A Coq-based Specification of a Monadic Second-Order version of Mereology

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We discuss here the encoding and reasoning about a logical system (mereology) and its related formally specified subsystems (geometry) which is a deductive system of interest in logic and computer science (e.g., formal ontologies). We first propose a sound framework whose objective consists of the specification of an automated and/or semi-automated reasoning process with the purpose of providing an expressive while decidable library. To that aim, we have interpreted Leśniewski's mereology in Monadic Second-Order (MSO) logic expressed in a typed framework using a single type, i.e., the type of names. This library which relies on an efficient interpretation of Leśniewski's basic operator ε as an indicator function allows for an efficient interpretation of all Leśniewski's definitions while involving only two axioms. Leśniewski's other axioms become theorems of our system under the MSO translation, and every theorem of Leśniewski's first-order theory of ontology is also a theorem of our system. The library has been expressed in the Coq theorem prover using CoqHammer, an automated reasoning hammer tool for Coq. This work can be extended to specify e.g., appropriate geometries.

1 Introduction

It is well-known that Leśniewski created the first formal theory of part-whole relations and called it mereology. Tarski has then developed among others, a geometry of solids based on Leśniewski's mereology. Alternatively, [9] following the idea of Tarski, has developed a formal proof that mereology based on a single axiom as stated by [23], is isomorphic to a boolean algebra without the least element (see also [33, 30]). Besides, according to Stone representation theorem [42], every field of sets can be embedded into some boolean algebra. A first consequence is that we have here another approach to ontological foundations with an unified representation of *part-whole* and *is-a* relations.

Many authors have reformulated mereology in a way most logicians today are familiar with ([36]). On the one hand, mereology has been reduced to a set-based theory based only on one binary predicate which stands for the relation "being a part of" [47, 7, 45]. However, this choice has generated multiple issues from the theoretical ([14, 31]) and from the practical side. Concerning the latter, while the generalization relation (*is-a* relation) is part of the OWL-DL suite, the *part-of* relation is not and that fact has resulted in some difficulties (see e.g., [1, 34]). Furthermore, having a decidable model of mereology is crucial if one expects to extend it with e.g., topology or geometry in order to take in account the spatial fragment of formal ontologies.

Whereas decidability of mereological theories has already been investigated, it put strong limitations over expressiveness [46]. From the perspective of a useful mereological theory, more expressiveness is required while decidability is a criteria to propose realizable versions. To address this issue, we present

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in this paper an expressive model of mereology in which (i) we first translate Leśniewski's Ontology (LO) in a set-based framework using Monadic Second Order logic (MSO) with the help of characteristic functions and following both the works of [9, 39], (ii) we extend the model initiated by [9] to provide a complete model of mereology and (iii) we have developed the complete model in a theorem prover (partly solved with automated theorem provers), i.e., with the achievement of a Coq implementation, a library which can be reused in the future for several purposes. MSO extends First Order logic (FOL) by the use of set variables ranging over sets of domain objects together while individual variables range over single domain objects. The choice of MSO is motivated by a number of arguments: (i) the ability to translate MSO formulas to finite state automata is a significant benefit for algorithmic applications, (ii) the availability of quantification over sets enhances the reasoning power and (iii) Monadic Second Order Logic theory of sets is decidable (this is a consequence of the theory of atomic boolean algebras which has a recursive set of decidable completion, see e.g., [28]).

2 Leśniewski's systems: an Overview

S. Leśniewski has proposed a complete system including three layers, (i) a logical layer composed of a two-valued propositional calculus (called protothetic) based on a single primitive, the equivalence construct, (ii) a second layer known as Leśniewski's Ontology (LO) which introduce names, and (iii) a third layer denoted as mereology [25], whose purpose was the description of the world with collective classes[1].

LO can be characterized as a general theory dealing with the logical relationships between names. The support of LO corresponds to the distributive interpretation of classes (like sets). It consists of (i) a primitive category N (i.e., names), (ii) a proposition forming function, i.e., the copula ε, which connect two variables in category N without imposing a type distinction between them, (iii) a single axiom controlling the behavior of terms in N, (iv) ontological definitions and (v) a rule for ontological extensionality. It results in a single axiom [25]:

$$\forall A\, a\, (A\,\varepsilon\, a \equiv (\exists B\, (B\,\varepsilon\, A) \wedge (\forall C\, D\, (C\,\varepsilon\, A \wedge D\,\varepsilon\, A) \rightarrow C\,\varepsilon\, D) \wedge (\forall C\, (C\,\varepsilon\, A \rightarrow C\,\varepsilon\, a)))) \tag{1}$$

The first conjunction of the right side of the equivalence prevents A from being an empty term, the second conjunction states the uniqueness of A while the last conjunction refers to a kind of convergence (anything which is A is also an a). Any well-formed expression can be assigned to exactly one category, one of two primitive categories (truth values or N) or to one of a potentially infinite number of categories that are a combination of the two primitive categories. Only two constant names are introduced through definitions, i.e., the empty term (Λ) and the universal term (Ω):

$$\forall A\ (A\ \varepsilon\ \Lambda \equiv (A\ \varepsilon\ A \wedge \neg(A\ \varepsilon\ A))) \tag{2}$$

$$\forall A\, (A\,\varepsilon\,\Omega \equiv A\,\varepsilon\, A) \tag{3}$$

Many relations are then introduced in LO such as inclusion with three forms, i.e., weak inclusion ($\subset$), strong inclusion ($\sqsubset$) and partial inclusion ($\triangle$), nominal negation ($\neg$), nominal conjunction ($\cap$), nominal disjunction ($\cup$). It is important to mention that definitions are not meta-theoretical abbreviations, but rather introduce new symbols into the object language (they are part of the language itself). More precisely each definition should follow meta-rules to be well-formed.

[1] A collective class does not require an unintuitive distinction between an individual and a totality.

Whereas LO relies on a single category N for names, two subcategories are specified by means of predicates, i.e., singular and plural names[2]. A singular name always occurs as the first argument of ε while the second argument can be either a plural or a singular. A formula like $A \;\varepsilon\; A$ is a predicate proving that A is a singular name, that is, it denotes a single object. As a consequence, two equalities are introduced, one for singular names called *singular_eq* and another for plural names referred to as *weak_eq*.

Mereology build on LO introduces few axioms which are not deducible from logical principles. It introduces collective classes, the name-forming functions "class of", "part of" and "element of" relations together with their properties. The most usual formalization introduces a mereological entity called part as a primitive. Mereology is developed on a minimal collection of axioms and new primitive functors are defined whose the most important is that of class, denoted Kl. Two name forming functors are required, i.e., *pt* (for "part of") and *el* (for "element of") which all belong to the category of functions taking an argument in N and returning a value in N. While the original theory starts with the *pt* primitive (i.e., proper part), it is also possible to start from the *el* (part) which is less restrictive. Therefore, Mereology can be formalized with a single axiom [23, 40] referred to as (M) (all primitive functors can be extracted from (M)):

$$\begin{aligned}
&\forall A\, B\, (A\;\varepsilon\;(el\; B) \;\equiv\; A\,\varepsilon\,A \;\wedge\; B\,\varepsilon\,B \;\wedge\; (B\;\varepsilon\;(el\;B) \;\rightarrow\; \forall a \forall b\, (B\,\varepsilon\,a \wedge \\
&\forall C\, (C\,\varepsilon\,b \equiv \forall D\, (D\,\varepsilon\,a \rightarrow D\,\varepsilon\,(el\,C)) \wedge \forall D\, (D\;\varepsilon\;(el\;C) \rightarrow \exists EF \\
&(E\,\varepsilon\,a \wedge F\,\varepsilon\,(el\,D) \wedge F\,\varepsilon\,(el\,E)))) \rightarrow A\,\varepsilon\,(el\,b))))
\end{aligned} \tag{4}$$

3 The Model of Mereology

It was stated by Tarski that Leśniewski's Mereology determine structures which bear a very strong resemblance to complete Boolean algebras [43, 9]. Every mereological structure can be transformed into complete Boolean lattice by adding the zero element (its non-existence is a consequence of axioms for Mereology). Alternatively, every complete Boolean lattice can be turned into a mereological structure by deleting the zero element. Following this idea, we build upon the early work of Clay [9], i.e., using a single axiom, we will be able to prove that Mereology which relies on the part relation is an atomless boolean algebra (i.e., with zero-deleted). We modify his work using characteristic functions, make a synthesis by relating to the work of [39], and embed it within Monadic Second Order calculus (MSO).

To construct a usable definition for the ε relation, we suggest to resort to characteristic functions[3]. They make it possible to translate statements about sets to statements about functions. Furthermore, recursive functions allow to represent recursive sets, i.e., sets for which there is an algorithm that decides whether or not they contain a given element: in other words, they are computable functions. They will be at the heart of definitions as will show hereafter, i.e., we will specifically use characteristic functions to convert definitions into lemmas.

The language of a MSO-theory contains all boolean connectives, first-order quantification and quantification over unary predicates. We assume a standard way for building the second order one place predicate calculus. Individual variables are described with capital letters, while variables that refer to sets will use lowercase letters. Atomic formulas belong to the sort o of boolean propositions. Complex formulas are constructed in a standard way using propositional binary connectives (i.e., $\in$, $\vee$, $=$) and

[2]Empty names are a case of plural names.

[3]A characteristic function is a function defined on a set X that indicates membership of an element in a subset A of X, having the value true for all elements of A and the value false for those that are not in A

quantifiers binding predicate variables. The formulation of MSO is based on a relational two-sorted language, with a sort of individuals called objects, with variables x, y, z, etc., and a sort of (monadic) predicates, with variables X, Y, Z, etc. denoting sets of objects. Formulae for MSO, denoted ϕ, are given according to the following syntax:

$$\begin{aligned} \texttt{Atoms}\ \tau &::= X \mid true \mid false \\ \texttt{MSO formulae}\ \phi &::= \ \tau = \tau \mid \tau \in x \mid \phi \vee \phi \mid \neg \phi \mid \exists X\, \phi(X) \mid \exists x\, \phi(x) \end{aligned} \tag{5}$$

Formulae are closed under disjunction, negation and quantification over first or second-order variables. Other connectives are defined on the basis of classical equivalences. For example, universal quantification is derived from existential quantification with: $\forall A\ \phi \equiv \neg \exists A\ \neg \phi$, implication $\phi \rightarrow \psi$, from $\neg\phi \vee \psi$, etc. Notice that binary predicate symbols may occur in MSO formulae, however, only the unary ones may be quantified over. MSO formulae are interpreted in the standard set-based model of names, where U denote the universe of names. Individual variables range over objects $A, B, C, \ldots$ such that singletons are such that $\{A\}, \{B\}, \ldots \subseteq U$ and predicate variables range over sets of objects $a, b, \ldots \subseteq U$. It follows that names (i.e., sets) are constrained with the definition:

$$N\,(a)\ \equiv\ a \subseteq U \tag{6}$$

3.1 Formalizing the Basic Ontology

First we need to prove that the mapping translates Leśniewski's single axiom for his ontology into a theorem of our present system. The language of LO includes all semantic categories and one primary constant ε (close to Latin word "est") whose arguments are names. All names (general, individual or empty) belong to the same category of names denoted N. The definable equality of MSO is such that [35] (it coincides with equality on objects):

$$(A\ =\ B)\ \equiv\ \forall a.\ (A \in a\ \rightarrow\ B\ \in\ a) \tag{7}$$

For any set a, we define a characteristic function as a predicate χ which depends on a and a given element A such that $\chi(a,A)$ is true if and only if A belongs to the set a:

$$\chi(a,\,A)\ \equiv\ A \in a\ \wedge\ N(a) \tag{8}$$

Set inclusion and set equality are defined as usual:

$$\begin{aligned} incl\,(a,\,b)\ &\equiv\ \forall A.\ \chi(a,\,A)\ \rightarrow\ \chi(b,\,A) \\ set\_eq\,(a,\,b)\ &\equiv\ \forall A.\ \chi(a,\,A)\ \leftrightarrow\ \chi(b,\,A) \end{aligned} \tag{9}$$

At the interplay between set theory and mereology, singletons play a crucial role [19, 26]. We introduce the *singleton* function which maps every object A to its associated set, $singleton(A)$. In all the remaining part of the paper we will use ι as a symbol to denote the *singleton* function "à la" Russell. Assuming axioms of set theory results in the following equivalence rule:

$$\forall A\, B.\ \chi(\iota\,(A),\,B)\ \equiv\ B = A \tag{10}$$

Then, individuals are introduced according to the following definition:

$$individual(a)\ \equiv \exists A.\ set\_eq(a,\,\iota(A)) \tag{11}$$

in which a denotes a second-order variable. Then, the ε relation of LO relating an individual and a plural name is expressed with the binary predicate η:

$$\begin{aligned}&\eta \ : N \to N \to bool \\ &\eta(a,b) \ \equiv \ individual(a) \ \wedge \ N(b) \ \wedge incl(a,b)\end{aligned} \tag{12}$$

Among the consequences of definitions (12), we can mention equality between names as:

Lemma 3.1. $\forall a.\, \eta(a,b) \ \wedge \ \eta(b,a) \leftrightarrow \ set\_eq(a,b)$

Proof. unfolding eta definition and using definition (9) □

In lemma 3.1, a and b are individuals and then, the set equality can be easily shown to imply an equality between objects X and Y such that $\exists X.\, set\_eq(a,\, \iota(X))$ and $\exists Y.\, set\_eq(b,\, \iota(Y))$.

Lemma 3.2. $\forall x\, y.\ set\_eq(\iota(x), \iota(y)) \leftrightarrow \ x = y$

Proof. in the first case, applying definition 7 and substituting the respective indicator functions we easily derive $A = x$ and $A = y$ and by transitivity, $x = y$. In the second case, rewriting equality and applying reflexivity solves the goal □

In [39] the author has proved that LO is embedded into MSO provided that the LO version does not use creative definitions. Definitions are said to be creative, if using them one can prove some formulas containing no defined terms, which can not be proved without the definitions. Since we have assumed a set-based background together with monadic second-order logic, and provided that all definitions rely on set theory's axioms, it results that MSO definitions satisfy the non-creativity condition. From these basic assumptions the foundational axiom of Leśniewski's ontology can be proved. For that purpose, we rephrase a set of lemmas from Clay's work without proofs according to the above statements:

Lemma 3.3. $\forall A.\, \eta(A,A) \ \leftrightarrow \ individual(A)$

Lemma 3.4. $\forall A\, b.\, \eta(A,b) \ \to \ \eta(A,A)$

Lemma 3.5. $\forall A\, a\, b.\ individual(a) \ \wedge \ A \in a \ \wedge \ \chi(b, A) \ \leftrightarrow \ incl(a,b)$

Lemma 3.6. $\forall a\, b\, c.\, \eta(a,b) \ \wedge \ \eta(c,a) \ \to \ set\_eq(c,a)$

Lemma 3.7. $\forall a\, b\, c.\, \eta(a,b) \ \wedge \ \eta(c,a) \ \to \ \eta(c,b)$

Lemma 3.8. $\forall a\, b\, c\, d.\, \eta(a,b) \ \wedge \ \eta(c,a) \ \wedge \ \eta(d,a) \to \ \eta(c,d)$

Lemma 3.9. $\forall A.\, N(\iota(A))$

Lemma 3.10. $\forall A.\ individual\,(\iota(A))$

Lemma 3.11. $\forall a A.\, \chi(a,\, A) \ \to \ \eta(\iota(A),\, a)$

Lemma 3.12. $\forall a A.\, \eta(\iota(A), a) \to \ A \ \in \ a$

Lemma 3.13. $\forall a A.\, \chi(a,\, A) \ \leftrightarrow \ \eta(\iota(A),\, a)$

Lemma 3.14. $\forall A.\, A \in \iota(A) \ \leftrightarrow \ \eta(\iota(A), \iota(A))$

Lemma 3.15. $\forall a A.\, N(a) \wedge (\forall x\, y.\, \eta(x,\, a) \ \wedge \ \eta(y,\, a) \to \eta(x,\, y)) \ \wedge \ A \ \in \ a \ \to \ \forall B.\, B \in \ a \ \to A \ = \ B$

Lemma 3.16. $\forall a\, b\, c.\, \eta(a,b) \ \wedge \ (\forall d.\, \eta(d,b) \ \to \ \eta(d,c)) \ \wedge (\forall e\, f.\, \eta(e,b) \ \wedge \ \eta(f,b) \ \to \ \eta(e,f)) \ \to \eta(a,b)$

Lemma 3.17. $\forall a\, b.\, set\_eq(a,b) \leftrightarrow \forall A.\, \eta(A,a) \ \leftrightarrow \ \eta(A,b)$

Theorem 3.1. $\forall ab.\, \eta(a,b) \leftrightarrow \exists c.\, \eta(c,a) \wedge (\forall d.\, \eta(d,a) \rightarrow \eta(d,b)) \wedge (\forall cd.\, \eta(c,a) \wedge \eta(d,a) \rightarrow \eta(c,d))$

Proof. using lemmas 3.4, 3.7, 3.8 and 3.16. □

The single axiom of LO has been proved in the MSO setting without any axiomatic assumptions other than usual set-based axioms.

We can generalize the definition of the indicator function as follows. A generic structure $\chi(\phi(a),A)$ with $\phi(a)$ any expression involving the set a as a free variable, and an individual variable A (i.e., an object) is an indicator function that is true only if A belongs to $\phi(a)$. We can show that the following theorem holds:

Theorem 3.2. $\forall a \forall A.\, \chi(\phi(a), A) \;\leftrightarrow\; \eta(\iota(A), \phi(a))$

Proof. from definition 8, we get: $\chi(\phi(a),A) \leftrightarrow A \in \phi(a) \wedge N(\phi(a))$ and from lemma 3.10: $individual(\iota(A)) \wedge A \in \phi(a) \wedge N(\phi(a))$. Since A denotes an individual, we can substitute $A \in \phi(a)$ in the right member with $incl(\iota(A), \phi(a))$. Then, using definition 12, the right member is easily rewritten as $\eta(\iota(A), \phi(a))$ □

We are now able to (i) provide definitions involving indicator functions instantiated with appropriate sets and (ii) apply theorem 3.2 to rewrite these definitions within lemmas. Some indicator functions can be introduced giving rise to powerful specifications. These are respectively nominal negation, nominal disjunction, nominal conjunction and weak inclusion.

$$\begin{gathered} neg : N \rightarrow N \\ \chi(neg\,(a),A) \;\equiv\; individual(\iota(A)) \wedge \neg\eta(\iota(A),a) \end{gathered} \tag{13}$$

$$\begin{gathered} disj : N \rightarrow N \rightarrow N \\ \chi(disj\,(a,b),\, A) \;\equiv\; \eta(\iota(A),a) \vee \eta(\iota(A),b) \end{gathered} \tag{14}$$

$$\begin{gathered} conj : N \rightarrow N \rightarrow N \\ \chi(conj\,(a,b),\, A) \;\equiv\; \eta(\iota(A),a) \wedge \eta(\iota(A),b) \end{gathered} \tag{15}$$

$$\begin{gathered} weak\_incl : N \rightarrow N \rightarrow N \\ \chi(weak\_incl\,(a,b),\, A) \;\equiv\; \eta(\iota(A),a) \;\rightarrow\; \eta(\iota(A),b) \end{gathered} \tag{16}$$

In the following, weak inclusion will be replaced with the symbol $\subseteq$ if no confusion exists. In addition, two functions express a minimal and a maximal existence:

$$\begin{gathered} exists\_at\_least : N \rightarrow bool \\ exists\_at\_least\,(a) \;\equiv\; \exists A\,.\, \eta(A,a) \end{gathered} \tag{17}$$

$$\begin{gathered} exists\_at\_most : N \rightarrow bool \\ exists\_at\_most\,(a) \;\equiv\; \forall BC.\, \eta(B,a) \wedge \eta(C,a) \;\rightarrow\; \eta(B,C) \end{gathered} \tag{18}$$

The empty name Λ is specified with a lemma which require several sub-lemmas not described here:

Lemma 3.18. $\forall A.\, \eta(A,\Lambda) \leftrightarrow \eta(A,A) \wedge \neg\eta(A,A)$

Using these definition a collection of lemmas are proved. Some examples are detailed below.

Lemma 3.19. $\forall A\, B\, a.\, \eta(A,B) \wedge \eta(B,a) \rightarrow \eta(A,a)$

Proof. by 3.7 □

We can deduce that not all individual names are empty and that the empty name does not denote an individual name:

Lemma 3.20. $\forall A.\, \neg\eta(A,\Lambda)$

Proof. by 3.18 □

Lemma 3.21. $\neg(exists\_at\_least\ \Lambda)$

Proof. unfolding definition 17 and applying lemma 3.20 □

Lemma 3.22. $\forall A\, a\, b.\, \eta(A, conj(a,b)) \rightarrow \eta(A,a) \wedge \eta(A,b)$

Proof. unfolding 12 and 15, then by 3.12 □

As observed in [**?**] the zero element of a boolean algebra is defined as the contradictory name in our LO translation. In this case, the boolean algebra provides a semantics for Mereology on top of this translation. More than an hundred of lemmas are inferred from these definitions.

3.2 Formalizing the part-of relation

For that purpose we extend the MSO syntax (5) with the le_o relation such that: $le\_o\ : object \rightarrow object \rightarrow bool$. For more clarity, we write $A \leqslant B$ instead of $le\_o(A,\, B)$. It is constrained by the single axiom for a boolean algebra without zero from Clay's paper which has the following form:

Axiom 3.1. $\forall AB.\, A \leqslant B \leftrightarrow (A \in U \wedge B \in U \wedge (B \leqslant B \rightarrow$
$(\forall \beta\, \alpha.\, incl(\alpha, U) \wedge incl(\beta, U) \wedge B \in \alpha \wedge (\forall C.\, C \in \beta \leftrightarrow$
$((\forall D.\, D \in \alpha \rightarrow D \leqslant C) \wedge (\forall D.\, D \leqslant C \rightarrow \exists EF.$
$E \in \alpha \wedge F \leqslant D \wedge F \leqslant E))) \rightarrow \exists L. set\_eq(\beta, \iota(L)) \wedge$
$A \leqslant L)))$

In addition, we assume that there exists at least two elements in the boolean algebra, axiom which is not too restrictive:

Axiom 3.2. $\exists AB.\, \neg(A = B)$

The first step turns out to prove that the introduced relation is a partial order. Proving that $\leqslant$, is reflexive follows from the specification of axiom 3.1.

Lemma 3.23. $\forall A.\, A \leqslant A$

The following requires the introduction of the supremum $sup(a)$ of a set a as a function together with a predicate definition as a characteristic function:

$$\begin{aligned} sup :\ & N \rightarrow N \\ \chi(sup\,(a),\, A) \ \equiv\ & (\forall D.\, D \in a \rightarrow D \leqslant A) \wedge (\forall D.\, D \leqslant A \rightarrow \\ & \exists EF.\, E \in a \wedge F \leqslant D \wedge F \leqslant E) \end{aligned} \tag{19}$$

in which $\chi(sup\,(a),\, A)$ checks if a given A belongs to the supremum. Using definition 8, lemma 3.24 can be easily established:

Lemma 3.24. $\forall Aa.\, A \in sup\,(a) \leftrightarrow (\forall D.\, D \in a \rightarrow D \leqslant A) \wedge (\forall D.\, D \leqslant A \rightarrow \exists EF.\, E \in a \wedge F \leqslant D \wedge F \leqslant E)$

Then, the following lemmas are derived:

Lemma 3.25. $\forall Aa. A \in a \rightarrow \exists L.\, set\_eq(sup(a), \iota(L))$

Lemma 3.26. $\forall A. A \in (sup\,(\iota(A)))$

Lemma 3.27. $\forall AC.\, (\forall G.\, C \leqslant G) \rightarrow A \in (sup\,(\iota(C)))$

Lemma 3.28. $\forall ABC.\, (\forall G. C \leqslant G) \rightarrow A = B$

Lemma 3.29. $\forall ABD.\, \neg(A = B) \rightarrow (\exists C.\, \neg(D \leqslant C))$

Lemma 3.30. $(\exists AB.\, \neg(A = B)) \rightarrow (\forall C.\, \exists D.\, \neg(C \leqslant D))$

Lemma 3.31. $(\exists AB.\, \neg(A = B)) \rightarrow \forall B.(\forall \beta\, \alpha.\, (B \in \alpha \wedge (\forall C.C \in \beta \leftrightarrow ((\forall D.D \in \alpha \rightarrow D \leqslant C) \wedge (\forall DH.D \leqslant C \wedge \neg(D \leqslant H) \rightarrow \exists EFG.\, E \in \alpha \wedge F \leqslant D \wedge F \leqslant E \wedge \neg(F \leqslant G))))) \leftrightarrow (B \in \alpha \wedge (\forall C.\, C \in \beta \leftrightarrow ((\forall D.\, D \in \alpha \rightarrow D \leqslant C) \wedge (\forall D.\, D \leqslant C \rightarrow \exists EF.\, E \in \alpha \wedge F \leqslant D \wedge F \leqslant E)))))$

Lemma 3.32. $\exists AB.\, \neg(A = B)) \rightarrow (\forall AB.\, A \leqslant B \leftrightarrow (B \leqslant B \rightarrow (\forall \beta\, \alpha.\, incl(\alpha, U) \wedge incl(\beta, U) \wedge B \in \alpha \wedge (\forall C.\, C \in \beta \leftrightarrow ((\forall D.D \in \alpha \rightarrow D \leqslant C)\, (\forall D.\, D \leqslant C \rightarrow \exists EF.E \in \alpha \wedge F \leqslant D \wedge F \leqslant E))) \rightarrow \exists L.\, set\_eq(\beta, \iota(L)) \wedge A \leqslant L))) \leftrightarrow \forall AB.\, A \leqslant B \leftrightarrow (B \leqslant B \rightarrow (\forall \beta\, \alpha.\, incl(\alpha, U) \wedge incl(\beta, U) \wedge B \in \alpha \wedge (\forall C.\, C \in \beta \leftrightarrow ((\forall D.\, D \in \alpha \rightarrow D \leqslant C) \wedge (\forall DH.\, D \leqslant C \wedge \neg(D \leqslant H) \rightarrow \exists EFG.\, E \in \alpha \wedge E \wedge F \leqslant D \wedge F \leqslant E \wedge \neg(F \leqslant G)))) \rightarrow \exists L.\, set\_eq(\beta, \iota(L)) \wedge A \leqslant L))$

Theorem 3.3. $\forall AB. A \leqslant B \leftrightarrow (B \leqslant B \rightarrow (\forall \beta\, \alpha.\, incl(\alpha, U) \wedge incl(\beta, U) \wedge B \in \alpha \wedge (\forall C.\, C \in \beta \leftrightarrow ((\forall D.\, D \in \alpha \rightarrow D \leqslant C) \wedge (\forall DH.\, D \leqslant C \wedge \neg(D \leqslant H) \rightarrow \exists EFG.\, E \in \alpha \wedge F \leqslant D \wedge F \leqslant E \wedge \neg(F \leqslant G)))) \rightarrow \exists L.\, set\_eq(\beta, \iota(L)) \wedge A \leqslant L))$

Proof. by axioms 3.2, 3.1 and lemma 3.32. □

Lemma 3.33. $\forall Aa.\, A \in (sup\, a) \leftrightarrow ((\forall D.\, D \in a \rightarrow D \leqslant A) \wedge \forall DH.\, D \leqslant A \wedge \neg(D \leqslant H) \rightarrow \exists EFG.\, E \in a \wedge F \leqslant D \wedge F \leqslant E \wedge \neg(F \leqslant G))$

Lemma 3.34. $\forall AB.\, (\forall K.\, B \leqslant K \rightarrow A \leqslant K) \rightarrow A \leqslant B$

Lemma 3.35. $\forall ABa.\, A \leqslant B \wedge B \in a \rightarrow \exists L.\, set\_eq(sup(a), \iota(L)) \wedge A \leqslant L$

Lemma 3.36. $\forall Aa.\, A \in a \rightarrow \exists L.set\_eq(sup\, a)(\iota\, L)$

The introduction of the lower bound *lowerBound* relies on the fact that the lower bound of a given object B is the set *lowerBound B* of all values A such that $A \leqslant B$ is true. This is specified as a function together with a predicate definition as a characteristic function:

$$\begin{gathered} lowerBound\ : object \rightarrow N \\ \phi(lowerBound\,(B),\, A)\ \equiv\ A \leqslant B \end{gathered} \tag{20}$$

From this definition, the following lemmas can be proved:

Lemma 3.37. $\forall AB.\, A \in lowerBound(B) \leftrightarrow A \leqslant B$

Lemma 3.38. $\forall A.\, A \in sup\ (lowerBound(A))$

Lemma 3.39. $\forall A.\, set\_eq(\iota(A),\, sup\ (lowerBound(A)))$

Theorem 3.4. $\forall ABC.\, A \leqslant B \,\wedge\, B \leqslant C\ \rightarrow A \leqslant C$

Proof. using lemmas 3.23, 3.37, 3.35, 3.38, 3.39, set extensionality for singletons (from rule (7), set_eq over singletons A and B is equivalent to object equality, $A = B$) and rule (10). □

Theorem 3.5. $\forall AB.\, A \leqslant B \,\wedge\, B \leqslant A\ \rightarrow\ A = B$

Proof. follows from 3.4, rule (7), lemmas 3.37, 3.39 and 3.24 □

According to theorems 3.23, 3.4 and 3.5, relation $\leqslant$ is a partial order relation. The introduction of the part-of relation requires the function definition, referred to as *el*:

$$\begin{aligned} & el:\ N \rightarrow N \\ \chi(el\,(a),\, A)\ \equiv\ & \eta(a,\, a)\ \wedge\ \exists BC.\, B\ \in\ \iota(A) \wedge \\ & C\ \in\ a\ \wedge\ B\ \leqslant\ C \end{aligned} \tag{21}$$

In this definition, χ denotes the characteristic function whose property is defined in the right member. Subsets of names $el(a)\ \subseteq\ U$ are built from all values of A which satisfy the right member. From theorem 3.2, it can be rewrited as $\eta(\iota(A), el(a))$. Following the original work of [**?**], some additional lemmas are required to state that this minimal model satisfies the structure of a boolean algebra with zero deleted. We only recall important lemmas for the sake of simplicity.

Lemma 3.40. $\forall AB.\, \eta(B,\, el(A)) \leftrightarrow (individual(B) \wedge individual(A) \wedge$
$\exists CD.\, C \in B \wedge D \in A \wedge C \leqslant D)$

Lemma 3.41. $\forall AB.\, A \leqslant B \leftrightarrow\ \eta(\iota(A), el(\iota(B)))$

The next lemma uses the above results to substitute objects with their related singletons in axiom 3.1.

Lemma 3.42. $\forall AB.\, \eta(\iota(A),\, el(\iota(B))) \leftrightarrow (A \in U \wedge$
$B \in U \wedge (\eta(\iota(B), el(\iota(B))) \rightarrow \forall\, b\, a.$
$(\eta(\iota(B), a) \wedge (\forall C.\, \eta(\iota(C), b) \leftrightarrow\ (C \in U \wedge$
$\forall D.\ \eta(\iota(D), a) \rightarrow\ \eta(\iota(D), el(\iota(C))) \wedge$
$\forall D.\, \eta(\iota(D), el\,(\iota(C))) \rightarrow \exists EF.\, \eta(\iota(E), a) \wedge$
$\eta(\iota(F), el(\iota(D)))\ \wedge\ \eta(\iota(F), el(\iota(E))) \rightarrow$
$\exists L.\ set\_eq(b, \iota(L))\ \wedge\ \eta(\iota(A), el(\iota(L))))))$

Lemma 3.43. $\forall ab.\ (\exists\, c\, d.\ \eta(c,b) \wedge \eta(d, el(a)) \wedge \eta(d, el(c)) \leftrightarrow\ \exists EF.$
$\eta(\iota(E), b)\ \wedge\ \eta(\iota(F), el(a))\ \wedge \eta(\iota(F), el(\iota(E)))$

Theorem 3.6. $\forall AB.\ \eta(A, el(B)) \leftrightarrow (\eta(A,A)\ \wedge\ \eta(B,B)\ \wedge$
$(\eta(B, el(B)) \rightarrow (\forall\, b\, a.\ N(b) \wedge\ \eta(B, a) \wedge (\forall D.\, \eta(D,\, b) \leftrightarrow$
$(\forall E.\ \eta(E, a) \rightarrow \eta(E, el(D))) \wedge (\forall E.\ \eta(E, el(D)) \rightarrow \exists FG.$
$\eta(F, a)\ \wedge \eta(G, el(E))\ \wedge\ \eta(G, el(F)))) \ \rightarrow\ \eta(A, el(b)))))$

Proof. by lemmas 3.3, 3.4, 3.40, 3.41, 3.42 and 3.43 □

Quantification occurs here over names, i.e., second-order variables. In such a way, theorem 3.6 states that mereology based on the *el* function has the structure of a complete boolean algebra without zero.

Theorem 3.7. *The structures of mereology and those of complete boolean algebra with zero deleted are identical.*

Proof. The first implication is proved according to [**?**].
The second implication has been proved in MSO as detailed above. It follows that the equivalence holds. □

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