Chapter 1

Quadratic Programming

In this chapter, we will present a brief introduction to Quadratic Programming.

Quadratic programming is a set of methods that solves a class of optimization problem with quadratic form. More specifically, we want to minimize the following objective function

$$c^{\intercal}x + \frac{1}{2}x^{\intercal}\Sigma x$$

subject to the following constraints:

$$a_i^{\intercal}x, \leq b_i \quad i = 1, 2, ..., m$$

In the above expression, $x \in \mathbb{R}^n$ is the variable, $c \in \mathbb{R}^n$ is a constant vector, and $\Sigma \in \mathbb{R}^{n \times n}$ is a constant matrix. The optimization problem has m constraints and for all i, $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$.

The constraints can be expressed in a more compact way:

$$Ax \leq b$$

where

$$A^{\intercal} = [a_1\, a_2 \, ... \, a_m] \text{ and}$$
$$b^{\intercal} = [b_1\, b_2 \, ... \, b_m]$$

Note here A is a matrix in $\mathbb{R}^{m \times n}$ and b is a vector in $\mathbb{R}^n$.

An constraint i is active if $a_i^\mathsf{T} x = b_i$. A related concept, active constraints indices, is defined as follows:

$$I(x_0) = \{i | a_i^\mathsf{T} x_0 = b_i\}$$

Note that the size of $I(x_0)$ is bounded by the number of constraints. In other words,

$$|I(x_0)| \leq m$$

One of the key concepts in quadratic programming is quasi-stationary point, which is defined as follows:

Definition 1.1 (Quasi-Stationary Point)**.** The point x_0 is quasi-stationary if $x_0 \in R$, and x_0 is optimal for the problem

$$\min\{c^\mathsf{T} x + \frac{1}{2} x^\mathsf{T} \Sigma x \mid a_i^\mathsf{T} x = b_i, \ \forall i \in I(x_0)\}$$

The point x_0 is a nondegenerate quasi-stationary point if it's a quasi-stationary point and the gradients of those constraints active at x_0 are linearly independent. The point x_0 is a degenerate quasi-stationary point if it's a quasi-stationary point and the gradients of those constraints active at x_0 are linear dependent.

Remark. An extreme point (in the context of linear programming) is a special case of quasi-stationary point. Recall that the x_0 is an extreme point if $x_0 \in R$ and there are n constraints having linearly independent gradients active at x_0. This reduces $I(x_0)$ to a single point, which is also the optimal point.

Proposition 1.1. *There are at most* 2^m *quasi-stationary points.*

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