Leśniewski's Foundations of Mathematics

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Introduction

During 1927–1931 Leśniewski published a series of articles (169 pages) entitled 'O podstawach matematyki' [On the Foundations of Mathematics] in the journal *Przegląd Filozoficzny* [Philosophical Review], and an abridged English translation of this series is presented here.[1] With the exception of this work, all of Leśniewski's publications appearing after the first World War were written in German, and hence accessible to scholars and logicians in the West.[2] This work, however, since written in Polish, has heretofore not been accessible to most Western readers, and it is hoped that this translation will encourage both the study of Leśniewski's works as well as the further development of his theories.

Leśniewski's foundations of mathematics consists of three theories: Protothetic, which, according to Leśniewski, roughly corresponds "to what is known in the discipline as the 'calculus of equivalent statements', 'Aussagenkalkul', 'theory of deduction' joined with the 'theory of apparent variables' "; Ontology, "which is a kind of modernized 'traditional logic', and with respect to its content and 'strength' most closely approaches Schröder's 'Klassenkalkul' considered as including a theory of 'individuals' "; and Mereology, an axiomatization of the part-whole relation, which Leśniewski initially called "general set theory". Mereology presupposed Ontology, which in turn is grounded in Protothetic. When Leśniewski began the publication of his 'O podstawach matematyki' he intended to present all three theories, but was able to present only Mereology. There is an informal discussion of Ontology in Chapter XI of this series; and the terminological explanations, as his rules were called, for Ontology were published in another journal in 1930. Protothetic was first presented in 1929 in *Fundamenta Mathematicae*, and in 1938 his last work on Protothetic appeared.[3] A year later, he died at the age of 53. During the Warsaw Insurrection of August, 1944 all of his unpublished manuscripts and lecture notes were destroyed.

A systematic account of Leśniewski's foundations of mathematics would require him to present Protothetic first, followed by Ontology, and to conclude with Mereology, and he had initially intended to present his theories in this order using a 'systematico-compendious method' like that of *Principia Mathematica*, but so doing, he said, would require many years. Since there was some urgency to publish his results, he decided to use what he called an 'autobiographico-synoptic method', i.e., to present his theories in the historical order of their development, which required that Mereology be presented at the outset. As Lejewski has pointed out, Leśniewski knew that Mereology required a prior theory of names as well as a propositional calculus, a theory and calculus he had been using informally in his presentation of Mereology. Having established his first axiomatization of Mereology in 1915, he axiomatized Ontology in 1920, and Protothetic was developed in 1923.[4]

Leśniewski developed Mereology as a direct result of his analysis of Russell's antinomy which he first encountered in 1911 in Łukasiewicz's book *O zasadzie sprzeczności u Arystotelesa. Studyum krytyczne* [The Principle of Contradiction in Aristotle. A Critical Study], (Kraków, 1910). This book strongly influenced many Polish philosophers of the period, and according to Sobociński, Leśniewski later admitted

> that it ended his wavering between philosophy and logic, liberating him from the influence of the philosophy of Cornelius and the psychological theories of Petrazycki; after this, he gave himself exclusively to the study of logic and the foundations of mathematics.[5]

This book also introduced Leśniewski to 'symbolic logic', and while he was so fascinated by the antinomy that it became his prime concern for eleven years, 'symbolic logic' filled him with aversion until the early 20's when he elaborated his own symbolism for his three systems. His reflections on the antinomy led him to consider those situations in which he considered an object to be a class or set of objects, and these considerations led him to the concept of a collective class. In 1914 he published his first of three analyses of Russell's antinomy in his article 'Czy klasa

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klas, niepodporządkowanych sobie, jest podporządkowana sobie?' [Is the Class of Classes that are not Subordinate to Themselves Subordinate to Itself?][6]. This analyses was informal, presented in colloquial language, and not based on any axiomatic theory of classes. Although Leśniewski spoke of it later as a poor article, it is important because it foreshadows his use of the Ontological epsilon and adumbrates the nuclear concept of Mereology, viz., the concept of a collective class, a class literally constituted by its members.[7] A year later he developed the first of several axiomatizations of Mereology which was published in 1916 in his monograph *Podstawy ogólnej teoryi mnogości. I* [Foundations of General Set Theory I], and which is recapitulated in the fourth chapter of his 'O podstawach matematyki'.

Leśniewski's long-standing aversion to 'symbolic logic' led him to use colloquial language not only in his first two analyses of Russell's antinomy but in the various axiomatizations of Mereology as well. However, his presentation is so perspicuous that translating his axioms, theorems, and definitions into his subsequently developed symbolic language presents no major problems. (Vide below.)

In the Introduction to 'O podstawach matematyki' Leśniewski explains why he was forced to use the 'autobiographico-synoptic method' in presenting his results, comments briefly on Russell's antinomy, notes that previous attempts to solve the antinomy depart considerably from the historico-intuitive basis from which it developed, and criticizes the attempts of Frege and Zermelo to solve the antinomy as lacking any intuitive justification; he saw Zermelo's various restrictions as devoid of such justification, and asserted "Non-intuitive mathematics does not contain within itself effective remedies for the shortcomings of intuition".

The comments of Whitehead and Russell in *Principia Mathematica* on their '$*1\cdot3 \vdash : q . \supset . p \vee q$' are carefully and devastatingly analyzed to illustrate the 'semantic doubts' that Leśniewski experienced during the first years of his acquaintance with 'symbolic logic', and in the next chapter he discusses his work on Russell's antinomy and some results he attained in 1913 and 1914, and presents arguments showing that no object is a class of classes that are not elements of themselves. In the third chapter he argues forcefully that his conception of a class is consistent with the common usage of the expressions 'class' and 'set' as well as with the usage of Cantor, and gives a vigorous analysis of the views of Frege, Dedekind, Hausdorff, Sierpiński, Fraenkel, and Whitehead and Russell on classes.

Having shown the deficiencies of these various set theories, he then recapitulates his 1916 monograph, *Podstawy ogólnej teoryi mnogości. I*, in Chapter IV in which he claims

> The set of definitions and axioms that I have laid down in this work dedicated to the most general problems of set theory has for me, in comparison with other hitherto known sets of definitions and axioms (Zermelo, Russell, etc.) the virtue that it clears away the 'antinomies' of general set theory without restricting the original Cantorian scope of the expression 'set' ... .

In this axiomatization 'part' is taken as primitive, with 'ingredient', 'class', and 'set' introduced by definitions. In the following chapter he presents additional theorems and definitions that were not included in the 1916 monograph but were obtained during the period prior to 1921, while in Chapter VI he presents the 1918 axiomatization of Mereology in which no defined terms appear in any axioms (as occurred in the earlier presentation), and he shows that it is equivalent to his earlier axiomatization. In the next chapter he presents a third axiomatization (again equivalent to the first) in which 'ingredient' rather than 'part' is taken as primitive, and in the following chapter he describes some conditions established by Kuratowski and Tarski as necessary and sufficient for an object to be the class of objects a, and he derives additional Mereological theorems. In Chapter IX he sets forth additional theorems that were obtained during 1921–1923, and shows that any of the terms '+', 'set', 'class', 'external', 'sum', and 'complement' can be taken as the single primitive term of Mereology. Finally, in Chapter X he shows that Mereology can be axiomatized by taking 'external' as primitive, and that this axiom set is equivalent to the earlier axiomatizations.

As noted earlier, the axioms, definitions, and theorems of 'O podstawach matematyki' are stated in colloquial Polish with a minimum of symbolism; no 'symbolic language' is used, and colloquial Polish has been translated into colloquial English rather than into a 'symbolic language'. However, for those who are more at ease with formalized, symbolic languages, we give a brief glossary for translating colloquial expressions into symbolic expressions, following the usage established by Sobociński and Lejewski. It should be noted that since Mereology presupposes Ontology, the use of 'is' in Mereology corresponds to the Ontological epsilon, so that, e.g. in the axiom 'If P is part of the object Q, then Q is not part of the object P' the two occurrences of 'is' are to be understood as being equivalent to 'ϵ' as characterized by the single axiom of Ontology:

$$(x)(y)[x \epsilon y \equiv ((\exists c)(c \epsilon x) \cdot (c)(d)((c \epsilon x . d \epsilon x) \supset c \epsilon d) \cdot (c)(c \epsilon x \supset c \epsilon y))]$$ [8]

Glossary

P is part of the object Q	$P \epsilon \mathrm{pt}(Q)$
Q is an object	$Q \epsilon Q$
Q is not part of the object P	$Q \epsilon \nu(\mathrm{pt}(P))$ [9]
P is an ingredient of the object Q	$P \epsilon \mathrm{ing}(Q)$
P is the same object as Q	$P = Q$ [10]
P is the class of objects *a*	$P \epsilon \mathrm{Kl}(a)$
no *a* is *b*	$(\exists c) c \epsilon a \cdot (c)(c \epsilon a \supset \sim(c \epsilon b))$ [11]
every *a* is *b*	$(\exists c) c \epsilon a \cdot (c)(c \epsilon a \supset c \epsilon b)$ [12]
P is Q	$P \epsilon Q$
P is a set of objects *a*	$P \epsilon \mathrm{st}(a)$
P is an element of the object Q	$P \epsilon \mathrm{el}(Q)$
P is a subset of the object Q	$P \epsilon \mathrm{sst}(Q)$
P is external to Q	$P \epsilon \mathrm{ex}(Q)$
P is the complement of the object Q with respect to R	$P \epsilon \mathrm{cmpl}(Q, R)$
P is the class of elements of the object R external to Q	$P \epsilon \mathrm{Kl}(\mathrm{el}(R) \cap \mathrm{ex}(Q))$
P is Q + R	$P \epsilon (Q + R)$
P is the class of objects (Q or R)	$P \epsilon \mathrm{Kl}(Q \cup R)$
P is the sum of the objects *a*	$P \epsilon \mathrm{sum}\,(a)$
P is $\phi\alpha\{a, b, \phi\}$	$P \epsilon \phi\alpha\{a, b, \phi\}$

In this translation I have tried to convey something of Leśniewski's style of writing, and I have not tried to somehow convey his meaning independently of his particular way of expressing his meaning. Kotarbiński has said that Leśniewski was "devoured by a passion for an absolute exactness of statement". Quine long ago said "Leśniewski was notable for the degree of prolixity which he was willing to admit in the interest of complete rigor and precision". These characteristics are manifested not only in the logical systems that Leśniewski created but in his manner of writing as well.

Polish, like Latin, has no definite or indefinite article, but idiomatic English requires their use. Rather than producing such expressions as, for example, 'if some object is class of objects *a*', I have preferred to follow the meaning dictated by the context, and have used instead 'if some object is the class of objects *a*'. I have avoided using such expressions as 'if some object is (the) class of objects *a*' since I believe that this device does not contribute enough to warrant using such unsightly and jarring expressions. It should be emphasized that in the statement of the axioms, theorems, and definitions in this translation definite and indefinite articles have no semantical role or status, as, for example, in Definition 1: 'P is an ingredient of the object Q if and only if P is the same object as Q, or is part of the object Q'. This is made clear when we translate the definition into a 'symbolic' language as '$((P \epsilon \mathrm{ing}(Q)) \equiv ((P = Q) \vee (P \epsilon \mathrm{pt}(Q)))$'.

Except in the annotation of proofs, text enclosed in square brackets, '[', ']', does not occur in the original, but is due to the translator. Additionally, footnotes in this translation have been numbered consecutively and placed at the end of the entire work, unlike the original wherein the notes were placed at the bottom of each page.

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Notes

[1] Stanisław Leśniewski, 'O podstawach matematyki', *Przegląd Filozoficzny* **30** (1927), 164–206; **31** (1928), 261–291; **32** (1929), 60–101; **33** (1930), 77–105; **34** (1931), 142–170. The omissions are these: (a) A number of theorems are proved by Leśniewski, but the proofs of only the first ten are translated; subsequently the proofs are indicated in square brackets by citing the previous axioms, theorems, or definitions of Mereology used in the proofs, e.g., '*Theorem CCIL*. If P is part of the object Q, then for some R – (R is an object, not (Q is the same object as R) and Q is a set of objects (P or R)). [Theorems CLX, CXLVIII, CLXIV, Definition VIII, Theorem XIV]'; (b) A long footnote about Whitehead's theory of events has been omitted since it has been translated and included in my article 'Leśniewski's Analysis of Whitehead's Theory of Events', *Notre Dame Journal of Formal Logic* **VII** (1966) 323–327; (c) The last chapter of the series, 'O zdaniach "jednostkowych" typu "A ε *b*"' ['Singular' Sentences of the type 'A ε *b*'], which is a digression from the main topic, Mereology, has been omitted since it has been summarized in detail in my article, 'The Development of Ontology'. This last chapter concluded with the note 'To be continued' but the continuation was never published.

[2] These articles are: (a) 'Über Funktionen, deren Felder Gruppen mit Rücksicht auf diese Funktionen sind', *Fundamenta Mathematicae* **13** (1929), 319–332; (b) 'Über Funktionen, deren Felder Abelsche Gruppen in Bezug auf diese Funktionen sind', *ibid*., **14** (1929), 242–251; (c) 'Grundzüge eines neuen Systems der Grundlagen der Mathematik', *ibid*., **14** (1929), 1–81; (d) 'Über die Grund-

lagen der Ontologie', *Comptes Rendus des séances de la Société des Sciences et des Lettres de Varsovie*, Classe III, **23** (1930), 111–132; (e) 'Uber Definitionen in der sogenannten Theorie der Deduktion', *ibid.*, **24** (1931), 289–309 (an English translation appears in *Polish Logic, 1920–1939*, edited by Storrs McCall, Oxford, Oxford University Press, 1967, pp. 170–187); (f) 'Grundzüge eines neuen Systems der Grundlagen der Mathematik, § 12', as if from *Collectanea Logica* **1** (1938), 61–144. (The journal was never published. According to Sobociński, "At the siege of Warsaw in September, 1939 the printing house of the periodical was completely burnt with all the prepared type blocks and off-prints. All the last proofs of the first volume, most of the prepared off-prints, the archives of the publication escaped in my flat. All this was destroyed in August 1944 during the Warsaw Insurrection". Vide Bolesław Sobociński, 'An Investigation of Protothetic', *Cahiers de l'Institut d'Études Polonaises en Belgique*, No. 5, Bruxelles, 1949, pp. 7–8); (g) 'Einleitende Bemerkungen zur Fortsetzung meiner Mitteilung u.d.T. "Grundzüge eines neuen Systems der Grundlagen der Mathematik" ', as if from *Collectanea Logica* **1** (1938), 1–60 (an English translation appears in McCall, *op. cit.*, pp. 116–169). Off-prints of the last two works mentioned survive in the Harvard College Library and in the library of the University of Münster in Westphalia.

[3] Expositions of Protothetic may be found in: (a) Leśniewski's 'Einleitende Bemerkungen ...'; (b) Jerzy Słupecki, 'St. Leśniewski's Protothetics', *Studia Logica* **1** (1953), 44–112; (c) Sobociński's 'An Investigation of Protothetic'; while expositions of Ontology are to be found in: (a) Tadeusz Kotarbiński, *Elementy teorji poznania, logiki formalnej i metodologji nauk* [Elements of Theory of Knowledge, Formal Logic, and Methodology of the Sciences], Lwów, 1929; translated into English as *Gnosiology – The Scientific Approach to the Theory of Knowledge*, London, 1966; (b) Czesław Lejewski, 'On Leśniewski's Ontology', *Ratio* **1** (1958), 150–176; (c) Jerzy Słupecki, 'S. Leśniewski's Calculus of Names', *Studia Logica* **3** (1955), 7–73; D. P. Henry, *Medieval Logic and Metaphysics*, London, 1972. Outlines of Mereology are in (a) Jerzy Słupecki, 'Towards a Generalized Mereology of Leśniewski', *Studia Logica* **8** (1958), 131–163; (b) Bolesław Sobociński, 'Studies in Leśniewski's Mereology', *Polish Society of Arts and Sciences Abroad* **5** (1954–55), 34–48.

[4] The background for the axiomatization of Ontology is narrated in my essay, 'The Development of Ontology'.

[5] Bolesław Sobociński, 'In Memoriam Jan Łukasiewicz', *Philosophical Studies* (Maynooth, Ireland), **VI** (1956), 11.

[6] *Przegląd Filozoficzny* **XVII** (1914), 63–75. Vide my 'Leśniewski's Analysis of Russell's Antinomy', *Notre Dame Journal of Formal Logic* **XVII** (1976), 19–34 for details.

[7] It is customary to distinguish collective classes from distributive classes. Roughly speaking, classes are construed in the distributive sense in, e.g., Zermelo–Fraenkel set theory, but in the collective sense in Mereology. Russell and Quine have called collective classes 'heaps', and Frege has called them 'wholes'. Quine insists that we must not confuse the two kinds of classes: a class of stones is not a heap of stones. "The heap is indeed a concrete object, as concrete as the stones that make it up; but the class of stones in the heap cannot properly be identified with the heap. For, if it could, then by the same token another class could be identified with the same heap, namely, the class of molecules of stones in the heap. But actually these classes have to be kept distinct; for we want to say that the one has just, say, a hundred members, while the other has trillions." (W. V. Quine, *From a Logical Point of View*, Cambridge, 1953, p. 114). In a letter to Russell (June 7, 1902) Frege made the distinction in the following way. "We regard every physical body as a whole, or system, consisting of parts. ...if we are given a whole, it is not yet determined what we are to envisage as its parts. As parts of a regiment I can regard the battalions, the companies or the individual soldiers, and as parts of a sand pile, the grains of sand or the silicon and oxygen atoms. On the other hand, if we are given a class, it is determined what objects are members of it. The only members of the class of prime numbers are the prime numbers, but not the class of prime numbers of the form $4n + 1$, for this class is not a prime number. The only members of the class of companies of a given regiment are the companies, but not the individual soldiers. For wholes or systems we have the proposition that a part of a part is a part of the whole. This proposition does not hold for classes as regards the objects that are members of them. The relation of a company to a class of companies is quite different from the relation of this company to the regiment of which it is a part." (Gottlob Frege, *Philosophical and Mathematical Correspondence*, edited by Gottfried Gabriel *et al.* Abridged from the German edition by Brian McGuinness, and translated by Hans Kaal, Chicago, 1980, p. 140).

[8] This is (in Peano-Russell notation, rather than in Leśniewski's) the 'long' axiom of Ontology, which may also be based on the shorter single axiom: '$(x)(y)(x \epsilon y \equiv (\exists z)(x \epsilon z \cdot z \epsilon y))$'. (For details vide Bolesław Sobociński, 'Successive Simplifications of the Axiom-System of Leśniewski's Ontology', a translation of his 'O kolejnych uproszczeniach aksjomatyki "ontologji" prof. St. Leśniewskiego', in *Polish Logic 1920–1939*, pp. 188–200.) For informal elucidations of the Ontological epsilon vide Lejewski's 'On Leśniewski's Ontology' and my 'The Development of Ontology'. It should be emphasized that the variables in the schema '$x \epsilon y$' are nominal variables, and that the constants that may be substituted for them are either singular names, terms designating one and only one object, e.g., 'The Vistula', 'Jan Sobieski III', or common names, terms designating more than one object, i.e., having multiple designation, e.g., 'planet', 'red', 'part of Europe', or, finally, empty names, expressions functioning like names but which fail to designate any object, e.g., 'The author of *Slawkenburgius on Noses*', 'centaur', 'Zeus'. Very roughly, if 'a' and 'b' are names in the above sense, then '$a \epsilon b$' is true provided that 'a' designates one and only one object, i.e., is a singular name, and that this object is also designated by 'b'.

[9] Vide Note 110 of the translation.

[10] In Ontology '$P = Q$' is defined as '$P \in Q \cdot Q \in P$'.

[11] Vide Note 112 of the translation.

[12] *Ibid.*

On the Foundations of Mathematics*

Stanisław Leśniewski

An apostate of philosophy but a grateful pupil offers this work in belated jubilee homage to his esteemed and dear professor of philosophy, Dr. Kazimierz Twardowski.

Introduction

The purpose of this work is to put an end to an awkward situation in which I have found myself for a number of years. The situation is that I have a considerable number of unpublished scientific results in various domains of the foundations of mathematics, that the number of these unpublished results continually grows, and that these results, meshing with one another and with the results of other scholars working in this field, continually pile up more technico-editorial difficulties connected with preparing them for print.

Attempting various ways of working up the scientific results I have obtained, I began, among other things, to set them forth by means of a systematico-compendious method, in this respect following on the lines of the famous work of Whitehead and Russell.[1] Yet such a work spreads out over a number of years, and it is difficult for me to determine precisely how much more time would be necessary in order to submit in this way to a broader expert discussion the complex of results which reflections on the foundations of mathematics have already taken me more than ten years.

The awkward situation gets more complicated by the fact that just as I formed some of my own views and reached some scientific results influenced by conversations with my professional colleagues as well as by their hitherto unpublished scientific results, so my views and remarks expressed over a number of years from a university lectern and in numerous scientific discussions contributed to the formation of some views and results of my professional colleagues, who, guided by a fine loyalty to me, put off publishing a number of their scientific results until my related results had been published.

Wishing to hasten the publication of the results of my own investigations into the foundations of mathematics, I have been compelled to change once more the way of presenting things. This time I have decided to use a method of writing which may be called an autobiographico-synoptic method, as opposed to a systematico-compendious method. I have decided for the time being to pass over in silence most of the consequences which previously I had intended to derive explicitly from my various assumptions, and to concentrate on the most perpicuous presentation possible of the foundations and fundamental contours of the several theories which I have constructed. In my exposition I shall do my best to enable the reader to be cognizant of the chronological order and mutual dependence of certain scientific facts, and to orient himself in particular as to how I base some of my theorems or constructions on the hitherto unpublished results of investigations of other researchers.

The system of the foundations of mathematics which is essentially and methodologically new in several respects and whose outline I wish to present in this work includes three deductive theories, the set of which I consider as one of the possible foundations for the whole of the system of the mathematical sciences. These theories are:

(1) the theory which I call protothetic[2], and which corresponds very roughly (with respect to the content of its theorems) to what is known in the discipline as the 'calculus of equivalent statements'[3], 'Aussagenkalkul'[4], 'theory of deduction'[5], joined with the 'theory of apparent variables'[6], etc.;

(2) the theory which I call *ontology*, and which is a kind of modernized 'traditional logic', and with respect to its content and 'strength' most closely approaches Schröder's 'Klassenkalkul'[7] considered as including a theory of 'individuals'[8];

(3) the theory which I call *mereology*, and whose first (and in many respects) imperfect outline I published in a work entitled *Podstawy ogólnej teoryi mnogości. I* [Foundations of General Set Theory I][9].

For me, thus far, Gottlob Frege's *Grundgesetze der Arithmetik*[10] has been the most impressive incorporation of the achievements obtained in the history of the foundations of mathematics with respect to the soundness of a deductive method, as well as the most valuable source of these achievements since the time of the Greeks. As is well known, Frege's system has been shown to be a contradictory system by Bertrand Russell, who constructed his famous 'antinomy' concerning the "class of classes which are not elements of themselves".[11]

Under the pervasive influence of Russell's investigation, the problem of the 'antinomies' has become a central problem in the intellectual efforts of a number of eminent mathematicians. Again and again these efforts depart significantly from the historico-intuitive basis from which the 'antinomies' have developed. This has contributed to the deterioration of the feeling of the difference between the mathematical sciences, construed as deductive theories serving to formulate the heterogeneous reality of the world in the most exact laws, and those noncontradictory deductive systems which indeed assure the possibility of obtaining on their basis a wealth of continually new theorems, but which, however, are simultaneously characterized by a lack of any intuitive-scientific advantages linking them to reality.

In the epilogue to the second volume of the above mentioned *Grundgesetze der Arithmetik* Frege presents a way of transforming his system so that Russell's 'antinomy' can no longer be constructed in it. This way requires the replacement of one of the axioms of the system by another axiom[12] which, on the basis of the general tone of the epilogue mentioned, it may be assumed does not have a sufficient intuitive basis even in the intuitions of the author himself. Ernest Zermelo's architectonically refined construction[13] introduces into 'set theory' a number of restrictions which are devoid of intuitive justification and are directed to removing the 'antinomies' from mathematics. The question as to whether Frege's system, as altered in the way mentioned above, or Zermelo's 'set theory' ever leads to a contradiction is a question which is completely uninteresting from the point of view of the states of intellectual torment faced with reality, of states flowing from an irrefutable, intuitive necessity of believing in the 'truth' of certain assumptions, and in the 'correctness' of certain arguments which together with these assumptions lead to a contradiction. From this point of view the only method for a real 'dissolving' of the 'antinomies' is the method of intuitively prizing open the arguments or assumptions which go to make up the contradiction.[14] Non-intuitive mathematics does not contain within itself effective remedies for the shortcomings of intuition.

Mr. Russell, creating his 'theory of types' to avoid the 'antinomies', refers, among other things, to considerations of an intuitive nature.[15] As is well known, the 'theory of types' is one of the cardinal elements of the above mentioned work of Messrs. Whitehead and Russell.[16] In the field of battle with the 'antinomies' it is hitherto the most representative synthesis. However, even Messrs. Whitehead and Russell are not satisfied with it in its present form.[17]

Both editions of the system of Messrs. Whitehead and Russell have glaring defects.[18] Specifically, the matter of establishing (upon the basis of this system) the conditions which must be satisfied by an expression in order that it be accepted as a definition or that it be added to the system as a new theorem is disastrous.[19]

Mr. Leon Chwistek, in his system of the foundations of mathematics[20], tries to formulate 'directives' for the introduction of definitions and for the addition of new theorems to the system more precisely than do Messrs. Whitehead and Russell.[21] In the present work I give a critique of Mr. Chwistek's system.

I have not encountered in the scientific literature any theoretical conception which both satisfies the requirements which I impose on a deductive theory and which 'dissolves' the existing 'antinomies' in a way I consider adequate. For the present the conception which I wish to set forth below satisfies me in several respects.

The system of the foundations of mathematics which I have constructed owes a number of significant improvements to Mr. Alfred Tarski, docent in the philosophy of mathematics in the University of Warsaw, my pupil at this university from 1919 to 1923, and 'my' doctor in 1924. I shall try explicitly to show this by considering the concrete results of the investigations which Mr. Tarski has conducted in connection with my system. However, by the very nature of things I cannot adequately indicate all of Mr. Tarski's extemporaneous critical comments which prized open some link in my theoretical conceptions during various states of development of the system, as well as all of the subtle and friendly advise and sometimes eschewed suggestions from which I had the opportunity to profit in numerous conversations with Mr. Tarski.

To a significant degree the only axiom of the theory which I call protothetic owes its present form (being the result of numerous successive simplifications in the primitive form of the axiom) to the results of investigations of Mr. Mordechaj Wajsberg, student in the University of Warsaw.

Chapter I: Some questions about the sense of 'logistic' propositions

In 1911 (during my student years) I came across Jan Łukasiewicz's book on the principle of contradiction in Aristotle.[22] From this book, which in its time exerted a significant influence on the intellectual development of a number of Polish 'philosophers' and 'philosophizing' scholars of my generation (for me personally it was a revelation in several respects), I first learned of the existence of 'symbolic logic'[23] and of Mr. Bertrand Russell and his 'antinomy' about the "class of classes which are not their own elements".[24]

The first contact with 'symbolic logic' filled me with an aversion for this discipline for a number of years afterwards. The exposition of its elements contained in Łukasiewicz's book as well as some other expositions of 'symbolic logic', which I took to in turn (wanting to familiarize myself one way or another with the results obtained by the representatives of this science), were for a long time unintelligible to me but it was not, as it seemed to me, my fault. Permeated by the effects of John Stuart Mill (on which, to begin with, I grew up), and 'tuned' to the 'general grammar' and logico-semantical problems à la Mr. Edmund Husserl and the representatives of the so-called Austrian School, I vainly approached the foundations of 'logistic' from this point of view. Incapable of empathy with alien ideas, I became estranged from the discipline, to a considerable degree influenced by the hazy and equivocal comments with which its exponents fit it out. The dominant note of the decidedly sceptical position which I assumed for a number of years with regard to 'symbolic logic' resulted from the fact that I could not understand what is the exact 'meaning' of the axioms and theorems of this theory or 'about what' and 'what' is one wishing to 'assert' by means of these axioms and theorems.

Anyone can easily understand the character of the semantic doubts which came over me through the fruitless attempts over a long period of time to read the works written by 'logicians' if, for example, he carefully analyzes the comments with which Messrs. Whitehead and Russell fit our particular types of expressions in the 'theory of deduction', and considers incidentally to what degree in these comments there inheres an exquisite cruelty with respect to the reader accustomed to applying some weight to what he reads.

Here is a reader of Messrs. Whitehead and Russell's work who wishes, let's say, to know the meaning of the expression '$\vdash : q . \supset . p \vee q$', which appears in their system as the 'primitive proposition' $*1 \cdot 3$.[25]

The authors comment: "This principle states: 'If q is true, then "p or q" is true' ".[26]

Wishing to find some general principle which would be authoritative for this particular interpretation of the 'primitive propositions' $*1 \cdot 3$, the reader might appeal to earlier comments concerning expressions of the type '$\vdash . p$', of the type '$p \vee q$', and of the type '$p \supset q$' since these very types of expression appear in the 'primitive proposition' $*1 \cdot 3$.

Reflecting on the meaning of expressions of the type '$\vdash . p$', the reader comes upon the following passage in Messrs. Whitehead and Russell: "The sign '$\vdash$', called the 'assertion-sign', means that what follows is asserted. It is required for distinguishing a complete proposition, which we assert, from any subordinate propositions contained in it but not asserted".[27] In another place he reads: "In symbols, if p is a proposition, p by itself will stand for the unasserted proposition, while the asserted proposition will be designated by

'$\vdash . p$'.

The sign '$\vdash$' is called the assertion-sign *; it may be read 'it is true that' (although philosophically this is not exactly what it means)".[28]

On the basis of the first of these two passages, the reader might assume that the propositions asserted by the authors in their system are the propositions following the sign of assertion and the dots standing immediately after it in expressions of the type '$\vdash . p$'; on the basis of the second, that the propositions asserted in the system are the very expressions of the type '$\vdash . p$'. The first of these hypotheses seems to be supported by the passage: "The dots after the assertion-sign indicate its range; that is to say, everything following is asserted until we reach either an equal number of dots preceding a sign of implication or the end of the sentence"[29]; the passage – "On all occasions where, in *Principia Mathematica*, we have an asserted proposition of the form '$\vdash . fx$' or '$\vdash . fp$', this is to be taken as meaning '$\vdash . (x) . fx$' or '$\vdash . (p) . fp$' " – seems to speak for the second hypothesis.[30]

After reading all the passages of the work commenting on the sign of assertion and expressions of the type '$\vdash . p$', the reader, not being particularly endowed with rampant thoughtlessness, will not be able, very likely, to say that he understands the authors' intentions. The reader, wishing to get his bearings on the basis of the comments of Messrs. Whitehead and Russell as to the sense which the authors give to expressions of the type '$\vdash . p$', as to how it might be possible to interpret these expressions using expressions of colloquial language without changing the sense of these expressions, finds in the authors' comments only the clear suggestion that it is not possible, using the counterpart expressions of the type 'it is true that p', to interpret the aforesaid expressions without changing their sense. The authors' remark quoted above is the suggestion according to which the assertion sign may be interpreted by using the expression 'it is true that', although 'philosophically' this expression does not mean exactly the same thing that the assertion sign means.

Here, not finding in Messrs. Whitehead and Russell adequate guidance regarding the sense of expressions of the type '$\vdash . p$', the reader is forced to spin out his own improvisations. The nature of these improvisations may become manifest by drawing up, among other things, the following questions:

(1) if an expression 'p' is a proposition, then is the corresponding expression of the type '$\vdash . p$' also a proposition?

(2) if an expression 'p', having a sense, is a proposition, then does the corresponding expression of the type '$\vdash . p$' have the same sense?

(3) what precisely should we consider as the axioms and theorems of the system of Messrs. Whitehead and Russell: the entire expression of the type '$\vdash . p$' or only the part of the expression following the assertion sign and the dots standing immediately after it? In particular, is the whole expression '$\vdash : q . \supset . p \vee q$' the axiom $*1 \cdot 3$, or is it only the expression '$q . \supset . p \vee q$', which constitutes a part of the preceding expression?

Three very diverse conceptions come to mind, and probably more than a few readers of Messrs. Whitehead and Russell's work have improvised something along these lines, more or less, to cope somehow with the sign of assertion and with expressions of the type '$\vdash . p$'. These conceptions are as follows.

Conception A. According to Messrs. Whitehead and Russell, the assertion sign means the same thing as the expression 'we assert that'. Expressions of the type '$\vdash . p$' may be read off by using the corresponding expressions of the type 'we assert that p', without changing their sense. If some expression 'p' is a proposition, then the corresponding expression of the type '$\vdash . p$' is also a proposition. The proposition '$\vdash . p$' has the same sense as the proposition 'we assert that p'. However, it does not have the same sense as the proposition 'p' itself. We must consider entire expressions of the type '$\vdash . p$' as axioms and theorems of Messrs. Whitehead and Russell's system. In particular, the entire expression '$\vdash : q . \supset . p \vee q$', which has the same sense as the expression 'we assert that $q . \supset . p \vee q$', is axiom $*1 \cdot 3$.

Conception B. For Messrs. Whitehead and Russell, the sign of assertion means the same thing as the expression 'that which follows is asserted'. Expressions of the type '$\vdash . p$' may be read off using the corresponding expressions of the type

'that which follows is asserted $\quad p$'

without changing their sense. If an expression 'p' is a proposition, then the corresponding expression of the type '$\vdash . p$' is not a proposition; it is a peculiar jigsaw puzzle of three successive parts, the first of which, the sign of assertion, is a one-term proposition which has the same sense as the proposition 'that which follows is asserted'. The second consists of dots, while the third, the expression 'p', is, as we have assumed, a proposition. This jigsaw puzzle, not being a proposition, does not have the same sense as the expression 'p', which is a proposition. We must take as axioms and theorems of Messrs. Whitehead and Russell's system not expressions of the type '$\vdash . p$', but only the parts of these expressions which follow the assertion sign and the dots standing immediately after it. In particular, not the expression '$\vdash : q . \supset . p \vee q$' but the expression '$q . \supset . p \vee q$', which is a part of the preceding expression, is axiom $*1 \cdot 3$. Contrasting their symbolism with 'ordinary written language'[31], and unconcerned that the reader will not know whether some proposition of 'ordinary written language' in *Principia Mathematica* is or is not a proposition asserted by the authors, Messrs. Whitehead and Russell do see such a danger when it involves a proposition formulated in the 'symbolism', and to avoid this danger they introduce the sign of assertion, which they wish to place before propositions formulated in the 'symbolism' in those and only those cases when they assert these propositions.

Conception C. If some expression 'p', having a sense, is a proposition, then the corresponding expression of the type '$\vdash . p$' has the same sense. Expressions of the type '$\vdash . p$' can, without changing their sense, be read off in the very same way as one reads off the part 'p' of these expressions. Thus, e.g., the expressions '$\vdash : q . \supset . p \vee q$' and '$q . \supset . p \vee q$' can be read in identically the same way, that is, as one reads off the second of these expressions. If some expression 'p' is a proposition, then the corresponding expression of the type '$\vdash . p$' is also a proposition. By having at their disposal in their 'symbolism' beside every meaningful expression 'p', which is a proposition, another and parallel form of the type '$\vdash . p$', which is also a proposition and has the same meaning, the authors wish to grant the form '$\vdash . p$' to those and only those propositions, formulated in the 'symbolism', which they assert. In just this way Messrs. Whitehead and Russell wish to show the reader whether a proposition formulated in the 'symbolism' is or is not asserted by them. In a similar way, by printing different groups of propositions in a system in different kinds of printed letters (which don't influence the sense of the propositions) one could show the reader that, e.g., certain propositions are borrowed from some author. As axioms and theorems of Messrs. Whitehead and Russell we must consider entire expressions of the type '$\vdash . p$'; in particular, axiom $*1 \cdot 3$ is the entire expression '$\vdash : q . \supset . p \vee q$'.

In improvising a conception about expressions of the type '$\vdash . p$' and about the sign of assertion in the system of Messrs. Whitehead and Russell, the reader will be forced to 'turn a blind eye' to various passages contained in the comments of the authors about their system. It would be difficult, e.g., to interpret the remarks of the authors cited above in such a way that the sign of assertion means that that which follows is asserted – and not have this remark 'at odds' with Conception C. Similarly, the forcing of the passage mentioned above in which it is said that "the asserted proposition will be designated by '$\vdash . p$'" to the requirements of Conception A or Conception B would not be easy.

Not prejudging here the question whether and in what way the reader is willing finally to seek the solution to the riddle of the sign of assertion in the system of Messrs. Whitehead and Russell, I note, incidentally, that reflecting on their work from the point of view of any of the three that I have outlined above in an attempt to get a more exact elucidation of the entire matter of the interpretational concepts, the reader may, for substantial reasons, assume a completely sceptical attitude toward the aforementioned work in the area discussed.

In connection with Conception A, if the axioms and theorems of a theory are expressions of the type '$\vdash . p$', and if expressions of the type '$\vdash . p$' have the same sense as the corresponding expressions of the type 'we assert that p', then the axioms and theorems discussed state, obviously, merely that the creators of the given theory assert so-and-so, and they are thusly very special propositions of the creators of the theory. The system composed of such propositions is certainly not a system of logic at all.[32] Rather, it may be considered as some deductive confession of the creators of the theory mentioned.

In connection with Conception B, if Messrs. Whitehead and Russell introduce the assertion sign as a preventative means for avoiding the danger that the reader will not see whether some particular proposition, formulated in the 'symbolism', is or is not a proposition asserted by them, then they are subject to the objection that they behave in practice in a way which is inconsistent with the purpose that they set themselves since they actually place the sign of assertion also before propositions which are formulated in the 'symbolism' but which they do not assert at all. For example, we find in their work the following expression:

'Similarly $\vdash : (y) : (\exists x) . f(x,y)$'.[33]

In this expression the expression '$(y) : (\exists x) . f(x,y)$' is most clearly a proposition formulated in the 'symbolism'. The sign of assertion standing before this expression must be, according to the comments of the authors, interpreted according to Conception B as indicating that the proposition '$(y) : (\exists x) . f(x,y)$' is a proposition asserted by the authors. Yet it is completely clear that Messrs. Whitehead and Russell do not at all assert the aforementioned proposition. If, relevant to the proposition '$(y) : (\exists x) . f(x,y)$' and in relation to a long series of additional propositions found in *Principia Mathematica* and formulated in the 'symbolism', we are certain that they are not propositions asserted by the authors, then this happens to a significant degree because even the sign of assertion, which is so excellently capable of misleading the reader, has not managed, in the given cases, to lead us astray.

With regard to Conception C, the objection brought forth in connection with Conception B may be raised *mutatis mutandis*.

Expanding further the nascent analysis of the 'primitive proposition' $*1 \cdot 3$, and reflecting on the question of the sense of propositions of the type '$q . \supset . p \vee r$', which has been represented in the 'primitive proposition' mentioned by the proposition '$q . \supset . p \vee q$' in the work of Messrs. Whitehead and Russell, the reader comes upon the following passage which establishes the sense of propositions of the type '$p \supset q$' generally: "we put:

$$*1 \cdot 01 . p \supset q . = . \sim p \vee q \text{ Df}.$$

Here the letters 'Df' stand for 'definition'. They, and the sign of equality together, are to be regarded as forming one symbol, standing for 'is defined to mean*'. Whatever comes to the left of the sign of equality is defined to mean the same as what comes to the right of it".[34] On the basis of this passage the reader is able to interpret propositions of the type '$q . \supset . p \vee r$' by means of the corresponding propositions of the type

(a) '$\sim q . \vee . p \vee r$'.

On the basis of the commentary of Messrs. Whitehead and Russell, the reader, wishing to get his bearings as to the sense the authors give to propositions of the type '$\sim p$', comes upon the following passage in their work: "If p is any proposition, the proposition 'not-p' or 'p is false', will be represented by '$\sim p$'.".[35] From this passage the reader can infer that propositions of the type '$\sim p$' may be, according to the position of the authors, interpreted (without changing the sense of these propositions) either by means of the corresponding propositions of the type 'not-p' or by means of the corresponding propositions of the type 'p is false'. Having noted that if some expression 'p' is a proposition, then the corresponding expression of the type 'p is α' may be treated as a proposition having a sense only on the assumption that the subject 'p' of this proposition is used in 'material supposition', and that, consequently, the cited proposition of the type 'p is α' is a proposition about the proposition 'p', and means the same thing as the corresponding proposition of the type " 'p' is α", whose subject " 'p' " is the name of the corresponding proposition 'p' and is not already used in 'material supposition' (just as, e.g., the expression 'between is a bisyllabic expression' may be treated as a meaningful proposition only on the assumption that the subject of the sentence, 'between', is used in 'material supposition', that, consequently, the proposition 'between is a bisyllabic expression' is a proposition about the expression 'between' and means the same thing as the proposition ' "between" is a bisyllabic expression', the subject of which, ' "between" ', is the name of the expression 'between' and is not already used in 'material supposition'), the reader, already accustomed to the sloppy and slapdash use of quotation marks characteristic of Messrs. Whitehead and Russell, as opposed, e.g., to Frege,[36] has the right to assume that in a case when some expression 'p' is a proposition, the authors use the corresponding proposition of the type 'p is false' in the same meaning as the corresponding proposition of the type ' "p" is false', that, consequently, propositions of the type '$\sim p$' may be, according to the position of the authors, interpreted by means of the corresponding propositions of the type 'not-p' and ' "p" is false'. Accordingly, interpreting the left side of any proposition of the type a mentioned above in two ways, the reader immediately obtains two new propositions which are interpretations of the corresponding propositions of the type '$q . \supset . p \vee r$'. These are the corresponding propositions of the types

(b) 'not-$q . \vee . p \vee r$'

and

(c) ' "q" is false $. \vee . p \vee r$'.

Wishing to understand the sense which the authors give to expressions of the type '$p \vee q$', the reader comes upon the following passage in their work: " If p and q are any propositions, the proposition 'p or q', i.e., 'either p is true or q is true', where the alternatives are to be not mutually exclusive, will be represented by

'$p \vee q$' ".[37]

Circumstances analogous to those which might induce the reader to interpret propositions of the type 'p is false' by the corresponding propositions of the type ' "p" is false' might also give rise to the reader's assumption that if some expression 'p' is a proposition, then the corresponding propositions of the type 'p is true' do not differ in sense, according to the authors, from the corresponding propositions of the type ' "p" is true'. Thus, on the strength of the passage just cited, the reader could interpret propositions of the type '$p \vee q$' by the corresponding propositions of the type 'p or q' or of the type ' "p" is true or "q" is true'. Accordingly, interpreting the right side of any proposition of the type (a) in two ways, the reader obtains immediately two new propositions which are two new interpretations of the corresponding proposition of the type '$q . \supset . p \vee r$'; these are the corresponding propositions of the types:

(d) '$\sim q . \vee . p$ or r',

and

(e) '$\sim q . \vee .$ "p" is true or "r" is true'.

On this basis, interpreting further, but in only one way, propositions of the types (a), (d) and (e) by the corresponding propositions of the types:

(f) ' "$\sim q$" is true or "$p \vee r$" is true',
(g) ' "$\sim q$" is true or "p or r" is true',
(h) ' "$\sim q$" is true or " "p" is true or "r" is true" is true',

the reader has three more types of propositions which are indirectly interpretations of the corresponding propositions of the type '$q . \supset . p \vee r$'. Interpreting the right side of propositions of the types (b) and (c) in two ways, the reader may maintain that propositions of the types

(i) 'not-q . ∨ . p or r',
(k) 'not-q . ∨ . "p" is true or "r" is true',
(l) ' "q" is false . ∨ . p or r',
(m) ' "q" is false . ∨ . "p" is true or "r" is true',

also do not, according to the authors, differ in sense from the corresponding propositions of the type 'q . ⊃ . $p \vee r$'. The reader obtains two more interpretations of propositions of this type by interpreting the corresponding propositions of the types b and c by the corresponding propositions of the types

(n) ' "not-q" is true or "$p \vee r$" is true'

and

(o) ' " "q" is false" is true or "$p \vee r$" is true'.

The two fold interpretation of propositions of each of the types (i)–(m) yields eight additional types of propositions which are interpretations of the corresponding propositions of the type 'q . ⊃ . $p \vee r$':

(p) 'not-q or (p or r)',
(q) ' "not-q" is true or "p or r" is true',
(r) 'not-q or ("p" is true or "r" is true)',
(s) ' "not-q" is true or " "p" is true or "r" is true" is true',
(t) ' "q" is false or (p or r)',
(u) ' " "q" is false" is true or "p or r" is true',
(v) ' "q" is false or ("p" is true or "r" is true)',
(x) ' " "q" is false" is true or " "p" is true or "r" is true" is true'.

The licence to interpret propositions of the type 'q . ⊃ . $p \vee r$' by corresponding propositions of each of the types (f)–(h) and (n)–(x) may fill the reader with complete resignation regarding the possibility of grasping the essential terminological proclivities of Messrs. Whitehead and Russell. Considering Propositions (F)–(H) and (N)–(X), belonging respectively to the types (f)–(h) and (n)–(x), and being provided with interpretations of the proposition 'Paris is on the Seine . ⊃ : Warsaw is on the Seine . ∨ . Warsaw is on the Vistula', the reader not only has no reasonable grounds for considering all those propositions as synonymous propositions, but may even quite reasonably, it would seem, believe that none of the Propositions (F)–(H) and (N)–(X) can be synonymous with any of the other propositions, and may ground this belief by referring to the fact that in at least one of the propositions of each pair of the Propositions (F)–(H) and (N)–(X) reference is made to some object (in particular to a certain proposition) about which nothing at all is said in the second of the propositions of the given pair. For example, in the proposition

(P) 'not-(Paris is on the Seine) or (Warsaw is on the Seine or Warsaw is on the Vistula)',

which belongs to the domain of geography, reference is made to Paris, the Seine, Warsaw, and the Vistula, but there is absolutely no reference made to any propositions, in particular to the proposition 'not-(Paris is on the Seine)' nor to the proposition 'Warsaw is on the Seine or Warsaw is on the Vistula', whereas in the proposition

(Q) ' "not-(Paris is on the Seine)" is true or "Warsaw is on the Seine or Warsaw is on the Vistula" is true',

which obviously does not belong to the domain of geography, and which, of course, is not concerned at all with the truth or falsity of these or other propositions, reference is made to the particular propositions 'not-(Paris is on the Seine)' and 'Warsaw is on the Seine or Warsaw is on the Vistula'; similarly, in the proposition

(T) ' "Paris is on the Seine" is false or (Warsaw is on the Seine or Warsaw is on the Vistula)'

reference is made to the proposition 'Paris is on the Seine', which is not referred to at all in either Proposition P or Proposition Q; etc.

Turning to the comments of Messrs. Whitehead and Russell on the 'primitive proposition' *1 · 3 cited on p. 9, and interpreting – according to the previously discussed interpretational schema for propositions of the type 'p is true', whose subject 'p' is a proposition – the proposition 'if q is true, then "p or q" is true' by the conditional sentence 'if "q" is true, then "p or q" is true', the reader may believe that the proposition 'q . ⊃ . $p \vee q$' does not differ in sense, according to the position of the authors, from the aforementioned conditional sentence. According to the passage 'The symbol employed for "p implies q", i.e., for "$\sim p \vee q$", is "$p \supset q$". This symbol may also be read "if p, then q" '[38] – the conditional sentence 'if "q" is true, then "p or q" is true' does not differ in sense from the proposition ' "q" is true . ⊃ . "p or q" is true', nor, consequently, from the proposition '$\sim$("q" is true) . ∨ . "p or q" is true', nor, thus, from its two interpretations resulting from the different ways of reading the left side of this proposition:

(α) 'not – ("q" is true) . ∨ . "p or q" is true'

and

(β) '"“q” is true" is false . ∨ . “p or q” is true'.

And thus it does not differ in sense from the propositions

(γ) 'not – (“q” is true) or “p or q” is true',
(δ) '“not – (“q” is true)” is true or "“p or q” is true" is true',
(ϵ) '"“q” is true" is false or “p or q” is true',
(ζ) '“"“q” is true" is false” is true or "“p or q” is true" is true',

which are interpretations of the two propositions α and β of the type '$p \vee q$', and which were provided above. The reader, consequently, has the basis for interpreting the proposition '$q . \supset . p \vee q$' by each of the Propositions (γ)–(ζ).

The reader, reflecting on the sort of propositions which are mentioned in the Propositions (γ)–(ζ), and not having in connection with this any reasonable grounds for determining which of the Propositions (γ)–(ζ) would have the same sense as any of the others of these propositions or have the same sense as any of the interpretations provided above of the proposition '$q . \supset . p \vee q$', belonging to the types (f)–(h) or (n)–(x) will continue to sag more under the burden of this interpretational tangle. The chaos, which it is impossible to muddle through, and which is the practical outcome of the reader's efforts to grasp the sense of the 'primitive proposition' $*1 \cdot 3$, would be considerably more aggravated if the reader tried to refer to a number of additional comments contained in the work of Messrs. Whitehead and Russell and relevant to the present discussion, but which are consistent neither with the comments of the authors already cited nor with each other.

I have here dwelled somewhat at length on the problem of interpreting the 'primitive proposition' $*1 \cdot 3$ as an illustrative example of the semantic doubts which I experienced during the first years of my association with 'symbolic logic' because I am inclined to believe that the far-reaching misunderstandings about the sense of the fundamental formulas of this discipline is even now an extremely live issue, and these misunderstandings can deter a great number of scientific workers from taking up 'logistic', for whom is insufficient the mere bliss of writing marks and transforming formulas, and who – contrary to the advocates of meaningless mathematics (after all, such people exist[39]) – want to understand the meanings of the formulas transformed, or to understand 'what' and 'about what' one wishes to assert by means of these formulas. I, for one, being in contact more or less regularly with the work of Messrs. Whitehead and Russell since 1914, only in the last four years have realized that the formulas of the so-called theory of deduction result by ignoring the assertion sign next to the intelligible formulas, and that they begin 'to make sense' when the propositions of the type '$\sim p$', '$p \vee q$', '$p \supset q$', etc., occurring in them, are interpreted consistently by the corresponding propositions of the type 'not p', 'p or q', 'if p, then q', etc., supplemented, whenever necessary to avoid misunderstanding, by parentheses adapted to the situation, but in no case – in spite the comments of the authors – to consider as an admissible reading of the aforementioned formulas propositions relating propositions and asserting some relation, as e.g., the relation of 'implication' between propositions. The proposition '$q . \supset . p \vee q$' analyzed above is interpreted precisely in the described manner of interpreting formulas of the 'theory of deduction' – by the proposition 'if q, then (p or q)'; the proposition '$p \vee p . \supset p$' – by the proposition 'if p or p, then p'; the proposition '$p \vee q . \supset . q \vee p$' – by the proposition 'if p or q, then (q or p)'; the proposition '$p \vee (q \vee r) . \supset . q \vee (p \vee r)$' – by the proposition 'if p or (q or r), then (q or (p or r))'; the proposition '$q \supset r . \supset : p \vee q . \supset . p \vee r$' – by the proposition 'if if q, then r, then if p or q, then (p or r)'; the proposition '$q . \supset . p \supset q$' – by the proposition 'if q, then if p, then q'; the proposition '$\sim p . \supset . p \supset q$' – by the proposition 'if not p, then if p, then q', etc. I note, parenthetically, that the interpretation given here of the formulas '$q . \supset . p \supset q$' and '$\sim p . \supset . p \supset q$' does not, obviously, match up with either that of 'philosophical logic' or with that splendid interpretation, à propos at just this point, which, although obscuring the proper sense of the formulas above, interprets them by the propositions “a true proposition is implied by any proposition” and “a false proposition implies any proposition”.[40]

Chapter II: Mr. Russell's antinomy regarding the 'Class of Classes That Are Not Elements of Themselves'

I come back to 'reminiscences' of the year 1911.

I came upon Mr. Łukasiewicz's book, mentioned above, just when my doctoral dissertation entitled 'Przyczynek do analizy zdań egzystencjalnych' [A Contribution to the Analysis of Existential Propositions] was prepared for print. It was written during 1910 and completed in 1911 under the direction of Dr. Kazimierz Twardowski, professor of philosophy at the University of Lwów and my esteemed professor conferring the doctor's degree upon me in 1912.[41] Mr. Łukasiewicz's book was the point of departure for my subsequent work entitled 'Próba dowodu ontologicznej zasady sprzeczności' [An Attempt to Prove the Ontological Principle of Contradiction] written in 1911.[42] In connection with a work of Mr. Tadeusz Kotarbiński entitled 'Zagadnienie istnienia przyszłości'[43] [The Problem of the Existence of the Future], I wrote an article in 1913 devoted to the problem 'Is Truth Only Eternal or is it Eternal and Sempiternal'.[44] At about the same time I was preparing for publication a work entitled 'Krytyka logicznej zasady wyłączonego środka' [Critique of the Logical Principle of Excluded Middle].[45] Living intellectually outside the sphere of the valuable scientific results attained by the exponents of 'mathematical logic', and yielding to numerous fatal addictions stemming from a culture that was unilaterally 'philosophico'-grammatical, I struggled helplessly in the works mentioned above with a series of problems beyond my strength, discovering occasionally the already discovered America. I have mentioned these works because I wish to indicate that I am very sorry that they have been published, and I herewith solemnly 'repudiate' these works (as I have already done from a university lectern) and assert the bankruptcy of the 'philosophico'-grammatical projects of the first period of my research.[46]

I eagerly took up the 'antinomies' during the same time that I was publishing these works one after the other. Since 1911, when I began to become acquanted with them by studying Mr. Russell's 'antinomy' concerning the "class of classes that are not elements of themselves", the problems associated with the 'antinomies' became the most persistant subject of my reflections for more than eleven years. In various sections of the present work I will set forth the particular results that I attained in this area. Here I wish to give an account of some results of an analysis of Mr. Russell's 'antinomy' that were attained in 1913 and 1914. The starting-point of my investigations of the 'antinomy' was its formulation (the earliest one I had encountered) in Mr. Łukasiewicz's book, cited above. However, my later reflections may also be easily applied *mutatis mutandis* to a number of other well-known formulations of Mr. Russell's 'antinomy'.

In his book Mr. Łukasiewicz writes as follows:

We say that the objects belonging to a given class are subordinate to this class.

Generally a class is not subordinate to itself, since as a set of elements it usually has different properties than each of the individual elements. The set of people is not a person, the set of triangles is not a triangle, etc. In some cases things are the other way round. Let us consider, e.g., the concept of a non-empty class, i.e., a class to which some individuals belong. However, not all classes are non-empty, for some are empty, e.g., the classes 'golden mountain', '*perpetuum mobile*', 'square circle' are empty since there are no individuals that could belong to these classes. Consequently, it is possible to distinguish from them those classes to which some individuals belong, and to form the concept of a 'non-empty class'. Non-empty classes fall under this concept as individuals, e.g., the class of people, the class of triangles, the class of even prime numbers (which contains only one element, the number 2), etc. The set of all these classes constitutes a new class, viz., the 'class of non-empty classes'. Now, this class of non-empty classes is also a non-empty class, and thus it is subordinate to itself.

Consequently, since some classes are subordinate to themselves and others not, it is possible (in order to distinguish the former classes from the latter) to form the concept 'class that is not subordinate to itself'. The class of people, of triangles, of even prime numbers, etc., fall under this concept as individuals. The set of all of these classes constitutes the 'class of classes that are not subordinate to themselves'. Let us abbreviate it as the class K.

The question arises: Is the class K subordinate to itself or not? If we assume that the class K is subordinate to itself, then since every class subordinate to the class K is not subordinate to itself, we arrive at the conclusion that the class K is not subordinate to itself. And thus a contradiction results, since from the fact that the class K is subordinate to itself it follows that it is not subordinate to itself.

Wishing to avoid this contradiction, we must assume that the class K is not subordinate to itself. But if it is not subordinate to itself, then it belongs to the class K, and consequently is subordinate to itself. And thus here too a contradiction results since from the fact that the class K is not subordinate to itself it follows that it is subordinate to itself. To whichever side we turn, we encounter a contradiction. What's to be done?[47]

Wishing 'to do something' and yet unable to cast anything sound against the putative assumptions sustaining this 'antinomy' or against the argument leading, on the basis of these assumptions, to the contradiction, I began to reflect on some of those situations where in practice I consider or do not consider some objects as classes or sets of objects (having available the expressions 'class' and 'set', I call only the set of *all* a's the class of a's[48]), and to critically analyze my belief in the particular assumptions of this 'antinomy' from this very perspective. The problem of 'empty classes' was not a subject of my investigations since I treated the concept of 'empty classes' from the very first time I encountered it as a 'mythological' concept, taking without any hesitation the position that

(1) if some object is the class[49] of objects a, then some object is a.[50]

In this way I came to believe that

(2) it constantly happens that some object is a class of such and such objects, and at the same time is a class of completely different objects (thus, e.g., the segment AB of Figure 1 is a class of segments being the segment AC or the segment CB, and at the same time a class of segments being the segment AD or the segment DB)[51], and that

(3) if one and only one object is P, then P is the class of objects P (so, e.g., the segment AB of Figure 1 is the class of segments AB of Figure 1).

Fig. 1.

Trying also to grasp the way in which I actually use expressions of the type 'P is subordinate to the class K' (which, following Łukasiewicz, at that time I used interchangeably with corresponding expressions of the type 'P is an element of the class K'[52]), I laid down a definition in which I stated

(4) P is subordinate to the class K if and only if in some sense of the expression 'a', the following conditions are satisfied: (α) K is the class of objects a, (β) P is an a.[53]

(At the time when I did not know how to use 'quantifiers'[54] I needed in the colloquial language that I was using an analogue of expressions of the types '$(\exists a) . f(a)$', '$(\exists X, a) . f(X, a)$', etc., which are familiar in a 'symbolic'[55] language, and I used corresponding expressions of the types 'in some sense of the expression "a" $f(a)$', 'in some senses of the expressions "X" and "a" $f(X, a)$', etc., treating in practice these complex expressions *mutatis mutandis* just as one treats the corresponding expressions of the types '$(\exists a) . f(a)$', ('$\exists X, a) . f(X, a)$', etc. Thus Proposition 4 was in my language of that period a practical analogue of the propositions written in a more 'symbolic' language: 'P is subordinate to the class K . ≡ . $(\exists a)$. K is the class of objects a . P is an a'[56], and 'P is an element of the class K . ≡ . $(\exists a)$. K is the class of objects a . P is an a', and because of this it harmonized completely with the standard way of using the expression 'element' in the practice of 'set theorists'.)

Using singular propositions of the type 'A is b' in a way permitting one to affirm that

(5) if P is a, then one and only one object is P,

and that

(6) if P is a, then P is P,

I asserted, in accordance with (5) and (3), that

(7) if P is a class, then P is the class of objects P,

and in accordance with (7) and (6), that

(8) if P is a class, then: (α) P is a class of objects P, (β) P is P.

From (8) I derived

(9) if P is a class, then in some sense of the expression 'a' the following conditions are satisfied: (α) P is a class of objects a, (β) P is a,

and from (9), in accordance with (4),

(10) if P is a class, then P is subordinate to the class P.

On the basis of (10) I asserted that

(11) no object is a class that is not subordinate to itself,[57]

and on the basis of (1) and (11) that

(12) no object is a class of classes that are not subordinate to themselves.[58]

Having the strongest belief in (12), I did not feel the slightest shadow of an 'antinomy' in the fact that both the assumption that the class of classes not subordinate to themselves is subordinate to itself, as well as the assumption that the class of classes that are not subordinate to themselves is not subordinate to itself lead to a contradiction, just as, seeing that no object is a round square I did not feel an 'antinomy' in the fact that both the assumption that a round square is a circle, as well as the assumption that a round square is not a circle lead to a contradiction. Then I no longer saw an 'antinomy' in Mr. Russell's construction, no longer believing in the existence of the class of classes that are not subordinate to themselves, and thus rejecting one of the fundamental positions of this construction.[59]

Nevertheless, I considered this construction defective for still another reason.

Considering a figure analogous to Figure 1, I asserted just as in the example of thesis 2, that

(13) AB is a class of segments that are the segment AC or the segment CB,

and

(14) AB is the class of segments that are the segment AD or the segment DB.

Considering that

(15) AC is a segment that is the segment AC or the segment CB,

I derived from (13) and (15) that

(16) in some sense of the expression 'a' the conditions (α) AB is a class of objects a, (β) AC is an a, are satisfied,

and from (4) and (16) that

(17) AC is subordinate to the class AB.

Noting that

(18) AC is not a segment that is the segment AD or the segment DB,

and in view of (14), (17), and (18), obviously now no longer believing that

> if K is the class of objects a, and P is subordinate to the class K, then P is an a,[60]

I rejected in this way the putative assumption of Mr. Russell's construction that would make it possible for me to affirm that

> if the class of classes that are not subordinate to themselves is the class of classes that are not subordinate to themselves, and the class of classes that are not subordinate to themselves is subordinate to the class of classes that are not subordinate to themselves, then the class of classes that are not subordinate to themselves is a class that is not subordinate to itself,

and to affirm, consequently – in view of (6) – that

> if the class of classes that are not subordinate to themselves is subordinate to the class of classes that are not subordinate to themselves, then the class of classes that are not subordinate to themselves is a class that is not subordinate to itself,

and given this it would be possible, from my position, to establish the basis (utilized in the construction of the 'antinomy') permitting the deduction of a contradiction from the assumption that the class of classes that are not subordinate to themselves is subordinate to itself. This fact was the reason why I could no longer see an 'antinomy' in Mr. Russell's construction.[61]

The 'dissolution' of the 'antinomy' concerning the 'class of classes that are not their own elements', which has been sketched in this section, is completely consistent with the theory developed below that is now based on clear, axiomatic foundations.

Chapter III: Various ways of understanding the expressions 'class' and 'set'

The point of departure for all my analyses of Mr. Russell's 'antinomy' has been the conception of a class (or set) that makes it possible to assert of any class (or of any set) of objects that it 'consists' (not necessarily disjointly) of exactly these objects, just as the above mentioned segment AB of Figure 1 (constituting for me, as I have pointed out above pursuant to this conception, the class of segments that are the segment AD or the segment DB) 'consists' of exactly those segments that are the segment AD or the segment DB. On the one hand, my conception is in this respect, in so far as I have been able to observe, completely consistent with the common usage of the expressions 'class' and 'set' in the ordinary language of those who have never been acquanted with any 'theory of classes' or 'theory of sets'. On the other hand, my conception is based on a strong scientific tradition represented more or less consistently by numerous past and present scholars, and particularly by Georg Cantor.

From among the host of Cantor's statements characterizing his position regarding the relation of a set of objects to exactly these objects I cite here only the following statement: "Jede Menge wohlunterschiedener Dinge kann *als ein einheitliches Ding für sich* angesehen werden, in welchem jene Dinge Bestandteile oder constitutive Elemente sind".[62] Thus, e.g., each of the sounds, the set of which is some musical composition, is, according to Cantor's position, a constitutent part of this set, and the musical composition itself consists of the sounds of which it is the set,[63] just as a painting consists of its various conforming parts, whose set it is.[64] While here I am in complete agreement with Cantor, nevertheless, since it transcends my abilities as an interpreter, I cannot settle the issue as to whether in fact some set of sounds that is a musical composition, e.g., *Beethoven's* Fifth Symphony, "verwandt ist mit dem *Platonischen* εἶδος oder ἰδέα, wie ach mit dem, was *Platon* in seinem Dialoge 'Philebos oder das höchste Gut' μιχτόν nennt",[65] nor also to determine to what degree my views on classes and sets are consistent with Cantor's declarations, saying exceedingly little, that by '*Mannichfaltigkeit*' or 'Menge' he understands "jedes Viele welches sich als Eines denken lässt, d.h. jeden Inbegriff bestimmter Elemente, welcher durch ein Gesetz zu einem Ganzen verbunden werden kann"[66], and that "unter einer 'Menge' " he comprehends "jede Zusammenfassung M von bestimmten wohlunterschiedenen Objekten m unserer Anschauung oder unseres Denkens (welche die 'Elemente' von M genannt werden) zu einem Ganzen".[67]

In his works Frege devoted a series of interesting remarks to the issue connected with the position that any class whatsoever of objects *consists* of these particular objects. I cite here a pair of quotations from this author since I wish to project my own views on the relevant matters against the historical background.

In the work cited above devoted to the foundations of arithmetic Frege writes: "müssen natürlich die Begriffe, deren man bedarf, scharf gefasst werden. Das gilt besonders von dem, was die Mathematiker mit dem Worte 'Menge' bezeichnen möchten. Dedekind" ..."braucht das Wort 'System' wohl in derselben Absicht".[68] And somewhat further on: "ist hier besonders deutlich, dass nach Dedekind die Elemente den eigentlichen Bestand des Systemes ausmachen".[69] And still further on:

> auch Schröder sieht im Grunde die Elemente als das an, was seine Klasse ausmacht. Eine leere Klasse dürfte eigentlich bei ihm ebensowenig vorkommen wie ein leeres System bei Dedekind; aber das aus dem Wesen der Sache entspringende Bedürfniss mach sich bei beiden Schriftstellern in verschiedener Weise geltend. Dedekind fährt an der oben abgebrochenen Stelle so fort: "Dagegen wollen wir das leere System, welches gar kein Element enthält, aus gewissen Gründen hier ganz ausschliessen, obwohl es für andere Untersuchungen bequem sein kann, ein solches zu erdichten". Danach wäre also eine solche Erdichtung erlaubt; es wird nur aus gewissen Gründen darauf verzichtet. Schröder wagt die Erdichtung einer leeren Klasse. Beide sind also darin, wie es scheint, mit vielen Mathematikern einig, man dürfe beliebig etwas erdichten, was nicht da ist, ja was sogar undenkbar ist; denn wenn die Elemente das System bilden, so wird das System mit den Elementen zugleich aufgehoben. Wo die Grenzen dieser Erdichtungswillkür liegen, und ob es überhaupt deren gebe, darüber wird wohl wenig Klarheit und Uebereinstimmung zu finden sein.[70]

Frege's remarks about the 'invention' by mathematicians of objects that do not exist are unfortunately and nowadays immensely timely, as may be seen if only from the following examples.

(I) Mr. Felix Hausdorff, in his well-known textbook on 'set theory', writes:

> Eine Menge entsteht durch Zusammenfassung von Einzeldingen zu einem Ganzen. Eine Menge ist eine Vielheit, als Einheit gedacht. Wenn diese oder ähnliche Sätze Definitionen sein wollten, so würde man mit Recht einwenden, dass sie idem per idem oder gar obscurum per obscurius definieren. Wir können sie aber als Demonstrationen gelten lassen, als Verweisungen auf einen primitiven, allen Menschen vertrauten Denkakt, der einer Auflösung in noch ursprünglichere Akte vielleicht weder fähig noch bedürftig ist. Wir wollen uns mit dieser Auffassung begnügen und es als Grundtatsache hinnehmen, dass ein Ding M in eigentümlicher, nicht definierbarer Weise gewisse andere Dinge *a*, *b*, *c*, ...und diese wiederum jenes bestimmen; eine Beziehung, die wir mit den Worten ausdrücken: die Menge M besteht aus den Dingen *a*, *b*, *c*,... .
>
> Eine Menge kann aus einer natürlichen Zahl von Dingen bestehen oder nicht; je nachdem heisst sie *endlich* oder *unendlich*. Beispiele sind einerseits die Menge der Einwohner einer Stadt, der Wasserstoffatome in der Sonne, der natürlichen Zahlen von 1 bis 1000, andererseits die Menge aller natürlichen Zahlen, aller Punkte einer Geraden, aller Kreise in einer Ebene.[71]

And somewhat further on: "Die fundamentale Beziehung eines Dinges *a* zu einer Menge A, der es angehört, bezeichnen wir mit G. Peano in Wort und Formel folgendermassen:

a is Element von A: $a \epsilon$ A".[72]

Having thus stated that a set results by 'Zusammenfassung' 'zu einem Ganzen' of certain objects called elements of this set, and citing examples of sets *consisting* of different kinds of elements, e.g., inhabitants of a town, atoms of hydrogen, etc., Mr. Hausdorff is ready to 'admit' (in the terminology of Dedekind and Frege we could have said 'invent') something that is supposedly a set although it does not have elements, and thus does not consist of them nor does it result by their 'Zusammenfassung' 'zu einem Ganzen'. The author declares: "Wir lassen aus Zweckmässigkeitsgründen auch eine Menge 0, die *Nullmenge* oder *leere Menge*, zu, die kein Element enthält".[73]

(II) In a textbook on 'set theory', whose author is Mr. Wacław Sierpiński, we come across the following definition: "We call the set composed of all and only those elements that belong simultaneously to both A and B the *product* of the sets A and B, and denote it by A × B, A . B, or simply AB".[74] Having noted that pursuant to this definition

(1) if some set X is AB, then the set X is composed of elements belonging simultaneously to both A and B,

and on the other hand, that

(2) if the set X is composed of elements belonging simultaneously to both A and B, then there is at least one element belonging simultaneously to both A and B,

since in the world we inhabit, which is not after all a mythological world, nothing whatsoever can be 'composed' of anything that does not in the least exist – we assert on the basis of (1) and (2) that if some set X is AB, then there exists at least one element belonging simultaneously to A and to B. Pursuant to the above, the fact that there is not in the world the product of any two sets that do not have common elements, is not for Mr. Sierpiński an obstacle to 'inventing' an object that is nevertheless said to be the product of precisely two such sets. Where Mr. Hausdorff has recourse to 'admission', Mr. Sierpiński applies 'introduction'. For the author writes:

> Every set of sets has obviously a designated sum. To make it possible to say the same thing for the product and difference, we must introduce the *empty set* which we will designate by 0. Thus, e.g., the formula
>
> $AB = 0$
>
> asserts that the sets A and B have no common element.[75]

(III) The object 'invented' by Mr. Adolf Fraenkel, if it were to exist at all, would present itself in an interesting manner. It would be possible, as we see, to be tempted to characterize it in the author's terminology as an improper set, that is not properly a set at all, although it is a set. In one of his works Mr. Fraenkel writes:

> der Durchschnitt $\mathscr{D}M = \mathscr{D}(N, P, R, \ldots)$ dagegen ist die Menge aller derjenigen Elemente, die gleichzeitig in *allen* Mengen N, P, R, ... enthalten sind. Die Vereinigungsmenge entspricht dem logischen 'entweder – oder', der Durchschnitt dem logischen 'sowohl – als auch'. Ist z.B. $M_1 = \{1, 2, 3, \ldots\}$, $M_2 = \{2, 3, 4, \ldots\}$, $M_3 = \{3, 4, 5, \ldots\}$ usw. und bedeutet $M = \{M_1, M_2, M_3, \ldots\}$ die abzählbare Menge all dieser (selbst sämtlich abzählbaren) Mengen, so ist die Verenigungsmenge $\mathfrak{S}M$ offenbar gleich $\{1, 2, 3, \ldots\}$ (also gleich M_1), dagegen der Durchschnitt $\mathscr{D}M$ gleich der Nullmenge; denn es gibt keine, wenn auch noch so grosse, natürliche Zahl, die gleichzeitig in *allen* Mengen M_1, $M_2, M_3, \ldots$ vorkommt.[76]

According to the first half of this quotation, we can say that in its second half

(1) the intersection $\mathscr{D}M$ is a set of elements contained simultaneously in each of the sets $M_1, M_2, M_3, \ldots$.

Considering the way in which Mr. Fraenkel uses expressions of the type '$\{a, b, c, \ldots\}$' we establish on the basis of the above quotation that

(2) if X is an element contained simultaneously in each of the sets $M_1, M_2, M_3, \ldots$, then X is a natural number.

According to the last part of this quotation we state that

(3) there is no natural number that is an element contained simultaneously in each of the sets M_1, M_2, $M_3, \ldots$.

On the basis of numerous contexts of his book I have no doubt that from (1) we may, pursuant to the author's position, infer

(4) if Y is an element of the intersection $\mathscr{D}M$, then Y is an element contained simultaneously in each of the sets $M_1, M_2, M_3, \ldots$.

From (2) and (3) it follows that

(5) there is no element contained simultaneously in each of the sets $M_1, M_2, M_3, \ldots$.

From (4) and (5) we see that

(6) the intersection $\mathscr{D}M$ does not contain any element.

For an appropriate use of what we already know, let us examine the following passage: "Aus rein formalen Gründen, namentlich um gewisse Tatsachen einfacher und bequemer aussprechen zu können, führen wir an dieser Stelle noch eine uneigentliche Menge ein, die sogenannte *Nullmenge*".... "Diese ist dadurch definiert, dass sie überhaupt kein Element enthält; sie ist also eigentlich gar keine Menge, soll aber als solche gelten und mit 0 bezeichnet werden".[77] From these passages it follows that

(7) if Z is 0, then Z is an improper set.

(8) if Z does not contain any element, then Z is 0,

and

(9) if Z is 0, then Z is not properly a set at all.

From (8) and (6) we deduce that

(10) the intersection $\mathscr{D}M$ is 0,

and from (7) and (10) that

(11) the intersection $\mathscr{D}M$ is an improper set,

and from (9) and (10) that

(12) the intersection $\mathscr{D}M$ is not properly a set at all.

Consequently, the intersection $\mathscr{D}M$, being, pursuant to (10), the '*Nullmenge*' 'introduced' by Mr. Fraenkel in the second of the cited passages from his work, is precisely – pursuant to (11), (12), and (1) – the above mentioned improper set that is not properly a set at all, although it is a set.[78]

I turn again to Frege. The objections of this scholar directed against all those who accept the existence of the 'empty class' although they themselves state on the other hand that classes *consist of* elements, do not apply, obviously, to my conception of a class that was sketched above in connection with Mr. Russell's 'antinomy'. Believing that if some object is the class of *a*'s (e.g., of people, of points, of square circles), then it consists of precisely these *a*'s, I have always rejected, pursuant to Thesis 1 of p. 16, the existence of theoretical monsters such as the class of square circles, fully understanding that nothing can consist of something which does not exist at all. In my life there has never been a time when I have not been in complete agreement with Frege's lapidary remark *à propos* Ernest Schröder's theory of classes: 'Wenn"...

> eine Klasse aus Gegenständen besteht, eine Sammlung, collective Verenigung von solchen ist, so muss sie verschwinden, wenn diese Gegenstände verschwinden. Wenn wir sämtliche Bäume eines Waldes verbrennen, so verbrennen wir damit den Wald. Eine leere Klasse kann es also nicht geben.[79]

In order to avoid possible misunderstandings I wish here to say a couple of more words relevant to my Thesis 3 of p. 16 and to Frege's article on Schröder's 'algebra of logic' that has been mentioned a couple of times already.

From Thesis 3 it follows that

(A) if one and only one object is an element of the class K, then the element of the class K is the class of elements of the class K.

Believing that

(B) if X is the class of elements of the class K, then the class K is the same object as X,

without arousing any doubts, just as in the considerations of Chapter II, from the point of view of the theory previously mentioned and developed below, but now on axiomatic foundations, I can infer from A and B that

(C) if one and only one object is an element of the class K, then the class K is the same object as the element of the class K.

Thus, if I were to use the expression 'unit class' in a manner permitting one to say that

(D) K is a unit class if and only if one and only one object is an element of the class K,[80]

then I could assert on the basis of D and C that

(E) if a class is a unit class, then it is the same object as its only element.

Being able to obtain Thesis E on the basis of my views on classes is apparently completely consistent with the position of Frege represented in the sentence "Nun ist unsere Annahme, dass singuläre Klassen mit Individuen zusammenfallen, eine notwendige Folge der Auffassung, dass die Klassen aus Individuen bestehen",[81] provided only that the expression appearing in this sentence, "singuläre Klassen mit Individuen zusammenfallen" (which is in the author's colloquial language apparently an equivalent of the form "eine Klasse, die nur aus einem Gegenstande besteht, mit diesem selbst zusammenfällt", that he uses a little earlier) may be, as I assume, actually rendered by accepting the above terminology laid down by means of my Thesis E.

While being able to express Thesis E with complete conviction, at the same time I absolutely reject the view according to which

(E*) every object is a class whose single element is this very object,

since (if only considering segment AB of Figure 1 and considering the fact that from the point of view of my conception of a class (cf., Thesis 17 of Chapter II) the segment AC is an element of the segment AB, which is, pursuant to Thesis 13, a class of segments that are the segment AC or the segment CB), I am clearly entitled to assert that although one and only one object is the segment AB of Figure 1, and although with respect to this, pursuant to Thesis 3, the segment AB is also the class of segments AB, this segment is not, however, in the least a class whose single element is just the segment AB.

Believing that if some class is a unit class, e.g., a class whose only element is some indivisible spatial and temporal 'point' (here I do not prejudge the question whether in general unit classes exist in the world), then it is the same object as its only element, and believing also that in no way is every object a class whose only element is this very object, it would seem that this time I come into glaring conflict with the position espoused by Frege. In a passage, which to facilitate its use I divide into three parts, the author writes:

> (a) Der Zweifel, ob jedes Individuum als Klasse betrachtet werden darf, die nur aus ihm besteht, wird durch folgende Überlegung verstärkt. Wir können für P in unserer vorhin angestellten Betrachtung auch eine Klasse nehmen, die selber eine Menge von Individuen umfasst; denn wie der Verfasser auf S. 148 sagt, kann eine solche Klasse als ein Gedankending und demgemäss auch als Individuum hingestellt werden.[83]
>
> (b) Ist nun Q wie oben die Klasse der mit P zusammenfallenden Gegenstände, so ist Q eine singuläre Klasse, die nur P als Individuum enthält.[84]
>
> (c) Wäre es nun richtig, dass eine singuläre Klasse mit dem Individuum zusammenfiele, das als einziges unter ihr begriffen wird, so fiele P mit Q zusammen. Nehmen wir nun an, es seien *a* und *b* verschiedene Gegenstände, die als Individuen unter P begriffen werden, so würden sie nun auch unter Q begriffen sein;

das hiesse, sowohl *a* als auch *b* fiele mit P zusammen. Folglich fiele auch *a* mit *b* zusammen gegen die erlaubte Annahme, sie seien verschieden.[85]

Thus, Frege oppugns the supposition that "jedes Individuum als Klasse betrachtet werden darf, die nur aus ihm besteht" by appealing to the contradiction that would be entailed by the supposition "dass eine singuläre Klasse mit dem Individuum zusammenfiele, das als einziges unter ihr begriffen wird". By and large it seems that in the article cited the author treats these two suppositions quite indiscriminately. Clearly, such a position cannot be maintained on the basis of my views on classes, if only, as I believe, the first of these suppositions may be expressed (with the admission of the terminology above that I have set up) by the sentence 'every object can be considered as a class whose single element is this very object' (cf. Thesis E*, which is inconsistent with my conception of classes), and the second – by Thesis E, which above I justified from the perspective of my conception of a class. The second supposition, formulated in this way on the basis of my conception of a class, emerges completely unscathed from the considerations of passage *c* because passage *b* (which permits Frege to assert in passage *c* that in the situation he indicates, "sowohl *a* als auch *b* fiele mit P zusammen") does not hold from the standpoint of this conception, and may be refuted from the perspective of my conception of a class if only by referring to the fact that segment AB of Figure 1, being the only object which is the same object as segment AB, is indeed (pursuant to thesis 3, p. 16) the class of objects that are the same object as the segment AB; however, it is not in the least (cf., Thesis D) a unit class because the segment AC as well as the segment AB itself are (cf., Theses 17 and 10 of Chapter II) elements of the segment AB.

I consider the author's peremptory assertion that "die Auffassung, wonach die Klasse aus Individuen besteht, und also das Einzelding mit der singulären Klasse zusammenfällt, in keinem Falle aufrecht erhalten werden kann"[86] as a completely unfounded assertion because from my point of view there is no justification for this passage nor force in its conclusive 'also'.

Opposing the position that every class of objects consists of precisely these objects, Frege sets against this position a conception of a class that he espouses, treating it as the "extension of a concept".[87] He expresses the following mysterious proposition, among others, about these 'extensions of concepts': "Der Umfang eines Begriffes besteht nicht aus den Gegenständen, die unter den Begriff fallen, etwa wie ein Wald aus Bäumen, sondern er hat an dem Begriffe selbst und nur an diesem seinen Halt".[88] In another place, on the subject of 'extensions of concepts' the author writes : "Im Laufe dieser Überlegungen sind wir nochmals darauf hingewiesen worden"

> ...dass der Umfang eines Begriffes seinen Bestand nicht in den Individuen hat, sondern in dem Begriffe selbst; d.h. in dem, was von einem Gegenstande ausgesagt wird, wenn er unter einen Begriff gebracht wird. Dann hat es kein Bedenken, von der Klasse der Gegenstände, die *b* sind, auch zu sprechen, wenn es kein *b* giebt. Und alle leeren Begriffe haben nun denselben Umfang. ...Wir können z.B. für *b* nehmen *sich selbst ungleicher Gegenstand*.[89]

At this point I do not give a complete analysis of Frege's conception that treats classes as 'extensions of concepts' because I have not thus far been able to understand, despite the most sincere efforts to do so, what various authors are actually talking about when they use the expression 'extension of a concept'. If the class of objects *a*, construed according to my conception of classes, consisting of the objects *a*, is not the 'extension of the concept *a*',[90] then, although unable to answer the question what this 'extension of the concept *a*' would have to be, when and where it is possible to encounter this 'extension', and whether something like this exists in the world at all, I am inclined, nevertheless, to cautiously conjecture that we are simply dealing with some object 'invented' by logicians to the vexation of numerous generations. From the above cited passage of Frege that the extension of a concept "hat an dem Begriffe selbst und nur an diesem seinen Halt" I do not grasp a particle more than from the most recondite enunciations of the representatives of 'romantic philosophy', which means that I understand simply nothing at all by this assertion. The assertion that

> Bei solchen Functionen, deren Werth immer ein Wahrheitswerth ist, kann man demnach statt 'Werthverlauf der Function' sagen 'Umfang des Begriffes' und es erscheint zweckmässig, Begriff geradezu eine Function zu nennen, deren Werth immer ein Wahrheitswerth ist[91]

does not in the least clarify 'extensions of concepts' for me because for me the expression 'Werthverlauf der Function' is not a bit more understandable than the expression 'Umfang des Begriffes'.[92] Someone who might hope that he would make it easier on himself in understanding Frege's views on the 'extensions of concepts' by analyzing the relevant views of 'traditional logicians' would be completely disappointed on that score by the following words written by our author in connection with the revision that he made of his earlier position concerning 'extensions of concepts' because of the appearance of Mr. Russell's 'antinomy', discussed above:

> Falls allgemein bei jedem Begriffe erster Stufe von dessen Umfange gesprochen werden darf, so kommt der Fall vor, dass Begriffe denselben Umfang haben, obwohl nicht alle Gegenstände, die unter den einen dieser Begriffe fallen, auch unter den andern fallen.
>
> Damit ist aber der Begriffsumfang im hergebrachten Sinne des Wortes eigentlich aufgehoben. Man darf nicht sagen, dass allgemein der Ausdruck "der Umfang eines ersten Begriffes fällt zusammen mit dem eines zweiten" gleichbedeutend sei mit dem Ausdrucke "alle unter den ersten Begriff fallenden Gegenstände fallen auch unter den zweiten und umgekehrt". Wir sehen aus dem Ergebnisse unserer Ableitung, dass es gar nicht möglich ist, mit den Worten "der Umfang des Begriffes $\Phi(\xi)$" einen solchen Sinn zu verbinden, dass allgemein aus der Gleichheit des Umfanges von Begriffen geschlossen werden könne, dass jeder unter den einen von ihnen fallende Gegenstande auch unter den andern falle.[93]

Mr. Zermelo seems to take the position that every set is the 'extension of a concept' although not by any means is every 'extension of a concept' a set. The author writes:

> Da nun andererseits 'Ordnungstypus einer wohlgeordneten Menge' gewiss ein logisch zulässiger *Begriff* ist, so folgt weiter, was allerdings viel einfacher schon aus der 'Russelschen Antinomie' hervorgeht, dass nicht jeder beliebige Begriffsumfang als Menge behandelt werden darf und dass somit die übliche Mengendefinition zu weit ist. Beschränkt man sich aber in der Mengenlehre auf gewisse feststehende Prinzipien wie die unserem Beweise zugrunde liegenden, einfache Mengen zu bilden und aus gegebenen neue abzuleiten, so lassen sich alle solchen Widersprüche vermeiden.[94]

If I am not mistaken in supposing that Mr. Zermelo takes sets as 'extensions of concepts', then my inability, noted above, to answer the question, what are various authors talking about when they use the expression 'extension of a concept', clearly applies also to Mr. Zermelo's 'set theory'. Although the question has to do with Mr. Zermelo's views on this subject, I can't even guess what the objects could be to which his axiomatization of 'set theory' would apply in even one sense of the expression 'Menge' and of the expression 'ϵ'.[95]

Messrs. Whitehead and Russell write in *Principia Mathematica*:

> The symbols for classes, like those for descriptions, are, in our system, incomplete symbols: their *uses* are defined, but they themselves are not assumed to mean anything at all. That is to say, the uses of such symbols are so defined that, when the *definiens* is substituted for the *definiendum*, there no longer remains any symbol which could be supposed to represent a class. Thus classes, so far as we introduce them, are merely symbolic or linguistic conveniences, not genuine objects as their members are if they are individuals.
>
> It is an old dispute whether formal logic should concern itself mainly with intensions or with extensions. In general, logicians whose training was mainly philosophical have decided for intensions, while those whose training was mainly mathematical have decided for extensions. The facts seem to be that, while mathematical logic requires extensions, philosophical logic refuses to supply anything except intensions. Our theory of classes recognizes and reconciles these two apparently opposite facts, by showing that an extension (which is the same as a class) is an incomplete symbol, whose use always acquires its meaning through a reference to intension.
>
> In the case of descriptions it was possible to *prove* that they are incomplete symbols. In the case of classes, we do not know of any equally definite proof, though arguments of more or less cogency can be elicited from the ancient problem of the One and the Many.* It is not necessary for our purposes, however, to assert dogmatically that there are no such things as classes. It is only necessary for us to show that the incomplete symbols which we introduce as representatives of classes yield all the propositions for the sake of which classes might be thought essential. When this has been shown, the mere principle of economy of primitive ideas leads to the nonintroduction of classes except as incomplete symbols.[96]

The asterisk contained in the passage above refers to a footnote which states:

> Briefly, these arguments reduce to the following: If there is such an object as a class, it must be in some sense *one* object. Yet it is only of classes that *many* can be predicated. Hence, if we admit classes as objects, we must suppose that the same object can be both one and many which seems impossible.[97]

In another place the authors declare: "The following theory of classes, although it provides a notation to represent them, avoids the assumption that there are such things as classes."[98]

On the basis of these statements of Messrs. Whitehead and Russell, I understand the situation to be as follows:

(1) the authors admittedly use in their system various expressions containing 'symbols for classes' as convenient 'symbolic or linguistic conveniences'; however, they do not at all have the intention of saying thereby something about objects that are classes in the sense assumed in the authors' comments on their system, which permits one to say that a class 'is the same' as an 'extension'. In their system Messrs. Whitehead and Russell do not say either that certain objects are classes or that no object is a class in the preceding sense;

(2) in the comments on their system considering the existence of objects that are classes in the above sense and not knowing how to prove their non-existence, when all is said and done, the authors do not believe that an object

could be a class in the sense mentioned, and they attach weight to the argument that "if we admit classes as objects, we must suppose that the same object can be both one and many which seems impossible";

(3) not believing in the existence of objects that are classes in the above sense, and at the same time not having any doubts about the existence of objects that are 'symbols for classes' and belonging like all 'symbols for classes' to 'incomplete symbols',[99] Messrs. Whitehead and Russell are free with their various *façons de parler* (that only obfuscate their real position) of the type contained in one of the cited passages above stating that "an extension (which is the same as a class) is an incomplete symbol", which means, according to the intentions of the authors, the same thing as the proposition "a symbol for an extension (which is the same as a class),[100] is an incomplete symbol", and which is capable of arousing in the reader, because of its literal interpretation, the suspicion, contrary to fact, that the authors at times consider not only 'symbols for classes' but also classes themselves as 'incomplete symbols', and thus as 'genuine objects' of some special kind.

In connection with the above I must add:

Note 1. In the comments to their system I do not know what Messrs. Whitehead and Russell mean by a class. The fact that a 'class' must be, pursuant to the authors' position, identical with an 'extension', does not help me here at all because I also do not know what the authors mean by an extension. So, when they are considering the problem of the existence or non-existence of objects that are 'classes', I also do not know what to make of the existence or non-existence of those objects they are considering. The collection of the authors remarks on their system does not supply enough information on that score. Not understanding the relevant terminology of Messrs. Whitehead and Russell, I cannot determine specifically whether and to what degree their doubts about the existence of objects that are classes, as they construe them, could apply to the specific positions of my conception of classes sketched above. In *Principia Mathematica* I have not found even one passage with respect to which I could feel even the faintest assumption that questions the existence of classes in my sense. Getting in the 'classes' of Messrs. Whitehead and Russell, just as in the 'extensions of concepts' of Frege, the scent of mythical specimens from the copious gallery of 'invented' objects, I cannot for my part shake off the inclination to string along, 'on credit', with the doubts of the authors that objects which are such 'classes' could exist in the world. Mr. Russell's remarks about 'heaps' may shed some light on the relation of my conception of classes to the views presented in the comments of Messrs. Whitehead and Russell on their system. In one of his works Mr. Russell writes as follows:

> We cannot take classes in the *pure* extensional way as simply heaps or conglomerations. If we were to attempt to do that, we should find it impossible to understand how there can be such a class as the null-class, which has no members at all and cannot be regarded as a 'heap'; we should also find it very hard to understand how it comes about that a class which has only one member is not identical with that one member. I do not mean to assert, or to deny, that there are such entities as 'heaps'. As a mathematical logician, I am not called upon to have an opinion on this point. All that I am maintaining is that, if there are such things as heaps, we cannot identify them with the classes composed of their constituents.[101]

If I rightly understand the quoted passage, then the fact that some object P is a 'heap' of a's, consists of all the a's, would, nevertheless, not be for Mr. Russell a sufficient basis for saying that the object P is a 'class' of objects a. Here Mr. Russell's terminology clearly would be in complete discord with my terminology; according to my way of using the expression 'set' and 'class', as well as to the accepted way of using the expression 'heap' in ordinary language (Mr. Russell does not fix the meaning of the expression 'heap' explicitly, taking this expression in a 'rough' condition from ordinary language), I can always say of a 'heap' of a's that it is a set of objects a, and of a 'heap' of objects a, consisting of all the a's, that it is the class of objects a. If I am not mistaken in assuming, on the basis of the quoted passage, that a class of a's in Mr. Russell's sense would consist, if it existed, of the objects a as its 'constitutents', then the class of objects a in Frege's sense (not, apparently, consisting of the objects a) could not also be the class of objects a in Mr. Russell's sense. The difficulty in understanding what would be necessary to ground the difference, from Mr. Russell's point of view, between a 'heap' of objects a and a 'class' of objects a, if both existed, and if each of them were to *consist* of all the a's – is a difficulty which I do not know how to surmount.

Note 2. I cannot, despite the most sincere wish to, treat seriously the thesis stating that "if we admit classes as objects, we must suppose that the same object can be both one and many which seems impossible", because I feel it contains some gross misunderstanding; even assuming that the expression 'many' does not induce any doubts regarding 'at least how many?', I cannot see any sense at all in saying of some object that it is 'many', though by assuming that the meaning of the expression 'many' is not quantitatively vacillating, that consequently, this expression means, for example, the same thing as 'at least two', I completely understand, for example, saying that 'many objects' exist in the world, or also saying that the segment AB of Figure 1 has 'many' parts. Not seeing in the quoted thesis any sense at all, I cannot consider it as an argument having even the least degree of 'cogency' regarding anything in the world.

Chapter IV: On *The Foundations of General Set Theory. I*

In 1915 I first attempted to give my conception of classes and sets (from whose perspective I have above analyzed Mr. Russell's 'antinomy') the form of a deductive theory. Based on four sentences that I called axioms, and using a number of other sentences that I called definitions, I obtained in what was then my own way a number of what I thought were interesting theorems. I published Part I of the outline of 'general set theory' that was constructed in this way in *Podstawy ogólnej teoryi mnogości. I* [The Foundations of General Set theory. I], mentioned in the introduction above. (The continuation never appeared in print.) In the preface to this work, broadly characterizing my 'general set theory', I wrote:

> The set of definitions and axioms that I have laid down in this work dedicated to the most general problems of set theory has for me, in comparison with other hitherto known sets of definitions and axioms (Zermelo, Russell, etc.), the virtue that it clears away the 'antinomies' of general set theory without restricting the original Cantorian scope of the expression 'set', as can be seen if only from my Axiom III, and on the other hand it does not lead to theorems that are in such glaring conflict with the intuitions of a 'totality', as are the theorems of hitherto 'non-naïve' set theory requiring differentiating an object from the set containing only this object as a single element. I readily admit that some of my theorems, as e.g., Theorem XXVII, may offend more or less the 'mathematical intuitions' of various thinkers contemplating the elegance of certain theoretical constructions independently of whether these constructions contribute to the scientific apprehension of reality in any way, or whether they only serve to justify the mathematical customs reigning in our epoch and to a great extent characterized by inertia. I cannot, however, deny myself the pleasure of stating that I tried to write my work so that it would apply not merely to all kinds of 'free creations' of various more or less Dedekinding creative spirits; thus, it comes about that I am more concerned that my theorems, having as precise a form as possible, should be consistent with the 'common sense' of representatives of 'esprit laique' involved in investigations of a reality that they themselves have not 'created' than I am that what I say should be consistent with those 'intuitions' of professional theoreticians of sets who marched out of the centrifuge (provided with the apparatus of 'free creation') of mathematical minds demoralized by speculative constructions 'detached from reality'.[102]

(Axiom III, mentioned in the above passage, reads: "If some object is m, then some object is the class of objects m".[103] The expression contained in this passage, "the theorems of hitherto 'non-naïve' set theory requiring differentiating an object from the set containing only this object as a single element", is unclear and admits different interpretations. Using this expression I had in mind a position, inconsistent with my understanding of the intuitions of a 'totality', according to which there exist two different objects such that one of these objects is a set containing the second as its only element. Theorem XXVII, which is mentioned in the passage above, has the following form: "the theorem 'if P is an element of the set of objects m, then P is an m' is false".[104]) Regarding the terms that I used in my work, I wrote in its preface:

> Regarding the usage of expressions, I should add that of the mathematical terms which I use I do not define only the expression 'part', assuming that this term cannot create misunderstandings considering that its intuitive character acquires considerable clarity in view of Axioms I and II. The terms 'set' and 'element', usually assumed without definition in set theory, are defined in the present work.[105]

(In my terminology of that period the expression 'mathematical term' had an extremely obscure semantic contour. With the exception of the term 'part' I considered all of the terms that I used without definition in my 'general set theory' as 'non-mathematical' terms taken over from the 'earlier' theories upon which 'general set theory' is based; unfortunately, I did not know how to characterize these 'earlier' theories adequately. Axioms I and II referred to in the passage cited read: Axiom I – 'If the object P is part of the object P_1, then the object P_1 is not part of the object P'[106]; Axiom II – 'If the object P is part of the object P_1, and the object P_1 is part of the object P_2, then the object P is part of the object P_2'.[107] I backed off from the position according to which the terms 'set' and 'element' are "usually assumed without definition in set theory", wishing to remove the possibility of interpretational misunderstandings associated with the term 'definition', even if by now someone could accept without any terminological 'shock' Cantor's statement in 'Beiträge zur Begründung der transfiniten Mengenlehre', cited in Chapter III[108] as a 'definition' of the term 'set' and 'element'[109]).

Since my *Podstawy ogólnej teoryi mnogości. I* is now a 'bibliographical rarity' I will here present the more important results contained in this work. Maintaining, on the whole, the 'style' of the original in the formulation of theses belonging to the system and in the construction of proofs, nevertheless, at the same time I insert into my exposition a number of minor improvements of the original (as to how the mentioned theory appears in its final form, the reader will be able to tell only after reading one of the later chapters of this work devoted to the foundations of 'mereology'.)

AXIOM I. If P is part of the object Q, then Q is not part of the object P.[110]

AXIOM II. If P is part of the object Q, and Q is part of the object R, then P is part of the object R.

DEFINITION I. P is an ingredient of the object Q if and only if P is the same object as Q, or is part of the object Q.[111]

DEFINITION II. P is the class of objects a if and only if the following conditions are satisfied:

(α) P is an object;
(β)[112] every a is an ingredient of the object P;
(γ) for all Q – if Q is an ingredient of the object P, then some ingredient of the object Q is an ingredient of some a.[113]

Examples. (I) The segment AB of Figure 2 is the class of parts of the segment AB because all three conditions indicated in Definition II are here satisfied. (II) Segment AB of Figure 2 is not the class of parts of segment AD because while conditions α and β are indeed here satisfied, condition γ is not satisfied (the segment EB is an ingredient of the segment AB, whereas not (some ingredient of the segment EB is an ingredient of some part of the segment AD)). (III) Segment AC of Figure 2 is not the class of ingredients of the segment AB because while conditions α and γ are indeed here satisfied, condition β is not satisfied (segment AB is an ingredient of the segment AB, but it is not an ingredient of the segment AC).[114]

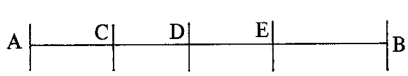

Fig. 2.

AXIOM III. If P is the class of objects a, and Q is the class of objects a, then P is Q.[115]

AXIOM IV. If some object is a, then some object is the class of objects a.[116]

THEOREM I. If P is an object, then P is not part of the object P.[117]

Proof. Let us assume that

(1) P is an object;

from Axiom I we see that[118]

(2) if P is part of the object P, then not(P is part of the object P),

and from (2) that

(3) not(P is part of the object P);

from (1) and (3) we infer that[119]

P is not part of the object P.

THEOREM II. If P is an object, then P is an ingredient of the object P.[120] (Follows from Definition I and the observation that if P is an object, then P is the same object as P.)

THEOREM III. If some object is part of the object P, and Q is an ingredient of the object P, then some ingredient of the object Q is an ingredient of some part of the object P.[121]

Proof. Let us assume that

(1) some object is part of the object P,

and

(2) Q is an ingredient of the object P;

from Definition I and (2) it follows that

(3) Q is the same object as P, or Q is part of the object P;

from (1) it follows that for some R –

(4) R is part of the object P;

and from Definition I and (4), that

(5) R is an ingredient of the object P;

from (4) we see that[122]

(6) R is an object,

and from Theorem II and (6) that

(7) R is an ingredient of the object R;

from (5), (4), and (7), we infer that

(8) some ingredient of the object P is an ingredient of some part of the object P;

from Theorem II and (2) it follows that[123]

(9) Q is an ingredient of the object Q;

from (3), (8), and (9) it follows that

some ingredient of the object Q is an ingredient of some part of the object P.

THEOREM IV. If P is an ingredient of the object Q, and Q is an ingredient of the object R, then P is an ingredient of the object R.[124]

Proof. Let us assume that

(1) P is an ingredient of the object Q,

and

(2) Q is an ingredient of the object R;

from Definition I and (1) we see that

(3) P is the same object as Q, or P is part of the object Q,

and from Definition I and (2) that

(4) Q is the same object as R, or Q is part of the object R;

from (3), (4), and Axiom II we infer that

(5) P is the same object as R, or P is part of the object R;

from Definition I and (5) it follows that

P is an ingredient of the object R.

THEOREM V. If (for all S – if S is an ingredient of the object P, then some ingredient of the object S is an ingredient of the object R) and Q is an ingredient of the object P, then some ingredient of the object Q is an ingredient of some ingredient of the object R, which is an ingredient of the object P.[125]

Proof. Let us assume that

(1) for all S – if S is an ingredient of the object P, then some ingredient of the object S is an ingredient of the object R,

and

(2) Q is an ingredient of the object P;

from (1) and (2) it follows that for some T –

(3) T is an ingredient of the object Q,

and

(4) T is an ingredient of the object R;

from Theorem IV, (3) and (2) we see that

(5) T is an ingredient of the object P;

from Theorem II and (3) we infer that

(6) T is an ingredient of the object T;

from (3), (4), (5), and (6) it follows that

some ingredient of the object Q is an ingredient of some ingredient of the object R, which is an ingredient of the object P.

THEOREM VI. If P is the class of objects *a*, and Q is an ingredient of the object P, then some ingredient of the object Q is an ingredient of some *a* that is an ingredient of the object P.[126]

Proof. Let us assume that

(1) P is the class of objects *a*,

and

(2) Q is an ingredient of the object P;

from Definition II, (1), and (2) it follows that

(3) some ingredient of the object Q is an ingredient of some *a*;

from Definition II and (1) we see that

(4) every *a* is an ingredient of the object P,

and from (3) and (4) that

some ingredient of the object Q is an ingredient of some *a* that is an ingredient of the object P.

THEOREM VII. If P is an object, then P is the class of ingredients of the object P.[127]

Proof. Let us assume that

(1) P is an object;

from Theorem II and (1) we infer that

(2) some object is an ingredient of the object P;

from Theorem II it follows that

(3) for all Q – if Q is an ingredient of the object P, then (Q is an ingredient of the object Q, and Q is an ingredient of the object P),

and from (3) that

(4) for all Q – if Q is an ingredient of the object P, then some ingredient of the object Q is an ingredient of some ingredient of the object P;

from Definition II, (1), (2), and (4) it follows that[128]

P is the class of ingredients of the object P.

THEOREM VIII. If P is an object, then P is the class of objects P.[129]

Proof. Let us assume that

(1) P is an object;

from (1) we infer pursuant to Thesis 6 of Chapter II, harmonizing completely with the system of 'ontology' developed below, that

(2) P is P;

from Theorem II and (1) it follows that

(3) P is an ingredient of the object P,

and from (3) that [130]

(4) for all Q – if Q is P, then Q is an ingredient of the object P;

from Theorem II it follows that

(5) for all Q – if Q is an ingredient of the object P, then (Q is an ingredient of the object Q and Q is an ingredient of the object P),

and from (5) and (2) that

(6) for all Q – if Q is an ingredient of the object P, then some ingredient of the object Q is an ingredient of some P;

from Definition II, (1), (2), (4), and (6) we see that

P is the class of objects P.

THEOREM IX. If some object is part of the object P, then P is the class of parts of the object P.[131]

Proof. Let us assume that

(1) some object is part of the object P;

from Axiom I and (1), we infer that[132]

(2) P is an object;

from (1) and Definition I it follows that

(3) every part of the object P is an ingredient of the object P;

from Theorem III and (1) it follows that

(4) for all Q – if Q is an ingredient of the object P, then some ingredient of the object Q is an ingredient of some part of the object P;

from Definition II, (2), (3), and (4) we see that

P is the class of parts of the object P.

THEOREM X. If P is an ingredient of the object Q, then Q is the class of ingredients of the object Q.[133]

Proof. Let us assume that

(1) P is an ingredient of the object Q;

from Definition I and (1) we infer that

(2) P is the same object as Q or P is part of the object Q,

and from (2) and Axiom I that

(3) Q is an object;

from Theorem VII and (3) it follows that

Q is the class of ingredients of the object Q.

DEFINITION III. P is a set of objects *a* if and only if the following conditions are satisfied:

(α) P is an object;
(β) for all Q – if Q is an ingredient of the object P, then some ingredient of the object Q is an ingredient of some *a* that is an ingredient of the object P.[134]

Examples. (I) The segement AC of Figure 2 is a set of parts of the segment AB, since in this case both conditions indicated in Definition III are satisfied. (II) The segment AC of Figure 2 is not a set of ingredients of the segment DB, since here while condition α is satisfied, condition β is not satisfied (the segment AC is an ingredient of the segment AC, yet not(some ingredient of the segment AC is an ingredient of some ingredient of the segment DB, that is an ingredient of the segment AC).) (III) Segment AE of Figure 2 is not a set of segments that are the segment AC or the segment AB because here once again condition β is not satisfied (segment DE is an ingredient of segment AE, yet not(some ingredient of the segment DE is an ingredient of some segment that is the segment AC or the segment AB, and that is also an ingredient of the segment AE)), although condition α is satisfied. (IV) Not(the round square is a set of round squares), because while condition β is indeed satisfied, condition α is not satisfied.

THEOREM XI. If P is a set of objects a, every a is b, and Q is an ingredient of the object P, then some ingredient of the object Q is an ingredient of some b that is an ingredient of the object P.[135] [Definition III]

THEOREM XII. If P is a set of objects a, and every a is b, then P is a set of objects b.[136] [Theorem XI, Definition III]

THEOREM XIII. If P is an a, then P is a set of objects a.[137] [Theorem II, Definition III]

THEOREM XIV. If P is the class of objects a, then P is a set of objects a.[138] [Theorem VI, Definition III]

THEOREM XV. If P is the class of sets of objects a, and Q is an ingredient of the object P, then some ingredient of the object Q is an ingredient of some a.[139] [Definition II, Definition III, Theorem IV]

DEFINITION IV. P is an element of the object Q if and only if for some a – (Q is the class of objects a, and P is an a).[140]

THEOREM XVI. If P is an element of the object Q, then P is an ingredient of the object Q.[141] [Definition IV, Definition II]

THEOREM XVII. If P is an ingredient of the object Q, then P is an element of the object Q.[142] [Theorem X, Definition IV]

THEOREM XVIII. If P is part of the object Q, then P is an element of the object Q.[143] (Follows from Definition I and Theorem XVII)

THEOREM XIX. If P is an object, then P is an element of the object P.[144] (Follows from Theorems II and XVII).

THEOREM XX. If P is the class of objects a, then every a is an element of the object P.[145] (Follows from Definition II and Theorem XVII)

THEOREM XXI. If P is a set of objects a, then some a is an element of the object P.[146] [Theorem II, Definition III, Theorem XVII]

THEOREM XXII. If P is part of the object Q, then not(for all R and a – if R is an element of some set of objects a, then R is an a).[147] [Theorems IX, XIII, XVIII, I] [148]

THEOREM XXIII. If P is an element of the object Q, and Q is an element of the object R, then P is an element of the object R.[149] [Theorems XVI, IV, XVII]

THEOREM XXIV. If P is the class of sets of objects a, then P is the class of objects a.[150] [Definition II, Theorems XXI, XIII, XV, Definition II] [151]

THEOREM XXV. If P is the class of objects a, then P is the class of sets of objects a.[152] [Theorem XIV, Axiom IV, Theorem XXIV, Axiom III] [153]

THEOREM XXVI. If P is a set of objects a, then P is an ingredient of the class of objects a.[154] [Theorem XXI, Axiom III, Axiom IV, Theorem XXV, Definition II] [155, 156]

THEOREM XXVII. If P is an object, and for all S – if S is an ingredient of the object P, then some ingredient of the object S is an ingredient of the object R, then P is an ingredient of the object R.[157] [Theorem V, Definition III, Theorems XXVI, XXI, X]

DEFINITION V. P is a subset of the object Q if and only if the following conditions are satisfied:
(α) P is an object;
(β) every element of the object P is an element of the object Q.[158]

THEOREM XXVIII. If P is a subset of the object Q, then P is an element of the object Q.[159] [Theorem XIX, Definition V]

THEOREM XXIX. If P is an element of the object Q, then P is a subset of the object Q.[160] [Theorems XIX, XXIII, Definition V] [161]

THEOREM XXX. If P is part of the object Q, then P is a subset of the object Q.[162] (Follows from Theorems XVIII and XXIX).

DEFINITION VI. P is external to Q if and only if the following conditions are satisfied:
(α) P is an object;
(β) no ingredient of the object Q is an ingredient of the object P.[163]

Examples. (I) Segment AC of Figure 2 is external to segment DB since here both conditions indicated in Definition VI are satisfied. (II) Segment AD of Figure 2 is not external to segment CB because although condition α is here indeed satisfied, condition β is not satisfied (segment CD is an ingredient of segment CB and is also an ingredient of segment AD). (III) Not (the round square is external to the sun), because while condition β is here indeed satisfied, condition α is not satisfied.

THEOREM XXXI. If P is an object, then P is not external to P.[164] [Definition VI][165]

THEOREM XXXII. If P is external to Q, then Q is external to P.[166] [Definition VI, Theorems X, II, Definition VI][167]

DEFINITION VII. P is the complement of the object Q with respect to R if and only if the following conditions are satisfied:

(α) Q is a subset of the object R;

(β) P is the class of elements of the object R external to Q.[168]

Examples. (I) Segment AC of Figure 2 is the complement of the class of parts of segment AB external to segment AC with respect to segment AB, because both conditions indicated in Definition VII are here satisfied. (II) Segment AD of Figure 2 is not the complement of the segment CB with respect to segment AB, since while condition α is indeed here satisfied, condition β is not satisfied. (III) Segment AC of Figure 2 is not the complement of segment DB with respect to segment AC, because while condition β, to be sure, is satisfied, condition α is not satisfied.

THEOREM XXXIII. If P is the complement of the object Q with respect to R, then Q is an ingredient of the object Q.[169] (Follows from Definition VII and Theorem II)

THEOREM XXXIV. If P is the complement of the object Q with respect to R, and S is an ingredient of the object P, then some ingredient of the object S is an ingredient of some element of the object R that is external to Q.[170] [Definition VII, Definition II]

THEOREM XXXV. If P is the complement of the object Q with respect to R, and S is an ingredient of the object P, then some ingredient of the object S is an ingredient of the object R.[171] [Theorems XXXIV, XVI, IV]

THEOREM XXXVI. If P is the complement of the object Q with respect to R, then P is external to Q.[172] [Theorems XXXIII, XXXIV, Definition VI, Theorem IV, Definition VI]

THEOREM XXXVII. If P is an object, then P is not the complement of the object P with respect to R.[173] (Follows from Theorems XXXI, XXXVI)

THEOREM XXXVIII. If P is the complement of the object Q with respect to R, and S is the complement of the object Q with respect to R, then P is S.[174] [Definition VII, Axiom III]

THEOREM XXXIX. If P is the complement of the object Q with respect to R, then P is an ingredient of the object R.[175] [Theorems XXXV, XXVII]

THEOREM XL. If P is the complement of the object Q with respect to R, S is an element of the object R, and not(S is an ingredient of the object Q), then not(S is external to P).[176] [Definition VII, Theorems XXXIII, XXVII, XVII, XXIII, Definitions VI, II]

THEOREM XLI. If P is the complement of the object Q with respect to R, and S is an ingredient of the object R, then some ingredient of the object S is an ingredient of some (Q or P).[177] [Theorems XVII, XL, II, Definitions VI, VII]

THEOREM XLII. If P is part of the object Q, then some object is the complement of the object P with respect to Q.[178] [Theorem XXX, Axiom I, Definition I, Theorems II, XXVII, XVII, Definition VI, Axiom IV, Definition VII]

THEOREM XLIII. If P is the complement of the object Q with respect to R, then P is part of the object R.[179] [Theorems XXXIX, XXXVI, Definition VI, Theorems XVII and XVI, Definitions V, VII, I]

THEOREM XLIV. If P is an object, then P is not the complement of the object Q with respect to P.[180] (Follows from Theorem I and Theorem XLIII)

THEOREM XLV. If P is the complement of the object Q with respect to R, then Q is the complement of the object P with respect to R.[181] [Theorems XLIII, XXX, Definition VII, Theorems XXVIII, XXXVI, XXXII, XL, II, Definitions II, VII][182]

THEOREM XLVI. If P is the complement of the object Q with respect to R, then Q is part of the object R.[183] (Follows from Theorems XLV and XLIII)

THEOREM XLVII. If P is an object, then P is not the complement of the object Q with respect to Q.[184] [Axiom I, Theorem XLVI]

THEOREM XLVIII. If P is the complement of the object Q with respect to R, then R is the class of objects (Q or P).[185] [Theorem XLVI, Definition I, Theorems X, XXXIX, XLI, Definition II].[186]

Chapter V: Further theorems and definitions of 'General Set Theory' from the period up to 1920 inclusively

I proceed now to reporting some results from my 'general set theory' that were not included in *Podstawy ogólnej teoryi mnogości. I*, discussed in Chapter IV, and that by and large have hitherto not appeared in print. I begin – pursuant to the title of this chapter – with results from the period to 1920 inclusively. All the results that I am familiar with from my 'general set theory' dating from this period are exclusively my results. In this regard, the situation began to change only in 1921. Merging into a distinctive, separate totality the various theorems and definitions according to the historical criterion of their date of origin, I aim here, as in other analogous cases, to satisfy in a possibly simple way the principle of the chronological '*suum cuique*' in the presentation of my own and other's scientific contributions. In order to preserve the uniformity of the presentation of 'general set theory', I shall formulate below the theses of the system and shall construct proofs in the 'style' that the reader has already encountered in the preceding chapter.

THEOREM IL. If (for all Q – if Q is the same object as P, or Q is part of the object P, then some object that is the same object as Q or part of the object Q is *a* or is part of some *a*) and R is an ingredient of the object P, then some ingredient of the object R is an ingredient of some *a*. [Definition I, Theorem II]

THEOREM L. P is part of the object Q if and only if (P is an ingredient of the object Q and not(P is the same object as Q)). (Follows from Definition I and Axiom I.[187])

THEOREM LI. If P is an ingredient of the object Q, and Q is an ingredient of the object P, then P is the same object as Q. [Definition I, Axiom I]

THEOREM LII. If some object is part of the object P, (for all R – if R is part of the object P, then some ingredient of the object R is an ingredient of the object Q) and S is an ingredient of the object P, then some ingredient of the object S is an ingredient of the object Q. [Definition I, Theorem IV]

THEOREM LIII. If P is an ingredient of the object Q, then every ingredient of the object P is an ingredient of the object Q. [Theorems II, IV]

THEOREM LIV. If P is the class of objects *a*, and Q is the same object as P, or is part of the object P, then some object that is the same object as Q or part of the object Q is an *a* or is part of some *a*. [Definitions I, II].

THEOREM LV. If P is an object, (for all Q – if Q is an *a*, then Q is an ingredient of the object P), and for all Q – if Q is an ingredient of the object P, then some ingredient of the object Q is an ingredient of some *a*, then P is the class of objects *a*. [Theorem II, Definition II]

THEOREM LVI. If P is an object, (for all Q – if Q is an *a*, then Q is the same object as P, or is part of the object P), and for all Q – if Q is the same object as P, or Q is part of the object P, then some object that is the same object as Q or part of the object Q, is an *a* or is part of some *a*, then P is the class of objects *a*. [Definition I, Theorems IL, LV]

THEOREM LVII. P is the class of objects *a* if and only if (P is an object, (for all Q – if Q is an *a*, then Q is an ingredient of the object P) and for all Q – if Q is an ingredient of the object P, then some ingredient of the object Q is an ingredient of some *a*). (Follows from Theorem LV and Definition II)

THEOREM LVIII. If every ingredient of the object P is an ingredient of the object Q, then P is an ingredient of the object Q. [Theorems X, II].

THEOREM LIX. If P is an ingredient of the object Q and not(Q is P), then Q is not an ingredient of the object P. [Theorems X, LI]

THEOREM LX. If every *a* is an ingredient of the object P, and for all Q – if Q is an ingredient of the object P, then some ingredient of the object Q is an ingredient of some *a*, then P is the class of objects *a*. [Theorem X, Definition II][188]

THEOREM LXI. If every *a* is the same object as P or part of the object P, and for all Q – if Q is part of the object P, then some object that is the same object as Q or part of the object Q, is an *a* or is part of some *a*, then P is the class of objects *a*. [Theorems IX, LVI]

THEOREM LXII. If P is the class of objects *a*, and for all Q – (Q is an *a* if and only if Q is a *b*), then P is the class of objects *b*. [Definition II, Theorem LX]

THEOREM LXIII. If P is the class of objects *a*, Q is an ingredient of the object R, and Q is an ingredient of the object P, then some ingredient of the object R is an ingredient of some *a*. [Definition II, Theorem IV]

THEOREM LXIV. If P is the class of objects *a*, Q is the class of objects *b*, and R is the class of objects (P or Q), and S is *a* or *b*, then S is an ingredient of the object R. [Theorems LVII, IV]

THEOREM LXV. If P is the class of objects *a*, Q is the class of objects *b*, R is the class of objects (P or Q), and S is an ingredient of the object R, then some ingredient of the object S is an ingredient of some (*a* or *b*). [Definition II, Theorem LXIII]

THEOREM LXVI. If P is the class of objects *a*, Q is the class of objects *b*, and R is the class of objects (P or Q), then R is the class of objects (*a* or *b*). [Theorems LXIV, LXV, LV]

THEOREM LXVII. P is an ingredient of the object Q if and only if every ingredient of the object P is an ingredient of the object Q. (Follows from Theorems LVIII and LIII)

THEOREM LXVIII. If P is the class of objects *a*, (for all R – if R is the class of objects *a*, then R is the class of objects *b*) and Q is the class of objects *b*, then Q is the class of objects *a*. [Axiom III]

THEOREM LXIX. If every *a* is an ingredient of the object P and an ingredient of the object Q, and for all R – if R is an ingredient of the object P or R is an ingredient of the object Q, then some ingredient of the object R is an ingredient of some *a*, then P is Q. [Theorem LX, Axiom III]

THEOREM LXX. If every *a* is the same object as P or part of the object P, every *a* is the same object as Q or part of the object Q, and for all R – if R is part of the object P or R is part of the object Q, then some object that is the same object as R or part of the object R is *a* or is part of some *a*, then P is Q. [Theorem LXI, Axiom III]

THEOREM LXXI. If P is the class of objects *a*, Q is the class of objects *b*, and for all R – (R is an *a* if and only if R is a *b*), then P is the same object as Q. [Theorem LXII, Axiom III]

THEOREM LXXII. If P is the class of objects *a*, then P is the class of classes of objects *a*. [Theorem VIII, Axiom III, Theorem LXII]

THEOREM LXXIII. If some object is an *a*, then for some P – ((for all Q – if Q is an *a*, then Q is an ingredient of the object P), and for all Q – if Q is an ingredient of the object P, then some ingredient of the object Q is an ingredient of some *a*). (Follows from Axiom IV and Theorem LVII)

THEOREM LXXIV. If some object is an *a*, then the class of objects *a* is an object. [Axioms IV, III] [189]

THEOREM LXXV. If P is the class of objects *a*, Q is the class of objects *b*, and R is the class of objects (*a* or *b*), then R is the class of objects (P or Q). [Axiom IV, Theorems LXVI, LXVIII]

THEOREM LXXVI. If P is the class of objects *a*, and Q is the class of objects *b*, then (R is the class of objects (P or Q) if and only if R is the class of objects (*a* or *b*)). (Follows from Theorems LXXV and LXVI)

THEOREM LXXVII. If (for all R and S – if R is a set of objects *a*, and S is a set of objects *a*, then R is the same object as S), P is an *a*, and Q is an *a*, then P is the same object as Q. [Theorem XIII]

THEOREM LXXVIII. If P is a set of sets of objects *a*, and Q is an ingredient of the object P, then some ingredient of the object Q is an ingredient of some *a* that is an ingredient of the object P. [Definition III, Theorem IV]

THEOREM LXXIX. If P is a set of sets of objects *a*, then P is a set of objects *a*. [Theorem LXXVIII, Definition III]

THEOREM LXXX. If P is a set of objects *a*, and for all Q – if Q is an *a*, then Q is an ingredient of the object P, then P is the class of objects *a*. [Definition III, Theorem LV]

THEOREM LXXXI. If P is a set of objects *a*, then P is the class of objects *a* that are ingredients of the object P. [Definition III, Theorem LV]

THEOREM LXXXII. P is the class of objects *a* if and only if (P is a set of objects *a*, and for all Q – if Q is an *a*, then Q is an ingredient of the object P). (Follows from Theorems LXXX, XIV, and LVII)

THEOREM LXXXIII. If P is a set of objects *a*, then for some *b* – (P is the class of objects *b*, and for all Q – if Q is a *b*, then Q is an *a*). (Follows from Theorem LXXXI)

THEOREM LXXXIV. P is an element of the object Q if and only if P is an ingredient of the object Q. (Follows from Theorems XVII and XVI).[190]

THEOREM LXXXV. P is an ingredient of the object Q if and only if for some *a* – (Q is the class of objects *a*, and P is an *a*). (Follows from Theorem LXXXIV and Definition IV)

THEOREM LXXXVI. If Q is the class of objects *a*, and P is an *a*, then P is the same object as Q, or is part of the object Q. (Follows from Theorem LXXXV and Definition I)

THEOREM LXXXVII. If P is the class of objects, then not(P is part of the object Q). [Theorem LXXXVI, Axiom I]

THEOREM LXXXVIII. P is the class of objects *a* if and only if (P is an object, (for all Q – if Q is an *a*, then Q is the same object as P, or is part of the object P), and for all Q – if Q is the same object as P, or Q is part of the object P, then some object that is the same object as Q, or part of the object Q, is an *a* or is part of some *a*). (Follows from Theorems LVI, LXXXVI, and LIV)

THEOREM LXXXIX. If P is the class of objects, and P is an ingredient of the object Q, then Q is the class of objects. [Theorem LXXXVII, Definition I]

THEOREM XC. If P is the class of objects *a*, and some *a* is the class of objects, then P is the class of objects. [Theorems LXXXV, LXXXIX]

THEOREM XCI. If P is a set of objects *a*, then some *a* is an ingredient of the object P.[191] (Follows from Theorems XXI, XVI)

THEOREM XCII. If P is a set of objects *a*, (for all Q and R – if Q is an *a*, and R is an *a*, then Q is the same object as R) and S is an *a*, then S is an ingredient of the object P. [Theorem XCI]

THEOREM XCIII. If P is a set of objects *a*, and for all Q – if Q is an *a*, then Q is a *b*, then P is a set of objects *b*. [Theorems XCI, XII]

THEOREM XCIV. If P is a set of objects *a*, and for all Q and R – if Q is an *a*, and R is an *a*, then Q is the same object as R, then P is the class of objects *a*. [Theorems XCII, LXXX]

THEOREM XCV. If P is the class of objects *a*, and for all Q – if Q is an *a*, then Q is a *b*, then P is a set of objects *b*. [Theorems XIV, XCIII]

THEOREM XCVI. P is a set of objects *a* if and only if for some *b* – (P is the class of objects *b*, and for all Q – if Q is a *b*, then Q is an *a*). (Follows from Theorems XCV, LXXXIII)

THEOREM XCVII. If P is the class of classes of objects *a*, then P is the class of objects *a*. [Theorems XX, LXXII, LXVIII]

THEOREM XCVIII. If (for all P and Q – if P is an *a*, and Q is an *a*, then P is the same object as Q), R is a set of objects *a*, and S is a set of objects *a*, then R is the same object as S. [Theorem XCIV, Axiom III]

THEOREM IC. If P is a set of classes of objects *a*, then P is the class of objects *a*. [Axiom III, Theorems XCIV, XCVII]

THEOREM C. (For all P and Q – if P is an *a*, and Q is an *a*, then P is the same object as Q) if and only if for all R and S – if R is a set of objects *a*, and S is a set of objects *a*, then R is the same object as S. (Follows from Theorems LXXVII, XCVIII)

THEOREM CI. If P is an object and for all Q – not(P is part of the object Q), then P is the class of objects. [Axiom IV, Theorem LXXXVI]

THEOREM CII. If some object is an *a*, then for some P – ((for all Q – if Q is an *a*, then Q is the same object as P, or is part of the object P) and for all Q – if Q is the same object as P, or Q is part of the object P, then some object that is the same object as Q or part of the object Q, is an *a* or part of some *a*). (Follows from Axiom IV and Theorem LXXXVIII)[192]

THEOREM CIII. If P is an object, and for all Q – if Q is an object, then Q is an ingredient of the object P, then P is the class of objects. [Axiom IV, Theorem LXXXIX]

THEOREM CIV. If P is the class of objects *a*, and Q is the class of objects not-*a*, then (P is the class of objects or Q is the class of objects). [Axiom IV, Theorem XC]

THEOREM CV. P is the class of objects if and only if (P is an object, and for all Q – not(P is part of the object Q)). (Follows from Theorems CI and LXXXVII)

THEOREM CVI. P is the class of objects if and only if (P is an object, and for all Q – if Q is an object, then Q is an ingredient of the object P). (Follows from Theorems CIII and LXXXV)

THEOREM CVII. If P is the class of objects *a*, and Q is the class of objects not-*a*, then (P is an ingredient of the object Q, or Q is an ingredient of the object P). [Theorems CIV, LXXXV]

THEOREM CVIII. If P is an object, and not(P is the same object as the class of objects), then some object is part of some object. [Theorems LXXIV, LXXXVI]

THEOREM CIX. If P is an object, Q is an object, and not(P is the same object as Q), then some object is part of some object. [Theorem CVIII][193]

THEOREM CX. If some object is part of the object P, and for all R – if R is part of the object P, then some ingredient of the object R is an ingredient of the object Q, then P is an ingredient of the object Q. [Theorems IX, LII, XXVII]

THEOREM CXI. If P is the class of objects *a*, Q is the class of objects *b*, and for all R – if R is an *a*, then R is a *b*, then P is an ingredient of the object Q. [Theorems XXV, XCV, LXXXV]

THEOREM CXII. If every part of the object P is an ingredient of the object Q, then P is an ingredient of the object Q. [Theorems II, CX]

THEOREM CXIII. P is a subset of the object Q if and only if P is an element of the object Q. (Follows from Theorems XXIX, XXVIII[194])

THEOREM CXIV. P is a subset of the object Q if and only if P is an ingredient of the object Q. (Follows from Theorems CXIII, LXXXIV[195])

THEOREM CXV. If P is an ingredient of the object Q, then not(P is external to Q). [Theorem II, Definition VI]

THEOREM CXVI. If P is an object, Q is an object, and not (some ingredient of the object P is an ingredient of the object Q), then P is external to Q. [Theorem II, Definition VI]

THEOREM CXVII. If P is part of the object Q, then not(P is external to Q). (Follows from Definition I, and Theorem CXV)

THEOREM CXVIII. If (for all Q – if Q is an ingredient of the object P, then for some R – (R is an *a* and not (Q is external to R))), and S is an ingredient of the object P, then some ingredient of the object S is an ingredient of some *a*. [Theorem CXVI]

THEOREM CXIX. If P is an object, Q is an object, and not(P is external to Q), then (P is the same object as Q, or P is part of the object Q, or Q is part of the object P, or some part of the object P is part of the object Q). [Theorem CXVI, Definition I]

THEOREM CXX. If (for all X and Y – if X is an *a*, and Y is an *a*, then X is the same object as Y, or is external to Y), (for all X – if X is a *b*, then X is an *a*), (for all X, Y, and Z – if X is a *b*, Y is a set of objects *a*, Z is a set of objects *a*, X is ϕ(Y), and X is ϕ(Z), then Y is the same object as Z), P is an *a*, Q is a set of objects *a*, R is an ingredient of the object Q, S is a *b*, S is ϕ(P), S is an ingredient of the object P, T is a *b*, T is ϕ(Q), U is an ingredient of the object T, V is P, and U is an ingredient of the object V, then R is an ingredient of the object P. [Theorem XIII, Definition VI, Theorem CXV]

THEOREM CXXI. If P is external to Q, R is an ingredient of the object Q, and S is an ingredient of the object P, then not(S is an ingredient of the object R). [Definition VI, Theorem IV]

THEOREM CXXII. If P is external to Q, and R is an ingredient of the object Q, then P is external to R. [Theorems CXXI, CXVI]

THEOREM CXXIII. If P is the class of objects a, and Q is an ingredient of the object P, then for some R – (R is an a and not(Q is external to R)). [Definitions II, VI]

THEOREM CXXIV. If P is an object, (for all Q – if Q is an a, then Q is an ingredient of the object P), and for all Q – if Q is an ingredient of the object P, then for some R – (R is an a and not(Q is external to R)), then P is the class of objects a. [Theorems CXVIII, LV]

THEOREM CXXV. P is an element of the object Q external to R if and only if P is an ingredient of the object Q external to R. (Follows from Theorem LXXXIV)

THEOREM CXXVI. S is the class of elements of the object Q external to R if and only if S is the class of ingredients of the object Q external to R. (Follows from Theorems LXII and CXXV)

THEOREM CXXVII. P is external to Q if and only if Q is external to P. (Follows from Theorem XXXII)

THEOREM CXXVIII. If P is an ingredient of the object Q, then not(Q is external to P). (Follows from Theorems CXV and XXXII)

THEOREM CXXIX. P is external to Q if and only if (P is an object, Q is an object, and not(some ingredient of the object P is an ingredient of the object Q)). (Follows from Theorems CXVI, XXXII, and Definition VI)

THEOREM CXXX. If P is part of the object Q, then not(Q is external to P). (Follows from Theorems CXVII and XXXII)

THEOREM CXXXI. If P is external to Q, then not(some part of the object P is part of of the object Q). (Follows from Theorem CXXIX and Definition I)

THEOREM CXXXII. P is external to Q if and only if (P is an object, Q is an object, not(P is the same object as Q), not(P is part of the object Q) not(Q is part of the object P), and not(some part of the object P is part of the object Q)). (Follows from Theorems CXIX, XXXII, XXXI, CXVII, CXXX, and CXXXI)

THEOREM CXXXIII. If P is the class of objects, then not(Q is external to P). [Theorems LXXXV, CXV]

THEOREM CXXXIV. P is the class of objects a if and only if (P is an object, (for all Q – if Q is an a, then Q is an ingredient of the object P), and for all Q – if Q is an ingredient of the object P, then for some R – (R is an a and not(Q is external to R))). (Follows from Theorems CXXIV, LXXXV, CXXIII)

THEOREM CXXXV. If P is the class of objects a, Q is an object and not (Q is external to P), then for some R – (R is an a and not(Q is external to R)). [Theorems CXVI, LXIII, Definition VI]

THEOREM CXXXVI. If R is an object, P is the class of objects (Q or R), S is an ingredient of the object P, S is external to Q, and T is an ingredient of the object S, then some ingredient of the object T is an ingredient of the object R. [Theorems IV, XXXII, Definition II, Theorem IV, Definition VI]

THEOREM CXXXVII. If P is the class of objects a, Q is external to P, and R is an a, then Q is external to R. [Theorems LXXXV, CXXII]

THEOREM CXXXVIII. If P is the class of objects a, and for all R – if R is an a, then Q is external to R, then Q is external to P. [Theorems XX, CXXXV]

THEOREM CXXXIX. If P is the class of objects a, then (Q is external to P if and only if for all R – if R is an a, then Q is external to R). (Follows from Theorems CXXXVIII, CXXXVII)

THEOREM CXL. If P is a set of objects a, and for all R – if R is an a, then Q is external to R, then Q is external to P. [Theorems LXXXI, CXXXVIII]

THEOREM CXLI. If P is an object, (for all Q – not(Q is external to P)), and R is an object, then R is an ingredient of the object P. [Theorems CXVI, XXVII]

THEOREM CXLII. If R is an object, P is the class of objects (Q or R), S is an ingredient of the object P, and S is external to Q, then S is an ingredient of the object R. [Theorems CXXXVI, XXVII]

THEOREM CXLIII. If P is an object and for all Q – not(Q is external to P), then P is the class of objects. [Theorems CXLI, CIII]

THEOREM CXLIV. P is the class of objects if and only if (P is an object and for all Q – not(Q is external to P)). (Follows from Theorems CXLIII, CXXXIII)

THEOREM CXLV. P is the complement of the object Q with respect to R if and only if (Q is an ingredient of the object R, and P is the class of ingredients of the object R external to Q). (Follows from Definition VII, Theorems CXIV and CXXVI)

THEOREM CXLVI. P is the complement of the object Q with respect to R if and only if Q is the complement of the object P with respect to R. (Follows from Theorem XLV)

DEFINITION VIII. P is Q+R if and only if the following conditions are satisfied:
(α) P is the class of objects (Q or R);
(β) Q is external to R.

Examples. (I) AB of Figure 2 is AC + the class of parts of the object AB external to AC because here both conditions indicated in Definition VIII are satisfied. (II) AB of Figure 2 is not AE+CB because while condition α is here satisfied, condition β is not satisfied. (III) AE of Figure 2 is not AC+DE because while condition β is here satisfied, condition α is not satisfied.

THEOREM CXLVII. Not(P is Q+Q). (Follows from Definition VIII and Definition VI)

THEOREM CXLVIII. If P is Q+R, then R is external to Q. (Follows from Definition VIII and Theorem XXXII).

THEOREM CIL. If P is Q+R, S is Q or R, and T is Q or R, then S is the same object as T or is external to T. [Definition VIII, Theorem XXXII]

THEOREM CL. If P is the class of objects, then not(R is Q+P). (Follows from Theorem CXXXIII and Definition VIII)

THEOREM CLI. If P is Q+R, and S is external to P, then S is external to Q. [Definition VIII, Theorem CXXXVII]

THEOREM CLII. If P is Q+R, S is external to Q, and S is external to R, then S is external to P. [Definition VIII, Theorems XXXII, CXXXVIII]

THEOREM CLIII. If P is Q+R, and S is Q+R, then P is the same object as S. [Definition VIII, Axiom III]

THEOREM CLIV. If (for all Q and R – if Q is an a, and R is the class of objects (a and not-Q), then P is Q+R), S is an a, T is an a, and not(S is the same object as T), then S is external to T. [Axiom IV, Definition VIII, Theorem CXXXVII]

THEOREM CLV. If P is external to Q, then P+Q is an object. [Axiom IV, Definition VIII, Theorem CLIII] [196]

THEOREM CLVI. If P is an object and for all Q and R – not(R is Q+P), then P is the class of objects. [Theorems CLV, CXLIII]

THEOREM CLVII. P is the class of objects if and only if (P is an object and for all Q and R – not(R is Q+P)). (Follows from Theorems CLVI, CL)

THEOREM CLVIII. If R is the complement of the object Q with respect to P, then P is Q+R. [Theorems XLVIII, XXXVI, XXXII, Definition VIII]

THEOREM CLIX. If P is Q+R, then R is the complement of the object Q with respect to P. [Definition VIII, Theorems LXXXV, XXXII, CXLII, LXXXV, II, LV, CXLV]

THEOREM CLX. If P is part of the object Q, then for some R – (Q is P+R). (Follows from Theorems XLII, CLVIII)

THEOREM CLXI. If P is Q+R, then R is part of the object P. (Follows from Theorems CLIX and XLIII)

THEOREM CLXII. If P is Q+R, then Q is part of the object P. (Follows from Theorems CLIX and XLVI)

THEOREM CLXIII. P is Q+R if and only if R is the complement of the object Q with respect to P. (Follows from Theorems CLVIII and CLIX)

THEOREM CLXIV. Not(P is Q+P). (Follows from Theorem CLXI and Axiom I)

THEOREM CLXV. Not(P is P+R). (Follows from Theorem CLXII and Axiom I)

THEOREM CLXVI. R is Q+P if and only if (Q is an ingredient of the object R, and P is the class of ingredients of the object R external to Q). (Follows from Theorems CLXIII and CXLV)

THEOREM CLXVII. R is Q+P if and only if R is P+Q. (Follows from Theorems CXLVI and CLXIII)

THEOREM CLXVIII. If R is Q+P, and R is Q+S, then P is the same object as S. [Theorem CLXVI, Axiom III]

THEOREM CLXIX. If P is Q+R, and S is external to P, then S is external to R. [Theorems CLXVII, CLI]

THEOREM CLXX. If P is Q+(R+S), then P is Q+(S+R). [Theorems CXLVIII, CLXVII, CLIII]

THEOREM CLXXI. If P is Q+R, then (S is external to P if and only if (S is external to Q, and S is external to R)). (Follows from Theorems CLII, CLI, and CLXIX)

THEOREM CLXXII. P is Q+(R+S) if and only if P is Q+(S+R). (Follows from Theorem CLXX)

THEOREM CLXXIII. If P is Q+(R+S), then P is S+(Q+R). [Theorems CXLVIII, VIII, Definition VIII, Theorems CLXXI, CLV, Definition VIII, Theorem VIII, Definition VIII, Theorems LXVI, LXII, LXXV, XXXII, CLII, Definition VIII]

THEOREM CLXXIV. If P is S+(Q+R), then P is Q+(R+S). [Theorems CLXX, CLXXIII, CLXX]

THEOREM CLXXV. P is Q+(R+S) if and only if P is S+(Q+R). (Follows from Theorems CLXXIV and CLXXIII)

DEFINITION IX. P is the sum of the objects *a* if and only if the following conditions are satisfied:

(α) P is the class of objects *a*;

(β) for all Q and R – if Q is an *a*, and R is an *a*, then Q is the same object as R, or is external to R.

Examples. (I) Segment AB of Figure 2 is the sum of the objects that are the segment AC or the class of parts of the segment AB external to the segment AC, because here both conditions indicated in Definition IX are satisfied. (II) Segment AB of Figure 2 is not the sum of the parts of segment AB because here while condition α is satisfied, condition β is not satisfied (segment AC is part of segment AB, segment AD is part of segment AB, but not(segment AC is the same object as segment AD, or is external to segment AD)). (III) AB of Figure 2 is not the sum of objects AC, because while here condition β is satisfied, condition α is not.

THEOREM CLXXVI. If P is the sum of the objects (Q or R), Q is an object, R is an object, and not(Q is the same object as R), then P is Q+R. [Definitions IX, VIII]

THEOREM CLXXVII. If P is an object, then P is the sum of objects P. [Theorem VIII, Thesis 5 of Chapter II, Definition IX] [197]

THEOREM CLXXVIII. If P is a set of objects *a*, and for all Q and R – if Q is an *a*, and R is an *a*, then Q is the same object as R, or is external to R, then P is the sum of the objects *a* that are ingredients of the object P. [Theorem LXXXI, Definition IX]

THEOREM CLXXIX. If P is the sum of the objects *a*, and for all Q – (Q is an *a* if and only if Q is a *b*), then P is the sum of the objects *b*. [Definition IX, Theorem LXII, Definition IX].

THEOREM CLXXX. If P is a set of objects *a*, and for all Q and R – if Q is an *a*, and R is an *a*, then Q is the same object as R, then P is the sum of objects *a*. [Theorem XCIV, Definition IX]

THEOREM CLXXXI. If P is Q+R, then P is the sum of objects (Q or R). [Definition VIII, Theorem CIL, Definition IX].

Theorem CLXXXII. If P is the sum of objects *a*, Q is the sum of objects *b*, Q is external to P, R is an *a*, and S is a *b*, then R is external to S. [Definition IX, Theorems CXXXVII, XXII, CXXXVII]

THEOREM CLXXXIII. If P is the sum of the objects *a*, Q is the sum of the objects *b*, P is external to Q, R is an *a* or a *b*, and S is an *a* or a *b*, then R is the same object as S, or is external to S. [Theorems XXXII, CLXXXII, Definition IX]

THEOREM CLXXXIV. If P is the sum of the objects *a*, Q is the sum of the objects *b*, and R is P+Q, then R is the sum of the objects (*a* or *b*). [Definitions IX, VIII, Theorems LXVI, CLXXXIII, Definition IX]

THEOREM CLXXXV. If P is the sum of the objects a, and Q is the sum of the objects a, then P is the same object as Q. (Follows from Definition IX and Axiom III)

THEOREM CLXXXVI. If P is the class of objects a, and for all Q and R – if Q is an a, and R is the class of objects (a and not Q), then P is Q+R, then P is the sum of the objects a. [Theorem CLIV, Definition IX]

THEOREM CLXXXVII. If some object is an a, and for all Q and R – if Q is an a, and R is an a, then Q is the same object as R, or is external to R, then the sum of the objects a is an object. [Axiom IV, Definition IX, Theorem CLXXXV] [198]

THEOREM CLXXXVIII. If P is the sum of the objects a, Q is an a, and R is the class of objects (a and not Q), then P is Q+R. [Theorem VIII, Definition IX, Theorems LXII, LXXV, Definition IX, Theorem CXXXVIII, Definition VIII]

THEOREM CLXXXIX. P is the sum of the objects a if and only if (P is the class of objects a, and for all Q and R – if Q is an a, and R is the class of objects (a and not Q), then P is Q+R). (Follows from Theorem CLXXXVI, Definition IX, and Theorem CLXXXVIII)

THEOREM CXC. P is Q+R if and only if (P is the sum of the objects (Q or R), Q is an object, R is an object, and not(Q is the same object as R)). (Follows from Theorems CLXXVI, CLXXXI, CLXII, CXLVIII, and CXLVII)

DEFINITION X. P is $\phi\alpha\{a, b, \phi\}$ if and only if (P is a b and for some Q – (Q is a set of objects a, P is ϕ(Q) and not(P is an ingredient of the object Q))).[199]

THEOREM CXCI. If (for all X and Y – if X is an a, and Y is an a, then X is the same object as Y, or is external to Y), (for all X – if X is a b, then X is an a), P is the class of objects $\phi\alpha\{a, b, \phi\}$, Q is an a, and Q is an ingredient of the object P, then Q is $\phi\alpha\{a, b, \phi\}$. [Theorem CXXIII, Definition X]

THEOREM CXCII. If (for all X – if X is a b, then X is an a) and P is the class of objects $\phi\alpha\{a, b, \phi\}$, then P is a set of objects a. [Theorem XCV, Definition X, Theorem XCIII]

THEOREM CXCIII. If (for all X and Y – if X is an a, and Y is an a, then X is the same object as Y, or is external to Y), (for all X – if X is a b, then X is an a), (for all X – if X is a set of objects a, then for some Y – (Y is a b, and Y is ϕ(X))) and P is the class of objects $\phi\alpha\{a, b, \phi\}$, then for some Q and R – (Q is a b, R is a set of objects a, Q is ϕ(P), Q is ϕ(R) and not(P is the same object as R)). [Theorems CXCII, CXCI, Definition X, Theorem LXXXV, Definition X].

THEOREM CXCIV. If P is a b, Q is a set of objects a, and P is ϕ(Q), then (P is an ingredient of the object Q, or some object is the class of objects $\phi\alpha\{a, b, \phi\}$). (Follows from Definition X and Axiom IV)

THEOREM CXCV. If P is an a, (for all X – if X is an a, then for some Y – (Y is a b, and Y is ϕ(X))) and for all Y – not(Y is the class of objects $\phi\alpha\{a, b, \phi\}$), then for some S – (S is a b, S is ϕ(P), and S is an ingredient of the object P). [Theorems XIII, CXCIV]

THEOREM CXCVI. If (for all X and Y – if X is an a, and Y is an a, then X is the same object as Y, or is external to Y), (for all X – if X is a b, then X is an a), P is an a, R is an a, and there are as many sets of objects a as objects b, then P is the same object as R.[200] [Theorems XIII, CXCII, CXCIII, Axiom IV, Theorems XCV, LXXXV, CXCV, CXCIV, Definition II, Theorems CXCV, CXX, CXXVIII, CXV] [201]

THEOREM CXCVII. If (for all X and Y – if X is an a, and Y is an a, then X is the same object as Y, or is external to Y), P is an a, R is an a, and not(P is the same object as R), then the objects a are less than the sets of objects a. [Theorems CXCVI, XIII] [202]

THEOREM CXCVIII. If some object is the sum of the objects a, P is an a, R is an a, and not(P is the same object as R), then the objects a are less than the sets of objects a. [Definition IX, Theorem CXCVII]

Chapter VI: The axioms of 'General Set Theory' dating from 1918

Axioms III and IV of my 'general set theory', stated above, were formulated using the term 'class', introduced previously in Definition II, while Definition II was formulated using the term 'ingredient', introduced previously in Definition I. In the course of time this situation began to offend me. In 1918 I considered various possible sets of axioms of 'general set theory' that would all be formulated using the fundamental term 'part' and would not contain any terms introduced into the framework of the theory by means of definitions. In the course of these deliberations I became convinced that I could obtain a 'general set theory' equivalent to the theory that I have set out in Chapters IV and V if I just took as the axioms of my theory (formulated using the fundamental term 'part' and not containing any terms introduced into the framework of the theory by means of definitions) theses asserting respectively that

(A) if P is part of the object Q, then Q is not part of the object P,
(B) if P is part of the object Q, and Q is part of the object R, then P is part of the object R,
(C) if every *a* is the same object as P, or part of the object P, every *a* is the same object as Q, or part of the object Q, and for all R – if R is part of the object P, or R is part of the object Q, then some object that is the same object as R, or part of the object R, is an *a* or is part of some *a*, then P is Q,

and

(D) if some object is an *a*, then for some P – ((for all Q – if Q is an *a*, then Q is the same object as P, or is part of the object P) and for all Q – if Q is part of the object P, then some object that is the same object as Q, or part of the object Q, is an *a* or is part of some *a*),

while the terms 'ingredient' and 'class' would be introduced by means of the definitions establishing, respectively, that

(E) P is an ingredient of the object Q if and only if P is the same object as Q, or is part of the object Q,

and

(F) P is the class of objects *a* if and only if (P is an object, (for all Q – if Q is an *a*, then Q is an ingredient of the object P) and for all Q – if Q is an ingredient of the object P, then some ingredient of the object Q is an ingredient of some *a*).[203]

We can easily establish that, on the basis of the theory which I have set out in Chapters IV and V, it is possible to obtain equivalents of all the theses that can be obtained on the basis of the new theory based on Axioms A–D and containing Definitions E–F, keeping in mind the fact that Theses A–C, E and F are only reiterations, respectively, of Axiom I, Axiom II, Theorem LXX, Definition I, and Theorem LVII of the theory that I set out in Chapters IV and V, while Thesis D follows immediately from Theorem CII of the aforementioned theory.[204] For that matter, we can convince ourselves that conversely as well the equivalents of all the theses that can be obtained on the basis of the theory that I set out in Chapters IV and V can be obtained within the framework of a theory based on Axioms A–D and containing Definitions E and F, keeping in mind the fact that Axiom I, Axiom II, and Definition I are only reiterations of Theses A, B and E, and by deriving explicitly the equivalents of Definition II, Axiom III, and Axiom IV from Theses A and C–F, which may be carried out as follows:

THEOREM G. If some object is an *a*, (for all Q – if Q is an *a*, then Q is the same object as P, or is part of the object P), (for all Q – if Q is part of the object P, then some object that is the same object as Q, or part of the object Q, is an *a* or is part of some *a*) and R is an ingredient of the object P, then some ingredient of the object R is an ingredient of some *a*. [E].[205]

THEOREM H. If P is the class of objects *a*, then every *a* is an ingredient of the object P. [E, F]

THEOREM K. If P is the class of objects *a*, and R is part of the object P, then some object that is the same object as R, or part of the object R, is an *a* or is part of some *a*. [E, F]

THEOREM M. If P is the class of objects *a*, then every *a* is the same object as P, or part of the object P. (Follows from Theorems H and E)

THEOREM N. (The equivalent of Definition II.) P is the class of objects *a* if and only if the following conditions are satisfied:

(α) P is an object;
(β) every *a* is an ingredient of the object P;
(γ) for all Q – if Q is an ingredient of the object P, then some ingredient of the object Q is an ingredient of some *a*.

(Follows from F and Theorem H)

THEOREM O. (The equivalent of Axiom III.) If P is the class of objects *a*, Q is the class of object *a*, then P is Q. [Theorems M, K, C]

THEOREM P. (The equivalent of Axiom IV). If some object is an *a*, then some object is the class of objects *a*. [D, E, Theorems G, A, F]

Chapter VII: The axioms of 'General Set Theory' dating from 1920

In 1920 I realized that I could make the term 'ingredient' the fundamental term of my 'general set theory' instead of the term 'part', and that I could obtain a theory equivalent to the theory that I set out in Chapters IV and V, if as axioms of the theory I took theses asserting respectively that

(a) if P is an ingredient of the object Q and not(Q is P), then Q is not an ingredient of the object P,

(b) if P is an ingredient of the object Q, and Q is an ingredient of the object R, then P is an ingredient of the object R,

(c) if every *a* is an ingredient of the object P and an ingredient of the object Q, and for all R – if R is an ingredient of the object P, or R is an ingredient of the object Q, then some ingredient of the object R is an ingredient of some *a*, then P is Q,

and

(d) if some object is an *a*, then for some P – ((for all Q – if Q is an *a*, then Q is an ingredient of the object P) and for all Q – if Q is an ingredient of the object P, then some ingredient of the object Q is an ingredient of some *a*),

while the terms 'part' and 'class' would be introduced by means of the definitions establishing respectively that

(e) P is part of the object Q if and only if (P is an ingredient of the object Q and not(P is the same object as Q)),

and

(f) P is the class of objects *a* if and only if (P is an object, (for all Q – if Q is an *a*, then Q is an ingredient of the object P) and for all Q – if Q is an ingredient of the object P, then some ingredient of the object Q is an ingredient of some *a*).[206]

We can easily establish that, on the basis of the theory which I have set out in Chapters IV and V, it is possible to obtain equivalents of all the theses that can obtained on the basis of the new theory based on Axioms (a)–(d) and containing definitions (e) and (f), keeping in mind the fact that Theses (a)–(f) are only reiterations, respectively, of Theorem LIX, Theorem IV, Theorem LXIX, Theorem LXXIII, Theorem L, and Theorem LVII of the theory that I set out in Chapters IV and V.[207] For that matter, we can convince ourselves that conversely as well the equivalents of all the theses that can be obtained on the basis of this very theory can be obtained within the framework of a theory based on Axioms (a)–(d) and containing Definitions (e) and (f) and by deriving explicitly the equivalents of Axiom I, Axiom II, Definition I, Definition II, Axiom III, and Axiom IV from Theses (a)–(f), which may be carried out as follows:

THEOREM (g). (The equivalent of Axiom I). If P is part of the object Q, then Q is not part of the object P, [(e), (a)].

THEOREM (h). (The equivalent of Axiom II). If P is part of the object Q, and Q is part of the object R, then P is part of the object R. [(e), (b), Theorem (g)].

THEOREM (i). If S is an object, then for some P – (S is an ingredient of the object P, and for all Q – if Q is an ingredient of the object P, then some ingredient of the object Q is an ingredient of some S). [(d)].

THEOREM (k). If S is an object, then for some T – (T is an ingredient of the object S). [Theorem (i)].

THEOREM (l). If S is an object, (for all Q – if Q is an ingredient of the object P, then some ingredient of the object Q is an ingredient of some S) and R is an ingredient of the object P, then some ingredient of the object R is an ingredient of some ingredient of the object S. [Theorem (k), (b)].

THEOREM (m). If S is an object, then S is an ingredient of the object S.[208] [Theorems (k), (i); (b), Theorem (l), (c)].

THEOREM (n). If P is the same object as Q, then P is an ingredient of the object Q. [Theorem (m)].

THEOREM (o). (The equivalent of Definition I). P is an ingredient of the object Q if and only if P is the same object as Q, or is part of the object Q. (Follows from Theorem (n) and (e)).

THEOREM (p). If P is the class of objects *a*, then every *a* is an ingredient of the object P.[209] [Theorem (m), (f)] [210]

THEOREM (q). (The equivalent of Definition II). P is the class of objects *a* if and only if the following conditions are satisfied:

(α) P is an object;
(β) every *a* is an ingredient of the object P;
(γ) for all Q – if Q if an ingredient of the object P, then some ingredient of the object Q is an ingredient of some *a*. (Follows from (f) and Theorem (p)).[211]

THEOREM (r). (The equivalent of Axiom III). If P is the class of objects *a*, and Q is the class of objects *a*, then P is Q. [Theorems (p), (q); (c)].

THEOREM (s). (The equivalent of Axiom IV). If some object is an *a*, then some object is the class of objects *a*. [(d), (a), (f)].

Chapter VIII: Some conditions established by Messrs. Kuratowski and Tarski as necessary and sufficient for P to be the class of objects *a*

In 1921 Mr. Kuratowski showed that on the basis of my 'general set theory'

(a) P is the class of objects *a* if and only if (every *a* is an ingredient of the object P, and for all Q – if every *a* is an ingredient of the object Q, and Q is an ingredient of the object P, then Q is the same object as P),

while Mr. Tarski proved that on this basis

(b) P is the class of objects *a* if and only if (every *a* is an ingredient of the object P, and for all Q – if every *a* is an ingredient of the object Q, then P is an ingredient of the object Q).

and

(c) P is the class of objects *a* if and only if (P is an object and for all Q – (every *a* is an ingredient of the object Q if and only if P is an ingredient of the object Q)).

Messrs. Kuratowski and Tarski obtained the results here formulated independently of each other.

Not wishing to proceed with the continuation of my own theorems from 'general set theory' without first showing the reader the chronologically earlier results of others in this field, in this chapter I take up Theses (a)–(c), presented a while back, and with the assistance of several auxiliary theorems I derive them in a simple way from theses already contained in Chapters IV and V. In my deductions I shall not consider the original proofs of Messrs. Kuratowski and Tarski; besides, after all these years I could not even reconstruct these proofs exactly.

The numbering of the following theses below continues the numbering that was interrupted in Chapter V.

THEOREM CIC. If (for all Q – if every *a* is an ingredient of the object Q, then P is an ingredient of the object Q), every *a* is an ingredient of the object R, and R is an ingredient of the object P, then R is the same object as P. [Theorem LI]

THEOREM CC. If P is the class of objects *a*, and P is an ingredient of the object Q, then every *a* is an ingredient of the object Q. [Definition II, Theorem IV]

THEOREM CCI. If P is the class of objects *a*, and every *a* is an ingredient of the object Q, then P is an ingredient of the object Q. [Theorems X, CXI]

THEOREM CCII. If every *a* is an ingredient of the object P, and for all Q – if every *a* is an ingredient of the object Q, and Q is an ingredient of the object P, then Q is the same object as P, then P is the class of objects *a*. [Axiom IV, Definition II, Theorem CCI]

THEOREM CCIII. If P is the class of objects *a*, every *a* is an ingredient of the object Q, and Q is an ingredient of the object P, then Q is the same object as P. [Theorems CCI, LI]

THEOREM CCIV. If P is the class of objects *a*, then (every *a* is an ingredient of the object Q if and only if P is an ingredient of the object Q). (Follows from Theorems CC and CCI)

THEOREM CCV. If every *a* is an ingredient of the object P, and for all Q – if every *a* is an ingredient of the object Q, then P is an ingredient of the object Q, then P is the class of objects *a*. [Theorems CIC, CCII]

THEOREM CCVI. (Theorem (a) of Mr. Kuratowski). P is the class of objects *a* if and only if (every *a* is an ingredient of the object P, and for all Q – if every *a* is an ingredient of the object Q, and Q is an ingredient of the object P, then Q is the same object as P). (Follows from Theorem CCII, Definition II, and Theorem CCIII)

THEOREM CCVII. If P is an object and for all Q) (every *a* is an ingredient of the object Q if and only if P is an ingredient of the object Q), then P is the class of objects *a*. [Theorems II, CCV]

THEOREM CCVIII. (Theorem (b) of Mr. Tarski). P is the class of objects a if and only if (every a is an ingredient of the object P, and for all Q – if every a is an ingredient of the object Q, then P is an ingredient of the object Q). (Follows from Theorem CCV, Definition II, and Theorem CCI)

THEOREM CCIX. (Theorem (c) of Mr. Tarski). P is the class of objects a if and only if (P is an object and for all Q – every a is an ingredient of the object Q if and only if P is an ingredient of the object Q)). (Follows from Theorems CCVII, CCIV)

Chapter IX: Further theorems of 'General Set Theory' dating from 1921–1923

Going on with the continuation of my own theorems from 'general set theory', I take up the reformation of the results I attained in this field in 1921–1923.

THEOREM CCX. If (for all S – if Q is part of the object S, and R is part of the object S, then P is an ingredient of the object S), not(Q is the same object as R), not(Q is part of the object R), not(R is part of the object Q), Q is an object, R is an object, and every (Q or R) is an ingredient of the object T, then P is an ingredient of the object T. [Definition I]

THEOREM CCXI. If P is an object, not(P is the same object as Q), R is an object, Q is the class of objects b, (for all S – if S is a b, then S is P or R), T is a b, and T is P, then (P is a b, and R is a b). [Theorem VIII, Axiom III, Theorem LXII]

THEOREM CCXII. If P is an object, then P is the same object as the set of objects P. [Theorems XIII, XCVIII]

THEOREM CCXIII. If P is an object, not(P is the same object as Q), R is an object, not(Q is the same object as R), and Q is a set of objects (P or R), then P is part of the object Q. [Theorems LXXXIII, XX, CCXI, LXXXVI]

THEOREM CCXIV. If every part of the object P is part of the object Q, then P is an ingredient of the object Q. (Follows from Definition I and Theorem CXII)

THEOREM CCXV. If P is the class of objects a, and for all R – if R is an a, then R is an ingredient of the object Q, then P is an ingredient of the object Q. [Theorems XX, CCI]

THEOREM CCXVI. If Q is an ingredient of the object P, R is an ingredient of the object P, and for all S – if every (Q or R) is an ingredient of the object S, then P is an ingredient of the object S, then P is the class of objects (Q or R). [Theorem CCV]

THEOREM CCXVII. If every part of the object P is part of the object Q, and every part of the object Q is part of the object P, then P is the same object as Q. [Theorems CCXIV, LI]

THEOREM CCXVIII. If P is a set of objects a, and for all R – if R is an a, then R is an ingredient of the object Q, then P is an ingredient of the object Q. [Theorems LXXXI, CCXV]

THEOREM CCXIX. If Q is an ingredient of the object P, then P is the class of objects (Q or P). [Theorems X, II, CCXVI]

THEOREM CCXX. If P is an object, (for all R – if R is external to P, then R is external to Q), and S is an ingredient of the object Q, then some ingredient of the object S is an ingredient of the object P. [Theorems CXV, CXVI]

THEOREM CCXXI. If R is an ingredient of the object P, and R is an ingredient of the object Q, then not(for all S – some object external to P or external to Q is not(external to S)). [Theorem CXXII]

THEOREM CCXXII. If for all S – some object external to P or external to Q is not(external to S), then P is an object. [Theorem XXXII]

THEOREM CCXXIII. If P is an object, Q is an object, and not(P is external to Q), then for some a and b – (P is the class of objects a, Q is the class of objects b, and some a is b). [Theorems CXVI, LXXXV]

THEOREM CCXXIV. If P is the class of objects a, Q is the class of objects b, and some a is b, then not(P is external to Q). [Theorems LXXXV, CXXIX]

THEOREM CCXXV. P is external to Q if and only if (P is an object, Q is an object, and for all a and b – if P is the class of objects a, and Q is the class of objects b, then not(some a is b)). (Follows from Theorems CCXXIII, XXXII, CCXXIV)

THEOREM CCXXVI. If for all S – some object external to P or external to Q is not(external to S), then P is external to Q. [Theorems CCXXII, CCXXI, CXVI]

THEOREM CCXXVII. If Q is an object, P is an object, and for all R – if R is external to P, then R is external to Q, then Q is an ingredient of the object P. [Theorems CCXX, XXVII]

THEOREM CCXXVIII. If P is an object, (for all Q and R – if Q is external to P, and R is an a, then Q is external to R), and S is an a, then S is an ingredient of the object P. [Theorem CCXXVII]

THEOREM CCXXIX. If P is external to Q, then some object external to P or external to Q is not(external to S). [Definition VI, Theorems XXXIII, CCXXVII, XXXII]

THEOREM CCXXX. If P is an object, Q is an object, and for all R – (R is external to P if and only if R is external to Q), then P is the same object as Q. [Theorems CCXXVII, LI]

THEOREM CCXXXI. Q is an ingredient of the object P if and only if (Q is an object, P is an object, and for all R – if R is external to P, then R is external to Q). (Follows from Theorems CCXXVII, X, CXXII)

THEOREM CCXXXII. If P is an object, and for all Q – (Q is external to P if and only if for all R – if R is an a, then Q is external to R), then P is the class of objects a. [Theorems CCXXVIII, CXV, CXVIII, LV]

THEOREM CCXXXIII. P is external to Q if and only if for all S – some object external to P or external to Q is not(external to S). (Follows from Theorems CCXXVI and CCXXIX)

THEOREM CCXXXIV. P is the class of objects a if and only if (P is an object and for all Q – (Q is external to P if and only if for all R – if R is an a, then Q is external to R)). (Follows from Theorems CCXXXII and CXXXIX)

THEOREM CCXXXV. P is the class of objects a if and only if (P is an object and for all Q – (P is external to Q if and only if for all R – if R is an a, then R is external to Q)). (Follows form Theorems CCXXXIV and CXXVII)

Theorem CCXXXVI. If (for all P – (P is $\phi(a)$ if and only if (P is an object and for all Q – (P is external to Q if and only if for all R – if R is an a, then R is external to Q)))), and some object is an a, then $\phi(a)$ is an object. [Theorem CCXXXV, Axioms III and IV] [212]

THEOREM CCXXXVII. If P is Q+R, then not(some ingredient of the object Q is an ingredient of the object R). (Follows from Definition VIII and Theorem CXXIX)

THEOREM CCXXXVIII. If P is Q+R, then not(some part of the object Q is part of the object R). (Follows from Definition VIII and Theorem CXXXI)

THEOREM CCXXXIX. If P is Q+R, then not(R is part of the object Q). (Follows from Theorems CXLVIII and CXVII)

THEOREM CCXL. If P is Q+R, then not(Q is part of the object R). (Follows from Theorems CXLVIII and CXXX)

THEOREM CCXLI. P is external to Q if and only if some object is P+Q. (Follows from Definition VIII and Theorem CLV)

THEOREM CCXLII. If P is Q+R, Q is an ingredient of the object S, and R is an ingredient of the object S, then P is an ingredient of the object S. [Definition VIII, Theorem CCXV]

THEOREM CCXLIII. If P is Q+R, Q is part of the object S, and R is part of the object S, then P is an ingredient of the object S. [Definition I, Theorem CCXLI]

THEOREM CCXLIV. If P is an object, (for all S – ((Q is an ingredient of the object S, and R is an ingredient of the object S) if and only if P is an ingredient of the object S)) and not(some ingredient of the object Q is an ingredient of the object R), then P is Q+R. [Theorems II, CCXVI, CXVI, Definition VIII]

THEOREM CCXLV. If P is an object, (for all S – ((Q is part of the object S, and R is part of the object S) if and only if P is an ingredient of the object S)), not(Q is the same object as R), not(Q is part of the object R), not(R is part of the object Q) and not (some part of the object Q is part of the object R), then P is Q+R. [Theorem II, Definition I, Theorems CCX, CCXVI, CXIX, Definition VIII]

THEOREM CCXLVI. If P is an object, (for all S – (S is external to P if and only if (S is external to Q, and S is external to R))) and Q is external to R, then P is Q+R. [Theorems XXXII, CCXXXII, Definition VIII]

THEOREM CCXLVII. If P is Q+R, and P is an ingredient of the object S, then Q is part of the object S. [Definition I, Theorem CLXII, Axiom II]

THEOREM CCXLVIII. P is part of the object Q if and only if for some R – (Q is P+R). (Follows from Theorems CLXII and CLX)

THEOREM CCIL. If P is part of the object Q, then for some R – (R is an object, not(Q is the same object as R) and Q is a set of objects (P or R)). [Theorems CLX, CXLVIII, CLXIV, Definition VIII, Theorem XIV]

THEOREM CCL. P is Q+R if and only if (P is an object, (for all S – (S is external to P if and only if (S is external to Q, and S is external to R))) and Q is external to R). (Follows from Theorems CCXLVI, CLXXI, and Definition VIII).

THEOREM CCLI. If P is Q+R, and P is an ingredient of the object S, then Q is an ingredient of the object S. (Follows from Theorem CCXLVII and Definition I).

THEOREM CCLII. If P is Q+R, and P is an ingredient of the object S, then R is part of the object S. [Theorems CLXVII, CCXLVII]

THEOREM CCLIII. P is part of the object Q if and only if (P is an object, not(P is the same object as Q) and for some R – (R is an object, not(Q is the same object as R) and Q is a set of objects (P or R))). (Follows from Theorems CCXIII, L, and CCIL)

THEOREM CCLIV. If P is Q+R, and P is an ingredient of the object S, then R is an ingredient of the object S. (Follows from Theorem CCLII and Definition I)

THEOREM CCLV. If P is Q+R, then ((Q is part of the object S, and R is part of the object S) if and only if P is an ingredient of the object S). (Follows from Theorems CCXLVII, CCLII, and CCXLIII)

THEOREM CCLVI. If P is Q+R, then ((Q is an ingredient of the object S, and R is an ingredient of the object S) if and only if P is an ingredient of the object S). (Follows from Theorems CCLI, CCLIV, and CCXLII)

THEOREM CCLVII. P is Q+R if and only if (P is an object, (for all S – ((Q is part of the object S, and R is part of the object S) if and only if P is an ingredient of the object S)), not(Q is the same object as R), not (Q is part of the object R), not(R is part of the object Q) and not(some part of the object Q is part of the object R)). (Follows from Theorems CCXLV, CCLV, CXLVII, CCXL, CCXXXIX, and CCXXXVIII)

THEOREM CCLVIII. P is Q+R if and only if (P is an object, (for all S – ((Q is an ingredient of the object S, and R is an ingredient of the object S) if and only if P is an ingredient of the object S)) and not(some ingredient of the object Q is an ingredient of the object R)). (Follows from Theorems CCXLIV, CCLVI, and CCXXXVII)

THEOREM CCLIX. P is Q+R if and only if (P is an object, (for all S – ((Q is part of the object S, and R is part of the object S) if and only if P is the same object as S, or is part of the object S)), not(Q is the same object as R), not(Q is part of the object R), not(R is part of the object Q) and not(some part of the object Q is part of the object R)). (Follows from Theorem CCLVII and Definition I)

THEOREM CCLX. If P is the same object as Q, then for some a – (Q is the sum of the objects a, and P is an a). [Theorem CLXXVII]

THEOREM CCLXI. If Q is the sum of the objects a, and P is an a, then P is an ingredient of the object Q. [Definition IX, Theorem LXXXV]

THEOREM CCLXII. If P is part of the object Q, then for some a – (Q is the sum of the objects a, and P is an a). [Theorems CLX, CLXXXI]

THEOREM CCLXIII. If P is an ingredient of the object Q, then for some a – (Q is the sum of the objects a, and P is an a). (Follows from Definition I, Theorems CCLX and CCLXII)

THEOREM CCLXIV. P is an ingredient of the object Q if and only if for some a – (Q is the sum of the objects a, and P is an a). (Follows from Theorems CCLXI, and CCLXIII)

THEOREMS CCXLVIII, CCLIII, LXXXV, CCXXXI, CCLXIV, and CLXIII show that instead of any of the terms 'part' and 'ingredient' I could have taken as the fundamental term of my 'general set theory' any of the terms '+', 'set', 'class', 'external', 'sum', and 'complement'.

Chapter X: Axioms of 'General Set Theory' dating from 1921

In 1921 I determined that I could obtain a theory equivalent to the theory that I have set out in Chapters IV, V, VIII, and IX and constructed by means of the fundamental term 'external'[213] if as the axioms of the theory I took theses asserting, respectively, that

(A) P is external to Q if and only if for all S – some object external to P or external to Q is not(external to S),

and

(B) if (for all P – (P is $\phi(a)$ if and only if (P is an object and for all Q – (P is external to Q if and only if for all R – if R is an *a*, then R is external to Q)))) and some object is an *a*, then $\phi(a)$ is an object,

while the terms 'class', 'ingredient', and 'part' are introduced by means of definitions establishing, respectively, that

(C) P is the class of objects *a* if and only if (P is an object and for all Q – (P is external to Q if and only if for all R – if R is an *a*, then R is external to Q)),

(D) P is an ingredient of the object Q if and only if for some *a* – (Q is the class of objects *a*, and P is an *a*),

and

(E) P is part of the object Q if and only if (P is an ingredient of the object Q and not(P is the same object as Q)).

We can easily establish that, on the basis of the theory which I have set out in Chapters IV, V, VIII, and IX, it is possible to obtain equivalents of all the theses that can be obtained on the basis of the new theory based on Axioms A and B and containing Definitions C–E, keeping in mind the fact that Theses A–E are only reiterations, respectively, of Theorems CCXXXIII, CCXXXVI, CCXXXV, LXXXV, and L of the theory that I set out in Chapters IV, V, VIII, and IX.[214] For that matter, we can convince ourselves that conversely as well the equivalents of all the theses that can be obtained on the basis of this very theory can be obtained within the framework of a theory based on Axioms A and B and containing Definitions C–E, and by deriving explicitly the equivalents of Axiom I, Axiom II, Definition I, Definition II, Axiom III, Axiom IV, and Definition VI from Theses A–E, which may be carried out as follows:

THEOREM F. If P is an object, then P is the class of objects P.[215] [C].

THEOREM G. If R is an object, and for all S and T – if R is external to S, and T is P or R, then T is external to S, then R is the class of objects (P or R). [C].

THEOREM H. If P is an ingredient of the object Q, and Q is external to R, then P is external to R. [D, C].

THEOREM J. If P is an object, then P is an ingredient of the object P.[216] [Theorem F, D].

THEOREM K. If every *a* is an ingredient of the object P, P is external to S, and R is an *a*, then R is external to S. [Theorem H].

THEOREM L. If P is the same object as Q, then P is an ingredient of the object Q. (Follows from Theorem J)[217]

THEOREM M. P is part of the object Q if and only if ((for some *a* – (Q is the class of objects *a*, and P is an *a*)) and not(P is the same object as Q)). (Follows from E and D).

THEOREM N. If P is part of the object Q, and Q is external to R, then P is external to R. [E, Theorem H].

THEOREM P. (The equivalent of Definition I). P is an ingredient of the object Q if and only if P is the same object as Q, or is part of the object Q. (Follows from Theorem L and E[218])

THEOREM Q. If P is part of the object Q, Q is external to S, and R is P, then R is external to S. [Theorem N]

THEOREM R. If P is part of the object Q, Q is part of the object R, R is external to S, and T is P or R, then T is external to S. [Theorem N]

THEOREM S. Not (P is external to P). (Follows from A and the observation that not(some object external to P or external to P is not(external to P)).)

THEOREM T. P is external to Q if and only if Q is external to P.[219] (Follows from A)

THEOREM U. If (for all Q – if Q is external to R, then Q is external to P), R is external to S, and T is P or R, then T is external to S. [Theorem T]

THEOREM V. If P is the class of objects a, then for some R – (R is an a). (Follows from C and Theorem S)

THEOREM W. If P is an ingredient of the object Q, then not(Q is external to P)[220]. (Follows from Theorems H and S)

THEOREM X. If P is an object, R is an object, and for all Q – if Q is external to R, then Q is external to P, then P is an ingredient of the object R.[221] [Theorems U, G; D]

THEOREM Y. If P is the class of objects a, then every a is an ingredient of the object P.[222] [Theorem V, D]

THEOREM Z. If P is external to Q, and R is an ingredient of the object Q, then not(R is an ingredient of the object P). [Theorems W, T, H]

THEOREM AA. If Q is an object, R is an object, and for all P – not(P is external to R), then some ingredient of the object Q is an ingredient of the object R. [Theorems J, X]

THEOREM AB. If Q is an object, R is an object, not(R is external to Q), and P is external to R, then some ingredient of the object Q is an ingredient of the object R. [A, Theorems T, X].

THEOREM AC. If (for all Q – if Q is an ingredient of the object P, then some ingredient of the object Q is an ingredient of some a), (for all R – if R is an a, then R is external to S) and U is an ingredient of the object P, then not(U is an ingredient of the object S). [Theorems T, Z, H]

THEOREM AD. If P is external to Q, then no ingredient of the object Q is an ingredient of the object P. [Theorems T, J, Z]

THEOREM AE. If Q is an object, R is an object and not(R is external to Q), then some ingredient of the object Q is an ingredient of the object R.[223] (Follows from Theorems AB and AA)

THEOREM AF. If P is an object, and no ingredient of the object Q is an ingredient of the object P, then P is external to Q. [D, Theorem AE]

THEOREM AG. If P is the class of objects a, and Q is an ingredient of the object P, then some ingredient of the object Q is an ingredient of some a. [Theorems W, AE; C]

THEOREM AH. If P is an object, (for all Q – if Q is an ingredient of the object P, then some ingredient of the object Q is an ingredient of some a), and for all R – if R is an a, then R is external to S, then P is external to S. [Theorems J, T, AC, AE]

THEOREM AJ. (The equivalent of Definition VI). P is external to Q if and only if the following conditions are satisfied:

(α) P is an object;
(β) no ingredient of the object Q is an ingredient of the object P.

(Follows from Theorems AF, AD).

THEOREM AK. If P is an object, every a is an ingredient of the object P, and for all Q – if Q is an ingredient of the object P, then some ingredient of the object Q is an ingredient of some a, then P is the class of objects a. [Theorems AH, K; C]

THEOREM AL. (The equivalent of Definition II.) P is the class of objects a if and only if the following conditions are satisfied:

(α) P is an object;
(β) every a is an ingredient of the object P;
(γ) for all Q – if Q is an ingredient of the object P, then some ingredient of the object Q is an ingredient of some a. (Follows from Theorems AK, Y, and AG)

Auxiliary Definition Δ. P is Kl(a) if and only if P is the class of objects a.

THEOREM AM. P is Kl(a) if and only if (P is an object and for all Q – (P is external to Q if and only if for all R – if R is an a, then R is external to Q)). (Follows from Definition Δ and C)

THEOREM AN. If some object is an a, then Kl(a) is an object. (Follows from B and Theorem AM)

THEOREM AO. (The equivalent of Axiom IV.) If some object is an a, then some object is the class of objects a. (Follows from Theorem AN and Definition Δ)

THEOREM AP. (The equivalent of Axiom III.) If P is the class of objects a and Q is the class of objects a, then P is Q. [Theorems V, AN, Definition Δ] [224, 225]

THEOREM AQ. (The equivalent of Axiom I.) If P is part of the object Q, then Q is not part of the object P. [Theorems M, F, AP, Q; C, Theorem N].

THEOREM AR (The equivalent of Axiom II.) If P is part of the object Q, and Q is part of the object R, then P is part of the object R. [Theorems M, R, G, AQ, M]

Notes

[* Translated from the Polish by Vito F. Sinisi]

[1] Alfred North Whitehead and Bertrand Russell, *Principia Mathematica*, Cambridge, Vol. I, 1910, Vol. II, 1912, Vol. III, 1913, Vol. I, Second edition, 1925.

[2] Formerly I used the expression 'logistic' to denote it; cf. Adolf Lindenbaum and Alfred Tarski, 'Komunikat o badaniach z zakresu teorji mnogości' [Communication on Investigations in Set Theory], presented by W. Sierpiński, *Comptes rendus de séances de la Société des Sciences et de Lettres de Varsovie* **XIX** (1926), Classe III, p. 322.

[3] Cf. Ernst Schröder, *Vorlesungen über die Algebra der Logik (exakte Logik)*, Erster Band, Leipzig, 1890, p. 161.

[4] Cf. Schröder, *op. cit.*, Zweiter Band, Erste Abteilung, Leipzig, 1891, pp. 1–84 and 256–276.

[5] Cf. Whitehead and Russell, *op. cit.*, Volume I, Second edition, pp. 90–126.

[6] Cf. *loc. cit.*, pp. 127–186.

[7] Cf. Schröder, *op. cit.*, Erster Band, pp. 160 and 161.

[8] Cf. *op. cit.*, Zweiter Band, Erste Abteilung, pp. 318–349.

[9] Stanisław Leśniewski, *Podstawy ogólnej teoryi mnogości. I* (Część. Ingredjens. Mnogość. Klasa. Element. Podmnogość. Niektóre ciekawe rodzaje klas.) [Part. Ingredient. Set. Class. Element. Subset. Some Interesting Types of Class], Prace Polskiego Koła Naukowego w Moskwie, Sekcya matematyczno-przyrodnicza, No. 2, Moscow, 1916.

[10] G. Frege, *Grundgesetze der Arithmetik, Begriffsschriftlich abgeleitet*, Jena, Erster Band, 1893, Zweiter Band, 1903.

[11] Cf. Frege, *op. cit.*, Zweiter Band, pp. 253 and 254.

[12] Cf. *loc. cit.*, pp. 262–265.

[13] E. Zermelo, 'Untersuchungen über die Grundlagen der Mengenlehre. I', *Mathematische Annalen* 65 (1908).

[14] Cf. K. Grelling and L. Nelson, 'Bemerkungen zu den Paradoxieen von Russell und Burali Forti', *Abhandlungen der Fries'schen Schule*, Neue Folge, Zweiter Band, 3, Heft, 1908, VIII, p. 314.

[15] Cf. Bertrand Russell, 'Mathematical Logic as Based on the Theory of Types', *American Journal of Mathematics* **XXX** (1908), 222. Cf. also Whitehead and Russell, *loc. cit.*, p. 37.

[16] Cf. *loc. cit.*, p. VII.

[17] Cf. *loc. cit.*, p. XIV.

[18] In connection with the first edition, cf., at least, Leon Chwistek, 'The Theory of Constructive Types (Principles of Logic and Mathematics), Part I, General Principles of Logic, Theory of Classes and Relations' (Extracted from the *Annales de la Société Mathématique de Pologne*, Cracow, 1923), p. 22, Footnote 3. Mr. Chwistek's work contains a number of interesting and objectively keen critical remarks on the first edition of the system of Messrs. Whitehead and Russell.

[19] Cf. Frege's 'Grundsätze' governing the establishment of definitions (Frege, *op. cit.*, Erster Band, pp. 51 and 52), as well as 'Regeln' governing the proving of theorems (*loc. cit.*, pp. 61–64). Cf. also *loc. cit.*, pp. VI and VII. Cf. also: (1) Bertrand Russell, *Introduction to Mathematical Philosophy*, London, New York, Second edition, April, 1920, p. 151; (2) Chwistek, *loc. cit.*, p. 21.

[20] Cf. *op. cit.* The continuation of this work has also appeared in print: Leon Chwistek, 'The Theory of Constructive Types (Principles of Logic and Mathematics), Part II, Cardinal Arithmetic', *Rocznik Polskiego Tow. Matematycznego, Annales de la Société Polonaise de Mathématique*, Vol. III, 1924, 1925. There are offprints of Part II with separate pagination which is the continuation of the pagination of the offprint of Part I. Referring to 'The Theory of Constructive Types' I shall always give the page numbers according to the pagination of the offprint.

[21] Cf. Chwistek, *op. cit.*, pp. 20–33.

[22] Jan Łukasiewicz, *O zasadzie sprzeczności u Arystotelesa. Studyum krytyczne* [The Principle of Contradiction in Aristotle. A Critical Study], Kraków, 1910.

[23] Cf. *op. cit.*, pp. 166–210.

[24] Cf. Łuksiewicz, *op. cit.*, pp. 128–132.

[25] Whitehead and Russell, *loc. cit.*, p. 96.

[26] *Loc. cit.*

[27] *Loc. cit.*, p. 8.

[28] *Loc. cit.*, p. 92.

[29] *Loc. cit.*

[30] *Loc. cit.*, p. XIII.

[31] Cf. *loc. cit.*, p. 8.

[32] Cf. *loc. cit.*, p. 85.

[33] *Loc. cit.*, p. XXIV.

[34] *Loc. cit.*, p. 94.

[35] *Loc. cit.*, p. 93.

[36] Cf. e.g., the expression 'denoted by $p \vee q$' (Whitehead and Russell, *loc. cit.*, p. 6) with the expression 'denoted by "$p \equiv q$" ' (*loc. cit.*, p. 7), or also the expression 'proposition p' (*loc. cit.*, p. 6) with the expression 'proposition "p" ' (*loc. cit.*, p. 8). In connection with quotation marks in Frege's works, cf. Frege, *loc. cit.*, p. 4.

[37] Whitehead and Russell, *loc. cit.*, p. 93.

[38] *Loc. cit.*, p. 7.

[39] Cf., e.g., J. v. Neumann, 'Zur Hilbertschen Beweistheorie', *Mathematische Zeitschrift*, Sonderabdruck aus Band 26, Heft 1, 1927, pp. 3–5. Regarding the meaningfulness of mathematical

propositions, cf. Frege, *op. cit.*, Zweiter Band, pp. 96–139.

[40] Cf. Whitehead and Russell, *loc. cit.*, p. 99.

[41] St. Leśniewski, 'Przyczynek do analizy zdań egzystencjalnych', *Przegląd Filozoficzny* [Philosophical Review] **XIV**, 3 (1911).

[42] Stanisław Leśniewski, 'Próba dowodu ontologicznej zasady sprzeczności', *Przegląd Filozoficzny* **XV**, No. 2 (1912). The two works just mentioned have appeared in Russian in a somewhat different form: S. I. Leśnévskij, *Logičéskiá razsuždéniá*, I: Opyt obosnovaniá ontologičéskago zakona protivoréčiá, II: K analizu ekzisténcialnyh prédloženij, St. Petersburg, 1913.

[43] Tadeusz Kotarbiński, 'Zagadnienie istnienia przyszłości', *Przegląd Filozoficzny* **XVI** No. 1 (1913).

[44] Stanisław Leśniewski, 'Czy prawda jest tylko wieczna czy tez i wieczna i odwieczna? Szkic popularno-polemiczny z zakresu teorji twórczości' [Is Truth Only Eternal or is it Both Eternal and Sempiternal? A Popular, Polemical Sketch in the Theory of Creation], *Nowe Tory* [New Paths] **VIII**, No. X (December, 1913).

[45] Stanisław Leśniewski, 'Krytyka logicznej zasady wyłączonego środku' (with precisely this atrocious printer's error in the title), *Przegląd Filozoficzny* **XVI**, Nos. II and III (1913). [In the title 'środku' should be 'środka'.]

[46] I devoted a section of the work entitled 'Krytyka logicznej zasady wyłączonego środka' (cf. *op. cit.*, pp. 317–320) to a critique of the conception of 'general objects'. (In the book *Logičéskiá razsuždéniá* the corresponding section is found on pp. 27–33.) In this section I tried to show that "no object is a 'general' object" (p. 320), stating "Regardless of the actual forms that they assume in various thinkers, the 'general objects' appearing in various systems, whether as 'concepts' in the sense of ancient or 'medieval' 'realism', or as Locke's 'general ideas', or as Professor Twardowski's 'objects of general representation' or as Husserl's 'ideal' objects existing 'outside of time' – these objects have, according to the authors who busy themselves with them, a unique, characteristic property that is based on the fact that an "object that is a so-called 'general object' relative to a certain group of 'individual' objects can have only those properties that are common to all the 'individual' objects corresponding to them" " (p. 319). When I wrote this section, I believed that so-called properties and so-called relations existed in the world as two special kinds of objects, and I did not have any scruples in using the expressions 'property' and 'relation'. Now I do not believe, as I once did, in the existence of objects that are properties nor in the existence of objects that are relations, since nothing impels me to a belief in the existence of such objects (cf., Tadeusz Kotarbiński, 'Sprawa istnienia przedmiotów idealnych' [On the Existence of Ideal Objects], off-print from *Przegląd Filozofiezny*, Lwów, 1921, pp. 7–11), and I have tried not to use the expressions 'property' and 'relation' in situations having any 'subtle' character without applying various extensive precautions and circumlocutions. Today, because of the possibility of various interpretational misunderstandings, I am not inclined to ascribe a view on 'general objects' to a particular author mentioned in the section cited above. Nevertheless, I wish here to state, linking to this section and having in mind all those who (in connection with the meaning that they would give to expressions of the type 'general object with respect to objects a') would be inclined to assert the proposition 'if X is a general object relative to the objects a, X is b, and Y is a, then Y is b', that this proposition entails the proposition 'if there exist at least two different a's, then there does not exist a general object relative to the objects a', as the following argument demonstrates:

(1) if X is a general object relative to the objects a, X is b, Y is a, then Y is b (assumption);

from (1) it follows that

(2) if X is a general object relative to the objects a, X is different from Z, and Z is a, then Z is different from Z,

and

(3) if X is a general object relative to the objects a, X is identical with Z, and Y is a, then Y is identical with Z;

from (2) it follows that

(4) if X is a general object relative to the objects a, and Z is a, then X is identical with Z;

and from (4) that

(5) if X is a general object relative to the objects a, Z is a, and Y is a, then (X is a general object relative to the objects a, X is identical with Z, and Y is a);

from (5) and (3) it follows that

(6) if X is a general object relative to the objects a, Z is a, and Y is a, then Y is identical with Z,

and from (6) that

if there exist at least two different a's, then a general object relative to the objects a does not exist.

(The argument would remain valid, *mutatis mutandis*, if one were to use analogously, instead of expressions of the type 'general object relative to the objects a', expressions of some other types, e.g., expressions of the type 'object of the general concept a'.) I consider my statement to be a careful formulation of theoretical tendencies inhering more or less explicitly in the arguments of the adversaries of various kinds of 'universals' during the various phases of the 'dispute' over them. If someone considers this statement a banal statement, I would appeal in its defence to the fact that nevertheless exponents of 'philosophy' unfortunately too often defend positions inconsistent with banal statements.

[47] Łukasiewicz, *op. cit.*, pp. 129–131.

[48] Cf., Stanisław Leśniewski, 'Czy klasa klas, nie podporządkowanych sobie, jest podporządkowana sobie? [Is the Class of Classes That Are Not Subordinate to Themselves Subordinate to Itself?], *Przegląd Filozoficzny*, **XVII** (1914), ·64. In this poor article I expressed my views on Mr. Russell's 'antinomy'. Not having at the time my own axiomatic theory of classes, I appealed there from case to case to various theses of this field in which I believed and which were necessary to my arguments. In this respect my procedure was just like the procedures of all those 'set theorists' who do not build their work on clear axiomatic foundations.

[49] When I wish to alleviate the reader's doubts regarding the grammatical case I am using in various expressions, I shall use the expression 'przedmiot' [object], in the appropriate grammatical case, before the expression whose case is questionable, and written in an abbreviated form ('p-tu' [of object], 'p-tów' [of objects], etc.), which I shall not use in any other contexts. I shall use the expressions 'p-tu P' [of object P], 'p-tów a' [of objects a], etc., which

have been created to alleviate grammatical doubts, as the appropriate grammatical cases of the expressions 'P', '*a*', etc. Earlier I used in analogous situations the expressions '(przedmiotu) P' [(of object) P], '(przedmiotów) *a*' [(of objects) *a*], etc. (cf., *loc. cit*). For uniformity of exposition and to avoid complicating it with historical details of little significance, I shall present matters as if I had used the expressions 'p-tu P', 'p-tów *a*', etc., in these situations.

[50] Cf., *loc. cit.*

[51] Cf. *loc. cit.*; cf. also G. Frege, 'Kritische Beleuchtung einiger Punkte in E. Schröders, *Vorlesungen über die Algebra der Logik*', *Archiv für Philosophie*, Zweite Abtheilung, Archiv für systematische Philosophie, Neue Folge der Philosophischen Monatshefte, I. Band, 1895, 4 Heft, pp. 434 and 435.

[52] Cf. Łukasiewicz, *loc. cit.*

[53] Cf. Łukasiewicz, *loc. cit.*

[54] Cf. C. S. Peirce, 'On the Algebra of Logic. A Contribution to the Philosophy of Notation', *American Journal of Mathematics* VII (1885), 197.

[55] Cf. Whitehead and Russell, *loc. cit.*, p. 15.

[56] Cf. Whitehead and Russell, *loc. cit.*, pp. 7 and 6.

[57] Cf. Leśniewski, *op. cit.*, p. 72. In this work, although I was cognizant of it at the time, I obtained a counterpart of the above formulated Thesis 11 in a different way than shown here; in the 1914 article I used a form of argumentation that was more complex and that seemed more 'interesting'. However, the wording of the relevant passages (pp. 71–73) is not, unfortunately, completely clear without additional comments.

[58] Cf. *op. cit.*, p. 73.

[59] Cf. Leśniewski, *op. cit.*, p. 74.

[60] Cf. *op. cit.*, p. 70.

[61] Cf. Leśniewski, *op. cit.*, p. 74.

[62] Georg Cantor, 'Mitteilungen zur Lehre vom Transfiniten', *Zeitschrift für Philosophie und philosophische Kritik*, Neue Folge, Einundneunzigster Band, 1887, p. 83.

[63] Cf. Georg Cantor, 'Mitteilungen zur Lehre vom Transfiniten (Fortsetzung des Abschnittes VIII und Schluss des Aufsatzes)', *Zeitschrift für Philosophie und philosophische Kritik*, Neue Folge, Zweiundneunzigster Band, 1888, pp. 242 and 243.

[64] Cf. *loc. cit.*, p. 243.

[65] Georg Cantor, *Grundlagen einer allgemeinen Mannichfaltigkeitslehre. Ein mathematisch-philosophischer Versuch in der Lehre des Unendlichen*, Leipzig, 1883, p. 43.

[66] *Loc. cit.*

[67] Georg Cantor, 'Beiträge zur Begründung der transfiniten Mengenlehre', (Erster Artikel), *Mathematische Annalen* 46, (Band 1895), 481.

[68] Frege, *Grundgesetze der Arithmetik*, Erster Band, p. 1.

[69] *Loc. cit.*, p. 2.

[70] *Loc. cit.*, pp. 2 and 3. Cf. Richard Dedekind, *Was sind und was sollen die Zahlen?*, Braunschweig, 1888, p. 2.

[71] F. Hausdorff, *Mengenlehre*, Zweite, neubearbeitete Auflage, Göschens Lehrbücherei. I. Gruppe. Reine Mathematik, Band 7, Berlin und Leipzig, 1927, p. 11.

[72] *Loc. cit.*

[73] *Op. cit.*, p. 12.

[74] Wacław Sierpiński, *Zarys teorji mnogości. Część pierwsza. Liczby pozaskończone*, Wydanie drugie, zmienione [Outline of Set Theory. First Part. Infinite Numbers, second revised edition], Warsaw, MCMXXIII, p. 4.

[75] Sierpiński, *op. cit.*, p. 5.

[76] Adolf Fraenkel, *Einleitung in die Mengenlehre. Eine elementare Einführung in das Reich des Unendlichgrossen*, Zweite erweiterte Auflage, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete, Band IX, Berlin, 1923, p. 55.

[77] *Op. cit.*, p. 15.

[78] Someone might say that on the basis of Mr. Fraenkel's statements concerning the 'Nullmenge' it would be possible to derive a contradiction, and consequently any sensational thesis in a considerably shorter way. I would reply to this that in my argument I have avoided as carefully as possible positions that are dubious because of factors involving interpretation. Thus, e.g., from the author's assertion that the 'Nullmenge', which is 'introduced' by him, is an improper set, I have not derived the conclusion that the 'Nullmenge' is a set, because I did not exclude the possibility that Mr. Fraenkel uses the expression 'improper set' after the fashion of the familiar expressions 'dead man', 'false diamond', 'painted fish', etc., of colloquial language. (Cf., e.g., *Hauptwerke der Philosophie in originalgetreuen Neudrucken*, Band IV: *Werke Bernard Bolzanos, I. Wissenschaftslehre* in vier Banden. Erster Band, Leipzig, 1914, p. 257, and also the citation from Savonarola on p. 79.) Although it is not very intelligible, I did not take advantage of the author's statement, which in itself is completely equivocal, that the 'Nullmenge', not being a set at all, nevertheless 'gelten soll' as a set.

[79] Frege, 'Kritische Beleuchtung einiger Punkte in E. Schröders *Vorlesungen über die Algebra der Logik*', pp. 436 and 437.

[80] Cf. Schröder, *op. cit.*, Erster Band, p. 247. Cf., also Frege, *op. cit.*, p. 443.

[81] *Op. cit.*, p. 445.

[82] *Op. cit.*, pp. 443 and 444.

[83] *Op. cit.*, p. 444.

[84] *Op. cit.*, pp. 444 and 445.

[85] *Op. cit.*, p. 445.

[86] *Loc. cit.*

[87] Cf., e.g., Frege, *Grundgesetze der Arithmetik*, Zweiter Band, pp. 253 and 254.

[88] Frege, 'Kritische Beleuchtung einiger Punkte in E. Schröders *Vorlesungen über die Algebra der Logik*', p. 455.

[89] *Op. cit.*, p. 451. The passages from the works of Frege that I have here cited should be an antidote against possible misapprehensions about this author that might be evoked by Mr. Weyl's vague report that Dedekind, Frege, and Russell "sich offenbar die 'Menge' als ein Kollektivum vorstellten". (Cf. Herm. Weyl, *Philosophie der Mathematik und Naturwissenschaft*, München und Berlin, 1927, p. 11.)

[90] Cf. Frege, *Grundgesetze der Arithmetik*, Erster Band, pp. 7 and 8.

[91] *Loc. cit.*, p. 8.

[92] Cf. Bertrand Russell, *The Principles of Mathematics*, Vol. 1, Cambridge, 1903, pp. 511–514.

[93] Cf. Frege, *op. cit.*, Zweiter Band, pp. 260 and 261.

[94] E. Zermelo, 'Neuer Beweis für die Möglichkeit einer Wohlordnung', *Mathematische Annalen* 65 (Band, 1908), 124.

[95] Cf. Zermelo, 'Untersuchungen über die Grundlagen der Mengenlehre', p. 262.

[96] Whitehead and Russell, *loc. cit.*, pp. 71 and 72.
[97] *Loc. cit.*, p. 72.
[98] *Loc. cit.*, p. 187. Cf., in connection with the third concluding quotation, B. Russell, 'Les paradoxes de la Logique', *Revue de Métaphysique et de Morale*, Quaterzième année, 1906, p. 636.
[99] Cf. Whitehead and Russell, *loc. cit.*, p. 66.
[100] The expression 'which' applies, in this proposition, to the expression 'extension', not to the expression 'symbol for an extension'.
[101] Russell, *Introduction to Mathematical Philosophy*, p. 183.
[102] Leśniewski, *Podstawy ogólnej teoryi mnogości. I*, pp. 5 and 6.
[103] *Op. cit.*, p. 12.
[104] *Op. cit.*, p. 24.
[105] *Op. cit.*, p. 7.
[106] *Op. cit.*, p. 9.
[107] *Loc. cit.*
[108] Stanisław Leśniewski, 'O podstawach matematyki', *Przegląd Filozoficzny*, XXX, Nos. II–III, 1927, p. 191. [This footnote refers to p. 15 of this translation.]
[109] Cf. Zermelo, *op. cit.*, p. 261.
[110] In writing the work cited I used and I use here sentences of the type 'A is not *b*' as equivalents of the corresponding sentences of the type 'A is an object, and not(A is *b*)' but not as equivalents of the corresponding sentences of the type 'not(A is *b*)'. Knowing that no object is a round square, and that consequently, pursuant to Thesis 5 of Chapter II (Leśniewski, *loc. cit.*, p. 14), not(a round square is an airplane), which is completely consistent with the system of 'ontology' developed below, as well as not(a round square is an object), yet I do not in the least have the grounds in the terminology above for saying that a round square is not an airplane, and what is more, I am completely entitled to say in the aforementioned terminology that not(a round square is not an airplane). Cf. Whitehead and Russell, *loc. cit.*, p. 69.
[111] In the original, sentences that I called 'definitions' were sentences about myself. Thus, e.g., 'Definition I' in the original read: "I use the expression 'ingredient of the object P' to denote the object P itself and every part of this object" (Leśniewski, *Podstawy ogólnej teoryi mnogości*, p. 9). Lucjan Zarzecki suggested the use of the expression 'ingredient' in this context.
[112] In writing the work cited I used and I use here sentences of the type 'every *a* is *b*' as equivalents of corresponding sentences of the type 'some object is *a*, and for all X – if X is *a*, then X is *b*', but not as equivalents of the corresponding sentences of the type 'for all X – if X is *a*, then X is *b*'; similarly I used and I use here sentences of the type 'no *a* is *b*' as equivalents of the corresponding sentences of the type 'some object is *a*, and for all X – if X is *a*, then X is not *b*' but not as equivalents of the corresponding sentences of the type 'for all X – if X is *a*, then X is not *b*'.
[113] Cf. Leśniewski, *op. cit.*, p. 12, Definition III. This definition reads as follows: "I use the expressions 'set of all objects *m*' and 'class of objects *m*' to denote every object P that satisfies the following two conditions:
(1) every *m* is an ingredient of the object P,
(2) if I is an ingredient of the object P, then some ingredient of the object I is an ingredient of some *m*".
[114] Below we shall find that conditions β and γ of Definition II, that I have formulated guided by the statement of Definition III of the original (mentioned in Footnote 113), entail, on the basis of my 'general set theory', condition α.
[115] Cf. *loc. cit.*, Axiom IV.
[116] Cf. *loc. cit.*, Axiom III.
[117] Theorem I of the original read: 'No object is part of itself' (*op. cit.*, p. 9). In the terminology introduced above in Note 112, this theorem states that some object is an object, and for all X – if X is an object, then X is not part of the object X. In the proof of this theorem I relied on the assumption, which I did not doubt, that some object is an object. The thesis stating that some object is an object cannot be obtained on the basis of the system of 'ontology' developed below. (To avoid possible misunderstandings I note here that in view of the fact that I use sentences of the type 'some *a* is *b*' as equivalents of corresponding sentences of the type 'for some X – (X is *a* and X is *b*)', in which the expression 'for some X' plays the role of the 'quantifier' '($\exists$X)', the sentence 'some object is an object' is for me equivalent to the sentence 'for some X – (X is an object and X is an object)', and hence also to the sentence 'for some X – X is an object'). Wishing to adapt my exposition of 'general set theory' to the aforementioned system of 'ontology', I shall not here use the assumption that some object is an object. The wish to avoid this very assumption causes me to formulate as Theorem I of the present exposition a weaker sentence than Theorem I of the original.
[118] Cf. the above Note 110.
[119] Cf. the above Note 110.
[120] Theorem II of the present exposition is a weaker sentence then Theorem II of the original, which reads: 'Every object is its own ingredient'. (*Loc. cit.*) Theorem II of the original was based on the assumption that some object is an object.
[121] Cf. *op. cit.*, p. 14, Thesis 10 in the proof of Theorem IX.
[122] In writing the work mentioned I used and here I also use the expression 'object' in a way that permits one to say that if A is *b*, then A is an object. This harmonizes completely with the system of 'ontology' developed below.
[123] Cf. the preceding Note 122.
[124] Cf. *op. cit.*, p. 9, Theorem IV.
[125] Cf. *op. cit.*, p. 22, Thesis 7 in the proof of Theorem XXIII.
[126] Cf. *op. cit.*, proof of Theorem VII, p. 13, lines 1–3.
[127] Using the clumsy sentence 'Every object P is the class of ingredients of this very object P', which is Theorem VIII of the original (*loc. cit.*), I wished to affirm that every object is the class of ingredients of this object. Theorem VII of the present exposition is a weaker sentence.
[128] Cf. the above Note 112.
[129] Using the clumsy sentence 'Every given object P is the class of objects P', which is Theorem X of the original (*op. cit.*, p. 14), I wished to express the idea that could be expressed more deftly in the sentence 'some object is an object, and for all P – if P is an object, then P is the class of objects P'; Theorem VIII of the present exposition is a weaker sentence.
[130] In writing the work mentioned I used and I use here sentences of the type 'A is *b*' in such a way as to permit one to say that if A is B, and B is *c*, then A is *c*. This harmonizes completely with the system of 'ontology' developed below.
[131] Cf. *op. cit.*, p. 13, Theorem IX.
[132] Cf. Note 110 above.
[133] Cf. *op. cit.*, p. 15, Thesis 2 in the proof of Theorem XI.
[134] Cf. *op. cit.*, p. 11, Definition II.

[135] Cf. *op. cit.*, p. 18, Thesis 4 in the proof of Theorem XIX.
[136] Cf. *loc. cit.*, Theorem XIX.
[137] Cf. *op. cit.*, p. 17, Theorem XVIII.
[138] Cf. *op. cit.*, p. 12, Theorem VII.
[139] Cf. *op. cit.*, p. 19, Thesis 6 in the proof of Theorem XX.
[140] Cf. Leśniewski, 'O podstawach matematyki', *loc. cit.*, p. 14. Cf. also Leśniewski, *Podstawy ogólnej teoryi mnogości. I*, p. 14, Definition IV.
[141] Cf. *op. cit.*, p. 15, Theorem XII.
[142] Cf. *loc. cit.*, Theorem XI.
[143] Cf. *op. cit.*, p. 16, Theorem XIII.
[144] Theorem XIX of the present exposition is a weaker sentence than Theorem XIV of the original, which reads: 'Every object is its own element' (*loc. cit.*)
[145] Cf. *loc. cit.*, Theorem XVI.
[146] Cf. *op. cit.*, p. 17, Theorem XVII.
[147] Cf. *op. cit.*, p. 24, Theorem XXVII. This theorem reads: "The theorem 'if P is an element of a set of objects *m*, then P is an *m*' is false". Today, in a similar situation, instead of this clumsy sentence I would say 'not(for all P and *m* – if P is an element of some set of objects *m*, then P is an *m*)'. In the proof of Theorem XXVII I relied on the assumption that some object is part of some object, which I did not doubt although I did not prove it in my 'general set theory'. In the present exposition I do not repeat this heretofore mistake, and I give my present Theorem XXII the form of a conditional sentence equivalent to the sentence 'if some object is part of some object, then not(for all P and *m* – if P is an element of some set of objects *m*, then P is an *m*)'. Below we shall find that by accepting the assumption that there exist at least two different objects, the thesis stating that some object is part of some object, can be easily proved.
[148] During the writing of the work mentioned I used and I use now sentences of the type 'A is the same object as B' in a way making it possible to say that if A is B, and B is an object, then A is the same object as B. This harmonizes completely with the system of 'ontology' developed below.
[149] Cf. *op. cit.*, p. 16, Theorem XV.
[150] Cf. *op. cit.*, p. 18, Theorem XX.
[151] Cf. Note 112 above.
[152] Cf. *op. cit.*, p. 20, Theorem XXII.
[153] Cf. Note 130 above.
[154] Cf. *op. cit.*, p. 20, Theorem XXI.
[155] Cf. Note 148 above.
[156] During the writing of the work mentioned I used and I use now sentences of the type 'A is the same object as B' in a way making it possible to say that if A is B, and for all C and D – if C is B, and D is B, then C is the same object as D, then A is the same object as B. This harmonizes completely with the system of 'ontology' developed below.
[157] Cf. *op. cit.*, p. 21, Theorem XXIII. This theorem reads: 'If it is true that if I is an ingredient of the object P_1, then some ingredient of the object I is an ingredient of the object P, then P_1 is an ingredient of the object P'. Using the awkward expression 'it is true that if I is an ingredient of the object P_1, then some ingredient of the object I is an ingredient of the object P' in the formulation of Theorem XXIII, I wished to express the idea that could be expressed more deftly using here the expression 'for all I – if I is an ingredient of the object P_1, then some ingredient of the object I is an ingredient of the object P'. In Theorem XXIII, through an oversight, I did not put the sentence 'P_1 is an object' as a second factor in the antecedent of this theorem, although I tacitly used this sentence in the proof of this theorem (cf. *loc. cit.*, Thesis 2). Through either a printer's or author's error in the proof of Theorem XXIII (p. 22, line 5 from the bottom) of the work cited, 'P_2' occurs instead of 'P', which should occur there.
[158] Cf. *op. cit.*, p. 25, Definition V.
[159] Cf. *op. cit.*, p. 30, Theorem XL.
[160] Cf. *loc. cit.*, Theorem XXXIX.
[161] Theorems XVI, XVII, XXVIII, and XXIX show that by introducing sentences of the types 'P is an ingredient of the object Q', 'P is an element of the object Q', and 'P is a subset of the object Q' on the basis of my 'general set theory' I had introduced three kinds of sentences that are equivalent to each other, and that thereby I had only complicated my terminology completely unnecessarily, since any one of these three kinds of sentences would suffice, in virtue of the mutual equivalence of these sentences, for all of my theoretical purposes. In 1918 the feeling of the ugliness of such a state of affairs was induced in me by the critical observations on my work by Dr. Kazimierz Kuratowski, Professor of Mathematics, Lwów Institute of Technology, who was then a student at the University of Warsaw. Mr. Kuratowski's remarks may be applied *mutatis mutandis* also to sentences of the type 'P is part of the object Q' and 'P is a proper subset of the object Q' (cf. *op. cit.*, p. 25, Definition VI, and p. 27, Theorems XXX and XXXI).
[162] Cf. *loc. cit.*, Thesis 3 in the proof of Theorem XXX.
[163] Cf. *op. cit.*, p. 32, Definition VIII.
[164] Theorem XXXI of the present exposition is a weaker sentence than Theorem XLVII of the original, which reads: 'No object is an object external to itself' (*op. cit.*, p. 33; cf. Note 112 above).
[165] Cf. Note 112 above.
[166] Cf. *op. cit.*, Theorem XLVI. Through either a printer's or author's error in the formulation of this theorem (p. 32, line 5 from the bottom) 'P_1' occurs instead of 'P', which should occur there.
[167] Cf. Note 112 above.
[168] Cf. *op. cit.*, p. 33, Definition IX.
[169] Cf. *op. cit.*, p. 37, Thesis 11 in the proof of Theorem L.
[170] Cf. *op. cit.*, p. 36, Thesis 4 in the proof of Theorem L.
[171] Cf. *loc. cit.*, Thesis 7.
[172] Cf. *op. cit.*, p. 35, Theorem IL.
[173] Cf. *op. cit.*, p. 40, Theorem LV. This theorem reads: 'No object is the complement of itself with respect to some object'.
[174] Cf. *op. cit.*, p. 39, Theorem LIV.
[175] Cf. *op. cit.*, p. 36, Thesis 8 in the proof of Theorem L.
[176] Cf. *op. cit.*, p. 37, Thesis 6 in the proof of Theorem LI.
[177] Cf. *op. cit.*, p. 41, Thesis 9 in the proof of Theorem LVIII.
[178] Cf. *op. cit.*, p. 34, Theorem XLVIII.
[179] Cf. *op. cit.*, p. 36, Theorem L.
[180] Cf. *op. cit.*, p. 40, Theorem LVI. This theorem has the following awkward form: 'No object P is the complement of any object with respect to the object P'.
[181] Cf. *op. cit.*, p. 37, Theorem LI.
[182] Cf. Theses 3 and 4 in the proof of Theorem XIII above.
[183] Cf. *op. cit.*, p. 39, Theorem LII.
[184] Cf. *op. cit.*, p. 40, Theorem LVII. This theorem has the following awkward form: 'No object is the complement of any object P with respect to that particular object P'.

185 Cf. *loc. cit.*, Theorem LVIII.
186 [In this extended note of about five pages, Leśniewski proves (following up a conjecture of Tarski) that Whitehead's axiomatic basis for the concept of an event, given in his *Enquiry Concerning the Principles of Natural Knowledge* (Cambridge, 1919), is an inadequate foundation for his theory of events. This proof is omitted here since it appears in an English translation in V. F. Sinisi, 'Leśniewski's Analysis of Whitehead's Theory of Events', *Notre Dame Journal of Formal Logic* VII (1966), 323–327.]
187 Cf. Theses 1, 3, and 4 above in the proof of Theorem XLII.
188 From Theorem LX we see that on the basis of my 'general set theory' conditions β and γ of Definition II have condition α as a consequence. Cf. Stanisław Leśniewski, 'O podstawach matematyki', *Przegląd Filozoficzny* **32** (1929), Footnote 1, p. 265. [This refers to Note 114 of this translation.]
189 Cf. Theses 3–5, and 8 in the proof of Theorem XXVI above.
190 Cf., Note 161.
191 Cf. Thesis 3 in the proof of Theorem XXI above.
192 Some further considerations of the present work are connected with Theorems LIX, LXIX, LXX, LXXIII, and CII presented in this chapter. Only this circumstance has induced me to include these theorems here, which in themselves I do not consider in the least interesting.
193 Theorem CIX shows that by admitting into 'general set theory' the assumption that at least two different objects exist, it would be possible to prove on the basis of the this theory the thesis asserting that some object is part of some object. Cf. Note 147.
194 Cf. *loc. cit.*, Footnote 161.
195 Cf. *loc. cit.*
196 Cf. Note 156.
197 Cf. Russell, *op. cit.*, pp. 176, 177. [It should be noted that in the proof of this theorem Leśniewski uses Thesis 5 of Chapter II, 'If P is an *a*, then one and only one object is P'.]
198 Cf. the above proof of Theorem CLV from Thesis 3 inclusively.
199 The reader will be able to realize the auxiliary roll of Definition X below.
200 In complete harmony with the orthodox statements of 'set theoreticians' in comparing 'sets' with respect to 'power', and at the same time in complete harmony with the system of 'ontology' developed below – I use here expressions of the type 'of the objects *a* there are as many as of the objects *b*' in a way permitting one to assert that

> of the objects *a* there are as many as of the objects *b* if and only if for some ϕ – ((for all X – if X is an *a*, then for some Y – (Y is a *b*, and Y is ϕ(X))), (for all X, Y, and Z – if X is an *a*, Y is a *b*, Z is a *b*, Y is ϕ(X), and Z is ϕ(X), then Y is the same object as Z), (for all X – if X is a *b*, then for some Y – (Y is an *a* and X is ϕ(Y))) and for all X, Y, and Z – if X is a *b*, Y is an *a*, Z is an *a*, X is ϕ(Y), and X is ϕ(Z), then Y is the same object as Z),

and I use expressions of the type 'of the objects *a* there are less than of the objects *b*' in a way permitting one to assert that

> of the objects *a* there are less than of the objects *b* if and only if ((for some *c* – ((for all X – if X is a *c*, then X is a *b*) and of the objects *a* there are as many as of the objects *c*)) and for all *c* – if for all X – if X is a *c*, then X is an *a*, then not(of the objects *b* there are as many as of the objects *c*)).

201 Cf., the above Note 200.
202 *Ibid.*
203 With regard to the fact, already affirmed, that Conditions β and γ of Definition II entail, on the basis of my 'general set theory', Condition α (cf. Note 188), I began to use Definition F instead of Definition II in introducing the term 'class'. Conditions α and γ are common to both definitions. In place of condition β of Definition II there appears in Definition F a weaker condition, for all Q – if Q is an *a*, then Q is an ingredient of the object P. This condition in conjunction with condition γ does not entail condition α, as may be seen, e.g., from the fact that

> for all Q – if Q is a round square, then Q is an ingredient of a round square,

and

> for all Q – if Q is an ingredient of a round square, then some ingredient of the object Q is an ingredient of some round square,

whereas

> not(a round square is an object).

(Cf. here the examples for Definition II.)
204 Cf. Note 192.
205 Cf. the justification above for Theorem II.
206 Cf. Thesis F of Chapter VI.
207 Cf. Note 192.
208 Cf. Theorem II.
209 Cf. the above Theorem H of Chapter VI.
210 Cf. the proof of Theorem H.
211 Cf. the above justification of Theorem N of Chapter VI.
212 Some further considerations of the present work are associated with Theorem CCXXXVI. Only this fact has induced me to include this theorem here, which in itself I do not consider in the least interesting.
213 [Cf., the short paragraph following Theorem CCLXIV above.]
214 Cf. Note 212.
215 Cf. Theorem VIII.
216 Cf. Theorem II.
217 Cf. the proof of Theorem *n* above in Chapter VII.
218 Cf. the justification above of Theorem *o* in Chapter VII.
219 Cf. Theorem CXXVII.
220 Cf. Theorem CXXVIII.
221 Cf. Theorem CCXXVII.
222 Cf. Theorem H above in Chapter VI, and Theorem *p* of Chapter VII.
223 Cf. Theorem CXVI.
224 Cf. the above justification of Thesis 3 in the proof of Theorem CLXXVII as well as the justification of Thesis 9 in the proof of Theorem CLXXXVIII.
225 Since 1920 I have used in the construction of 'general set theory' shorter equivalent expressions of the type 'pt(A)', 'ingr(A)', 'Kl(*a*)', 'ex(A)', etc., instead of the expressions of the type 'part of the object A', 'ingredient of the object A', 'class of objects *a*', 'external to A', etc. Considering that in the exposition of my 'general set theory' beginning in Chapter IV above I used expressions of the latter types just mentioned, I shall use these expressions as I have formerly in order to preserve the uniformity of the exposition. If I were to systematically use expressions of the type 'Kl(*a*)' instead of equivalent expressions of the type 'class of objects *a*'

both in the theses of the theory that I set out in Chapters IV, V, VIII and IX as well as in the theses of the present chapter, I would not have had the least reason to resort to any auxiliary definitions in the derivation of the equivalents of Theses AO and AP. Instead of Definition Δ, given above, and Theorems AM–AP I could have, in this case, formulated, for example, only three theorems, asserting, respectively that:

THEOREM AN. If some object is an *a*, then Kl(*a*) is an object. (Follows from the equivalent of thesis B and the equivalent of thesis C.)

THE EQUIVALENT OF THEOREM AO. If some object is an *a*, then some object is Kl(*a*). (Follows from Theorem AN).

THE EQUIVALENT OF THEOREM AP. If P is Kl(*a*), and Q is Kl(*a*), then P is Q.

(*Proof.* let us assume that:

(1) P is Kl(*a*)

and

(2) Q is Kl(*a*);

(3) some object is an *a* (from the equivalent of Theorem V, and (1))

(4) Kl(*a*) is an object (Theorem AN, and (3))

P is Q ((4), (1), (2)).)

In my original deductions of 1921 I used this very argument schema in the derivation of the equivalents of Theorems AO and AP. If I had wished throughout to argue analogously (and thus also without the aid of an auxiliary definition) also in those cases where in the theses of 'general set theory' I use, as I had formerly, not expressions of the type 'Kl(*a*)' but equivalent expressions of the type 'class of objects *a*', I might, in the derivation of the equivalent of Theorem AN from Theses B and C, come up against the reader's doubts as to if and why I am free to 'substitute' the expression 'class of objects *a*' for the whole expression '$\phi(a)$' in the Thesis B. (Here cf. (1) Kazimierz Ajdukiewicz, 'Założenia logiki tradycyjnej' [The Assumptions of Traditional Logic], *Przegląd Filozoficzny* 29 (for the year 1926), Nos. III–IV (1927), 209, 210, 213, and 214; (2) Kazimierz Ajdukiewicz, 'Sprostowanie ważniejszych błędów zawartych w artykule p.t. 'Założenia logiki tradycyjnej' [Corrections of the More Important Errors Contained in the Article entitled 'The Assumptions of Traditional Logic'], *Przegląd Filozoficzny* 30, Nos. II–III (1927), 252; (3) V. Neumann, *op. cit.*, p. 42; (4) Stanisław Leśniewski, 'Gründzuge eines neuen Systems der Grundlagen der Mathematik. Einleitung und §§ 1–11', Sonderabdruck (mit unveränderter Pagination) aus dem XIV Bande der *Fundamenta Mathematicae*, Warszawa, 1929, p. 77). Wishing not to enter into the subtleties of 'substitution' and to remove even the possibility of such doubts, I have preferred to conduct a somewhat longer argument by introducing the *ad hoc* auxiliary Definition Δ. In the derivation of Theorem AN from Thesis B and Theorem AM, I have used the customary 'substitution' of the 'constant' 'Kl' for the 'variable' 'ϕ' in thesis B, which should not raise any doubts of the kinds just now mentioned.