Math 120A: Homework 8

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Problem 1

We can parameterize $C = \{|z-2| = 2\}$ as

$$z(t) = 2e^{it} + 2 \qquad (0 \le t < 2\pi)$$

$$\begin{aligned}
\int_C \frac{1}{z-2}\,dz &= \int_0^{2\pi} \frac{1}{2e^{it}}\left(2ie^{it}\right)\,dz \\
&= \int_0^{2\pi} o\,dz \\
&= 2\pi i
\end{aligned}$$

Problem 2: Page 138, 1(a)

If z is on C then

$$|z+4| \leq |z| + |4| = 6$$

$$|z^3 - 1| \geq ||z|^3 - |1|| = 7.$$

Therefore

$$\left|\frac{z+4}{z^3-1}\right| \leq \frac{6}{7}$$

so $M = \frac{6}{7}$ and the length of C is $L = \pi$.

$$\left|\int_C \frac{z+4}{z^3-1}\,dz\right| \leq ML = \frac{6\pi}{7}.$$

Problem 3: Page 139, 4

If z is on C_R then

$$|2z^2 - 1| \leq 2|z|^2 + |-1| = 2R^2 + 1$$

$$|z^4 + 5z^2 + 4| = |(z^2 - 1)||(z^2 - 4)| \geq ||z|^2 - |1|||z|^2 - |4|| = (R^2 - 1)(R^2 - 4)$$

Therefore

$$\left|\frac{2z^2 - 1}{z^4 + 5z^2 + 4}\right| \leq \frac{2R^2 + 1}{(R^2 - 1)(R^2 - 4)}$$

so $M_R = \frac{2R^2+1}{(R^2-1)(R^2-4)}$ and the length of C is $L = \pi R$.

$$\left|\int_C \frac{2z^2 - 1}{z^4 + 5z^2 + 4}\right| \leq M_R L = \frac{\pi R(2R^2 + 1)}{(R^2 - 1)(R^2 - 4)}$$

If we divide the numerator and denominator by R^4, we see that the integral goes to 0 as $R \to \infty$ since $M_R L \to 0$.

$$\frac{\pi R(2R^2 + 1)}{(R^2 - 1)(R^2 - 4)} \cdot \frac{\frac{1}{R^4}}{\frac{1}{R^4}} = \frac{\pi(\frac{2}{R} + \frac{1}{R^3})}{(1 - \frac{1}{R^2})(1 - \frac{4}{R^2})}$$

Problem 4: Page 147, 1

$$\begin{aligned}
\int_C z^n \, dz &= \left[\frac{1}{n+1} z^{n+1}\right]_{z_1}^{z_2} \\
&= \frac{1}{n+1} z_2^{n+1} - \frac{1}{n+1} z_1^{n+1} \\
&= \frac{1}{n+1} (z_2^{n+1} - z_1^{n+1})
\end{aligned}$$

Problem 5: Page 147, 2(b)(c)

$$\begin{aligned}
\int_0^{\pi+2i} \cos\left(\frac{z}{2}\right)\, dz &= \left[2\sin\left(\frac{z}{2}\right)\right]_0^{\pi+2i} \\
&= 2\sin\left(\frac{\pi}{2}+i\right) \\
&= \frac{e^{i\frac{\pi}{2}-1} - e^{-i\frac{\pi}{2}+1}}{i} \\
&= \frac{e^{-1}i + ei}{i} \\
&= e + \frac{1}{e}
\end{aligned}$$

$$\begin{aligned}
\int_1^3 (z-2)^3\, dz &= \left[\frac{1}{4}(z-2)^4\right]_1^3 \\
&= \frac{1}{4} - \frac{1}{4} \\
&= 0
\end{aligned}$$

Problem 6

Note that the term we are integrating has an antiderivative.

$$\frac{d}{dz}\left(-\frac{1}{3}(z-5)^{-3}\right) = (z-5)^{-4}$$

Therefore for any closed contour which does not pass through 5, the integral is 0.