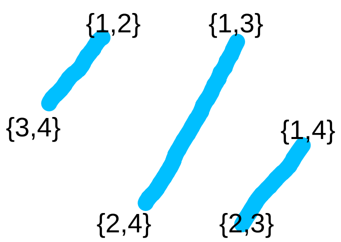
Math 158 Homework 1

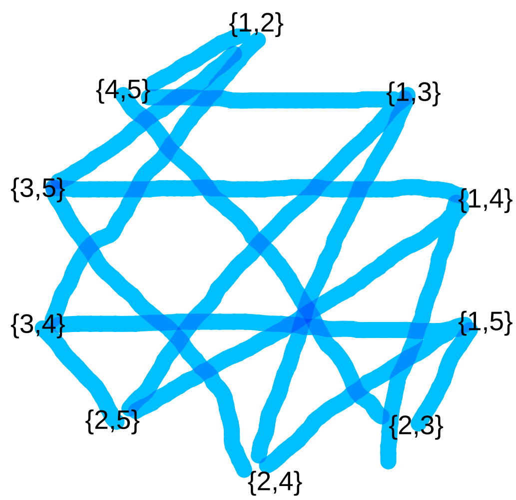
Homework 1 is to be handed in using this template in gradescope, by 4.00pm PST on Friday January 20th. Late homework will not be accepted.

Question 1.7.2° Let $K_{n:r}$ denote the ***Kneser graph***, whose vertex set is the set of r-element subsets of an n-element sets, and where two vertices form an edge if the corresponding sets are disjoint.

(a) Describe $K_{n:1}$ for $n \geq 1$.
(b) Draw $K_{4:2}$ and $K_{5:2}$.
(c) Determine $|E(K_{n:r})|$ for $n \geq 2r \geq 1$. [6]

a) The vertex set consists of all subsets of size 1 of [n], so the vertex set has n elements.
These subsets are all disjoint with each other, so K_{n:1} would be isomorphic to the complete graph of n vertices, K_n.
b)The left graph is K_{4:2} and the right graph is K_{5:2}





c) There are (n choose r) vertices and (n-r choose r) vertices that match with each vertex.
This will double count the number of edges, so the total number of edges is
1/2 (n choose r)(n-r choose r).

Question 1.7.4° Let G be a digraph such that every vertex has positive in-degree. Prove that G contains a directed cycle.

[3]

Let P be a directed path from v_1 to v_r.
Since every vertex has positive in-degree, there must always be a vertex, v_i, that connects to v_1.
If v_i if not on the path, then adding v_i to the path results in a new longer path.
If v_i is on the path, then G contains a directed cycle.
Repeating this process, v_i must at some point be on the path since G contains finite vertices.
Thus, G contains a directed cycle.

Question 1.7.12. Let G be an n-vertex graph with $n \geq 2$ and $\delta(G) \geq (n-1)/2$. Prove that G is connected and that the diameter of G is at most two.

[3]

Picking two arbitrary vertices as the ends of a path, the neighborhood of both vertices must overlap by the pigeonhole principle. There are n-2 vertices that are not the ends of the path, yet each end of the path must be connected to at least (n-1)/2 other vertices.
Thus each node is connected by a path of at most size two and the diameter of G is at most two.
Since the diameter is finite, G must be connected.

Question 1.7.14.

(a) Let P and Q be longest paths in a connected graph G. Prove that

$$V(P) \cap V(Q) \neq \emptyset.$$

[4]

Let P be a path from v_p1 to v_pk and let Q be a path from v_q1 to v_qk.

Assume that V(P) and V(Q) are disjoint.
Since G is a connected graph, it must be possible to traverse from any vertex to any other vertex.
Thus, there must exist a path that connects some v_pi to some v_qj whose edge set is disjoint with the edge set of P and Q.
If i > k/2 let P_max be the path from v_p1 to v_pi and otherwise, v_pi to v_pk.
Similarly, if j > k/2 let Q_max be the path from v_q1 to v_qj and otherwise, v_qj to v_qk.

Then the concatenation of the paths P_max, v_pi to v_qj, and Q_max yields a path that is longer than P and Q. Thus V(P) and V(Q) are disjoint.

Question 2.5.7. Prove that a graph of minimum degree at least $k \geq 2$ containing no triangles contains a cycle of length at least $2k$.

[4]

Let P be a longest path from v_1 to v_r. Since this is already the longest path, the neighborhood of v_r must only contain vertices on the path.
Let us induct on k.

For k=2, v_r must have a neighbor v_i such that i<=r-3, which satisfies the base case since the cycle v_i v_{i+1} ... v_r v_i must have a size of at least 4.

Assume that a graph of minimum degree k+1(meaning it is also minimum degree k) has a cycle,v_i v_{i+1} ... v_r v_i, of length at least 2k.
Since the graph contains no triangles, the neighbors of v_r cannot be neighbors of each other.
Thus there must be an edge from v_r to v_j, where j <= i-2.
The cycle v_j v_{j+1} ... v_r v_j thus has length at least 2k+2, completing the inductive step.