STA 721 HW 7

1. Suppose $\mathbf{Y} \sim \mathsf{N}(\mathbf{X}\boldsymbol{\beta}, \sigma^2 \mathbf{I}_n)$. Consider finding an estimator $\mathbf{a}$ for $\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta}$ to minimize squared error loss, $(\boldsymbol{\mu} - \mathbf{a})^T(\boldsymbol{\mu} - \mathbf{a})$. Show that the posterior mean of $\boldsymbol{\mu}$ minimizes the posterior expected loss:

$$\mathsf{E}[(\boldsymbol{\mu} - \mathbf{a})^T(\boldsymbol{\mu} - \mathbf{a})]$$

   where the expectation is taken with respect to the posterior distribution of $p(\boldsymbol{\mu} \mid \mathbf{Y})$.

2. Suppose that you are using a $g$-prior for $\boldsymbol{\beta}$: $\boldsymbol{\beta} \mid \phi, g \sim \mathsf{N}(\mathbf{a}_0, \frac{g}{\phi}(\mathbf{X}^T\mathbf{X})^{-1}$ for the model $\mathbf{Y} \sim \mathsf{N}(\mathbf{X}\boldsymbol{\beta}, \mathbf{I}_n/\phi)$ where $\phi \sim G(\nu_0/2, \mathsf{SS}_0/2)$ and $\mathbf{X}$ is of rank $p$.

   (a) Find the posterior distribution for $\boldsymbol{\mu} = \mathbf{X}\boldsymbol{\beta}$.

   (b) Suppose that you decide to reparametrize your model, $\mathbf{X}\boldsymbol{\beta} = \mathbf{X}\mathbf{U}\mathbf{U}^{-1}\boldsymbol{\beta} = \mathbf{Z}\boldsymbol{\alpha} = \boldsymbol{\mu}$ where $\mathbf{U}$ is a $p \times p$ matrix that is full rank. What is the implied prior distribution for $\boldsymbol{\alpha} \mid \phi$?

   (c) Using the prior above, show that posterior distribution for $\mathbf{Z}\boldsymbol{\alpha}$ is the same as the posterior distribution for $\mathbf{X}\boldsymbol{\beta}$.

   (d) Community Problem [post on Piazza] Write an `R` function to compute $(1 - \alpha)100\%$ credible intervals for each $\beta_j$ using the output from `lm`. (replacement for confint) (extra for any linear combination $\boldsymbol{\lambda}\boldsymbol{\beta}$.)