STA 721 HW 8

1. For the Independent Jeffreys Reference prior $p(\boldsymbol{\beta}, \phi) \propto 1$ and model $\mathbf{Y} \mid \boldsymbol{\beta}, \phi \sim \mathsf{N}(\mathbf{X}\boldsymbol{\beta}, \phi^{-1}\mathbf{I}_n)$ where $\mathbf{X}$ is of full column rank p, derive/validate the results from class
   (a) $p(\boldsymbol{\beta} \mid \phi, \mathbf{Y})$
   (b) $p(\phi \mid \mathbf{Y})$
   (c) $p(\boldsymbol{\beta} \mid \mathbf{Y})$
2. If $p(\log(\sigma^p)) \propto 1$, find $p(\sigma^p)$ (up to proportionality)
3. If $p(\phi) \propto \phi^{-1}$, find $p(\sigma^2)$ (upto proportionality) (can you use the previous result?)
4. Consider the model

$$\mathbf{Y} \sim \mathsf{N}(\mathbf{X}_0\alpha + \mathbf{X}\boldsymbol{\beta}, \phi^{-1}\mathbf{I}_n)$$

   with the blocked Zellner g-prior:

$$\begin{aligned} p(\alpha, \phi) &\propto \phi^{-1} && (1) \\ \boldsymbol{\beta} \mid \phi &\sim \mathsf{N}(0, \frac{g}{\phi}(\mathbf{X}^T(\mathbf{I} - \mathsf{P}_{\mathbf{X}_0})\mathbf{X})^{-1}) && (2) \end{aligned}$$

   where $\mathbf{X}_0$ and $\mathbf{X}$ are not orthogonal. Find $p(\alpha \mid \mathbf{Y}, \boldsymbol{\beta}\phi$, $p(\boldsymbol{\beta} \mid \mathbf{Y}, \phi)$ and $\phi \mid \mathbf{Y}$ (hint: decompose $\mathbf{Y}$ and $\mathbf{X}$ into parts that are in $C(\mathbf{X}_0)$ and the orthgonal complement $C(\mathbf{X}_0)^\perp$ using appropriate projections and decompose the likelihood into three terms. Are α and $\boldsymbol{\beta}$ independent given $\mathbf{Y}$ and ϕ? Find the distribution of α given $\mathbf{Y}$ and ϕ. (Hint if they are not independent use iterated expectations!)