Discrete Structures

Syed Faisal Bukhari, PhD
Associate Professor
Department of Data Science (DDS), Faculty of Computing and Information Technology (FCIT), University of the Punjab (PU)

Text book

Discrete Mathematics and Its Application, 7th Edition

Kenneth H. Rosen

References

Chapter 5

1. Discrete Mathematics and Its Application, 7^h Edition

By Kenneth H. Rose

2. Discrete Mathematics with Applications

By Thomas Koshy

These slides contain material from the above resources.

Principle of Mathematical Induction

To prove that $P(n)$ is true for all positive integers n , where $P(n)$ is a propositional function, we complete two steps:

Basis Step: We verify that $P(1)$ is true.

Inductive Step: We show that the conditional statement

$P(k) \rightarrow P(k+1)$ is true for all positive integers k.

Expressed as a **rule of inference**, this proof technique can be stated as

$(P(1) \wedge \forall k\,(P(k) \rightarrow P(k+1))) \rightarrow \forall n\, P(n)$, when the domain is the set of positive integers.

Example: Conjecture a formula for the sum of the first n positive odd integers. Then prove your conjecture using mathematical induction.

Solution

$1 + 3 + 5 + \ldots + (2n - 1) = n^2$

Let P(n) be the proposition that the sum sums of the first n positive odd integers is n^2

Basis Step: P(1) is true

$\because 2(1) - 1 = (1)^2 \Rightarrow 1 = 1$

Inductive Step: Let it will be true for k

$1 + 3 + 5 + \ldots + (2k - 1) = k^2$

Under this assumption, it must be shown that **P(k + 1) is true**

Adding **2(K+1) -1** on both sides

$1 + 3 + 5 + \ldots + (2k - 1) + \mathbf{2(k+1) - 1} = k^2 + \mathbf{2(k+1) - 1}$

$1 + 3 + 5 + \ldots + (2k - 1) + 2(k + 1) - 1 = k^2 + 2k + 1$

$1 + 3 + 5 + \ldots + (2k - 1) + 2(\overline{\boldsymbol{k} + \mathbf{1}}) - 1 = (\overline{\boldsymbol{k} + \mathbf{1}})^2$

Consequently, by the principle of mathematical induction we can conclude that P (n) is true for all positive integers n. That is, we know that $1 + 3 + 5 + \ldots + (2n - 1) = n^2$ for all positive integers n.

Example Use mathematical induction to show that $1 + 2 + 2^2 + . . . + 2^n = 2^{n+1} - 1$ for all **nonnegative** integers n.

Solution:

Let P(n) be the proposition that $1 + 2 + 2^2 + . . . + 2^n = 2^{n+1} – 1$ for the integer n.

Basis Step: P(0) is true

$\because 2^0 = 2^{0+1} – 1 \Rightarrow 1 = 1$

Inductive Step: Let it will be true for k

$1 + 2 + 2^2 + . . . + \mathbf{2^k} = 2^{k+1} – 1$

Under this assumption, it must be shown that P(k + 1) is true

Adding $\mathbf{2^{k+1}}$ on both sides

Cont.

$1 + 2 + 2^2 + . . . + 2^k \mathbf{+ 2^{k+1}} = 2^{k+1} – 1 \mathbf{+ 2^{k+1}}$

$1 + 2 + 2^2 + . . . + 2^k + 2^{k+1} = 2.2^{k+1} – 1$

$1 + 2 + 2^2 + . . . + 2^k + \mathbf{2^{\overline{K+1}}} = \mathbf{2^{\overline{K+1}+1}} – 1$

We have completed the **basis step** and the **inductive step**, by mathematical induction we know that **P(n)** is true for all nonnegative integers n.

That is, $1 + 2 + 2^2 + . . . + 2^n = 2^{n+1} – 1$ for all nonnegative integers n.

Example Sums of **Geometric Progressions.** Use mathematical induction to prove this formula for the sum of a finite number of terms of a geometric progression:

$\sum_{j=0}^{r} ar^{j} = a + ar + ar^2 + \ldots + ar^n = \frac{ar^{n+1} - a}{r-1}$ where $r \neq 1$,

where n is a nonnegative integer.

Solution

Let P(n) be the statement that the sum of the first n + 1 terms of a geometric progression in this formula is correct.

Basis Step: P(0) is true, because

$$ar^{o} = \frac{ar^{o+1} - a}{r-1} \Rightarrow a = a$$

Inductive Step: Let it will be true for k

$$a + ar + ar^2 + \ldots + ar^k = \frac{ar^{k+1} - a}{r-1}$$

Under this assumption, it must be shown that P(k + 1) is true

Adding ar^{k+1} on both sides

$$a + ar + ar^2 + \ldots + ar^k + ar^{k+1} = \frac{ar^{k+1} - a}{r - 1} + ar^{k+1}$$

$$a + ar + ar^2 + \ldots + ar^k + ar^{k+1} = \frac{ar^{k+1} - a + (r - 1)ar^{k+1}}{r - 1}$$

$$a + ar + ar^2 + \ldots + ar^k + ar^{k+1} = \frac{ar^{k+1} - a + ar^{k+1+1} - ar^{k+1}}{r - 1}$$

$$a + ar + ar^2 + \ldots + ar^k + ar^{\overline{K+1}} = \frac{ar^{\overline{K+1}+1} - a}{r - 1}$$

We have completed the **basis step** and the **inductive step**, so by mathematical induction P(n) is true for all nonnegative integers n. This shows that the formula for the sum of the terms of a geometric series is correct.

Example Use mathematical induction to prove the inequality **$n < 2^n$** for all positive integers n.

Solution

$n < 2^n$

Basis Step: P (1) is true, because $1 < 2^1 \Rightarrow 1 < 2$

Inductive Step: Let it will be true for n = k

$k < 2^k$

Under this assumption, it must be shown that P(k + 1) is true

Adding 1 on both sides

$k \mathbf{+ 1} < 2^k \mathbf{+ 1}$

$\Rightarrow k + 1 < 2^k + 2^k$ $\because \mathbf{1 \le 2^k}$

$\Rightarrow k + 1 < 2.2^k$

$\Rightarrow \overline{\boldsymbol{k+1}} < \mathbf{2}^{\overline{\boldsymbol{k+1}}}$

We have completed both the basis step and the inductive step, by the principle of mathematical induction we have shown that n < 2^n is true for all positive integers n

Example Use mathematical induction to prove that $\mathbf{2^n < n!}$ for every positive integer n with $\mathbf{n \geq 4}$. (Note that this inequality is false for $n = 1$, 2, and 3.)

Let P(n) be the proposition that $2^n < n!$

Basis Step: To prove the inequality for $n \geq 4$ requires that the basis step be

P (4). Note that P (4) is true, because

$2^4 < 4!$

$16 < 24$

Inductive Step: For the inductive step, we assume that P(k) is true for the positive integer k with **$k \geq 4$**.

$2^k < k!$ ---------------------------------------(1)

We have to show to that $2^{k+1} < (k + 1)!$. Multiply (1) by 2

$2 \times 2^k < 2 \times k!$

$2^{k+1} < 2 \times k!$ $\quad\because 2^{k+1} = 2 \times 2^k$

$2^{k+1} < (k+1)k!$ $\quad\because 2 < k + l$

$2^{k+1} < (k+1)!$ $\quad\because (k + 1)! = (k + 1)k!$

This shows that P(k + 1) is true when P (k) is true. This completes the inductive step of the proof. Hence P(n) is true for positive integers greater than equal to 4.

Example Use mathematical induction to prove that $n^3 - n$ is divisible by 3 whenever n is a positive integer.

Solution

Let $P(n) = n^3 - n$

Basis Step: P(1) is true

$\because P(1) = 1^3 - 1 = 0$, which is divisible by 3

Inductive Step: Let it will be true for n= k

$P(k) = \mathbf{k^3 - k}$

We have to show that $(k + 1)^3 - (k + 1)$ is divisible by 3

$P(k + 1) = (k + 1)^3 - (k + 1)$

$P(k + 1) = k^3 + 3k^2 + 3k + 1 - k - 1$

$P(k + 1) = k^3 - k + 3k^2 + 3k$

$P(k + 1) = \mathbf{k^3 - k} + \mathbf{3k(k+1)}$

P(k + 1) = first term is divisible by 3 + second term is divisible by 3

P(k + 1) = sum is divisible by 3

We have completed the basis step and the inductive step, so P(n) is divisible by 3 for all positive integral values of n.

Suggested Readings

5.1 Mathematical Induction