Checking Assumptions: Residuals and Influential Observations

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STA721 Linear Models

Duke University

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Outline

Topics

- Distribution of Residuals
- Leverage
- Standardized Residuals
- Cook's Distance
- Example: Stackloss data

Readings: Christensen Chapter 13

Linear Model

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Mean will not be zero if we have left out terms

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Parameterization

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$$
= \mathbf{1}_n\alpha_0^* + (\mathbf{Z} - \mathbf{1}_n\bar{\mathbf{Z}}^T)\boldsymbol{\alpha} + \boldsymbol{\epsilon}
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Leverage is a function of the estimated Mahalanobis distance D_i^2 of z_i from $\bar{\mathbf{Z}}$

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h_{ii}(1 - h_{ii}) &\geq 0
\end{aligned}$$

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Illustration

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Same as before!

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May still miss extreme points with high leverage, but will pick up unusual y_is

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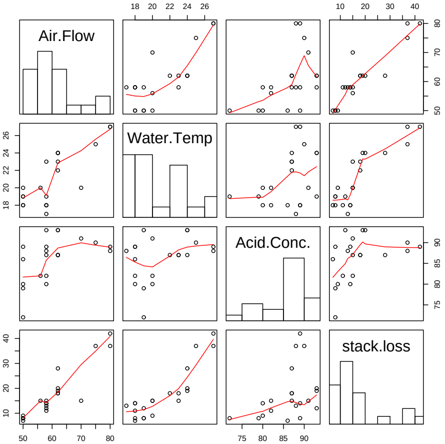
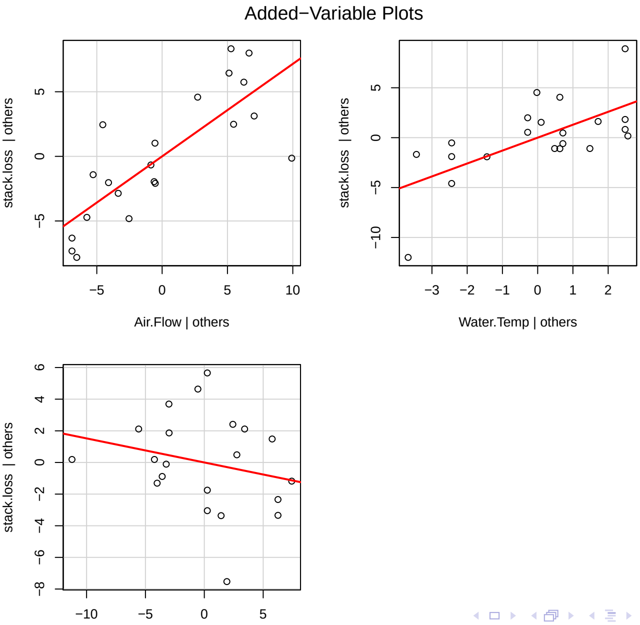
Cases

Stackloss Data



Stackloss Added Variable Plot



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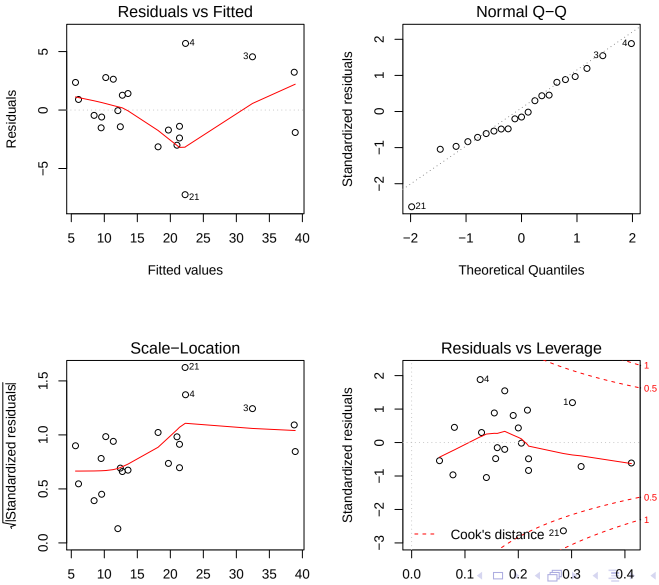

Case 21

- Leverage 0.285
- Cooks'd Distance .69
- p-value t_{21} is 0.0042
- Bonferonni adjusted p-value is 0.0024
- Other points? Masking?
- Refit without Case 21

Others have suggested that cases (1, 3, 4, 21) are outliers

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Are there outliers in the Stackloss Data?