BMA

Hoff Chapter 9, Liang et al 2007, Hoeting et al (1999), Clyde & George (2004) Statistical Science

October 29, 2015

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Trade-off of model complexity versus goodness of fit

Lastly, assign distribution to space of models

Priors on Model Space

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  $p_{\boldsymbol{\gamma}} \sim \text{Bin}(p, .5)$
- $\gamma_j \mid \pi \overset{\text{iid}}{\sim} \text{Ber}(\pi)$ and $\pi \sim \text{Beta}(a, b)$ then $p_{\boldsymbol{\gamma}} \sim \text{BB}(p, a, b)$

$$p(p_{\boldsymbol{\gamma}} \mid p, a, b) = \frac{\Gamma(p+1)\Gamma(p_{\boldsymbol{\gamma}} + a)\Gamma(p - p_{\boldsymbol{\gamma}} + b)\Gamma(a+b)}{\Gamma(p_{\boldsymbol{\gamma}} + 1)\Gamma(p - p_{\boldsymbol{\gamma}} + 1)\Gamma(p + a + b)\Gamma(a)\Gamma(b)}$$

- $p_{\boldsymbol{\gamma}} \sim \text{BB}(p, 1, 1) \sim \text{Unif}(0, p)$
- For sparsity. want b to be a function of p typically.

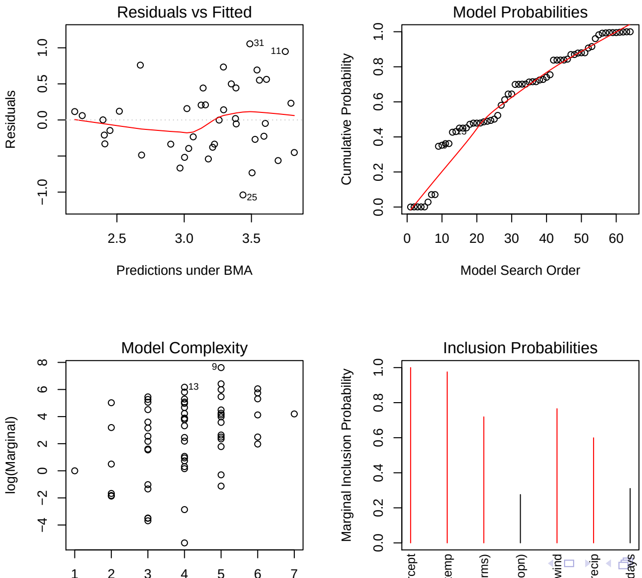
USair Data

```
library(BAS)
poll.bma = bas.lm(log(SO2) ~ temp + log(mgffirms) +
                             log(popn) + wind +
                             precip+ raindays,
                  data=pollution,
                  prior="g-prior",
                  alpha=41, # n
                  n.models=2^6,
                  modelprior = uniform(),
                  initprobs="Uniform")
> poll.bma

 Marginal Posterior Inclusion Probabilities:
 Intercept   temp   log(mfgfirms)  log(popn)   wind  precip
   1.0000   0.9755        0.7190      0.2757  0.7654  0.5994
```

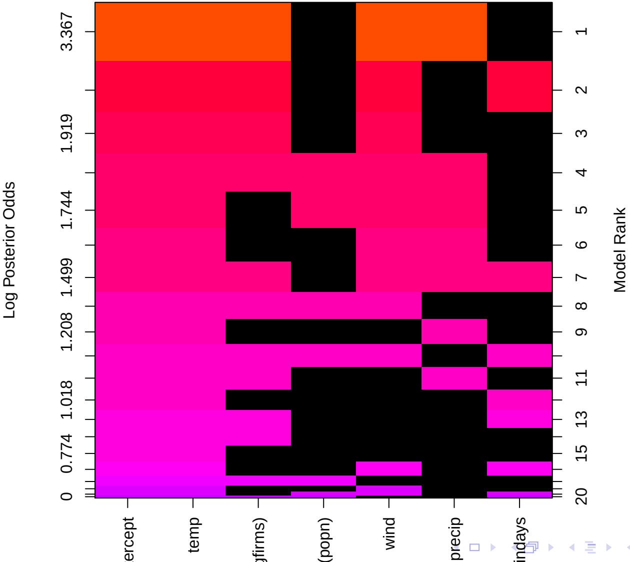
Plots

plot(poll.bma, ask=F)



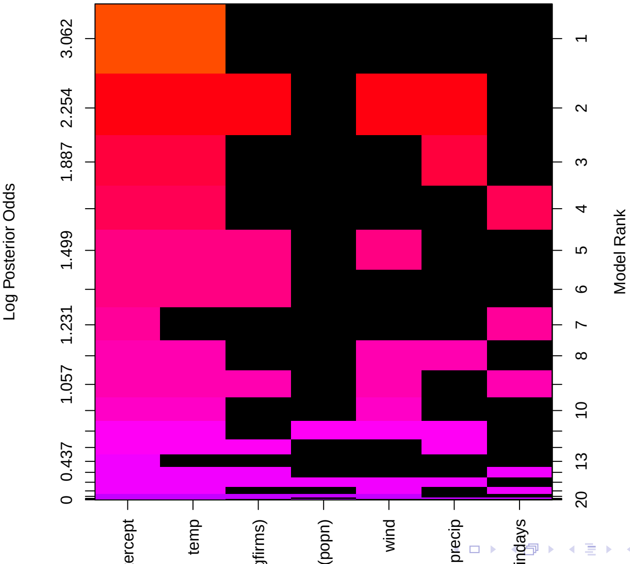
Posterior Distribution with Uniform Prior on Model Space

image(poll.bma)



Posterior Distribution with BB(1,p) Prior on Model Space

image(poll-bb.bma)



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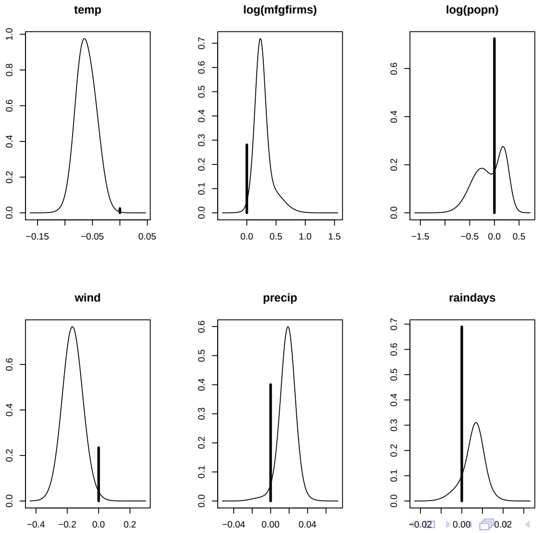
in context

of testing one hypothesis with equal prior odds

Coefficients

```
beta = coef(poll.bma)
par(mfrow=c(2,3));  plot(beta, subset=2:7,ask=F)
```



Information Paradox

The Bayes factor for comparing $\mathcal{M}_\gamma$ to the null model:

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- BF converges to a fixed constant $(1+g)^{-p_\gamma/2}$ (does not go to infinity

"Information Inconsistency" see Liang et al JASA 2008

Mixtures of g priors & Information consistency

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All have tails that behave like a Cauchy distribution

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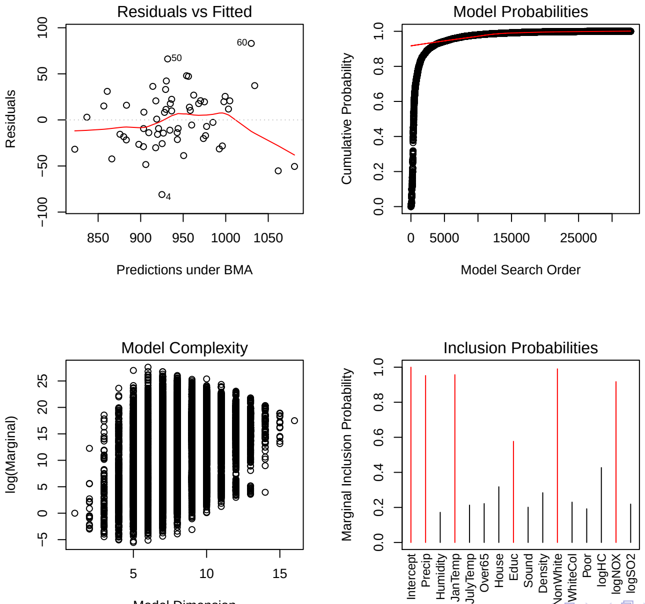
- Is pollution associated with mortality after adjusting for other socio-economic and meteorological factors?
- 15 predictor variables implies $2^{15} = 32,768$ possible models
- Use Zellner-Siow Cauchy prior $1/g \sim G(1/2, n/2)$

```
mort.bma = bas.lm(MORTALITY ~ ., data=mortality,
                  prior="ZS-null",
                  alpha=60,  n.models=2^15,
                  update=100, initprobs="eplogp")
```

Posterior Distributions



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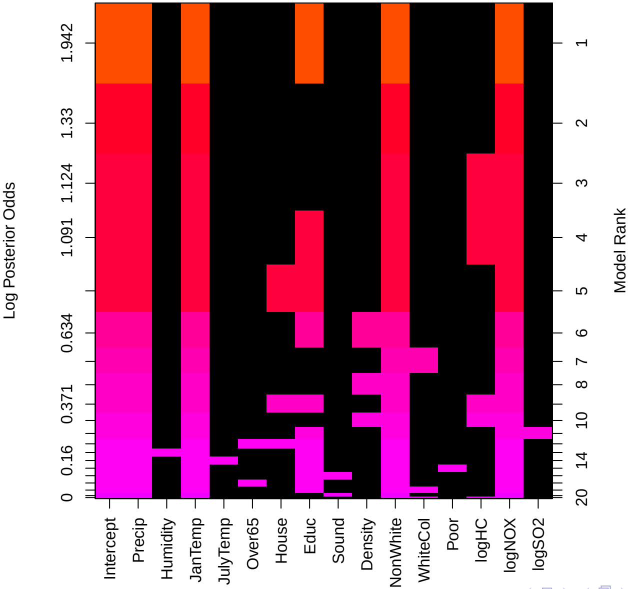
- Marginal inclusion probability for logHC $= 0.427144$
- Marginal inclusion probability for logSO2 $= 0.218978$

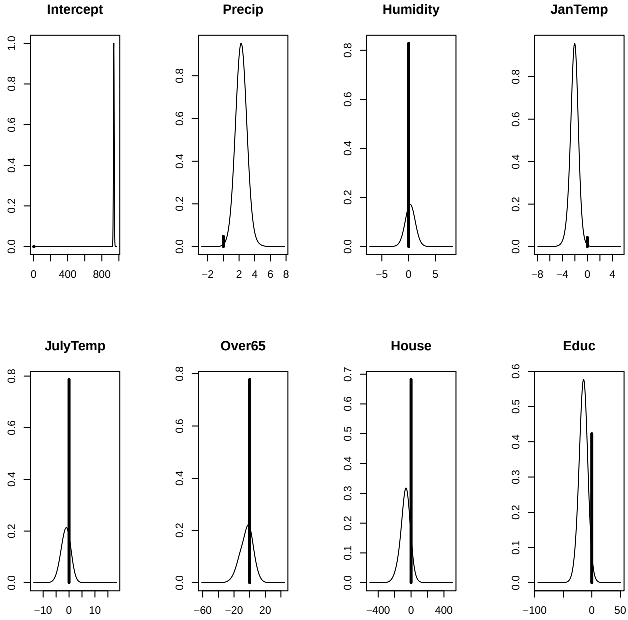
Bayes Factors are not additive! Better to work with probabilities.

Model Space



Log Posterior Odds
1.942
1.33
1.124
1.091
0.634
0.371
0.16
0
Model Rank
1
2
3
4
5
6
7
8
10
14
20
Intercept
Precip
Humidity
JanTemp
JulyTemp
Over65
House
Educ
Sound
Density
NonWhite
WhiteCol
Poor
logHC
logNOX
logSO2

Coefficients



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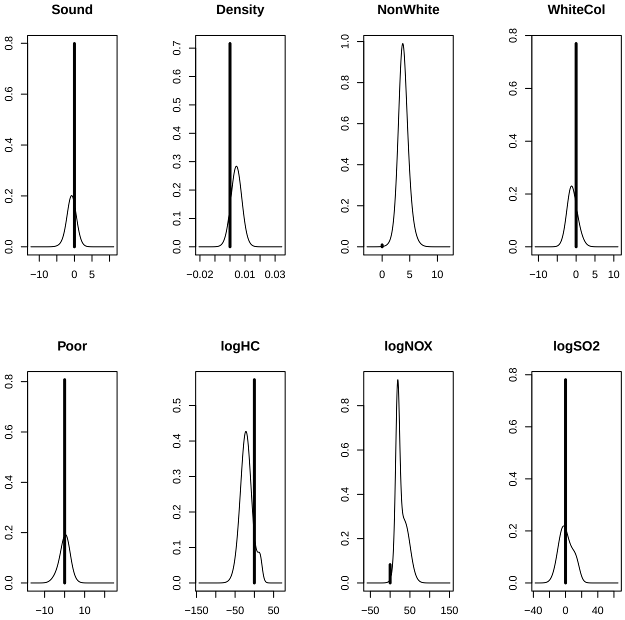

Effect Estimation

- Coefficients in each model are adjusted for other variables in the model
- OLS: leave out a predictor with a non-zero coefficient then estimates are biased!
- Model Selection in the presence of high correlation, may leave out "redundant" variables;
- improved MSE for prediction (Bias-variance tradeoff)
- Bayes is biased anyway so should we care?
- What is meaning of $\sum_{\gamma} \beta_{j\gamma} \gamma_j P(\mathcal{M}_\gamma \mid \mathbf{Y})$

Problem with confounding! Need to change prior?

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- Computational

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Model averaging versus Model Selection – what are objectives?