Math 31AH: Spring 2021
Homework 3 Solutions
Due 5:00pm on Friday 10/15/2021

Problem 1: Direct sum. Let V and W be an $\mathbb{F}$-vector spaces. The *direct sum* of V and W, denoted $V \oplus W$, is the $\mathbb{F}$-vector space

$$V \oplus W := \{(\mathbf{v}, \mathbf{w}) \, : \, \mathbf{v} \in V, \; \mathbf{w} \in W\}$$

with addition and scalar multiplication defined by

$$(\mathbf{v}, \mathbf{w}) + (\mathbf{v}', \mathbf{w}') := (\mathbf{v} + \mathbf{v}', \mathbf{w} + \mathbf{w}') \qquad c \cdot (\mathbf{v}, \mathbf{w}) := (c \cdot \mathbf{v}, c \cdot \mathbf{w})$$

where $\mathbf{v}, \mathbf{v}' \in V, \mathbf{w}, \mathbf{w}' \in W$, and $c \in \mathbb{F}$. Suppose $\mathcal{B}$ is a basis of V and $\mathcal{C}$ is a basis of W. Prove that

$$\mathcal{B} \oplus \mathcal{C} := \{(\mathbf{v}, \mathbf{0}) \, : \, \mathbf{v} \in \mathcal{B}\} \cup \{(\mathbf{0}, \mathbf{w}) \, : \, \mathbf{w} \in \mathcal{C}\}$$

is a basis of $V \oplus W$. In particular, we have the useful formula

$$\dim V \oplus W = \dim V + \dim W$$

whenever V and W are finite-dimensional.

Solution: We show first that $\mathcal{B} \oplus \mathcal{C}$ spans $V \oplus W$. Let $(\mathbf{v}, \mathbf{w}) \in V \oplus W$. Since $\mathcal{B}$ spans V, there exist $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathcal{B}$ and $c_1, \dots, c_n \in \mathbb{F}$ such that $c_1\mathbf{v}_1 + \cdots c_n\mathbf{v}_n = \mathbf{v}$. Since $\mathcal{C}$ spans W, there exist $\mathbf{w}_1, \dots, \mathbf{w}_m \in \mathcal{C}$ and $d_1, \dots, d_m \in \mathbb{F}$ such that $d_1\mathbf{w}_1 + \cdots + d_m\mathbf{w}_m = \mathbf{w}$. We have

$$\sum_{i=1}^{n} c_i(\mathbf{v}_i, \mathbf{0}) + \sum_{j=1}^{m} d_j(\mathbf{0}, \mathbf{w}_j) = (\mathbf{v}, \mathbf{w})$$

so that $\mathcal{B} \oplus \mathcal{C}$ spans $V \oplus W$.

We now show that $\mathcal{B} \oplus \mathcal{C}$ is linearly independent. Indeed, suppose $(\mathbf{v}_1, \mathbf{0}), \dots, (\mathbf{v}_n, \mathbf{0}), (\mathbf{0}, \mathbf{w}_1), \dots, (\mathbf{0}, \mathbf{w}_m) \in \mathcal{B} \oplus \mathcal{C}$ and

$$\sum_{i=1}^{n} c_i(\mathbf{v}_i, \mathbf{0}) + \sum_{j=1}^{m} d_j(\mathbf{0}, \mathbf{w}_j) = (\mathbf{0}, \mathbf{0})$$

for some scalars $c_i, d_j \in \mathbb{F}$. This implies

$$\sum_{i=1}^{n} c_i\mathbf{v}_i = \mathbf{0} \quad \text{and} \quad \sum_{j=1}^{m} d_j\mathbf{w}_j = \mathbf{0}$$

Since $\mathcal{B}$ and $\mathcal{C}$ are linearly independent, this gives $c_i, d_j = 0$ for all i, j. We conclude that $\mathcal{B} \oplus \mathcal{C}$ is linearly independent, so that $\mathcal{B} \oplus \mathcal{C}$ is a basis of $V \oplus W$.

Problem 2: Real sequences. Let V be the $\mathbb{R}$-vector space of all infinite sequences $(a_1, a_2, \dots)$ of real numbers, under the operations described in Homework 2. For any $i \geq 1$, let $\mathbf{e}_i \in V$ be the sequence

$$\mathbf{e}_i := (0, 0, \dots, 0, 1, 0, 0, \dots)$$

with a unique 1 in the i^{th} position and 0's elsewhere. Let

$$S = \{\mathbf{e}_1, \mathbf{e}_2, \dots\} \subseteq V$$

Does S span V? If not, describe the subspace $\text{span}(S)$ of V. Prove your claims.

Solution: No! We claim that $\text{span}(S)$ is the subspace $W \subseteq V$ consisting of eventually zero sequences (which is not equal to V since the sequence $(1, 1, 1, \dots)$ is not eventually zero).

Indeed, any element in $\text{span}(S)$ is a **finite** linear combination of the form

$$\mathbf{a} = a_{i_1}\mathbf{e}_{i_1} + \cdots + a_{i_r}\mathbf{e}_{i_r}$$

where $a_{i_j} \in \mathbb{R}$. If $N = \max(i_1, \dots, i_r)$, then $n > N$ implies $a_n = 0$ where $\mathbf{a} = (a_1, a_2, \dots)$. So, every element of $\text{span}(S)$ is eventually zero and $\text{span}(S) \subseteq W$.

On the other hand, let $\mathbf{a} = (a_1, a_2, \dots) \in W$. There exists N such that $n > N$ implies $a_n = 0$. We have

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + \cdots + a_N\mathbf{e}_N \in \text{span}(S)$$

so that $\text{span}(S) = W$.

Problem 3: A basis for polynomials. Let V be the $\mathbb{R}$-vector space of polynomials $f(t)$ in a variable t with degree $\leq n$. Show that the set

$$\mathcal{B} := \{1, (t+1), (t+1)^2, \dots, (t+1)^n\}$$

is a basis of V.

Solution: We know that V is $(n+1)$-dimensional (indeed, the set $\{1, t, t^2, \dots, t^n\}$ is a basis of V) and $\mathcal{B}$ has $n+1$ elements. Therefore, it suffices to show that $\mathcal{B}$ is linearly independent. To this end, suppose

$$\sum_{i=0}^{n} c_i(t+1)^i = 0$$

for some real numbers $c_0, c_1, \dots, c_{n+1}$. If the c_i are not all zero, choose d maximal so that $c_d \neq 0$. We have

$$c_d(t+1)^d + \cdots + c_1(t+1) + c_0 = 0$$

The coefficient of t^d on the left hand side is c_d, so we have $c_d = 0$. This is a contradiction, so that $c_i = 0$ for all i and $\mathcal{B}$ is linearly independent.

Problem 4: Real-valued functions. Let V be the $\mathbb{R}$-vector space of differentiable functions $f : \mathbb{R} \to \mathbb{R}$. Find an infinite linearly independent subset $I \subseteq V$. (Hint: You may use knowledge from calculus about growth rates.)

Solution: Let $I = \{x, x^2, x^3, x^4, \dots\}$ be the set of functions given by the powers of x. We claim that I is linearly independent. Indeed, suppose we have a linear relation

$$c_n x^n + \cdots + c_1 x = 0$$

with $c_n \neq 0$. From calculus, we know that

$$\lim_{x \to \infty} c_n x^n + \cdots + c_1 x = \begin{cases} +\infty & c_n > 0 \\ -\infty & c_n < 0 \end{cases}$$

In particular, the function $c_n x^n + \cdots + c_1 x$ is not the zero function, which is a contradiction. We conclude that I is linearly independent.

Problem 5: Homogeneous systems. Let $\mathbb{F}$ be a field, let A be an $m \times n$ matrix over $\mathbb{F}$, and let $A\mathbf{x} = \mathbf{0}$ be the associated homogeneous system of linear equations. Prove that the solution set of this system is a subspace of $\mathbb{F}^n$.

Solution: Let $W = \{\mathbf{w} \in \mathbb{F}^n : A\mathbf{w} = \mathbf{0}\}$ be the solution set in question. Since $A\mathbf{0} = \mathbf{0}$ we have $\mathbf{0} \in W$. If $\mathbf{w}, \mathbf{w}' \in W$ we have $A(\mathbf{w} + \mathbf{w}') = A\mathbf{w} + A\mathbf{w}' = \mathbf{0} + \mathbf{0} = \mathbf{0}$ so that $\mathbf{w} + \mathbf{w}' \in W$. Finally, if $\mathbf{w} \in W$ and $c \in \mathbb{F}$ we have $A(c\mathbf{w}) + c(A\mathbf{w}) = c \cdot \mathbf{0} = \mathbf{0}$ so that $c\mathbf{w} \in W$. We conclude that W is a subspace of $\mathbb{F}^n$.

Problem 6: Particular solutions. Let $\mathbb{F}$ be a field, let A be an $m \times n$ matrix over $\mathbb{F}$, let $\mathbf{b} \in \mathbb{F}^m$, and let $A\mathbf{x} = \mathbf{b}$ be the associated not-necessarily-homogeneous system of linear equations. Let $\mathbf{x}_0 \in \mathbb{F}^n$ be a vector such that

$$A\mathbf{x}_0 = \mathbf{b}$$

(The vector $\mathbf{x}_0$ is sometimes called a 'particular solution' of $A\mathbf{x} = \mathbf{b}$.) Let $W \subseteq \mathbb{F}^n$ be the solution set of the associated homogeneous system, i.e.

$$W := \{\mathbf{w} \in \mathbb{F}^n : A\mathbf{w} = \mathbf{0}\}$$

Prove that the full solution set to the original system $A\mathbf{x} = \mathbf{b}$ is

$$\{\mathbf{x} \in \mathbb{F}^n : A\mathbf{x} = \mathbf{b}\} = \{\mathbf{x}_0 + \mathbf{w} : \mathbf{w} \in W\}$$

Solution: First suppose $\mathbf{x} \in \mathbb{F}^n$ and $A\mathbf{x} = \mathbf{b}$. We claim that $\mathbf{x} - \mathbf{x}_0 \in W$. Indeed, we have $A(\mathbf{x} - \mathbf{x}_0) = A\mathbf{x} - A\mathbf{x}_0 = \mathbf{b} - \mathbf{b} = \mathbf{0}$. This shows that $\mathbf{x}-\mathbf{x}_0 \in W$ so that $\mathbf{x} = \mathbf{x}_0 + (\mathbf{x}-\mathbf{x}_0) \in \{\mathbf{x}_0 + \mathbf{w} : \mathbf{w} \in W\}$. This establishes the containment $\{\mathbf{x} \in \mathbb{F}^n : A\mathbf{x} = \mathbf{b}\} \subseteq \{\mathbf{x}_0 + \mathbf{w} : \mathbf{w} \in W\}$.

On the other hand, suppose $\mathbf{w} \in W$. We have $A(\mathbf{x}_0 + \mathbf{w}) = A\mathbf{x}_0 + A\mathbf{w} = \mathbf{b} + \mathbf{0} = \mathbf{b}$, so that $\mathbf{x}_0 + \mathbf{w} \in \{\mathbf{x} \in \mathbb{F}^n : A\mathbf{x} = \mathbf{b}\}$. This shows $\{\mathbf{x} \in \mathbb{F}^n : A\mathbf{x} = \mathbf{b}\} \supseteq \{\mathbf{x}_0 + \mathbf{w} : \mathbf{w} \in W\}$ and by the last paragraph $\{\mathbf{x} \in \mathbb{F}^n : A\mathbf{x} = \mathbf{b}\} = \{\mathbf{x}_0 + \mathbf{w} : \mathbf{w} \in W\}$.

Problem 7: Completing a basis. Let $n \geq 1$ be a positive integer and let V be an n-dimensional vector space over a field $\mathbb{F}$. Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$ be a finite linearly independent subset of V. Prove that there exist $n-s$ vectors $\mathbf{v}_{s+1}, \mathbf{v}_{s+2}, \ldots, \mathbf{v}_n \in V$ such that the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_s, \mathbf{v}_{s+1}, \ldots, \mathbf{v}_n\}$ is a basis of V.

Solution: We proceed by induction on the quantity $m := n - s$. Observe that $m \geq 0$ since $\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$ is linearly independent. If $m = 0$, then V is s-dimensional and the s-element linearly independent set $\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$ is already a basis.

Now assume $m > 0$, so that $\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$ does not span V. Choose an element $\mathbf{v}_{s+1} \in V - \mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_s)$ arbitrarily. We claim that $\{\mathbf{v}_1, \ldots, \mathbf{v}_s, \mathbf{v}_{s+1}\}$ is linearly independent. Indeed, if $c_i \in \mathbb{F}$ are not all zero and

$$c_1\mathbf{v}_1 + \cdots + c_s\mathbf{v}_s + c_{s+1}\mathbf{v}_{s+1} = \mathbf{0}$$

we must have $c_{s+1} \neq 0$ (since $\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$ is linearly independent so that

$$\mathbf{v}_{s+1} = c_{s+1}^{-1} \cdot (c_1\mathbf{v}_1 + \cdots + c_s\mathbf{v}_s) \in \mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_s)$$

which contradicts the assumption $\mathbf{v}_{s+1} \in V - \mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_s)$. Therefore, the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_s, \mathbf{v}_{s+1}\}$ is linearly independent. By induction on m, there exist $v_{s+2}, \ldots, v_n \in V$ such that $\{\mathbf{v}_1, \ldots, \mathbf{v}_s, \mathbf{v}_{s+1}, \mathbf{v}_{s+2}, \ldots, \mathbf{v}_n\}$ is a basis of V. Since the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_s, \mathbf{v}_{s+1}, \mathbf{v}_{s+2}, \ldots, \mathbf{v}_n\}$ contains $\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$, this completes the proof.

Problem 8: Trimming down to a basis. Let V be an $\mathbb{F}$-vector space where $\mathbb{F}$ is a field and $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\} \subseteq V$ be a finite subset which spans V. Prove that there exist $1 \leq i_1 < \cdots < i_n \leq m$ such that $\{\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_n}\}$ is a basis of V.

Solution: Since V is spanned by a finite set of vectors, V is finite dimensional; let $n := \dim V$. We induct on the (nonnegative!) number $r := m - n$. If $r = 0$, then $\dim V = m$ so that the spanning set $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ must be a basis.

Now suppose $r > 0$. Since $m > n = \dim V$, the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is not linearly independent. Thus, there exist scalars $c_1, \ldots, c_m \in \mathbb{F}$ not all zero so that $c_1\mathbf{v}_1 + \cdots + c_m\mathbf{v}_m = \mathbf{0}$. Choose $1 \leq i \leq n$ so that $c_i \neq 0$.

Now

$$\mathbf{v}_i = c_i^{-1}(c_1\mathbf{v}_1 + \cdots + c_{i-1}\mathbf{v}_{i-1} + c_{i+1}\mathbf{v}_{i+1} + \cdots + c_n\mathbf{v}_n),$$

so that $\mathbf{v}_i \in \mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n)$ and the smaller set

$$\{\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n\}$$

spans V. By induction on r, a subset of $\{\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n\}$ is a basis of V. This completes the proof.

Problem 9: (Optional; not to be handed in.) Let $n \geq 0$ and let

$$V = \{S \,:\, S \subseteq [n]\}$$

Define addition on V by

$$S + T := S \Delta T$$

where $S \Delta T := (S - T) \cup (T - S)$ is the *symmetric difference* of S and T. Give a definition of scalar multiplication which turns V into an $\mathbb{F}_2$-vector space.