Online Supplementary Materials for "Designing High-Frequency Market Liquidity Measures with Applications to Monetary Policy"

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Abstract

This note contains supplementary materials for Li et al. (2026). Appendix SA provides additional simulation results to complement those in the main paper. Appendix SB presents the detailed proofs of the main theoretical results in Li et al. (2026). References to equations, theorems, lemmas, and figures prefixed with SA or SB (e.g., Figure SA.1, Lemma SB.3) pertain to objects within this supplement; unprefixed numbering denotes corresponding objects in the main paper.

Appendix SA Additional Simulations

First, we present additional simulation results to examine the finite-sample performance of the spot liquidity estimator $\mathcal{S}(m)_t^n$ under different choices of the local bandwidth l_n. Specifically, we consider $l_n \in \{2000, 3000\}$ while keeping other simulation settings identical to those in Appendix H. Figure SA.1 and Figure SA.2 illustrate the estimation results for $m \in \{1, 3, 5\}$ with $l_n = 2000$ and $l_n = 3000$, respectively. The accuracy of the spot liquidity estimator $\mathcal{S}(m)_t^n$ remains satisfactory across different

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choices of l_n, demonstrating the robustness of our estimation method to the selection of the local bandwidth l_n.

Next, we examine the finite-sample performance of the spot liquidity estimator $\mathcal{S}(m)_t^n$ and $\mathrm{DOFI}(m)_t^n$ when employing the Toeplitz correction method proposed in Appendix F. Figure SA.3 and Figure SA.4 illustrate the estimation results for $\mathcal{S}(m)_t^n$ ($m \in \{1,3,5\}$), and the Q-Q plots for $\mathcal{S}(m)_t^n$ ($m \in \{1,5\}$) and $\mathrm{DOFI}(m)_t^n$ ($m \in \{5,10\}$), respectively. All the parameter settings remain the same as those in Appendix H, except that both the spot estimation and the estimation of the asymptotic variance of $\mathrm{DOFI}(m)_t^n$ incorporate the Toeplitz correction with $\overline{m} = m + 5$. We can see that the spot liquidity estimator $\mathcal{S}(m)_t^n$ performs well with the Toeplitz correction, and the finite-sample distributions of both $\mathcal{S}(m)_t^n$ and $\mathrm{DOFI}(m)_t^n$ align closely with the standard normal distribution after applying the Toeplitz correction, demonstrating the robustness of the proposed correction method.

Lastly, to deal with the time-varying autocorrelation in order flows, we propose a data-driven method to select $k_{n,p}$ based on the pre-estimated value of $\mathrm{DOFI}(5)_t^n$. The selection rule is described in Algorithm 2. The order flow follows a time-varying AR(1), defined by $\chi_i^n = \rho_i \chi_{i-1}^n + \sqrt{1-\rho_i^2} e_i$ with $\{e_i\}_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$ and the autoregressive coefficient $\rho_i = 0.35 + 0.15\cos(2\pi t_i^n)$. We further set $\gamma_t' = 1 + 0.05\cos(2\pi t)$. All other parametric settings remain the same as those in Appendix H. Figure SA.5 illustrates the estimation results for $\mathcal{S}(m)_t^n$ ($m \in \{1,3,5\}$). It shows that our spot liquidity estimator $\mathcal{S}(m)_t^n$ performs well even when the order flow autocorrelation is time-varying using a data-driven selection method for the differencing parameter $k_{n,p}$.

Appendix SB Mathematical Proofs

In the following proofs, we follow Jacod et al. (2017) and Li and Linton (2022a) to let $\Omega = \Omega^{(0)} \times \Omega^{(1)}$, $\mathcal{F} = \mathcal{F}^{(0)} \otimes \mathcal{G}$ and $\mathbb{P} = \mathbb{P}^{(0)} \otimes \mathbb{P}^{(1)}$, where $(\Omega^{(0)}, \mathcal{F}^{(0)}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}^{(0)})$ is the filtered probability space where X, α, γ and the observation times are defined. Let K denote a constant independent of n, which can change across lines or within one line.

For any processes V and U, we define $f(V,U,j;k)_i^n := \left(V_i^n - V_{i-k}^n\right)\left(U_{i+j}^n - U_{i+j+k}^n\right)$ and $\overline{f}(V,U,j)_i^n := \frac{1}{p_n}\sum_{p=0}^{p_n-1} f(V,U,j;k_{p,n})_i^n$, where $k_{p,n} = k_n + p$. When $V = U$, we further simplify the notations to $f(V,j;k)_i^n$ and $\overline{f}(V,j)_i^n$, respectively. Note that $f(V,U,j;k)_i^n$ reduces to $f(j;k)_i^n$ defined in the main text for $V = U = Y$. For integer $l_n \geq 2\overline{k}_n + j$, let $\mathbb{I}(j)_t^n := \{n_t + \overline{k}_n, \ldots, n_t + l_n - \overline{k}_n - j\}$ be an index set, where $\overline{k}_n = k_n + p_n$. We also denote the set of indices $\mathbb{I}_t^n := \{n_t + 1, \ldots, n_t + l_n\}$.

Assumption (K)**.** *Let Z be an Itô semimartingale represented by (20). The component processes b^Z, σ^Z, and δ^Z are bounded with $\delta^Z(\omega,t,z) \leq J^Z(z)$ for some bounded function $J^Z(z)$*

satisfying $\int_{\mathbb{R}}(J^Z(z))^2\lambda(dz) < \infty$.

Using a classical localization procedure, it suffices to assume the following stronger assumption to replace Assumption (H), (NL-v), and (O):

Assumption (S-HONL). *Let Assumption (H), Assumption (O) and Assumption (NL-v) hold, and we further assume that Assumption (K) holds for processes X, α and γ, and the process $1/\alpha$ is bounded.*

SB.1 Proof of Technical Lemmas

Let Q be a real-valued function defined on $\mathbb{R}^h$. It satisfies the Lipschitz condition with polynomial growth of degree M if there exists a constant $K > 0$ such that for any $x = (x_1, \dots, x_h), y = (y_1, \dots, y_h) \in \mathbb{R}^h$, we have $|Q(x) - Q(y)| \leq K\left(1 + |x|_1^M + |y|_1^M\right)|x - y|_1$, where $|x|_1 := \sum_{i=1}^h |x_i|$. Examples of such functions include $Q(x_1, x_2) = x_1 x_2$ and $Q(x) = x^4$. In the subsequent Lemma SB.1, we assume the two functions Q and Q' are locally Lipschitz with polynomial growth of degree M, with $M \leq 3$, which is sufficient for our analysis.

We introduce some extra notations. Let $\mathbf{I}_h = \{k_q \in \mathbb{Z} : q = 1, 2, \dots, h\}$ be a tuple of integers of size h in an increasing order. Denote $\overline{q}_h := k_h - k_1$. For any integer i, we define $i \oplus \mathbf{I}_h := \{i + k_q : k_q \in \mathbf{I}_h, q = 1, 2, \dots, h\}$. Denote $\mathbb{Q}(\mathbf{I}_h)_{i-} := (i - k_h) \oplus \mathbf{I}_h, \mathbb{Q}(\mathbf{I}_h)_{i+} := (i - k_1) \oplus \mathbf{I}_h$. Thus, $\mathbb{Q}(\mathbf{I}_h)_{i-}$ (resp. $\mathbb{Q}(\mathbf{I}_h)_{i+}$) shifts the indices of $\mathbf{I}_h$ such that the largest (resp. smallest) index equals i.

Lemma SB.1. *Given two positive integers h, h', let $\xi_i^n := Q(\chi_k^n : k \in \mathbb{Q}(\mathbf{I}_h)_{i-})$ and $\xi_i'^n := Q'(\chi_k^n : k \in \mathbb{Q}(\mathbf{I}_{h'})_{i+})$. Then, for any positive integer j, we have*

$$|\mathbb{E}\big(\xi_i^n \xi_{i+j}'^n\big) - \mathbb{E}(\xi_i^n)\mathbb{E}\big(\xi_{i+j}'^n\big)| \leq K(j^{-v} + (j \vee \overline{q}_h \vee \overline{q}_{h'})\Delta_n).$$

Proof of Lemma SB.1. We assume ξ^n, ξ'^n have mean zero without loss of generality. Denote their stationary approximations by $\widetilde{\xi}_i^n := Q(\widetilde{\chi}(k)_{t_i^n} : k \in \mathbb{Q}(\mathbf{I}_h)_{i-})$; $\widetilde{\xi}_{i+j}'^n := Q'(\widetilde{\chi}(k)_{t_i^n} : k \in \mathbb{Q}(\mathbf{I}_{h'})_{(i+j)+})$. According to Dahlhaus et al. (2019), Assumption (NL-v) can be applied to $\xi_i^n, \xi_{i+j}^n, \widetilde{\xi}_i^n$ and $\widetilde{\xi}_{i+j}^n$. Thus, we have $\|\xi_i^n\|_2 \vee \|\widetilde{\xi}_{i+j}'^n\|_2 \leq K$, $\|\xi_{i+j}'^n - \widetilde{\xi}_{i+j}'^n\|_2 \leq K(j \vee \overline{q}_{h'})\Delta_n$, and $\|\xi_i^n - \widetilde{\xi}_i^n\|_2 \leq K\overline{q}_h\Delta_n$. Then, apply Cauchy-Schwartz inequality, we have

$$|\mathbb{E}\big(\xi_i^n \xi_{i+j}'^n\big) - \mathbb{E}(\widetilde{\xi}_i^n \widetilde{\xi}_{i+j}')| \leq \|\xi_i^n\|_2 \|\xi_{i+j}'^n - \widetilde{\xi}_{i+j}'^n\|_2 + \|\widetilde{\xi}_{i+j}'^n\|_2 \|\xi_i^n - \widetilde{\xi}_i^n\|_2 \leq K(j \vee \overline{q}_h \vee \overline{q}_{h'})\Delta_n.$$

Next, let $\mathcal{G}_i' := \sigma(\dots, e_{i-1}, e_i)$ be the filtration generated by the innovations upon the i-th observation. Let $\mathcal{P}_k(\cdot) := \mathbb{E}(\cdot|\mathcal{G}_k') - \mathbb{E}\big(\cdot|\mathcal{G}_{k-1}'\big)$ be the martingale difference operator.

Let $\widetilde{\xi}(k)_i^{n,*}$ be a coupled version of $\widetilde{\xi}_i^n$ with e_k replaced by e_k^* which is an independent copy of e_k. Since e_k^* is independent of e_k, we have $\mathbb{E}\left(\widetilde{\xi}(k)_i^{n,*}|\mathcal{G}'_k\right) = \mathbb{E}\left(\widetilde{\xi}_i^n|\mathcal{G}'_{k-1}\right)$. Thus, by Jensen's inequality, we have

$$\|\mathcal{P}_k(\widetilde{\xi}_i^n)\|_2 = \left\|\mathbb{E}\left(\widetilde{\xi}_i^n - \widetilde{\xi}(k)_i^{n,*}|\mathcal{G}'_k\right)\right\|_2 \le \|\widetilde{\xi}_i^n - \widetilde{\xi}(k)_i^{n,*}\|_2 \le Kd(i-k,2).$$

Similarly, we have $\|\mathcal{P}_k(\widetilde{\xi}'^n_{i+j})\|_2 \le Kd(i+j-k,2)$. Then, use the projection decomposition and the Cauchy-Schwarz inequality and let $m = i-k$, we have

$$\begin{aligned}\left|\mathbb{E}\left(\widetilde{\xi}_i^n\widetilde{\xi}'^n_{i+j}\right)\right| &= \left|\sum_{k=-\infty}^{i}\mathbb{E}\left(\mathcal{P}_k(\widetilde{\xi}_i^n)\mathcal{P}_k(\widetilde{\xi}'^n_{i+j})\right)\right| \le \sum_{k=-\infty}^{i}\|\mathcal{P}_k(\widetilde{\xi}_i^n)\|_2\|\mathcal{P}_k(\widetilde{\xi}'^n_{i+j})\|_2 \\ &\le K\sum_{m=0}^{\infty}d(m,2)d(m+j,2) \le Kj^{-v}.\end{aligned}$$

This completes the proof. $\square$

Now, we make the following decomposition:

$$\widehat{\Gamma}(j)_t^n - \Gamma(j)_t := G(j)_t^n + G'(j)_t^n + H(j)_t^n + D(j)_t^n,$$

where

$$\begin{aligned}G(j)_t^n &:= \frac{1}{l_n}\sum_{i\in\mathbb{I}_t^n}\gamma_t^2\left(\chi_i^n\chi_{i+j}^n - r(j)_t\right), \quad G'(j)_t^n := \frac{1}{l_n}\sum_{i\in\mathbb{I}_t^n}\left(\gamma_i^n\gamma_{i+j}^n - (\gamma_i^n)^2\right)\chi_i^n\chi_{i+j}^n, \\ H(j)_t^n &= \frac{1}{l_n}\sum_{i\in\mathbb{I}_t^n}\left((\gamma_i^n)^2 - \gamma_t^2\right)r(j)_t, \quad D(j)_t^n = \frac{1}{l_n}\sum_{i\in\mathbb{I}_t^n}((\gamma_i^n)^2 - \gamma_t^2)(\chi_i^n\chi_{i+j}^n - r(j)_t).\end{aligned} \tag{SB.1}$$

Next, we present a key limiting theorem. Its proof is an immediate consequence of Lemma SB.2, Lemma SB.3, and Lemma SB.4, which we state and prove later.

Theorem SB.1. *Let Assumption (S-HONL) hold, and further assume Condition B.4. Define that $\mathbf{G}_t^n := (G(j)_t^n, G(j')_t^n)$, $\mathbf{H}_t^n := (H(j)_t^n, H(j')_t^n)$, we have*

$$u_n\left(\mathbf{G}_t^n, \mathbf{H}_t^n\right) \xrightarrow{\mathcal{L}_s} \left(\mathbf{G}_t, \mathbf{H}_t\right), \tag{SB.2}$$

where $\mathbf{G}_t := (G(j)_t, G(j')_t)$, $\mathbf{H}_t := (H(j)_t, H(j')_t)$ are defined on an extension of $(\Omega, \mathcal{F}, \mathbb{P})$, which are two centered Gaussian random vectors, conditional on $\mathcal{F}$, having the following covariance structure,

$$\widetilde{\mathbb{E}}(G(j)_tG(j')_t|\mathcal{F}) = \frac{1}{1+c}\cdot\gamma_t^4\cdot s(j,j')_t, \quad \widetilde{\mathbb{E}}(H(j)_tH(j')_t|\mathcal{F}) = \frac{c}{1+c}\cdot\frac{\widetilde{\sigma}_t^2 r(j)_t r(j)'_t}{3\alpha_t}, \tag{SB.3}$$

$$\widetilde{\mathbb{E}}(G(j)_t H(j)_t|\mathcal{F}) = \widetilde{\mathbb{E}}(G(j)_t H(j')_t|\mathcal{F}) = \widetilde{\mathbb{E}}(G(j')_t H(j)_t|\mathcal{F}) = \widetilde{\mathbb{E}}(G(j')_t H(j')_t|\mathcal{F}) = 0. \tag{SB.4}$$

Lemma SB.2. *Let Assumption (S-HONL) hold. Assume $l_n \to \infty$, and $l_n\Delta_n \to 0$ as $n \to \infty$. We have*

$$u_n(H(j)_t^n - H'(j)_t^n) \xrightarrow{\mathbb{P}} 0, \tag{SB.5}$$

where

$$H'(j)_t^n := \frac{1}{l_n}\sum_{i\in\mathbb{I}_t^n}\left(\int_t^{t_i^n}\widetilde{\sigma}_u \mathrm{d}W_u\right) r(j)_t. \tag{SB.6}$$

Proof of Lemma SB.2. The process γ^2 can be expressed as follows:

$$\gamma_t^2 = \gamma_0^2 + \int_0^t \widetilde{b}_s \mathrm{d}s + \int_0^t \widetilde{\sigma}_s \mathrm{d}W_s + \left(\widetilde{\vartheta}\mathbf{1}_{\{|\widetilde{\vartheta}|\le 1\}}\right)\star\left(\mathfrak{p}-\mathfrak{q}\right)_t + \left(\widetilde{\vartheta}\mathbf{1}_{\{|\widetilde{\vartheta}|>1\}}\right)\star\mathfrak{p}_t.$$

Under Assumption (H), it can also be represented as $\gamma_t^2 = \gamma_0^2 + \int_0^t \widetilde{b}'_s \mathrm{d}s + \int_0^t \widetilde{\sigma}_s \mathrm{d}W_s + M_t$, where the bounded process $\widetilde{b}'_s$ is defined as $\widetilde{b}'_s := \widetilde{b}_s + \int_{|\widetilde{\vartheta}|>1}\widetilde{\vartheta}(s,z)\lambda(\mathrm{d}z)$, and $M := \widetilde{\vartheta}\star(\mathfrak{p}-\mathfrak{q})$.

Next, for $\epsilon \in (0,1]$, we define $\Omega(\epsilon)_t^n := \left\{|\Delta M_s| \le \epsilon, \forall\, s \in \left(t, t_{n_t+l_n}^n\right]\right\}$. Since the interval $\left(t, t_{n_t+l_n}^n\right]$ becomes empty as $n \to \infty$, we have $\mathbb{P}(\Omega(\epsilon)_t^n) \to 1$ as $n \to \infty$ for any $\epsilon \in (0,1]$. On $\Omega(\epsilon)_t^n$, we further have $\gamma_t^2 = \gamma_0^2 + \int_0^t \widetilde{b}'(\epsilon)_s \mathrm{d}s + \int_0^t \widetilde{\sigma}_s \mathrm{d}W_s + M(\epsilon)_t + J(\epsilon)_t$, with

$$\widetilde{b}'(\epsilon)_s := \widetilde{b}'_s - \int_{\{\Gamma(z)>\epsilon\}}\widetilde{\vartheta}(s,z)\lambda(\mathrm{d}z),\;\; M(\epsilon)_t := \widetilde{\vartheta}\mathbf{1}_{\{\Gamma(z)\le\epsilon\}}\star(\mathfrak{p}-\mathfrak{q})_t,\;\; J(\epsilon)_t := \widetilde{\vartheta}\mathbf{1}_{\{\Gamma(z)>\epsilon\}}\star\mathfrak{p}_t.$$

Now, we make the decomposition $(\gamma_i^n)^2 - \gamma_t^2 - \int_t^{t_i^n}\widetilde{\sigma}_u \mathrm{d}W_u = \sum_{\ell=1}^3 \mathfrak{A}(\ell)_i^n$ for any $i \in \mathbb{I}_t^n$, where

$$\mathfrak{A}(1)_i^n := \int_t^{t_i^n}\widetilde{b}'(\epsilon)_s \mathrm{d}s, \quad \mathfrak{A}(2)_i^n := M(\epsilon)_{t_i^n} - M(\epsilon)_t, \quad \mathfrak{A}(3)_i^n := J(\epsilon)_{t_i^n} - J(\epsilon)_t.$$

Denote $\mathcal{K}_i^n := \sigma\{\Delta(n,m) : n_t+1 \le m \le i\}$. Apply the estimate (8.9) in Jacod and Todorov (2010), we have $\mathbb{E}\left(|\mathfrak{A}(1)_i^n|\mathbf{1}_{\{\Omega(\epsilon)_t^n\}}|\mathcal{K}_i^n\right) \le K(i-n_t)\Delta_n\epsilon^{-((r-1)\vee 0)}$ for any $r \in [0,2]$. Let $\phi(\epsilon) := \int_{\{\Gamma(z)\le\epsilon\}}\Gamma^2(z)\lambda(\mathrm{d}z)$. Then, by the Burkholder-Davis-Gundy inequality, we have

$$\mathbb{E}\left((\mathfrak{A}(2)_i^n)^2\mathbf{1}_{\{\Omega(\epsilon)_t^n\}}|\mathcal{F}_t \vee \mathcal{K}_i^n\right) \le K(i-n_t)\Delta_n\phi(\epsilon).$$

Next, we have for any $r \in [0,2]$,

$$\mathbb{E}\big(|\mathfrak{A}(3)_i^n| \mathbf{1}_{\{\Omega(\epsilon)_t^n\}} |\mathcal{K}_i^n\big) \leq K(i-n_t)\Delta_n \int_{\{\Gamma(z)>\epsilon\}} \lambda(\mathrm{d}z) \leq K(i-n_t)\Delta_n \epsilon^{-r}.$$

The above estimates yield the following

$$\frac{u_n}{l_n} \sum_{i\in\mathbb{I}_t^n} \sum_{\ell=1}^{3} \mathbb{E}\big(|\mathfrak{A}(\ell)_i^n| \mathbf{1}_{\{\Omega(\epsilon)_t^n\}}\big) \leq K\sqrt{l_n\Delta_n}(\epsilon^{-((r-1)\vee 0)} \vee \epsilon^{-r}) \bigvee \sqrt{\phi(\epsilon)}.$$

Since $l_n\Delta_n \to 0$, $\mathbb{P}(\Omega(\epsilon)_t^n) \to 1$ as $n\to\infty$ for any $\epsilon \in (0,1]$, and $\phi(\epsilon) \to 0$ as $\epsilon \to 0$, we have $\lim_{\epsilon\to 0}\limsup_{n\to\infty} u_n \mathbb{E}\big(|H(j)_t^n - H'(j)_t^n|\, \mathbf{1}_{\{\Omega(\epsilon)_t^n\}}\big) = 0$. The proof is complete. □

Lemma SB.3. *Let the assumptions of Theorem SB.1 hold. If $v > 1$, $l_n \to \infty$ and $l_n^{3/2}\Delta_n \to 0$, we have*

$$u_n\left(\frac{\widetilde{\sigma}_t}{\sqrt{\alpha_t}}\widetilde{H}(j)_t^n - H(j)_t^n\right) \xrightarrow{\mathbb{P}} 0, \quad u_n D(j)_t^n \xrightarrow{\mathbb{P}} 0, \quad u_n G'(j)_t^n \xrightarrow{\mathbb{P}} 0, \qquad \text{(SB.7)}$$

where

$$\widetilde{H}(j)_t^n := \frac{\sqrt{\alpha_t}}{l_n}\sum_{i\in\mathbb{I}_t^n}\left(W_{t_i^n} - W_t\right) r(j)_t. \qquad \text{(SB.8)}$$

Proof of Lemma SB.3. Let $\overline{H}'(j)_t^n := \frac{1}{l_n}\sum_{i\in\mathbb{I}_t^n}\widetilde{\sigma}_t\left(W_{t_i^n} - W_t\right) r(j)_t$. Thus, we have (recall $H'(j)_t^n$ is defined in (SB.6))

$$H'(j)_t^n - \overline{H}'(j)_t^n = \frac{r(j)_t}{l_n}\sum_{i\in\mathbb{I}_t^n}(n_t + l_n - i)\int_{t_i^n}^{t_{i+1}^n}(\widetilde{\sigma}_s - \widetilde{\sigma}_t)\mathrm{d}W_s,$$

which, by the orthogonality of the summands, yields

$$\mathbb{E}\Big(u_n^2(H'(j)_t^n - \overline{H}'(j)_t^n)^2\Big) \leq K l_n\Delta_n(l_n^2\Delta_n \wedge 1).$$

Thus, $u_n\left(H'(j)_t^n - \overline{H}'(j)_t^n\right) \xrightarrow{\mathbb{P}} 0$, or equivalently, $u_n\left(H'(j)_t^n - \frac{\widetilde{\sigma}_t}{\sqrt{\alpha_t}}\widetilde{H}(j)_t^n\right) \xrightarrow{\mathbb{P}} 0$, which, combined with (SB.5), leads to the first convergence in (SB.7).

Now let $\mathfrak{B}_i^n := \left((\gamma_i^n)^2 - \gamma_t^2\right)\left(\chi_i^n\chi_{i+j}^n - r(j)_t\right)$, we have

$$(D(j)_t^n)^2 = \frac{1}{l_n^2}\sum_{i\in\mathbb{I}_t^n}(\mathfrak{B}_i^n)^2 + \frac{2}{l_n^2}\sum_{k=1}^{l_n-1}\sum_{i=n_t+1}^{n_t+l_n-k}\mathfrak{B}_i^n\mathfrak{B}_{i+k}^n. \qquad \text{(SB.9)}$$

For $k \leq l_n - 1$, by the independence of $\mathcal{F}^{(0)}$ and $\mathcal{G}$, successive conditioning, Lemma SB.1, and the estimate (A.6) in Jacod et al. (2017), we have

$$\sup_{i\in\mathbb{I}_t^n}\left|\mathbb{E}\left(\mathfrak{B}_i^n\mathfrak{B}_{i+k}^n|\mathcal{F}_t\right)\right|\leq K\left((k^{-v}\vee l_n\Delta_n)l_n\Delta_n+k(k^{-v}\vee l_n\Delta_n)l_n^{1/2}\Delta_n^{3/2}\right),$$

which yields $u_n^2\mathbb{E}((D(j)_t^n)^2)\leq Ku_n^2(\Delta_n\vee l_n^{-v+\frac{3}{2}}\Delta_n^{\frac{3}{2}}\vee l_n^2\Delta_n^2)\leq Ku_n^2(\Delta_n\vee l_n^2\Delta_n^2)\to 0$, under the conditions $l_n\to\infty$, $l_n^{3/2}\Delta_n\to 0$, and $v>1$. Now let $\mathfrak{C}_i^n:=(\gamma_i^n\gamma_{i+j}^n-(\gamma_i^n)^2)\chi_i^n\chi_{i+j}^n$, by successive conditioning and the Cauchy-Schwartz inequality, we have $\left|\mathbb{E}\left(\mathfrak{C}_i^n\mathfrak{C}_{i+k}^n\right)\right|\leq K\Delta_n^{3/2}$ for $k\geq j$ and $\left|\mathbb{E}\left(\mathfrak{C}_i^n\mathfrak{C}_{i+k}^n\right)\right|\leq K\Delta_n$ for $k<j$. Using a similar decomposition as in (SB.9), we obtain $\mathbb{E}((G'(j)_t^n)^2)\leq K\left(l_n^{-1}\Delta_n\bigvee\Delta_n^{3/2}\right)$, which yields the last convergence in (SB.7). $\square$

Lemma SB.4. *For any pairs of $(j_1,j_2),(m_1,m_2)\in\mathbb{N}^*\times\mathbb{N}^*$, let*

$$\widetilde{\mathbf{H}}_t^n:=\left(\widetilde{H}(j_1)_t^n,\widetilde{H}(j_2)_t^n\right),\quad \widetilde{\mathbf{G}}_t^n:=\left(\widetilde{G}(m_1)_t^n,\widetilde{G}(m_2)_t^n\right),$$

where $\widetilde{H}(j)_t^n$ is defined in (SB.8) and $\widetilde{G}(j)_t^n:=\frac{1}{l_n}\sum_{i\in\mathbb{I}_t^n}\left(\chi_i^n\chi_{i+j}^n-r(j)_t\right)$. Then, $u_n(\widetilde{\mathbf{H}}_t^n,\widetilde{\mathbf{G}}_t^n)$ jointly converges stably in law to a pair of centered Gaussian vectors $(\widetilde{\mathbf{H}}_t,\widetilde{\mathbf{G}}_t)$ defined on an extension $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$ of the original probability space $(\Omega,\mathcal{F},\mathbb{P})$. Conditional on $\mathcal{F}$, the limiting variables $\widetilde{\mathbf{H}}_t:=(\widetilde{H}(j_1)_t,\widetilde{H}(j_2)_t)$ and $\widetilde{\mathbf{G}}_t:=(\widetilde{G}(m_1)_t,\widetilde{G}(m_2)_t)$ have the following covariance structure:

$$\widetilde{\mathbb{E}}\left(\widetilde{H}(j_1)_t\widetilde{H}(j_2)_t\right)=\frac{c}{1+c}\cdot\frac{r(j_1)_tr(j_2)_t}{3},\quad \widetilde{\mathbb{E}}\left(\widetilde{G}(m_1)_t\widetilde{G}(m_2)_t\right)=\frac{1}{1+c}\cdot s(m_1,m_2)_t;$$
$$\widetilde{\mathbb{E}}\left(\widetilde{H}(j_\ell)_t\widetilde{G}(m_{\ell'})_t\right)=0;\quad \text{for } \ell,\ell'\in\{1,2\},$$

where $s(m_1,m_2)_t$ is defined in Theorem 4.2.

Proof of Lemma SB.4. Let $\overline{H}(j)_t^n:=\frac{1}{l_n}\sum_{i=n_t+1}^{n_t+l_n}(n_t+l_n-i)\sqrt{\Delta_n}U_ir(j)_t$, where $\{U_i\}_i$ is a sequence of independent standard Gaussian random variables defined on an auxiliary space $(\Omega',\mathcal{F}',\mathbb{P}')$, such that the extension $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$ is the product filtered extension of $(\Omega,\mathcal{F},\mathbb{P})$ and $(\Omega',\mathcal{F}',\mathbb{P}')$. We first establish a *convergence in law* of $u_n\overline{\mathbf{H}}_t^n$ to $\widetilde{\mathbf{H}}_t$, where $\overline{\mathbf{H}}_t^n:=\left(\overline{H}(j_1)_t^n,\overline{H}(j_2)_t^n\right)$. By Lemma 13.3.14 in Jacod and Protter (2011), it suffices to check, for any $j_1,j_2\in\mathbb{N}^*$,

$$\mathbb{E}'\left(u_n^2\left(\overline{H}(j_1)_t^n\overline{H}(j_2)_t^n\right)\right)\to\frac{cr(j_1)_tr(j_2)_t}{3(1+c)},\quad \mathbb{E}'\left(\left(u_n\overline{H}(j_\ell)_t^n\right)^4\right)\to 0.$$

Since $l_n\to\infty$ and $c_n\to c$ as $n\to\infty$, the first convergence above follows from the trivial convergence that $\mathbb{E}'\left(\overline{H}(j_1)_t^n\overline{H}(j_2)_t^n/(l_n\Delta_n)\right)\to\frac{r(j_1)_tr(j_2)_t}{3}$. The second convergence can be obtained from the estimate that $\left(\frac{1}{l_n\sqrt{l_n\Delta_n}}\right)^4\sum_i(n_t+l_n-i)^4\mathbb{E}'(U_i^4)\Delta_n^2\leq Kl_n^{-1}\to 0$, which follows from the i.i.d. property of $\{U_i\}_i$, and the fact $\sum_i (n_t + l_n - i)^4 \le K(l_n^5)$.

Now let $\widehat{H}(j)_t^n := \frac{1}{l_n}\sum_{i\in\mathbb{I}_t^n}(n_t + l_n - i)\sqrt{\Delta_n}\widehat{U}_i^n r(j)_t$, where $\widehat{U}_i^n := \sqrt{\frac{\alpha_t \Delta(n,i)}{\Delta_n}}U_i$. Apply the arguments and estimates right before (B.97) in Li and Linton (2024), we have

$$\widetilde{\mathbb{E}}\Big(u_n^2(\widehat{H}(j)_t^n - \overline{H}(j)_t^n)^2\Big) \le K u_n^2 l_n \Delta_n (\Delta_n^{1+(\eta\wedge\eta')} \vee l_n\Delta_n) \to 0,$$

thus $u_n(\widehat{H}(j)_t^n - \overline{H}(j)_t^n) \xrightarrow{\mathbb{P}} 0$, which implies the convergence $u_n\widehat{\mathbf{H}}_t^n \xrightarrow{\mathcal{L}} \widetilde{\mathbf{H}}_t$, where $\widehat{\mathbf{H}}_t^n := \Big(\widehat{H}(j_1)_t^n, \widehat{H}(j_2)_t^n\Big)$.

Next, using the *convergence in law* of $u_n\widehat{\mathbf{H}}_t^n$ to $\widetilde{\mathbf{H}}_t$, we demonstrate that $u_n\widetilde{\mathbf{H}}_t^n$ *converges stably in law* to $\widetilde{\mathbf{H}}_t$. This amounts to showing that for any bounded random variable V defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and any bounded, continuous function $f : \mathbb{R}^2 \mapsto \mathbb{R}$,

$$u_n\mathbb{E}\Big(Vf(\widetilde{\mathbf{H}}_t^n)\Big) \to \widetilde{\mathbb{E}}\Big(Vf(\widetilde{\mathbf{H}}_t)\Big) = \mathbb{E}(V)\mathbb{E}'\Big(f(\widetilde{\mathbf{H}}_t)\Big). \tag{SB.10}$$

Let $\mathcal{H}_t := \sigma(W_s : s \ge t)$. Then, $\widetilde{\mathbf{H}}_t^n$ becomes $\mathcal{H}_t$ measurable; moreover, we can always assume that V is $\mathcal{H}_t$ measurable—otherwise, we can replace it by $\mathbb{E}\left(V|\,\mathcal{H}_t\right)$. Let $B_{t,k} = (t, t+\frac{1}{k}]$, $\overline{W}(k)_{t,s} := \int_t^s \mathbf{1}_{\{B_{t,k}^c\}}(u)\mathrm{d}W_u$, and we define a sequence of σ-fields $\mathcal{H}_{t,k} := \sigma(\overline{W}(k)_{t,s} : s \ge 0)$. Note that $\bigvee_k \mathcal{H}_{t,k} = \mathcal{H}_t$. Thus, we can replace V by $V_k := \mathbb{E}\left(V|\,\mathcal{H}_{t,k}\right)$ in (SB.10) since $V_k \to V$ in $\mathbb{L}^1$ given that V is bounded and $\mathcal{H}_t$-measurable. Now we take n large enough so that $l_n\Delta_n < \frac{1}{k}$, then $\widetilde{\mathbf{H}}_t^n$ is independent of the σ-field $\mathcal{H}_{t,k}$. Therefore, successive conditioning yields the first equality in the following chain of equalities:

$$u_n\mathbb{E}\Big(Vf(\widetilde{\mathbf{H}}_t^n)\Big) = u_n\mathbb{E}(V)\mathbb{E}'\Big(f(\widetilde{\mathbf{H}}_t^n)\Big) = u_n\mathbb{E}(V)\mathbb{E}'\Big(f(\widehat{\mathbf{H}}_t^n)\Big) \to \mathbb{E}(V)\mathbb{E}'(f(\mathbf{H}_t)),$$

where the second equality follows from the scaling and symmetric properties of the Wiener process that yield distributional equivalence, and the last convergence is implied by the convergence in law of $u_n\widehat{\mathbf{H}}_t^n$ to $\widetilde{\mathbf{H}}_t$. This proves (SB.10) hence the stable convergence of $u_n\widetilde{\mathbf{H}}_t^n$ to $\widetilde{\mathbf{H}}_t$. Then, apply Theorem 2.10 and Proposition 3.3 in Dahlhaus et al. (2019), we have $u_n\widetilde{\mathbf{G}}_t^n$ converge in distribution to $\widetilde{\mathbf{G}}_t$ as $l_n^{3/2}\Delta_n \to 0$. The $\mathcal{F}^{(0)}$-stable convergence of $u_n\widetilde{\mathbf{G}}_t^n$ to $\widetilde{\mathbf{G}}_t$ follows immediately from the independence of $\mathcal{F}^{(0)}$ and $\mathcal{G}$. The joint convergence can be proved using the same arguments to prove Theorem A.4 of Jacod et al. (2017). Now the proof is complete. □

Lemma SB.5. *Let Assumption (S-HONL) hold, and further assume $(k_n \vee l_n)\Delta_n \to 0$. Then, we have*

$$\mathbb{E}\Big(|\Gamma(j)_t^n - \overline{\Gamma}(j)_t^n|\Big) \le K l_n^{-1/2}\overline{k}_n\Delta_n^{1/2},$$

where $\overline{\Gamma}(j)_t^n := \frac{1}{l_n}\sum_{i\in\mathbb{I}(j)_t^n}(\gamma_i^n)^2\overline{f}(\chi^n,j)_i^n$.

Proof of Lemma SB.5. For any nonnegative integer p, we have the following decomposition (recall $k_{p,n} = k_n + p$): $f(Y,j;k_{p,n})_i^n - (\gamma_i^n)^2 f(\chi^n,j;k_{p,n})_i^n = \sum_{\ell=1}^{9}\mathfrak{D}(\ell)_i^n$, where

$$\begin{aligned}
&\mathfrak{D}(1)_i^n := f(X,j;k_{p,n})_i^n,\ \mathfrak{D}(2)_i^n := \gamma_{i+j}^n f(X,\chi^n,j;k_{p,n})_i^n,\ \mathfrak{D}(3)_i^n := \chi_{i+j+k_{p,n}}^n f(X,\gamma,j;k_{p,n})_i^n,\\
&\mathfrak{D}(4)_i^n := \gamma_i^n f(\chi^n,X,j;k_{p,n})_i^n,\ \mathfrak{D}(5)_i^n := \gamma_i^n(\gamma_{i+j}^n-\gamma_i^n)f(\chi^n,j;k_{p,n})_i^n,\\
&\mathfrak{D}(6)_i^n := \gamma_i^n\chi_{i+j+k_{p,n}}^n f(\chi^n,\gamma,j;k_{p,n})_i^n,\ \mathfrak{D}(7)_i^n := \chi_{i-k_{p,n}}^n f(\gamma,X,j;k_{p,n})_i^n,\\
&\mathfrak{D}(8)_i^n := \gamma_{i+j}^n\chi_{i-k_{p,n}}^n f(\gamma,\chi^n,j;k_{p,n})_i^n,\ \mathfrak{D}(9)_i^n := \chi_{i-k_{p,n}}^n\chi_{i+j+k_{p,n}}^n f(\gamma,j;k_{p,n})_i^n.
\end{aligned}$$

Define the filtration $\mathcal{H}_i^n := \mathcal{F}_{i-k_{p,n}}^n\otimes\mathcal{G}$, thus $\mathfrak{D}(\ell)_i^n$ is always $\mathcal{H}_{i+2k_{p,n}}^n$ measurable. By successive conditioning, applying (A.6) and (A.7) from Jacod et al. (2017), and considering the boundedness of the fourth moments of χ^n, we have

$$\mathbb{E}(|\mathbb{E}(\mathfrak{D}(\ell)_i^n|\mathcal{H}_i^n)|) \leq \begin{cases} Kk_{p,n}^{3/2}\Delta_n^{3/2}, & \text{if } \ell = 1,3,7,9;\\ Kk_{p,n}\Delta_n, & \text{if } \ell = 2,4,6,8;\\ Kj\Delta_n, & \text{if } \ell = 5;\end{cases}$$

and

$$\mathbb{E}\big(|\mathfrak{D}(\ell)_i^n|^2\big) \leq \begin{cases} Kk_{p,n}^2\Delta_n^2, & \text{if } \ell = 1,3,7,9;\\ Kk_{p,n}\Delta_n, & \text{if } \ell = 2,4,6,8;\\ Kj\Delta_n. & \text{if } \ell = 5.\end{cases}$$

Then, Lemma A.6 in Jacod et al. (2017) yields

$$\mathbb{E}\Big(\big|l_n^{-1}\sum\nolimits_{i\in\mathbb{I}(j)_t^n}\mathfrak{D}(\ell)_i^n\big|\Big) \leq \begin{cases} Kl_n^{-1/2}k_{p,n}^{3/2}\Delta_n(l_n^{1/2}\Delta_n^{1/2}\vee 1), & \text{if } \ell = 1,3,7,9;\\ Kl_n^{-1/2}k_{p,n}\Delta_n^{1/2}(l_n^{1/2}\Delta_n^{1/2}\vee 1), & \text{if } \ell = 2,4,6,8;\\ Kl_n^{-1/2}k_{p,n}^{1/2}\Delta_n^{1/2}(l_n^{1/2}\Delta_n^{1/2}k_{p,n}^{-1/2}\vee 1), & \text{if } \ell = 5.\end{cases}$$

The result now readily follows since $(k_n\vee l_n)\Delta_n\to 0$. □

Lemma SB.6. *Let Assumption (S-HONL) hold, and $v>1$. We have*

$$\mathbb{E}\Big((\widetilde{\Gamma}(j)_t^n - \overline{\Gamma}(j)_t^n)^2\Big) \leq K\left(\frac{1}{p_nl_n}\bigvee k_n^{-v}\bigvee \overline{k}_n\Delta_n\right), \tag{SB.11}$$

where $\widetilde{\Gamma}(j)_t^n := \frac{1}{l_n}\sum_{i\in\mathbb{I}(j)_t^n}(\gamma_i^n)^2\chi_i^n\chi_{i+j}^n$.

Proof of Lemma SB.6. We first decompose $\overline{f}(\chi^n,j)_i^n - \chi_i^n\chi_{i+j}^n = \mathfrak{E}(1)_i^n + \mathfrak{E}(2)_i^n + \mathfrak{E}(3)_i^n$, where $\mathfrak{E}(1)_i^n := -\chi_i^n \overline{\chi}_{i+j+k_n}^n$, $\mathfrak{E}(2)_i^n := -\chi_{i+j}^n \overline{\chi}_{i-k_n}^n$, $\mathfrak{E}(3)_i^n := \frac{1}{p_n}\sum_{p=0}^{p_n-1} \chi_{i-k_{p,n}}^n \chi_{i+j+k_{p,n}}^n$, with $\overline{\chi}_{i+j+k_n}^n := \frac{1}{p_n}\sum_{p=0}^{p_n-1} \chi_{i+j+k_{p,n}}^n$, $\overline{\chi}_{i-k_n}^n := \frac{1}{p_n}\sum_{p=0}^{p_n-1} \chi_{i-k_{p,n}}^n$. Then, by Hölder's inequality, we have

$$\mathbb{E}\Big(|\widetilde{\Gamma}(j)_t^n - \overline{\Gamma}(j)_t^n|^2\Big) \leq K\sum_{\ell=1}^{3}\mathbb{E}\Big(\sum\nolimits_{i\in\mathbb{I}(j)_t^n}(\gamma_i^n)^2\mathfrak{E}(\ell)_i^n/l_n\Big)^2.$$

By the boundedness of γ, it suffices to show that $\mathbb{E}\Big(\sum_{i\in\mathbb{I}(j)_t^n}\mathfrak{E}(\ell)_i^n/l_n\Big)^2$ has the same bound as in (SB.11) for $\ell = 1,2,3$. Next, by a direct application of Lemma SB.1, we have for $1 \leq k \leq k_n$, $\Big|\mathbb{E}\big(\mathfrak{E}(1)_i^n\mathfrak{E}(1)_{i+k}^n\big) - \frac{r(k)_{t_i^n}}{p_n}\Big| \leq K(k_n^{-v}\vee \overline{k}_n\Delta_n)$; and for $k > k_n$, we have an upper bound for $\big|\mathbb{E}\big(\mathfrak{E}(1)_i^n\mathfrak{E}(1)_{i+k}^n\big)\big| \leq K(k_n^{-v}\vee\overline{k}_n\Delta_n)$, which follows from the separating techniques used in the proof of Lemma S5 in Li and Linton (2022b) and applying Lemma SB.1. The two estimates immediately lead to $\mathbb{E}\Big(\sum_{i\in\mathbb{I}(j)_t^n}\mathfrak{E}(1)_i^n/l_n\Big)^2 \leq K\left(\frac{1}{p_nl_n}\bigvee k_n^{-v}\bigvee \overline{k}_n\Delta_n\right)$. Similarly, we have the same bound for $\mathbb{E}\Big(\sum_{i\in\mathbb{I}(j)_t^n}\mathfrak{E}(2)_i^n/l_n\Big)^2$. Next, we consider the case where $\ell = 3$. Let's denote $\overline{\chi}(j;p)_i^n := \chi_{i-k_n-p}^n\chi_{i+j+k_n+p}^n$, and $m := |p-p'|$ for any $0 \leq p,p' \leq p_n$. By Lemma SB.1, for $k < \overline{k}_n$, we have

$$\Big|\mathbb{E}\big(\overline{\chi}(j;p)_i^n\overline{\chi}(j;p')_{i+k}^n\big) - r(|k-m|)_{t_{i-k_n-p}^n}\cdot r(|k+m|)_{t_{i-k_n-p}^n}\Big| \leq K(k_n^{-v}\vee\overline{k}_n\Delta_n),$$

which implies

$$\begin{aligned}&\Big|\mathbb{E}\big(\mathfrak{E}(3)_i^n\mathfrak{E}(3)_{i+k}^n\big) - \frac{r(k)^2_{t_{i-k_n-p}^n}}{p_n} - \frac{2}{p_n^2}\sum_{m=1}^{p_n-1}(p_n-m-1)r(|k-m|)_{t_{i-k_n-p}^n}\cdot r(|k+m|)_{t_{i-k_n-p}^n}\Big|\\ &\leq K(k_n^{-v}\vee\overline{k}_n\Delta_n).\end{aligned}$$

Since $v > 1$, we have the following bound

$$\begin{aligned}&\Big|\frac{r(k)^2_{t_{i-k_n-p}^n}}{p_n} + \frac{2}{p_n^2}\sum_{m=1}^{p_n-1}(p_n-m-1)r(|k-m|)_{t_{i-k_n-p}^n}\cdot r(|k+m|)_{t_{i-k_n-p}^n}\Big|\\ &\leq K\frac{k^{-2v}}{p_n} + K\frac{k^{-v}}{p_n}\sum_{m=1}^{p_n-1}|k-m|^{-v} \leq Kk^{-v}/p_n.\end{aligned}$$

Therefore, we get $\big|\mathbb{E}\big(\mathfrak{E}(3)_i^n\mathfrak{E}(3)_{i+k}^n\big)\big| \leq K(\frac{k^{-v}}{p_n}\vee k_n^{-v}\vee\overline{k}_n\Delta_n)$ for any $k \leq \overline{k}_n - 1$. For $k \geq \overline{k}_n$, using similar arguments in Lemma S6 of Li and Linton (2022b), we obtain an upper bound for $\big|\mathbb{E}\big(\overline{\chi}(j;p)_i^n\overline{\chi}(j;p')_{i+k}^n\big)\big| \leq K(k_n^{-v}\vee\overline{k}_n\Delta_n)$, which leads to $\big|\mathbb{E}\big(\mathfrak{E}(3)_i^n\mathfrak{E}(3)_{i+k}^n\big)\big| \leq K(k_n^{-v}\vee\overline{k}_n\Delta_n)$. Now, we obtain two similar bounds as we have for $\big|\mathbb{E}\big(\mathfrak{E}(1)_i^n\mathfrak{E}(1)_{i+k}^n\big)\big|$. Hence, we can derive the same upper bound for $\mathbb{E}\Big(\sum_{i\in\mathbb{I}(j)_t^n}\mathfrak{E}(3)_i^n/l_n\Big)^2$ as in (SB.11).

The proof is complete. □

Lemma SB.7. *Let $l_n\Delta_n \to 0$ and $(k_n \vee p_n)/\sqrt{l_n} \to 0$. We have*

$$u_n\mathbb{E}\Big(|\widehat{\Gamma}(j)_t^n - \widetilde{\Gamma}(j)_t^n|\Big) \to 0.$$

Proof of Lemma SB.7. Let $\widehat{\Gamma}(j)_t^n - \widetilde{\Gamma}(j)_t^n =: \mathfrak{F}(1)_t^n + \mathfrak{F}(2)_t^n$, where

$$\mathfrak{F}(1)_t^n := \frac{1}{l_n}\sum_{i\in\mathbb{I}(j)_t^n}\left(\gamma_i^n\gamma_{i+j}^n - (\gamma_i^n)^2\right)\chi_i^n\chi_{i+j}^n, \quad \mathfrak{F}(2)_t^n := \frac{1}{l_n}\sum_{i\in\mathbb{I}_t^n\setminus\mathbb{I}(j)_t^n}\varepsilon_i^n\varepsilon_{i+j}^n.$$

By (A.6) in Jacod et al. (2017), the independence of $\mathcal{F}^{(0)}$ and $\mathcal{G}$, we have $\mathbb{E}(|\mathfrak{F}(1)_t^n|) \le K\sqrt{\Delta_n}$. Then, since the fourth moment of χ^n is finite, we get $\mathbb{E}(|\mathfrak{F}(2)_t^n|) \le K(\overline{k}_n/l_n)$. Thus, we have $u_n\mathbb{E}\Big(|\widehat{\Gamma}(j)_t^n - \widetilde{\Gamma}(j)_t^n|\Big) \le Ku_n(\Delta_n^{1/2} \vee \frac{\overline{k}_n}{l_n}) \to 0$. □

We now state some auxiliary lemmas used to prove the consistency of the asymptotic variance estimator. We define the following auxiliary quantities and filtration: for $q \in \{1,\ldots,q_n\}$, let $\mathcal{H}(d_n)_q^n := \mathcal{F}^n_{n_t+2qd_n} \otimes \mathcal{G}_{n_t+(2q-1)d_n}$, $\mathbb{I}_{j,q}^n := \{n_t+2qd_n+\overline{k}_n,\ldots,n_t+(2q+1)d_n - \overline{k}_n - j\}$, $\widetilde{\Gamma}(j;d_n)^n_{t(q)_t^n} := \frac{1}{d_n}\sum_{i\in\mathbb{I}_{j,q}^n}(\gamma_i^n)^2\chi_i^n\chi_{i+j}^n$ and $\widetilde{\Theta}(j;d_n)^n_{t(q)_t^n} := \widetilde{\Gamma}(j;d_n)^n_{t(q)_t^n} - \widetilde{\Gamma}(j;d_n)^n_{t(q-1)_t^n}$. In the subsequent proof, we use shorthand notations $\widetilde{\Gamma}(j)_q^n \equiv \widetilde{\Gamma}(j;d_n)^n_{t(q)_t^n}$ and $\widetilde{\Theta}(j)_q^n \equiv \widetilde{\Theta}(j;d_n)^n_{t(q)_t^n}$ for brevity. We now introduce the 'oracle' estimator for the asymptotic variance as $\widetilde{\sigma}^2(j,j')_t^n := \frac{d_n}{2q_n}\sum_{q=1}^{q_n-1}\widetilde{\Theta}(j)_q^n\widetilde{\Theta}(j')_q^n$. The following lemma establishes the validity of this approximation.

Lemma SB.8. *Let $k_n \wedge p_n \to \infty$, and for any $v > 2$, $d_n\overline{k}_n\Delta_n \to 0$, $d_nk_n^{-v} \to 0$ and $d_nL_n\Delta_n \to 0$. We have*

$$\mathbb{E}\Big(\big|\sigma^2(j,j')_t^n - \widetilde{\sigma}^2(j,j')_t^n\big|\Big) \to 0. \tag{SB.12}$$

Proof of Lemma SB.8. Let $\mathfrak{G}(j;1)_q^n = \Gamma(j)^n_{t(q)_t^n} - \overline{\Gamma}(j)^n_{t(q)_t^n}$ and $\mathfrak{G}(j;2)_q^n = \overline{\Gamma}(j)^n_{t(q)_t^n} - \widetilde{\Gamma}(j)^n_{t(q)_t^n}$ where $\overline{\Gamma}(j)_t^n$ is defined in Lemma SB.5 (with bandwidth d_n here). We make the following decomposition

$$\sigma^2(j,j')_t^n = \frac{d_n}{2q_n}\sum_{q=1}^{q_n-1}\Big(\sum_{\ell=1}^{2}(\mathfrak{G}(j;\ell)_q^n - \mathfrak{G}(j;1)_{q-\ell}^n) + \widetilde{\Theta}(j)_q^n\Big)\Big(\sum_{\ell=1}^{2}(\mathfrak{G}(j';\ell)_q^n - \mathfrak{G}(j';1)_{q-\ell}^n) + \widetilde{\Theta}(j')_q^n\Big).$$

First, since we have $\mathbb{E}((\mathfrak{G}(j;1)_q^n)^2) \le K\sum_{\ell=1}^{9}\sum_{\ell'=1}^{9}|\mathbb{E}(\mathfrak{D}(\ell)_i^n\mathfrak{D}(\ell')_i^n)|$ and $\mathbb{E}((\mathfrak{D}(\ell)_i^n)^2) \le K\overline{k}_n\Delta_n$ for any $\ell = 1,\ldots,9$ and $i \in \mathbb{I}_{j,q}^n$ (see Lemma SB.5), we obtain that $\mathbb{E}\big((\mathfrak{G}(j;1)_q^n)^2\big) \le K\overline{k}_n\Delta_n$. By the estimate above, Lemma SB.6, and the Cauchy-Schwarz inequality, for

$q, m \in \{1, \ldots, q_n\}$, we have

$$\mathbb{E}\big(\big|\mathfrak{G}(j;1)_q^n \mathfrak{G}(j';1)_m^n\big|\big) \leq K\overline{k}_n\Delta_n, \; \mathbb{E}\big(\big|\mathfrak{G}(j;2)_q^n \mathfrak{G}(j';2)_m^n\big|\big) \leq K\left(\frac{1}{p_n d_n} \bigvee \frac{1}{k_n^v} \bigvee \overline{k}_n\Delta_n\right),$$

and $\mathbb{E}\big(\big|\mathfrak{G}(j;1)_q^n \mathfrak{G}(j';2)_m^n\big|\big) \leq K(\overline{k}_n\Delta_n)^{1/2}\left(p_n^{-1/2}d_n^{-1/2} \vee k_n^{-v/2} \vee (\overline{k}_n\Delta_n)^{1/2}\right)$. Next, apply the Cauchy-Schwarz inequality to the decomposition of $(\widetilde{\Theta}(j)_q^n)^2$, we obtain if $d_n L_n \Delta_n \to 0$, $\mathbb{E}\left((\widetilde{\Theta}(j)_q^n)^2\right) \leq K d_n^{-1}$. Lastly, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\mathbb{E}\left(\left|\mathfrak{G}(j;1)_q^n \widetilde{\Theta}(j)_m^n\right|\right) \leq K(\overline{k}_n\Delta_n)^{1/2} d_n^{-1/2}, \\
&\mathbb{E}\left(\left|\mathfrak{G}(j;2)_q^n \widetilde{\Theta}(j)_m^n\right|\right) \leq K\left(p_n^{-1/2}d_n^{-1/2} \vee k_n^{-v/2} \vee (\overline{k}_n\Delta_n)^{1/2}\right) d_n^{-1/2}.
\end{aligned}$$

Therefore, (SB.12) readily follows given all the asymptotic conditions. □

Then, we make the following decomposition

$$\widetilde{\sigma}^2(j,j')_t^n = \frac{d_n}{2q_n}\sum_{q=1}^{q_n-1}\left(\overline{Z}(d_n)_q^n + \overline{Z}(d_n)_{q-1}^n + U(j)_q^n U(j')_q^n + \sum_{l=1}^{6}\psi_q^{n,l}\right),$$

where

$$\begin{aligned}
&Z(j)_q^n := \widetilde{\Gamma}(j)_q^n - \gamma_{t(q)_t^n}^2 r(j)_t, \; \overline{Z}(d_n)_q^n := Z(j)_q^n Z(j')_q^n, \; U(j)_q^n := \left(\gamma_{t(q)_t^n}^2 - \gamma_{t(q-1)_t^n}^2\right) r(j)_t; \\
&\psi_q^{n,1} := -Z(j)_q^n Z(j')_{q-1}^n, \; \psi_q^{n,2} := -Z(j)_{q-1}^n Z(j')_q^n, \; \psi_q^{n,3} := Z(j)_q^n U(j')_q^n; \\
&\psi_q^{n,4} := -Z(j)_{q-1}^n U(j')_q^n, \; \psi_q^{n,5} := Z(j')_q^n U(j)_q^n, \; \psi_q^{n,6} := -Z(j')_{q-1}^n U(j)_q^n.
\end{aligned}$$

After some tedious but elementary calculations, we can establish the following estimates that will be used in the proof for Theorem D.1:

$$\mathbb{E}\left(\left|d_n\mathbb{E}\big(\overline{Z}(d_n)_q^n|\mathcal{H}(d_n)_q^n\big) - \gamma_{t(q)_t^n}^4 s(j,j')_{t(q)_t^n}\right|\right) \leq K\left(d_n^2\Delta_n \vee d_n^{-1}\right), \tag{SB.13}$$

$$d_n\mathbb{E}\big(\big|U(j)_q^n U(j')_q^n\big|\big) \leq K d_n L_n \Delta_n. \tag{SB.14}$$

$$\mathbb{E}\big((\overline{Z}(d_n)_q^n)^2\big) \leq K(d_n^{-2} \vee d_n\Delta_n) \tag{SB.15}$$

$$\mathbb{E}\big(\big|\mathbb{E}\big(\psi_q^{n,l}|\mathcal{H}(d_n)_q^n\big)\big|\big) \leq K(d_n^{-1/2-v} \vee d_n^{1/2}\Delta_n), \quad \text{if} \quad l = 1, 2. \tag{SB.16}$$

$$\mathbb{E}\big(\big|\mathbb{E}\big(\psi_q^{n,l}|\mathcal{H}(d_n)_q^n\big)\big|\big) \leq K\Delta_n^{1/2}, \quad \text{if} \quad l = 3, 4, 5, 6. \tag{SB.17}$$

$$\mathbb{E}\big((\psi_q^{n,l})^2\big) \leq K(d_n^{-2} \vee d_n\Delta_n), \quad \text{if} \quad l = 1, 2. \tag{SB.18}$$

$$\mathbb{E}\big((\psi_q^{n,l})^2\big) \leq K d_n^{1/2}\Delta_n^{1/2}(d_n^{-1} \vee d_n^{1/2}\Delta_n^{1/2}), \quad \text{if} \quad l = 3, 4, 5, 6. \tag{SB.19}$$

SB.2 Proof of Main Theorems

Proof of Theorem 4.1. We have the decomposition that $\widehat{\Gamma}(j)_t^n - \Gamma(j)_t := G(j)_t^n + G'(j)_t^n + G''(j)_t^n$, where $G(j)_t^n$ and $G'(j)_t^n$ are defined in (SB.1), and $G''(j)_t^n := H(j)_t^n + D(j)_t^n$. Cauchy-Schwarz inequality and the bounded fourth moment of χ^n yield an upper bound for $\mathbb{E}(|G'(j)_t^n| \vee |G''(j)_t^n|) \leq K\sqrt{l_n \Delta_n}$. By (ii) of Theorem 2.7 (the local LLN) in Dahlhaus et al. (2019), we have if $l_n \Delta_n \to 0$, $G(j)_t^n \xrightarrow{\mathbb{P}} 0$. Thus, we have the convergence $\widehat{\Gamma}(j)_t^n \xrightarrow{\mathbb{P}} \Gamma(j)_t$. The convergences of $\widehat{\mathcal{S}}(m)_t^n$ and $\widehat{\mathrm{DOFI}}(m)_t^n$ readily follow. $\square$

Proof of Theorem 4.2. The results follows from Theorem SB.1 and the second and third convergence in Lemma SB.3. $\square$

Proof of Corollary 4.1. Theorem 4.2 admits a natural extension to a multivariate version. Define $\widehat{\mathbf{\Gamma}}(m)_t^n := (\widehat{\Gamma}(\ell)_t^n)_{\ell=0}^m$ and $\mathbf{\Gamma}(m)_t := (\Gamma(\ell)_t)_{\ell=0}^m$. Then we have

$$u_n(\widehat{\mathbf{\Gamma}}(m)_t^n - \mathbf{\Gamma}(m)_t) \xrightarrow{\mathcal{L}_s} \mathbf{Z}_t,$$

where $\mathbf{Z}_t$ is an $(m+1)$-dimensional centered Gaussian vector with $(m+1) \times (m+1)$-dimensional conditional covariance matrix $\mathbf{\Sigma}_t$ whose (i,j)-th entry is given by $\mathbf{\Sigma}(i,j)_t = \sigma^2(i-1, j-1)_t$. Let $g : \mathbb{R}^{m+1} \to \mathbb{R}$ with $g(x_0, \ldots, x_m) := 2\sqrt{\sum_{|j| \leq m} \theta(j;m) x_j}$. We have that the the $(i+1)$-th entry of $\nabla g(\mathbf{\Gamma}(m)_t)$ is given by $\frac{\theta(i;m)}{\sqrt{\sum_{|j|\leq m} \theta(j;m)\Gamma(j)_t}}$ for $i = 0, \ldots, m$. Note that we have $\mathrm{DOFI}(m)_t := \frac{\sum_{|j|\leq m} \theta(j;m)\Gamma(j)_t}{\Gamma(0)_t} - 1 = 2\sum_{j=1}^m \theta(m)_j \frac{\Gamma(j)_t}{\Gamma(0)_t}$. Define $g' : \mathbb{R}^{m+1} \to \mathbb{R}$ by $g'(x_0, \ldots, x_m) := 2\sum_{j=1}^m \theta(j;m) \frac{x_j}{x_0}$, and let $\mathfrak{S}'(i;m)_t$ denote the $(i+1)$-th entry of $\nabla g'(\mathbf{\Gamma}(m)_t)$. We obtain that $\mathfrak{S}'(0;m)_t = -\frac{2}{\Gamma(0)_t^2} \sum_{j=1}^m \theta(j;m)\Gamma(j)_t$ and $\mathfrak{S}'(j;m)_t = \frac{2\theta(j;m)}{\Gamma(0)_t}$ for $j = 1, \ldots, m$. Now, the first and second part of Corollary 4.1 follow from applications of the Delta method to the function g and g', respectively. The proof is now complete. $\square$

Now, we make the following decomposition:

$$Z(j)_t^n := \widetilde{\Gamma}(j)_t^n - \Gamma(j)_t = G(j)_t^n + H(j)_t^n + D(j)_t^n + G'(j)_t^n + \sum_{\ell=1}^3 \mathfrak{R}(j;\ell)_t^n, \tag{SB.20}$$

where $\mathfrak{R}(j;1)_t^n := \Gamma(j)_t^n - \overline{\Gamma}(j)_t^n$, $\mathfrak{R}(j;2)_t^n := \overline{\Gamma}(j)_t^n - \widetilde{\Gamma}(j)_t^n$, and $\mathfrak{R}(j;3)_t^n := \widetilde{\Gamma}(j)_t^n - \widehat{\Gamma}(j)_t^n$.

Proof of Theorem C.1. First, we show that

$$u_n \sum_{\ell=1}^3 \mathfrak{R}(j;\ell)_t^n \xrightarrow{\mathbb{P}} 0 \tag{SB.21}$$

under Condition B.4. By Lemma SB.5, we have $u_n\mathbb{E}(|\mathfrak{R}(j;1)_t^n|) \to 0$ if $(k_n \vee l_n)\Delta_n \to 0$ and $u_n l_n^{-1/2}\overline{k}_n\Delta_n^{1/2} \to 0$. Next, by Lemma SB.6, if $p_n \to \infty$, $u_n^2 k_n^{-v} \to 0$, and $u_n^2\overline{k}_n\Delta_n \to 0$, we get $u_n^2\mathbb{E}\big((\mathfrak{R}(j;2)_t^n)^2\big) \to 0$. Lastly, by Lemma SB.7, if $l_n\Delta_n \to 0$ and $\overline{k}_n/\sqrt{l_n} \to 0$, we have $u_n\mathbb{E}(|\mathfrak{R}(j;3)_t^n|) \to 0$. (SB.21) is proved. Now, Theorem C.1 is a consequence of Theorem SB.1, Lemma SB.3 and (SB.21). This concludes the proof. $\square$

Proof for Theorem D.1. In view of (SB.13) and (SB.14), under the conditions $d_n L_n \Delta_n \to 0$ and $v > 2$, it follows that, as $n \to \infty$, we have

$$
\begin{aligned}
&\frac{1}{q_n}\sum_{q=1}^{q_n-1}\mathbb{E}\Big(\Big|d_n\mathbb{E}\big(\overline{Z}(d_n)_q^n|\mathcal{H}(d_n)_q^n\big) - \gamma_{t(q)_t^n}^4 s(j,j')_{t(q)_t^n}\Big|\Big) \le K\left(d_n^2\Delta_n \vee d_n^{-1}\right) \to 0,\\
&\frac{1}{q_n}\sum_{q=1}^{q_n-1}\mathbb{E}\Big(\Big|d_n\mathbb{E}\big(\overline{Z}(d_n)_{q-1}^n|\mathcal{H}(d_n)_{q-1}^n\big) - \gamma_{t(q-1)_t^n}^4 s(j,j')_{t(q)_t^n}\Big|\Big) \le K\left(d_n^2\Delta_n \vee d_n^{-1}\right) \to 0,\\
&\frac{d_n}{q_n}\sum_{q=1}^{q_n-1}\mathbb{E}\big(\big|U(j)_q^n U(j')_q^n\big|\big) \le K d_n L_n \Delta_n \to 0.
\end{aligned}
$$

Using the estimates above, Proposition 3.3 in Dahlhaus et al. (2019) and (B.80) in Li and Linton (2024), we deduce that

$$
\frac{d_n}{q_n}\sum_{q=1}^{q_n-1}\mathbb{E}\big(\overline{Z}(d_n)_q^n|\mathcal{H}(d_n)_q^n\big) \xrightarrow{\mathbb{P}} \gamma_t^4 s(j,j')_t, \quad \frac{d_n}{q_n}\sum_{q=1}^{q_n-1}\mathbb{E}\big(\overline{Z}(d_n)_{q-1}^n|\mathcal{H}(d_n)_{q-1}^n\big) \xrightarrow{\mathbb{P}} \gamma_t^4 s(j,j')_t \tag{SB.22}
$$

$$
\frac{d_n}{q_n}\sum_{q=1}^{q_n-1} U(j)_q^n U(j')_q^n \xrightarrow{\mathbb{P}} 0. \tag{SB.23}
$$

Let $\Psi(1;d_n)_q^n := \overline{Z}(d_n)_q^n - \mathbb{E}\big(\overline{Z}(d_n)_q^n|\mathcal{H}(d_n)_q^n\big)$, $\Psi(2;d_n)_q^n := \psi_q^{n,1} + \psi_q^{n,2}$, and $\Psi(3;d_n)_q^n := \psi_q^{n,3} + \psi_q^{n,4} + \psi_q^{n,5} + \psi_q^{n,6}$, which are $\mathcal{H}(d_n)_{q+1}^n$ measurable. First, by successive conditioning, for all $q \in \{1,\dots,q_n\}$, we have $\mathbb{E}(|\mathbb{E}(\Psi(1;d_n)_q^n|\mathcal{H}(d_n)_q^n)|) = 0$. Next, by (SB.16) and (SB.17), for $q \in \{1,\dots,q_n\}$, we obtain $\mathbb{E}\big(\big|\mathbb{E}\big(\Psi(2;d_n)_q^n|\mathcal{H}(d_n)_q^n\big)\big|\big) \le K(d_n^{-1/2-v} + d_n^{1/2}\Delta_n)$ and $\mathbb{E}\big(\big|\mathbb{E}\big(\Psi(3;d_n)_q^n|\mathcal{H}(d_n)_q^n\big)\big|\big) \le K\Delta_n^{1/2}$. Then, by (SB.15), (SB.18), and (SB.19), we have, $\mathbb{E}\big((\Psi(1;d_n)_q^n)^2\big) + \mathbb{E}\big((\Psi(2;d_n)_q^n)^2\big) \le K(d_n^{-2} \vee d_n\Delta_n)$ and $\mathbb{E}\big((\Psi(3;d_n)_q^n)^2\big) \le K d_n^{1/2}\Delta_n^{1/2}(d_n^{-1} \vee d_n^{1/2}\Delta_n^{1/2})$. Thus, by Lemma A.6 in Jacod et al. (2017), we obtain $\mathbb{E}\Big(\Big|\sum_{q=1}^{q_n-1}\Psi(1;d_n)_q^n\Big|\Big) \le K\left(q_n^{\frac{1}{2}} d_n^{-\frac{1}{2}} + q_n^{\frac{1}{2}} d_n \Delta_n^{\frac{1}{2}}\right)$, $\mathbb{E}\Big(\Big|\sum_{q=1}^{q_n-1}\Psi(2;d_n)_q^n\Big|\Big) \le K(q_n d_n^{-\frac{1}{2}-v} + q_n d_n^{\frac{1}{2}}\Delta_n + q_n^{\frac{1}{2}} d_n^{-\frac{1}{2}} + q_n^{\frac{1}{2}} d_n \Delta_n^{\frac{1}{2}})$, and $\mathbb{E}\Big(\Big|\sum_{q=1}^{q_n-1}\Psi(3;d_n)_q^n\Big|\Big) \le K\left(q_n\Delta_n^{\frac{1}{2}} + q_n^{\frac{1}{2}} d_n^{\frac{3}{2}}\Delta_n + q_n^{\frac{3}{4}} d_n^{\frac{1}{4}}\Delta_n^{\frac{1}{4}}\right)$.

The estimates above imply that, under Condition B.5, as $n \to \infty$, for $\ell = 1, 2, 3$, we have

$$\frac{d_n}{q_n}\mathbb{E}\left(\left|\sum_{q=1}^{q_n-1} \Psi(\ell; d_n)_q^n\right|\right) \to 0. \tag{SB.24}$$

Apply the same argument as above for $\Psi(1; d_n)_{q-1}^n$, we get $\frac{d_n}{q_n}\mathbb{E}(|\sum_{q=1}^{q_n-1}\Psi(1; d_n)_{q-1}^n|) \to 0$. Now, in view of (SB.22), (SB.23), and (SB.24), we have $\widetilde{\sigma}^2(j, j')_t^n \xrightarrow{\mathbb{P}} \gamma_t^4 s(j, j')_t$. Finally, Theorem D.1 follows from (SB.12). The proof is now complete. □

Proof for Theorem 4.3. We first show that $Z(j)_t^n \xrightarrow{\mathbb{P}} 0$. By Lemma SB.5, if $(k_n \vee l_n)\Delta_n \to 0$ and $\overline{k}_n l_n^{-1/2}\Delta_n^{1/2} \to 0$, we have $\mathbb{E}(|\mathfrak{R}(j; 1)_t^n|) \to 0$. Then, by Lemma SB.6, when $k_n, l_n \to \infty$ and $v > 1$, we obtain $\mathbb{E}\big((\mathfrak{R}(j; 2)_t^n)^2\big) \to 0$. Lastly, by Lemma SB.7, if $l_n\Delta_n \to 0$ and $\overline{k}_n/l_n \to 0$, we have $\mathbb{E}(|\mathfrak{R}(j; 3)_t^n|) \to 0$. Now, together with Theorem 4.1, the first part of (12) is proved, and the second and third part of it immediately follow by the Slutsky theorem. (13) is an immediate consequence of Theorem C.1, specialized to the degenerate case with a regular observation scheme, combined with the proof technique of Corollary 4.1 and Theorem D.1. The proof is now complete. □

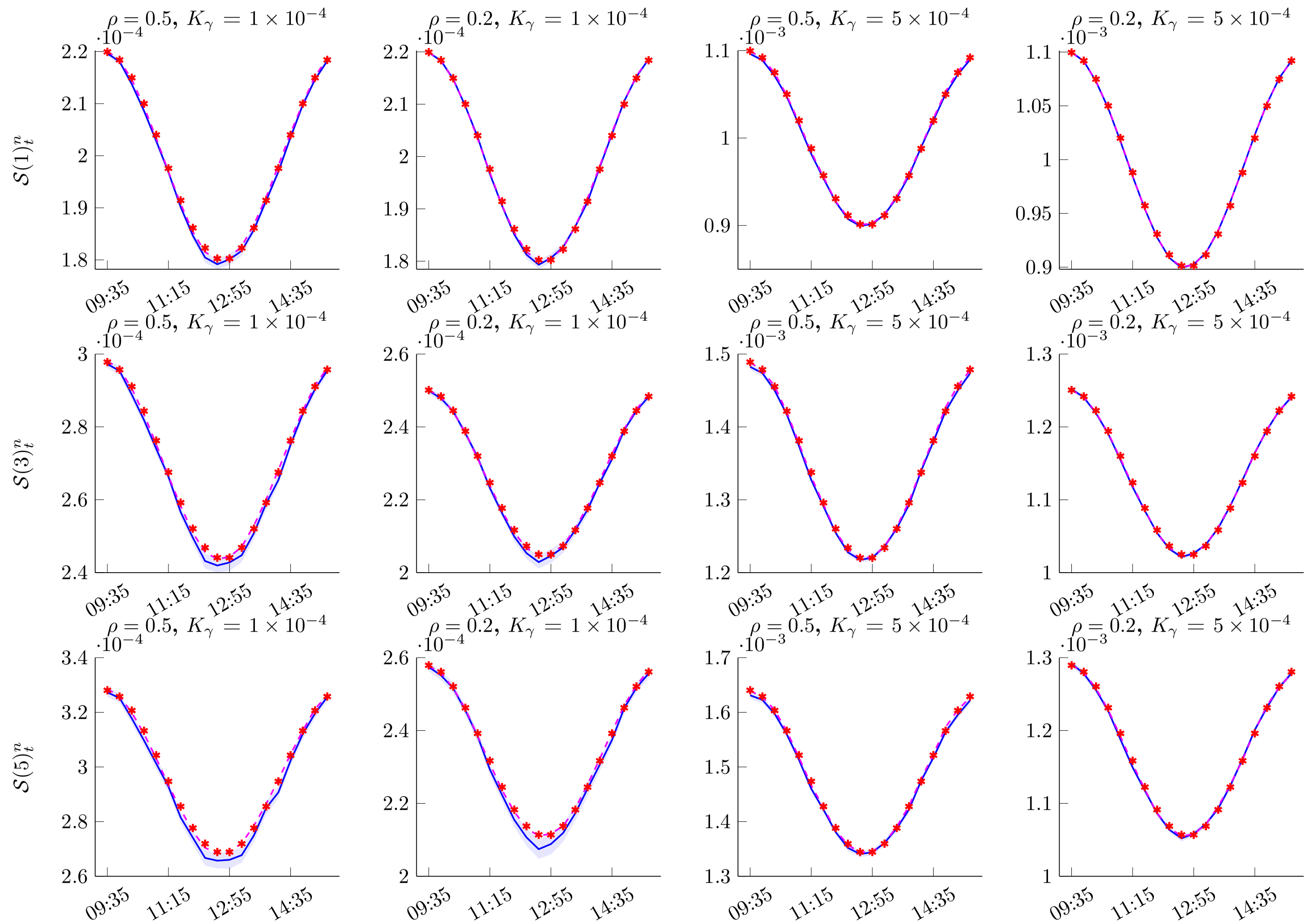

Figure SA.1: The solid blue lines plot the mean across 1,000 replications of the spot estimators $\mathcal{S}(m)^n_t$ (11) for $m \in \{1, 3, 5\}$. Blue shading denotes the corresponding 95% simulated confidence intervals. Red stars mark the true liquidity values $\mathcal{S}(m)_t$ defined in (4). Pink dashed lines report the means of the infeasible sample-analogue estimates $\widehat{\mathcal{S}}(m)^n_t$ defined in (9). We consider four transitory-component specifications formed by crossing scale and serial dependence: strong ($K_\gamma = 5 \times 10^{-4}$) versus weak ($K_\gamma = 1 \times 10^{-4}$) scales, and low ($\rho = 0.2$) versus high ($\rho = 0.5$) autocorrelation coefficients in order flows. Each subplot labels the specification. We let l_n is set to 2000 for all estimations. For $\rho = 0.2$, we choose $k_{n,p} \in \{6, \dots, 10\}$, while for $\rho = 0.5$, we choose $k_{n,p} \in \{8, \dots, 12\}$.

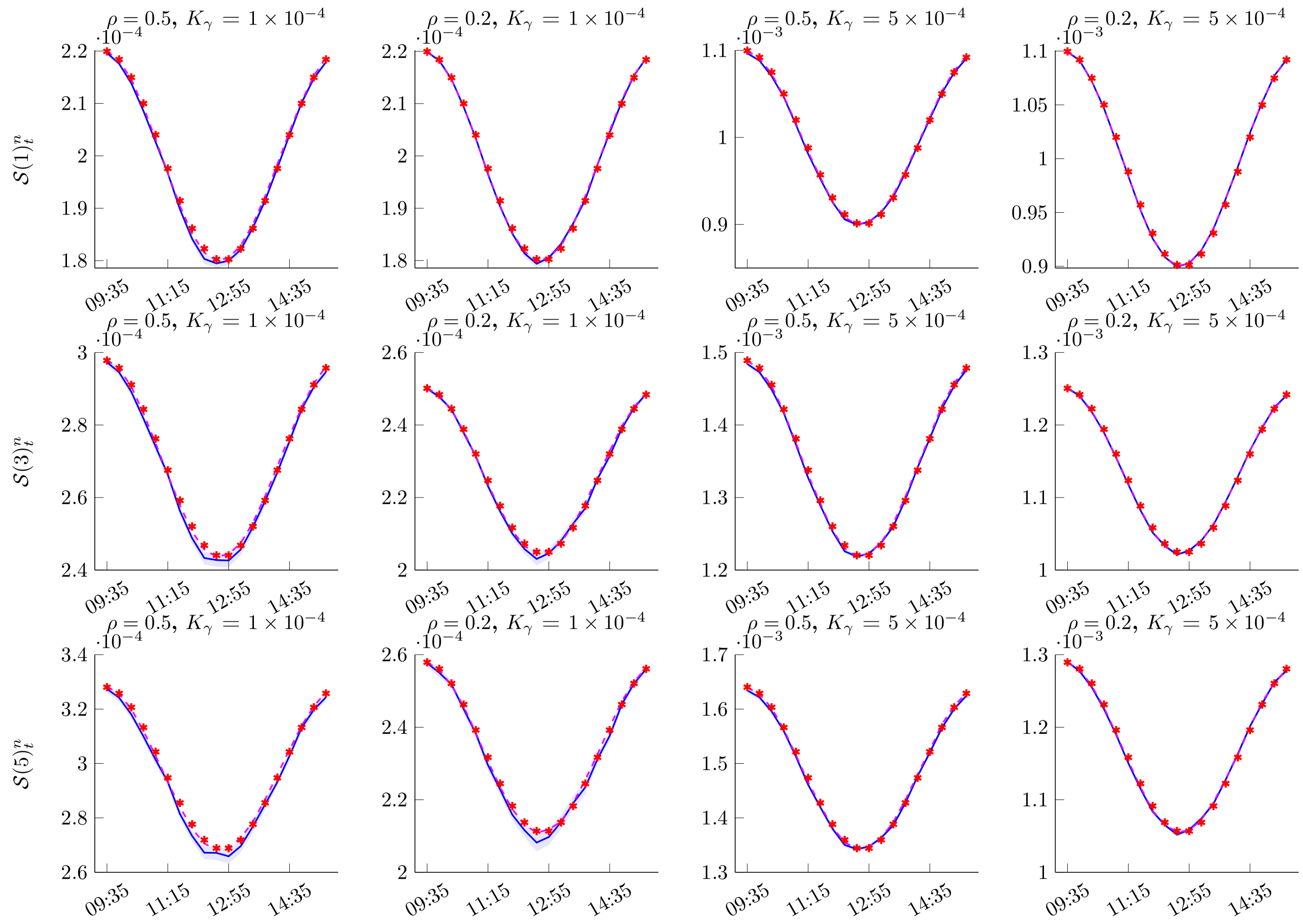

Figure SA.2: The solid blue lines plot the mean across 1,000 replications of the spot estimators $\mathcal{S}(m)_t^n$ (11) for $m \in \{1, 3, 5\}$. Blue shading denotes the corresponding 95% simulated confidence intervals. Red stars mark the true liquidity values $\mathcal{S}(m)_t$ defined in (4). Pink dashed lines report the means of the infeasible sample-analogue estimates $\widehat{\mathcal{S}}(m)_t^n$ defined in (9). We consider four transitory-component specifications formed by crossing scale and serial dependence: strong ($K_\gamma = 5 \times 10^{-4}$) versus weak ($K_\gamma = 1 \times 10^{-4}$) scales, and low ($\rho = 0.2$) versus high ($\rho = 0.5$) autocorrelation coefficients in order flows. Each subplot labels the specification. We let l_n is set to 3000 for all estimations. For $\rho = 0.2$, we choose $k_{n,p} \in \{6, \ldots, 10\}$, while for $\rho = 0.5$, we choose $k_{n,p} \in \{8, \ldots, 12\}$.

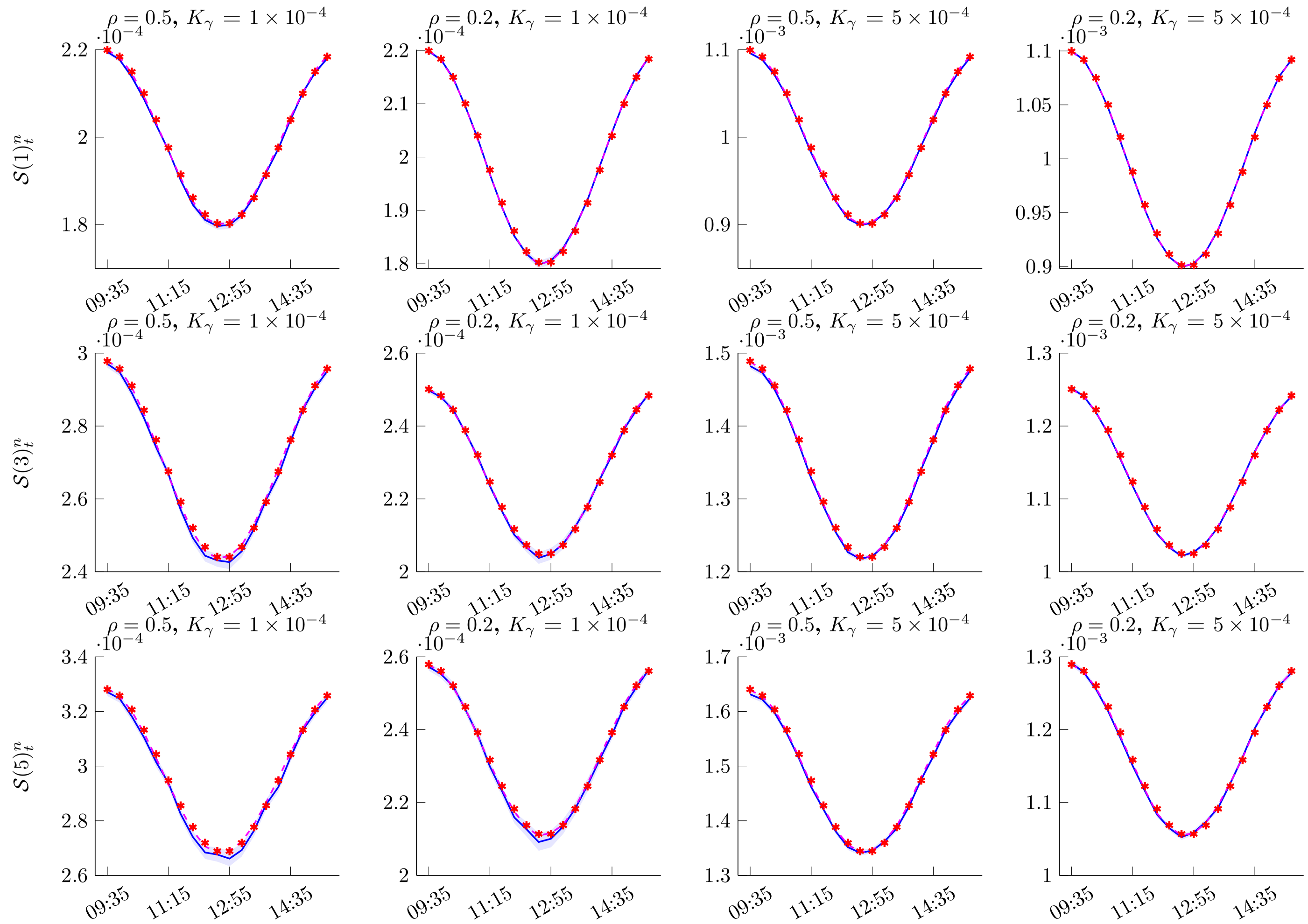

Figure SA.3: The solid blue lines plot the mean across 1,000 replications of the spot estimators $\mathcal{S}(m)^n_t$ (11) for $m \in \{1, 3, 5\}$. Blue shading denotes the corresponding 95% simulated confidence intervals. Red stars mark the true liquidity values $\mathcal{S}(m)_t$ defined in (4). Pink dashed lines report the means of the infeasible sample-analogue estimates $\widehat{\mathcal{S}}(m)^n_t$ defined in (9). We consider four transitory-component specifications formed by crossing scale and serial dependence: strong ($K_\gamma = 5 \times 10^{-4}$) versus weak ($K_\gamma = 1 \times 10^{-4}$) scales, and low ($\rho = 0.2$) versus high ($\rho = 0.5$) autocorrelation coefficients in order flows. Each subplot labels the specification. We let l_n is set to 2500 for all estimations. For $\rho = 0.2$, we choose $k_{n,p} \in \{6, \ldots, 10\}$, while for $\rho = 0.5$, we choose $k_{n,p} \in \{8, \ldots, 12\}$. Both the spot estimation and the estimation of the asymptotic variance of $\mathrm{DOFI}(m)^n_t$ incorporate the retoeplitz correction with $\overline{m} = m + 5$.

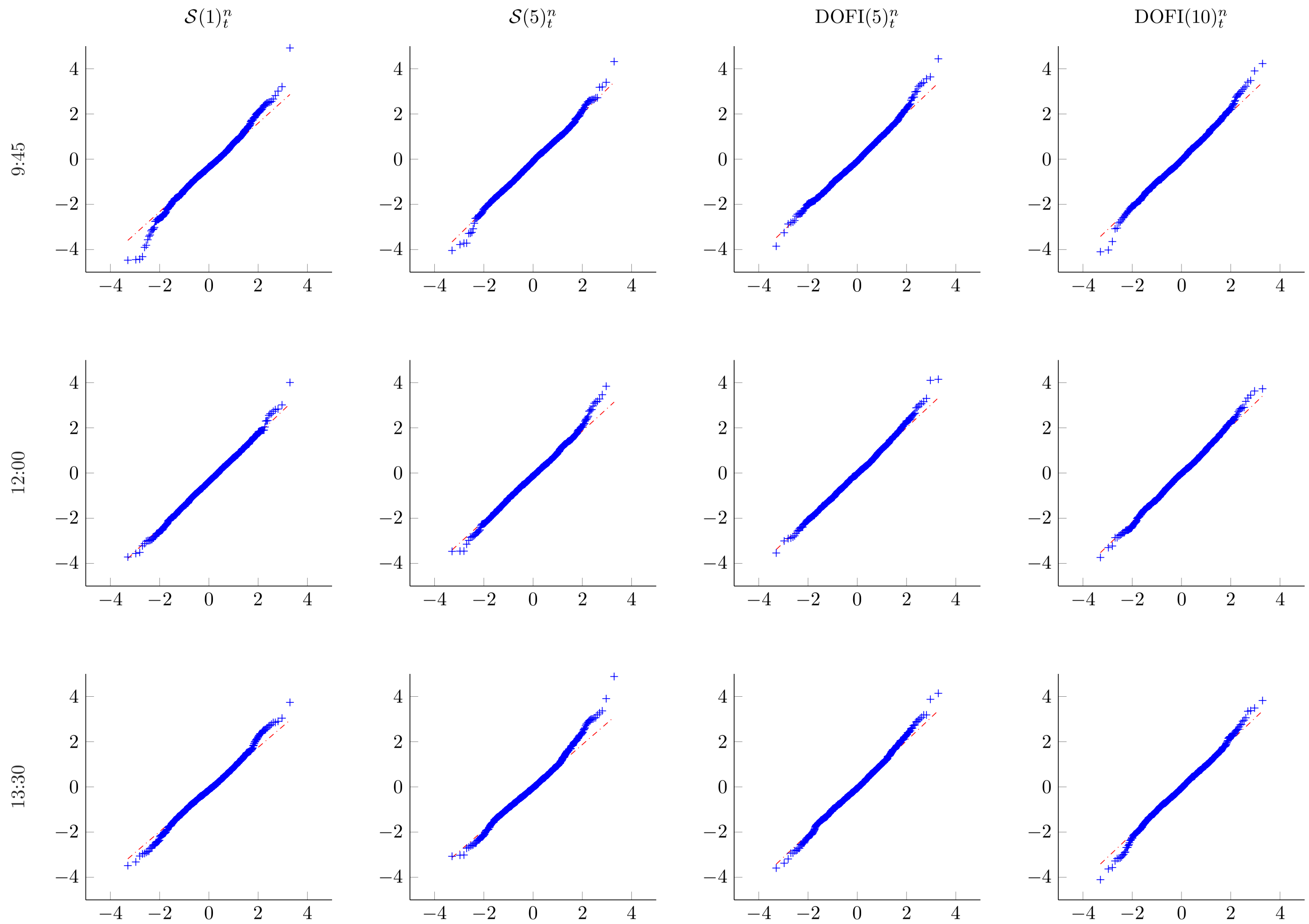

Figure SA.4: This figure shows Q-Q plots for $\mathcal{S}(m)^n_t$ $(m = 1, 5)$ and $\mathrm{DOFI}(m)^n_t$ $(m = 5, 10)$. Each column presents a different normalized estimator, and each row presents results evaluated at times 9:45, 12:00, and 13:30, respectively.We specify $K_\gamma = 5 \times 10^{-4}$ and $\rho = 0.5$. For the spot estimator, we let $k_{n,p}$ ranges from 6 to 10 and $l_n = 2500$. For the asymptotic variance estimator, $k_{n,p}$ ranges from 6 to 10 when $m = 1$ and from 8 to 12 when $m = 5$ or 10. The bandwidth parameters are chosen as $L_n = \lfloor 20\sqrt{n} \rfloor$ and $d_n = \lfloor L_n/50 \rfloor$. Both the spot estimation and the estimation of the asymptotic variance of $\mathrm{DOFI}(m)^n_t$ incorporate the retoeplitz correction with $\overline{m} = m + 5$.

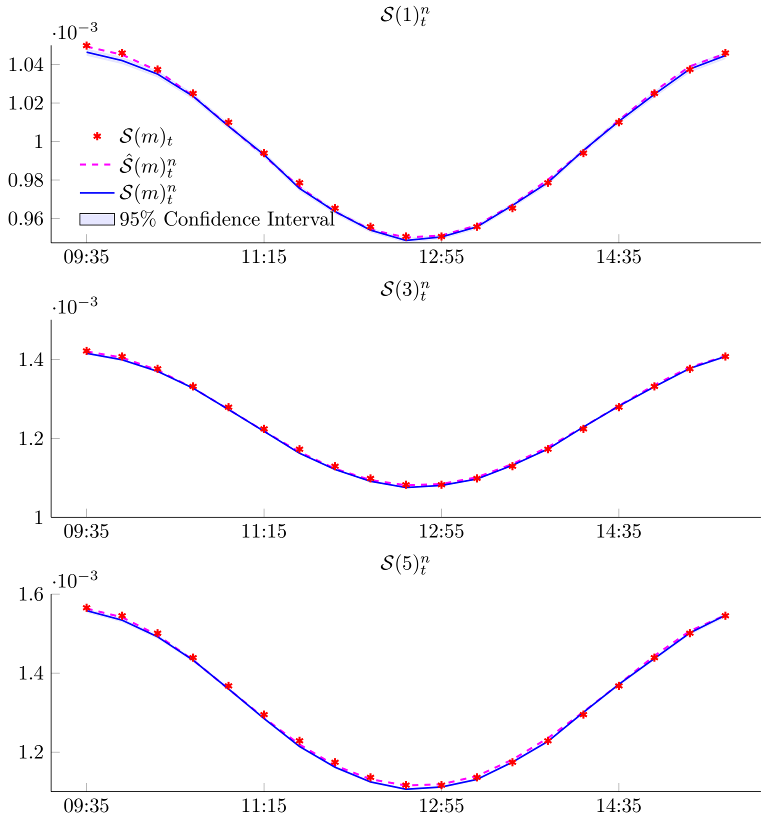

Figure SA.5: The solid blue lines plot the mean across 1,000 replications of the spot estimators $\mathcal{S}(m)_t^n$ (11) for $m \in \{1, 3, 5\}$. Blue shading denotes the corresponding 95% simulated confidence intervals. Red stars mark the true liquidity values $\mathcal{S}(m)_t$ defined in (4). Pink dashed lines report the means of the infeasible sample-analogue estimates $\widehat{\mathcal{S}}(m)_t^n$ defined in (9). We consider order flows χ follows a tvAR(1) model defined by $\chi_i^n = \rho_i \chi_{i-1}^n + \sqrt{1-\rho_i^2} e_i$ with $\{e_i\}_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$ and the autoregressive coefficient $\rho_i = 0.35 + 0.15\cos(2\pi t_i^n)$. We further let $\gamma_t' = 1 + 0.05\cos(2\pi t)$. We set the scale of the noise as $K_\gamma = 5 \times 10^{-4}$. We let l_n is set to 2500 for all estimations, and adopt the data-driven selection method for $k_{n,p}$ as shown in Algorithm 2.

References

DAHLHAUS, R., S. RICHTER, AND W. B. WU (2019): "Towards a general theory for nonlinear locally stationary processes," *Bernoulli*, 25, 1013 – 1044.

JACOD, J., Y. LI, AND X. ZHENG (2017): "Statistical properties of microstructure noise," *Econometrica*, 85, 1133–1174.

JACOD, J. AND P. E. PROTTER (2011): *Discretization of Processes*, vol. 67, Springer Science & Business Media.

JACOD, J. AND V. TODOROV (2010): "Do price and volatility jump together?" *Annals of Applied Probability*, 20, 1425–1469.

LI, Z. M. AND O. LINTON (2022a): "A ReMeDI for microstructure noise," *Econometrica*, 90, 367–389.

——— (2022b): "Supplementary Material for "A REMEDI FOR MICROSTRUCTURE NOISE"," *Econometrica*, 90, 367–389, supplementary Material.

——— (2024): "Robust estimation of integrated and spot volatility," *Journal of Econometrics*.

LI, Z. M., O. LINTON, Y. ZHAI, AND H. ZHANG (2026): "Designing High-Frequency Market Liquidity Measures with Applications to Monetary Policy," Working Paper.