Math 31CH HW 7
Due May 31 at 11:59 pm by Gradescope Submission
6.6.5 (skip part b), 6.7.3, 6.7.4, 6.7.6, 6.7.9, 6.8.6, 6.8.7, 6.8.12

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EXERCISES FOR SECTION 6.6

Exercise 6.6.5 (skip part b): Consider the region $X = P \cap B \subset \mathbb{R}^3$, where P is the plane of equation $x + y + z = 0$ and B is the ball $x^2 + y^2 + z^2 \leq 1$. Orient P by the normal $\vec{N} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and orient the sphere $x^2 + y^2 + z^2 = 1$ by the outward-pointing normal.

a. Which of $\operatorname{sgn} dx \wedge dy$, $\operatorname{sgn} dx \wedge dz$, $\operatorname{sgn} dy \wedge dz$ give the same orientation of P as $\vec{N}$?

c. Is the parametrization

$$t \mapsto \begin{pmatrix} \dfrac{\cos(t)}{\sqrt{2}} - \dfrac{\sin(t)}{\sqrt{6}} \\ -\dfrac{\cos(t)}{\sqrt{2}} - \dfrac{\sin(t)}{\sqrt{6}} \\ 2\dfrac{\sin(t)}{\sqrt{6}} \end{pmatrix}$$

compatible with the boundary orientation of ∂X?

d. Do any of $\operatorname{sgn} dx$, $\operatorname{sgn} dy$, $\operatorname{sgn} dz$ define the orientation of ∂X at every point?

e. Do any of $\operatorname{sgn} x\,dy - y\,dx$, $\operatorname{sgn} x\,dz - z\,dx$, $\operatorname{sgn} y\,dz - z\,dy$ define the orientation of ∂X at every point?

Part A: $\operatorname{sgn} dx \wedge dy$ and $\operatorname{sgn} dy \wedge dz$ give the same orientation of P as $\vec{N}$.

$$\begin{aligned}
\det \begin{bmatrix} 1 & v_x & w_x \\ 1 & v_y & w_y \\ 1 & v_z & w_z \end{bmatrix} &= (v_y w_z - v_z w_y) - (v_x w_z - v_z w_x) + (v_x w_y - v_y w_x) \\
&= (-v_y(w_x + w_y) + (v_x + v_y)w_y) - (-v_x(w_x + w_y) + (v_x + v_y)w_x) + (v_x w_y - v_y w_x) \\
&= -v_y w_x - v_y w_y + v_x w_y + v_y w_y + v_x w_x + v_x w_y - v_x w_x - v_y w_x + v_x w_y - v_y w_x \\
&= 3v_x w_y - 3v_y w_x = 3dx \wedge dy(\vec{v}, \vec{w}) \\
&= -3(v_x(-w_x - w_y) - (-v_x - v_y)w_x) = -3dx \wedge dy(\vec{v}, \vec{w}) \\
&= 3(v_y(-w_x - w_y) - (-v_x - v_y)w_y) = 3dy \wedge dz(\vec{v}, \vec{w})
\end{aligned}$$

Part C: $\gamma(t)$ is an outward pointing normal of X and the tangent vector of the parameterization is

$$\gamma'(t) = \begin{bmatrix} -\frac{\sin(t)}{\sqrt{2}} - \frac{\cos(t)}{\sqrt{6}} \\ \frac{\sin(t)}{\sqrt{2}} - \frac{\cos(t)}{\sqrt{6}} \\ 2\frac{\cos(t)}{\sqrt{6}} \end{bmatrix}$$

The parameterization is not consistent since

$$\begin{aligned}
\det\left[\vec{N}, \gamma(t), \gamma'(t)\right] &= 3dy \wedge dz\left[\gamma(t), \gamma'(t)\right] \\
&= 3\det\begin{bmatrix} -\frac{\cos(t)}{\sqrt{2}} - \frac{\sin(t)}{\sqrt{6}} & \frac{\sin(t)}{\sqrt{2}} - \frac{\cos(t)}{\sqrt{6}} \\ 2\frac{\sin(t)}{\sqrt{6}} & 2\frac{\cos(t)}{\sqrt{6}} \end{bmatrix} \\
&= 6\left(\left(-\frac{\cos^2(t)}{2\sqrt{3}} - \frac{\sin(t)\cos(t)}{6}\right) - \left(\frac{\sin^2(t)}{2\sqrt{3}} - \frac{\sin(t)\cos(t)}{6}\right)\right) \quad = -\sqrt{3}
\end{aligned}$$

Part D Since γ is orientation reversing, any form directly orienting ∂X must yield a negative yalue on $\gamma'(t)$ Since none of the components of $\gamma'(t)$ are always negative, $\operatorname{sgn} dx$, $\operatorname{sgn} dy$, and $\operatorname{sgn} dz$ do not define the orientation for ∂X.

Part E Calculating the forms on $\gamma'(t)$ yields that only $x\,dy - y\,dx$ and $y\,dz - z\,dy$ define the orientation on ∂X.

$$\begin{aligned}
x\,dy - y\,dx \implies & \left(\frac{\cos(t)}{\sqrt{2}} - \frac{\sin(t)}{\sqrt{6}}\right)\left(\frac{\sin(t)}{\sqrt{2}} - \frac{\cos(t)}{\sqrt{6}}\right) - \left(-\frac{\cos(t)}{\sqrt{2}} - \frac{\sin(t)}{\sqrt{6}}\right)\left(-\frac{\sin(t)}{\sqrt{2}} - \frac{\cos(t)}{\sqrt{6}}\right) \\
&= -\frac{\cos^2(t)}{\sqrt{3}} - \frac{\sin^2(t)}{\sqrt{3}} \\
&= -\frac{1}{\sqrt{3}}
\end{aligned}$$

$$\begin{aligned}
x\,dz - z\,dx \implies & \left(\frac{\cos(t)}{\sqrt{2}} - \frac{\sin(t)}{\sqrt{6}}\right)\left(2\frac{\cos(t)}{\sqrt{6}}\right) - \left(2\frac{\sin(t)}{\sqrt{6}}\right)\left(-\frac{\sin(t)}{\sqrt{2}} - \frac{\cos(t)}{\sqrt{6}}\right) \\
&= \frac{\cos^2(t)}{\sqrt{3}} + \frac{\sin^2(t)}{\sqrt{3}} \\
&= \frac{1}{\sqrt{3}}
\end{aligned}$$

$$\begin{aligned}
y\,dz - z\,dy \implies & \left(-\frac{\cos(t)}{\sqrt{2}} - \frac{\sin(t)}{\sqrt{6}}\right)\left(2\frac{\cos(t)}{\sqrt{6}}\right) - \left(2\frac{\sin(t)}{\sqrt{6}}\right)\left(\frac{\sin(t)}{\sqrt{2}} - \frac{\cos(t)}{\sqrt{6}}\right) \\
&= \frac{-\cos^2(t)}{\sqrt{3}} - \frac{\sin^2(t)}{\sqrt{3}} \\
&= -\frac{1}{\sqrt{3}}
\end{aligned}$$

EXERCISES FOR SECTION 6.7

Exercise 6.7.3: In Example 6.7.7, confirm that:
a. $d\Phi_{\vec{F}_2} = 0$.
b. $d\Phi_{\vec{F}_3} = 0$.
Part A:

$$\begin{aligned}
d\Phi_{\vec{F}_2} &= d\left(\frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy\right) \\
&= \left(D_1\frac{-y}{x^2+y^2}\,dx + D_2\frac{-y}{x^2+y^2}\,dy\right) \wedge dx + \left(D_1\frac{x}{x^2+y^2}\,dx + D_2\frac{x}{x^2+y^2}\,dy\right) \wedge dy \\
&= \left(\frac{2xy}{(x^2+y^2)^2}\,dx + \frac{y^2-x^2}{(x^2+y^2)^2}\,dy\right) \wedge dx + \left(\frac{y^2-x^2}{(x^2+y^2)^2}\,dx + \frac{-2xy}{(x^2+y^2)^2}\,dy\right) \wedge dy \\
&= 0
\end{aligned}$$

Part B:

$$\begin{aligned}
d\Phi_{\vec{F}_3} &= d\left(\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\,dy\wedge dz + \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}\,dz\wedge dx + \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\,dx\wedge dy\right) \\
&= \frac{-2x^2+y^2+z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}\,dx\wedge dy\wedge dz + \frac{-2y^2+x^2+z^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}\,dy\wedge dz\wedge dx + \frac{-2z^2+x^2+y^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}\,dz\wedge dx\wedge dy \\
&= 0
\end{aligned}$$

Exercise 6.7.4: Let φ be the 2-form on $\mathbb{R}^4$ given by

$$\varphi = x_1^2x_3\,dx_2\wedge dx_3 + x_1x_3\,dx_1\wedge dx_4.$$

Compute $d\varphi$.
Solution:

$$\begin{aligned}
d(x_1^2x_3\,dx_2\wedge dx_3 + x_1x_3\,dx_1\wedge dx_4) &= d(x_1^2x_3\,dx_2\wedge dx_3) + d(x_1x_3\,dx_1\wedge dx_4) \\
&= (D_1(x_1^2x_3)\,dx_1)\wedge dx_2\wedge dx_3 + (D_3(x_1x_3)\,dx_3)\wedge dx_1\wedge dx_4 \\
&= 2x_1x_3\,dx_1\wedge dx_2\wedge dx_3 - x_1\,dx_1\wedge dx_3\wedge dx_4
\end{aligned}$$

Exercise 6.7.6: Let f be a function from $\mathbb{R}^3$ to $\mathbb{R}$. Compute the exterior derivatives:
a. $d(f\,dx\wedge dz)$.
b. $d(f\,dy\wedge dz)$.
Part A:

$$\begin{aligned}
d(f\,dx\wedge dz) &= (D_1(f)dx + D_2(f)dy + D_3(f)dz)\wedge dx\wedge dz \\
&= -D_2(f)\,dx\wedge dy\wedge dz
\end{aligned}$$

Part B:

$$\begin{aligned}
d(f\,dy\wedge dz) &= (D_1(f)dx + D_2(f)dy + D_3(f)dz)\wedge dy\wedge dz \\
&= D_1(f)\,dx\wedge dy\wedge dz
\end{aligned}$$

Exercise 6.7.9: Find all the 1-forms $\omega = p(y,z)\,dx + q(x,z)\,dy$ such that

$$d\omega = x\,dy \wedge dz + y\,dx \wedge dz.$$

Solution:

$$\begin{aligned} d(p\,dx + q\,dy) &= (D_2 p\,dy + D_3 p\,dz) \wedge dx + (D_1 q\,dx + D_3 q\,dz) \wedge dy \\ &= (-D_3 q)\,dy \wedge dz + (-D_3 p)\,dx \wedge dz + (D_1 q - D_2 p)\,dx \wedge dy \\ &= x\,dy \wedge dz + y\,dx \wedge dz + 0\,dx \wedge dy \end{aligned}$$

From this,

$$D_3 q = -x$$

$$D_3 p = -y$$

$$D_1 q = D_2 p$$

Thus, ω must be in the form

$$\omega = (-yz + F(y))\,dx + (-xz + G(x))\,dy$$

where F and G are arbitrary differentiable functions.

EXERCISES FOR SECTION 6.8

Exercise 6.8.6: Show that $df = W_{\text{grad}\, f}$ when $f\begin{pmatrix}x\\y\\z\end{pmatrix} = xyz$ by computing both from the definitions and evaluating on a vector $\vec{v} = \begin{pmatrix}a\\b\\c\end{pmatrix}$.

Solution: The exterior derivative of f is

$$\begin{aligned} df(\vec{v}) &= D_1(xyz)\, dx + D_2(xyz)\, dy + D_3(xyz)\, dz \\ &= yz\, dx + xz\, dy + xy\, dz \end{aligned}$$

The work form of the gradient is

$$W_{\text{grad}\, f} = W_{\begin{bmatrix}yz\\xz\\xy\end{bmatrix}} = yz\, dx + xz\, dy + xy\, dz$$

Thus $df = W_{\text{grad}\, f}$.

Exercise 6.8.7: Let $\varphi = xy\, dx + z\, dy + yz\, dz$ be a 1-form on $\mathbb{R}^3$. For what vector field $\vec{F}$ can φ be written $W_{\vec{F}}$? Show the equivalence of $dW_{\vec{F}}$ and $\Phi_{\vec{\nabla}\times\vec{F}}$ by computing both from the definitions.

Solution: The 1-form can be rewritten as a work form with vector field

$$\vec{F} = \begin{bmatrix}xy\\z\\yz\end{bmatrix}$$

The exterior derivative of the work form is

$$\begin{aligned} d(xy\, dx + z\, dy + yz\, dz) &= d(xy\, dx) + d(z\, dy) + d(yz\, dz) \\ &= (D_2(xy)\, dy) \wedge dx + (D_3(z)\, dz) \wedge dy + (D_2(yz)\, dy) \wedge dz \\ &= -x\, dx \wedge dy + (z-1)\, dy \wedge dz \end{aligned}$$

The curl is

$$\vec{\nabla} \times \vec{F} = \begin{bmatrix}D_1\\D_2\\D_3\end{bmatrix} \times \begin{bmatrix}xy\\z\\yz\end{bmatrix} = \begin{bmatrix}z-1\\0\\-x\end{bmatrix}$$

The flux of the curl is equal to the exterior derivative of the work form

$$\Phi_{\vec{\nabla}\times\vec{F}} = (z-1)\, dy \wedge dz - 0\, dx \wedge dz - x\, dx \wedge dy = W_{\vec{F}}$$

Exercise 6.8.12: **a.** What is the divergence of $\vec{F}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{bmatrix}x^2\\y^2\\yz\end{bmatrix}$?

b. Use part (a) to compute $d\Phi_{\vec{F}} P_{\begin{pmatrix}1\\1\\2\end{pmatrix}}(\vec{e}_1, \vec{e}_2, \vec{e}_3)$.

c. Compute it again, directly form the definition of exterior derivative.

Part A:

$$\operatorname{div} \vec{F} = \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} \cdot \begin{bmatrix} x^2 \\ y^2 \\ z^2 \end{bmatrix} = 2x + 3y$$

Part B: Using part A, the derivative of the flux form of $\vec{F}$ is

$$d\Phi_{\vec{F}} = M_{\operatorname{div} \vec{F}} = (2x + 3y)\, dx \wedge dy \wedge dz$$

Therefore,

$$d\Phi_{\vec{F}} P_{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}(\vec{e}_1, \vec{e}_2, \vec{e}_3) = (2 + 3) \det(\vec{e}_1, \vec{e}_2, \vec{e}_3) = 5$$

Part C: The exterior derivative is

$$\begin{aligned} d\Phi_{\vec{F}} &= d(x^2\, dy \wedge dz + y^2\, dz \wedge dx + yz\, dx \wedge dy) \\ &= (2x\, dx) \wedge dy \wedge dz + (2y\, dy) \wedge dz \wedge dx + (y\, dz) \wedge dx \wedge dy \\ &= (2x + 3y)\, dx \wedge dy \wedge dz \end{aligned}$$

This is the same form as from part b, so evaluating it at the same parallelogram would also yield 5.