On the Parametrized Complexity of Red-Blue Points Separation [1]

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The Problem

Red-blue Points Separation

Let $\mathcal{R}, \mathcal{B} \subset \mathbb{R}^2$ two point-sets in the plane. We need to find some set of lines $\mathcal{L}$, that separates $\mathcal{R}$ from $\mathcal{B}$.

Results

- **Lower Bound**: In the general case, there is *no $n^{o(k)}$ algorithm*, unless ETH is false.
- **Upper Bound**: Axis-Parallel case, there is an $O(n \log n + n|\mathcal{B}|9^{|\mathcal{B}|})$ algorithm.
- **Conjecture**: Axis-Parallel case, there is an FPT algorithm, in the number of lines $|\mathcal{L}|$.

Exponential Time Hypothesis

Exponential Time Hypothesis (ETH)

3-SAT *cannot be solved in subexponential time.*

- Stated by Impagliazzo & Paturi in 1999 [3]
- If ETH holds, then $\mathbf{P} \neq \mathbf{NP}$
- *Stronger conjecture*, than $\mathbf{P} \neq \mathbf{NP}$

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2-Track Hitting Set

Definition

Input:

1. A, B two *totally ordered* sets.
2. $\mathcal{S}_A, \mathcal{S}_B$ two collections of *intervals* of A, B, respectively.
3. An *1-1 correspondence* $\phi \colon A \to B$

Parameter:

- The size of the solution $k = |S|$.

Output:

1. A set $S = \{(a, \phi(a)) \mid a \in A, \phi(a) \in B\}$
2. The 1st projection of S *is a hitting set for A*
3. The 2nd projection of S *is a hitting set for B*.

Structured 2-Track Hitting Set

Definition

1. The definition of the sets $A, B, \mathcal{S}_A, \mathcal{S}_B$ and the solution S is the *same* as 2-Track Hitting Set.
2. Additional *constraints in the 1-1 correspondence*.
3. $\mathcal{C}$ is a *partition* of A into k *color classes*, each containing t elements.
4. For each $a \in C_i$, $\phi(a) = \sigma \circ \sigma_i$, where σ *maps* C_i *to a color class* C_i' *of* B, σ_i "shuffles" the elements of C_i in C_i'

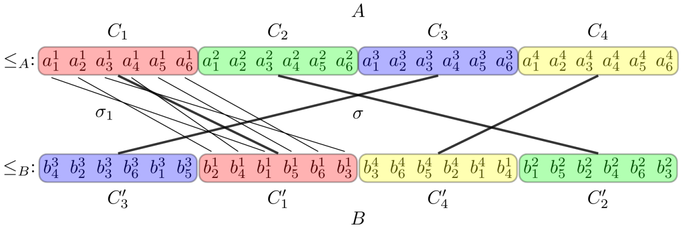


Figure 1: Constraints on the 1-1 correspondence of the Structured 2-Track Hitting Set.

The Reduction

Theorem 1 (Bonnet and Miltzow) [2]

STRUCTURED 2-TRACK HITTING SET is W[1]-hard and, unless the ETH fails, cannot be solved in $f(k)n^{o(k/\log k)}$ time for any computable function f.

1. We use RED-BLUE POINT SEPARATION as a "black box", in order to solve an instance of STRUCTURED 2-TRACK HITTING SET.
2. STRUCTURED 2-TRACK HITTING SET $\preccurlyeq$ RED-BLUE POINT SEPARATION
3. From Theorem 1, and the reduction (Theorem 2) we have the desired lower bound (Theorem 3).

Theorem 3

RED-BLUE SEPARATION is W[1]-hard w.r.t. the number of lines k, and unless ETH fails, cannot be solved in $f(k)n^{o(k/\log k)}$ time for any computable function f.

Strategy

- Given an instance $\mathcal{I}_{S2-THS}$ of the Structured 2-Track Hitting Set problem, we need to construct an instance $\mathcal{J}_{R-BS}$ of the Red-Blue Separation problem, such that

$$\mathcal{I}_{S2-THS} \in \mathcal{L}_{S2-THS} \Leftrightarrow \mathcal{J}_{R-BS} \in \mathcal{L}_{R-BS}$$

- We need to encode the following structures:
  1. The sets A, B.
  2. The collections of intervals $\mathcal{S}_A, \mathcal{S}_B$.
  3. The permutation of the color classes σ (inter-class permutation).
  4. The permutation of the elements of each class σ_i (*intra*-class permutation).
- The *positions* of red and blue points on the plane will be the restrictions.
- The *lines* selected by the R-BS algorithm, will correspond to the elements of a hitting set in S2-THS.

Goal

Theorem 2

Let $\mathcal{I}_{S2-THS}$ be an instance of the $\mathrm{S2\text{-}THS}$ problem, and $\mathcal{J}_{R-BS}$ be the corresponding instance (of our construction) of the $\mathrm{R\text{-}BS}$ problem. Then, $\mathcal{I}_{S2-THS} \in \mathcal{L}_{S2-THS}$, *if and only if $\mathcal{J}_{R-BS}$ can be solved with $6k + 14$ lines*.

Note: *From Theorems 1, 2 we obtain Theorem 3 as a Corollary.*

Wire Diagram

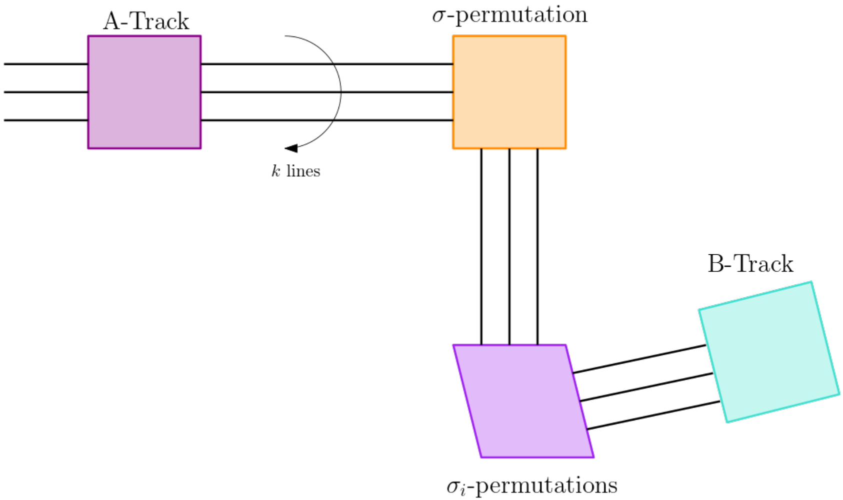
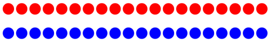
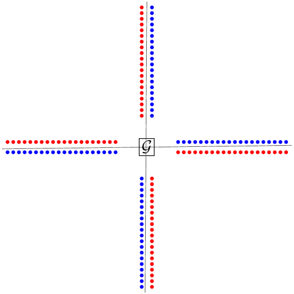


Figure 2: The wire diagram of the gadgets.

Long Alley Gadgets



(a) Long alley

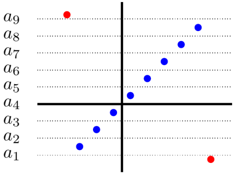


(b) Surrounding a gadget with long alleys.

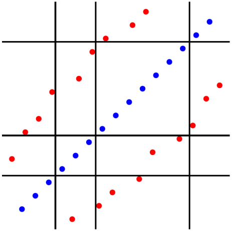
Figure 3: Long alleys and their use.

- Enforcing (almost) *horizontal* or *vertical* lines.
- We surround gadgets with long alleys in order to force only *axis parallel* lines to pass through a gadget.
- There are infinitely many possible lines passing through a long alley, even in $\mathbb{Q}^2$.

Interval Gadgets & Encoding A-Track



(a) Interval gadget corresponding to $[a_1, a_9] = \{a_1, a_2, \dots, a_9\}$.



(b) Interval gadgets put together. A representation of a track.

- Figure 4a. The algorithm must choose *at least one* horizontal line, corresponding to some $a_i \in [a_1, a_9]$.
- Figure 4a. In order to separate the red points from the red *with 2 axis-parallel lines*, the choice of the former imposes the later.
- Figure 4b. A representation of a track.
- Figure 4b. Separating these points requires taking lines associated to a *minimum hitting set*.
- Note 1. We include the *color classes* as intervals w.l.o.g.
- Note 2. We surround track-A with $4k$ long alleys.

Encoding *Inter*-class Permutation σ

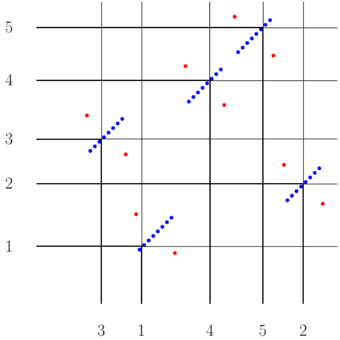


Figure 5: Encoding the permutation $\sigma = 31452$.

- We place the σ-gadget at the *right-hand side* of the A-tack gadget.
- The choices within the color classes are transferred from almost horizontal, to almost vertical.
- Given the k choices of horizontal lines originating from the A-track gadget, it results in a vertical propagation accompanied by the desired *reordering* of the color classes.

Encoding *Intra*-class Permutations σ_i

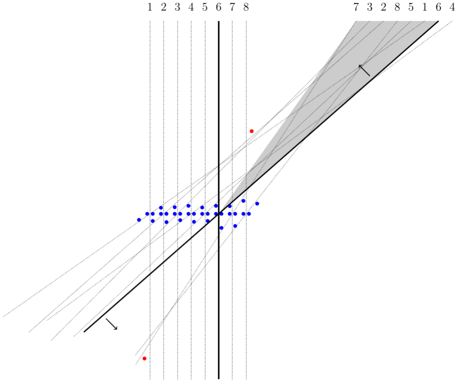


Figure 6: Half encoding of the permutation $\sigma_i = 73285164$, of the i-th color class.

- We place a σ_i-gadget, for each color class.
- All σ_i-gadgets are bellow the σ-gadget.
- We utilize the arbitrary slopes of the lines, in order to keep the gadget "small" and maintain the parameter.
- The gadget on the left enforces that *the choice of a_i^j is linked with an element smaller or equal to b_i^j.*
- We use a symmetrical gadget to enforce the other inequality.

Encoding Track B

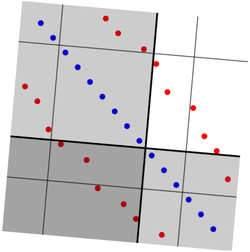


Figure 7: The encoding of the B-Track.

- We set the encoding of the B-Track around 5° to the right.
- The rotation facilitates the arbitrary slopes of the σ_i-gadgets.
- From the prespective of the B-Track the "incoming" lines behave like *vertical*.

The Big Picture

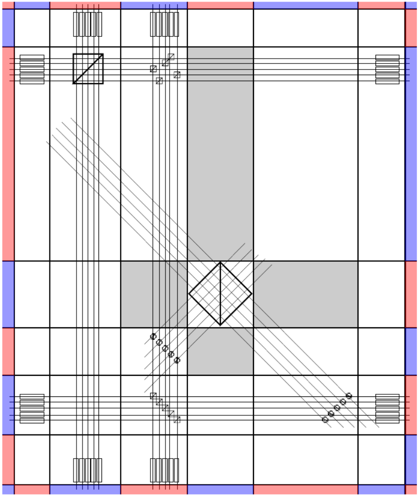


Figure 8: The Big Picture.

- The thin rectangles are the *long alleys*.
- The bold square in the with the diagonal *top left corner* is the encoding of *A-track*.
- The other bold square *in the middle* is the encoding of *B-track*.
- The smaller squares with the diagonal are encodings of the *σ-permutation*.
- The small round gadgets are the *half-encodings* of the *σ_i-permutations*.

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Description & Strategy

- We present an FPT algorithm for Axis-Parallel Red-Blue Separation.
- Parameter: $\min\{|\mathcal{R}, \mathcal{B}|\}$, w.l.o.g. we assume $\mathcal{B}$ is the smaller set.

Theorem 4

An optimal solution of Axis-Parallel Red-Blue Separation can be computed in $O(n \log n + n|\mathcal{B}|9^{|\mathcal{B}|})$ time.

1. We divide the plane into $|\mathcal{B}| + 1$ *vertical stripes*.
2. Each stripe contains only red points, hence an optimal solution uses *at most two lines inside a single strip*.
3. We enumerate the number of lines used in each strip in $9^{|\mathcal{B}|}$ time.
4. We check if a specification holds, by verifying a 2-CNF formula.

2-CNF: Definitions & Variables

Definitions:

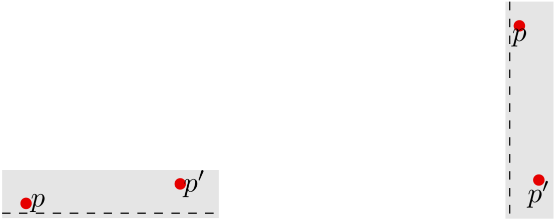
- Let X, Y two list of the points $\mathcal{R} \cup \mathcal{B}$, sorted by x and y coordinate, respectively.
- $V_i = \{p \in \mathbb{R}^2 \mid X[i] \leq p(x) \leq X[i+1]\}$
- $H_i = \{p \in \mathbb{R}^2 \mid Y[i] \leq p(y) \leq Y[i+1]\}$
- Let S be the *solution*.

Variables:

- For each red point $p \in H_i$, let y_p^i denote "*the line of S in H_i is below p*".
- For each red point $p \in V_i$, let x_p^i denote "*the line of S in S_i is to left of p*".

Constraints:

1) For each pair of red points $p, p' \in H_i$, that are in consecutive lexicographical (y, x) order, we add the restriction $y^i_p \rightarrow y^i_{p'}$.
2) For each pair of red points $p, p' \in V_i$, that are consecutive in lexicographical (x, y) order, we add the restriction $x^i_p \rightarrow x^i_{p'}$



(a) Constraint 1: $y^i_p \rightarrow y^i_{p'}$.

(b) Constraint 2: $x^i_p \rightarrow x^i_{p'}$.

Figure 9: Constraints 1, 2.

2-CNF: C_{ij}-separable & Interesting Cells

Definition: C_{ij}-separable

Let $C_{ij} = H_i \cap V_j$ be a *cell*. A red point $p \in C_{ij}$ is called *C_{ij}-separable* *for a point $p_b \in \mathcal{B}$*, if p can be separated from p_b by a vertical r horizontal line *running through the interior of C_{ij}*.

Definition: Interesting Cell

We say that C_{ij} is *interesting* for a point $p_b \in \mathcal{B}$ if the following hold:

1. C_{ij} contains at least one red point that is *C_{ij}-separable* for p_b.
2. At least one of H_i or V_j contains at most one horizontal or vertical line from S respectively.
3. There is no vertical line from S in a strip *strictly between* p_b and V_j
4. There is no horizontal line from S in a strip *strictly between* p_b and H_i

▶ **Note:** The interesting cells for p_b are the cells that contain lines, whose position *cannot be predetermined*.

2-CNF: Constraint 3

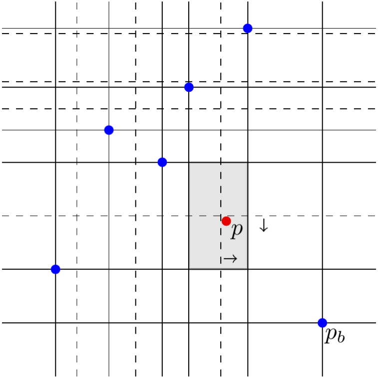


Figure 10: The *gray cell* is interesting for p_b. Here, we construct the clause $y_p^2 \vee \neg x_p^4$. We would have $y_p^2 = 0$, $\neg x_p^4 = 1$.

3) For each $p_b \in \mathcal{B}$, each interesting cell C_{ij} and each red point $p \in C_{ij}$ we construct a clause:

1. If $y(p_b) \geq Y(i+1)$, add $\neg y_p^i$, else add y_p^i.
2. If $x(p_b) \geq X(i+1)$, add $\neg x_p^i$, else add x_p^i.
3. We construct $O(|\mathcal{B}||\mathcal{R}|)$ clauses.
4. In an infeasible instance, we will have empty (unsatisfiable) clauses.

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Conclusions & Future Work

Conclusions

- We presented a reduction from Structured 2-Track Hitting Set to Red-Blue Points Separation.
- We proved the R-BPS cannot have an FPT algorithm, n the general case.
- We presented an FPT algorithm for the special case of Axis Parallel R-BPS.

Future Work

- The complexity of Axis Parallel R-BPS. Is it FPT, w.r.t. the number of lines?
- The complexity of R-BPS, when lines are allowed to have 3 slopes.
- The complexity of Axis Parallel R-BPS in 3-dimensions.

Thank you for your time!

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Bibliography

É. Bonnet, P. Giannopoulos, and M. Lampis, *On the parameterized complexity of red-blue points separation*, CoRR, abs/1710.00637 (2017).

E. Bonnet and T. Miltzow, *Parameterized hardness of art gallery problems*, ACM Trans. Algorithms, 16 (2020).

R. Impagliazzo and R. Paturi, *Complexity of k-sat*, in Proceedings. Fourteenth Annual IEEE Conference on Computational Complexity (Formerly: Structure in Complexity Theory Conference) (Cat.No.99CB36317), 1999, pp. 237–240.