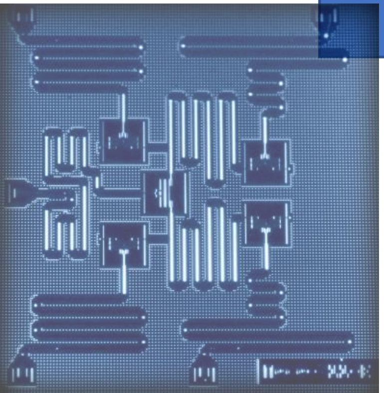
7. From the code-world to reality: physical implementation

Quantum Computing



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Requirements: DiVincenzo criteria

1. A **scalable** system with well characterized **qubits**
2. The ability to **initialize** the system in a simple fiducial state, such as $|00 \cdots 0\rangle$
3. **Long decoherence times**, much longer than the gate operation time
4. A **universal set** of quantum gates
5. A qubit-specific **measurement** capability

https://arxiv.org/abs/quant-ph/0002077

Examples of architectures

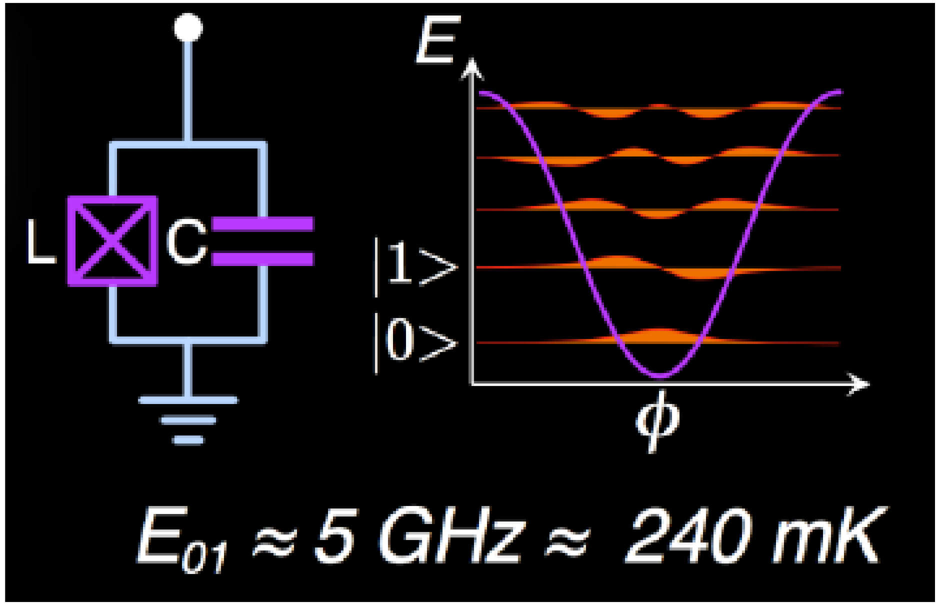
Trapped-ions quantum computer

https://ionq.com/technology

Superconducting circuits

https://www.rigetti.com/

https://www.ibm.com/quantum-computing/learn/what-is-ibm-q/

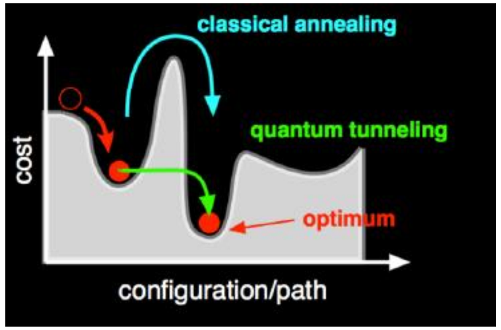


Steps of development

Quantum Annealer

- Optimization problems

- No clear quantum advantage
- Large number of noisy qubits



Approximate NISQ QC

- Material discovery
- Quantum chemistry
- Quantum simulation
- Optimization

- Hybrid quantum-classical approach
- Good advantage with 50-100 qubits
- Need to **understand and mitigate errors**

Fault-tolerant Universal QC

- Arbitrary quantum algorithms:
  Binary combinatorial problems
  Cryptography
  Digital quantum simulation

- Quantum error correction needed
- Significant hardware overhead

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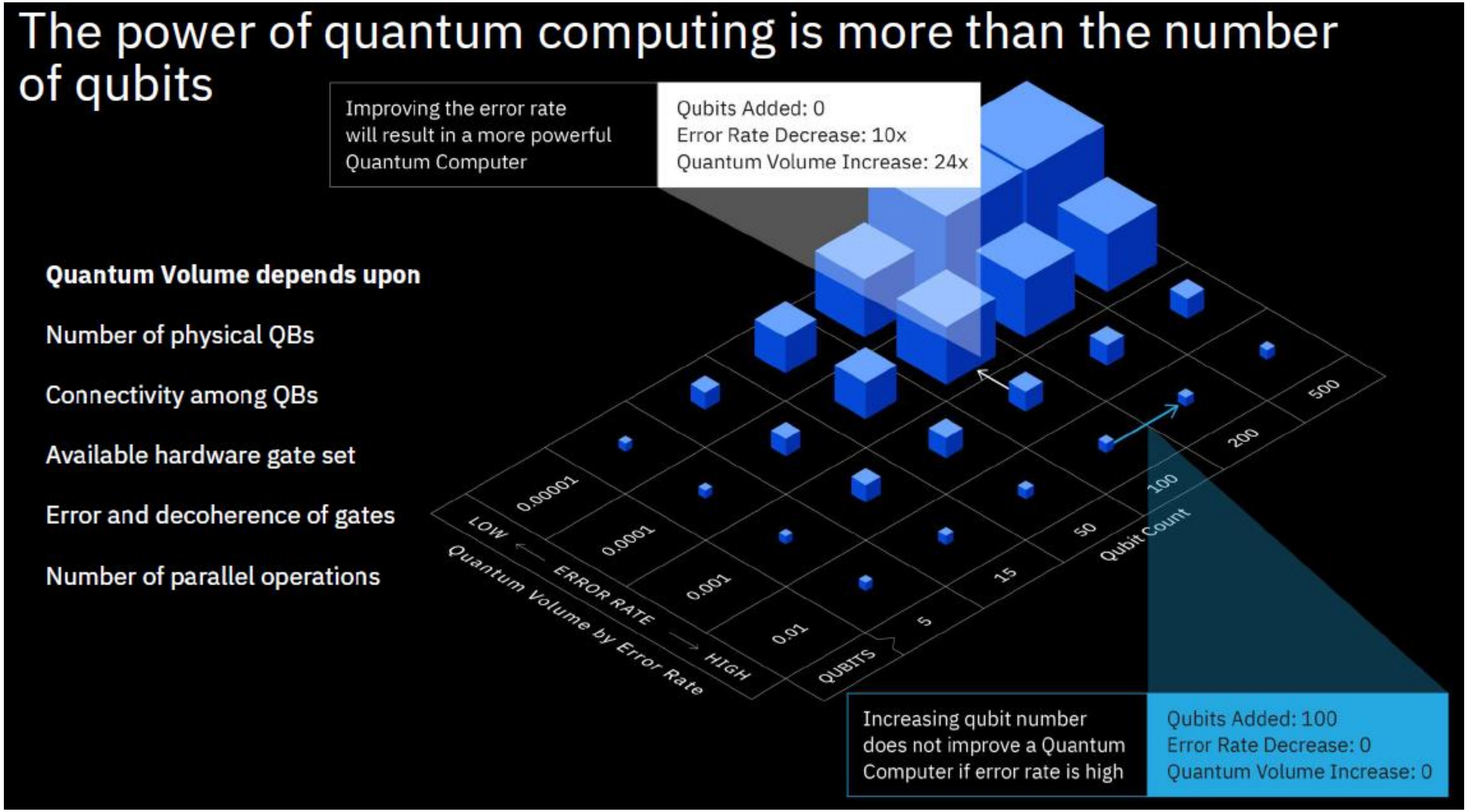
Quantum Computing

Quantum Volume



Mixed states

Sometimes we do **not** have **enough information** to specify the state vector of a quantum system (e.g. in presence of **environmental decoherence**), but we know the probabilities $\mathcal{P}_n$ to be in a state $|\psi_n\rangle$. We can then express the mean value of an operator A as

$$\bar{A} = \sum_n \mathcal{P}_n \langle\psi_n|A|\psi_n\rangle$$

Different from the expectation value $\langle\psi|A|\psi\rangle$, in which case we **know** precisely $|\psi\rangle$

Density operator $\rho = \sum_n \mathcal{P}_n |\psi_n\rangle\langle\psi_n|$ **mixed** state $\xrightarrow[\mathcal{P}_n = \delta_{nk}]{}$ $\rho = |\psi_k\rangle\langle\psi_k|$ **pure** state $|\psi_k\rangle$

$$\boxed{\mathrm{Tr}[\rho A]} = \sum_m \langle\psi_m| \textstyle\sum_n \mathcal{P}_n |\psi_n\rangle\langle\psi_n|A|\psi_m\rangle =$$
$$= \textstyle\sum_{n,m} \mathcal{P}_n \langle\psi_m|\psi_n\rangle \langle\psi_n|A|\psi_m\rangle =$$
$$= \textstyle\sum_n \mathcal{P}_n \langle\psi_n|A|\psi_n\rangle = \boxed{\bar{A}}$$

Properties of the density operator:

- $\mathrm{Tr}[\rho] = \sum_n \mathcal{P}_n = 1$
- Positive: $\langle\chi|\rho|\chi\rangle = \sum_n \mathcal{P}_n |\langle\chi|\psi_n\rangle|^2 \geq 0$
- For **pure** states $\mathrm{Tr}[\rho^2] = 1$

Density operator

The postulates of quantum mechanics can be re-formulated in terms of the density operator. Density operators allow us to describe **ensembles of quantum states**. This is particularly useful if

(i) the state of the system is unknown

(ii) we aim to describe a subsystem of a composite quantum system (see chapter 4).

In a closed quantum system, the time evolution of ρ is described by the unitary operator $U(t, t_0)$. Indeed,

$$\rho(t_0) = \sum_n \mathcal{P}_n \, |\psi_n(t_0)\rangle\langle\psi_n(t_0)|$$

$$\rho(t) = \sum_n \mathcal{P}_n \, U(t,t_0)|\psi_n(t_0)\rangle\langle\psi_n(t_0)|U^\dagger(t,t_0) = U(t,t_0)\rho(t_0)U^\dagger(t,t_0)$$

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H, \rho(t)]$$

Density operator

Quantum measurements are described by a set of measurement operators P_k projecting the state of the system onto subspaces corresponding to the measurement outcomes a_k. The probability that the result a_k occurs is given by

$$p_k = \mathrm{Tr}[P_k \rho]$$

And the state of the system after the measurement is $$\frac{P_k \rho P_k^\dagger}{\mathrm{Tr}[P_k \rho]}$$

with measurement (projector) operators satisfying the completeness relation $\sum_k P_k^\dagger P_k = \mathbb{I}$

We have introduced two completely different time evolutions for the quantum system: measurements induce an instantaneous, irreversible projection of the state, whereas the dynamics described by Schrödinger equation is unitary and reversible.

Partial trace

It is not possible to describe part of a physical system by as state vector.
Let AB be a composite quantum system consisting of two subsystems A and B and described by state operator ρ^{AB} in state space $\mathcal{H}_A \otimes \mathcal{H}_B$. If $\mathcal{M}$ is an observable on subsystem A represented by the operator $M = M_A \otimes \mathbb{I}_B$

$$\langle M \rangle = \mathrm{Tr}[\rho^{AB} M] = \sum_{ij\in A} \sum_{\mu\nu\in B} \langle i,\mu|\rho^{AB}|j,\nu\rangle \underbrace{\langle j,\nu|M_A \otimes \mathbb{I}_B|i,\mu\rangle}_{\langle j|M_A|i\rangle \delta_{\mu\nu}} = \sum_{ij\in A} \sum_{\mu\in B} \langle j|M_A|i\rangle\langle i,\mu|\rho^{AB}|j,\mu\rangle$$

$$\langle i|\rho^A|j\rangle = \sum_{\mu\in B} \langle i,\mu|\rho^{AB}|j,\mu\rangle \qquad \rho^A = \mathrm{Tr}_B[\rho^{AB}] \qquad \Longrightarrow \langle M \rangle = \mathrm{Tr}[\rho^A M_A]$$

Reduced density operator

Partial trace with respect to B

Decoherence

$$|\Psi\rangle = \alpha|0_A 1_B\rangle + \beta|1_A 0_B\rangle \qquad \rho^A = \mathrm{Tr}_B|\Psi\rangle\langle\Psi| = |\alpha|^2\,|0_A\rangle\langle 0_A| + |\beta|^2\,|1_A\rangle\langle 1_A| = \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix}$$

All information about phases of complex numbers α, β lost!

if a pair of states of the system of interest becomes correlated with mutually orthogonal states of another system, then all the phase coherence between the orthogonal states of the first system is lost

$\Rightarrow$ **DECOHERENCE**

REMARKS:

1. In another basis set ρ^A could be not diagonal. Try to write it in the basis $|\pm_A\rangle = (|0_A\rangle \pm |1_A\rangle)/\sqrt{2}$
2. Phase information is only **locally** lost.
3. Coherences can be dynamically recovered, unless we lose control on some of the quantum variables, e.g. if one subsystem is an environment containing many degrees of freedom.

Decoherence

We can model the interaction of the quantum system with the environment by adding a term to the equation of motion for ρ:

$$\frac{d\rho(t)}{dt} = -\frac{i}{\hbar}[H, \rho(t)] + \mathcal{D}[\rho]$$

Here $\rho = \mathrm{Tr}_E \rho^{SE}$ is the **REDUCED** density matrix on the system, having traced the environmental degrees of freedom.

Unitary evolution

Non-unitary evolution due to system-environment interaction

It makes our computer **QUANTUM**

HARMFUL: It destroys the quantumness of our computer on a time-scale set by the *decoherence time*

It must be reduced as much as possible on devices

Phase damping channel

$$|0_A 0_E\rangle \longrightarrow \sqrt{1-p}|0_A 0_E\rangle + \sqrt{p}|0_A 1_E\rangle = |0_A\rangle \otimes \left(\sqrt{1-p}|0_E\rangle + \sqrt{p}|1_E\rangle\right)$$

$$|1_A 0_E\rangle \longrightarrow \sqrt{1-p}|1_A 0_E\rangle + \sqrt{p}|1_A 2_E\rangle = |1_A\rangle \otimes \left(\sqrt{1-p}|0_E\rangle + \sqrt{p}|2_E\rangle\right)$$

The state of the qubit does not change, but the state of the environment changes depending on the state of the qubit. $|0_A\rangle$ and $|1_A\rangle$ do not become entangled with the environment (pointer states) but a superposition does.

$$|\Psi\rangle = (\alpha|0_A\rangle + \beta|1_A\rangle) \otimes |0_E\rangle \longrightarrow \rho_0^A = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* \\ \beta\alpha^* & |\beta|^2 \end{pmatrix}$$

$$U: \mathcal{H}_A \otimes \mathcal{H}_E \rightarrow \mathcal{H}_A \otimes \mathcal{H}_E \qquad U|\Psi\rangle = \alpha\sqrt{1-p}|0_A 0_E\rangle + \alpha\sqrt{p}|0_A 1_E\rangle + \beta\sqrt{1-p}|1_A 0_E\rangle + \beta\sqrt{p}|1_A 2_E\rangle$$

$$\rho^A = \mathrm{Tr}_E\left[U|\Psi\rangle\langle\Psi|U^\dagger\right] = |\alpha|^2|0_A\rangle\langle 0_A| + |\beta|^2|1_A\rangle\langle 1_A| + \alpha\beta^*(1-p)|0_A\rangle\langle 1_A| + \mathrm{h.c.}$$

$$\rho^A = \begin{pmatrix} |\alpha|^2 & \alpha\beta^*(1-p) \\ \beta\alpha^*(1-p) & |\beta|^2 \end{pmatrix} = \begin{pmatrix} |\alpha|^2 & \alpha\beta^* e^{-t/T_2} \\ \beta\alpha^* e^{-t/T_2} & |\beta|^2 \end{pmatrix} \xrightarrow[t\to\infty]{} \begin{pmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{pmatrix}$$

$$p = (1 - e^{-t/T_2})$$

Decoherence time

Quantum operations

In general we cannot describe incoherent processes by unitary matrices acting on the whole Hilbert space, but we need to focus on a rather small subsystem described by a reduced density matrix. Within this framework, the evolution of ρ at discrete time steps can be given expressed by the quantum operation

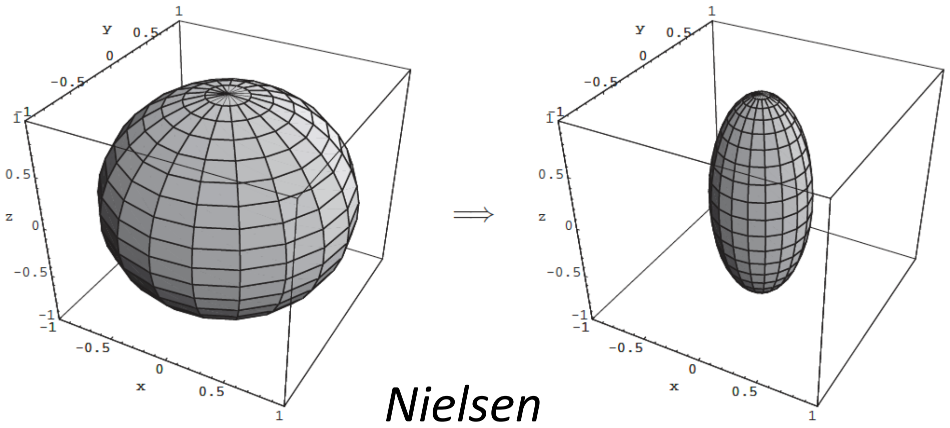
$$\mathcal{E}(\rho) = \sum_k E_k \rho E_k^\dagger \qquad \sum_k E_k E_k^\dagger = \mathbb{I}$$

Kraus operators, which can be expressed in terms of Paulis

Example: **phase flip** $\quad E_0 = \sqrt{1-p}\,\mathbb{I} \qquad E_1 = \sqrt{p}\,Z$

$$\rho \rightarrow \mathcal{E}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger = (1-p)\rho + pZ\rho Z$$

You can check this is equivalent to the previous slide calculation (continuous phase damping, with $1-2p = e^{-t/T_2}$).
The corresponding Bloch vector is projected along z

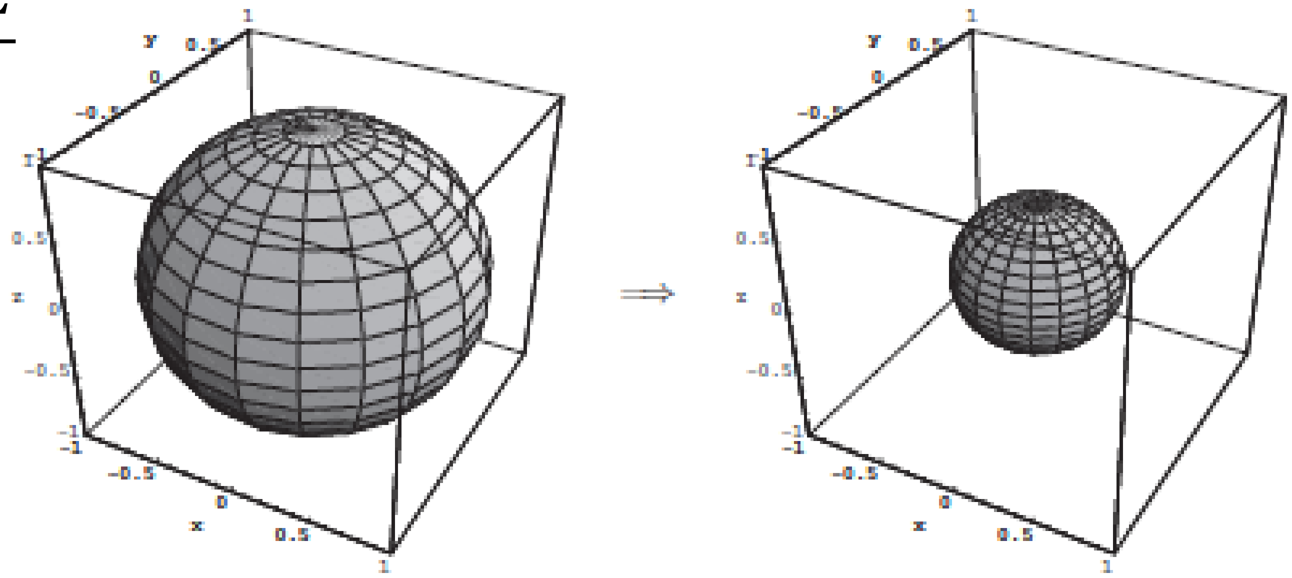


Nielsen

Depolarizing channel

$$\mathcal{E}(\rho) = (1-p)\rho + p\frac{\mathbb{I}}{2}$$
$$= (1-p)\rho + p\frac{\rho + X\rho X + Y\rho Y + Z\rho Z}{4}$$
$$= \left(1 - \frac{3p}{4}\right)\rho + p\frac{X\rho X + Y\rho Y + Z\rho Z}{4}$$

Uniform Pauli error channel



$$\mathcal{E}(\rho) = (1-p)\rho + p\frac{\mathbb{I}}{2^n}$$

On n qubits

M. A. Nielsen, I. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000)

Relaxation

Amplitude damping (i.e. relaxation at $T = 0$) approximately modeled by

$$E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix} \qquad E_1 = \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}$$

$$\rho \to \mathcal{E}(\rho) = E_0 \rho E_0^\dagger + E_1 \rho E_1^\dagger$$

Decay of excited diagonal state with rate $1 - p = e^{-t/T_1}$

$$\rho(t) = \begin{pmatrix} 1 - |\beta|^2(1-p) & \alpha\beta^*\sqrt{1-p} \\ \beta\alpha^*\sqrt{1-p} & |\beta|^2(1-p) \end{pmatrix} = \begin{pmatrix} 1 - |\beta|^2 e^{-t/T_1} & \alpha\beta^* e^{-t/2T_1} \\ \beta\alpha^* e^{-t/2T_1} & |\beta|^2 e^{-t/T_1} \end{pmatrix} \xrightarrow[t \to \infty]{} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Bloch sphere for mixed states

An arbitrary density matrix for a qubit in a mixed state can be written as

$$\rho = \frac{\mathbb{I} + \mathbf{r}\cdot\boldsymbol{\sigma}}{2} \qquad \text{with } \mathbf{r} = (u, v, w),\ \boldsymbol{\sigma} = (X, Y, Z) \qquad \|\mathbf{r}\|^2 = u^2 + v^2 + w^2 \leq 1$$

Bloch vector for state ρ

$\mathrm{Tr}\rho = 1$ ✓

$$\mathrm{Tr}\rho^2 = \frac{1}{2}(1 + \|\mathbf{r}\|^2)$$

Pure state $\|\mathbf{r}\|^2 = 1$

Mixed state $\|\mathbf{r}\|^2 < 1$

Maximally mixed: $\|\mathbf{r}\|^2 = 0$

If we diagonlize ρ we get:

$$\rho = \frac{1}{2}(1 + \|\mathbf{r}\|)|\rho_+\rangle\langle\rho_+| + \frac{1}{2}(1 - \|\mathbf{r}\|)|\rho_-\rangle\langle\rho_-|$$

It reduces to the pure state $\rho = |\rho_+\rangle\langle\rho_+|$ if $\|\mathbf{r}\| = 1$.

Decoherence = shrinking of Bloch sphere (see Nielsen, Chuang section 8.3)

Qiskit: noise models

Errors

Systematic (unitary) errors

Reduced by calibrating the hardware

Gate errors

Depolarizing channel

Increases with number of operations (circuit depth)

Relaxation and dephasing

Modeled by T_1 and T_2 dechorence times

Increases with time

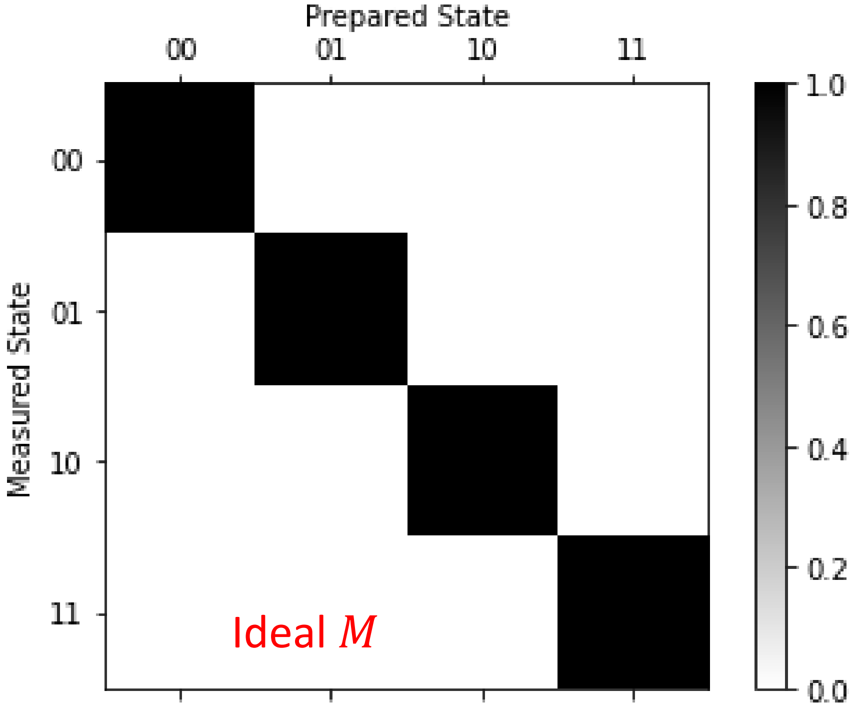
Measurement errors

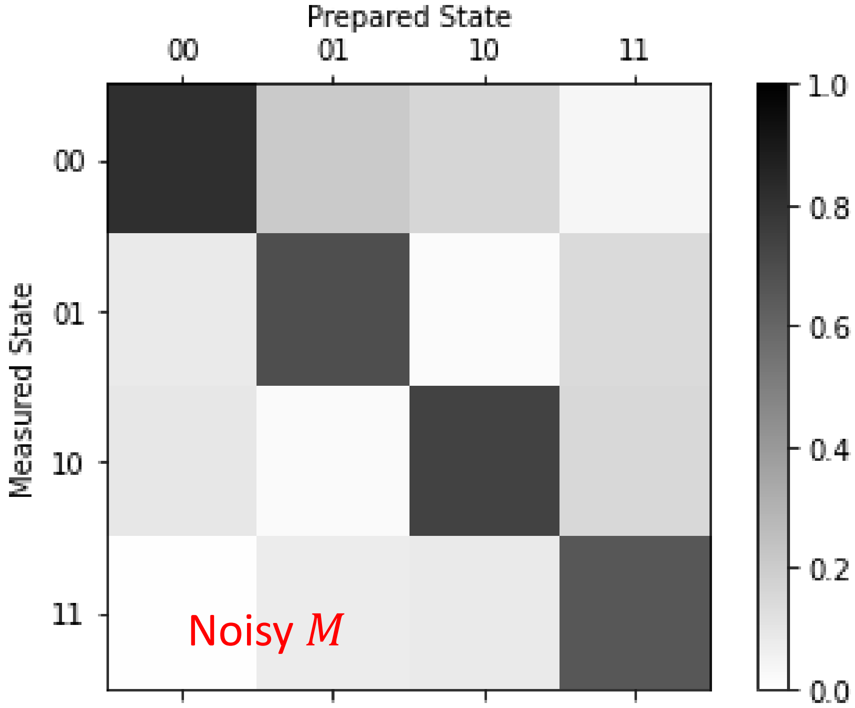
Assignement error at the qubit readout. Modeled by a deformation of the ideal projectors $P_0 = |0\rangle\langle 0|, P_1 = |1\rangle\langle 1|$

$$\pi_0 = (1-p)\ P_0 + p\ P_1$$
$$\pi_1 = (1-p)\ P_1 + p\ P_0$$

Error mitigation: measurement errors

The effect of noise occurring during computation can be complex. A simpler form of noise occurs during final readout. We can compute a **calibration matrix** $\langle y|M|x\rangle$ containing the probability of measuring output $|y\rangle$ for a bit string prepared in state $|x\rangle$ and then use it to **correct the output** C of a calculation.



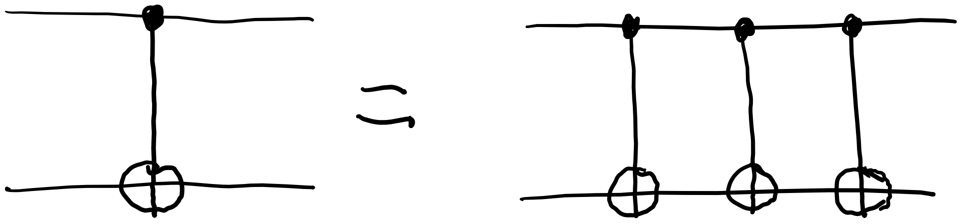


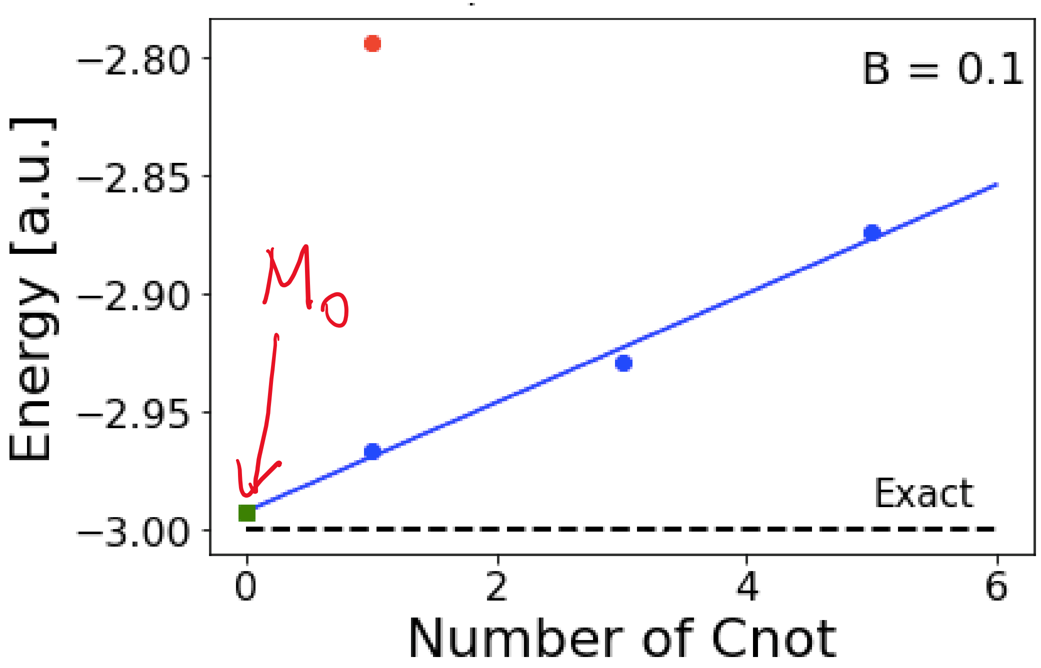
$$C_{noisy} = M\, C_{ideal} \quad \rightarrow \quad C_{ideal} = M^{-1}\, C_{noisy}$$

Error mitigation: gate errors

Extrapolation to zero noise

$$M(\lambda) = M_0 + \sum_k a_k \lambda^k$$





The expectation value of a given observable can be expressed in a power series of a noise parameter λ. Often the behavior is approximately linear and hence we can extract the zero-noise expectation value M_0 by linear extrapolation.

Quantum Error Correction

"We have learned that it is possible to fight entanglement with entanglement"

John Preskill

Example: three-qubit **bit-flip repetition code**

We can detect bit flip error on any of the three qubits by parity measurements of Z_1Z_2 and Z_2Z_3
Parity measurements can be implemented by two *additional ancillae* (Rep. Prog. Phys. **76**, 076001)
These measurement detect errors while **preserving the superposition**.
Two or more errors are not corrected.

Initial state $(\alpha|0\rangle + \beta|1\rangle) \otimes |00\rangle$

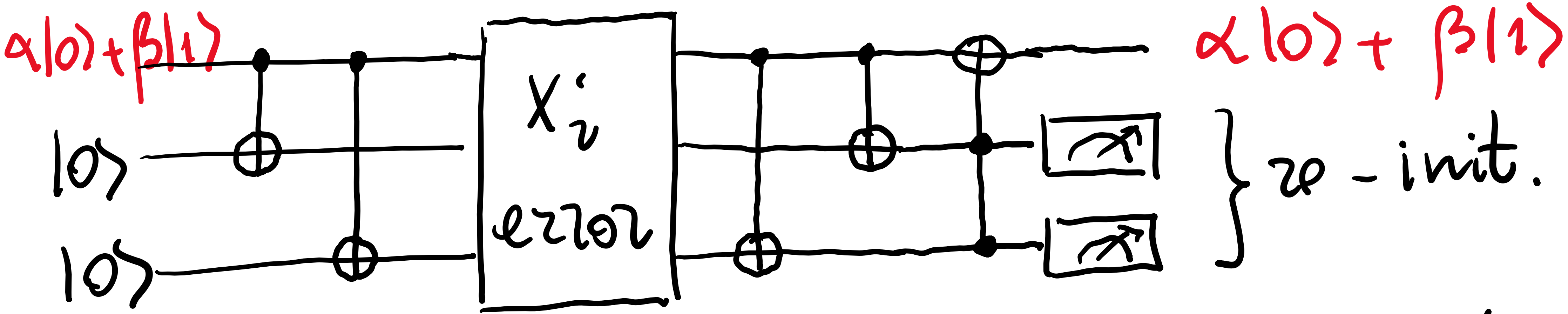
Encoding ↓

$\alpha|000\rangle + \beta|111\rangle$ ⟶

Output	$\langle Z_1Z_2\rangle$	$\langle Z_2Z_3\rangle$	Error
$\alpha|000\rangle + \beta|111\rangle$	1	1	No
$\alpha|100\rangle + \beta|011\rangle$	-1	1	1°
$\alpha|010\rangle + \beta|101\rangle$	-1	-1	2°
$\alpha|001\rangle + \beta|110\rangle$	1	-1	3°

Design a circuit
- to encode the protected state
- Correct the state of the 1° qubit without parity checks

3-qubit bit-flip repetition code



- This circuit detects an X error on any of the 3 qubits but corrects only the 1st one
- It can be adapted to correct Z errors
- It preserves superposition of the 1st qubit

Qubit calibration using Pulse