(a)

$$\begin{aligned}\sum_{x=0}^{N} e^{\frac{-2\pi i k x}{N}} &= \sum_{x=0}^{N} \left(e^{\frac{-2\pi i k}{N}}\right)^x \\ &= \frac{1-\left(e^{\frac{-2\pi i k}{N}}\right)^N}{1-e^{\frac{-2\pi i k}{N}}} \\ &= \frac{1-e^{-2\pi i k}}{1-e^{\frac{-2\pi i k}{N}}}\end{aligned}$$

(b) Now by l'hoptial rule we take the limit (this is an indeterminate form as $k \to 0$)

$$\begin{aligned}\lim_{k\to 0} \frac{1-e^{-2\pi i k}}{1-e^{\frac{-2\pi i k}{N}}} &= \lim_{k\to 0} \frac{2\pi i e^{-2\pi i k}}{2\pi i e^{\frac{-2\pi i k}{N}}/N} \\ &= N\end{aligned}$$

if k is integer then $e^{-2\pi i k} = 1$ and since k is not an integer multiple of N we have that $k/N \notin \mathbb{N}$ and thus $e^{-2\pi i k/N} \neq 1$. So plugging into the expression:

$$\frac{1-e^{-2\pi i k}}{1-e^{\frac{-2\pi i k}{N}}} = \frac{1-1}{1-e^{\frac{-2\pi i k}{N}}} = 0$$

since as argued the denominator is not zero.

(c) Taking the DFT of something with a non integer wave length:

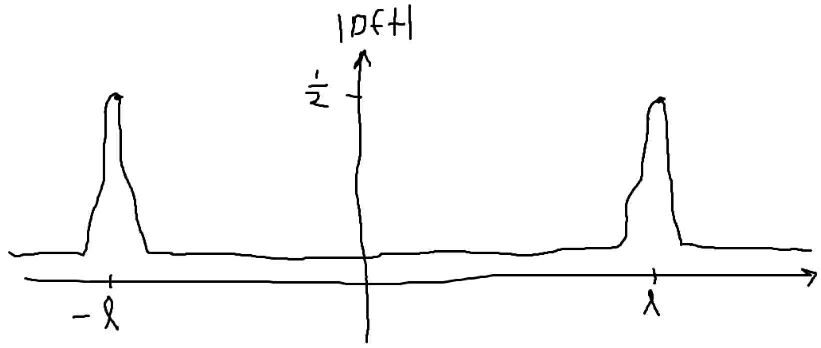
$$f(x) = \sin(2\pi l x) = \frac{e^{2\pi i l x} - e^{-2\pi i l x}}{2i}$$

and thus

$$\begin{aligned}\text{DFT} &= \sum_{x=0}^{N} \frac{e^{2\pi i l x} - e^{-2\pi i l x}}{2i} e^{\frac{-2\pi i k x}{N}} \\ &= \frac{1}{2i}\sum_{x=0}^{N} e^{\frac{-2\pi i (k-l) x}{N}} - e^{\frac{-2\pi i (l+k) x}{N}} \\ &= \frac{1}{2i}\left(\frac{1-e^{-2\pi i (k-l)}}{1-e^{\frac{-2\pi i (k-l)}{N}}} - \frac{1-e^{-2\pi i (k+l)}}{1-e^{\frac{-2\pi i (k+l)}{N}}}\right)\end{aligned}$$

So we expect this thing to be never zero! Since $(k-l) \notin \mathbb{Z}$ and $(k+l) \notin \mathbb{Z}$. In addition we would expect to attain some form of maximum at when $k \to l$ and $k \to -l$. Drawing the conclusion from that the graph should look something like:

[h]



(d) see python for the numerical stuffs

(e) Taking the DFT of the window as follows:

$$
\begin{aligned}
\mathrm{DFT}(W) &= \sum_{x=0}^{N} 0.5 - 0.5\cos(2\pi x/N)e^{\frac{-2\pi ikx}{N}} \\
&= 0.5\sum_{x=0}^{N} e^{\frac{-2\pi ikx}{N}} - 0.5\sum_{x=0}^{N}\cos(2\pi x/N)e^{\frac{-2\pi ikx}{N}} \\
&= 0.5\cdot\mathrm{DFT}(1) - 0.5\cdot\mathrm{DFT}(\cos(2\pi x/N)) \\
&= 0.5N\delta(k) - 0.25N\delta(k-1) - 0.25N\delta(k+1) \\
\implies &= [N/2, -N/4, 0, \cdots, 0, -N/4]
\end{aligned}
$$

the last line is shifted as such because of the circulant nature of the DFT so -1 ends up being the last term. So by multiplying in real space it is equivalent to convolving in Fourier space so we want to convolve DFT(f) with DFT(W) but since it is mostly zeros all we need to do is mix the 3 adjacent terms. So the value at a given point is that value multiplied by $N/2$ plus the neighboring values multiplied by $N/4$