First Steps in Motivic Homotopy Theory

for LaTeX Class

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Contents

Part I

A Part Heading

Chapter 1

What is in this book?

Chapter 2

Lecture 1: Introduction

The famous *Milnor conjecture* was stated in 1969 by Milnor in his seminal paper [Mil70]. In it he defined the *Milnor K-theory* $K_*^M(F)$ of a field F and related this graded ring to Galois cohomology H^*_{Gal} as well as the associated graded Witt ring

$$\mathrm{Gr}_I(W(F)) := \bigoplus_{n\geq 0} I^n(F)/I^{n+1}(F) = \mathbf{Z}/2 \oplus I(F)/I^2(F) \oplus \cdots ,$$

where $I(F) \subseteq W(F)$ is the fundamental ideal. Establishing isomorphisms in degrees 0, 1 and 2, Milnor asked whether this holds in every degree. By the invent of motivic homotopy theory, Voevodsky and his collaborators settled the conjecture in 2003. Every known proof of the Milnor conjecture uses motivic homotopy theory.

Theorem 2.0.1 (Milnor's conjecture)**.** *Suppose F is a field of characteristic different from 2. Then there exist graded ring isomorphisms*

$$\begin{tikzcd} & K_*^M(F)/2K_*^M(F) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ \mathrm{Gr}_I(W(F)) \arrow[rr] & & H^*_{\mathrm{Gal}}(F, \mathbf{Z}/2), \end{tikzcd}$$

where

$$K_n^M(F) := (F^\times)^{\otimes n}/\langle a_1 \otimes \cdots \otimes a_n : a_i + a_{i+1} = 1 \textit{ for some } i\rangle$$

is the n-th Milnor K-group.

As opposed to the situation in topology, in motivic homotopy theory there are not one but two spheres: The simplicial sphere S^1, known to topologists; and the *Tate sphere* $S^\alpha := \mathbf{G}_m$. This results in a bigrading on several important theories in motivic homotopy—most notably on the motivic homotopy groups and motivic cohomology. In fact, in the proof of Milnor's conjecture, the Milnor K-theory K_*^M is replaced by motivic cohomology. Furthermore, the Galois cohomology H^*_{Gal} is replaced by étale motivic cohomology, i.e., motivic cohomology on an étale site.

2.1 The idea of motivic homotopy theory

Techniques from topology are used extensively, but in an algebro-geometric way. Most notably is the use of:

- Thom-spaces
- Spectra
- Steenrod operations, i.e., operations on motivic cohomology
- Objects representing (co)homology theories
- The Hopf map, which in the geometric case is induced by the canonical map $\mathbf{A}^2 \setminus \{0\} \to \mathbf{P}^1$. Taking complex points yields the topological Hopf fibration $S^3 \to S^2$.

In topology, homotopies are parametrized by the unit interval $I = [0,1]$. In motivic homotopy, the algebro-geometric object corresponding to I is the affine line $\mathbf{A}^1 := \mathbf{A}^1_F$. Evaluating polynomials in 0 and 1 gives us paths.

Definition 2.1.1. Let $f, g : X \to Y$ be two morphisms of schemes over F. We say that f and g are *elementary* $\mathbf{A}^1$*-homotopic*, writing $f \simeq g$, if there is a map

$$H : X \times_F \mathbf{A}^1 \longrightarrow Y$$

such that

$$H \circ i_0 = f, \quad H \circ i_1 = g,$$

where $i_j : X \to X \times \{j\}$ $(j = 0, 1)$ are the canonical inclusions.

This definition is the "naive" lift of the homotopy relation in topology. However, the situation is here quite different: As opposed to topology, the relation of elementary $\mathbf{A}^1$-homotopy is *not* an equivalence relation, because it is not transitive. However, we can consider the equivalence relation generated by elementary $\mathbf{A}^1$-homotopy. We will call two maps in the same equivalence class $\mathbf{A}^1$*-homotopic.*

2.1.1 Setting

There are several possible settings for motivic homotopy. One possibility is the following:

Start with the category Sm_F of smooth schemes of finite type over F (in fact, there is no need to restrict ourselves to working over a field). In order to do homotopy theory we frequently consider quotient spaces. However, the category Sm_F is poorly behaved under colimits:

Example 2.1.2. The colimit of the diagram $* \leftarrow \{0,1\} \hookrightarrow \mathbf{A}^1$ is a nodal curve, hence the quotient of two smooth F-schemes need not be smooth.

We will therefore embed the category Sm_F in a larger category with better categorical properties. Consider the embedding

$$\mathrm{Sm}_F \hookrightarrow [\mathrm{Sm}_F^{\mathrm{op}}, \mathrm{Set}] =: \mathrm{Pre}(\mathrm{Sm}_F).$$

Here $\mathrm{Pre}(\mathrm{Sm}_F)$ is the category of presheaves on Sm_F, i.e., the functor category having as its objects functors $\mathrm{Sm}_F^{\mathrm{op}} \to \mathrm{Set}$, and the morphisms are natural transformations. The embedding $\mathrm{Sm}_F \hookrightarrow \mathrm{Pre}(\mathrm{Sm}_F)$ is the Yoneda embedding $X \mapsto \mathrm{Hom}_{\mathrm{Sm}_F}(-, X)$.

Since our aim is to find a suitable category for doing homotopy theory, we need to consider a simplicial version of the category $\mathrm{Pre}(\mathrm{Sm}_F)$. Let

$$\mathcal{S} := \Delta^{\mathrm{op}}\mathrm{Set} := [\Delta^{\mathrm{op}}, \mathrm{Set}]$$

denote the category of simplicial sets. Thus Δ is the simplex category, whose objects are finite ordered sets

$$[n] := \{0 < 1 < \cdots < n\}$$

for $n \geq 0$; and the morphisms are maps preserving the ordering. The functor category $\Delta^{\mathrm{op}}\mathrm{Set}$ is a combinatorial model for topological spaces, in which we can do homotopy theory—i.e., there are model structures on $\Delta^{\mathrm{op}}\mathrm{Set}$.

Finally, we arrive at the definition of *motivic spaces*: We set

$$\mathcal{MS}_F := \Delta^{\mathrm{op}}\mathrm{Sm}_F = [\mathrm{Sm}_F^{\mathrm{op}}, \mathcal{S}].$$

Note that we have a canonical embedding $\mathrm{Sm}_F \hookrightarrow \mathcal{MS}_F$ by considering a scheme X as a constant motivic space. We think of a motivic space as the analog in motivic homotopy theory to a topological space in ordinary homotopy.

2.1.2 Algebro-geometric versions of topological simplices

Example 2.1.3 (Simplicial sets)**.** A topological space X yields a singular chain complex.

The standard topological n-simplex is given by

$$\Delta_n := \left\{ (x_0, \ldots, x_n) \in \mathbf{R}^{n+1} : \sum_i x_i = 1, x_i \geq 0 \right\}.$$

The face maps $d^i : \Delta_{n-1} \to \Delta_n$ inserts a 0 in the ith coordinate, and the degeneracy maps $s^i : \Delta_{n+1} \to \Delta_n$ adds the coordinates x_i and x_{i+1}. Geometrically, d^i inserts Δ_{n-1} as the ith face of Δ_n, and s^i projects Δ_{n+1} onto the n-simplex orthogonal to its ith face.

Define the simplicial set $\mathrm{Sing}(X)$ as

$$\mathrm{Sing}(X)_n := \mathrm{Hom}_{\mathrm{Top}}(\Delta_n, X);$$

the elements of $\mathrm{Sing}(X)_n$ are called n-simplices of X. Precomposing with d^i or s^i yields respectively face maps

$$d_i : \mathrm{Sing}(X)_{n+1} \to \mathrm{Sing}(X)_n$$

and degeneracy maps

$$s_i : \mathrm{Sing}(X)_{n-1} \to \mathrm{Sing}(X)_n.$$

Using the functor Sing we can factor the n-th singular homology functor $H_n(-; \mathbf{Z})$ as

$$H_n(-; \mathbf{Z}) : \mathrm{Top} \xrightarrow{\mathrm{Sing}} \mathcal{S} \xrightarrow{\mathbf{Z}} \Delta^{\mathrm{op}}\mathrm{Ab} \xrightarrow{\sum (-1)^i d_i} \mathrm{Ch}_{\mathbf{Z}} \xrightarrow{H_n} \mathrm{Ab}.$$

Here $\mathbf{Z} : \mathcal{S} \to \Delta^{\mathrm{op}}\mathrm{Ab}$ is the free functor in each degree, and $\mathrm{Ch}_{\mathbf{Z}}$ is the category of chain complexes.

We now have the diagram

$$\begin{tikzcd} \Delta \arrow[r, "\text{Yoneda}"] \arrow[dr, "\Delta_\bullet"'] & \mathcal{S} \arrow[d, "\mathrm{Re}"] \\ & \mathrm{Top} \end{tikzcd}$$

where $\Delta_\bullet([n]) := \Delta_n$, and the "realization functor" Re is the left Kan extension of $\Delta_\bullet$ along the Yoneda embedding. Defining $\mathrm{Re}(\Delta(-,[n])) := \Delta_n$ determines the functor. Furthermore, the functor Re has Sing as its right adjoint,

$$\mathrm{Re} : \mathcal{S} \rightleftarrows \mathrm{Top} : \mathrm{Sing}\,.$$

We can put a model structure on $\mathcal{S}$ such that this adjoint pair becomes a Quillen equivalence.

Example 2.1.4 (Standard simplicial ring)**.** For any $n \geq 0$, let

$$F_n := \frac{F[x_0, \ldots, x_n]}{(\sum_i x_i - 1)}.$$

For each i, define face maps $F_{n+1} \to F_n$ by $x_i \mapsto 0$, and degeneracy maps $F_{n-1} \to F_n$ by $x_i \mapsto x_i + x_{i+1}$. Define

$$\Delta_F^n := \operatorname{Spec} F_n.$$

There is a noncanonical isomorphism $\mathbf{A}_F^n \cong \Delta_F^n$. Letting n vary we obtain a functor

$$\Delta_F^\bullet : \Delta \to \mathrm{Sm}_F,$$

i.e., $\Delta_F^\bullet$ is a cosimplicial object.

Example 2.1.5. As a first attempt to define singular homology for schemes, we can try to follow the recipe from topology. Suppose $F = \mathbf{Q}$, so that $\Delta_{\mathbf{Q}}^0 = \operatorname{Spec} \mathbf{Q}$. Consider the affine scheme $\operatorname{Spec} \mathbf{Q}(\sqrt{-1})$. Since there are no ring maps $\mathbf{Q}(\sqrt{-1}) \to \mathbf{Q}$,

$$\mathrm{Hom}_{\mathrm{Sm}_{\mathbf{Q}}}(\Delta_{\mathbf{Q}}^0, \operatorname{Spec} \mathbf{Q}(\sqrt{-1})) = \varnothing.$$

This approach would therefore yield the uninteresting result $H_0(\mathbf{Q}(\sqrt{-1}); \mathbf{Z}) = 0$.

The problem of defining singular homology for schemes was solved by Suslin-Voevodsky. They proved an algebraic version of the Dold-Thom theorem, which states that if X is a pointed CW-complex, there is a weak equivalence

$$\mathrm{Sym}^\infty(X) \xrightarrow{\ \sim\ } \mathrm{Re}(\mathbf{Z}\{\mathrm{Sing}(X)_\bullet\}),$$

where $\mathrm{Sym}^\infty = \mathrm{colim}_n \, \mathrm{Sym}^n(X)$.

Suslin-Voevodsky replaces the free functor $\mathbf{Z} : \mathcal{S} \to \Delta^{\mathrm{op}}\mathrm{Ab}$ with the functor Cor_F of correspondences, which we will come back to shortly. For X a smooth F-scheme, define

$$H_p^{\mathrm{Suslin}}(X; \mathbf{Z}) := H_p(\mathrm{Cor}(\Delta_F^\bullet, X))$$

as the *Suslin homology* of X over F. This homology theory has several good properties. For example, if X is a complex variety,

$$H_p^{\mathrm{Suslin}}(X; \mathbf{Z}/n) \cong H_p^{\mathrm{sing}}(X^{\mathrm{an}}; \mathbf{Z}/n),$$

where X^{an} is the analytic space associated with X.

2.1.3 Correspondences

Correspondences are in some sense "more sensitive" than taking free abelian groups via the functor $\mathbf{Z} : \mathcal{S} \to \Delta^{\mathrm{op}}\mathrm{Ab}$ mentioned above.

Definition 2.1.6. Let $X, Y \in \mathrm{Sm}_F$. An *elementary correspondence* is a closed irreducible subset $W \subseteq X \times Y$, with a finite surjective map $W \to X$. Define $\mathrm{Cor}_F(X, Y)$ as the free abelian group on elementary correspondences.

We then have $\mathrm{Cor}(X, \mathrm{Spec}\, F) = \bigoplus_{X_i} \mathbf{Z}$, where the sum ranges over all connected components X_i of X.

Correspondences give rise to a category of motives over F,

$$\begin{tikzcd} \mathrm{Sm}_F \arrow[rr] \arrow[dr] & & \mathcal{MS}_F \arrow[dl] \\ & \mathrm{DM}_F & \end{tikzcd}$$

Here DM_F is Voevodsky's derived category of motives, whose objects are chain complexes of presheaves on Sm_F with transfers.

2.2 Motives

Let $\mathcal{V}_F$ denote the category of smooth projective varieties over F. Grothendieck and his collaborators speculated on the existence of a category $\mathcal{M}_F$ of so-called pure motives, through which any Weil cohomology theory should factor through.

Example 2.2.1. The following are examples of Weil cohomology theories:

- Betti cohomology: For $F \subseteq \mathbf{C}$, $H^*_B(X) := H^*_{\mathrm{sing}}(X^{\mathrm{an}}; \mathbf{Q})$.
- ℓ-adic cohomology: For ℓ different from the characteristic of F,

$$H^*_\ell(X) := H^*_{\acute{e}t}(X; \mathbf{Q}_\ell) = \varprojlim H^*_{\acute{e}t}(X; \mathbf{Z}/\ell^n) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell.$$

- Crystalline cohomology $H^*_{\mathrm{crys}}(X; \mathbf{Q}_{\mathrm{char} F})$.
- Algebraic de Rham cohomology: For $\mathrm{char}(F) = 0$, $H^*_{\mathrm{dR}}(X) := \mathbf{H}^*(X; \Omega^\bullet_{X/F})$, where $\Omega^\bullet_{X/F}$ is the de Rham complex of X, and $\mathbf{H}$ denotes hypercohomology.

We see that $H^*_?$ takes values in different vector spaces, and that some theories impose restrictions on F. However, there are perhaps more similarities than differences between the above theories:

1. All cohomology theories are contravariant functors.
2. $\dim H^0_?(X) = 1 = \dim H^{2\dim X}_?(X)$.
3. $H^i_?(X) = 0$ for $i < 0$ or $i > 2\dim X$.
4. $H^*_?$ satisfies Poincaré duality and Künneth theorems.

A *pure motive* should thus mean a universal theory satisfying the properties 1-4 above.

Theorem 2.2.2 (Grothendieck)**.** *The category $\mathcal{M}_F$ exists if and only if the standard conjecture on algebraic cycles holds.*

The main idea for constructing a category of motives is as follows. We start out by a geometric category of some kind, e.g., $\mathcal{V}_F$ or Sm_F, and then form a linear category (i.e., the Hom-objects are abelian groups). This linear category can be for example an abelian tensor-category; a triangulated category or a stable ∞-category. For example, Voevodsky's DM_F is a derived version of $\mathcal{M}_F$.

This is inspired by the Beilinson conjectures, which says that there exist complexes of Zariski sheaves $\mathbf{Z}(n)$, for $n \geq 0$, such that:

1. $\mathbf{Z}(0) = \mathbf{Z}$; $\mathbf{Z}(1) = \mathcal{O}^*[-1]$.
2. $\mathbf{H}^n(F; \mathbf{Z}(n)) = H^m_M(F; \mathbf{Z}(n)) \cong K^M_n(F)$.
3. $H^{2n}(X; \mathbf{Z}(n)) \cong \mathrm{CH}^n(X)$, where $\mathrm{CH}^n(X)$ is the Chow group of X.
4. $H^p(X; \mathbf{Z}(n)) = 0$ for all $p < 0$ (Beilinson-Soulé vanishing conjecture).
5. There is a spectral sequence
$$E_2^{p,q} = H^{p,q}(X; \mathbf{Z}(-q)) \implies K_{-p-q}(X),$$
where the righthand side is algebraic K-theory of X.
6. $\mathbf{Z}(n) \otimes^{\mathbf{L}} \mathbf{Z}/\ell \cong \tau_{\leq n} R\pi_* \mu_\ell^{\otimes n}$, where $\pi : (\mathrm{Sm}_F)_{\text{ét}} \to (\mathrm{Sm}_F)_{\mathrm{Zar}}$ is the forgetful functor, τ is a truncation functor and $\mu_\ell^{\otimes n}$ is the étale sheaf of roots of unity on Sm_F. This is known as the Beilinson-Lichtenbaum conjecture.

The Beilinson-Lichtenbaum conjecture is equivalent to the Bloch-Kato conjecture, which contains Milnor's conjecture as the special case $\ell = 2$. Voevodsky et al. have proven every point above except 4.

Chapter 3

Lecture 2

In topology, Brown representability tells us that cohomology theories H^* (i.e., functors satisfying homotopy invariance, wedge axiom and Mayer-Vietoris) are represented by Ω-spectra. This means that, for X a pointed CW-complex, there is an Ω-spectrum $E = (E_n)_{n\geq 0}$ such that

$$H^n(X) = [X, E_n]_*.$$

The Ω-spectra are the fibrant objects in the stable model structure on spectra.

It is desirable to have such a universality also on the motivic side. From the category $\mathcal{MS}_F$ we build the *stable motivic homotopy category* $\mathcal{SH}(F)$ as the category of motivic spectra stabilized with respect to $\mathbf{P}^1$. Note that we have an embedding

$$\Sigma^\infty_{\mathbf{P}^1} : \mathcal{MS}_F \to \mathcal{SH}(F)$$

defined as follows. For $X \in \mathcal{MS}_F$, let $X_+ := X \amalg \operatorname{Spec} F$ be the associated pointed space. Then the spectrum $\Sigma^\infty_{\mathbf{P}^1}(X)$ is defined as having constituent spaces $(\Sigma^\infty_{\mathbf{P}^1}(X))_n := \Sigma^n_{\mathbf{P}^1}(X)$ and identity structure maps. We mention that there is a weak equivalence $\mathbf{P}^1 \cong S^1 \wedge \mathbf{G}_m$.

Several important cohomology theories are represented by objects of $\mathcal{SH}(F)$:

Motivic cohomology. There is a motivic spectrum $\mathsf{M}\mathbf{Z}$ which yields the bigraded cohomology theory known as *motivic cohomology* as follows. Let X be a smooth F-scheme, regarded as a representable sheaf in $\mathcal{MS}_F$. Then we put

$$\mathsf{M}\mathbf{Z}^{p,q}(X) := \operatorname{Hom}_{\mathcal{SH}(F)}(\Sigma^\infty_{\mathbf{P}^1} X_+, S^{p-q} \wedge \mathbf{G}_m^q \wedge \mathsf{M}\mathbf{Z}).$$

Algebraic K-theory. The spectrum KGL represents algebraic K-theory, in the sense of Quillen. More precisely, we have an isomorphism

$$\mathsf{KGL}^{p,q}(X) \cong K_{2q-p}(X).$$

for any $X \in \mathrm{Sm}_F$.

Algebraic cobordism. The spectrum MGL represents the bigraded theory of algebraic cobordism. First introduced by Voevodsky, this algebraic cobordism is thought of as "the universal orientable theory". For $F \subseteq \mathbf{C}$, there is a realization functor $\mathrm{Re}_{\mathbf{C}} : \mathcal{SH}(F) \to \mathcal{SH}$, where $\mathcal{SH}$ is the ordinary stable homotopy category. This functor satisfies $\mathrm{Re}_{\mathbf{C}}(\mathsf{MGL}) = \mathsf{MU}$, where MU is complex cobordism.

3.1 Properties of $\mathcal{SH}(F)$

The categegory $\mathcal{SH}(F)$ has the following properties.

- $\mathcal{SH}(F)$ obtained by inverting $\Sigma_{\mathbf{P}^1}$.
- $\mathcal{SH}(F)$ is a triangulated category, with shift functor $\Sigma_s := - \wedge S^1$.
- There are realization functors

$$\begin{tikzcd} & \mathcal{SH} \\ \mathcal{SH}(F) \arrow[ur, "\mathrm{Re}_{\mathbf{C}}", "F\subseteq\mathbf{C}"'] \arrow[dr, "\mathrm{Re}_{\mathbf{R}}", "F\subseteq\mathbf{R}"'] & \\ & \mathcal{SH}^{\mathbf{Z}/2} \end{tikzcd}$$

  Here $\mathcal{SH}^{\mathbf{Z}/2}$ is the $\mathbf{Z}/2$-equivariant stable homotopy category. The categories $\mathcal{SH}$ and $\mathcal{SH}^{\mathbf{Z}/2}$ are sometimes thought of as "test objects": If we want to investigate whether the category $\mathcal{SH}(F)$ has a certain property, it can often be helpful to first check if the property holds for $\mathcal{SH}$ and $\mathcal{SH}^{\mathbf{Z}/2}$.
- There is a six functor formalism.
- We have a slice filtration, i.e., a filtration

$$\cdots \subseteq \Sigma_{\mathbf{P}^1}\mathcal{SH}(F)^{\mathrm{eff}} \subseteq \mathcal{SH}(F)^{\mathrm{eff}} \subseteq \Sigma_{\mathbf{P}^1}^{-1}\mathcal{SH}(F)^{\mathrm{eff}} \subseteq \cdots \subseteq \mathcal{SH}(F)$$

  of triangulated subcategories of $\mathcal{SH}(F)$. This gives meaning to $\mathsf{M}\mathbf{Z}$ being a universal object, because the slice 0 of the motivic sphere spectrum $\mathbf{1} = \Sigma_{\mathbf{P}^1}^{\infty}(\mathrm{Spec}(F)_+)$, $s_0\mathbf{1}$, satisfies

$$s_0\mathbf{1} = \mathsf{M}\mathbf{Z}.$$

  Via the realization functor $\mathrm{Re}_{\mathbf{C}}$, the motivic sphere spectrum $\mathbf{1}$ is mapped to the topological sphere spectrum. Furthermore, if $\mathcal{E} \in \mathcal{SH}(F)$, then $s_*\mathcal{E} \in \mathsf{M}\mathbf{Z}$-mod.

3.2 Grothendieck's six functor formalism

The category $\mathcal{SH}(F)$ satisfies a six functor formalism. This means that given a scheme map $f : Y \to X$, the following holds:

1. There exist three adjoint functor pairs:

$$f^* : \mathcal{SH}(X) \rightleftarrows \mathcal{SH}(Y) : f_* \text{ (no restrictions on } f\text{);}$$
$$f_! : \mathcal{SH}(Y) \rightleftarrows \mathcal{SH}(X) : f^! \text{ (for } f \text{ separated and of finite type);}$$
$$(\otimes, \underline{\mathrm{Hom}}),\ \text{giving a closed symmetric monoidal structure on } \mathcal{SH}(X).$$

2. There is a natural transformation $\alpha_f : f_! \to f_*$ which is an isomorphism if f is proper.

3. Given a pullback square

$$\begin{array}{ccc} Y' & \xrightarrow{f'} & X' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{f} & X \end{array}$$

with f separated of finite type, there are natural isomorphisms

$$g^* f_! \xrightarrow{\cong} f'_! g'^*,$$

$$g'_* f'^! \xrightarrow{\cong} f^! g_*.$$

4. For any closed immersion $i : Z \to S$ with complementary open immersion j, there is a distinguished triangle of natural transformations

$$j_! j^! \xrightarrow{\alpha'_j} \mathrm{id} \xrightarrow{\alpha_i} i_* i^* \longrightarrow j_! j^! [1],$$

where α'_j is the counit of the adjunction, and α_i denotes the unit.

In addition to the above properties there are purity, duality and exchange isomorphisms, but we will not elaborate on those here.

3.3 $\mathbf{A}^1$-chain connectedness

Back to geometry, there is a notion of "path connectedness" in motivic homotopy theory:

Definition 3.3.1 (Asok-Morel)**.** A scheme $X \in \mathrm{Sm}_F$ is $\mathbf{A}^1$-*chain connected* if for every finitely generated separable field extension E/F we have $X(E) \neq \varnothing$, and given any pair $x, y \in X(E)$, there is a finite sequence of

- E-rational points
$$x = x_0, x_1, \ldots, x_{N-1}, x_N = y \quad \in X(E)$$
- morphisms $f_i : \mathbf{A}^1_E \to X$ such that $f_i(0) = x_{i-1}$ and $f_i(1) = x_i$, $i = 1, \ldots, N$.

We think of this definition as the property that "any two points can be connected by the images of a chain of maps from the affine line".

Remark 3.3.2. The extension E/F need *not* be finite, so this definition is meaningful also for e.g., $\mathbf{C}$. Thus we can for example ask for $\mathbf{C}(t)$ points.

Example 3.3.3. Stable F-rational schemes X (i.e., $X \times \mathbf{P}^N$ is birational to a projective space) are $\mathbf{A}^1$-chain connected.

Theorem 3.3.4 (Asok-Morel)**.** *Suppose that $X \in \mathrm{Sm}_F$ is projective and that E/F is a finitely generated separable extension. Let $\sim$ denote $\mathbf{A}^1$-chain equivalence. Then*

$$\pi_0^{\mathbf{A}^1}(X)(E) \cong X(E)/\sim$$

where $\pi_0^{\mathbf{A}^1}(X)$ *is the Nisnevich sheaf on* Sm_F *assocoiated with the presheaf* $U \mapsto [U, X]_{\mathbf{A}^1}$.

In particular, $\mathbf{A}^1$*-chain connectedness implies* $\mathbf{A}^1$*-connectedness (where* X *is* $\mathbf{A}^1$-connected *if* $\pi_0^{\mathbf{A}^1}(X) \to \operatorname{Spec} F$ *is an isomorphism).*

Remark 3.3.5. Whether $\mathbf{A}^1$-connectedness implies $\mathbf{A}^1$-chain connectedness or not is an open problem.

Example 3.3.6 (Russell cubic)**.** Let X be the Russell cubic in $\mathbf{A}^4_{\mathbf{C}}$, given by

$$x + x^2y + z^2 + t^3 = 0.$$

There is a $\mathbf{C}^* := \mathbf{C} \setminus \{0\}$-action on X, defined by

$$\begin{aligned} \mathbf{C}^* \times X &\to X \\ (\lambda, (x, y, z, t)) &\mapsto (\lambda^6 x, \lambda^{-6} y, \lambda^3 z, \lambda^2 t). \end{aligned}$$

The functions xy and yz^2 are $\mathbf{C}^*$-invariant, so the map

$$\begin{aligned} \phi : X &\to \mathbf{A}^2 \\ (x, y, z, t) &\mapsto (xy, yz^2) \end{aligned}$$

are constant on the $\mathbf{C}^*$-orbits. In fact, ϕ defines a GIT-quotient of X by this action.

The following is known about X:

- X is topologically contractible (this does *not* imply that X is $\mathbf{A}^1$-contractible).
- X has trivial vector bundles (it a theorem that $\mathbf{A}^1$-contractible implies trivial vector bundles).
- X has trivial motivic cohomology. This implies, via the slice filtration, that $\Sigma^\infty_{\mathbf{P}^1} X$ is contractible, i.e., X is stably contractible.

Fasel showed in 2015 that X is $\mathbf{A}^1$-contractible. It is not known if X is $\mathbf{A}^1$-chain connected, or, more generally, which classes of topologically contractible schemes are $\mathbf{A}^1$-chain connected.

3.4 The Yoneda embedding

We recall some category theory that is needed later on.

Definition 3.4.1 (Yoneda embedding, 1954)**.** Suppose $\mathcal{C}$ be a locally small category (i.e., the Hom-objects are sets). The *Yoneda-embedding* is the functor

$$r : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$$

which sends an object in $\mathcal{C}$ to the corresponding representable functor:

$$\mathcal{C} \ni C \mapsto rC := \mathrm{Hom}_{\mathcal{C}}(-, C) \in [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}].$$

Note that a morphism $f : C \to D$ in $\mathcal{C}$ yields a natural transformation

$$rf = \mathrm{Hom}_{\mathcal{C}}(-, f) : \mathrm{Hom}_{\mathcal{C}}(-, C) \to \mathrm{Hom}_{\mathcal{C}}(-, D)$$

by composing with f. It follows that r is indeed a functor.

Lemma 3.4.2 (Yoneda)**.** *Let $\mathcal{C}$ be a locally small category. For any object $C \in \mathcal{C}$ and any functor $F \in [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$, there is an isomorphism*

$$\mathrm{Hom}_{[\mathcal{C}^{\mathrm{op}},\mathrm{Set}]}(rC, F) \cong FC,$$

which is natural in both C and F.

Remark 3.4.3. Naturality in C means that, given any map $f : C \to D$ in $\mathcal{C}$, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Hom}(rC, F) \arrow[r, "\cong"] & FC \\
\mathrm{Hom}(rD, F) \arrow[u, "\mathrm{Hom}(rf,F)"] \arrow[r, "\cong"'] & FD \arrow[u, "Ff"']
\end{tikzcd}$$

Similarly, naturality in F means that given any map $\theta : F \to G$ in $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$, we have a commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(rC, F) \arrow[r, "\cong"] \arrow[d, "\mathrm{Hom}(rC,\theta)"'] & FC \arrow[d, "\theta_C"] \\
\mathrm{Hom}(rC, G) \arrow[r, "\cong"'] & GC
\end{tikzcd}$$

Proof of Lemma 3.4.2. We define an isomorphism

$$\eta_{C,F} : \mathrm{Hom}(rC, F) \to FC,$$

by

$$\eta_{C,F}(\theta) := \theta_C(1_C) \in FC,$$

for $\theta \in \mathrm{Hom}(rC, F)$. To ease notation, let $x_\theta := \theta_C(1_C)$.

We aim to define an inverse to $\eta_{C,F}$. To this end, suppose $a \in FC$, and define a natural transformation $\theta_a : rC \to F$ as follows. Given any $C' \in \mathcal{C}$, define:

$$\begin{aligned}
(\theta_a)_{C'} : \mathrm{Hom}(C', C) &\to FC' \\
f &\mapsto (\theta_a)_{C'}(f) := F(f)(a).
\end{aligned}$$

We must show that θ_a is a natural transformation, i.e., given $f : C'' \to C'$ in $\mathcal{C}$, we must show that the diagram

$$\begin{tikzcd}
\mathrm{Hom}(C'', C) \arrow[r, "(\theta_a)_{C''}"] & FC'' \\
\mathrm{Hom}(C', C) \arrow[u, "\mathrm{Hom}(f,C)"] \arrow[r, "(\theta_a)_{C'}"'] & FC' \arrow[u, "F(f)"']
\end{tikzcd}$$

commutes. So take $g \in (rC)(C') = \mathrm{Hom}(C', C)$, then

$$\begin{aligned}
((\theta_a)_{C''} \circ \mathrm{Hom}(f, C))(g) &= (\theta_a)_{C''}(gf) \\
&= F(gf)(a) \\
&= (F(f) \circ F(g))(a) \\
&= F(f)(\theta_a)_{C'}(g),
\end{aligned}$$

as desired.

We must show that θ_a and x_θ are mutually inverse. Given a natural transformation $\theta : rC \to F$ we have

$$(\theta_{x_\theta})_{C'}(g) = F(g)(x_\theta) = F(g)(\theta_C(1_C))$$

by definition. Since

$$(F(g)) \circ \theta_C = \theta_{C'} \circ rC(g) \tag{3.1}$$

by naturality of θ, we have

$$(\theta_{x_\theta})_{C'}(g) = F(g)(\theta_C(1_C)) = \theta_{C'} \circ rC(g)(1_C) = \theta_{C'}(g),$$

hence $\theta_{x_\theta} = \theta$.

Similarly, for $a \in FC$ we have

$$\begin{aligned} x_{\theta_a} &= (\theta_a)_C(1_C) \\ &= F(1_C)(a) \\ &= 1_{FC}(a) = a, \end{aligned}$$

where the second equality is the definition of θ_a, and the third inequality holds since F is a functor. This shows that $\mathrm{Hom}(rC, F) \cong FC$.

We proceed to show the naturality in both variables. Let $\phi : F \to F'$, then

$$\begin{aligned} \phi_C(x_\theta) &= \phi_C(\theta_C(1_C)) \\ &= (\phi\theta)_C(1_C) \\ &= x_{\phi\theta} \\ &= \eta_{C,F'}(\mathrm{Hom}(rC, \phi)(\theta)), \end{aligned}$$

hence the diagram

$$\begin{tikzcd} \mathrm{Hom}(rC, F) \arrow[r, "\eta_{C,F}"] \arrow[d, "\mathrm{Hom}(rC,\phi)"'] & FC \arrow[d, "\phi_C"] \\ \mathrm{Hom}(rC, F') \arrow[r, "\eta_{C,F'}"'] & F'C \end{tikzcd}$$

is commutative.

For naturality in the variable C, take a morphism $f : C' \to C$. Then

$$\begin{aligned} \eta_{C',F} \circ \mathrm{Hom}(rf, F)(\theta) &= \eta_{C',F}(\theta \circ rf) \\ &= (\theta \circ rf)_{C'}(1_{C'}) \\ &= (\theta_{C'} \circ (rf)_{C'})(1_{C'}) \\ &= \theta_{C'}(f \circ 1_{C'}) \\ &= \theta_{C'}(f) \\ &= \theta_{C'}(1_C \circ f) \\ &= \theta_{C'} \circ (rC)(f)(1_C) \end{aligned}$$

Using Equation (3.1) again, we can write this as

$$\begin{aligned} \theta_{C'} \circ (rC)(f)(1_C) &= F(f) \circ \theta_C(1_C) \\ &= F(f) \circ \eta_{C,F}(\theta). \end{aligned}$$

This shows that the diagram

$$\begin{tikzcd} \mathrm{Hom}(rC', F) \arrow[r, "\eta_{C',F}"] & FC' \\ \mathrm{Hom}(rC, F) \arrow[u, "\mathrm{Hom}(rf,F)"] \arrow[r, "\eta_{C,F}"'] & FC \arrow[u, "Ff"'] \end{tikzcd}$$

is commutative, as desired. □

Corollary 3.4.4. *The Yoneda embedding is full and faithful.*

Proof. Let $C, D \in \mathcal{C}$. Taking $F = rD$ in the Yoneda lemma we immediately have

$$\mathrm{Hom}_{\mathcal{C}}(C, D) = (rD)(C) \cong \mathrm{Hom}_{[\mathcal{C}^{\mathrm{op}},\mathrm{Set}]}(rC, rD).$$

We must show that this isomorphism is induced by r. Let $f \in \mathrm{Hom}_{\mathcal{C}}(C, D)$. Then, by the proof of the Yoneda lemma, f is sent under the above bijection to the natural transformation

$$\theta_f \in \mathrm{Hom}_{[\mathcal{C}^{\mathrm{op}},\mathrm{Set}]}(rC, rD)$$

defined by

$$(\theta_f)_{C'}(g) = f \circ g$$

for $g \in (rC)(C') = \mathrm{Hom}_{\mathcal{C}}(C', C)$. But this says that

$$(\theta_f)_{C'} = (rf)_{C'} : \mathrm{Hom}_{\mathcal{C}}(C', C) \to \mathrm{Hom}_{\mathcal{C}}(C', D),$$

hence $\theta_f = rf$. □

Remark 3.4.5. That $\mathcal{C}$ is locally small does not imply that $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ is locally small. However, the Yoneda lemma ensures that $\mathrm{Hom}_{[\mathcal{C}^{\mathrm{op}},\mathrm{Set}]}(rC, F)$ is always a set.

Corollary 3.4.6 (Yoneda principle)**.** *Suppose C and D are objects in the locally small category $\mathcal{C}$. If $rC \cong rD$ in $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$, then $C \cong D$ in $\mathcal{C}$.*

Thus r is a representation of $\mathcal{C}$ in the presheaf category $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$.

Example 3.4.7. Important for us will be the case when $\mathcal{C} = \mathrm{Sm}_F$. Later on, we will often consider a scheme $X \in \mathrm{Sm}_F$ as a presheaf, i.e., we identify X with its image in $[\mathrm{Sm}_F^{\mathrm{op}}, \mathrm{Set}]$ under the Yoneda embedding.

Example 3.4.8. A category $\mathcal{C}$ is *Cartesian closed* if

- $\mathcal{C}$ has a terminal object
- The product $X \times Y$ of two objects X, Y of $\mathcal{C}$ exists in $\mathcal{C}$
- The exponential X^Y of two objects X, Y of $\mathcal{C}$ exists in $\mathcal{C}$.

The third axiom means that the functor $- \times Y : \mathcal{C} \to \mathcal{C}$ has a right adjoint.

Using the Yoneda principle, we can show that for any objects A, B and C in a Cartesian closed category $\mathcal{C}$, there is an isomorphism

$$(A^B)^C \cong A^{(B \times C)}.$$

Indeed, if $X \in \mathcal{C}$, then

$$\mathrm{Hom}_{\mathcal{C}}(X,(A^B)^C) \cong \mathrm{Hom}_{\mathcal{C}}(X \times C, A^B)$$

by adjointness. Continuing, we obtain:

$$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(X \times C, A^B) &\cong \mathrm{Hom}_{\mathcal{C}}((X \times C) \times B, A)\\ &\cong \mathrm{Hom}_{\mathcal{C}}(X \times (B \times C), A)\\ &\cong \mathrm{Hom}_{\mathcal{C}}(X, A^{(B\times C)}).\end{aligned}$$

This yields $r(A^B)^C \cong rA^{(B\times C)}$, hence $(A^B)^C \cong A^{(B\times C)}$ by the Yoneda principle.

3.4.1 Limits and colimits in functor categories

Let us briefly recall the definition of limits and colimits. Let $\mathcal{C}$ and J be categories, and let $F : J \to \mathcal{C}$ be a diagram in $\mathcal{C}$ (that is, F is a functor). We think of J as the *index category*, and the objects of J are often written as i, $j \in J$. Furthermore, we will write $F_j := F(j) \in \mathcal{C}$ for the values of F.

Definition 3.4.9 (Cones)**.** Given a diagram $F : J \to \mathcal{C}$, a *cone* to F consists of an object $C \in \mathcal{C}$ and a collection of morphisms

$$\psi_j : C \to F_j$$

for each $j \in J$, such that for any morphism $\alpha_{ij} : i \to j$ in J, there following diagram is commutative:

$$\begin{tikzcd} & C \arrow[dl, "\psi_i"'] \arrow[dr, "\psi_j"] & \\ F_i \arrow[rr, "F_{\alpha_{ij}}"'] & & F_j \end{tikzcd}$$

There is a category of cones to F: A morphism

$$\theta : (C, \psi_j) \to (C', \psi'_j)$$

of cones is a morphism $\theta : C \to C'$ in $\mathcal{C}$ making each diagram

$$\begin{tikzcd} C \arrow[rr, "\theta"] \arrow[dr, "\psi_j"'] & & C' \arrow[dl, "\psi'_j"] \\ & F_j & \end{tikzcd}$$

commutative.

Definition 3.4.10. A *limit* of a diagram $F : J \to \mathcal{C}$, written $\phi_i : \lim_{j\in J} F_j \to F_i$, is a terminal object in the category of cones to F.

Thus a limit of F has the universal property that given any cone (C, ψ_j) to F, there is a unique morphism $u : C \to \lim F_j$ rendering the following diagram commutative:

$$\begin{tikzcd} & C \arrow[d, dotted, "\exists! u"] \arrow[ddl, bend right, "\psi_i"'] \arrow[ddr, bend left, "\psi_k"] & \\ & \lim_{j\in J} F_j \arrow[dl, "\phi_i"] \arrow[dr, "\phi_k"'] & \\ F_i \arrow[rr, "F_{\alpha_{ik}}"] & & F_k \end{tikzcd}$$

A limit of F, if it exists, is hence unique up to unique isomorphism.

Exercise 3.4.11. Let $\mathcal{C}$ be a category and C an object of $\mathcal{C}$. Show that the functor $\mathrm{Hom}(C, -)$ preserves limits, and hence that representable functors preserves limits.

Dual to the notion of limits we give the definition of colimits:

Definition 3.4.12. Given a diagram $F : J \to \mathcal{C}$, a *cocone* to F is consists of an object $C \in \mathcal{C}$ and morphisms $\psi_j : F_j \to C$ such that for each $\alpha_{ij} : i \to j$ in J, the diagram

$$\begin{tikzcd} & C & \\ F_i \arrow[ur, "\psi_i"] \arrow[rr, "F_{\alpha_{ij}}"'] & & F_j \arrow[ul, "\psi_j"'] \end{tikzcd}$$

commutes. A morphism of cocones $\theta : (C, \psi_j) \to (C', \psi'_j)$ is a morphism $\theta : C \to C'$ in $\mathcal{C}$ such that $\theta \circ \psi_j = \psi'_j$ for all $j \in J$.

Definition 3.4.13. A *colimit* of a diagram $F : J \to \mathcal{C}$, written $\phi_i : F_i \to \mathrm{colim}_{j\in J} F_j$, is an initial object in the category of cocones to F.

In other words, a colimit of F has the universal property that given any cocone (C, ψ_j) to F, there is a unique morphism $u : \mathrm{colim}\, F_j \to C$ such that the diagram below commutes. Moreover, a colimit of F, if it exists, is unique up to unique isomorphism.

$$\begin{tikzcd} & C & \\ & \mathrm{colim}_{j\in J} F_j \arrow[u, dotted, "\exists! u"] & \\ F_i \arrow[uur, bend left, "\psi_i"] \arrow[ur, "\phi_i"] \arrow[rr, "F_{\alpha_{ik}}"] & & F_k \arrow[uul, bend right, "\psi_k"'] \arrow[ul, "\phi_k"'] \end{tikzcd}$$

Definition 3.4.14. A limit or colimit for $F : J \to \mathcal{C}$ is *small* if the index category J is small.

A category $\mathcal{C}$ is

- *complete* if $\mathcal{C}$ has all small limits
- *cocomplete* if $\mathcal{C}$ has all small colimits
- *bicomplete* if $\mathcal{C}$ is both complete and cocomplete.

Later we wish to put model structures on the category of motivic spaces, but to do so we must assure that our category is bicomplete.

Proposition 3.4.15. *If $\mathcal{C}$ is a locally small category, then $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ has all small limits and colimits.*

Moreover, for any $C \in \mathcal{C}$, the evaluation functor

$$\begin{aligned} \mathrm{ev}_C : [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}] &\to \mathrm{Set} \\ F &\mapsto F(C) \end{aligned}$$

preserves all limits and colimits.

Proof. Let J be a small category and suppose we are given a functor $F : J \to [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$. If the limit of F exists, it is a functor

$$\lim_{j\in J} F_j : C^{\mathrm{op}} \to \mathrm{Set}.$$

We show that *if* the limit exists as an object in $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$, then it is defined pointwise.

By the Yoneda lemma we have, for any $C \in \mathcal{C}$,

$$\left(\lim_{j\in J} F_j\right)(C) \cong \mathrm{Hom}(rC, \lim F_j).$$

Since representable functors preserve limits by Exercise 3.4.11 we have

$$\begin{aligned}\mathrm{Hom}(rC, \lim F_j) &\cong \lim \mathrm{Hom}(rC, F_j)\\ &= \lim(F_j(C)).\end{aligned}$$

Therefore, we are forced to define

$$\left(\lim_{j\in J} F_j\right)(C) := \lim_{j\in J}(F_j(C)).$$

Thus the completeness of $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ follows from the completeness of Set.

From this it follows also that ev_C preserves limits, since

$$\mathrm{ev}_C(\lim F_j) = (\lim F_j)(C) = \lim(F_j C).$$

$\square$

Example 3.4.16 (The Grothendieck-construction)**.** Suppose $\mathcal{C}$ is a small category. Given a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, there is a small index category known as the *category of elements*, written $\int_{\mathcal{C}} F$. It is constructed as follows:

- The objects of $\int_{\mathcal{C}} F$ are pairs (x, C), for $C \in \mathcal{C}$ and $x \in F(C)$.
- The morphisms $h : (x, C) \to (x', C')$ are morphisms $h : C' \to C$ in $\mathcal{C}$ satisfying

$$F(h)(x) = x'.$$

The category $\int_{\mathcal{C}} F$ is small because $\mathcal{C}$ is small. Furthermore, there is a "projection functor"

$$\pi : \int_{\mathcal{C}} F \to \mathcal{C}$$

defined by

$$\begin{aligned}(x, C) &\mapsto C,\\ (h : (x, C) \to (x', C')) &\mapsto h.\end{aligned}$$

Proposition 3.4.17. *If $\mathcal{C}$ is a small category, then every object $F \in [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ is a colimit of representable functors, i.e., there are objects $C_j \in \mathcal{C}$ $(j \in J)$ such that*

$$\mathrm{colim}_{j\in J}\, rC_j \cong F.$$

More precisely, there is a canonical choice of index category J and a functor $\pi : J \to \mathcal{C}$ such that there is a natural isomorphism $\mathrm{colim}_J\, r \circ \pi \cong F$.

Proof. Our index category J will be the category of elements of Example 3.4.16, i.e., we let

$$J := \int_{\mathcal{C}} F.$$

Note that, by the Yoneda lemma, there is a bijection between elements $x \in F(C)$ and natural transformations $x : rC \to F$. We wish to define a cocone $(F, \phi_{(x,C)})$ to $r \circ \pi$. Note that, for $(x, C) \in \int_{\mathcal{C}} F$, the $\phi_{(x,C)}$ will be morphisms

$$\phi_{(x,C)} : (r \circ \pi)(x, C) \to F.$$

But $(r \circ \pi)(x, C) = rC$, so we can simply define

$$\phi_{(x,C)} := x : rC \to F,$$

where we identify $x \in \mathrm{Hom}(rC, F)$ with $x \in FC$. We show that $(F, \phi_{(x,C)})$ is initial in the category of cocones to $r\pi$. So assume $(G, \psi_{(x,C)})$ is another cocone to $r\pi$. We must construct a unique natural transformation $\theta\colon F \to G$ making the standard diagrams commutative. For any $C \in \mathcal{C}$, define

$$\theta_C : FC \to GC$$

by $\theta_C(x) := \psi_{(x,C)}$, where we again have identified $\psi_{(x,C)} : rC \to G$ with $\psi_{(x,C)} \in GC$. To show that this map is unique, let $\eta : F \to G$ be a natural transformation commuting with the maps $x : yC \to F$. Then the Yoneda lemma yields that $\eta \circ x = \eta_{(x,C)} = \eta \circ x$. □

Proposition 3.4.18. *If $\mathcal{C}$ is a small category, the Yoneda embedding $r : \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ is the free cocompletion of $\mathcal{C}$. This means that given any cocomplete category $\mathcal{D}$ with a functor $F : \mathcal{C} \to \mathcal{D}$, there is a limit preserving functor*

$$F_! : [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}] \to \mathcal{D}$$

such that the following diagram commutes up to natural isomorphism:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "r"] \arrow[d, "F"'] & {[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]} \arrow[dl, dotted, "\exists F_!"] \\ \mathcal{D} & \end{tikzcd}$$

Moreover, the functor $F_!$ is unique up to natural isomorphism.

Remark 3.4.19. Since F is a composition of two functors that preserve colimits, it follows that F preserves colimits.

Thus $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ is thought of as the "initial functor category".

Proof of Proposition 3.4.18. We only define the functor $F_!$. For $G \in [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$, by Proposition 3.4.17 we may write

$$G \cong \mathrm{colim}_{j \in J}\, rA_j,$$

for $J = \int_{\mathcal{C}} G$ and $A_j \in \mathcal{C}$. We then define

$$F_!(G) := \mathrm{colim}_{j \in J}\, F(A_j).$$

Since $\mathcal{D}$ is cocomplete, $F_!(G) \in \mathcal{D}$. □

Bibliography

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