Math 31BH: Midterm

Due 02/07 at 16:00

Merrick Qiu

1. (a) For any arbitrary $t \in \mathbb{R}$,

$$\begin{aligned} f(t) &= (2\cos t, \sin t) \\ &= (2\cos(t+2\pi), \sin(t+2\pi)) \\ &= f(t+2\pi) \end{aligned}$$

Therefore, $f(t) = f(t+2\pi)$ for all $t \in \mathbb{R}$.

(b) Let $v \in \operatorname{Im} f$ be arbitrary. Therefore $v = (2\cos t, \sin t)$ for some t. Since

$$\frac{1}{4}(2\cos t)^2 + \sin^2 t = \cos^2 t + \sin^2 t = 1$$

$v \in C$, so $\operatorname{Im} f \subseteq C$ since v was arbitrary.

Let $v \in C$ be arbitrary in the form $v = (x, y)$. Let $t = \arccos \frac{x}{2}$; t exists since $\|\frac{x}{2}\|$ cannot be greater than 1. Since $\frac{1}{4}x^2 + y^2 = 1$, then $\frac{1}{2}x = \sqrt{1-y^2}$. Using the identity that $\cos(\arcsin(x)) = \sqrt{1-x^2}$,

$$\arccos \frac{x}{2} = \arccos \sqrt{1-y^2} = \arcsin y$$

Therefore $t = \arccos \frac{x}{2} = \arcsin y$ meaning that $f(t) = (x, y)$ and v is in the image of f. Since $v \in \operatorname{Im} f$ and v was arbitrary, $C \subseteq \operatorname{Im} f$. Since $\operatorname{Im} f \subseteq C$ and $C \subseteq \operatorname{Im} f$, $\operatorname{Im} f = C$. The curve of f is an ellipse.

(c) The function $g(t) = (t, \sqrt{1 - \frac{1}{4}t^2})$ is a non-periodic parameterization of C since

$$\frac{1}{4}x^2 + \sqrt{1 - \frac{1}{4}x^2}^2 = 1$$

and

$$(t, \sqrt{1 - \frac{1}{4}x^2}) \neq (t+p, \sqrt{1 - \frac{1}{4}(x+p)^2})$$

for any nonzero $p \in \mathbb{R}$.

2. (a) The component functions for the basis of elementary matrices are

$$\begin{aligned} f_{11}(t) &= e^t & f_{12}(t) &= e^{2t} \\ f_{21}(t) &= e^{3t} & f_{22}(t) &= e^{4t} \end{aligned}$$

(b) $f(t)$ is smooth if all of its component functions are smooth. Since all the component functions are in the form e^{kt} for some constant k, the component functions have derivative ke^{kt}. Therefore the derivative exists for all $t \in \mathbb{R}$ and the derivative is continuous since the second derivative, k^2e^{kt}, exists. Thus, the component functions of f are continuously differentiable.

The component function derivatives are nonvanishing since ke^{kt} does not ever equal zero. Therefore the component functions are smooth and thus $f(t)$ is a smooth curve.

(c)

$$L(f) = \int_0^1 \|f'(t)\|\, dt = \int_0^1 \sqrt{(e^t)^2 + (2e^{2t})^2 + (3e^{3t})^2 + (4e^{4t})^2}\, dt$$

3. (a) Using the chain rule on the component functions,

$$\begin{aligned} g'(t) &= (f(t) \cdot f(t))' \\ &= (\sum_{i=1}^{m} f_i(t)^2)' \\ &= \sum_{i=1}^{m} 2f_i(t)f_i'(t) \\ &= 2f(t) \cdot f'(t) \end{aligned}$$

(b) Let $f : \mathbb{R} \to \mathbb{R}^3$ represent the position of a particle on a sphere through time. Let $g(t) = f(t) \cdot f(t)$. Since the particle is moving in a sphere, the norm of the position must be a constant, so $g'(t) = 0$. Since $g'(t) = 2f(t) \cdot f'(t)$, then $f(t) \cdot f'(t) = 0$, so position and velocity are always orthogonal.

(c) Let $f : \mathbb{R} \to \mathbb{R}^3$ represent the velocity of a particle moving at a constant speed. Let $g(t) = f(t) \cdot f(t)$. Since the speed is constant, the norm of the velocity must be a constant, so $g'(t) = 0$. Since $g'(t) = 2f(t) \cdot f'(t)$, then $f(t) \cdot f'(t) = 0$, so velocity and acceleration are always orthogonal.