MATH 31AH - Homework 8

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1 Quotients and matrices

Proof. $\overline{T} : V/W \to V/W$ with $\overline{T}(v + W) = T(v) + W$ is well defined.

Let $\tilde{T} : V \to V/W$ with $\tilde{T}(v) := T(v) + W$. If $w \in W$, then $T(w) \in W$ since W is T-invariant. For all $w \in W$, $\tilde{T}(w) = T(w) + W = 0 + W$, so $W \subseteq \ker \tilde{T}$. Because of the universal property of quotient spaces, the function $\overline{T} : V/W \to V/W$ with $\overline{T}(v+W) = \tilde{T}(v) = T(v)+W$ is well-defined.

Since the first m indexes are an ordered basis for W and W is T-invariant, A represents T restricted to W. $\mathcal{B}-\mathcal{C}$ is a basis of V/W and $\overline{T}(v+W) = T(v)+W$, so C represents what $\overline{T}$ does to $v \in V - W$. $\square$

2 Quotients and Direct Sums

Proof. $(V \oplus W)/\mathcal{W} \cong V$.

To avoid notational abuse, we will use $\mathcal{W} := \{(0, w) : w \in W\}$. Let $T : V \to (V \oplus W)/\mathcal{W}$ with $T(v) := (v, 0) + \mathcal{W}$ be a function. Every coset in $(V \oplus W)/\mathcal{W}$ can uniquely be written as $(v, w) + \mathcal{W} = (v, 0) + \mathcal{W}$, so T is surjective. Let $v_1 \in V$ and $v_2 \in V$ be different vectors.

$$T(v_1) = (v_1, 0) + \mathcal{W} \text{ and } T(v_2) = (v_2, 0) + \mathcal{W}$$

Since $(v_2 - v_1, 0) \notin \mathcal{W}$, T is injective. Since T is injective and surjective, T is bijective.

T is linear since

$$\begin{aligned} T(c_1v_1 + c_2v_2) &= (c_1v_1 + c_2v_2, 0) + \mathcal{W} \\ &= c_1((v_1, 0) + \mathcal{W}) + c_2((v_2, 0) + \mathcal{W}) \\ &= c_1T(v_1) + c_2T(v_2) \end{aligned}$$

Since T is bijective and linear, it is an isomorphism. Therefore, $(V \oplus W)/\mathcal{W} \cong V$. $\square$

3 Quotients and Duals

Proof. U is a subspace of V^*

The zero functional has $\lambda_0(w) = 0$ for all $w \in W$, so $\lambda_0 \in U$.

For all $\lambda, \mu \in U$ and for all $w \in W$,

$$\begin{aligned}(\lambda + \mu)(w) &= \lambda(w) + \mu(w) \\ &= 0 + 0 \\ &= 0\end{aligned}$$

Therefore, $\lambda + \mu \in U$.

For all $\lambda \in U$, for all $c \in \mathbb{F}$, and for all $w \in W$,

$$\begin{aligned}(c\lambda)(w) &= c\lambda(w) \\ &= c \cdot 0 \\ &= 0\end{aligned}$$

Therefore, $c\lambda \in U$.

Since U has a zero, it is closed under addition, and it is closed under scalar multiplication, U is a subspace of V^*. $\square$

Proof. W^* and V^*/U are isomorphic.

Let $T : W^* \to V^*/U$ with $T(w) := w + U$ be a function. From the definition of U, W^* and U span V and $W^* \cap U = 0$ so $W^* \oplus U = V^*$. Therefore each coset in V^*/U can be uniquely written as $(w + u) + U = w + U$ with $w \in W^*$ and $u \in U$, so T is surjective. Let $w_1 \in W^*$ and $w_2 \in W^*$ be different functionals.

$$T(w_1) = w_1 + U \text{ and } T(w_2) = w_2 + U$$

Since $w_2 - w_1 \notin U$, T is injective. Since T is injective and surjective, T is bijective.

T is linear since

$$\begin{aligned}T(c_1 w_1 + c_2 w_2) &= (c_1 w_1 + c_2 w_2) + U \\ &= c_1(w_1 + U) + c_2(w_2 + U) \\ &= c_1 T(w_1) + c_2 T(w_2)\end{aligned}$$

Since T is bijective and linear, it is an isomorphism. Therefore, $W^* \cong V^*/U$. $\square$

Proof. $(V/W)^*$ and U are isomorphic.

Let $T : U \to (V/W)^*$ with $T(\lambda)(v + W) := \lambda(v)$ with $v \in V$ be a function. T is well-defined since for $v = v' + w$.

$$\begin{aligned}T(\lambda)(v + W) &= \lambda(v) \\ &= \lambda(v' + w) \\ &= \lambda(v') + \lambda(w) \\ &= \lambda(v') \\ &= T(\lambda)(v' + W)\end{aligned}$$

T has an inverse $T^{-1} : (V/W)^* \to U$ with $T^{-1}(\lambda)(v) := \lambda(v + W)$ for $\lambda \in (V/W)^*$ and $v \in V$, so T is bijective.

T is also linear since

$$\begin{aligned} T(c_1\lambda + c_2\mu)(v+W) &= (c_1\lambda + c_2\mu)(v) \\ &= c_1\lambda(v) + c_2\mu(v) \\ &= c_1 T(\lambda)(v+W) + c_2 T(\mu)(v+W) \end{aligned}$$

Since T is bijective and linear, T is a isomorphism. Therefore, $(V/W)^* \cong U$. □

4 Matrix Direct Sum

If A is a $m \times n$ matrix and B is a $p \times q$ matrix, then the first m entries of $(A \oplus B)x$ are $Ax_{1:n}$ and the last p entries of $(A \oplus B)x$ are $Bx_{n+1:q}$. This "segregates" the linear transformations of A and B to only acting on the first n entires and last q entries in the same way that T only acts on v and T' only acts on v' in $T \oplus T'$.

5 Matrix Tensor Product

Proof. $(A \otimes B) \cdot (C \otimes D) = (AC) \otimes (BD)$

The ijth block of $(A \otimes B) \cdot (C \otimes D)$ is

$$\sum_{k=1}^{n} (a_{ik}B)(c_{kj}D) = \sum_{k=1}^{n} a_{ik}c_{kj}BD$$

The ijth block of $(AC) \otimes (BD)$ is

$$(\sum_{k=1}^{n} a_{ik}c_{kj})BD = \sum_{k=1}^{n} a_{ik}c_{kj}BD$$

Since both expressions result in the same summation, they are equal. □

6 Representing Tensor Transformations

Let $\mathcal{B} = \{(1,0),(0,1)\}$ be the standard basis of $\mathbb{R}^2$, $\mathcal{C} = \{x^2, x, 1\}$ be basis for V_2, and $\mathcal{D} = \{x, 1\}$ be a basis for V_1. The basis of $\mathbb{R}^2 \otimes V_2$ is $\mathcal{B} \otimes \mathcal{C}$ and the basis of $\mathbb{R}^2 \otimes V_1$ is $\mathcal{B} \otimes \mathcal{D}$. Since a tensor transformation can be defined by what it does to the basis vectors, we have that

$$[T \oplus U]_{\mathcal{B}\otimes\mathcal{D}}^{\mathcal{B}\otimes\mathcal{C}} = \begin{array}{cc} \begin{array}{cccccc} (1,0)\otimes x^2 & (1,0)\otimes x & (1,0)\otimes 1 & (0,1)\otimes x^2 & (0,1)\otimes x & (0,1)\otimes 1 \end{array} & \\ \begin{pmatrix} 2 & 0 & 0 & 4 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{c} (1,0)\otimes x \\ (1,0)\otimes 1 \\ (0,1)\otimes x \\ (0,1)\otimes 1 \end{array} \end{array}$$

7 Tensors and Duals

Proof. φ is a well-defined linear map.

Let $\psi : V \times V^* \to \mathbb{F}$ be a linear map with $\psi(v, \lambda) := \lambda(v)$ for $v \in V$ and $\lambda \in V^*$. ψ is linear in the first term since

$$\begin{aligned}\psi((v+v'), \lambda) &= \lambda(v+v') \\ &= \lambda(v) + \lambda(v') \\ &= \psi(v, \lambda) + \psi(v', \lambda)\end{aligned}$$

ψ is linear in the second term since

$$\begin{aligned}\psi(v, \lambda + \lambda') &= (\lambda + \lambda')(v) \\ &= \lambda(v) + \lambda'(v) \\ &= \psi(v, \lambda) + \psi(v, \lambda')\end{aligned}$$

From the universal property of tensor spaces, $\varphi : V \otimes V^*$ is well-defined with $\varphi(v \otimes \lambda) := \psi(v, \lambda) = \lambda(v)$. □

8 Determinants and Tensors

Proof. ψ is well defined.

Let $\varphi : V \times ... \times V \to \mathbb{F}$ be a linear map with $\varphi(v_1, ..., v_n) = \det(v_1 ... v_n)$. Since the determinant is multilinear, φ is multilinear. By the universal property of tensor spaces, $\psi : V \otimes ... \otimes V \to \mathbb{F}$ is well-defined with $\psi(v_1 \otimes ... \otimes v_n) := \varphi(v_1, ..., v_n) = \det(v_1 ... v_n)$. □