Homework 6

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Exercise 6.4.1: If the cone M of equation $f(x,y,z) = x^2 + y^2 - z^2 = 0$ (Example 5.2.4) is oriented by ∇f, does the parametrization $\gamma : (r, \theta) \mapsto (rcos\theta, rsin\theta, r)$ preserve orientation?

Solution: The gradient of the locus is

$$\nabla f = \begin{bmatrix} 2x \\ 2y \\ -2z \end{bmatrix} = \begin{bmatrix} 2r\cos\theta \\ 2r\sin\theta \\ -2r \end{bmatrix}$$

The derivative of the parameterization is

$$D\gamma(r,\theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \\ 1 & 0 \end{bmatrix}$$

The orientation of the transformation is orientation reversing since

$$\begin{aligned} \det \begin{bmatrix} 2r\cos\theta & \cos\theta & -r\sin\theta \\ 2r\sin\theta & \sin\theta & r\cos\theta \\ -2r & 1 & 0 \end{bmatrix} &= -2r(r\cos^2\theta + r\sin^2\theta) - (2r^2\cos^2\theta + 2r^2\sin^2\theta) \\ &= -2r^2 - 2r^2 \\ &= -4r^2 \\ &\leq 0 \end{aligned}$$

Exercise 6.4.4: What is the integral $\int_S x_3 dx_1 \wedge dx_2 \wedge dx_4$, where S is the part of the 3-dimensional manifold of equation

$$x_4 = x_1 x_2 x_3 \quad \textit{where } 0 \leq x_1, x_2, x_3 \leq 1$$

oriented by $\Omega = sgn\ dx_1 \wedge dx_2 \wedge dx_3$? Hint: This surface is a graph, so it is easy to parametrize.

Solution: The manifold can be parameterized as

$$\gamma(x_1, x_2, x_3) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_1 x_2 x_3 \end{bmatrix}$$

The derivative of the parameterization is

$$D\gamma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ x_2 x_3 & x_1 x_3 & x_1 x_2 \end{bmatrix}$$

This parameterization is oriented since

$$dx_1 \wedge dx_2 \wedge dx_3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ x_2 x_3 & x_1 x_3 & x_1 x_2 \end{bmatrix} = 1$$

The integral is

$$\begin{aligned}
\int_S x_3 dx_1 \wedge dx_2 \wedge dx_4 &= \int_0^1 \int_0^1 \int_0^1 (x_3 dx_1 \wedge dx_2 \wedge dx_4) \left(P_{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_1 x_2 x_3 \end{pmatrix}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ x_2 x_3 & x_1 x_3 & x_1 x_2 \end{bmatrix} \right) dx_1\, dx_2\, dx_3 \\
&= \int_0^1 \int_0^1 \int_0^1 (x_1 x_2 x_3)\, dx_1\, dx_2\, dx_3 \\
&= \frac{1}{8}
\end{aligned}$$

Exercise 6.5.4: Show that $\Phi_{\vec{F}\times\vec{G}} = W_{\vec{F}} \wedge W_{\vec{G}}$.

Solution:
Using the fact that the wedge product is distributive,

$$\begin{aligned} W_F \wedge W_G &= (F_1 dx_1 + F_2 dx_2 + F_3 dx_3) \wedge (G_1 dx_1 + G_2 dx_2 + G_3 dx_3) \\ &= (F_2 G_3 dx_2 \wedge dx_3 + F_3 G_2 dx_3 \wedge dx_2) + (F_3 G_1 dx_3 \wedge dx_1 + F_1 G_3 dx_1 \wedge dx_3) \\ &\qquad + (F_1 G_2 dx_1 \wedge dx_2 + F_2 G_1 dx_2 \wedge dx_1) \\ &= (F_2 G_3 - F_3 G_2) dx_2 \wedge dx_3 + (F_3 G_1 - F_1 G_3) dx_3 \wedge dx_1 \\ &\qquad + (F_1 G_2 - F_2 G_1) dx_1 \wedge dx_2 \\ &= \Phi_{F\times G} \end{aligned}$$

Exercise 6.5.5: Show that $M_{\vec{F}\cdot\vec{G}} = W_{\vec{F}} \wedge \Phi_{\vec{G}} = W_{\vec{G}} \wedge \Phi_{\vec{F}}$.

Solution:
Distributing and factoring shows $M_{\vec{F}\cdot\vec{G}} = W_{\vec{F}} \wedge \Phi_{\vec{G}}$

$$\begin{aligned}
W_F \wedge \Phi_G &= (F_1 dx_1 + F_2 dx_2 + F_3 dx_3) \wedge (G_1 dx_2 \wedge dx_3 + G_2 dx_3 \wedge dx_1 + G_3 dx_1 \wedge dx_2) \\
&= F_1 G_1 dx_1 \wedge dx_2 \wedge dx_3 + F_2 G_2 dx_2 \wedge dx_3 \wedge dx_1 + F_3 G_3 dx_3 \wedge dx_1 \wedge dx_2 \\
&= (F_1 G_1 + F_2 G_2 + F_3 G_3) dx_1 \wedge dx_2 \wedge dx_3 \\
&= M_{F \cdot G}
\end{aligned}$$

Since the dot product is commutative, $M_{\vec{F}\cdot\vec{G}} = M_{\vec{G}\cdot\vec{F}} = W_{\vec{G}} \wedge \Phi_{\vec{F}}$.

Exercise 6.5.6: What is the work form field $W_{\vec{F}}(P_a(\vec{u}))$ of the vector field

$$\vec{F}(x, y, z) = (x^2 y, x - y, -z)$$

at $\vec{a} = (0, 1, 2)$, evaluated on the vector $\vec{u} = (1, -1, 1)$.

Solution: $F(a)$ is

$$\vec{F}(a) = (0, -1, -2)$$

The work form evaluated at $\vec{u}$ is

$$W_{\vec{F}}(P_a(\vec{u})) = (0, -1, -2) \cdot (1, -1, 1) = -1$$

Exercise 6.5.9:

a. Construct an oriented parallelogram anchored at $(1,1,0)$ to which the 2-form $\Phi = y dy \wedge dz + x dx \wedge dz - z dx \wedge dy$ of Example 6.5.3 will assign a positive number.

b. At what point x might you anchor $P_x(\vec{e}_1, \vec{e}_2)$ if you wanted Φ evaluated on the parallelogram to return a positive number? A negative number?

Solution:

The parallelogram $[\vec{e}_2, \vec{e}_3]$ yields a positive number

$$y dy \wedge dz + x dx \wedge dz - z dx \wedge dy \left(P_{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}} \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 1 + 0 - 0 = 1$$

Evaluating at point $(0,0,-1)$ would yield a positive number

$$y dy \wedge dz + x dx \wedge dz - z dx \wedge dy \left(P_{\begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \right) = 0 + 0 - (-1) = 1$$

Evaluating at point $(0,0,1)$ would yield a negative number

$$y dy \wedge dz + x dx \wedge dz - z dx \wedge dy \left(P_{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \right) = 0 + 0 - 1 = -1$$

Exercise 6.5.13: Verify that $\det(\vec{F}(x), \vec{v}_1, \ldots, \vec{v}_{n-1})$ is an $(n-1)$-form field, so that Definition 6.5.10 of the flux form on $\mathbb{R}^n$ makes sense.

Solution: Through development of the first column,

$$\det(\vec{F}, \vec{v}_1, \ldots, \vec{v}_{n-1}) = \sum_{i=1}^{n} (-1)^{i-1} F_i \, dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_n$$

Since the flux form is a linear combination of elementary $(n-1)$-forms, $\det(\vec{F}(x), \vec{v}_1, \ldots, \vec{v}_{n-1})$ is a $(n-1)$-form field.

Exercise 6.5.15: Given $\vec{F}(x,y,z) = (y^2, x+z, xz)$ and $f(x,y,z) = xz+zy$, the point $x = (1,1,-1)$, and the vectors $\vec{v}_1 = (0,1,1), \vec{v}_2 = (1,1,0), \vec{v}_3 = (-1,1,1)$, what is

a. the work form $W_{\vec{F}}(P_x(\vec{v}_1))$?

b. the flux form $\Phi_{\vec{F}}(P_x(\vec{v}_1, \vec{v}_2))$?

c. the mass form $M_f(P_x(\vec{v}_1, \vec{v}_2, \vec{v}_3))$?

Solution:

$F(x)$ is

$$\vec{F}(x) = (1, 0, -1)$$

The work form is

$$W_{\vec{F}}(P_x(\vec{v}_1)) = (1,0,-1) \cdot (0,1,1) = -1$$

The flux form is

$$\Phi_{\vec{F}}(P_x(\vec{v}_1, \vec{v}_2)) = \det \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} = -1 + 1 = 0$$

Since $f(x) = -2$, the mass form is

$$M_f(P_x(\vec{v}_1, \vec{v}_2, \vec{v}_3)) = -2 \det \begin{bmatrix} 0 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = -2(1) = -2$$

Exercise 6.5.17: Let R be the rectangle with vertices $(0,0),(0,a),(b,a),(b,0)$, with $a,b>0$, and oriented so that these vertices appear in that order. Find the work of the vector field $\vec{F}(x,y)=(xy,ye^x)$ around the boundary of R.

Solution:
The work from $(0,0)$ to $(0,a)$ can be parameterized with $\gamma(t)=(0,t)$

$$\int_0^a F(0,t)\cdot(0,1)\,dt=\int_0^a(0,t)\cdot(0,1)\,dt=\int_0^a t\,dt=\frac{a^2}{2}$$

The work from $(0,a)$ to (b,a) can be parameterized with $\gamma(t)=(t,a)$

$$\int_0^b F(t,a)\cdot(1,0)\,dt=\int_0^b(at,ae^t)\cdot(1,0)\,dt=\int_0^b at\,dt=\frac{ab^2}{2}$$

The work from (b,a) to $(b,0)$ can be parameterized with $\gamma(t)=(b,t)$

$$\int_a^0 F(b,t)\cdot(0,1)\,dt=-\int_0^a(bt,te^b)\cdot(0,1)\,dt=-\int_0^a te^b\,dt=-\frac{a^2e^b}{2}$$

The work from $(b,0)$ to $(0,0)$ can be parameterized with $\gamma(t)=(t,0)$

$$\int_b^0 F(t,0)\cdot(1,0)\,dt=-\int_0^b(0,e^t)\cdot(1,0)\,dt=0$$

The total work is

$$\frac{a^2}{2}+\frac{ab^2}{2}-\frac{a^2e^b}{2}$$

Exercise 6.5.18: Find the work of $\vec{F}(x,y,z) = (x^2, y^2, z^2)$ over the arc of helix parametrized by $\gamma(t) = (cost, sint, at)$ for $0 \leq t \leq \alpha$, and oriented so that γ is orientation preserving.

Solution: Evaluating the integral gives

$$\begin{aligned}
\int_0^\alpha F(\cos t, \sin t, at) \cdot (-\sin t, \cos t, a)\, dt &= \int_0^\alpha (\cos^2 t, \sin^2 t, a^2t^2) \cdot (-\sin t, \cos t, a)\, dt \\
&= \int_0^\alpha -\sin t \cos^2 t + \sin^2 t \cos t + a^3 t^2\, dt \\
&= \frac{1}{3} \left[\cos^3 t + \sin^3 t + a^3 t^3\right]_0^\alpha \\
&= \frac{\cos^3 \alpha + \sin^3 \alpha + a^3 \alpha^3}{3} - \frac{1}{3}
\end{aligned}$$

Exercise 6.5.20: What is the flux of the vector field $\vec{F}(x,y,z) = (x,-y,xy)$ through the surface $z = \sqrt{x^2+y^2}$, $x^2+y^2 \leq 1$, oriented by the outward normal?

Solution: The surface can be parameterized as

$$\gamma(r,\theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ r \end{bmatrix}$$

The derivative of the parameterization is

$$D\gamma = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \\ 1 & 0 \end{bmatrix}$$

The outward normal vector is

$$\vec{n} = \nabla F = \begin{bmatrix} 2x \\ 2y \\ -2z \end{bmatrix} = 2\begin{bmatrix} r\cos\theta \\ r\sin\theta \\ -r \end{bmatrix}$$

γ is orientation reversing since

$$\begin{aligned} \det\begin{bmatrix} r\cos\theta & \cos\theta & -r\sin\theta \\ r\sin\theta & \sin\theta & r\cos\theta \\ -r & 1 & 0 \end{bmatrix} &= r(r\cos^2\theta + r\sin^2\theta) - 1(r^2\cos^2\theta + r^2\sin^2\theta) \\ &= -2r^2 \\ &\leq 0 \end{aligned}$$

Integrating the flux through the surface yields

$$\begin{aligned} -\int_0^{2\pi}\int_0^1 \det\begin{bmatrix} r\cos\theta & \cos\theta & -r\sin\theta \\ -r\sin\theta & \sin\theta & r\cos\theta \\ r^2\sin\theta\cos\theta & 1 & 0 \end{bmatrix} dr\,d\theta &= -\int_0^{2\pi}\int_0^1 \frac{1}{2}r^3\sin(2\theta) - r^2\cos(2\theta)\,dr\,d\theta \\ &= -\int_0^{2\pi} \frac{1}{6}\sin(2\theta) - \frac{1}{2}\cos(2\theta)\,d\theta \\ &= 0 \end{aligned}$$