Introduction to Linear Models

STA721 Linear Models Duke University

Merlise Clyde

August 25, 2015

Coordinates

- Instructor: Merlise Clyde
  214 Old Chemistry
  Office Hours MWF 1:00-2:0 or right after class (or by appointment)
- Teaching Assistants: Nicole Dalzell & Shin Shirota
- Course: Theory and Application of linear models from both a frequentist (classical) and Bayesian perspective

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- more info on Course Website
  `http://stat.duke.edu/courses/Fall15/sta721`

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  - Causal interpretation
  - Testing of hypotheses
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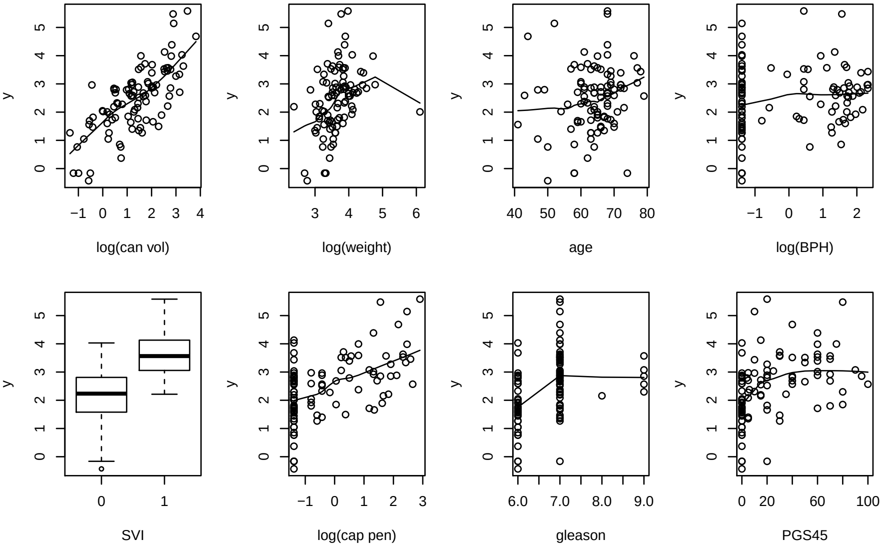
Prostate Example



Simple Linear Regression

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$$y_i = \beta_0 + \beta_1 x_i + \epsilon_i \text{ for } i = 1, \ldots, n$$

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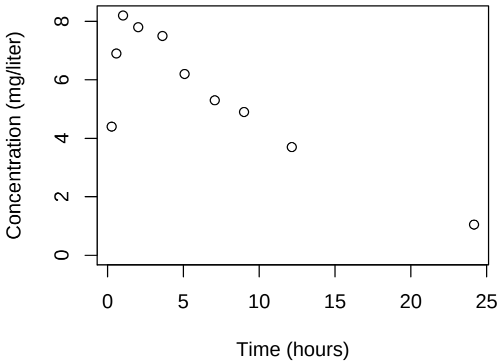
what should go into $\mathbf{X}$ and do we need all columns of $\mathbf{X}$ for inference about $\mathbf{Y}$?

Nonlinear Models

Mean function may be an intrinsically nonlinear function of t

$$\mathsf{E}[Y_i] = f(t_i, \boldsymbol{\theta})$$



Quadratic Linear Regression

Taylor's Theorem:

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Quadratic in x, but linear in β's, but remainder term is in errors $\boldsymbol{\epsilon}$

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Use Nonlinear Regression or other Nonparametric models

Kernel Regression

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$$y_i = \beta_0 + \sum_{j=1}^{J} \beta_j e^{-\lambda(x_i - k_j)^d} + \epsilon_i \text{ for } i = 1, \ldots, n$$

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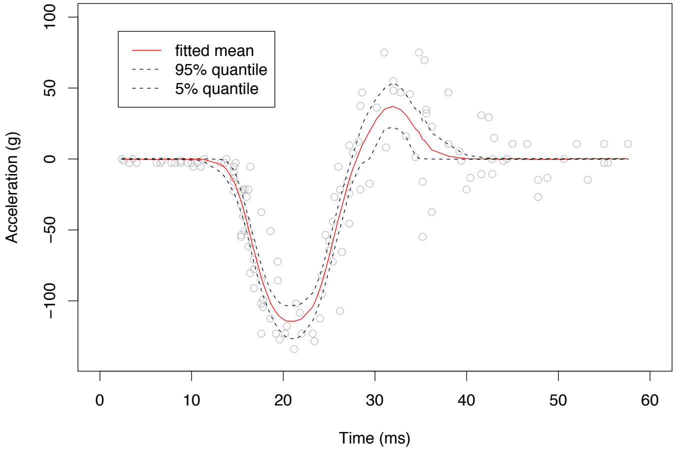
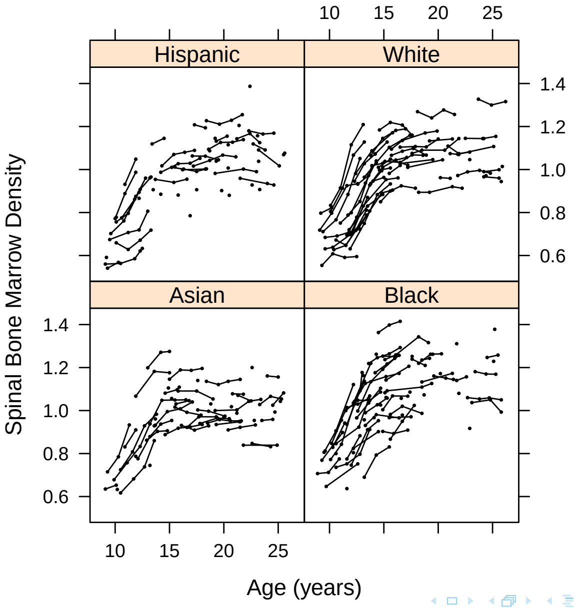
Linear in β given λ
Learn λ and J

Kernel Regression Example



duke.eps

Hierarchical Models - Spinal Bone Density



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All models are wrong, but some may be useful (George Box)

Ordinary Least Squares

Goal: Find the best fitting "line" or "hyper-plane" that minimizes

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Philosophy

- for many problems frequentist and Bayesian methods will give similar answers (more a matter of taste in interpretation)
- For small problems, Bayesian methods allow us to incorporate prior information which provides better calibrated answers
- for problems with complex designs and/or missing data Bayesian methods are often easier to implement (do not need to rely on asymptotics)
- For problems involving hypothesis testing or model selection frequentist and Bayesian methods can be strikingly different.
- Frequentist methods often faster (particularly with "big data") so great for exploratory analysis and for building a "data-sense"
- Bayesian methods sit on top of Frequentist Likelihood

Important to understand advantages and problems of each perspective!