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The Dynamics of Earth's Climate: Understanding Energy Balance and Greenhouse Gas Effects

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Abstract

The current climate change, unlike past occurrences, is primarily driven by human activities, leading to a rapid rise in surface temperatures and an increase in extreme weather events. To adapt and mitigate these impacts, society must prepare by understanding future climate changes. Mathematical modeling remains the key method to predict how the climate will evolve under natural and human-induced influences. This study explores various mathematical methods, particularly modeling and sensitivity analysis, to understand Earth's climate system and how mankind generated GHG emissions can influence these changes. The model relies on fundamental physics principles to establish the foundation for equations. The model focuses on examining the interactions between *albedo*, *solar radiation*, and the Earth's energy balance, particularly how greenhouse gasses influence temperature by trapping outgoing radiation.

It considers feedback mechanisms(*or positive/negative feedback loops*), such as changes in *albedo*, that can amplify or dampen the effects of these processes on global temperatures., The study's findings are validated by comparing the model's results with observations and climate models, ensuring the accuracy of the conclusions drawn.

1. Introduction

Mathematics serves as a powerful tool in addressing complex problems, providing a systematic framework for analyzing and understanding phenomena in various scientific aspects. In the context of modeling greenhouse gas emissions (GHG) and their impact on climate change, mathematics allows us to quantify and simulate the interactions between different components of the Earth's climate system.

By borrowing principles from physics, such as the conservation law of energy, we can develop mathematical models that capture the fundamental processes behind earth's energy balance. These models enable us to explore the potential consequences of GHG emissions, assess the effectiveness of mitigation strategies, and inform policy decisions aimed at mitigating the impacts of climate change.

1.1 Climate as a Complex Dynamical System

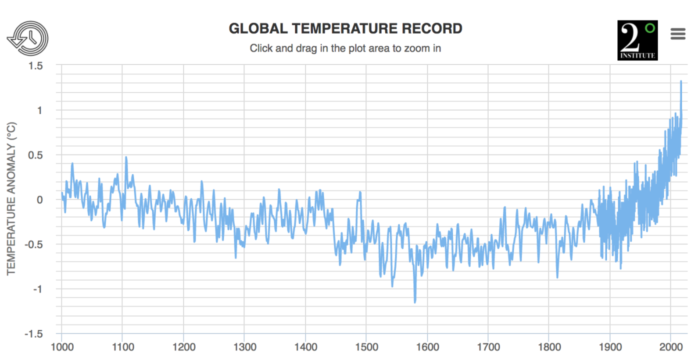
The Earth is no stranger to climate change, and in fact, has been experiencing drastic shifts in temperatures due to natural occurrences, shifts large enough to transition between glacial and interglacial periods.These shifts are attributed to factors such as changes in the Earth's orbit, solar radiation, volcanic activity, greenhouse gas concentrations and many other factors.It is vital to understand that the Earth's climate behavior is a complex dynamical system.
A system is understood as a set of interrelated and interdependent elements that work together to achieve certain goals[1]It is called complex if it has properties such as emergent behavior, self-organization, a hierarchical structure, nonlinearity and spontaneous order.

Climate change is one of the most controversial and pressing challenges facing humanity today, with far-reaching impacts on ecosystems, economies and societies. Despite the overwhelming scientific evidence on human-induced climate change, there remains a significant level of controversy and skepticism among some segments of the population. This skepticism is often fueled by various factors, including ideological beliefs, misinformation, and vested interests in maintaining the status quo.

1.2 Pre industrial era of Earth's climate

The pre-industrial era refers to the period in history before the intervention of mankind's industrial machinery, which is the main cause of fossil fuel combustion. This era is considered to span from ancient times until the 18th century. This however, does not imply stable emissions. Natural processes such as volcanic activity, ocean atmospheric interactions and other natural occurrences influence CO_2 levels greatly, leading to large fluctuations over geological timescales.

In fact, these variations in CO_2 concentration played a crucial role in Earth's climate dynamics, driving the earth between shifts of glacial and interglacial period and ultimately how the earth transitioned out of the Ice Age. Today, we are in what is known as a warm interglacial period.



The plot above illustrates deviations in global temperatures from the long term average over the last 1000 years. There is a clear uptick in the gradient of temperature deviations from the year 1800 onwards, with greater and more frequent deviations from the mean. This suggests a significant shift in global climate patterns.

The long-term average refers to the statistical mean or average calculated over a substantial period, typically spanning decades to centuries, capturing a stable representation of a specific parameter or variable over an extended timeframe. In the context of climate science, the long-term average often pertains to parameters such as temperature, precipitation, or sea level, providing a baseline reference for understanding variability and trends over time.[1]

2.1 Earth's Energy Balance

The Earth's atmospheric energy balance refers to the equilibrium between incoming solar radiation from the Sun and outgoing energy from the Earth. This balance is influenced by various factors, including absorption of solar radiation by the Earth's surface, reflection of solar radiation back into space, and the retention of heat by greenhouse gasses(GHG).

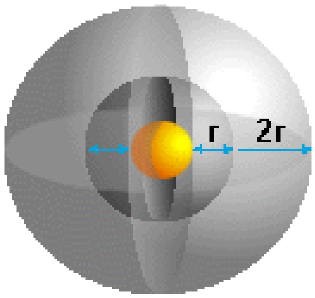
It is important to note that while these factors play a significant role in the Earth's energy balance, there are other contributing factors that will not be considered in the model due to the highly dimensional dynamics of the system. While achieving more accurate results remains vital for mathematical modeling, it is important to maintain a relative amount of simplicity without compromising precision.

When studying the pre-industrial era of Earth's climate, we aim to understand the planet's temperature before significant human-induced changes. Utilizing the Stefan-Boltzmann Law, we can estimate the Earth's pre-industrial temperature by considering it as a black body radiator. By analyzing factors like solar radiation input and the Earth's albedo (reflectivity), we can calculate the amount of energy absorbed and emitted by the planet. This allows us to approximate the equilibrium temperature consistent with the energy balance of incoming solar radiation and outgoing thermal radiation, providing insights into historical climate conditions.

2.2 R-Squared Law

The R squared law can help us determine the amount of solar radiation coming from the sun. The R-Squared law states that the further you are from an emitting object, the less light you receive.From a physical standpoint, this inference aligns with the law's principle, which dictates that objects situated farther away from a radiating source (such as the Sun) receive reduced illumination, resulting in lower temperatures.

Mathematically, it can be seen in the figure below



The amount of energy received by an object r distance from the sun would be four times the amount of energy by an object of distance $2r$. Using only the following information we can dedicate a simpler(but less accurate) estimation of the temperatures of any planets just using the distance. Since these planets are in constant orbit, we will be using the mean distance.

For example, an unrelated sidetrack but interesting finding is how we can calculate other planets' temperatures based on the r squared law solely

Using only the following information we can dedicate a simpler(but less accurate) estimation of the temperatures of any planets just using the distance. Since these planets are in constant orbit, we will be using the mean distance.

$$\frac{(T_1)^4}{(T_2)^4} = \frac{r_2^2}{r_1^1}$$

- **T1 : Temperature of the Earth(288K)**
- **T2 : Temperature of Mercury**
- r_2^2 **: mean distance from earth to sun(1 AU)**
- r_1^2 **: mean distance from mercury to sun(0.37AU)**
  ******1 AU approx* $1.5 * 10^8 km$

$$T_2 = T_1 \sqrt{\frac{r_1^2}{r_2^2}}$$

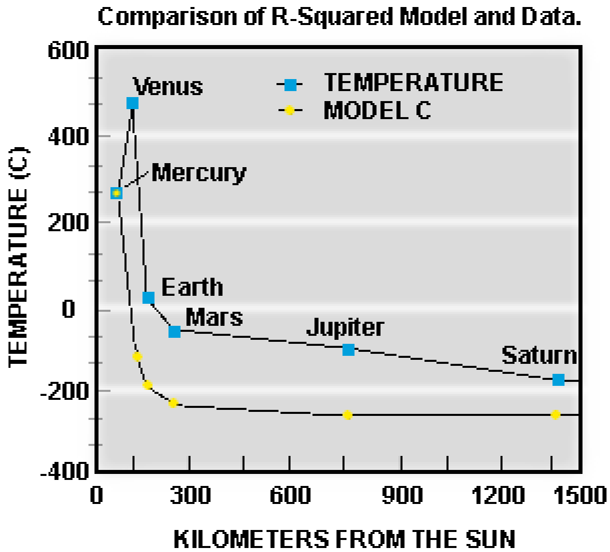
$$T_2 = 288 \left(\frac{0.387}{1}\right)^{0.5}$$
$$T_2 = 288 \times 0.622$$
$$T_2 \approx 179.8 \text{ K}$$

The estimated temperature of Mercury, in degrees celsius is

$$T_{\text{Celsius}} = T_{\text{Kelvin}} - 273.15$$

Approximately -93.35 celsius.

Similarly we can apply this to all the planets(see figure below)



It is evident that the over simplified model is not sufficient as the temperatures of the planets calculated by the model(marked yellow) are lower than reality. The model needs improvements.

2.2 Law of conservation of energy

The Law of Conservation of Energy tells us that the amount lost (in the infrared) has to equal the amount received (in the visible). Another way of viewing this is to say that the temperature of the Early Earth has to rise until exactly as much energy is lost through radiation as is gained from solar absorption. This equilibrium sets the temperature of any planet.[2] We can conclude that the energy entering the earth's atmosphere can never be destroyed, so it is either absorbed as heat or reflected back into space.

To transition this into a concept in a mathematical model, we can start by considering the energy balance of earth. Let E_in be the total energy coming into the atmosphere from the Sun, and E_out be the total energy radiated back into space. According to the law of conservation of energy, these two must be at equal equilibrium

$$E_{in} = E_{out}$$

$$\text{Energy received from the Sun} = \text{Energy emitted by the Earth}$$

In order to find the energy coming into the earth, we need to calculate the total energy being radiated from the sun.

2.3 Stefan-Boltzmann law

The Stefan-Boltzmann Law describes the power radiated per unit area by a black body in terms of its temperature. A blackbody is a representation of an object that perfectly absorbs all the radiation incident upon it, and emits radiation depending solely on its temperature.

It states that the total energy radiated per unit surface area of a black body across all wavelengths per unit time (also known as the black-body radiation flux) is directly proportional to the fourth power of the black body's thermodynamic temperature.[x] In the context of our model, for the Earth and Sun it can be expressed as follows:

$$P_s = \epsilon \sigma A_s T^4$$

$$P_e = \epsilon\sigma A_e T^4$$

Where ϵ is the **emissivity** of an object, which is a measure of how efficiently an object emits thermal radiation compared to a perfect **blackbody**. Our first assumption in this model is that since the sun is considered to be the closest approximation of a perfect black body, we will set this to 1, and for the earth we will assume that it emits energy as thermal radiation relatively well, at a value of 0.605.

σ is the **Stefan-Boltzmann** constant in units of joules per second per square meter $(5.67 \times 10^{-8}) W/(m(^2) K(^4))$

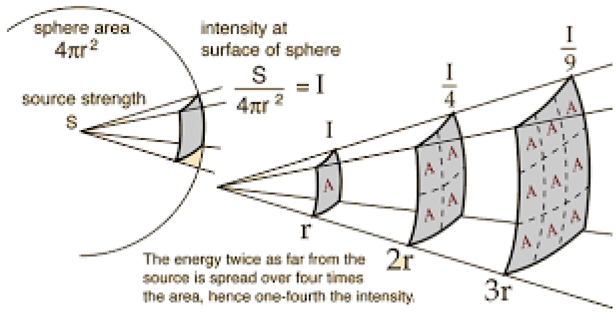
A_s is the surface area of the sun $4 * \pi * r^2$

A_e is the surface area of the earth $4\pi 2$

T is the surface temperature of the object, assume 10,000 F for the Sun. This model will be used to predict the temperature of the Earth in its pre-industrial era.

Using this equation, the radiated power for the Sun is approximately $3.85 * 10^2 0W$. We can call this the Luminosity of the sun, the raw amount of radiation being emitted from the sun.

In order to determine how much of this radiated power from the Sun is actually reaching the atmosphere and received by the Earth, we need to consider several factors that affect the amount of energy reaching earth. These factors include the distance, and the surface area of the earth which intercepts the solar radiation.



[6]

We need to recognize that the amount of radiation gets 'diluted' by the time it reaches earth's orbit, as confirmed by the r squared law. This dilution can be represented as

$\frac{r_s}{R_s}$ where R_s is the mean distance between the earth and the sun(in meters), and r_s is the radius of the Sun.

The amount of radiated energy that reaches the Earth's atmosphere per unit area is the Solar constant S.

$$S = \frac{r_s}{R_s} \frac{L}{4\pi r^2} = 1367W/m^-2$$

Using the Law of Conservation of Energy, Equating the two sides, and solving for T

Solar constant = P_e

$$S = \epsilon\sigma A_e T^4$$

$$1367\,\mathrm{W/m}^{-2} = 0.61 \times 5.67 \times 10^{-8}\,\mathrm{W/m}\,T^4$$

$$T = \sqrt[4]{\frac{1367\,\mathrm{W/m}^{-2}}{0.61 \times 5.67 \times 10^{-8}\,\mathrm{W/m}}}$$

After performing the calculations, we determine the temperature of the pre-industrial Earth to be approximately 288 Kelvin (which is roughly 14 degrees Celsius).While this value provides an estimate, it's important to note that there isn't a definitive way to verify this data due to the lack of direct measurements from that era.

We observe that this temperature seems somewhat high for a pre-industrial Earth. This discrepancy can be attributed to the model's oversight in not factoring in albedo, the Earth's reflectivity, which can significantly influence its temperature by affecting the balance of incoming and outgoing radiation. Incorporating albedo into the model would likely result in a more accurate representation of the pre-industrial Earth's temperature.

2.4 Albedo

The albedo of Earth is formally defined as the measure of the fraction of incident solar radiation reflected by the Earth's surface, atmosphere, and clouds back into space. It is expressed as a dimensionless value between 0 and 1, where 0 indicates a surface that absorbs all incoming solar radiation, and 1 indicates a surface that reflects all incoming solar radiation.[x] We can denote albedo as α

With this information, we can deduce that a certain amount of the energy is reflected, and we can represent the earth's energy balance in this new equation.

$$P_e = (1 - \alpha)S = \sigma A T^4$$

This equation aligns with the concept that when considering an Earth with an albedo value of 1, indicative of a surface that reflects all incoming solar radiation, the energy at the atmosphere would equate to zero. We can assume that some of the energy is reflected, universally, albedo is estimated to be 0.3 .

After the new equation, the temperature is 260K, which is about -13.15 celsius. This is known as the Radiative Equilibrium Temperature of the Earth, when the emissions exactly balance the radiation received by the sun.

It is crucial to acknowledge the dynamic nature of albedo, particularly in response to temperature increases. As temperatures rise, the melting of ice and snow reduces surface albedo, leading to a decrease in the ratio of reflection to absorption of solar radiation. Notably, this reduction in albedo exacerbates the imbalance in energy exchange, amplifying the warming effect beyond the direct impact of albedo reduction. This phenomenon creates a recursive loop within Earth's climate system, where higher temperatures induce albedo reductions, further enhancing the absorption of solar radiation and consequent warming. Albedo can be expressed mathematically as a recursive function

$$\text{Albedo} = \text{Albedo}_0 - \alpha \cdot \Delta T$$

A negative change in temperature is indicative of colder climate, can result to freezing conditions. This can increase the surface area of the Earth that is covered by reflective surfaces such as snow and ice, which results in the increase of Albedo in our equation. As more of Earth's surface reflects incoming radiation, less heat is absorbed, further lowering temperatures and potentially leading to a feedback loop that reinforces this cooling effect. Therefore we can introduce albedo into the model, reducing the Solar constant S.

$$S(1-\alpha)$$

The universal standard for albedo is $\alpha = 0.3$, resulting in a total energy of

$$Q = 956.9W \quad .$$

We can attempt to verify the albedo of Earth using our model and see if the result is close to the universal standard of **0.3.**

$$\epsilon\sigma A_e T^4 = (1 - \alpha)\, S$$

$$T = \sqrt[4]{\frac{S(1-\alpha)}{\sigma}}$$

Given that we know the current global temperature to be 15 degrees celsius, we can update our equation and solve for albedo.

$$x,\ 288.15 = \sqrt[4]{\frac{1361(1-x)}{(5.67 \cdot 10^{-8})}}$$

Solution

$$x = 0.71279\ldots$$

After calculating the albedo to be 0.7, we can evaluate its likelihood by comparing it to known values and theoretical constraints. Albedo values typically range from 0 to 1, with 0 representing a perfectly absorbing surface and 1 representing a perfectly reflecting surface. A value of 0.7 would imply an exceptionally high reflectivity, which could be false based on the properties of the surface of our Earth.

As mentioned earlier, Earth is a complex dynamical system due to the multitude of processes simultaneously occurring, and it’s clear that the model does not capture all of these occurrences. Particularly, the greenhouse effect is not accounted in this model.

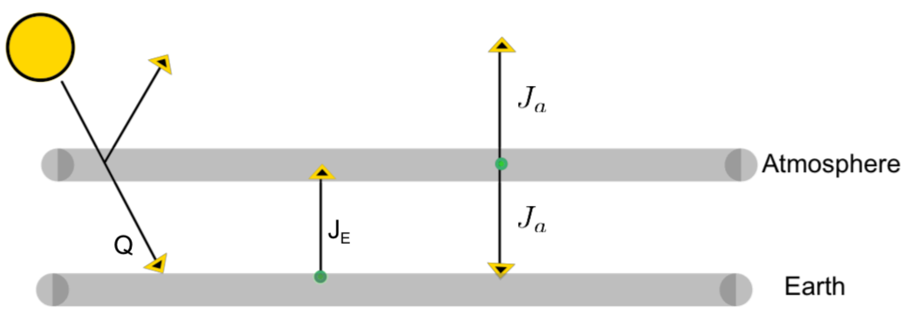
3.0 Greenhouse Gasses

Greenhouse gasses (GHGs) affect climates by trapping the reflected energy in the Earth's atmosphere, leading to an increase in temperatures at the surface. This phenomenon, known as the greenhouse effect, occurs when certain gasses such as carbon dioxide (CO2), methane (CH4), and water vapor (H2O) absorb and re-radiate infrared radiation emitted by the Earth's surface. According to the Intergovernmental Panel on Climate Change (IPCC), the enhanced greenhouse effect resulting from human activities, primarily the burning of fossil fuels and deforestation, has intensified the warming of the Earth's atmosphere, contributing to climate change (IPCC, 2021). This trapping of energy alters global temperature patterns, precipitation regimes, and weather extremes, highlighting the significant impact of GHGs on Earth's climate system.

3.1 GHG model for emissivity

In this model, we have our Sun, the Earth's surface and the atmosphere. Solar Energy(S) reaches the Earth's surface. In this model, we want to calculate the emissivity ϵ as this is the main indicator of greenhouse gas activity. The higher our emissivity , the more radiation our atmosphere absorbs or traps.

The **emissivity**, of a radiating object is the ratio of the amount of radiation the object emits compared to amount of radiation a blackbody at the same temperature, emits.This value is always between 0 and 1.



[6]

Every arrow indicates an initial starting point from a radiating body, and arrives to an absorbing body.

J_a is the energy radiated per unit area by the atmosphere

J_e is the energy radiated per unit area by Earths surface

Q is the amount of solar energy reaching Earth's surface.

Recall that our solar system is in a state of equilibrium, meaning the radiation absorbed by each body is equal to the amount it emits.

$Earth$: $Q + J_a = J_e$

$Atmosphere$: $\epsilon_a J_e = 2J_a$

Now lets incorporate the Stefan-Boltzman law to rewrite the radiation of the Earth and atmosphere in terms of temperature.

$Earth : Q + \epsilon_a (T^4)_a = \sigma(T_E)^4$

$atmosphere : \epsilon_a (T^4)_E = 2\epsilon_a (T^4)_a$

We can use the 2nd equation to relate the temperatures of the earth and the atmosphere. After simplifying the last equation we get

$$T_E^4 = 2T_a^4$$

Substitute this into the earth equation we get

$$Q + \epsilon_a \sigma T_a^4 = 2\sigma T_a^4$$

Solve for T_a

$$T_a^4 = \frac{Q}{\sigma(2 - \epsilon_a)}$$

And

$T_e^4 = \frac{2Q}{(2-\epsilon_a)}$

Where

Q=956.9W(in albedo section)

$\sigma = (5.67 \times 10^{-8}) W/(m(^2)K(^4))$

After calculating for T_a^4 & T_e^4

$$T_a = \frac{254.9}{(2-\epsilon_a)} \ K \ and \qquad T_E = \frac{303.3}{(2-\epsilon_a)} \ K$$

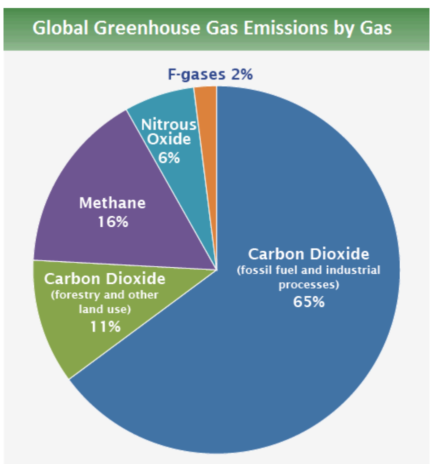
As ϵ increases, the temperatures of the earth and atmosphere both increase. According to this model, since emissivity ranges between 0 and 1, the possible temperatures for T_A range from 127.45 to 254.9K and for T_e from 151.65 to 303.3K. This can give us reasonable insight into climate patterns.

Currently, we know that the surface temperature of the Earth is 15 celsius, or 288K. By plugging in the value of the T_e we get that $\epsilon = 0.947$. This indicates that on average, Earth emits 94.7% of the radiation that a perfect black body at the same temperature would emit.

We can use this model to test how GHG can affect global temperatures given that we know the $\delta\epsilon_a$.

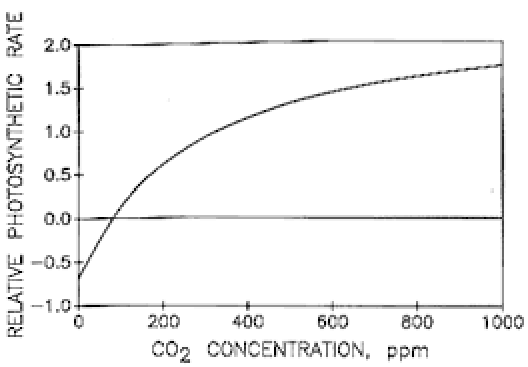
3.2 Model validation through CO2 effects on temperature

Model validation is the last step in the iterative cycle of mathematical modeling. One common approach to validation is to compare model-derived values with observed data. In this study, we aim to validate the model's predictions with real world observations. Given that CO2 is the highest in emissions, we can deduce temperature change based on its emissivity $$\epsilon_a$$.



The climate of the planet Earth is maintained by the nonlinear coupling between various processes. Thus, it is clear that one cannot obtain a definitive conclusion, using a globally averaged model, concerning the effect of an increase in $$CO_2$$ upon climate.[7]

Doubling the concentration of carbon dioxide (CO2) in the Earth's atmosphere, often referred to as "CO2 doubling," is a scenario frequently used in climate science to understand the potential impacts of increased greenhouse gas emissions. This scenario is of particular interest because it allows scientists to study the sensitivity of the climate system to changes in CO2 levels and to make projections about future climate change.



[4]

We have empirical data illustrating the growth of CO2 in ppm, exhibiting a logarithmic nature. This means that as CO2 concentrations increase, the additional warming effect of each unit of CO2 becomes smaller.

According to the IPCC, if the amount of carbon dioxide in the atmosphere doubles, it will cause atmospheric emissivity ϵ to increase by 0.019.

Let's use our model to see the effect of this increase on temperatures.

$$T_E = \frac{303.3}{(2-\epsilon_a)} \quad K$$

An increase of 0.019 to ϵ from our initial calculated value of 0.947 results in

$\epsilon_a = 0.966$, and our model predicts that T_e will increase from 288.3K to 293K, or 5 degrees celsius. This finding aligns with the literature review.

Conclusion

This study demonstrates the significance of mathematical modeling in comprehending the complex dynamics of not just earth but also other planets in our universe through the help of simplified models.Through mathematical modeling and rational assumptions we can gain a deeper understanding of the intricate relationships among key variables such as temperature, solar radiation, greenhouse gas emissions, and albedo.

Initially, we explored a simplistic model, the R-squared law, to estimate planetary temperatures based on their distance from the Sun. Upon comparing this model with empirical data, we revised and improved it by introducing dilution factors to more accurately approximate the solar radiation entering Earth's atmosphere and accurately calculated Solar Radiation. Despite these improvements, the model still showed discrepancies, prompting us to incorporate albedo into our calculations.

Subsequently, we delved into the effects of doubling CO2 levels and developed a mathematical model linking gas emissivity to the Stefan-Boltzmann constant, providing deeper insights into the mechanisms of climate change. Using this model, we verified our results with IPCC findings.

It’s important to recognize that mathematical modeling is an iterative process, a continuous cycle. This iterative process of model refinement and validation helps us get increasingly more accurate data.

THANK YOU FOR YOUR TIME

REFERENCES

1 Soldatenko S, Bogomolov A, Ronzhin A. Mathematical Modelling of Climate Change and Variability in the Context of Outdoor Ergonomics. Mathematics. 2021; 9(22):2920. https://doi.org/10.3390/math9222920

2 IPCC, 2014: Climate Change 2014: Synthesis Report. Contribution of Working Groups I, II and III to the Fifth Assessment Report of the Intergovernmental Panel on Climate Change [Core Writing Team, R.K. Pachauri and L.A. Meyer (eds.)]. IPCC, Geneva, Switzerland, 151 pp.

3 IPCC (Intergovernmental Panel on Climate Change). "Climate Change 2014: Synthesis Report. Contribution of Working Groups I, II, and III to the Fifth Assessment Report of the Intergovernmental Panel on Climate Change." IPCC, 2014.

4 Manabe, S., & Wetherald, R. T. (1975). The effects of doubling the CO2 concentration on the climate of a general circulation model. Journal of the Atmospheric Sciences, 32(1), 3-15.

5 D. Halliday, R. Resnick, J. Walker, Fundamentals of Physics (John Wiley and Sons, 2005)

6 Branson, M. (n.d.). Greenhouse Gas Math Topic: Mathematical Models. Retrieved from https://web.stevenson.edu/mbranson/m4tp/version1/greenhouse-gas-math-topic-mathematical-models.html

7Manabe, S., & Wetherald, R. T. (1975). The Effects of Doubling the CO2 Concentration on the climate of a General Circulation Model. *Journal of Atmospheric Sciences*, *32*(1), 3-15. https://doi.org/10.1175/1520-0469(1975)032<0003:TEODTC>2.0.CO;2