Sampling Distributions Continued

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STA721 Linear Models

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September 13, 2016

Outline

Topics

- Student t Distributions
- Chi-squared Distributions

Readings: Christensen Apendix C, Chapter 1-2

Sampling Distribution of β

If $\mathbf{Y} \sim \mathsf{N}(\mathbf{X}\beta, \sigma^2 \mathbf{I}_n)$
Then $\hat{\beta} \mid \sigma^2, \beta \sim \mathsf{N}(\beta, \sigma^2(\mathbf{X}^T\mathbf{X})^{-1})$

Unknown σ^2

$$\hat{\beta}_j \mid \beta_j, \sigma^2 \sim \mathsf{N}(\beta, \sigma^2[(\mathbf{X}^T\mathbf{X})^{-1}]_{jj})$$

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If we substitute $\hat{\sigma}^2 = \mathbf{e}^t\mathbf{e}/(n - r(\mathbf{X}))$ in the above?

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If we substitute $\hat{\sigma}^2 = \mathbf{e}^t\mathbf{e}/(n - r(\mathbf{X}))$ in the above?

$$\frac{(\hat{\beta}_j - \beta_j)/\sigma\sqrt{[(\mathbf{X}^T\mathbf{X})^{-1}]_{jj}}}{\sqrt{\mathbf{e}^T\mathbf{e}/(\sigma^2(n - r(\mathbf{X}))}} \stackrel{\mathrm{D}}{=} \frac{N(0,1)}{\sqrt{\chi^2_{n-r(\mathbf{X})}/(n - r(\mathbf{X})}} \sim t(n-r(\mathbf{X}), 0, 1)$$

Need to show that $\mathbf{e}^T\mathbf{e}/\sigma^2$ has a χ^2 distribution and is independent of the numerator!

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See Casella & Berger or DeGroot & Schervish for derivation - nice change of variables and marginalization problem!

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- Characteristic Function

$$\mathrm{E}[e^{itZ^2}] = \varphi(t) = (1-2it)^{-1/2}$$

Chi-Squared Distribution with p Degrees of Freedom

If $Z_j \overset{\text{iid}}{\sim} \mathsf{N}(0,1)$ $j = 1, \ldots p$ then $X \equiv \mathbf{Z}^T\mathbf{Z} = \sum_j^p Z_j^2 \sim \chi^2_p$

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A Gamma distribution with shape $p/2$ and rate $1/2$, $G(p/2, 1/2)$

$$f(x) = \frac{1}{\Gamma(p/2)} (1/2)^{-p/2} x^{p/2-1} e^{-x/2} \qquad x > 0$$

Quadratic Forms

Theorem

Let $\mathbf{Y} \sim N(\boldsymbol{\mu}, \sigma^2 \mathbf{I}_n)$ *with* $\boldsymbol{\mu} \in C(\mathbf{X})$ *then if* $\mathbf{Q}$ *is a rank* k *orthogonal projection on to* $C(\mathbf{X})^\perp$, $(\mathbf{Y}^T \mathbf{Q} \mathbf{Y})/\sigma^2 \sim \chi^2_k$

Proof.

For an orthogonal projection $\mathbf{Q} = \mathbf{U} \boldsymbol{\Lambda} \mathbf{U}^T = \mathbf{U}_k \mathbf{U}_k^T$ where $C(\mathbf{Q}) = C(\mathbf{U}_k)$ and $\mathbf{U}_k^T \mathbf{U}_k = \mathbf{I}_k$ (Spectral Theorem)

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\mathbf{Z}^T \mathbf{Z} &\sim \chi^2_k
\end{aligned}$$

Since $U^T \mathbf{Y}/\sigma \stackrel{\mathrm{D}}{=} \mathbf{Z}$, $\frac{\mathbf{Y}^T \mathbf{Q} \mathbf{Y}}{\sigma^2} \sim \chi^2_k$

$\square$

Sum of Squares Error (SSE)

Let $\mathbf{Y} \sim \mathrm{N}(\boldsymbol{\mu}, \sigma^2 \mathbf{I}_n)$ with $\boldsymbol{\mu} \in C(\mathbf{X})$.
Because $\boldsymbol{\mu} \in C(\mathbf{X})$, $\mathbf{I} - \mathbf{P}_{\mathbf{X}}$ is a projection on $C(\mathbf{X})^{\perp}$

Residual Sum of Squares Example

Sum of Squares Error (SSE)

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$$\frac{\mathbf{e}^T\mathbf{e}}{\sigma^2} = \mathbf{Y}^T \frac{(\mathbf{I}_n - \mathbf{P}_{\mathbf{X}})}{\sigma}^2 \mathbf{Y} \sim \chi^2_{n-r(\mathbf{X})}$$

Estimated Coefficients and Residuals are Independent

If $\mathbf{Y} \sim \mathrm{N}(\mathbf{X}\beta, \sigma^2 \mathbf{I}_n)$
Then $\mathrm{Cov}(\hat{\beta}, \mathbf{e}) = \mathbf{0}$ which implies independence

Functions of independent random variables are independent (show characteristic functions or densities factor)

Putting it all together

$\hat{\boldsymbol{\beta}} \sim \mathsf{N}(\boldsymbol{\beta}, \sigma^2(\mathbf{X}^T\mathbf{X})^{-1})$

- $(\hat{\beta}_j - \beta_j)/\sigma[(\mathbf{X}^T\mathbf{X})^{-1}]_{jj} \sim \mathsf{N}(0,1)$
- $\mathbf{e}^T\mathbf{e}/\sigma^2 \sim \chi^2_{n-r(\mathbf{X})}$
- $\hat{\beta}$ and $\mathbf{e}$ are independent

$$\frac{(\hat{\beta}_j - \beta_j)/\sigma[(\mathbf{X}^T\mathbf{X})^{-1}]_{jj}}{\sqrt{\mathbf{e}^T\mathbf{e}/(\sigma^2(n-r(\mathbf{X})))}} \sim t(n - r(\mathbf{X}), 0, 1)$$

Inference

- 95% Confidence interval: $\hat{\beta}_j \pm t_{\alpha/2}\text{SE}(\hat{\beta}_j)$

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- 95% Confidence interval: $\hat{\beta}_j \pm t_{\alpha/2}\mathsf{SE}(\hat{\beta}_j)$ use `qt(a, df)` for t_a quantile
- derive from pivotal quantity $t = (\hat{\beta}_j - \beta_j)/\mathsf{SE}(\hat{\beta}_j)$ where $P(t \in (t_{\alpha/2}, t_{1-\alpha/2})) = 1 - \alpha$

Pivotal Quantities and CI

Other Quantities

Linear Combinations: $\lambda^T \hat{\beta}$

Prostate Example

`xtable(confint(prostate.lm))` from library(MASS) and library(xtable)

	2.5 %	97.5 %
(Intercept)	-1.91	3.25
lcavol	0.41	0.76
lweight	0.12	0.79
age	-0.04	0.00
lbph	-0.01	0.22
svi	0.28	1.25
lcp	-0.29	0.08
gleason	-0.27	0.36
pgg45	-0.00	0.01

- For a "1" unit increase in $\mathbf{X}_j$, expect $\mathbf{Y}$ to increase by $\hat{\beta}_j \pm t_{\alpha/2}\text{SE}(\hat{\beta}_j)$
- for log transforms

$$\mathbf{Y} = \exp(\mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}) = \prod \exp(\mathbf{X}_j \beta_j) \exp(\boldsymbol{\epsilon})$$

- if $\mathbf{X} = \log(\mathbf{W}_j)$ then look at 2-fold or % increases in $\mathbf{W}$ to look at multiplicative increase in median of $\mathbf{Y}$
- `ifcavol` increases by 10% then we expect `PSA` to increase by $1.10^{(CI)} = (1.0398\%, 1.0751\%)$ or by 3.98 to 7.51 percent

For a 10% increase in cancer volume, we are 95% confident that the PSA levels will increase by approximately 4 to 7.5 percent.

Derivation

Fitted Values

Unknown Mean: $\mu_i = \mathbf{x}_i^T \boldsymbol{\beta}$

Use `predict` function in `R`