Math 31BH: Assignment 1

Due 01/09 at 23:59

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1. Let $(\mathbf{V}, \langle\cdot,\cdot\rangle)$ be a Euclidean space, and let $\mathcal{B} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be an orthonormal basis of $\mathbf{V}$. Show that the $n \times 1$ coordinate matrix of any $\mathbf{v} \in \mathbf{V}$ relative to $\mathcal{B}$ is given by

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} \langle \mathbf{e}_1, \mathbf{v} \rangle \\ \vdots \\ \langle \mathbf{e}_n, \mathbf{v} \rangle \end{bmatrix}.$$

Solution: Let $v = a_1e_1 + \cdots + a_ne_n$. Using the bilinearity of the inner product and the definition of an orthonormal basis, we have that for a arbitrary index i

$$\begin{aligned} \langle e_i, v \rangle &= \langle e_i, a_1e_1 + \cdots + a_ie_i + \cdots + a_ne_n \rangle \\ &= a_1\langle e_i, e_1 \rangle + \cdots + a_i\langle e_i, e_i \rangle + \cdots + a_n\langle e_i, e_n \rangle \\ &= a_i \end{aligned}$$

Since $\langle e_i, v \rangle = a_i$ for all indexes,

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} \langle \mathbf{e}_1, \mathbf{v} \rangle \\ \vdots \\ \langle \mathbf{e}_n, \mathbf{v} \rangle \end{bmatrix}$$

2. With the same notation as in the previous problem, let $A \in \operatorname{End}\mathbf{V}$ be a linear operator. Show that the $n \times n$ matrix of A relative to $\mathcal{B}$ is

$$[A]_{\mathcal{B}} = \begin{bmatrix} \langle \mathbf{e}_1, A\mathbf{e}_1 \rangle & \dots & \langle \mathbf{e}_1, A\mathbf{e}_n \rangle \\ \vdots & & \vdots \\ \langle \mathbf{e}_n, A\mathbf{e}_1 \rangle & \dots & \langle \mathbf{e}_n, A\mathbf{e}_n \rangle \end{bmatrix}.$$

Solution: Let the ith row and jth column of $[A]_{\mathcal{B}}$ be a_{ij}.

$$\begin{aligned} \langle e_i, Ae_j \rangle &= \langle e_i, a_{1j}e_1 + \cdots + a_{ij}e_i + \cdots + a_{nj}e_n \rangle \\ &= a_{1j}\langle e_i, e_1 \rangle + \cdots + a_{ij}\langle e_i, e_i \rangle + \cdots + a_{nj}\langle e_i, e_n \rangle \\ &= a_{ij} \end{aligned}$$

Since $\langle e_i, Ae_j \rangle = a_{ij}$ for all indexes,

$$[A]_{\mathcal{B}} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} = \begin{bmatrix} \langle \mathbf{e}_1, A\mathbf{e}_1 \rangle & \dots & \langle \mathbf{e}_1, A\mathbf{e}_n \rangle \\ \vdots & & \vdots \\ \langle \mathbf{e}_n, A\mathbf{e}_1 \rangle & \dots & \langle \mathbf{e}_n, A\mathbf{e}_n \rangle \end{bmatrix}$$

3. With the same notation as in the previous problems, for each $1 \leq i, j \leq n$ let $E_{ij} \in \mathrm{End}\mathbf{V}$ be the linear operator defined by

$$E_{ij}\mathbf{e}_k = \langle \mathbf{e}_j, \mathbf{e}_k \rangle \mathbf{e}_i, \quad 1 \leq k \leq n.$$

What is the matrix of E_{ij} relative to $\mathcal{B}$?

Solution: Since the basis is orthonormal, the expression $E_{ij}e_k$ is equal to e_i when $j = k$ and 0 when $j \neq k$. In other words, E_{ij} sends the basis vector e_j to the basis vector e_i. Thus the matrix of E_{ij} will contain a 1 at the index (i, j) and 0 everywhere else.

4. With the same notation as in the previous problems, prove that

$$E_{ij}E_{kl} = \langle \mathbf{e}_j, \mathbf{e}_k \rangle E_{il}$$

Deduce from this that $\mathcal{E} = \{E_{ij} \colon 1 \leq i, j \leq n\}$ is an orthonormal basis of $\mathrm{End}\mathbf{V}$, where by definition the scalar product of two operators $A, B \in \mathrm{End}\mathbf{V}$ is $\langle A, B \rangle = \operatorname{Tr} A^*B$, with Tr the trace and A^* the adjoint (aka transpose) of A. What is $\dim \mathrm{End}\mathbf{V}$?

Solution: Let m be a arbitrary index. We have that

$$\begin{aligned} E_{ij}E_{kl}e_m &= E_{ij}\langle e_l, e_m \rangle e_k \\ &= \langle e_l, e_m \rangle E_{ij}e_k \\ &= \langle e_l, e_m \rangle \langle e_j, e_k \rangle e_i \\ &= \langle e_j, e_k \rangle \langle e_l, e_m \rangle e_i \\ &= \langle e_j, e_k \rangle E_{il}e_m \end{aligned}$$

Since $E_{ij}E_{kl}$ is a linear transformation and it sends all basis vectors to the same element as $\langle e_j, e_k \rangle E_{il}$, we have that $E_{ij}E_{kl} = \langle e_j, e_k \rangle E_{il}$.

The elements in $\mathcal{E}$ can be represented by $n \times n$ matrices with one 1 and 0 everywhere else; these matrices are linearly independent and also span all endomorphisms of V since $\mathrm{End}\mathbf{V}$ can be represented as all $n \times n$ matrices. Since

$$\begin{aligned} \langle E_{ij}, E_{kl} \rangle &= \operatorname{Tr} E_{ij}^* E_{kl} \\ &= \operatorname{Tr} E_{ji}E_{kl} \\ &= \operatorname{Tr} \langle e_i, e_k \rangle E_{jl} \end{aligned}$$

$\langle E_{ij}, E_{kl}\rangle$ is equal to 1 iff $i = k$ and $j = l$ and 0 otherwise. Therefore $\mathcal{E}$ is an orthonormal basis. The dimension of End$\mathbf{V}$ is $n \cdot n = n^2$.

5. With the same notation as in the previous problems, prove that $\mathcal{S} = \{E_{ij} + E_{ji} \colon 1 \le i \le j \le n\}$ is an orthogonal basis of the subspace Sym$\mathbf{V}$ of End$\mathbf{V}$ consisting of symmetric operators. What is $\dim \mathrm{Sym}\mathbf{V}$?

   Solution: The elements in $\mathcal{S}$ can be represented by symmetric $n \times n$ matrices with two 1s off the diagonal and 0 everywhere else or a 2 on the diagonal and 0 everywhere else. Every $n \times n$ symmetric matrix $A \in \mathrm{Sym}\mathbf{V}$ with entries a_{ij} can then be written as

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} a_{ij}(E_{ij} + E_{ji}) + \sum_{i=1}^{n} \frac{a_{ii}}{2}(E_{ii} + E_{ii})$$

   Therefore, $\mathcal{S}$ spans Sym$\mathbf{V}$.

   Since

$$\begin{aligned}
\langle E_{ij} + E_{ji}, E_{kl} + E_{lk}\rangle &= \mathrm{Tr}(E_{ij} + E_{ji})^*(E_{kl} + E_{lk}) \\
&= \mathrm{Tr}(E_{ij} + E_{ji})(E_{kl} + E_{lk}) \\
&= \mathrm{Tr}\, E_{ij}E_{kl} + E_{ij}E_{lk} + E_{ji}E_{kl} + E_{ji}E_{lk} \\
&= \mathrm{Tr}\langle e_j, e_k\rangle E_{il} + \langle e_j, e_l\rangle E_{ik} + \langle e_i, e_k\rangle E_{jl} + \langle e_i, e_l\rangle E_{jk}
\end{aligned}$$

   $\langle E_{ij} + E_{ji}, E_{kl} + E_{lk}\rangle$ is nonzero iff $i = k$ and $j = l$ or $i = l$ and $j = k$, both of which would indicated that $E_{ij} + E_{ji} = E_{kl} + E_{lk}$. Therefore $\mathcal{S}$ is orthogonal, which implies it is also linearly independent. Since, $\mathcal{S}$ also spans Sym$\mathbf{V}$, $\mathcal{S}$ is an orthogonal basis for Sym$\mathbf{V}$.

   The dimension of Sym$\mathbf{V}$ is the number of upper triangular elements in the $n \times n$ matrix representation, so the dimension is $1 + \cdots + n = \frac{n(n+1)}{2}$.