Systems with distributed mass

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This document briefly explains how the equivalent SDOF mass and stiffness of a system with distributed mass can be calculated for a simply supported beam.

The frequency is calculated as:

$\boldsymbol{f} = \frac{\boldsymbol{\pi}}{\boldsymbol{2*L^2}} * \sqrt{\frac{\boldsymbol{EI}}{\boldsymbol{\mu}}}$ where L is the span of the beam, EI the bending stiffness and μ the mas of the beam per meter length.

Treated system

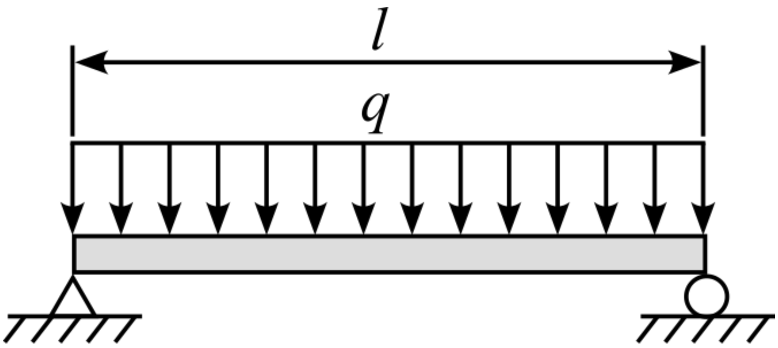

Figure 1 Simply supported beam

The system we are looking at is the one shown in the figure above. It is a simply supported beam with the following properties:

- l the length of the beam
- x is the longitudinal coordinate of the beam starting at the left support (0) and ending at the right support (l)
- $m(x)$ is the mass of the beam per unit length
- $EI(x)$ is the bending stiffness of the beam calculated from the Young's modulus of the material and the sectional inertia (that could be variable over the length of the beam)

Deformation of the system

The deformation of this beam is a simple bending. We can approximate the bending shape of the bending by a sine function. If we want to describe the deformation over time we can do so as follows:

$$u(x,t) = \psi(x) * z(t)$$

Where:

- $z(t)$ is the displacement over time at the characteristic point. In the case of the simply supported beam this would correspond to the vertical displacement at midspan.
- $\psi(x)$ corresponds to the deformed shape. We can "guess" the deformation as:

$$\psi(x) = \sin(x * \frac{\pi}{L})$$

  This complies with the support conditions:

- $\psi(x=0)=0$
- $\psi(x=L)=0$
- $\psi\left(x=\frac{L}{2}\right)=1$ (The deformation is characterized by the maximum at midspan.)

Note: for other cases not treated in this document, another estimation of the deformation of the system must be made.

Equivalent mass and stiffness

The first step is to find the first and second derivation of the shape function, as we will need them later.

$$\psi'(x)=\cos\left(x*\frac{\pi}{L}\right)*\frac{\pi}{L}$$
$$\psi''(x)=-\sin\left(x*\frac{\pi}{L}\right)*\frac{\pi^2}{L^2}$$

Knowing the shape of the deformed structure, the equivalent mass, noted as m^* can be calculated as follows:

$$m^*=\int_0^L m(x)\,\psi^2(x)\,dx=\int_0^L m(x)\sin^2\left(x*\frac{\pi}{L}\right)\,dx$$

The stiffness of the equivalent SDOF is calculated as follows:

$$k^*=\int_0^L EI(x)\big(\psi''(x)\big)^2dx-N*\int_0^L(\psi'(x))^2\,dx$$

In this case however we don't have a normal force in the beam. Thus, the equation simplifies as follows:

$$k^*=\int_0^L EI(x)\big(\psi''(x)\big)^2dx=\int_0^L EI(x)\left(-\sin\left(x*\frac{\pi}{L}\right)*\frac{\pi^2}{L^2}\right)^2\,dx=\int_0^L EI(x)\sin^2\left(x*\frac{\pi}{L}\right)*\frac{\pi^4}{L^4}dx$$

The equivalent force acting on the system can be calculated as follows:

$$q^*=-\ddot{u}_g(t)*\int_0^L m(x)\,\psi(x)\,dx=-\ddot{u}_g(t)*\int_0^L m(x)\sin(x*\frac{\pi}{L})\,dx$$

Where $\ddot{u}_g$ is the acceleration acting on the structure.

Finally, we can write the equation of motion of the system as follows:

$$m^*\,\ddot{z}+k^*\,z=q^*$$

To find the eigenvalues we need to assume $q^*=0$

Hence the circular period is: $\omega=\sqrt{k^*/m^*}$