Estimation & Decisions

STA721 Linear Models Duke University

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Bayesian Estimation

Model

$$\mathbf{Y} \sim \mathsf{N}(\mathbf{X}\beta, \mathbf{I}_n/\phi)$$

with precision $\phi = 1/\sigma^2$.

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More Prior Choices:

- More on g-priors
- Zellner-Siow Cauchy Prior
- Utility and choice of Estimators

Another Version of Zellner's g-prior

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\mathbf{Y} &= \mathbf{1}\beta_0 + \mathbf{X}_1\boldsymbol{\beta} + \boldsymbol{\epsilon} \\
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\boldsymbol{\beta} \mid \phi &\sim \mathsf{N}(\mathbf{0}, \frac{g}{\phi}(\mathbf{X}^T(\mathbf{I}_n - \mathsf{P}_{\mathbf{1}})\mathbf{X})^{-1})
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$$(\mathbf{Y}_c - \mathbf{X}_c\boldsymbol{\beta})^T(\mathbf{Y}_c - \mathbf{X}_c\boldsymbol{\beta}) + (\boldsymbol{\beta}^T\frac{\mathbf{U}^T\mathbf{U}}{g}\boldsymbol{\beta})$$

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Example

In SLR it is like an extra $Y_0 = 0$ at $\mathbf{X}_o = \sqrt{\frac{SS_x}{g}}$:

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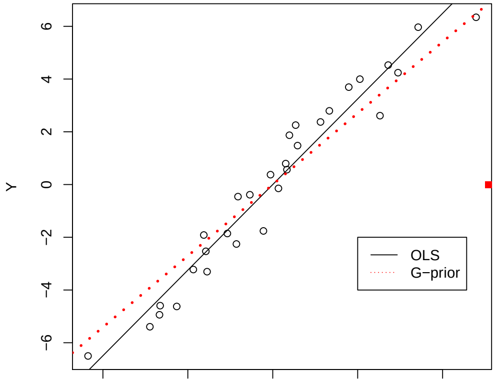

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- Mixtures of Conjugate Priors

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Theorem (Diaconis & Ylivisaker 1985)

Given a sampling model $p(y \mid \boldsymbol{\theta})$ from an exponential family, any prior distribution can be expressed as a mixture of conjugate prior distributions

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&\propto \int p(\boldsymbol{\theta} \mid \mathbf{Y}, \omega) p(\mathbf{Y} \mid \omega) p(\omega)\, d\omega
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&\propto \int p(\boldsymbol{\theta} \mid \mathbf{Y}, \omega) p(\mathbf{Y} \mid \omega) p(\omega)\, d\omega \\
p(\boldsymbol{\theta} \mid \mathbf{Y}) &= \frac{\int p(\boldsymbol{\theta} \mid \mathbf{Y}, \omega) p(\mathbf{Y} \mid \omega) p(\omega)\, d\omega}{\int p(\mathbf{Y} \mid \omega) p(\omega)\, d\omega}
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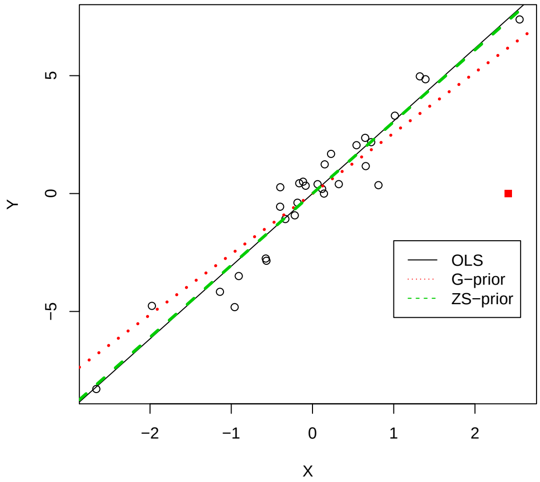
- Choice of g?
- $\frac{g}{1+g}$ weight given to the data
- Let $\tau = 1/g$ assign $\tau \sim G(1/2, 1/2)$
- Find prior distribution
- Can expres posterior as a mixture of g-priors

Example Again

From JAGS:



```
model = function(){
  for (i in 1:n) {
      Y[i] ~ dnorm(X[i]*beta, phi)
  }
  beta ~ dnorm(0, SSX/(n*phi*lambda))
  phi ~ dgamma(.05, .05)
  lambda ~ dgamma(.5, .5)
}

write.model(model, "ZSmodel")
model.file="ZSmodel"
data = list(Y=Y, X=X, n =length(Y), SSX=sum(X^2) )
ZSout = jags(data,inits=NULL,
             parameters.to.save=c("beta", "lambda", "phi"),
             model=model.file, n.iter=10000)
```

How Good are these Estimators?

Quadratic loss for estimating $\boldsymbol{\beta}$ using estimator $\mathbf{a}$

$$L(\boldsymbol{\beta}, \mathbf{a}) = (\boldsymbol{\beta} - \mathbf{a})^T (\boldsymbol{\beta} - \mathbf{a})$$

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  where λ_j are eigenvalues of $\mathbf{X}^T\mathbf{X}$.
- If smallest $\lambda_j \to 0$ then MSE $\to \infty$
- Note: estimate is unbiased!

Is the g-prior better?

Explore Frequentist properties of using a Bayesian estimator

$$\mathsf{E}_{\mathbf{Y}}[(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_g)^T(\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_g)$$

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when is the g prior better than the Reference prior of OLS?

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