Parametrized Two-Player Nash Equilibrium [5]

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Results

Problems	Computation Time
ℓ-sparse Games	$\ell^{O(k\ell)} \cdot n^{O(1)}$
Locally Bounded Treewidth Games	$f(k,\ell) \cdot n^{O(1)}$
k-unbalanced Games	$\ell^{O(k^2)} \cdot n^{O(1)}$

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Probability Simplex

d-Simplex

Let $\langle e^1, e^2, \ldots, e^{d+1} \rangle$ be the *normal* base of $\mathbb{R}^{d+1}$. We call the set $\Delta_d = convex(\langle e^1, e^2, \ldots, e^{d+1} \rangle) \subset \mathbb{R}^{d+1}$, a d-Simplex.

1. A d-Simplex is a d-dimensional object in $d+1$-dimensional space.
2. For all $x \in \Delta_d$, we have $x \geq 0$.
3. For all $x \in \Delta_d$, we have $\sum_{i=1}^{d+1} x_i = 1$.
4. Each $x \in \Delta_d$ defines a *probability distribution* on d possibilities.
5. For some $x \in \Delta_d$, we call *support* the set $S(x) = \{i \in [d+1] \mid x_i > 0\}$

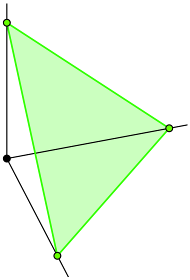

Figure 1: The 2-simplex in $\mathbb{R}^3$.

2-player Normal Form Games

Bi-matrix Game

1. 2-player game.
2. Player 1 has m strategies. while player 2 has n strategies.
3. Let $A, B \in \mathbb{R}^{m \times n}$ two real matrices.
4. The two players choose a vector of their respective simplices, *simultaneously*:
    1. Player 1 chooses some $x \in \Delta_{m-1}$.
    2. player 2 chooses some $y \in \Delta_{n-1}$
5. The players obtain u_1, u_2 units of utility, respectively, where:
    1. $u_1 = x^t A y$
    2. $u_2 = x^t B y$

Notes:

1. *"Every"* 2-player game can be represented as a Bi-matrix game.
2. We assume that $A = B$ [6].
3. We assume $A \in \mathbb{R}^{n \times n}_{\geq 0}$ [6].

Nash Equilibrium

Definition

Let $(x^\star, y^\star) \in \Delta_{m-1} \times \Delta_{n-1}$ be the two strategies of the two players. We say that the *strategy profile* $(x^\star, y^\star)$ is a *Nash Equilibrium* (NE), iff,

1. For all $x \in \Delta_m$, we have $(x^\star)^t A y^\star \geq x^t A y^\star$, *and*
2. for all $y \in \Delta_n$, we have $(x^\star)^t A y^\star \geq (x^\star)^t A y$.

I.e. my strategy is optimal, for a fixed strategy of the other player; and this is true for both players.

Notes:

1. The Nash Equilibrium is the most acceptable *solution concept* for a Normal Form Game.
2. John Nash proved (1950) that *every* Normal Form Game has a NE.

Computing a Nash Equilibrium

- Remember: For some $x \in \Delta_d$ we call *support* of x the set $S(x) = \{i \in [d+1] \mid x_i > 0\}$ containing the indices of the positive coordinates of x.

Support Lemma

The pair of strategy vectors (x, y) is a Nash equilibrium for the Bi-matrix game (A, B) iff:

1. $x_s > 0 \Rightarrow (Ay)_s \geq (Ay)_j$, for all $j \neq s$
2. $y_t > 0 \Rightarrow (x^t B)_t \geq (x^t B)$, for all $i \neq t$

Algorithm: Support Enumeration

1. "Guess" a support vector for each player.
2. Determine whether the inequalities of Support Lemma hold.

Complexity of Computing a NE

1. Computing a NE is PPAD-complete, *even* for a 2-player game [3, 1, 2].
   Hence, we do *not* expect to find a polynomial-time algorithm fro NE.
2. Computing a NE is *as difficult as* computing the support of a NE [6].
3. We can compute a NE of a Bi-matrix game, where the *support is bounded by k*, in $n^{O(k)}$ time (see Support Enum.).
4. Unless $\mathrm{FPT} = \mathrm{W}[1]$, there is no $n^{o(k)}$ time algorithm for computing a NE with support of size at most k in a Bi-matrix game [4].

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Towards a FPT Algorithm

1. Finding a proper parameter:
   - The obvious parameters, i.e. the number of players or the size of the support, *are not suitable*.
   - We consider ℓ-sparse Bi-matrix games, which have *at most* ℓ *non-zero values* in each row or column.
   - We, also, will consider (A, B) Bi-matrix games, where the matrices have at most ℓ different values.
   - Parameter (in both cases): $\ell + k$, where k is an upper bound to the size of the support.
2. Finding a structure:
   - It's not obvious which discrete structure to utilize.
   - We will extrapolate a *Game Graph*, by considering the non-zero entries of the payoff matrices.
   - In ℓ-sparse Bi-matrix games we will have $\Delta(G) \leq \ell$, where G is the Game Graph.
   - In games with ℓ different values, the Game Graph will exploit the *Locally Bounded Treewidth*.

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Game Graph

Game Graph

Let (A, B) be a Bi-matrix game. Let $\mathcal{G} = A \vee B$ be the matrix, where,

$$\mathcal{G}[i][j] = \begin{cases} 1, & A[i][j] \neq 0 \text{ or } B[i][j] \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

We call the *bipartite* $G = (S^1 \cup S^2, E)$, induced by $\mathcal{G}$ the *game graph* of (A, B). Where S^1, S^2 the set of strategies of each player, respectively.

Note: For ease of the presentation, we (often) assume $A = B$, and $A, B \geq 0$.

Game Graph: Example

$$A = \begin{bmatrix} 0 & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \\ 0 & 0 & a_{53} & a_{54} \end{bmatrix}$$

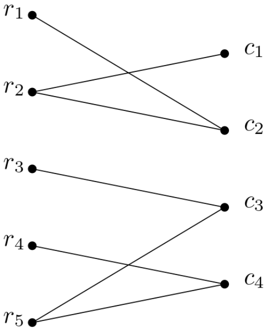


Figure 2: The game graph induced by the A.

Utilizing the Game Graph

Minimal Nash Equilibria

A Nash Equilibrium (x, y) is *minimal*, if for any Nash Equilibrium (x', y'), with $S(x') \subseteq S(x)$ and $S(y') \subseteq S(y)$, we have $S(x') = S(x)$ and $S(y') = S(y)$.

Lemma

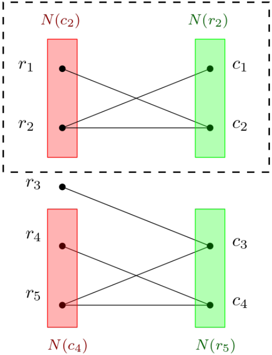
If *minimal* NE $\Rightarrow$ *connected*

Game Graph: Example 2 (a)

Lemma

If *not* connected $\Rightarrow$ *not* minimal NE

$$[0\ x_1\ 0\ 0\ x_2]\begin{bmatrix} 0 & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \\ 0 & 0 & a_{53} & a_{54} \end{bmatrix}\begin{bmatrix} 0 \\ y_1 \\ 0 \\ y_2 \end{bmatrix}$$



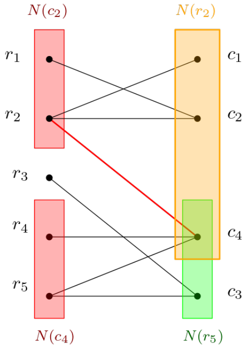
- **Remarks** Since the induced graph is *not connected*, then the NE (x, y) is not minimal. In deed the strategy profile $x' = (0, 1, 0, 0, 0)$, $y' = (0, 1, 0, 0)$ *is* a NE.

Game Graph: Example 2 (b)

Lemma

If minimal NE $\Rightarrow$ connected

$$[0\ x_1\ 0\ 0\ x_2] \begin{bmatrix} 0 & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & \color{red}{a_{24}} \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \\ 0 & 0 & a_{53} & a_{54} \end{bmatrix} \begin{bmatrix} 0 \\ y_1 \\ 0 \\ y_2 \end{bmatrix}$$



- Remarks If we assume that $a_{24} > 0$, then the same trick *won't work*, (x', y') will not be a NE.
- Note: That Lemma 2 is *not a characterization*. Hence, we could have a connected induced graph, that does not correspond to a NE.

An FPT Algorithm for ℓ-sparse Games (1)

Lemma

We can enumerate all the induced subgraphs on t vectices, with c connected components in,

$$(\Delta + 1)^{2t} \cdot n^{c+O(1)}$$

time.

Algorithm

1. Enumerate all induced *connected* subgraphs on k vertices.
2. Let V^1, V^2 be the corresponding support from the previous graph.
3. Check the Support Lemma for the given support.

An FPT Algorithm for ℓ-sparse Games (2)

Theorem

A Nash Equilibrium in a ℓ-sparse Bi-matrix game, where the support sizes is bounded by k, can be computed in $\ell^{O(k\ell)} \cdot n^{O(1)}$.

Corollary

The problem of determining whether a Nah Equilibrium with support sizes at most k exists in an ℓ-sparse Bi-matrix game, admits *no polynomial kernel*.

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Locally Bounded Treewidth

Definition

A graph class has *locally bounded treewidth* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that for every graph $G = (V, E)$ of the class, any vertex v, and any $d \in \mathbb{N}$, the subgraph of G induced by all vertices within distance *at most* d from v has treewidth at most $f(d)$.

- Note: *The crucial property of locally bounded treewidth graphs is that first-order queries can be answered in FPT time on such graphs, when the parameter is the size of the first-order formula.*

A New Class of Games: Bounded Valued Matrices

Bounded Valued Bi-matrix Games

Let $P \subset \mathbb{Q}$, and $|P| \leq \ell$, for some value ℓ. We consider a subclass of Bi-matrix games (A, B), where $A, B \in P^{m \times n}$.

- Note: *In this section we will present an algorithm that finds an equilibrium with support sizes at most k, when the game come from a graph with locally bounded treewidth.*
- Parameter: $\ell + k$

Theorem

A Nash Equilibrium in a *locally bounded treewidth* game, where the support sizes are bounded by k, and the payoff matrices have *at most ℓ different values*, can be computed in $f(k, \ell) \cdot n^{O(1)}$ time for some computable function $f(\cdot)$.

Equilibrium Patterns

Definition

(a) Let I, J be two subsets of k *elements* in $[n]$. We say that two matrices $A^\star, B^\star \in \mathbb{Q}^{k\times k}$ *occur* in the Bi-matrix game (A, B) if $A^\star = A_{I,J}$ and $B^\star = B_{I,J}$.
(b) The pair $(A^\star, B^\star)$ *forms an equilibrium pattern* if there exists an equilibrium (x, y) where $(A^\star, B^\star)$ occurs in the game (A, B), at $(S(x), S(y))$.

1. Our algorithm will try *all possible* ℓ^{2k^2} pairs of matrices $(A^\star, B^\star)$.
2. For each pair we determine whether it is an *equilibrium pattern*.
3. When does a pair of matrices $(A^\star, B^\star)$ form an equilibrium pattern?
   a) It must *occur* in the game (A, B) for some indices sets I, J.
   b) There must be an equilibrium (x, y) with $S(x) = I$ and $S(y) = J$, *such that neither player has better alternative*.

Quering an Equilibrium Pattern with a First-Order Formula

▶ Example: Consider a win-lose, $P = \{0,1\}$, game (A, B), encoded into relations $A/2, B/2$, such that $A(r,c)$ is *true* iff $A[r][c] = 1$. Likewise for B.

$$A^\star = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$B^\star = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The pattern $(A^\star, B^\star)$.

$$\begin{aligned} &\exists\, r_1, r_2, c_1, c_2 \\ &A(r_1,c_1) \wedge \neg A(r_1,c_2) \wedge \neg A(r_2,c_1) \wedge A(r_2,c_2) \wedge \\ &B(r_1,c_1) \wedge \neg B(r_1,c_2) \wedge \neg B(r_2,c_1) \wedge B(r_2,c_2) \wedge \\ &\forall r'(\neg A(r',c_1) \vee \neg A(r',c_2)) \wedge \forall c'(\neg B(r_1,c') \vee \neg B(r_2,c')) \end{aligned}$$

The corresponding first-order formula.

▶ Note: In general, with ℓ different values, there would be $\ell - 1$ relations A_i and B_i encoding the game, where $A_i(r,c)$ is true if $A_{r,c} = z_i$, for every $z_i \in P$ except the zero value.

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Conclusions & Future Work

Conclusions

- We presented parametrised algorithms to compute a Nash Equilibrium in 2-player games.
- We defined an underlying graph structure in Bi-matrix Games.
- We utilized the graph-theoretical structure inherited in Bi-matrix Games.

Future Work

- Is there a polynomial-time algorithm for computing Nash equilibria in games of bounded treewidth?
- Can we remove the assumption regarding the number of different values in the algorithm for Locally Bounded Treewidth Games?

Thank you for your time!

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