Appendix A

The Appendix for Chapter 2

A.1 MCMC Algorithm

MCMC Algorithms: We use $\pi()$ to denote prior distribution, $q()$ to denote Metropolis Hastings proposal distribution and $l()$ to denote likelihood function. Let p_+, p_- and $p_=$ be three positive numbers that satisfy $p_+ + p_- + p_= = 1$.

We use superscript t on the parameters to denote the posterior samples at iteration t and let N_b to denote the burn-in period, N_r to denote the run length, and Δ to denote the thinning rate (to reduce correlations among samples). The algorithm we use to draw posterior samples can be summarized in the following manner:

1. Start the chain at $t = 0$ by initializing the parameters $(\beta_0^0, \alpha^0, \sigma^{2^0}, J^0, \boldsymbol{\beta}^0, \boldsymbol{\omega}^0)$.

2. With probability p_f we update the fixed dimensional parameters and with probability $1 - p_f$ we implement one of the three moves (BIRTH, DEATH, UPDATE) to update varing dimensional parameters.

3. If $t \geq N_r - N_b$ and $(t - N_b) \mod \Delta = 0$, save state for later analysis.

4. Increase t by one and return to step 2 above.

Following the above procedures, we obtain $\lfloor (N_r - N_b)/\Delta \rfloor$ posterior samples of the parameters which can be used to draw inference on the parameters or the functions of the parameters.

The trans-dimensional steps may be summarized as follows:

- BIRTH Step

  With probability p_+, we set $J^* = J^{t-1}+1$ and generate a random index r uniformly from $1, \cdots, J^{t-1}+1$ and sample a new point (β^*, ω^*) from proposal distribution $q^b(\beta, \omega)$, where $q^b()$ ensures that $\beta^* > \epsilon$. Set $(\boldsymbol{\beta}^*, \boldsymbol{\omega}^*)$ by letting: $(\beta_j^*, \omega_j^*) = (\beta_j^{t-1}, \omega_j^{t-1})$, for $j = 1, \cdots, r-1$; $(\beta_r^*, \omega_r^*) = (\beta^*, \omega^*)$; $(\beta_j^*, \omega_j^*) = (\beta_{j-1}^{t-1}, \omega_{j-1}^{t-1})$, for $j = r+1, \cdots, J^*$. Let $\boldsymbol{\Theta}^* = (\boldsymbol{\beta}^*, \boldsymbol{\omega}^*, J^*, \beta_0^{t-1}, \sigma^{2^{t-1}}, \alpha^{t-1})$, $\boldsymbol{\Theta}^{t-1} = (\boldsymbol{\beta}^{t-1}, \boldsymbol{\omega}^{t-1}, J^{t-1}, \beta_0^{t-1}, \sigma^{2^{t-1}}, \alpha^{t-1})$ and $\boldsymbol{\theta}^t = (\boldsymbol{\beta}^t, \boldsymbol{\omega}^t, J^t, \beta_0^t, \sigma^{2^t}, \alpha^t)$. With probability $\min(1, H)$, we accept the proposal and set $\boldsymbol{\Theta}^t \equiv \boldsymbol{\Theta}^*$; with probability $1 - \min(1, H)$, we reject the proposal and set $\boldsymbol{\Theta}^t \equiv \boldsymbol{\Theta}^{t-1}$. The Hastings ratio H for this move is:

$$\begin{aligned} H &= \frac{l(\mathbf{y}|\boldsymbol{\Theta}^*)}{l(\mathbf{y}|\boldsymbol{\Theta}^{t-1})} \times \frac{\pi(\beta^*, \omega^*|J^*)\pi(J^*)}{\pi(\beta^{t-1}, \omega^{t-1}|J^{t-1})\pi(J^{t-1})} \\ &\times \frac{\left(p_- + p_= \int_{-\infty}^{\epsilon} q^d(\beta|\beta^*)d\beta\right)/J^*}{p_+/J^*} \times \frac{1}{q^b(\beta^*, \omega^*)} \end{aligned}$$

- UPDATE Step

  With probability $1-p_+$, generate a random index r uniformly from $1, \cdots, J_{t-1}$. With probability $p_=$, we propose a new point β_r^* from proposal distribution $q^d(\beta|\beta_r^{t-1})$. If $\beta_r^* \geq \epsilon$, we implement UPDATE step; otherwise we implement DEATH step which we specify below. We first update β_r. Let $\boldsymbol{\Theta}_{(-\beta_r)}$ denote the rest of the parameters in the model. With probability $\min(1, H)$, we accept the proposal and set $\beta_r^t \equiv \beta_r^*$; with probability $1 - \min(1, H)$, we reject it and set $\beta_r^t \equiv \beta_r^{t-1}$. The Hastings ratio H is:

$$H = \frac{l\left(\mathbf{y}|\beta_r^*, \mathbf{\Theta}_{(-\beta_r)}^{t-1}\right)\pi(\beta_r^*)q^d(\beta_r^{t-1}|\beta_r^*)}{l\left(\mathbf{y}|\beta_r^{t-1}, \mathbf{\Theta}_{(-\beta_r)}^{t-1}\right)\pi(\beta_r^{t-1})q^d(\beta_r^*|\beta_r^{t-1})}$$

We then update every component of ω_r in a similar fashion.

- DEATH Step

  Using the random index r and β_r^* generated in the UPDATE step, with probability $p_- + p_= \times \Pr(\beta_r^* < \epsilon)$, set $J^* \equiv J^{t-1} - 1$ and generate $(\mathbf{u}^*, \boldsymbol{\theta}^*)$ by deleting the r-th component from $(\boldsymbol{\beta}, \boldsymbol{\omega})$, i.e. let $(\beta_j^*, \omega_j^*) = (\beta_j^{t-1}, \omega_j^{t-1})$, for $j = 1, 2, \cdots, r-1$ and $(\beta_j^*, \omega_j^*) = (\beta_{j+1}^{t-1}, \omega_{j+1}^{t-1})$ for $j = r, r+1, \cdots, J^*$. Let $\mathbf{\Theta}^* = (\boldsymbol{\beta}^*, \boldsymbol{\omega}^*, J^*, \beta_0^{t-1}, {\sigma^2}^{t-1}, \alpha^{t-1})$, $\mathbf{\Theta}^{t-1} = (\boldsymbol{\beta}^{t-1}, \boldsymbol{\omega}^{t-1}, J^{t-1}, \beta_0^{t-1}, {\sigma^2}^{t-1}, \alpha^{t-1})$ and $\mathbf{\Theta}^t = (\boldsymbol{\beta}^t, \boldsymbol{\omega}^t, J^t, \beta_0^t, {\sigma^2}^t, \alpha^t)$. With probability $\min(1, H)$, we accept the DEATH move and set $\mathbf{\Theta}^t = \mathbf{\Theta}^*$; with probability $1 - \min(1, H)$, we reject the DEATH move and set $\mathbf{\Theta}^t = \mathbf{\Theta}^{t-1}$. The Hastings ratio for this move is:

$$\begin{aligned} H &= \frac{l(\mathbf{y}|\mathbf{\Theta}^*)}{l(\mathbf{y}|\mathbf{\Theta}^{t-1})} \times \frac{\pi(\boldsymbol{\beta}^*, \boldsymbol{\omega}^*|J^*)\pi(J^*)}{\pi(\boldsymbol{\beta}^{t-1}, \boldsymbol{\omega}^{t-1}|J^{t-1})\pi(J^{t-1})} \\ &\times \frac{p_+/J^*}{\left(p_- + p_= \int_{-\infty}^{\epsilon} q^d(\beta|\beta_r^{t-1})d\beta\right)/J^*} \times q^b(\beta_r^{t-1}, \omega_r^{t-1}) \end{aligned}$$

- Update $(\beta_0, \sigma^2, \alpha)$ We update $(\beta_0, \sigma^2, \alpha)$ element by element.

  Sample a candidate point β_0^* from $q(\beta_0^*|\beta_0^{t-1})$. With probability $\min(1, H)$, we accept the proposal and set $\beta_0^t = \beta_0^*$; with probability $1 - \min(1, H)$, we reject the proposal and set $\beta_0^t = \beta_0^{t-1}$. The Hastings ratio for this move is:

$$H = \frac{l(\mathbf{y}|\beta_0^*)\pi(\beta_0^*)q(\beta_0^{t-1}|\beta_0^*)}{l(\mathbf{y}|\beta_0^{t-1})\pi(\beta_0^{t-1})q(\beta_0^*|\beta_0^{t-1})}$$

Updating parameter α depends on the choice of prior Lévy process. For certain Lévy process, there exists conjugate prior for α and we can use a Gibbs step to update α but for certain Lévy process there is no conjugate prior for *alpha* and under this circumstance, we update α in a similar fashion to the update of β_0.

Assuming an independent normal error model and a prior $\pi(\sigma^2) \propto \frac{1}{\sigma^2}$, the conditional distribution of σ^2 is InverseGamma, and easily updated using a Gibbs step. Note that if $x \sim$ Inv-Gamma(α, β), with density function $\frac{\beta^\alpha}{\Gamma(\alpha)} x^{-(\alpha+1)} e^{-\beta/x}$, then the full conditional posterior distribution for σ^2 is

$$\sigma^2|\mathbf{y}, \boldsymbol{\Theta} \quad \sim \quad \text{Inv-Gamma}\left(\frac{n}{2},\ \frac{1}{2}\sum_{i=1}^{n}\left[y_i - \sum_{j=1}^{J}\beta_j k(x_i;\omega_j)\right]^2\right)$$