Four-Color Theorem [1, 2]

A problem that remained open for over a century.

Merkouris Papamichail†

†Computer Science Department
University of Crete

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The Problem

Problem 1 (Francis Guthrie, 1852)

Can a map be drawn on a sheet of paper, using only *four* colors, in such a way that countries sharing a common border have different colors?

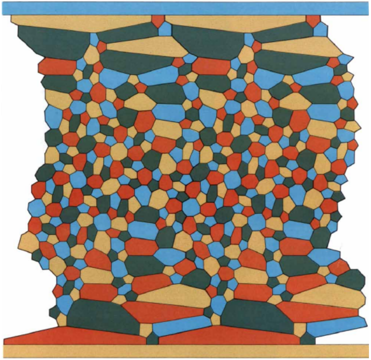

Figure 1: A map of 846 countries colored with four colors, by Edward F. Moore

History

Correcting a century old proof..

1. First stated by Francis Guthrie, 1852.
2. *"There is no map s.t. there are five neighbor countries"*, Augustus De Morgan (around the same time).
3. Arthur Cayley presents the problem to the London Mathematical Society, 1878.
4. First (false) "proof" by Alfred Bray Kempe, 1879.
5. Percy John Headwood finds a mistake in Kempe's proof, 1890.
6. Improvments on Kempe's arguments, by George D. Birkhoff, 1913.
7. Further work by Heinrich Heesch, 1936.
8. The problem solved with the use of computers, by Kenneth Appel and Wolfrang Haken, 1976.

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Graph Theory

Definition

Graph Let V be a finite set. Also, let E be a set of *undirected* pairs on V, i.e.,

$$E \subseteq \{\{v, u\} \mid v, u \in V\}.$$

We call the pair $G = (V, E)$ a *graph* on V. We call the elements of V *nodes* (or vertices), and the elements of E *edges*.

- $N(v)$ denotes the *neighborhood* of v.
- $d(v) = |N(v)|$ denotes the *degree* of v.

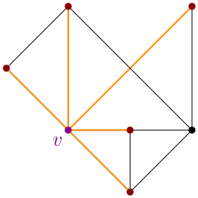


Figure 2: Example of a graph.

Planar Graphs

Definition

A graph $G = (V, E)$ is called *planar* if it can be drawn on the plane $\mathbb{R}^2$, with *no crossing edges*.

- We call the bounded areas of a planar graph *faces*.
- Euler's formula for planar graphs,

$$n - m + f = 2$$

- Planar Graphs are sparse,

$$m \leq 3n - 6$$

- There is some $v \in V$, with $d(v) \leq 5$.

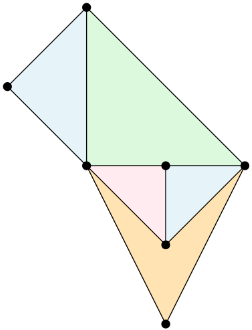
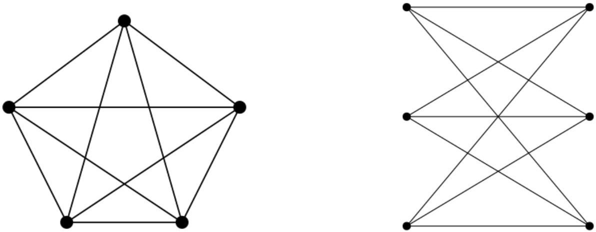


Figure 3: A planar embeding of the graph of Figure 2.

Kuratowski's Theorem (1930)

If a graph G contains the *5-clique* K_5 or the *complete bipartite graph* $K_{3,3}$, then it is *not* planar.



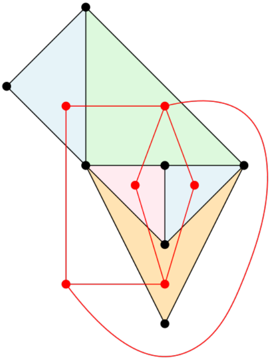
(a) The 5-clique K_5.

(b) The 3,3-clique $K_{3,3}$.

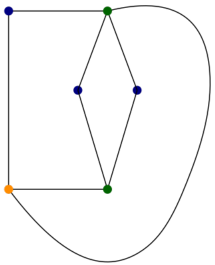
Figure 4: The *forbiden graphs* of Kuratowski's Theorem.

Dual Graphs

From a planar graph, we can construct the *dual graph*. The vertices correspond to faces of the primal graph, while two vartices of the dual are connected, if the corresponding faces share a common edge at their border.



(a) The dual graph of the Figure 2.



(b) The dual graph of Figure 2 is 3-colorable.

A graph-theoretical definition

Definition Chromatic Number of a Graph

Let $G = (V, E)$ be a graph. Let $\phi\colon V \to \{1, \ldots, k\}$ be a function such that $\phi(v) \neq \phi(u)$, if $\{v, u\} \in E$. If such function exist, we call G *k-colorable*. The *smallest* number k such that G is k-colorable is called the *chromatic number* of G and denoted with $\chi(G)$.

Problem 2.

Let $G = (V, E)$ be an arbitrary *simple*, *planar* graph. Does it hold,

$$\chi(G) \leq 4?$$

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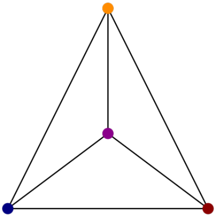
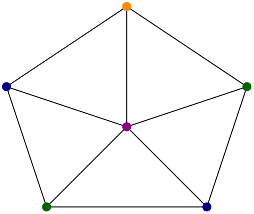
Preliminary Results

1. The is a planar graph that is *not* 3-colorable.
2. Kuratowski's Theorem (that forbids K_5 from planar graphs) doesn't guarantees 4-colorability.



(a) A planar graph with $\chi(G) > 3$.



(b) The number of mutual adjacent nodes, is not the same are the chromatic number. Here we have at most 3 mutually adjacent nodes, but $\chi(G) = 4$.

Figure 6: Examples that suggest that the four color conjecture should hold.

Five-color Theorem (1/4)

Five-color Theorem (Percy John Headwood, 1890)

Let $G = (V, E)$ an arbitrary simple, planar graph. Then $\chi(G) \leq 5$.

Proof:

- Let, for sake of *contradiction*, that there is a graph G with $\chi(G) > 5$.
- Let G be the *smallest* such graph, with respect to the number of vertices.
- From Euler's formula we know that $\exists v,\ d(v) \leq 5$.

Five-color Theorem (2/4)

We consider the following cases:

- We have $d(v) \leq 4$. (easier case)
  1. Then $G \setminus v$ can be colored with five colors.
  2. There are *at most* 4 colors that have been used on neighbors of v.
  3. Hence, there is *at least* one color available for v.
  4. Therefore, G can be colored using five colors.
  5. A contradiction!

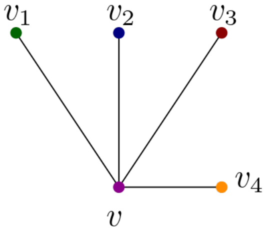


Figure 7: The first case of the five-color theorem.

Five-color Theorem (3/4)

From the above discussion..

1. v_1, v_3 should be on the same connected component.
2. Hence, there is a v_1, v_3-path with colors $1, 3$.

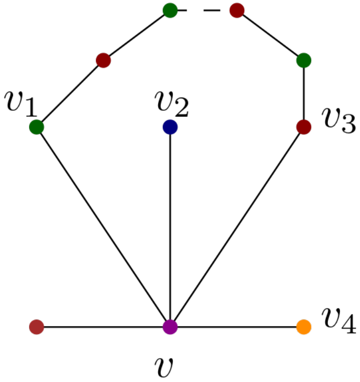


Figure 8: The second case of the five-color theorem, for colors 1,3.

Five-color Theorem (4/4)

Regarding the nodes v_2, v_4 and the colors $2, 4$, following the same reasoning, there is a path between v_2, v_4 using only the colors $2, 4$.

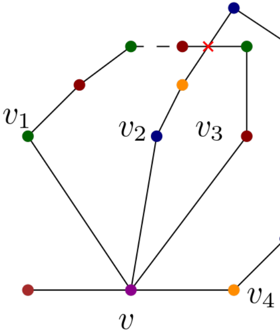


Figure 9: The second case of the five-color theorem, for colors 2, 4.

Therefore, there must be two edges the cross each other. This is absurd, since G is planar.

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Triangulations, Configurations & Unavoidable Sets

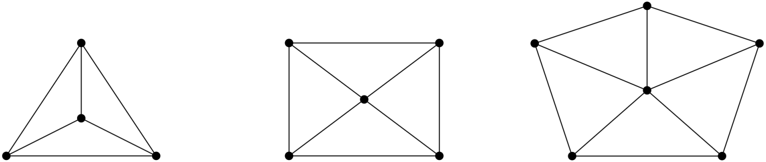
Planar Triangulations

A planar graph $G = (V, E)$ is a graph, that does *not* contain cycles with length *larger than three*.

- A *configuration* in a planar triangulation is a *separation cycle* C (also called a *ring*) together with the portion of the graph *inside* C.
- A set of configurations is *unavoidable* if a minimal counter-example must contain a member of it.
- A configuration is *reducible* if a planar graph containing it cannot be a minimal counter-example.

- We know that in planar graphs, there is a vertex v, with $d(v) \leq 5$.
- In a *triangulation* every vertex has degree $d(v) \geq 3$.

▶ From the above, the three configurations below form an unavoidable set.



(a) 3-ring configuration. (b) 4-ring configuration. (c) 5-ring configuration.

Figure 10: An unavoidable set for the four-color theorem.

Kempe's Proof (1/2)

- Kempe proof follows a technique similar to the Five-color's Theorem proof.
- Consider the center vertex, as the vertex v of the proof.
- From a 4-coloring in $G \setminus v$, we want to construct a 4-coloring for G.
- The 3-ring configuration is trivial.
- For the 4-ring we use an argument similar to the Five-color Theorem.
- for the 5-ring..

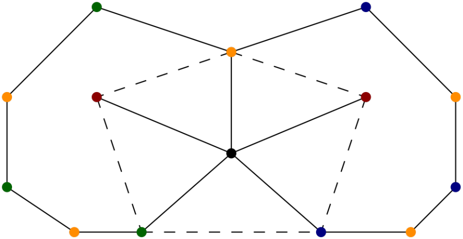
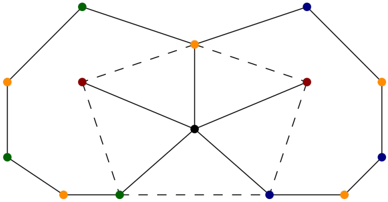
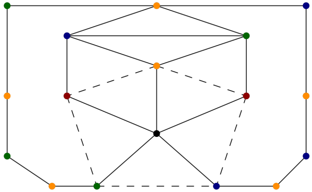


Figure 11: Alternating color paths for the 5-ring.

Kempe's Proof (2/2)



(a) The 5-ring of Kempe's Proof.



(b) A counter-example to Kempe's proof.

Figure 12

- So we can change the red and blue colors, on the right, and red and green colors on the left, thus releasing red for the vertex v right?
- Unfortunately, no!
- Thus, Kempe's proof collapses.

A Computer Aided Proof

- We have not shown that the 5-ring is reducible, thus we must consider *larger configurations*.
- The first proof (Appel, Hanken) used configurations with ring size up to 14.
- A ring os size 13 has 66.430 different colorings.
- Reducibility requires showing that *each* leads to a 4-coloring of the full graph.
- Appel and Hanken, working with Koch, found an unavoidable set of 1936 reducible configurations.
- They used 1000 hours of computation time.
- In 1996 Robetson et al. simplified the proof to consist of 633 configurations.
- In 1997 a desktop workstation could prove the Four-color Theorem in 3 hours!

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Conclusions

In this presentation..

- We examined the historic outline of the Four-color Theorem.
- We made a brief introduction in Planar Graphs and Chromatic Numbers.
- We proved the Five-color Theorem.
- We outlined the main points of the Four-color theorem by Appel and Hanken.
- The Four-color Theorem was the first (and only so far) theorem that proved with the aid of computer.
- Appel and Hanken's proof *cannot* be verified by hand from a mathematician.
- Until this day this novel approach raises questions regarding the acceptance of computers as a mathematical instrument.

Since 1976..

- A shorter more elegant proof of the Four-color Theorem has not be found yet!
- It is unknown is such a proof is possible.
- If it can be proven that such a proof is impossible it would raise many existential questions regarding the nature of mathematical proofs.
- In 1995 Robertson et al. improve on the proof by Appel and Hanken, but their proof still uses a computer [4].
- In 2005 George Gonthier formalizes Robertson's et al. proof in order to be checked by `Coq v7.3.1` proof assistant [3].

Thank you for your time!

Figure 13: The IBM 370-158 computer. One of the computers used by Appel and Hanken for the proof of the four color theorem. The computer was installed at the University of Illinois.

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