s_{jy})^2}$

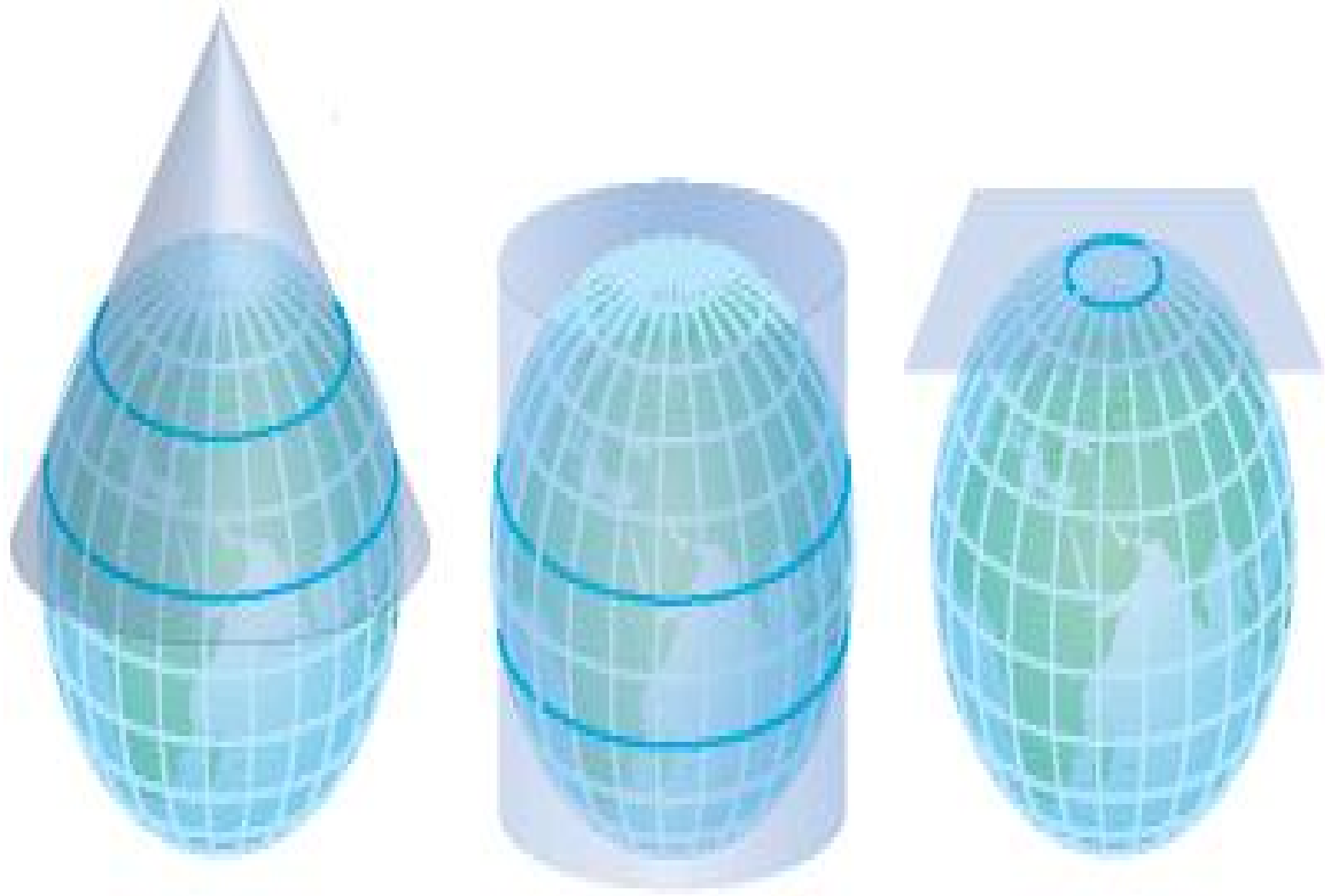
Geostatistical Data

- Must project geographic coordinates from the sphere to a planar projection. Latitude and longitude are not planar. A local projection preserves distances. Examples: the UTM zone system, state plane system, albers equal area
- Manhattan distance: functions on a fixed grid rather than shortest path
- Great circle distance: shortest path on a sphere, good for large distances

Geostatistical Data



Geostatistical Data



Geostatistical Data

- The most widely used quantification of spatial autocorrelation is the **semivariogram**.
- It measures the similarity of values as a function of the distance between their locations.
- Traditional geostatisticians tend to favor the semivariogram over the covariogram/correlogram for historical reasons and because the empirical semivariogram is an unbiased estimator of the true semivariogram, while the covariogram is biased.

Geostatistical Data

- $Var[Z(s+h) - Z(s)] = E[(Z(s+h) - Z(s))^2]$
- This is the expected squared difference between values, which generally increases as a function of the distance between the locations.
- $Var[Z(s+h) - Z(s)] = 2\gamma((s+h) - s) = 2\gamma(h)$
- $2\gamma(h)$ is the variogram and $\gamma(h)$ is the semivariogram

Geostatistical Data

Some additional properties of the semivariogram

- An assumption that is made in spatial analysis is that the spatial process under study repeats itself over the domain D. Such a spatial process is said to be stationary. For a stationary process the absolute coordinates at which we observe the process are unimportant. All that matters are the orientated distances between the points. In a stationary process if we translate the entire set of coordinates by a specific amount in a specified direction, the entire process remains the same.

Geostatistical Data

- It is useful to view spatial data as multivariate, despite it being the same measurement (i.e. $PM_{2.5}$ concentration) at multiple locations
- We have a joint probability density:
  $F(Z(\mathbf{s})) = P(Z(s_1) \leq z_1, Z(s_2) \leq z_2, ..., Z(s_n) \leq z_n)$
- Strong stationarity means that the joint density is invariant under translation: $P(Z(s_1) \leq z_1, Z(s_2) \leq z_2, ..., Z(s_n) \leq z_n) = P(Z(s_1 + h) \leq z_1, Z(s_2 + h) \leq z_2, ..., Z(s_n + h) \leq z_n)$

Geostatistical Data

- A weaker form of stationarity assumes that the moments (mean, variance) of the joint density are invariant.
- Called second-order stationarity
- $E(Z(s)) = \mu$
- $Cov(Z(s+h), Z(s)) = C(h)$
- $C(h)$ only depends on distance, h where C is a covariogram

Geostatistical Data

Some additional properties of the semivariogram

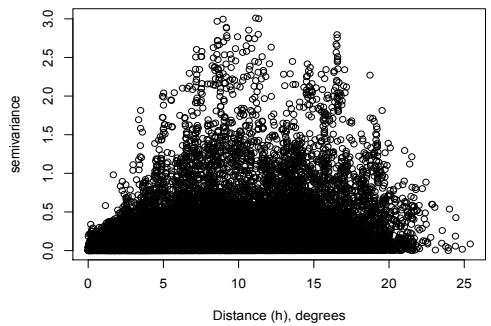
- Stationarity: when the random (spatial) process is stationary it is a function of the spatial lag, or distance only ($\gamma(h)$).
- $\gamma(-h) = \gamma(h)$
- $\gamma(0) = 0$ since $Var(Z(s) - Z(s)) = 0$.
- the spatial process is **isotropic** if $\gamma(h) = \gamma(||h||)$
- the semivariogram and the covariance function are related by:

$$\begin{aligned}
\gamma(h) &= \frac{1}{2}E[(Z(s+h) - Z(s))^2] \\
&= \frac{1}{2}E[((Z(s+h) - \mu) - (Z(s) - \mu))^2] \\
&= -E[(Z(s+h) - \mu)(Z(s) - \mu)] + \frac{1}{2}E[(Z(s+h) - \mu)^2] \\
&\quad + \frac{1}{2}E[(Z(s) - \mu)^2] \\
&= -C(h) + C(0)
\end{aligned}$$

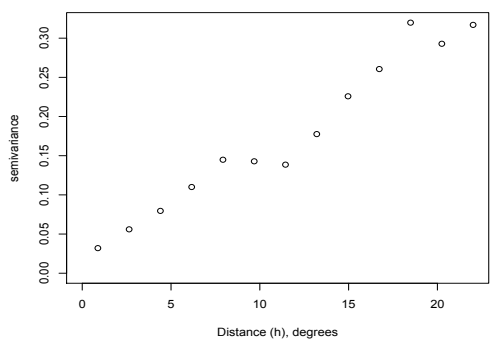
Geostatistical Data

- Spatial processes have the assumption of stationarity, i.e. $E[Z(\mathbf{s})] = \mu$ for all $\mathbf{s} \in D$. This means the mean of the process does not depend on location.
- Stationarity also states that $Cov(Z(s_i), Z(s_j)) = C(s_i - s_j)$. This means that the covariance depends only on the difference between locations s_i and s_j and not on the locations themselves (stationarity). $C(\cdot)$ is the covariance function.

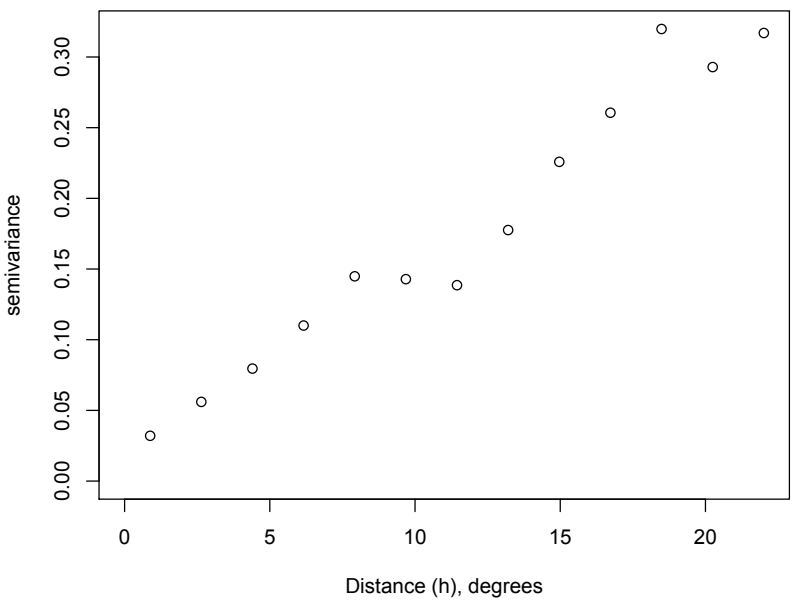
Plot the separation distance $||h||$ vs $\gamma(h)$ for all pairs of points, but this is difficult to interpret.

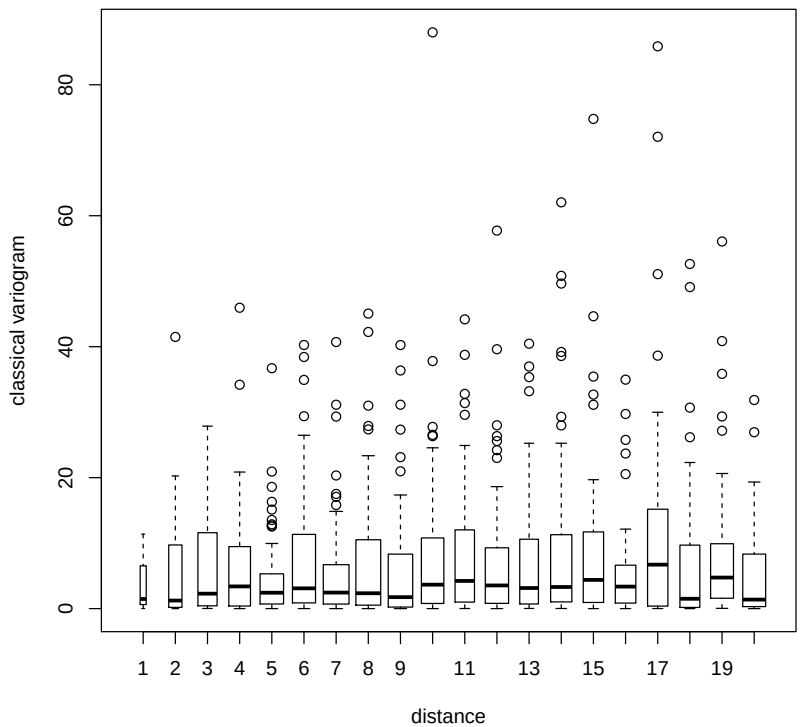


Instead an empirical estimate can be calculated by binning the distances: $\hat{\gamma} = \frac{1}{2N(h)} \sum_{N(h)} [Z(s_i) - Z(s_j)]^2$



- Divide distance into K intervals
  $I_1 = (0, h_1), ..., I_K = (h_{K-1}, h_K)$
- $\hat{\gamma}(h_k) = \frac{1}{2N(h_k)} \sum_{N(h)} [Z(s_i) - Z(s_j)]^2$
- $N(h_k)$ is the set of pairs in the interval I_k

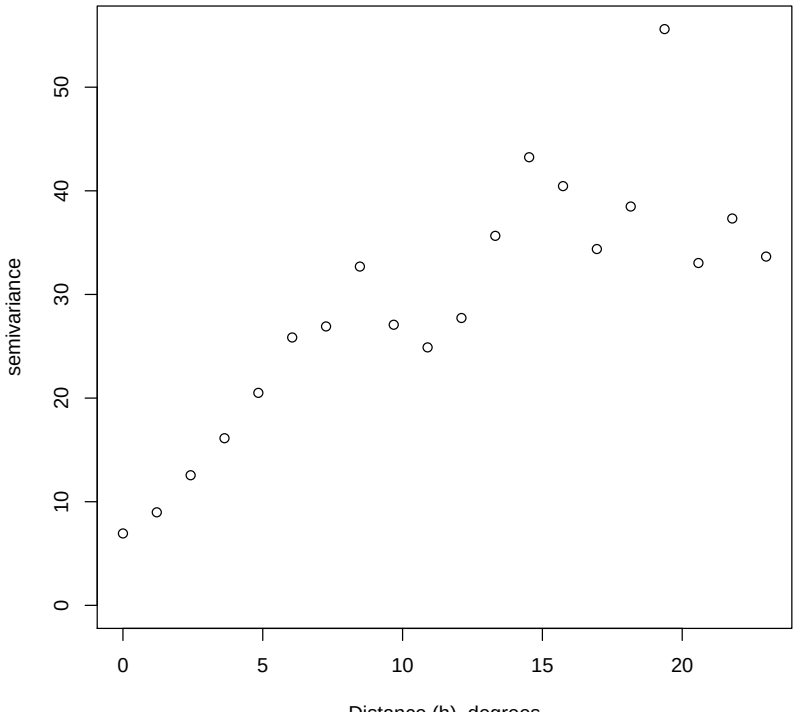


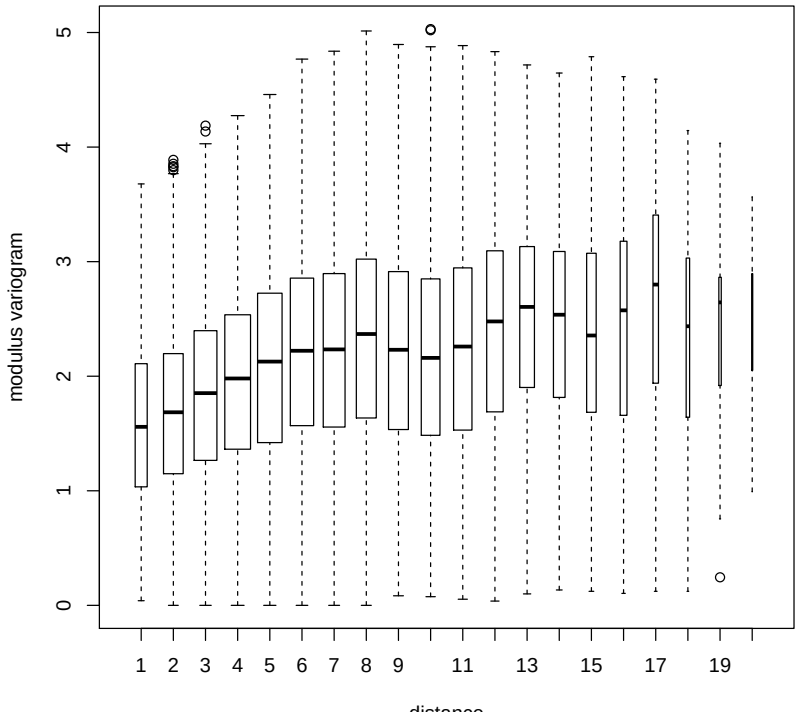


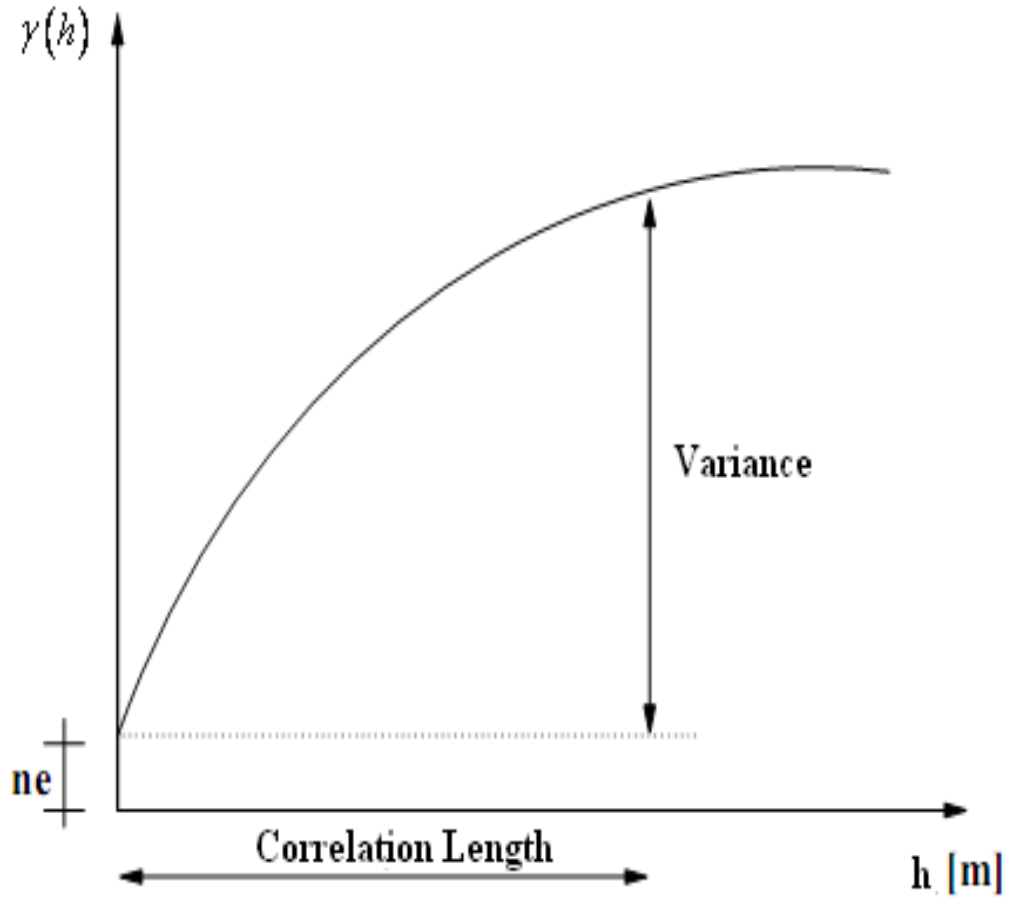
Geostatistical Data

Empirical Semivariogram

- There is a more robust estimate of the variogram by Cressie and Hawkins, 1980
- Less sensitive to outliers
- $\hat{\gamma}(h_k) = \frac{1}{N(h_k)} \sum_{N(h)} |Z(s_i) - Z(s_j)|^{1/2}$
- Again, $N(h_k)$ is the set of pairs in the interval I_k



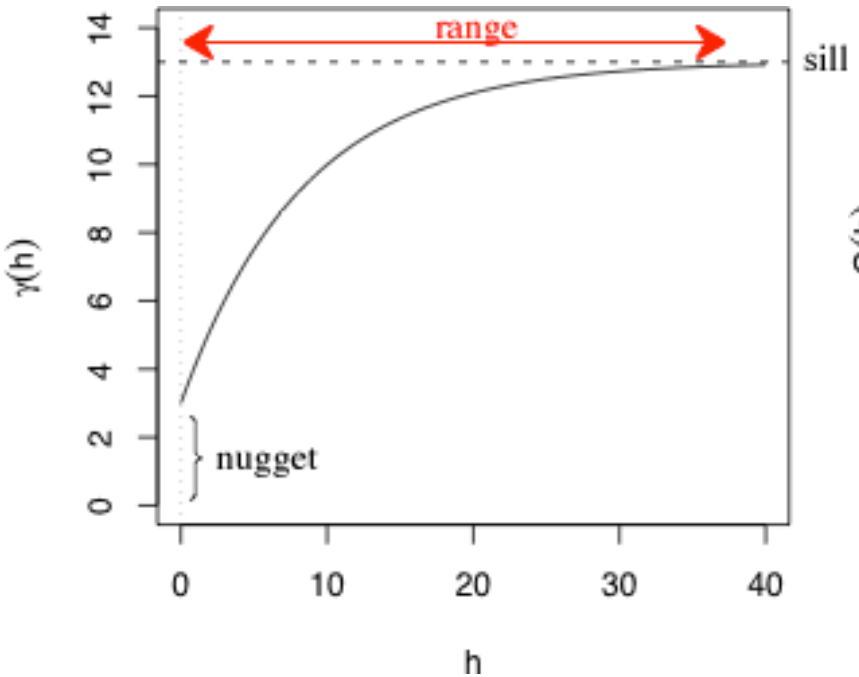


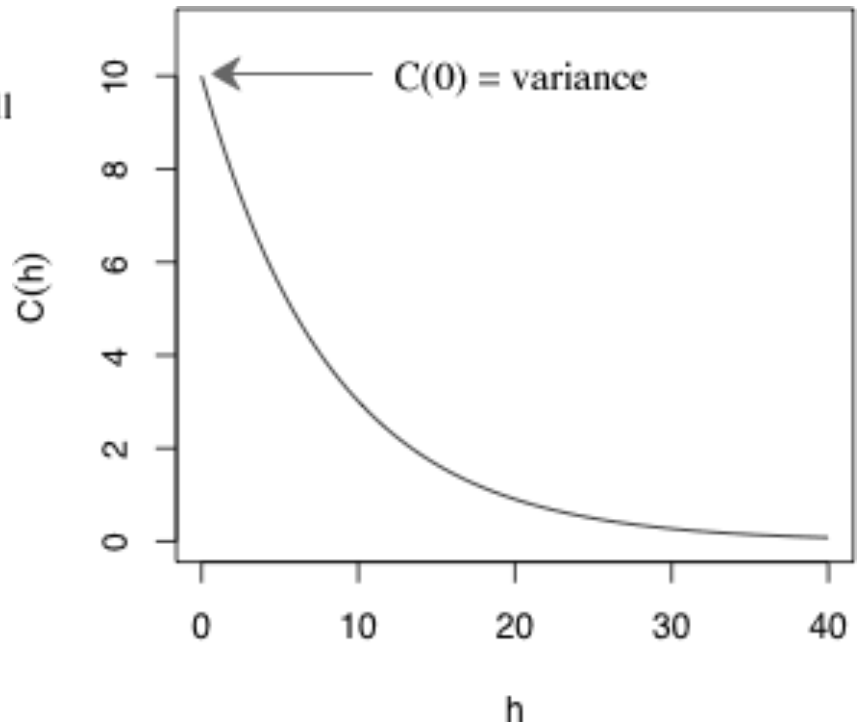


- Observations that are close together are more alike than those far apart: increasing variance in pairwise difference with increasing h means decreasing autocorrelation.

Geostatistical Data

Semivariogram interpretation





- Strength of spatial structure is based on where the semivariogram reaches an asymptote. This distance is called the **range**, ρ.
- The semivariance where the asymptote is reached is the **sill**, σ^2.
- The discontinuity at the origin is called the **nugget**, τ^2.
- The process is stationary if there is stabilization around a value
- Recall that if the process is not stationary $C(h)$ doesn't exist.

We want to fit a theoretical model to our "empirical" semivariogram to describe the shape of the spatial process.
Common parametric semivariogram functions:

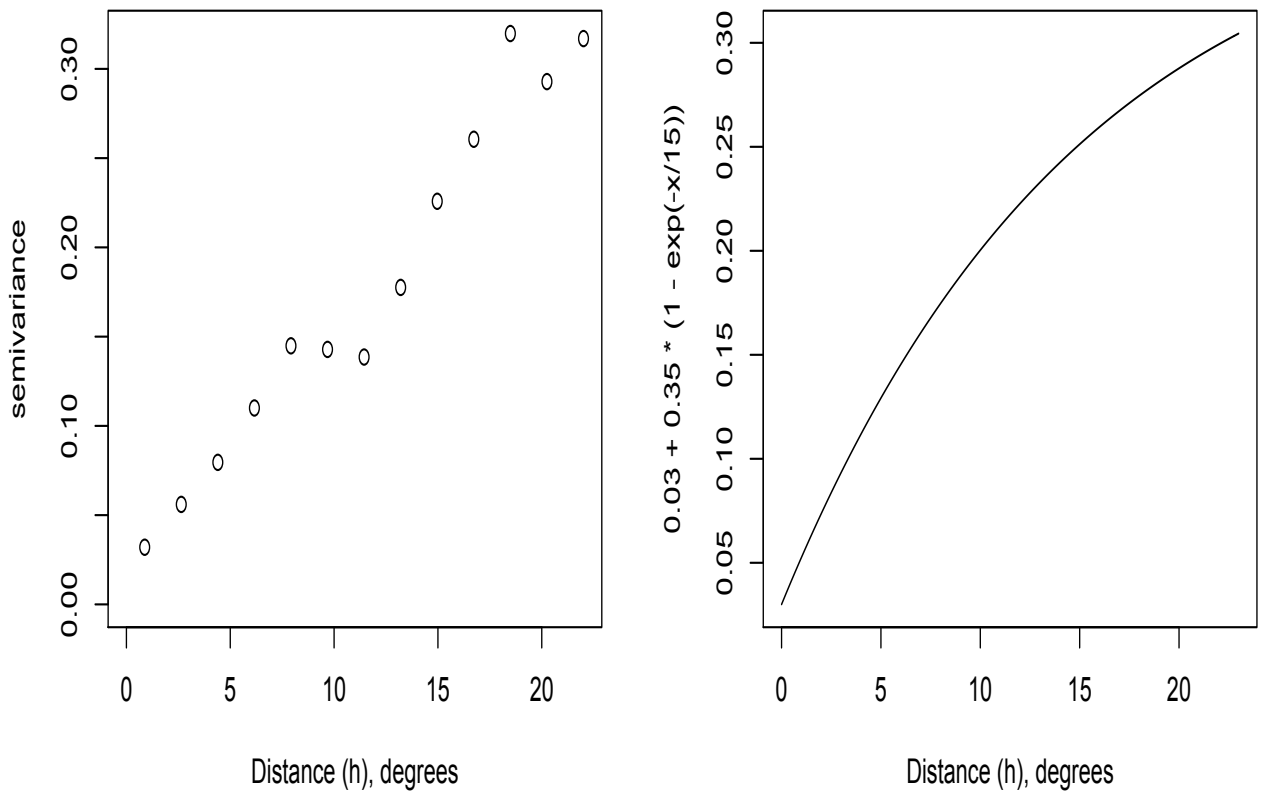
- Exponential
- Spherical
- Gaussian
- Matern

We want to fit a theoretical model to our "empirical" semivariogram to describe the shape of the spatial process. Common parametric semivariogram functions:

- Exponential
- Spherical
- Gaussian
- Matern
- Linear (beware!)

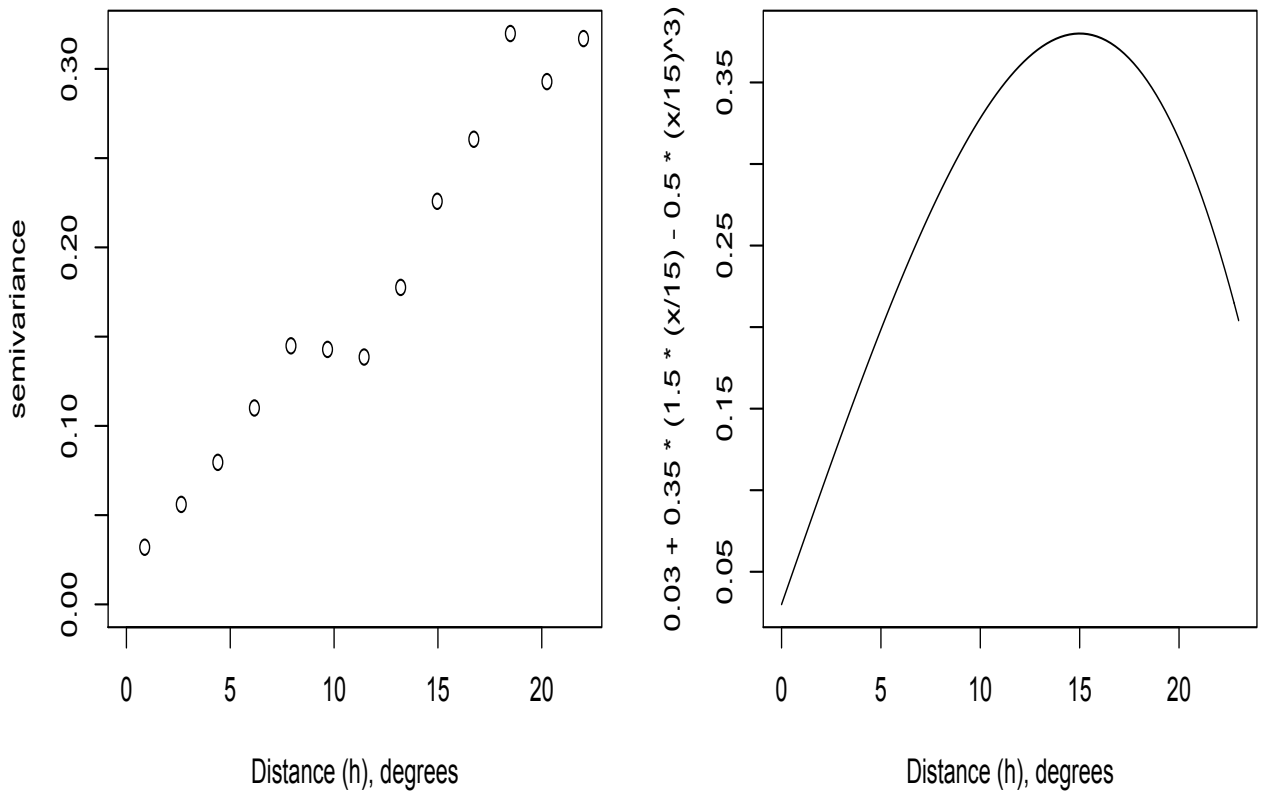
Exponential:

$$\gamma(h) = \tau^2 + \sigma^2(1 - exp(-h/\rho))$$



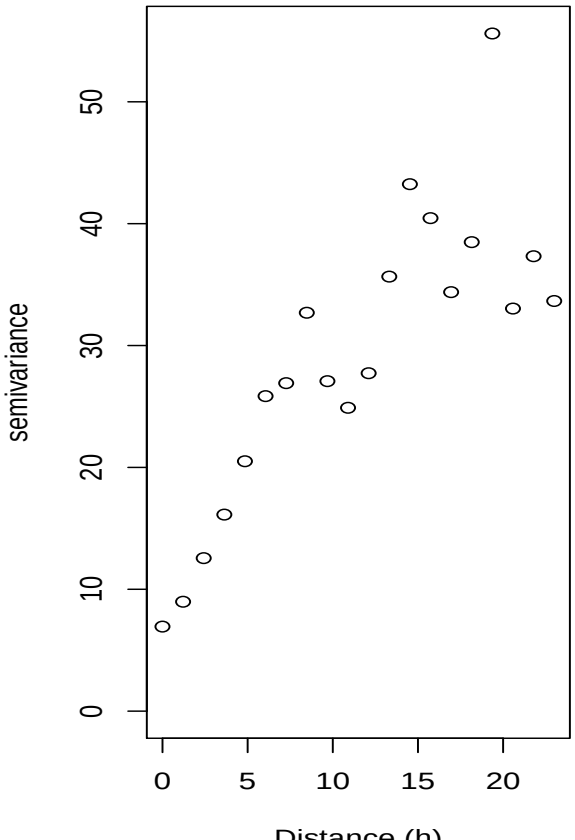
Spherical:

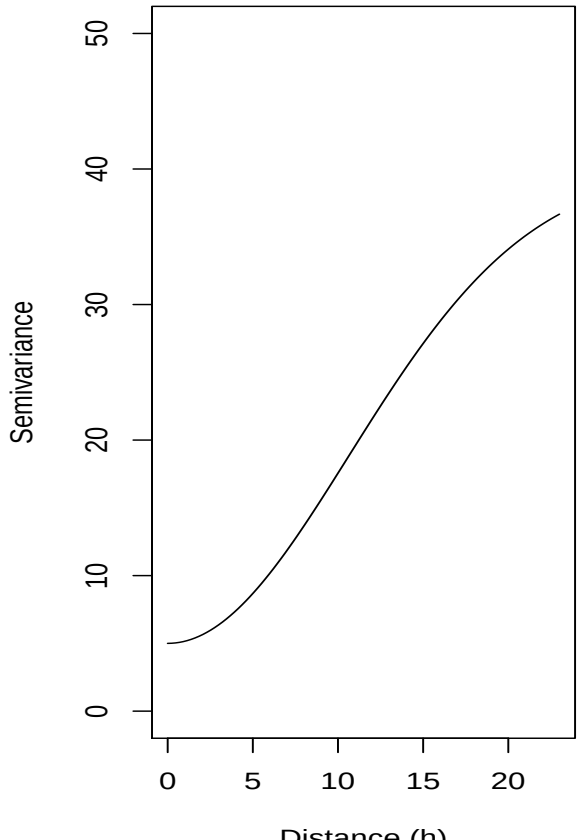
$$\gamma(h) = \tau^2 + \sigma^2(3/2(h/\rho) - 1/2(h/\rho)^3)$$



Gaussian:

$$\gamma(h) = \tau^2 + \sigma^2(1 - exp(-\frac{h^2}{\rho^2}))$$





Eyeballing the semivariogram is useful for exploratory purposes and to find the approximate shape of the spatial process, but we would rather find a valid theoretical semivariogram function that reflects the empirical semivariogram.
Ordinary Least Squares: find the parameters $\theta = (\tau^2, \sigma^2, \rho)$ that minimize the squared vertical distance between the empirical and theoretical semivariograms.

$$(\hat{\gamma}(h) - \gamma(h;\theta))^T (\hat{\gamma}(h) - \gamma(h;\theta))$$

where $\hat{\gamma}(h)$ is the empirical semivariogram and $\gamma(h;\theta)$ is the theoretical semivariogram with parameters θ
We will cover this in the next lecture.

Isotropy means that the semivariance depends only on the distance between points, not direction.
Anisotropy means the semivariance also depends on direction as well as distance.
We can examine anisotropy with a **directional semivariogram**.