The four, the threes and the twos

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Abstract

Your abstract.

1 Axioms

Given $n \in \mathbb{N} \geq 1$ and $x \in \mathbb{N} \in [1, n-1]$

1.1 4 = two 2

$$\sum\{4, n-4\} = \sum\{n-4, 2, 2\}$$
$$\prod\{4, n-4\} = \prod\{n-4, 2, 2\}$$

This means that for any solution that contains $\{4\}$ there also exists a "twin" solution that contains $\{2, 2\}$ instead and vice-versa.

1.2 1 = bad

$$\sum\{1, n-1\} = \sum\{n\}$$
$$\prod\{1, n-1\} = \prod\{n\}$$

Choosing 1 as an element of the partition is never optimal (except when forced, $n = 1$) because it never increases the total product but wastes potential for the other factors (only $n-1$ of the sum left).

1.3 Factor 3 is preferable over 2 (as long as Axiom 2 is not violated)

$$\begin{aligned}
\sum\{3, n-3\} &= \sum\{n\} \\
\prod\{3, n-3\} &\geq \prod\{n\} \quad | \; n \geq 5 \\
3 * (n-3) &> n \\
3n - 9 &> n \\
2n &> 9 \rightarrow n \geq 5
\end{aligned}$$

$$\begin{aligned}
\sum\{2, n-2\} &= \sum\{n\} \\
\prod\{2, n-2\} &\geq \prod\{n\} \quad | \; n \geq 4 \\
2 * (n-2) &\geq n \\
2n - 4 &\geq n \\
n &\geq 4
\end{aligned}$$

$$\begin{aligned}
3n - 9 &> 2n - 4 \\
n &> 5
\end{aligned}$$

Ergo, for every $n > 5$ it is better to choose 3 as the next factor instead of 2 For $n = 5$ it doesn't matter, the only solutions are $\{3, 2\}$ and $\{2, 3\}$. All other solutions (as well as choosing $x = 3$ for $n = 4$) would have to contain a 1 in the resulting partition.

1.4 Recursion / More factors = better (except too small)

$$\sum\{x', n'-x'\} = \sum\{x1, x2, n-x1-x2\} = \sum\{x, n-x\} = \sum\{n\}$$
$$\prod\{x', n'-x'\} = \prod\{x1, x2, n-x1-x2\} > \prod\{x, n-x\} > \prod\{n\} \quad | \; x > 1, n > 3$$

$$\begin{aligned} x * (n-x) &> n \\ nx - x^2 &> n \quad // \div n \quad (\text{ok}, n \geq 1) \\ x - \frac{x^2}{n} &> 1 \\ x * (1 - \frac{x}{n}) &> 1 \quad | \; \frac{x}{n} \in (0,1) \end{aligned}$$

This is the reason why we shouldn't pick a factor $x > 3$ (unless forced), it is worth more to split that factor into multiple smaller factors (as long as we don't violate any other Axioms, hence $n > 3$).

2 Conclusion

Every $n \in \mathbb{N}$ can be reduced using a combination of any of the four Axioms.

Example $n = 8$:

$$\begin{aligned} n = 8 \rightarrow x = 3 &\quad //(3) \\ n' = 5 \rightarrow x = 3 &\quad //(3) \\ n'' = 2 \rightarrow x = 2 &\quad //(2/3/4) \end{aligned}$$

Resulting partition = $\{3, 3, 2\}$

Simplified:

1. While $n > 4$ choose a factor $x = 3$
2. If $n = 4$ create two solutions, one using $x = 4$ and one with two factors $(2, 2)$
3. If $n < 4$ choose $x = n$, and call it a day

References