STA 721 HW 6

1. If $\mathbf{Y} \sim \mathsf{N}(\mathbf{X}\boldsymbol{\beta}, \sigma^2 \mathbf{I}_n)$, show that the likelihood function for $\boldsymbol{\beta}, \phi$ where $\phi = 1/\sigma^2$ can be written as

$$\mathcal{L}(\beta, \phi) \propto \phi^{n/2} e^{-\phi \frac{\mathsf{SSE}}{2}} e^{-\frac{\phi}{2}(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})^T(\mathbf{X}^T\mathbf{X})(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})}.$$

   Do you need to consider a Jacobian term for a change of variables? (explain)

2. Consider the prior data dependent prior $\boldsymbol{\beta} \mid \phi \sim N(\hat{\boldsymbol{\beta}}, \sigma^2 n(\mathbf{X}^T\mathbf{X})^{-1})$ and $\phi \sim G((n+2)/(2n), \mathsf{SSE}/(2n)$ where $\hat{\boldsymbol{\beta}}$ is the MLE of $\boldsymbol{\beta}$, $\mathbf{X}$ is $n \times p$ and rank p and SSE is the residual sum of squares.

   (a) Find the prior mean of σ and σ^2. (For those who like abstraction, find $\mathsf{E}(\sigma^j)$ - when will this exist?) Please do not use the inverse gamma distribution, but use the distribution of ϕ.

   (b) Using the likelihood above, find the conditional posterior distribution of $\boldsymbol{\beta}$ given ϕ and the marginal posterior distribution for ϕ, simplifying as much as possible. What is the posterior mean for $\boldsymbol{\beta}$ and σ^2?

   (c) Find the marginal posterior distribution of β_j.

   (d) Suppose that $\beta_j \mid \mathbf{Y}, \phi$ are independent (and uncorrelated). What does that imply about $\mathbf{X}$? Will the β_j be independent after marginalizing over ϕ? Are they uncorrelated? (show your work)