On the Parametrized Complexity of Red-Blue Points Separation [1]

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Abstract

In this report we present a synopsis of Édouard Bonnet's et al. paper on *The Parametrized Complexity of Red-Blue Points Separation.* The authors provide a lower bound to the problem by a reduction from STRUCTURED 2-TRACK HITTING SET PROBLEM, proving that there is no FPT algorithm, unless the ETH fails.

1 Introduction

Bonnet's et al. paper is divided in two parts, the first the first regarding the reduction that establishes the lower bound for RED-BLUE POINT SEPARATION problem, while the second part dedicated to the to the FPT algorithm for AXIS-PARALLEL RED-BLUE SEPARATION. Here we will present only the result regarding the lower bound for the problem, which is the constitutes the main focus of the paper.

We begin this section with a formal definition of the RED-BLUE POINTS SEPARATION (R-BPS) problem. Next we briefly discuss Exponential Time Hypothesis and its consequences in complexity theory. Lastly, we define the 2-TRACK HITTING SET problem, along with its more sophisticated variant STRUCTURED 2-TRACK HITTING SET (S2-THS). We will be using the latter problem in the next section regarding the reduction.

Definition 1 (RED-BLUE POINTS SEPARATION Problem)**.** Let $\mathcal{R}, \mathcal{B} \subset \mathbb{R}^2$ be the two point-sets in the plane, We need to find the *minimum* set of lines $\mathcal{L} \subset \mathbb{R}^2$, that separates $\mathcal{R}$ from $\mathcal{B}$.

An other way to interpreter Definition 1 is to consider the "cells" that are formed by the lines in $\mathcal{L}$. For a feasible solution, we forbid a cell to contain both a red point and a blue point. In Section 2 we will establish that there is no $n^{o(k)}$ algorithm for the above problem, unless the ETH fails. Now we give a formal definition of the Exponential Time Hypothesis.

Definition 2 (Exponential Time Hypothesis)**.** The 3-SATISFYABILITY problem (3-SAT) cannot be solved in *subexponential* time.

The Exponential Time Hypothesis (ETH) was stated by Impagliazzo and Paturi in 1999 [3]. It is a *stronger* hypothesis than $\mathbf{P} \neq \mathbf{NP}$, while if ETH holds, then we also have $\mathbf{P} \neq \mathbf{NP}$, while the opposite is not true.

1.1 2-Track Hitting Set Poblems

We give now the definition of the 2-TRACK HITTING SET problem.

Definition 3 (2-TRACK HITTING SET)**.** Let A, B be two *totally ordered sets*, and $\mathcal{S}_A, \mathcal{S}_B$ be a *collection of intervals* from A, B respectively. We also assume that there is a *one-to-one correspondence* $\phi\colon A \to B$ from A to B. We need to find some set of pairs S, where $S = \{(a, \phi(a)) \mid a \in A, \phi(a) \in B\}$, where the first projection is a *hitting set* of $(A, \mathcal{S}_A)$, while the second projection is a hitting set of $(A, \mathcal{S}_B)$. We consider the parameter of the problem to be the size of the solution $k = |S|$.

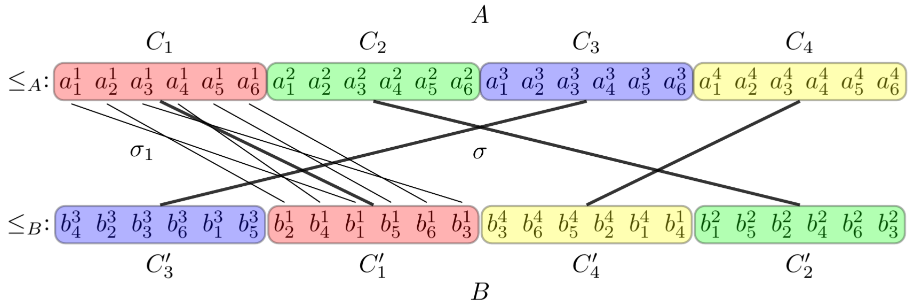


Figure 1: Constraints on the one-to-one correspondence of the STRUCTURED 2-TRACK HITTING SET.

We will often be referring to the pair $(A, \mathcal{S}_A)$ as the A-track, while to the pair $(B, \mathcal{S}_B)$ as the B-track. Note, that the actual choices we make are from set A, which are transferred via ϕ to B. For our reduction we will be using a more sophisticated version of the above problem, where we add some restrictions regarding the correspondence ϕ. We give the following definition.

Definition 4 (STRUCTURED 2-TRACK HITTING SET)**.** Let the sets $A, B, \mathcal{S}_A, \mathcal{S}_B$ be defined as in Definition 3. We will consider some additional constraints for the one-to-one correspondence $\phi\colon A \to B$. We consider a partition of A, B into k color classes $\mathcal{C}, \mathcal{C}'$ respectively. Each color class is an *interval* in A and B, respectively; while it consists of t elements. We define a permutation $\sigma\colon \mathcal{C} \to \mathcal{C}'$, which "shuffles" the color classes of A, we denote $C_i' = \sigma(C_i)$ the image of the i-th color class. We will call $\sigma(\cdot)$ the *inter-class* permutation. On the other hand, we define for each color class $C_i \in \mathcal{C}$ a permutation $\sigma_i\colon C_i \to C_i'$, where $C_i' = \sigma(C_i)$. We will call the $\sigma_i(\cdot)$, *intra-class* permutations. Lastly, for each $a \in C_i$ we assume that $\phi(a) = \sigma \circ \sigma_i(a)$.

While somewhat elaborate, Definition 4 can be better explained in Figure 1. There, we observe the four color classes of A-tack be "shuffled" into the color classes of B-track by the σ-permutation. Also, observe the permutation enforced by $\sigma_1(\cdot)$ on the elements of C_1, which results on the ordering of the elements in C_1'.

2 The Reduction

In this Section we highlight the the most important steps of the reduction that establishes the desired lower bound of the RED-BLUE POINTS SEPARATION problem. Our goal is to prove the following Theorem.

Theorem 5. RED-BLUE POINTS SEPARATION problem is W[1]-hard with respect to the number of lines k, and unless the ETH fails, cannot be solved in $f(k)n^{o(k/\log k)}$ time.

In this direction, we will be utilizing a previous result by Bonnet and Miltzow, regarding the hardness of STRUCTURED 2-TRACK HITTING SET.

Theorem 6 (Bonnet and Miltzow [2])**.** STRUCTURED 2-TRACK HITTING SET is W[1]-hard and, unless the ETH fails, cannot be solved in $f(k)n^{o(k/\log k)}$ time for any computable function f.

Observe that from Theorem 6, in order to prove Theorem 5, we only need to construct a reduction from S2-THS to R-BPS. This way, we would have established that R-BPS is *as hard as* S2-THS. In order to do this, we assume an arbitrary instance $\mathcal{I}$ of S2-THS and construct an instance $\mathcal{J}$ of R-BPS. Naturally, we have to show that $\mathcal{I}$ is a "YES"-instance of S2-THS *if and only if* $\mathcal{J}$ is a "YES"-instance of R-BPS. For this purpose, we need to encode all of the structures and sets of S2-THS,

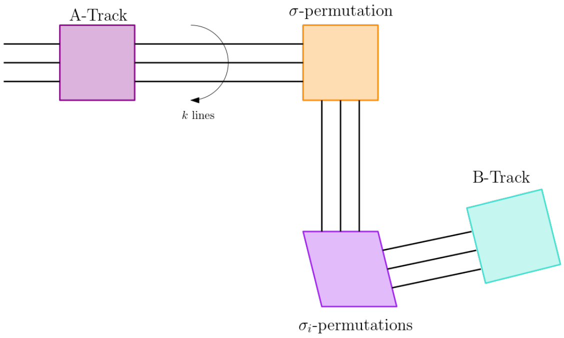


Figure 2: A wire-diagram depicting the relative positions of the gadgets in the plane.

namely the sets A, B, the collections of ntervals $\mathcal{S}_A, \mathcal{S}_B$, the inter-class permutation σ and the intra-class permutations σ_i for each of the k color classes. All the aforementioned classes will be encoded as sets of red and blue points on the plane, while the solution will corespond to the lines selected by the arbitrary R-BPS algorithm. The reduction is stated at the theorem bellow.

Theorem 7. Let $\mathcal{I}$ be an instance of the S2-THS problem, and $\mathcal{J}$ be the corresponding instance (of our construction) of the R-BS problem. Then $\mathcal{I}$ is a "YES"-instance *if and only if* $\mathcal{J}$ can be solved with $6k + 14$ lines, where k is the parameter of S2-THS.

Note that from Theorems 6 and 7, we obtain Theorem 5 as a corollary.

3 Gadgets

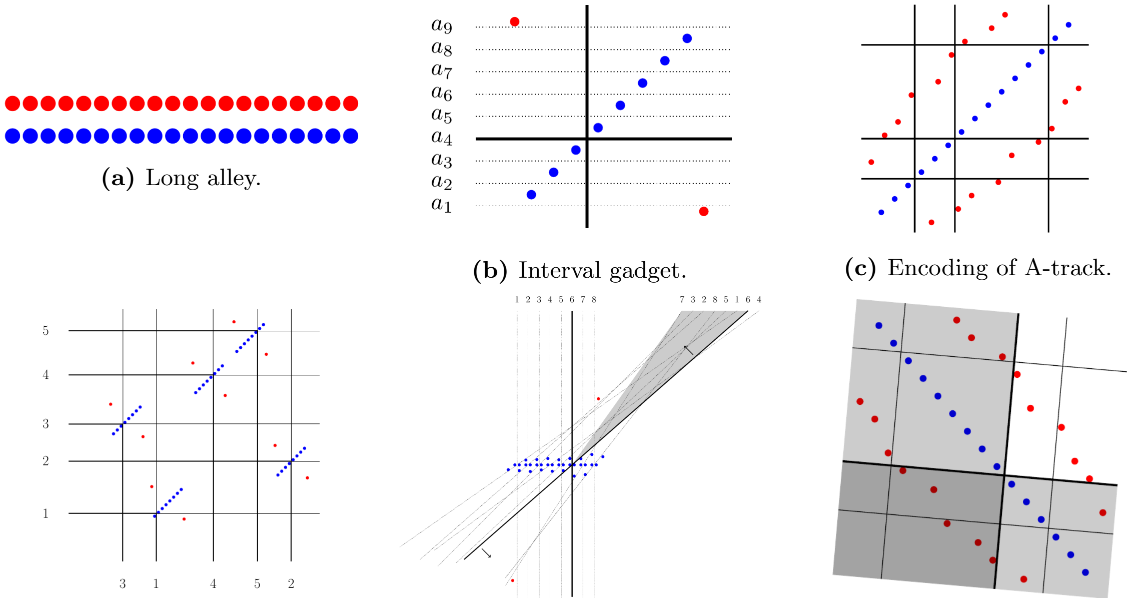
The main focus of our wok will be encoding the above structures with gadgets in the plane. In Figure 2 we present a sketch of the relative positions of the gadgets in the plane. We start from the top-left and end to the bottom-right. We begin with the encoding of the A-track, that is we encode the set A, along with the intervals of $\mathcal{S}_A$. We ensure that a selection of lines separating the ed from the blue points in this gadget results on a hitting set of the A-track $(A, \mathcal{S}_A)$. Next, we need to transfer this selection to B-track is a way that respects the ϕ-correspondence. Notice that the selected lines will change slope and direction as we proceed from a gadget to the next. The exact way that this is achieved will be clear in the sequel.

3.1 Long Alleys

Before we begin with the encoding of the structures of S2-THS, we need to define an utility gadget that we will be needing later on. In the gadget encoding the intervals, we will need to restrict the selected separating lines to be almost horizontal or almost vertical. To enforce that, we use the *long alley gadgets*. A horizontal long alley gadget consists of ℓ horizontally consecutive red points and ℓ horizontally consecutive blue points (see Figure 3a). A *vertical long alley* is defined analogously.

3.2 Interval Gadgets & Encoding A-track

In Figure 3b we depict the gadget for the interval $[a_1, a_9]$. For each $a \in [a_1, a_2]$ we draw a blue point in the plane. All the blue points are arranged diagonally. Each line passing bellow the $i-th$ blue element, translates to the selection of the A set. In order to enforce that a selected line will intersect each interval of $\mathcal{S}_A$ we use these blue points depicted in Figure 3b. We place a red point bellow a_1 and to the right of a_9. Symmetrically, we place another red point above a_9 and to the left of a_1. In order to separate the red points from the blue efficiently, the algorithm will use two lines vertical to each



(a) Long alley. **(b)** Interval gadget. **(c)** Encoding of A-track.

(d) σ-permutation. **(e)** σ_i-permutation. **(f)** B-track encoding.

Figure 3: The gadgets of the reduction.

other. One almost parallel and one almost horizontal. The choice of the former impose the latter. For technical reasons we add each of the k color classes to the collection of the intervals $\mathcal{S}_A$, without loss of generality.

In Figure 3c we put together all the points of the A-track along with the interval gadgets of $\mathcal{S}_A$. Separating these points with the fewest axis-parallel lines requires taking the hoizontal and vertical lines *associated to a minimum hitting set*. To enforce that the algorithm will chose only horizontal and vertical lines, we surround the gadget with $4k$ long alleys.

3.3 Encoding σ-permutation

In Figure 3d we present the gadget encoding the inter-class σ-permutation. Note that we place this gadget to the right of the A-track (see Figure 2). The choices within the k color classes of the A-track are transferred from almost horizontal lines to almost vertical ones. This way, we obtain the desired reordering of the color classes.

3.4 Encoding σ_i-permutations & B-track

In Figure 3e we present the gadget for the intra-class permutations σ_i. Since we wish to encode a permutation σ_i, for every $i \in [k]$ on t elements, we cannot use the same mechanism, as we did with the inter-class permutation, for it would blow-up out parameter dramatically and would not result in an FPT reduction. We will crucially use the fact that separating lines can have arbitrary slopes. Note that we placed the σ_i-permutation gadgets bellow the σ-permutation gadget (see Figure 2). Hence the selection of k lines will be entering the gadget vertically. The ”outgoing” lines will be leaving the gadget with a slight rotation to the right, about 5°. What we achieve this way is to link the choice of a^i_j, the j-th element of the i-th color class, with the *choice of an element smaller or equal to* b^i_j. We use a symmetric gadget to get the other inequality, so that choosing some lines corresponding to a^i_j actually forces to take some lines corresponding to b^i_j (see Figure 4).

In Figure 3f we present the gadget for the B-track. Note that the B-track is slightly rotated to the right, about 5°, in order to compensate for the incoming lines from the σ_i-permutations' gadgets.

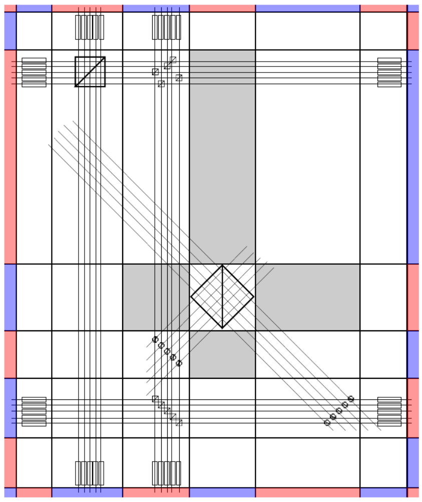


Figure 4: The big picture of the $\mathcal{J}$ instance of the R-BS problem.

Since the slope of the entering lines agrees with the rotations of the B-track, the incoming lines are *almost horizontal*, from the point of view of the gadget.

3.5 The Big Picture

In Figure 4 we give the big picture of our construction. The thin rectangles represent the long alleys. The bold square with the diagonal, in the top left corner represents the encoding of the A-track. The other bold square in the middle repesents the encoding of the B-track, note the we rotated the B-track 45° *only for presentation purposes*. The smaller squares with the diagonal are the encoding of the σ-permutation. Lastly, the small round gadgets are the *half-encodings* of the σ_i-permutations.

4 Conclusions & Future Work

We presented a main result of the Bonnet et al. [1] paper *On the Parametrized Complexity of the Red-Blue Points Separation*. We highlighted the main steps of the reduction from the Structured 2-Track Hitting Set, to Red-Blue Points Separation problem. This way we proved that the R-BPS *cannot* have an FPT algorithm, unless the ETH fails. An open problem that is proposed by the authors for future work regards the complexity of R-BPS when the lines have three slopes.

References

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