```
n=  2 prod=       2 [[2]]
n=  3 prod=       3 [[3]]
n=  4 prod=       4 [[4], [2, 2]]
n=  5 prod=       6 [[3, 2]]
n=  6 prod=       9 [[3, 3]]
n=  7 prod=      12 [[4, 3], [3, 2, 2]]
n=  8 prod=      18 [[3, 3, 2]]
n=  9 prod=      27 [[3, 3, 3]]
n= 10 prod=      36 [[4, 3, 3], [3, 3, 2, 2]]
n= 11 prod=      54 [[3, 3, 3, 2]]
n= 12 prod=      81 [[3, 3, 3, 3]]
n= 13 prod=     108 [[4, 3, 3, 3], [3, 3, 3, 2, 2]]
n= 14 prod=     162 [[3, 3, 3, 3, 2]]
n= 15 prod=     243 [[3, 3, 3, 3, 3]]
n= 16 prod=     324 [[4, 3, 3, 3, 3], [3, 3, 3, 3, 2, 2]]
n= 17 prod=     486 [[3, 3, 3, 3, 3, 2]]
n= 18 prod=     729 [[3, 3, 3, 3, 3, 3]]
n= 19 prod=     972 [[4, 3, 3, 3, 3, 3], [3, 3, 3, 3, 3, 2, 2]]
n= 20 prod=    1458 [[3, 3, 3, 3, 3, 3, 2]]
n= 21 prod=    2187 [[3, 3, 3, 3, 3, 3, 3]]
n= 22 prod=    2916 [[4, 3, 3, 3, 3, 3, 3], [3, 3, 3, 3, 3, 3, 2, 2]]
n= 23 prod=    4374 [[3, 3, 3, 3, 3, 3, 3, 2]]
n= 24 prod=    6561 [[3, 3, 3, 3, 3, 3, 3, 3]]
n= 25 prod=    8748 [[4, 3, 3, 3, 3, 3, 3, 3], [3, 3, 3, 3, 3, 3, 3, 2, 2]]
n= 26 prod=   13122 [[3, 3, 3, 3, 3, 3, 3, 3, 2]]
n= 27 prod=   19683 [[3, 3, 3, 3, 3, 3, 3, 3, 3]]
n= 28 prod=   26244 [[4, 3, 3, 3, 3, 3, 3, 3, 3], [3, 3, 3, 3, 3, 3, 3, 3, 2, 2]]
n= 29 prod=   39366 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 2]]
n= 30 prod=   59049 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3]]
n= 31 prod=   78732 [[4, 3, 3, 3, 3, 3, 3, 3, 3, 3], [3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2]]
n= 32 prod=  118098 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2]]
n= 33 prod=  177147 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3]]
n= 34 prod=  236196 [[4, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3], [3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2]]
n= 35 prod=  354294 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2]]
n= 36 prod=  531441 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3]]
n= 37 prod=  708588 [[4, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3], [3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2]]
n= 38 prod= 1062882 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2]]
n= 39 prod= 1594323 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3]]
n= 40 prod= 2125764 [[4, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3], [3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2]]
n= 41 prod= 3188646 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2]]
n= 42 prod= 4782969 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3]]
n= 43 prod= 6377292 [[4, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3], [3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2]]
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n= 45 prod=14348907 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3]]
n= 46 prod=19131876 [[4, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3], [3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2]]
n= 47 prod=28697814 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2]]
n= 48 prod=43046721 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3]]
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n= 50 prod=86093442 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2]]
n= 51 prod=129140163 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3]]
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n= 53 prod=258280326 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2]]
n= 54 prod=387420489 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3]]
n= 55 prod=516560652 [[4, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3], [3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2]]
n= 56 prod=774840978 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2]]
n= 57 prod=1162261467 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3]]
n= 58 prod=1549681956 [[4, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3], [3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2]]
n= 59 prod=2324522934 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2]]
n= 60 prod=3486784401 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3]]
n= 61 prod=4649045868 [[4, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3], [3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2]]
n= 62 prod=6973568802 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2]]
n= 63 prod=10460353203 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3]]
n= 64 prod=13947137604 [[4, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3], [3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2]]
n= 65 prod=20920706406 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2]]
n= 66 prod=31381059609 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3]]
n= 67 prod=41841412812 [[4, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3], [3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2]]
n= 68 prod=62762119218 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2]]
n= 69 prod=94143178827 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3]]
n= 70 prod=125524238436 [[4, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3], [3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2]]
n= 71 prod=188286357654 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2]]
n= 72 prod=282429536481 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3]]
n= 73 prod=376572715308 [[4, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3], [3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2]]
n= 74 prod=564859072962 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2]]
n= 75 prod=847288609443 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3]]
n= 76 prod=1129718145924 [[4, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3], [3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2]]
n= 77 prod=1694577218886 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2]]
n= 78 prod=2541865828329 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3]]
n= 79 prod=3389154437772 [[4, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3], [3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2]]
n= 80 prod=5083731656658 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2]]
n= 81 prod=7625597484987 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3]]
n= 82 prod=10167463313316 [[4, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3], [3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2, 2]]
n= 83 prod=15251194969974 [[3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 3, 2]]
```

Ramblings: $f(\mathbb{N}) \longmapsto \mathbb{N}$

1. $f([4, x])$ or $f([x, 2, 2])$ is equivalent

$\rightarrow \sum[4, x] = 4 + x = x + 2 + 2 = \sum[x, 2, 2]$

$\rightarrow \prod[4, x] = 4 \cdot x = x \cdot 2 \cdot 2 = \prod[x, 2, 2]$

2. a '1' is never good (except $n = 1$)

$\rightarrow$ it wastes '1' of the sum and doesn't increase the product

3. '3' is preferrable over '2' (except $n = 4$ because it would result in $[3, 1]$, violating 2)

$\rightarrow \sum[3, x] = 3 + x = 2 + 1 + x = \sum[2, 1, x]$

$\prod[3, x] = 3x > 2x = \prod[2, 1, x]$

4. smaller factors should be preferred

$(x \leq n-2)$ $\sum[x, n-x] = n = \sum[x-2, 2, n-x]$

-1 would not make sense (2)

$\prod[x, n-x] = nx - x^2$

$\prod[x-2, 2, n-x] = (2x-4) \cdot (n-x) = 2nx - 2x^2 - 4n + 4x$

$= 2(nx - x^2) + 4(-n+x) \overset{?}{\geq} nx - x^2 \quad // -(nx - x^2), -4(-n+x)$

$x(n-x) = nx - x^2 \geq -4(-n+x) = 4n - 4x = 4(n-x) \quad // \div (n-x)$

ok because $n \neq x \rightarrow \neq 0$

$x \geq 4$

5. every $n \in \mathbb{N}^+$ can be reduced using 1, 2, 3 & 4

$n = 1 \rightarrow 2 \rightarrow 1$

$n = 2 \rightarrow 2 \rightarrow 2$ (2 is the only option left)

$n = 3 \rightarrow 3 \rightarrow 3$

$n = 4 \rightarrow 1\ 3 \rightarrow (4, [2, 2])$

$n = 5 \rightarrow 3 \rightarrow [3, f(n' = 2)] = [3, 2]$

....

Given $n \in \mathbb{N}^+$, $x \in [1, n-1]$, $y \in [1, x-1]$

$[x, n-x]$ $\quad \sum \cdot = n$

$\prod \cdot = x(n-x) = nx - x^2$ $\quad | \, x \leq n-1$

Becomes maximal for:

$(x \leq n-2)$ $\sum [x, n-x] = n = \sum [x-3, 3, n-x]$

$\prod [x, n-x] = nx - x^2 = x(n-x)$

$\prod [x-3, 3, n-x] = 3((x-3)(n-x)) = 3(nx - x^2 - 3n + 3x)$

$= 3(x(n-x) + 3(-n+x)) = 3x(n-x) + 9(-n+x)$

$3x(n-x) + 9(-n+x) \overset{?}{\geq} x(n-x)$ $\quad // -(x(n-x)), -9(-n+x)$

$2x(n-x) \geq -9(-n+x) = 9(n-x)$ $\quad // \div (n-x) \neq 0$ ✓

$2x > 9 \longrightarrow x \geq 4.5 \xrightarrow{\mathbb{N}} x \geq 5$