Math 188: Homework 1

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Problem 1: Closed Form of a Recurrence Relation.

The characteristic equation of $a_n = 5a_{n-1} - 8a_{n-2} + 4a_{n-3}$ is

$$t^3 - 5t^2 + 8t - 4 = (t-2)^2(t-1)$$

The closed form solution is in the form

$$a_n = \alpha_1 2^n + \alpha_2 n 2^n + \alpha_3$$

Plugging in the values of the starting conditions yields

$$\begin{aligned}
1 &= \alpha_1 + \alpha_3 \\
1 &= 2\alpha_1 + 2\alpha_2 + \alpha_3 \\
2 &= 4\alpha_1 + 8\alpha_2 + \alpha_3
\end{aligned}$$

Solving for the constants yields $\alpha_1 = -1, \alpha_2 = \frac{1}{2}, \alpha_3 = 2$. The closed formula for the recurrence relation is

$$a_n = -2^n + \frac{n2^n}{2} + 2$$

I used Wolfram Alpha for factoring the characteristic equation and solving the system of equations for the constants.

Problem 2: Vandermonde Matrix Determinant is Nonzero.

We will induct on d to prove that the determinant is nonzero. For $d = 1$, the determinant is 1. For $d = n$, suppose that $(r_i^{j-1})_{i,j=1,\ldots,n}$ has nonzero determinant for all $r_1, \ldots, r_n$ distinct. Let $V = (r_i^{j-1})_{i,j=1,\ldots,n+1}$ for $r_1, \ldots, r_{n+1}$ distinct.

$$V = \begin{bmatrix} 1 & r_1 & r_1^2 & \ldots & r_1^n \\ 1 & r_2 & r_2^2 & \ldots & r_2^n \\ 1 & r_3 & r_3^2 & \ldots & r_3^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & r_{n+1} & r_{n+1}^2 & \ldots & r_{n+1}^n \end{bmatrix}$$

Starting from the last column and ending at the second column, subtract each column by r_1 times the previous column. Then use the laplace expansion formula along the first row, and then factor $(r_i - r_1)$ out every ith row.

$$\begin{aligned}
\det \begin{bmatrix} 1 & r_1 & r_1^2 & \ldots & r_1^n \\ 1 & r_2 & r_2^2 & \ldots & r_2^n \\ 1 & r_3 & r_3^2 & \ldots & r_3^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & r_{n+1} & r_{n+1}^2 & \ldots & r_{n+1}^n \end{bmatrix} &= \det \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 1 & r_2 - r_1 & r_2^2 - r_2 r_1 & \ldots & r_2^n - r_2^{n-1} r_1 \\ 1 & r_3 - r_1 & r_3^2 - r_3 r_1 & \ldots & r_3^n - r_3^{n-1} r_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & r_{n+1} - r_1 & r_{n+1}^2 - r_{n+1} r_1 & \ldots & r_{n+1}^n - r_{n+1}^{n-1} r_1 \end{bmatrix} \\
&= \det \begin{bmatrix} r_2 - r_1 & r_2(r_2 - r_1) & \ldots & r_2^{n-1}(r_2 - r_1) \\ r_3 - r_1 & r_3(r_3 - r_1) & \ldots & r_3^{n-1}(r_3 - r_1) \\ \vdots & \vdots & \ddots & \vdots \\ r_{n+1} - r_1 & r_{n+1}(r_{n+1} - r_1) & \ldots & r_{n+1}^{n-1}(r_{n+1} - r_1) \end{bmatrix} \\
&= (r_2 - r_1)(r_3 - r_1) \ldots (r_{n+1} - r_1) \det \begin{bmatrix} 1 & r_2 & \ldots & r_2^{n-1} \\ 1 & r_3 & \ldots & r_3^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & r_{n+1} & \ldots & r_{n+1}^{n-1} \end{bmatrix} \\
&\neq 0
\end{aligned}$$

Since all the r_i terms are distinct, the $(r_i - r_1)$ terms are nonzero. By our inductive hypothesis, the determinant of the matrix on the right must be nonzero. Thus, the determinant of $(r_i^{j-1})_{i,j=1,\ldots,d}$ is nonzero.

A nonzero determinant implies that each of the rows are linearly independent, meaning that the sequences $(r_1^n)_{n \geq 0}, \ldots, (r_d^n)_{n \geq 0}$ that make up the rows of the matrix are linearly independent.

Proof adapted from Wikipedia.

Problem 3: Recurrence Relation as Piecewise Polynomials

The characacteristic polynomial, $(t^2-1)^d = (t-1)^d(t+1)^d$, implies that for constants α and β, the closed form solution of the recurrence relation is

$$a_n = \sum_{i=0}^{d-1} \alpha_i n^i (-1)^n + \sum_{i=0}^{d-1} \beta_i n^i$$

$(-1)^n = 1$ for even n and $(-1)^n = -1$ for odd n so

$$a_n = \begin{cases} \sum_{i=0}^{d-1} (\alpha_i + \beta_i) n^i & \text{n is even} \\ \sum_{i=0}^{d-1} (\beta_i - \alpha_i) n^i & \text{n is odd} \end{cases}$$

Since the equations for each case are polynomials of degree $d-1$, the statement is true.

Problem 4: Equal Linear Recurrence Sequences

1. Since both sequences are of order d and share the same d terms after k, all the terms $n \geq k+d$ can be uniquely determined through the linear recurrence relation equation. The a_{k-1}th term is determined through the equation,

$$a_{k+d-1} = c_1 a_{k+d-2} + c_2 a_{k+d-3} + \ldots + c_d a_{k-1}$$

   Solving out for a_{k-1} yields an equation that only depends on constants and a_k through a_{k+d-1}.

$$a_{k-1} = \frac{a_{k+d-1} - c_1 a_{k+d-2} - c_2 a_{k+d-3} - \ldots - c_{d-1} a_k}{c_d}$$

   Repeating this proceedure uniquely determines all terms from 0 to $k-1$ to be the same for both sequences. Since both sequences have the same terms, they are equal.

2. The -1st term can be found similarly to the $(k-1)$ term from the previous part by the equation

$$b_{-1} = \frac{b_{d-1} - c_1 b_{d-2} - c_2 b_{d-3} - \ldots - c_{d-1} b_0}{c_d}$$

   Repeating this process uniquely determines the values of the sequence for negative indexes since the next negative index solely depends on the d larger values in the sequence.

3. The next negative fibonacci number can be determined using

$$f_{n-2} = f_n - f_{n-1}$$

   This gives the value that are the same as the fibonacci values but with alternating signs.

$$\begin{aligned} f_{-1} &= 1 \\ f_{-2} &= -1 \\ f_{-3} &= 2 \\ f_{-4} &= -3 \\ f_{-5} &= 5 \\ f_{-6} &= -8 \\ &\vdots \end{aligned}$$

Problem 5: Limits of Sums and Products of Power Series

1. For all n, if $([x^n]A_i(x))_i = [x^n]A(x)$ for some $i \geq N_a(n)$ and $([x^n]B_i(x))_i = [x^n]B(x)$ for some $i \geq N_b(n)$, then

$$([x^n](A_i(x) + B_i(x)))_i = [x^n](A(x) + B(x))$$

   is true for $i \geq \max(N_a(n), N_b(n))$. Thus, $\lim_{i\to\infty}(A_i(x)+B_i(x)) = A(x)+B(x)$.

2. For all n, if $([x^n]A_i(x))_i = [x^n]A(x)$ for some $i \geq N_a(n)$ and $([x^n]B_i(x))_i = [x^n]B(x)$ for some $i \geq N_b(n)$, then

$$A(x)B(x) = \sum_{n\geq 0}\left(\sum_{i=0}^{n} a_i b_{n-i}\right)x^n$$

   Thus the coefficient in the nth term depends on the coefficients of powers from 0 to n from A and B. So the nth term in the product will converge at $i \geq \max_{0\leq m\leq n}(\max(N_a(m), N_b(m)))$. Thus, $\lim_{i\to\infty}(A_i(x)B_i(x)) = A(x)B(x)$.