McGill University

Experimental Methods 1

PHYS-257

Assignment 2: Textbook Questions

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September 22, 2018

1 Problem (3.2) Uniform distribution

(i) We want to show that

$$P_{\mathrm{U}}(x;\bar{x},a) = \begin{cases} 1/a & \text{if } \bar{x}-a/2 \leq x \leq \bar{x}+a/2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

is normalized. Note that the probablity of finding an x which is not in the interval $\bar{x}-a/2 \leq x \leq \bar{x}+a/2$ is 0 as P_{U} is 0 for $x \notin [\bar{x}-a/2, \bar{x}+a/2]$. Now, the probability of finding an x in the interval $\bar{x}-a/2 \leq x \leq \bar{x}+a/2$ is given by the following integral

$$\begin{aligned} P(\bar{x}-a/2 \leq x \leq \bar{x}+a/2) &= \int_{\bar{x}-a/2}^{\bar{x}+a/2} \frac{1}{a}\,dx && (2)\\ &= \frac{x}{a}\bigg|_{\bar{x}-a/2}^{\bar{x}+a/2} && (3)\\ &= \frac{(\bar{x}+a/2)}{a} - \frac{(\bar{x}-a/2)}{a} && (4)\\ &= 1. && (5) \end{aligned}$$

Hence, the probablity of finding an x at any value is given by adding the probability for $x \notin [\bar{x}-a/2, \bar{x}+a/2]$ and $x \in [\bar{x}-a/2, \bar{x}+a/2]$: $0+1=1$. P_{U} is thereby normalized.

(ii) Similarly, the mean can be calculated with the following integral:

$$\begin{aligned} \bar{x} &= \int_{-\infty}^{\infty} P_{\mathrm{U}} x\,dx && (6)\\ &= \int_{-\infty}^{\bar{x}-a/2} P_{\mathrm{U}} x\,dx + \int_{\bar{x}-a/2}^{\bar{x}+a/2} P_{\mathrm{U}} x\,dx + \int_{\bar{x}+a/2}^{\infty} P_{\mathrm{U}} x\,dx && (7)\\ &= \int_{-\infty}^{\bar{x}-a/2} 0\cdot x\,dx + \int_{\bar{x}-a/2}^{\bar{x}+a/2} \frac{1}{a} x\,dx + \int_{\bar{x}+a/2}^{\infty} 0\cdot x\,dx && (8)\\ &= \int_{\bar{x}-a/2}^{\bar{x}+a/2} \frac{1}{a} x\,dx && (9)\\ &= \int_{\bar{x}-a/2}^{\bar{x}+a/2} \frac{1}{a} x\,dx && (10)\\ &= \frac{x^2}{2a}\bigg|_{\bar{x}-a/2}^{\bar{x}+a/2} && (11)\\ &= \frac{(\bar{x}+a/2)^2}{2a} - \frac{(\bar{x}-a/2)^2}{2a} && (12)\\ &= \frac{\bar{x}^2 + a\bar{x} + a^2/4 - \bar{x}^2 + a\bar{x} - a^2/4}{2a} && (13)\\ &= \frac{2a\bar{x}}{2a} && (14)\\ &= \bar{x}. && (15) \end{aligned}$$

Indeed, the mean of the distribution is $\bar{x}$.

(iii) The variance σ^2 is given by the integral

$$\begin{aligned}
\sigma^2 &= \int_{-\infty}^{\infty} P_{\mathrm{U}}(x-\bar{x})^2\,dx \\
&= \int_{-\infty}^{\bar{x}-a/2} P_{\mathrm{U}}(x-\bar{x})^2\,dx + \int_{\bar{x}-a/2}^{\bar{x}+a/2} P_{\mathrm{U}}(x-\bar{x})^2\,dx + \int_{\bar{x}+a/2}^{\infty} P_{\mathrm{U}}(x-\bar{x})^2\,dx \\
&= \int_{-\infty}^{\bar{x}-a/2} 0\cdot(x-\bar{x})^2\,dx + \int_{\bar{x}-a/2}^{\bar{x}+a/2} \frac{1}{a}\cdot(x-\bar{x})^2\,dx + \int_{\bar{x}+a/2}^{\infty} 0\cdot(x-\bar{x})^2\,dx \\
&= \int_{\bar{x}-a/2}^{\bar{x}+a/2} \frac{1}{a}\cdot(x^2+\bar{x}^2-2x\bar{x})\,dx \\
&= \frac{1}{a}\left[\frac{x^3}{3}+x\bar{x}^2-\frac{2x^2}{2}\bar{x}\right]\Big|_{\bar{x}-a/2}^{\bar{x}+a/2} \\
&= \frac{1}{a}\left[\frac{x^3}{3}+x\bar{x}^2-x^2\bar{x}\right]\Big|_{\bar{x}-a/2}^{\bar{x}+a/2} \\
&= \frac{1}{a}\left[\frac{(\bar{x}+a/2)^3}{3}-\frac{(\bar{x}-a/2)^3}{3}+(\bar{x}+a/2)\bar{x}^2-(\bar{x}-a/2)\bar{x}^2-(\bar{x}+a/2)^2\bar{x}+(\bar{x}-a/2)^2\bar{x}\right] \\
&= \frac{1}{a}\left[\frac{3a\bar{x}^2+a^3/4}{3}+a\bar{x}^2-2a\bar{x}\right] \\
&= \frac{a^2}{12}
\end{aligned}$$

Hence, the standard deviation is indeed $\sigma = a/\sqrt{12}$.

2 Problem (3.5) Gaussian distribution

The probability of finding an $x \in [x_1, x_2]$ if x is distributed in a Gaussian fashion is given by the area under the Guassian curve between the bounds x_1 and x_2:

$$P(x_1 \le x \le x_2) = \frac{1}{\sigma\sqrt{2\pi}} \int_{x_1}^{x_2} \exp\left[-\frac{(x-\bar{x})^2}{2\sigma^2}\right]. \tag{16}$$

The variable x in this case is the mass of bags of pasta. We have $\bar{x} = 502$ g and $\sigma = 14$ g.

First, we want to find the probability $P(-\infty \le x \le 500 \text{ g})$, which is given by (note that all numbers plugged into the probability equation have units of grams)

$$\begin{aligned}
P(-\infty \le x \le 500) &= \frac{1}{14\sqrt{2\pi}} \int_{-\infty}^{500} \exp\left[-\frac{(x-502)^2}{392}\right] && (17)\\
&= \mathrm{Erf}(500; 502, 14) - \mathrm{Erf}(-\infty; 502, 14) && (18)\\
&= -\frac{\mathrm{Erf}\left(\frac{1}{7\sqrt{2}}\right)-1}{2} && (19)\\
&\approx 0.4432 && (20)
\end{aligned}$$

Hence, the probability of getting a bag which weighs less than 500 g is 0.4432 or $\approx 44.32\%$.

Now, we want to find how many bags out of a sample of 1000 should contain at least 530 g. We compute the probability $P(530 \le x \le \infty \text{ g})$, which is given by (note that all numbers plugged into the probability equation have units of grams)

$$
\begin{aligned}
P(530 \leq x \leq \infty) &= \frac{1}{14\sqrt{2\pi}} \int_{530}^{\infty} \exp\left[-\frac{(x-502)^2}{392}\right] && (21)\\
&= \mathrm{Erf}(\infty; 502, 14) - \mathrm{Erf}(530; 502, 14) && (22)\\
&= -\frac{\mathrm{Erf}\left(\sqrt{2}\right) - 1}{2} && (23)\\
&\approx 0.02275 && (24)
\end{aligned}
$$

Hence, the probability of getting a bag which weighs at least 530 g is ≈ 0.02275. Hence, $1000 \cdot 0.02275 = 22.75 \approx 23$ bags out of 1000 will contain at least 530 g.