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RELATION OF LEŚNIEWSKI'S MEREOLOGY TO BOOLEAN ALGEBRA

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It has been stated by Tarski [5] and "proved" by Grzegorczyk [3] that:

(A) The models of mereology and the models of complete Boolean algebra with zero deleted[1] are identical.

Proved has been put in quotes, not because Grzegorczyk's proof is faulty but because the system he describes as mereology is in fact not Leśniewski's mereology.

Leśniewski's first attempt at describing the collective class, i.e. mereology, was done in ordinary language with no rigorous logical foundation. In describing the collective class, he needed to use the notion of distributive class. So as to clearly distinguish and expose the interplay between the two notions of class, he introduced his calculus of name (name being the distributive notion), which is also called ontology, since he used the primitive term "is." At this stage, mereology included ontology. Then, in order to have a logically rigorous system, he developed as a basis, a propositional calculus with quantifiers and semantical categories (types), called protothetic. At this final stage, what is properly called mereology includes both protothetic and ontology.

What Grzegorczyk describes as mereology is even weaker than Leśniewski's initial version. To quote from [3]:

> "In order to emphasize these formal relations let us consider the systems of axioms of mereology for another of its primitive terms, namely for the term "*ingr*" defined as follows:
>
> A *ingr* $B. \equiv .A$ is a part $B \vee A$ is identical to B.
>
> The proposition "A *ingr* B" can be read "A is contained in B" or after Leśniewski, "A is ingredient of B"."

Notice that by replacing Leśniewski's "A is an ingredient of B" by "A *ingr* B," Grzegorczyk has eliminated the "is." In doing so, he has eliminated the notion of distributive class and consequently the interplay between the two notions of class—aspects of mereology which Leśniewski thought were so important that he constructed his ontology in order to describe them clearly. This elimination of "is" is the most important deficiency in Grzegorczyk's system. Next he gives the system the weakest possible logical base by making it an elementary theory. It is this weak system that Grzegorczyk proves shares the same models with complete Boolean algebra with zero deleted.

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[1] Deleting zero from a Boolean algebra results in a system without a zero except in the case when the Boolean algebra has exactly two elements.

Słupecki [4] also suffers under the misapprehension that mereology is formally nothing more than a particular elementary theory of partial ordering. He remedies the weak logic by replacing the first-order logic by Chwistek's simple theory of types, but he fails to include ontology or any other theory of the distributive class to use as counterpoint for the notion of collective class. Słupecki unwittingly demonstrates the need for this counterpoint by saying:

> "... a consequence which disagrees with intuition is the nonexistence of finite sets whose number of elements differs from each of the numbers $2^2 - 1, 2^3 - 1, 2^4 - 1, \cdots$."

Since cardinality is primarily a distributive notion, one's intuition should not be violated if the collective class cannot describe it.

Since Leśniewski's mereology includes protothetic and ontology and Boolean algebra is usually given some other logical base, statement (A) needs to be put into a precise context. There is also need for a formal definition of complete Boolean algebra with zero deleted. Since statement (A) is in some sense not completely true, we break it up into the following two statements:

(1) Mereology is a complete Boolean algebra with zero deleted.

(2) Complete Boolean algebra with zero deleted is a mereology.

We shall present two alternative ways of introducing partial ordering into Leśniewski's logic and show that in order for (2) to hold we must be unreasonably restrictive in our definition of complete Boolean algebra with zero deleted.

It is commonly required in a definition of Boolean algebra that there exist two distinct elements, namely 0 and 1. Such a definition, together with any reasonable definition of complete Boolean algebra with zero deleted, would require the existence of at least one element, namely 1. Since Leśniewski, due to strong philosophical beliefs, carefully avoids postulating the existence of anything, (1) cannot hold unless we use a definition of Boolean algebra that does not postulate the existence of any nonzero element, but is otherwise equivalent to the usual definition. We use such a definition in what follows.

There seem to be two roads to making (1) precise:

(a) Construct an axiom system, $\mathfrak{A}$ with the following properties: {$\mathfrak{A}$, zero exists} is an axiom system for complete Boolean algebra and "zero exists" is independent of $\mathfrak{A}$. Thus $\mathfrak{A}$ makes no claim about the existence or nonexistence of zero. We then call the system {$\mathfrak{A}$, if two elements exist then zero does not exist} a complete Boolean algebra with zero deleted. Then show that mereology satisfies this axiom system.

Using a set-theoretical base such an axiom system is constructed in [1]. The following single axiom for a complete Boolean algebra with zero deleted is derived:

$$\mathbf{BD'}.\ [AB] \mathbin{\vdots\cdot\vdots} A \leq B. \equiv \mathbin{\vdots}\ \mathbin{\vdots} A \in U.\ B \in U \mathbin{\vdots}\ \mathbin{\vdots} B \leq B. \supset \mathbin{:\cdot:} [ab] \mathbin{:\cdot:} a \subset U.\ b \subset U.\ B \in a \mathbin{.\cdot.} [C] \mathbin{.\cdot.} C \in b. \equiv : [D] : D \in a\ .\supset.\ D \leq C : [D] : D \leq C\ .\supset.\ [\exists EF].\ E \in a.\ F \leq D.\ F \leq E :: \supset.\ [\exists L].\ b = \{L\}.\ A \leq L,$$ [2]

where $\leq$ is a relation on U.

---

[2] This axiom is patterned after a single axiom of mereology due to Sobociński and Lejewski.

(b) Extend the field of mereology to include exactly one more name and extend the ordering so as to yield a complete Boolean algebra.

Before either road can be taken we must decide how to introduce partial ordering into Leśniewski's logic.

(i) The first method is to introduce $\leq$ as a name-forming functor of one name argument and to formalize the statement "A is less than or equal to B" by

$$A \, \varepsilon \leq\!(B).$$

Since every element in the field of the partial ordering must be able to occur to the left of ε, the field must be contained in $\mathbf{V}$. Therefore the field is a general name which we shall designate by f. The translation of $\mathbf{BD}'$ into Leśniewski's logic then is:

$$\begin{aligned}&\mathbf{BDL}.\ [AB] \vdots\cdot\vdots\ A \,\varepsilon \leq\!(B).\ \equiv\ \vdots\ \vdots\ A \,\varepsilon f.\ B \,\varepsilon f \vdots\ \vdots\ B \,\varepsilon \leq\!(B).\ \supset\ :\cdot:\ [ab]\\ &:\cdot: a \subset f.\ b \subset f.\ B \,\varepsilon\, a\ .\cdot.\ [C]\ .\cdot.\ C \,\varepsilon\, b.\ \equiv : [D] : D \,\varepsilon\, a.\ \supset.\ D \,\varepsilon \leq\!(C) :\\ &[D] : D \,\varepsilon \leq\!(C)\ .\supset.\ [\exists EF].\ E \,\varepsilon\, a.\ F \,\varepsilon \leq\!(D).\ F \,\varepsilon \leq\!(E) :\ : \supset.\ A \,\varepsilon \leq\!(b).\end{aligned}$$

A single axiom for mereology is:

$$\begin{aligned}&\mathbf{M}.\ [AB] \vdots\cdot\vdots\ A \,\varepsilon\, \mathbf{el}(B).\ \equiv\ \vdots\ \vdots\ A \,\varepsilon\, A.\ B \,\varepsilon\, B \vdots\ \vdots\ B \,\varepsilon\, \mathbf{el}(B) \supset\ :\cdot:\ [ab] :\ :\\ &B \,\varepsilon\, a\ .\cdot.\ [C]\ .\cdot.\ C \,\varepsilon\, b.\ \equiv : [D] : D \,\varepsilon\, a.\ \supset.\ D \,\varepsilon\, \mathbf{el}(C) : [D] : D \,\varepsilon\, \mathbf{el}(C)\\ &.\supset.\ [\exists EF].\ E \,\varepsilon\, a.\ F \,\varepsilon\, \mathbf{el}(D).\ F \,\varepsilon\, \mathbf{el}(E) :\ : \supset.\ A \,\varepsilon\, \mathbf{el}(b).\end{aligned}$$

Remembering that $[A] : A \,\varepsilon\, A\ .\equiv.\ A \,\varepsilon\, \mathbf{V}$, if one replaces f by $\mathbf{V}$ and $\leq$ by **el** in **BDL** one arrives at **M**. If one considers $\leq$ as **el** restricted to f. one gets **BDL** from **M**. Therefore using road (a) we have shown that complete Boolean algebra with zero deleted and mereology have the same models.

Road (b) fails in this method since any extension of **el** to $\leq$ would require that we add another individual name to the field of **el**, i.e. to $\mathbf{V}$. But $\mathbf{V}$ is the totality of individual names, so there is no other individual name to add.

REMARK. Słupecki, in his above-mentioned system, does add a $\mathbf{0}$ or empty object. This empty object's existence is guaranteed by the addition of the axiom $[\exists X](X = \mathbf{0})$. However, in Leśniewski's mereology, the particular quantifier has no existential import. Here the existence of the name a requires a thesis of the form $[\exists A].A \,\varepsilon\, a$. Using Leśniewski's notation of Λ for the empty name, the ontological thesis $[A].\ \sim (A \,\varepsilon\, \Lambda)$ precludes the existence of Λ, so it cannot be added to the individual (i.e. unique and existing) names to yield a complete Boolean algebra.

However this method is too restrictive. It is so restrictive that it excludes names under $\subset$ (the containment relation) from being a partially ordered system, much less a complete Boolean algebra. Even if one excludes the difficulty of the empty name, we still have that the nonempty names under $\sqsubset$ (containment for nonempty names) fail to be a partially ordered system. To show this let us suppose we are able to define ε^* and $\leq^*$, analogs for ε and $\leq$ so that the following statement holds:

$$* \qquad [ab] : a \sqsubset b.\ \equiv\ .a \,\varepsilon^* \leq^*\!(b).$$

For ε^*, the whole category of names (except for the empty name) must play the role of individual names; so whatever are to play the roles of nonindividual general names must be of a different semantical category. Since all ε^*-names, both individual and general, must be able to fill the second argument place of ε^*, we would have to have two semantical categories being able to fill the same argument place. But this is prohibited in Leśniewski's logic. Therefore, * cannot hold. Therefore $\sqsubset$ cannot be considered a partial ordering.

Therefore method (i) must be discarded and with it our proof that mereology and complete Boolean algebra with zero deleted are equivalent.

(ii) The second method is to introduce $\leq$ as a predicate forming functor of two arguments, both from the same semantical category, $\leq(AB)$.[3] We need to be able to specify the field of the partial ordering. But Leśniewski's logic is not a set theory. It does not have a general collection-forming mechanism. It only has a collection-forming mechanism if the elements we are collecting can be considered as individuals relative to some analog of ε. So the simplest way it has of referring to a collection of elements from a given category is to use the predicate forming functor of one argument which is true precisely of these elements, i.e. the characteristic functor. Thus, the field will be specified by a predicate U. In this situation the translation of **BD′** becomes

$$\mathbf{BD}.\ [AB] \vdots\cdot\vdots\ A \leq B.\ \equiv\ \vdots\ \vdots\ U(A).\ U(B)\ \vdots\ \vdots\ B \leq B.\ \supset\ \vdots\cdot\vdots\ [\alpha, \beta]\ \vdots\cdot\vdots$$
$$\alpha \subset U.\ \beta \subset U.\ \alpha(B)\ .\cdot.\ [C]\ .\cdot.\ \beta(C).\ \equiv [D]: \alpha(D)\ .\supset.\ D \leq C: [D]:$$
$$D \leq C\ .\supset.\ [\exists EF].\ \alpha(E).\ F \leq D.\ F \leq E:\ :\supset [\exists L].\ \beta = \circ((L))^{4}.\ A \leq L.^{5}$$

Now using either (a) or (b) we can prove mereology is a complete Boolean algebra with zero deleted.

For (a) let $A \ \varepsilon\ \mathbf{el}(B)$ be $A \leq B$. We need the following definition:

$$[Aa]: \varepsilon\,((a))(A)\ .\equiv.\ A\ \varepsilon\ a.$$

Let $\varepsilon((\mathrm{V}))$ be U. The $\alpha \subset U$ requires that α equals $\varepsilon((a))$ for some name a. Thus $A\ \varepsilon\ a$ transforms to $\alpha(A)$. Also $A\ \varepsilon\ A$ transforms to $U(A)$. So axiom **M** is easily seen to be an instance of **BD**.

For (b) we shall only give the necessary definitions and interpretations. The proof though straightforward is somewhat tedious.

Let U be $\rightharpoonup$ and $\leq$ be φ as they are defined below.

$$[a]\ .\cdot.\ {\rightharpoonup}\{a\}.\ \equiv: [BC]: B\ \varepsilon\ a.\ C\ \varepsilon\ a\ .\supset.\ B = C.$$
$$[AB]: \varphi\{AB\}\ .\equiv.\ {\rightharpoonup}\{A\}.\ {\rightharpoonup}\{B\}.\ A \subset \mathbf{el}(B).$$

Returning to (a), one can easily see that nonempty names form a complete Boolean algebra with zero deleted. Let U be ! and $\leq$ be $\sqsubset$ as they are defined below.

$$[a]: !\{a\}\ .\equiv.\ [\exists A].\ A\ \varepsilon\ a,$$
$$[ab]: a \sqsubset b\ .\equiv.\ !\{a\}.\ a \subset b.$$

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[3] For ease of reading we shall write $A \leq B$ for $\leq(AB)$.

[4] $[ab]$: $\circ((a))\{b\}.\ \equiv\ .a \circ b.$

[5] The parentheses in this axiom are meant to be ambiguous so that in reality we have an axiom for each semantical category.

However, the nonempty names are not a model for mereology since this would require definitions of ε^* and **el*** such that

$$(**) \qquad [ab] : a \sqsubset b . \equiv . a\, \varepsilon^*\, \mathbf{el}^*(b)$$

holds.

The same argument that showed (*) is impossible, also applies here. Therefore (2) is false.

However, there is a sense in which (2) is true; we can construct a model of mereology which reflects the structure of any given complete Boolean algebra with zero deleted. In this model the role of a given A such that $U(A)$ is played by the characteristic functor of A, so that the universe of the model is not in the semantical category of A, but in the category of predicate forming functors with arguments in the category of A. We now give this construction.

For the analogs in the category of names for the definitions involved see [2]. We shall use ( ) to enclose arguments to the same semantical category as A, and $\langle\ \rangle$ to enclose arguments belonging to the same category as U. For ease of reading we shall employ the following conventions: $\Sigma \subset \sigma$ for $\subset\langle\Sigma\sigma\rangle$, $B \circ C$ for $\circ(BC)$, $\Phi \circ \Sigma$ for $\circ\langle\Phi\Sigma\rangle$ ($\circ$ stands for equality, i.e. mutual containment.), $\Sigma\ \eta\ \sigma$ for $\eta\langle\Sigma\sigma\rangle$, η will be the analog of ε. The following three definitions give the analogs of general names, individuals, and ε respectively.

DN1. $[\sigma] : \mathcal{N}\langle\sigma\rangle . \equiv . \sigma \subset U.$

DN2. $[\Sigma] : . \mathcal{I}\langle\Sigma\rangle . \equiv : [\exists A] . \Sigma(A) . \mathcal{N}\langle\Sigma\rangle : [AB] : \Sigma(B) . \supset . A \circ B.$

DN3. $[\Sigma\sigma] : \Sigma\ \eta\ \sigma . \equiv . \mathcal{I}\langle\Sigma\rangle . \mathcal{N}\langle\sigma\rangle . \Sigma \subset \sigma.$

N1. $[\Sigma] : \Sigma\ \eta\ \Sigma . \equiv . \mathcal{I}\langle\Sigma\rangle.$ [DN3, DN2]

N2. $[\Sigma\sigma] : \Sigma\ \eta\ \sigma . \supset . \Sigma\ \eta\ \Sigma.$ [DN3, DN2]

N3. $[A\Sigma\sigma] : \mathcal{I}\langle\Sigma\rangle . \Sigma(A) . \sigma(A) . \supset . \Sigma \subset \sigma$

Hyp (3). $\supset$:

(4) $[BC] : \Sigma(B) . \Sigma(C) . \supset . B \circ C$: [DN2, 1]

(5) $[B] : \Sigma(B) . \supset . \sigma(B)$: [4, 2, 3]

$\Sigma \subset \sigma.$ [5]

N4. $[\Sigma\ \Phi\ \sigma] : \Sigma\ \eta\ \sigma . \Phi\ \eta\ \Sigma . \supset . \Phi \circ \Sigma.$

Hyp (2). $\supset$.

(3) $\mathcal{I}\langle\Sigma\rangle.$

(4) $\Phi \subset \Sigma.$ [DN3, 1]

(5) $\mathcal{I}\langle\Phi\rangle.$ [DN3, 2]

$[\exists A].$

(6) $\Phi(A).$ [DN2, 5]

(7) $\Sigma(A).$ [4, 6]

(8) $\Sigma \subset \Phi.$ [N3, 3, 7, 6]

$\Phi \circ \Sigma.$ [4, 8]

N5. $[\Sigma\ \Phi\ \sigma] : \Sigma\ \eta\ \sigma . \Phi\ \eta\ \Sigma . \supset . \Phi\ \eta\ \sigma.$ [N4]

N6. $[\Sigma\ \Phi\ \Psi\sigma] : \Sigma\ \eta\ \sigma . \Phi\ \eta\ \Sigma . \Psi\ \eta\ \Sigma . \supset . \Phi\ \eta\ \Psi.$ [N4, Φ/Ψ]

DN4. $[AB] : \chi((A))(B) . \equiv . U(A) . A \circ B.$

N7. $[A] : \chi((A))(A) . \equiv . U(A).$ [DN4]

Thus $\chi((A))$ is the characteristic function for an A to which U applies.

N8. $[A]. \mathscr{N}\langle\chi((A))\rangle.$ [DN1, DN4]

N9. $[A]: U(A) . \equiv . \mathscr{I}\langle\chi((A))\rangle$ [DN2, N7, N8, DN4, DN1]

N10. $[\sigma A]: \mathscr{N}\langle\sigma\rangle. \sigma(A) . \supset . \chi((A)) \eta \sigma.$

Hyp (2) . ⊃ .

(3) $U(A).$ [DN1, 1, 2]

(4) $\mathscr{I}\langle\chi((A))\rangle.$ [N9, 3]

(5) $\chi((A)) \subset \sigma.$ [DN4, 2]

$\chi((A)) \eta \sigma.$ [DN3, 4, 1, 5]

N11. $[\sigma A]: \chi((A)) \eta \sigma . \supset . \sigma(A).$

Hyp (1) . ⊃ .

(2) $\mathscr{I}\langle\chi((A))\rangle.$

(3) $\chi((A)) \subset \sigma.$ [DN3, 1]

$\chi((A))(A).$ [N9, 2, N7]

N12. $[\sigma A]: \mathscr{N}\langle\sigma\rangle. \sigma(A) . \equiv . \chi((A)) \eta \sigma.$ [N10, N11, DN3]

N13. $[A]: \chi((A))(A) . \equiv . \chi((A)) \eta \chi((A)).$ [N12, N8]

N14. $[\Sigma AB]: . \mathscr{N}\langle\Sigma\rangle: [\Phi\Psi]: \Phi \eta \Sigma. \Psi \eta \Sigma . \supset . \Phi \eta \Psi: \Sigma(A). \Sigma(B): \supset . A \circ B.$

Hyp (4) . ⊃ .

(5) $\chi((A)) \eta \Sigma.$ [N12, 1, 3]

(6) $\chi((B)) \eta \Sigma.$ [N12, 1, 4]

(7) $\chi((A)) \eta \chi((B))$ [2, 5, 6]

(8) $\chi((B))(A).$ [N12, 7]

$A \circ B.$ [DN4, 8]

N15. $[\theta \Sigma \sigma] . \cdot . \theta \eta \Sigma: [\Phi]: \Phi \eta \Sigma . \supset . \Phi \eta \sigma: [\Phi\Psi]: \Phi \eta \Sigma. \Psi \eta \Sigma . \supset .$
$\Phi \eta \Psi: \supset . \Sigma \eta \sigma.$

Hyp (3) . ⊃ :

(4) $\mathscr{I}\langle\theta\rangle.$

(5) $\mathscr{N}\langle\Sigma\rangle$

(6) $\theta \subset \Sigma:$ [DN3, 1]

(7) $[AB]: \Sigma(A). \Sigma(B) . \supset . A \circ B:$ [N14, 5, 3]

$[\exists C].$

(8) $\theta(C).$ [DN2, 4]

(9) $\Sigma(C).$ [8, 6]

(10) $\mathscr{I}\langle\Sigma\rangle.$ [DN2, 9, 5, 7]

(11) $\Sigma \eta \Sigma.$ [N1, 6]

$\Sigma \eta \sigma$ [2, 11]

N16. $[\Sigma\sigma]: . \Sigma \eta \sigma . \equiv : [\exists\theta]. \theta \eta \Sigma: [\Phi]: \Phi \eta \Sigma . \supset .$
$\Phi \eta \sigma: [\Phi\Psi]: \Phi \eta \Sigma. \Psi \eta \Sigma . \supset . \Phi \eta \Psi.$ [N2, N5, N6; N15]

This is an analog for a single axiom of ontology. In order to complete the ontological portion of the model we need to prove a thesis corresponding to the law of extensionality and a metatheorem corresponding to the rule of definition.

N17. $[\sigma \tau \Sigma]: \sigma \subset \tau. \mathscr{N}\langle\tau\rangle. \Sigma \eta \sigma . \supset . \Sigma \eta \tau.$

Hyp (3) . ⊃ .

(4) $\mathscr{I}\langle\Sigma\rangle.$

(5) $\Sigma \subset \sigma.$ [DN3, 3]

(6) $\Sigma \subset \tau.$ [1, 5]

$\Sigma\ \eta\ \tau.$ [DN3, 4, 2, 6]

N18. $[\sigma\tau] :\ : \mathscr{N}\langle\tau\rangle . \supset :. \sigma \subset \tau . \supset : [\Sigma] : \Sigma\ \eta\ \sigma . \supset . \Sigma\ \eta\ \tau.$ [N17]

N19. $[A\ \sigma\ \tau] .\cdot. \mathscr{N}\langle\sigma\rangle : [\Sigma] : \Sigma\ \eta\ \sigma . \supset . \Sigma\ \eta\ \tau : \sigma(A) : \supset . \tau(A).$

Hyp (3) . $\supset$.

(4) $\chi((A))\ \eta\ \sigma.$ [N12, 1, 3]

(5) $\chi((A))\ \eta\ \tau.$ [2, 4]

$\tau(A).$ [N12, 5]

N20. $[\sigma\tau] .\cdot. \mathscr{N}\langle\sigma\rangle : [\Sigma] : \Sigma\ \eta\ \sigma . \supset . \Sigma\ \eta\ \tau : \supset . \sigma \subset \tau.$ [N19]

N21. $[\sigma\tau] :\ : \mathscr{N}\langle\sigma\rangle . \mathscr{N}\langle\tau\rangle . \supset .\cdot. \sigma \subset \tau . \equiv : [\Sigma] : \Sigma\ \eta\ \sigma . \supset . \Sigma\ \eta\ \tau.$ [N18, N20]

N22. $[\sigma\tau] :\ : \mathscr{N}\langle\sigma\rangle . \mathscr{N}\langle\tau\rangle . \supset .\cdot. \sigma \circ \tau \equiv : [\Sigma] : \Sigma\ \eta\ \sigma . \equiv . \Sigma\ \eta\ \tau$ [N21]

In what follows, $a_1 \cdots a_n$ represent variables for previously introduced semantical categories.

DN5. $[a_1 \cdots a_n A \sigma] : \mathscr{K}\langle\langle\sigma\rangle\rangle \nleqq a_1 \cdots a_n A \ngeqq . = . \sigma\{a_1 \cdots a_n\}(A).$

DN6. $[\alpha\lambda] : \mathscr{L}\{\{\alpha\}\} \nless \lambda \ngtr . = . [\exists\sigma] . \lambda \circ \mathscr{K}\langle\langle\sigma\rangle\rangle . \alpha\langle\langle\sigma\rangle\rangle.$

N23. $[\alpha\sigma] : \mathscr{L}\{\{\alpha\}\} \nless \mathscr{K}\langle\langle\sigma\rangle\rangle \ngtr . \equiv . \alpha\langle\langle\sigma\rangle\rangle.$ [DN6]

N24. $[\sigma\ \tau\ a_1 \cdots a_n] .\cdot. \mathscr{N}\langle\sigma\{a_1 \cdots a_n\}\rangle . \mathscr{N}\langle\tau\{a_1 \cdots a_n\}\rangle : [\Sigma] :$
$\Sigma\ \eta\ \sigma\{a_1 \cdots a_n\} . \equiv . \Sigma\ \eta\ \tau\{a_1 \cdots a_n\} : \supset : [\alpha] : \alpha\langle\langle\sigma\rangle\rangle . \equiv . \alpha\langle\langle\tau\rangle\rangle.$

Hyp (3) . $\supset$:.

(4) $\sigma\{a_1 \cdots a_n\} \circ \tau\{a_1 \cdots a_n\}.$ [N22, 1, 2, 3]

(5) $\mathscr{K}\langle\langle\sigma\rangle\rangle \circ \mathscr{K}\langle\langle\tau\rangle\rangle :$ [DN5, 4]

(6) $[\alpha] : \mathscr{L}\{\{\alpha\}\} \nless \mathscr{K}\langle\langle\sigma\rangle\rangle \ngtr . \equiv . \mathscr{L}\{\{\alpha\}\} \nless \mathscr{K}\langle\langle\tau\rangle\rangle \ngtr :$ [5]

$[\alpha] : \alpha\langle\langle\sigma\rangle\rangle . \equiv . \alpha\langle\langle\tau\rangle\rangle.$ [6, N23]

DN7. $[a_1 \cdots a_n\ \Sigma\ \sigma] : \mathscr{M}(|a_1 \cdots a_n \Sigma|)\langle\sigma\rangle . \equiv . \Sigma\ \eta\ \sigma\{a_1 \cdots a_n\}.$

N25. $[a_1 \cdots a_n \sigma\ \tau\ \Sigma] .\cdot. [\alpha] : \alpha\langle\sigma\rangle . \equiv . \alpha\langle\tau\rangle : \supset : \Sigma\ \eta\ \sigma\{a_1 \cdots a_n\}.$
$\equiv . \Sigma\ \eta\ \tau\{a_1 \cdots a_n\}.$ [DN7]

N26. $[a_1 \cdots a_n\ \sigma\ \tau] :\ : \mathscr{N}\langle\sigma\{a_1 \cdots a_n\}\rangle . \mathscr{N}\langle\tau\{a_1 \cdots a_n\}\rangle . \supset .\cdot. [\Sigma] :$
$\Sigma\ \eta\ \sigma\{a_1 \cdots a_n\} . \equiv . \Sigma\ \eta\ \tau\{a_1 \cdots a_n\} : \equiv : [\alpha] : \alpha\langle\langle\sigma\rangle\rangle . \equiv . \alpha\langle\langle\tau\rangle\rangle$ [N24, N25]

N26 is the law of ontological extensionality for η. Next we prove a metatheorem which corresponds to the ontological rule of definition for η.

N27. $[\Sigma] : \mathscr{I}\langle\Sigma\rangle . \supset . [\exists A] . \Sigma(A) . \Sigma \circ \chi((A)) . U(A).$

Hyp (1) . $\supset$.

(2) $\mathscr{N}\langle\Sigma\rangle.$ [DN2, 1]

$[\exists A].$

(3) $\Sigma(A).$

(4) $\Sigma \subset U.$ [DN2, 1]

(5) $U(A).$ [3, 4]

(6) $\chi((A))(A).$ [N7, 5]

(7) $\mathscr{I}\langle\chi((A))\rangle.$ [N9, 4]

(8) $\Sigma \subset \chi((A)).$ [N3, 1, 3, 6]

(9) $\chi((A)) \subset \Sigma.$ [N3, 7, 6, 3]

(10) $\Sigma \circ \chi((A)).$ [8, 9]

$[\exists A] . \Sigma(A) . \Sigma \circ \chi((A)) . U(A).$ [3, 10, 5]

METATHEOREM. *An expression of the form*

$$[a_1 \cdots a_n] \, . \cdot . \, \mathscr{N}\langle \sigma\{a_1 \cdots a_n\}\rangle \, . \supset : [\Sigma] : \Sigma \, \eta \, \sigma\{\{a_1 \cdots a_n\}\} \, . \equiv . \, \Sigma \, \eta \, \Sigma \, . \, \sigma^*\{\{a_1 \cdots a_n\}\}\langle\Sigma\rangle$$

can be introduced as a thesis of the system, provided that the functor $\sigma\{a_1 \cdots a_n\}$ *has been defined by*

(*) $$[a_1 \cdots a_n A] : \sigma\{\{a_1 \cdots a_n\}\}(A) \, . \equiv . \, \sigma^*\{\{a_1 \cdots a_n\}\}\langle\chi((A))\rangle .$$

PROOF.

*1. $[a_1 \cdots a_n \Sigma] : \Sigma \, \eta \, \sigma\{a_1 \cdots a_n\} \, . \supset . \, \sigma^*\{a_1 \cdots a_n\}\langle\Sigma\rangle$

Hyp $\supset$.

(2) $\mathscr{I}\langle\Sigma\rangle .$

(3) $\Sigma \subset \sigma\{a_1 \cdots a_n\} .$ [DN3, 1]

$[\exists A] .$

(4) $\Sigma(A) .$

(5) $\Sigma \circ \chi((A)) .$ [N27, 2]

(6) $\sigma\{a_1 \cdots a_n\}(A) .$ [3, 4]

(7) $\sigma^*\{\{a_1 \cdots a_n\}\}\langle\chi((A))\rangle$ [*, 6]

$\sigma^*\{\{a_1 \cdots a_n\}\}\langle\Sigma\rangle .$ [5, 7]

*2. $[a_1 \cdots a_n \Sigma] : \mathscr{N}\langle\sigma\{a_1 \cdots a_n\}\rangle . \, \Sigma \, \eta \, \Sigma . \, \sigma^*\{\{a_1 \cdots a_n\}\}\langle\Sigma\rangle \, . \supset . \, \Sigma \, \eta \, \sigma\{a_1 \cdots a_n\} .$

Hyp (3) . $\supset$.

(4) $\mathscr{I}\langle\Sigma\rangle .$ [N1, 2]

$[\exists A] .$

(5) $\Sigma(A) .$

(6) $\Sigma \circ \chi((A)) .$ [N27, 4]

(7) $\sigma^*\{\{a_1 \cdots a_n\}\}\langle\chi((A))\rangle .$ [3, 6]

(8) $\sigma\{a_1 \cdots a_n\}(A) .$ [*, 7]

(9) $\Sigma \subset \sigma\{a_1 \cdots a_n\} .$ [N3, 4, 5, 8]

$\Sigma \, \eta \, \sigma\{a_1 \cdots a_n\} .$ [DN3, 4, 1, 9]

*3. $[a_1 \cdots a_n] \, . \cdot . \, \mathscr{N}\langle\sigma\{a_1 \cdots a_n\}\rangle \, . \supset : [\Sigma] : \Sigma \, \eta \, \sigma\{a_1 \cdots a_n\} \, . \equiv . \, \Sigma \, \eta \, \Sigma .$
$\sigma^*\{\{a_1 \cdots a_n\}\}\langle\Sigma\rangle .$ [N2, *1; *2]

This completes the proof of the ontological portion of the model. Within this model, we have the expected characterization of the η-individuals as given by the following theorem.

N24. $[\Sigma] : \mathscr{I}\langle\Sigma\rangle \, . = [\exists A] . \, U(A) . \, \Sigma \circ \chi((A)) .$ [N27, N9]

We now construct the portion of the model involving "element." We wish a definition of element so that one η-individual is an element of another if the argument of the first is less than or equal to the argument of the second, i.e.

$$[\Sigma\Phi] : \Sigma \, \eta \, \mathbf{el}\langle\langle\Phi\rangle\rangle \, . \equiv . \, \Sigma \, \eta \, \Sigma . \, \Phi \, \eta \, \Phi . \, [\exists BC] . \, \Sigma(B) . \, \Phi(C) . \, B \leq C .$$

According to the metatheorem we must introduce

DN8. $[\Phi A] : \mathbf{el}\langle\langle\Phi\rangle\rangle(A) \, . \equiv . \, \Phi \, \eta \, \Phi . \, [\exists BC] . \, \chi((A))(B) . \, \Phi(C) . \, B \leq C .$

So that the following may be introduced as a thesis

N29. $[\Phi] \therefore \mathscr{N}\langle \mathbf{el}\langle\langle\Phi\rangle\rangle\rangle . \supset : [\Sigma] : \Sigma\ \eta\ \mathbf{el}\langle\langle\Phi\rangle\rangle . \equiv . \Sigma\ \eta\ \Sigma .\ \Phi\ \eta\ \Phi .$
$[\exists BC] .\ \Sigma(B) .\ \Phi(C) .\ B \leq C.$ [Metatheorem, DN8]

N30. $[\Phi A] : \mathbf{el}\langle\langle\Phi\rangle\rangle(A) . \supset . U(A).$
Hyp (1) $. \supset .$
$[\exists B].$
(2) $\chi((A))(B).$ [DN8, 1]
(3) $\chi((A))(A).$ [DN4, 2]
$U(A).$ [N7, 3]

N31. $[\Phi] .\ \mathscr{N}\langle \mathbf{el}\langle\langle\Phi\rangle\rangle\rangle.$ [N30, DN1]

N32. $[\Sigma\Phi] : \Sigma\ \eta\ \mathbf{el}\langle\langle\Phi\rangle\rangle . \equiv . \Sigma\ \eta\ \Sigma .\ \Phi\ \eta\ \Phi .\ [\exists BC] .\ \Sigma(B) .\ \Phi(C).$
$B \leq C.$ [N29, N31]

N33. $[AB] : \chi((A))\ \eta\ \mathbf{el}\langle\langle\chi((B))\rangle\rangle . \supset . A \leq B.$
Hyp (1) $. \supset .$
$[\exists CD].$
(2) $\chi((A))(C).$
(3) $\chi((B))(D).$ [N32, 1]
(4) $C \leq D.$
(5) $A \circ C.$ [DN4, 2]
(6) $B \circ D.$ [DN4, 3]
$A \leq B.$ [4, 5, 6]

N34. $[AB] : A \leq B . \supset . \chi((A))\ \eta\ \mathbf{el}\langle\langle\chi((B))\rangle\rangle$
Hyp (1) $. \supset .$
(2) $U(A).$
(3) $U(B).$ [BD, 1]
(4) $\chi((A))\ \eta\ \chi((A)).$ [N9, 2, N1]
(5) $\chi((B))\ \eta\ \chi((B)).$ [N9, 3, N1]
(6) $\chi((A))(A).$ [N7, 2]
(7) $\chi((B))(B).$ [N7, 3]
$\chi((A))\ \eta\ \mathbf{el}\langle\langle\chi((B))\rangle\rangle.$ [N32, 4, 5, 6, 7, 1]

N35. $[AB] : A \leq B . \equiv . \chi((A))\ \eta\ \mathbf{el}\langle\langle\chi((B))\rangle\rangle.$ [N33, N34]

N36. $[\sigma BC] \therefore \sigma(B) : [D] : \sigma(D) . \supset . D \leq C : \supset . U(C).$ [BD]

Starting with **BD** and using the following theses, DN1, N36, N12, N35, in order, we arrive at

N37. $[AB] :: \chi((A))\ \eta\ \mathbf{el}\langle\langle\chi((B))\rangle\rangle . \equiv :: U(A) .\ U(B) ::$
$\chi((B))\ \eta\ \mathbf{el}\langle\langle\chi((B))\rangle\rangle . \supset :.: [\sigma\tau] :.: \mathscr{N}\langle\tau\rangle .\ \chi((B))\ \eta\ \sigma \therefore [C] \therefore$
$\chi((C))\ \eta\ \tau . \equiv : U(C) : [D] : \chi((D))\ \eta\ \sigma . \supset . \chi((D))\ \eta\ \mathbf{el}\langle\chi((C))\rangle : [D] :$
$\chi((D))\ \eta\ \mathbf{el}\langle\langle\chi((C))\rangle\rangle . \supset . [\exists EF] .\ \chi((E))\ \eta\ \sigma .\ \chi((F))\ \eta\ \mathbf{el}\langle\langle\chi((D))\rangle\rangle.$
$\chi((F))\ \eta\ \mathbf{el}\langle\langle\chi((E))\rangle\rangle : : \supset . [\exists L] .\ \tau \circ \chi((L)) .\ \chi((A))\ \eta\ \mathbf{el}\langle\langle\chi((L))\rangle\rangle.$

N38. $[\Delta] \therefore [\exists\Psi\theta] .\ \Psi\ \eta\ \sigma .\ \theta\ \eta\ \mathbf{el}\langle\langle\Delta\rangle\rangle .\ \theta\ \eta\ \mathbf{el}\langle\langle\Psi\rangle\rangle : \supset \therefore$
$[\exists EF] .\ \chi((E))\ \eta\ \sigma .\ \chi((F))\ \eta\ \mathbf{el}\langle\langle\Delta\rangle\rangle .\ \chi((F))\ \eta\ \mathbf{el}\langle\langle\chi((E))\rangle\rangle$
Hyp (1) $. \supset .$
(2) $\mathscr{I}\langle\Psi\rangle.$ [DN3, 1]
(3) $\mathscr{I}\langle\theta\rangle.$ [DN3, 2]
$[\exists EF].$

(4) $\Psi \circ \chi((E))$. [N27, 2]
(5) $\theta \circ \chi((F))$. [N27, 3]
$[\exists EF].\ \chi((E))\ \eta\ \sigma.\ \chi((F))\ \eta\ \mathbf{el}\langle\langle\Delta\rangle\rangle.\ \chi((F))\ \eta\ \mathbf{el}\langle\langle\chi((E))\rangle\rangle.$ [1, 4, 5]

N39. $[\Delta] \therefore [\exists EF].\ \chi((E))\ \eta\ \sigma.\ \chi((F))\ \eta\ \mathbf{el}\langle\langle\Delta\rangle\rangle.\ \chi((F))\ \eta\ \mathbf{el}\langle\langle\chi((E))\rangle\rangle :$
$\equiv : [\exists\Psi\theta].\ \Psi\ \eta\ \sigma.\ \theta\ \eta\ \mathbf{el}\langle\langle\Delta\rangle\rangle.\ \theta\ \eta\ \mathbf{el}\langle\langle\Psi\rangle\rangle.$ [N38]

In what follows $\mathscr{E}\langle\langle\chi((A))\rangle\rangle$ will represent any expression involving $\chi((A))$, but not otherwise involving A. $\mathscr{E}\langle\Sigma\rangle$ represents the expression obtained by replacing $\chi((A))$ by Σ. Thus A does not occur in $\mathscr{E}\langle\Sigma\rangle$.

N40. $[\Delta\sigma] \therefore [D] : \chi((D))\ \eta\ \sigma\ .\supset .\ \mathscr{E}\langle\chi((D))\rangle : \Delta\ \eta\ \sigma : \supset .\ \mathscr{E}\langle\Delta\rangle$
Hyp (2) . ⊃ .
(3) $\mathscr{I}\langle\Delta\rangle$. [DN3, 2]
$[\exists D]$.
(4) $\Delta \circ \chi((D))$. [N27, 3]
(5) $\chi((D))\ \eta\ \sigma$. [2, 4]
(6) $\mathscr{E}\langle\chi((D))\rangle$. [1, 5]
$\mathscr{E}\langle\Delta\rangle$. [6, 4]

N41. $[\sigma] \therefore [D] : \chi((D))\ \eta\ \sigma\ .\supset .\ \mathscr{E}\langle\chi((D))\rangle :\equiv : [\Delta] : \Delta\ \eta\ \sigma\ .\supset .\ \mathscr{E}\langle\Delta\rangle.$ [N40]

N42. $[\sigma\Gamma] \therefore [C] : \chi((C))\ \eta\ \sigma\ .\equiv .\ U(C).\ \mathscr{E}\langle\chi((C))\rangle : \Gamma\ \eta\ \sigma : \supset .\ \Gamma\ \eta\ \Gamma.$
$\mathscr{E}\langle\Gamma\rangle.$ [N2, N40]

N43. $[\sigma\Gamma] \therefore [C] : \chi((C))\ \eta\ \sigma\ .\equiv .\ U(C).\ \mathscr{E}\langle\chi((C))\rangle : \Gamma\ \eta\ \Gamma.\ \mathscr{E}\langle\Gamma\rangle : \supset .\ \Gamma\ \eta\ \sigma.$
Hyp (3) . ⊃ .
(4) $\mathscr{I}\langle\Gamma\rangle$. [N1, 2]
$[\exists C]$.
(5) $\Gamma(C)$.
(6) $\Gamma \circ \chi((C))$. [N27, 4]
(7) $U(C)$.
(8) $\mathscr{E}\langle\chi((C))\rangle$. [3, 6]
(9) $\chi((C))\ \eta\ \sigma$. [1, 7, 8]
$\Gamma\ \eta\ \sigma$. [6, 9]

N44. $[\sigma C] \therefore [\Gamma] : \Gamma\ \eta\ \sigma \equiv .\ \Gamma\ \eta\ \Gamma.\ \mathscr{E}\langle\Gamma\rangle : \chi((C))\ \eta\ \sigma : \supset .\ U(C).\ \mathscr{E}\langle\chi((C))\rangle$
Hyp (2) . ⊃
(3) $\mathscr{I}\langle\chi((C))\rangle$. [DN3, 2]
(4) $U(C)$. [N9, 3]
(5) $\mathscr{E}\langle\chi((C))\rangle$. [1, 2]
$U(C).\ \mathscr{E}\langle\chi((C))\rangle$. [4, 5]

N45. $[\sigma C] \therefore [\Gamma] : \Gamma\ \eta\ \sigma\ .\equiv .\ \Gamma\ \eta\ \Gamma.\ \mathscr{E}\langle\Gamma\rangle : U(C).\ \mathscr{E}\langle\chi((C))\rangle : \supset .\ \chi((C))\ \eta\ \sigma$
Hyp (3) . ⊃
(4) $\mathscr{I}\langle\chi((C))\rangle$. [N9, 2]
(5) $\chi((C))\ \eta\ \chi((C))$. [N1, 4]
$\chi((C))\ \eta\ \sigma$. [1, 5, 3]

N46. $[\sigma] \therefore [C] : \chi((C))\ \eta\ \sigma\ .\equiv .\ U(C).\ \mathscr{E}\langle\chi((C))\rangle :\equiv : [\Gamma] :$
$\Gamma\ \eta\ \sigma\ .\equiv .\ \Gamma\ \eta\ \Gamma.\ \mathscr{E}\langle\Gamma\rangle.$ [N42, N43, N44, N45]

N47. $[\Sigma] \therefore [\Phi] : \Sigma\ \eta\ \mathbf{el}\langle\langle\Phi\rangle\rangle\ .\equiv .\ \Phi\ \eta\ \Phi.\ \mathscr{E}\langle\Phi\rangle : \supset : [B] : \Sigma\ \eta\ \mathbf{el}\langle\langle\chi((B))\rangle\rangle.$
$\equiv .\ U(B).\ \mathscr{E}\langle\chi((B))\rangle.$ [N1, N9]

N48. $[\Sigma]\ .\cdot.\ [B]:\Sigma\ \eta\ \mathbf{el}\langle\langle\chi((B))\rangle\rangle\ .\equiv.\ U(B).\ \mathscr{E}\langle\chi((B))\rangle :\supset: [\Phi]:$
$\Sigma\ \eta\ \mathbf{el}\langle\langle\Phi\rangle\rangle\ .\equiv.\ \Phi\ \eta\ \Phi.\ \mathscr{E}\langle\Phi\rangle.$ [N32, N1, N27]

N49. $[\Sigma]\ .\cdot.\ [B]:\Sigma\ \eta\ \mathbf{el}\langle\langle\chi((B))\rangle\rangle\ .\equiv.\ U(B).\ \mathscr{E}\langle\chi((B))\rangle :\equiv: [\Phi]:$
$\Sigma\ \eta\ \mathbf{el}\langle\langle\Phi\rangle\rangle\ .\equiv.\ \Phi\ \eta\ \Phi.\ \mathscr{E}\langle\Phi\rangle.$ [N47, N48]

N50. $[\Gamma\ \Phi\ \sigma]\ .\cdot.\ \Phi\ \eta\ \sigma: [\Delta]: \Delta\ \eta\ \sigma\ .\supset.\ \Delta\ \eta\ \mathbf{el}\langle\langle\Gamma\rangle\rangle :\supset.\ \Gamma\ \eta\ \Gamma.$ [N32]

Starting with N37, use N39 to eliminate E and F, then N41 twice to eliminate D, then N46 first to eliminate C and then to eliminate A, then N49 to eliminate B and finally N50 to eliminate $\Gamma\ \eta\ \Gamma$, we arrive at

N51. $[\Sigma\Phi] \vdots\cdot\vdots\ \Sigma\ \eta\ \mathbf{el}\langle\langle\Phi\rangle\rangle\ .\equiv\vdots\ \vdots\ \Sigma\ \eta\ \Sigma: \Phi\ \eta\ \Phi\ \vdots\ \vdots\ \Phi\ \eta\ \mathbf{el}\langle\langle\Phi\rangle\rangle\ .\supset:\cdot: [\sigma\tau] :\cdot:$
$\mathscr{N}\langle\tau\rangle.\ \Phi\ \eta\ \sigma\ .\cdot.\ [\Gamma]\ .\cdot.\ \Gamma\ \eta\ \tau\ .\equiv: [\Delta]: \Delta\ \eta\ \sigma\ .\supset.\ \Delta\ \eta\ \mathbf{el}\langle\langle\Gamma\rangle\rangle : [\Delta]:$
$\Delta\ \eta\ \mathbf{el}\langle\langle\Gamma\rangle\rangle\ .\supset.\ [\exists\Psi\theta].\ \Psi\ \eta\ \sigma.\ \theta\ \eta\ \mathbf{el}\langle\langle\Delta\rangle\rangle.\ \theta\ \eta\ \mathbf{el}\langle\langle\Psi\rangle\rangle : :\supset.$
$\Sigma\ \eta\ \mathbf{el}\langle\langle\tau\rangle\rangle.$

In N51 note that the only relativisation of quantifiers to $\mathscr{N}$ needed is for τ. Comparing N51 to **M** we see that the construction of the model is finished.

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