Lasso & Bayesian Lasso

Readings Chapter 15 Christensen

STA721 Linear Models Duke University

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Lasso

Tibshirani (JRSS B 1996) proposed estimating coefficients through L_1 constrained least squares "Least Absolute Shrinkage and Selection Operator"

- Control how large coefficients may grow

$$\min_{\boldsymbol{\beta}} (\mathbf{Y}^c - \mathbf{X}^c \boldsymbol{\beta}^c)^T (\mathbf{Y}^c - \mathbf{X}^c \boldsymbol{\beta}^c)$$

  subject to

$$\sum |\beta_j^c| \leq t$$

- Equivalent Quadratic Programming Problem for "penalized" Likelihood

$$\min_{\boldsymbol{\beta}^c} \|\mathbf{Y}^c - \mathbf{X}^c \boldsymbol{\beta}^c\|^2 + \lambda \|\boldsymbol{\beta}^c\|_1$$

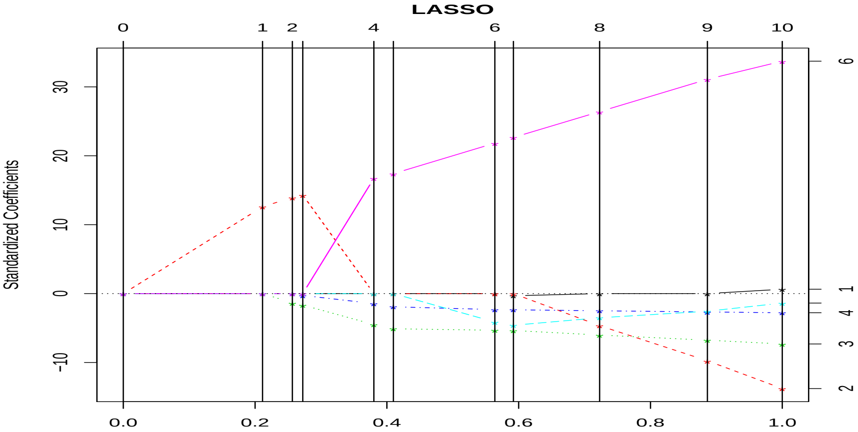
- Posterior mode

$$\max_{\boldsymbol{\beta}} -\frac{\phi}{2} \{\|\mathbf{Y}^c - \mathbf{X}^c \boldsymbol{\beta}^c\|^2 + \lambda \|\boldsymbol{\beta}^c\|_1\}$$

Picture

R Code

```
> library(lars)
> longley.lars = lars(as.matrix(longley[,-7]), longley[,7],
                      type="lasso")
> plot(longley.lars)
```



Solutions

```
> round(coef(longley.lars),5)
     GNP.deflator      GNP Unemployed Armed.Forces Population    Year
 [1,]     0.00000  0.00000    0.00000      0.00000    0.00000 0.00000
 [2,]     0.00000  0.03273    0.00000      0.00000    0.00000 0.00000
 [3,]     0.00000  0.03623   -0.00372      0.00000    0.00000 0.00000
 [4,]     0.00000  0.03717   -0.00459     -0.00099    0.00000 0.00000
 [5,]     0.00000  0.00000   -0.01242     -0.00539    0.00000 0.90681
 [6,]     0.00000  0.00000   -0.01412     -0.00713    0.00000 0.94375
 [7,]     0.00000  0.00000   -0.01471     -0.00861   -0.15337 1.18430
 [8,]    -0.00770  0.00000   -0.01481     -0.00873   -0.17076 1.22888
 [9,]     0.00000 -0.01212   -0.01663     -0.00927   -0.13029 1.43192
[10,]     0.00000 -0.02534   -0.01869     -0.00989   -0.09514 1.68655
[11,]     0.01506 -0.03582   -0.02020     -0.01033   -0.05110 1.82915
```

Cp Solution

Min $C_p = SSE_p/\hat{\sigma}^2_F - n + 2p$

```
> summary(longley.lars)
LARS/LASSO
Call: lars(x = as.matrix(longley[, -7]), y = longley[, 7], type
   Df     Rss        Cp
0   1 185.009 1976.7120
1   2   6.642   59.4712
2   3   3.883   31.7832
3   4   3.468   29.3165
4   5   1.563   10.8183
5   4   1.339    6.4068
6   5   1.024    5.0186
7   6   0.998    6.7388
8   7   0.907    7.7615
9   6   0.847    5.1128
10  7   0.836    7.0000

     GNP.deflator      GNP Unemployed Armed.Forces Population    Year
[7,]      0.00000  0.00000   -0.01471     -0.00861   -0.15337 1.18430
```

Features

Combines shrinkage (like Ridge Regression) with Selection (like stepwise selection)

Uncertainty? Interval estimates?

Park & Casella (JASA 2008) and Hans (Biometrika 2010) propose Bayesian versions of the Lasso

$$\begin{aligned}
\mathbf{Y} \mid \alpha, \boldsymbol{\beta}, \phi &\sim \mathsf{N}(\mathbf{1}_n\alpha + \mathbf{X}^c\boldsymbol{\beta}, \mathbf{I}_n/\phi) \\
\boldsymbol{\beta} \mid \alpha, \phi, \boldsymbol{\tau} &\sim \mathsf{N}(\mathbf{0}, \mathrm{diag}(\boldsymbol{\tau}^2)/\phi) \\
\tau_1^2 \ldots, \tau_p^2 \mid \alpha, \phi &\overset{\text{iid}}{\sim} \mathrm{Exp}(\lambda^2/2) \\
p(\alpha, \phi) &\propto 1/\phi
\end{aligned}$$

Can show that $\beta_j \mid \phi, \lambda \overset{\text{iid}}{\sim} DE(\lambda\sqrt{\phi})$

$$\int_0^\infty \frac{1}{\sqrt{2\pi s}} e^{-\frac{1}{2}\phi\frac{\beta^2}{s}} \, \frac{\lambda^2}{2} e^{-\frac{\lambda^2 s}{2}} \, ds = \frac{\lambda\phi^{1/2}}{2} e^{-\lambda\phi^{1/2}|\beta|}$$

Scale Mixture of Normals (Andrews and Mallows 1974)

Gibbs Sampling

- Integrate out α: $\alpha \mid \mathbf{Y}, \phi \sim \mathsf{N}(\bar{y}, 1/(n\phi))$
- $\boldsymbol{\beta} \mid \boldsymbol{\tau}, \phi, \lambda, \mathbf{Y} \sim \mathsf{N}(,)$
- $\phi \mid \boldsymbol{\tau}, \boldsymbol{\beta}, \lambda, \mathbf{Y} \sim \mathbf{G}(,)$
- $1/\tau_j^2 \mid \boldsymbol{\beta}, \phi, \lambda, \mathbf{Y} \sim \mathsf{InvGaussian}(,)$

$X \sim \mathsf{InvGaussian}(\mu, \lambda)$

$$f(x) = \sqrt{\frac{\lambda^2}{2\pi}} x^{-3/2} e^{-\frac{1}{2}\frac{\lambda^2 (x-\mu)^2}{\mu^2 x}} \qquad x > 0$$

Homework: Derive the full conditionals for $\boldsymbol{\beta}$, ϕ, $1/\tau^2$ see
`http://www.stat.ufl.edu/~casella/Papers/Lasso.pdf`

Other Options

Range of other scale mixtures used

- Horseshoe (Carvalho, Polson & Scott)
- Generalized Double Pareto (Armagan, Dunson & Lee)
- Normal-Exponenetial-Gamma (Griffen & Brown)
- Bridge - Power Exponential Priors

Properties of Prior?

Horseshoe

Carvalho, Polson & Scott propose

- Prior Distribution on

$$\boldsymbol{\beta} \mid \phi \sim \mathsf{N}(\mathbf{0}_p, \frac{\mathsf{diag}(\tau^2)}{\phi})$$

- $\tau_j^2 \mid \lambda \overset{\text{iid}}{\sim} C^+(0, \lambda)$
- $\lambda \sim \mathsf{C}^+(0, 1/\phi)$
- $p(\alpha, \phi) \propto 1/\phi)$

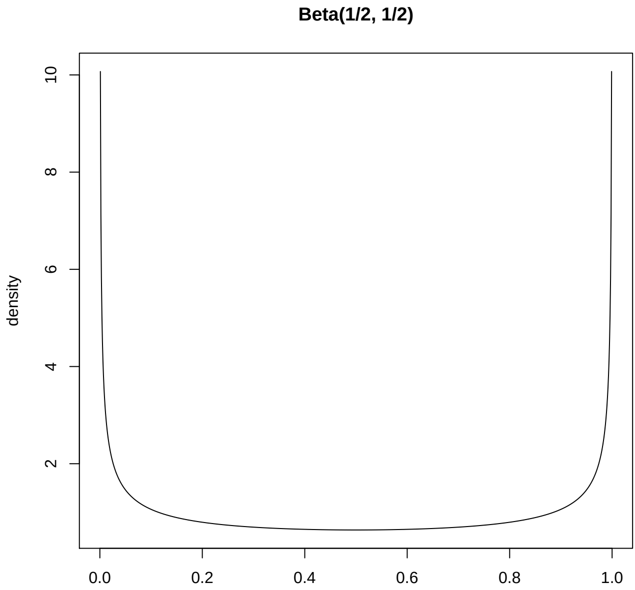
In the case $\lambda = \phi = 1$ and with $\mathbf{X}^t\mathbf{X} = \mathbf{I}$ $\mathbf{Y}^* = \mathbf{X}^T\mathbf{Y}$

$$E[\beta_i \mid \mathbf{Y}] = \int_0^1 (1 - \kappa_i) y_i^* p(\kappa_i \mid \mathbf{Y}) \; d\kappa_i = (1 - \mathsf{E}[\kappa \mid y_i^*]) y_i^*$$

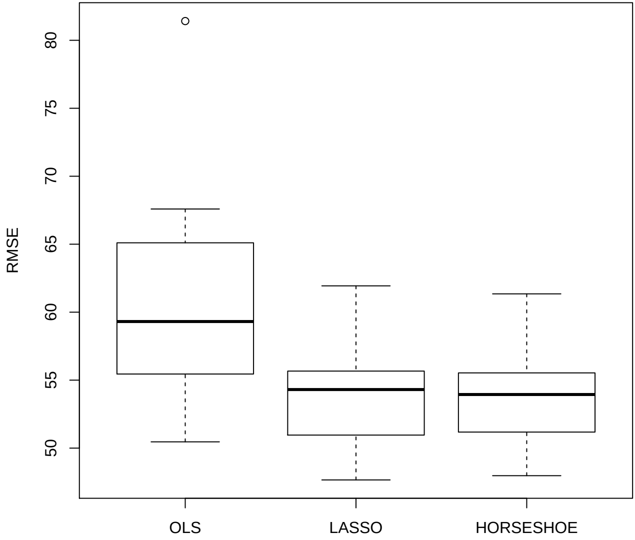
where $\kappa_i = 1/(1 + \tau_i^2)$ shrinkage factor

Half-Cauchy prior induces a Beta(1/2, 1/2) distribution on κ_i a priori

Horseshoe



Simulation Study with Diabetes Data



Other Options

Range of other scale mixtures used

- Generalized Double Pareto (Armagan, Dunson & Lee)
  $\lambda \sim \text{Gamma}(\alpha, \eta)$ then $\beta_j \sim \text{GDP}(\xi = \eta/\alpha, \alpha)$

  $$f(\beta_j) = \frac{1}{2\xi}(1 + \frac{|\beta_j|}{\xi\alpha})^{-(1+\alpha)}$$

  see `http://arxiv.org/pdf/1104.0861.pdf`
- Normal-Exponenetial-Gamma (Griffen & Brown 2005)
  $\lambda^2 \sim \text{Gamma}(\alpha, \eta)$
- Bridge - Power Exponential Priors (Stable mixing density)

See the monomvn package on CRAN

Choice of prior? Properties? Fan & Li (JASA 2001) discuss Variable selection via nonconcave penalties and oracle properties

Choice of Estimator & Selection?

- Posterior Mode (may set some coefficients to zero)
- Posterior Mean (no selection)

Bayesian Posterior does not assign any probability to $\beta_j = 0$

- selection based on posterior mode ad hoc rule - Select if $\kappa_i < .5$)
  See article by Datta & Ghosh `http://ba.stat.cmu.edu/journal/forthcoming/datta.pdf`
- Selection solved as a post-analysis decision problem
- Selection part of model uncertainty $\Rightarrow$ add prior probability that $\beta_j = 0$ and combine with decision problem