COSC326 Etude 12 Explanation

K = house number, N = numbers of houses on street

We are trying to find the integers 'k' and 'n' such that $k < n$ and $1 + .... +(k-1) = (k+1) + ... + n$.

This equation is equivalent to:

$k(k-1)/2 = n(n+1)/2 - k(k+1)/2$

Which can be rewritten as:

$(2n+1)^2 - 2(2k)^2 = 1$

If we have $x = 2n + 1$ and $y = 2k$ , we have Pell's Equation, $x^2 - 2y^2 = (+ \text{ or } -) 1$( original equation is $x^2 - ny^2$ and in this case the n is 2 as it is a nonsquare positive integer). We want to get the whole number solutions of this equation. In this case of n = 2, we can get derive our solutions from the fact that the $(\sqrt{2}-1) \times (\sqrt{2}+1) = 1$.

This can also be written as $\sqrt{2} - 1 = 1/(1/\sqrt{2})$ which is equivalent to $\sqrt{2} = 1 + 1/(1 + \sqrt{2})$

Since the right hand side of that equivalent equation is equal to the $\sqrt{2}$, we can substitute it in anywhere we see $\sqrt{2}$.

$\sqrt{2} = 1 + 1/(1 + (1 + 1 / (1 + \sqrt{2})))$ we can continue doing this substituting of the right hand side of the first $\sqrt{2}$ equation, until we decide to stop and work out our fractions. This is called an infinite continued fraction and for this particular one we are using, the first few convergents are 1, 3/2, 7/5, 17/12, 41/29, 99/70, 239/169.

If we make a table of these numerators and denominators it looks like this

| Num | Denom |
|---|---|
| 1 | 1 |
| 3 | 2 |
| 7 | 5 |
| 17 | 12 |
| 41 | 29 |
| 99 | 70 |
| 239 | 169 |

From this we can see that each denominator is sum of the 2 numbers in the row above and each numerator is the row above added to twice the denominator in that row. If we keep looking at this we can see that the square of the numerator - 2 * denominator is either + or - 1. This means that our table are the whole number solutions of $x^2 - 2y^2$ = positive or negative 1.

We now look back to our original rewritten equation, $(2n+1)^2 - 2(2k)^2 = (+ \text{ or } -) 1$. We can pluck our 2n + 1 values and 2k values from the table above.

For example If we look at the numerator and denominator 17 and 12 as 2n + 1 and 2k, we get our n as 8 and our k as 6. The conditions of our house problem are satisfied for this case.

In my code I calculate this continued fractions table until the numbers it calculates are larger than the biggest accepted Integer value.

I then check through all these table values starting from 2 (0,0 and 1,1 don't fit the specifications of the etude) for the whole number solutions that satisfy our problems specifications of house numbers being equal to the end of the street from the house left and right.