Math 170B: Homework 3

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Problem 1

Each polynomial is of degree <= 2. Note that $f(1) = 1$ is continuous and $f(2) = \frac{3}{2}$ is continuous.

$$f(x) = \begin{cases} x & x \in (-\infty, 1] \\ -\frac{1}{2}(2-x)^2 + \frac{3}{2} & x \in [1, 2] \\ \frac{3}{2} & x \in [2, \infty) \end{cases}$$

Also note that $f'(1) = 1$ is continuous and $f'(2) = 0$ is continuous.

$$f'(x) = \begin{cases} 1 & x \in (-\infty, 1] \\ 2 - x & x \in [1, 2] \\ 0 & x \in [2, \infty) \end{cases}$$

Thus, the function is a quadratic spline.

Problem 2

Evaluating the derivatives yields

$$f(x) = \begin{cases} a(x-2)^2 + b(x-1)^3 & x \in (-\infty, 1] \\ c(x-2)^2 & x \in [1,3] \\ d(x-2)^2 + e(x-3)^3 & x \in [3, \infty) \end{cases}$$

$$f'(x) = \begin{cases} 2a(x-2) + 3b(x-1)^2 & x \in (-\infty, 1] \\ 2c(x-2) & x \in [1,3] \\ 2d(x-2) + 3e(x-3)^2 & x \in [3, \infty) \end{cases}$$

$$f''(x) = \begin{cases} 2a + 6b(x-1) & x \in (-\infty, 1] \\ 2c & x \in [1,3] \\ 2d + 6e(x-3) & x \in [3, \infty) \end{cases}$$

For continuity at the knots,

$$a = c \quad c = d$$

For continuity of the derivatives,

$$-2a = -2c \quad 2c = 2d$$

For continuity of the second derivative

$$2a = 2c \quad 2c = 2d$$

Thus the function is a cubic spline whenever $a = c = d$.

Problem 3

We have that a spline is composed of many polynomial pieces

$$S(x) = \begin{cases} S_i(x) & x \in [t_i, t_{i+1}] \\ & \vdots \end{cases}$$

where each piece is a quadratic polynomial

$$S_i(x) = a_i x^2 + b_i x + c_i.$$

There are $n-1$ equations, which yields a total of $3(n-1)$ unknowns. Each polynomial must pass through its endpoints so

$$S_i(t_i) = a_i t_i^2 + b_i t_i + c_i = y_i$$
$$S_i(t_{i+1}) = a_i t_{i+1}^2 + b_i t_{i+1} + c_i = y_{i+1}$$

which gives $2(n-1)$ equations.

The derivatives in the inner knots must be equal so

$$S_i'(t_{i+1}) = 2a_i t_{i+1} + b_i = 2a_{i+1} t_{i+1} + b_{i+1} = S_{i+1}'(t_{i+1})$$

This yields $n-2$ more equations.

To get the last equation, we can impose that $z_0 = 0$.

Problem 4

This problem is similar to Hermite interpolation except the derivatives dont match at x_0. In order so that $l_0(x_0) = 1$ and $l_0(x_i) = l_0'(x_i) = 0$ when $i > 0$ it must be that

$$l_0(x) = \prod_{i=1}^{n} \frac{(x - x_i)^2}{(x_0 - x_i)^2}$$

We want that(but only for $j > 0$ for the derivative)

$$h_i(x_j) = \delta_{ij} \qquad h_i'(x_j) = 0.$$

Let $L_i(x)$ be

$$L_i(x) = \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}.$$

Then we can construct $h_i(x)$ to have the properties we want with

$$h_i(x) = \frac{x - x_0}{x_i - x_0} L_k(x)^2 [1 + C(x - x_i)]$$

for some constant C. Note that $h_i(x_i) = 1$ and when $i \neq j$, $h_i(x_j) = 0$. so $h_i(x)$ satisfies our first property. The derivative is also always 0 The derivative of $h_i'(x)$ is

$$h_i'(x_i) = \frac{1}{x_i - x_0} + 2L_i'(x_i) + C = 0$$

so we can set

$$C = -\frac{1}{x_i - x_0} - 2L_i'(x_i).$$

Similarly we want that

$$k_i(x_j) = 0 \qquad k_i'(x_j) = \delta_{ij}.$$

The polynomial that satisfies these conditions is

$$k_i(x) = \frac{x - x_0}{x_i - x_0} L_k(x)^2 (x - x_i)$$

Note that $h_i(x_i) = 0$ and the derivative is only 1 when $i = j$

Since this polynomial can be thought of as interpolation where each point is repeated twice except for x_0, which yields a total of $2n + 1$. By the Lagrange error formula we can write that

$$f(x) - p_{2n}(x) = \frac{(x - x_0) \prod_{i=1}^{n} (x - x_i)^2}{(2n + 1)!} f^{(2n+1)}(\eta)$$

Quadratic Spline

```
function coefficients = QuadFit(x,y)
n = size(x,2);
coefficients = zeros(n-1,3);

prevDeriv = 0;
for i = 1:n-1
    % Solve the system of equations Ax = b
    % ax_i^2      + bx_i      + c = y_i
    % ax_{i+1}^2 + bx_{i+1} + c = y_{i+1}
    % 2ax_i       + b             = f'(x_i)
    A = [x(i)^2,  x(i),   1;
         x(i+1)^2, x(i+1), 1;
         2*x(i),   1,      0];
    b = [y(i); y(i+1); prevDeriv];
    currCoefficients =  (A\b)';
    coefficients(i, :) = currCoefficients;
    prevDeriv = 2*currCoefficients(1)*x(i+1) + currCoefficients(2);
end
```

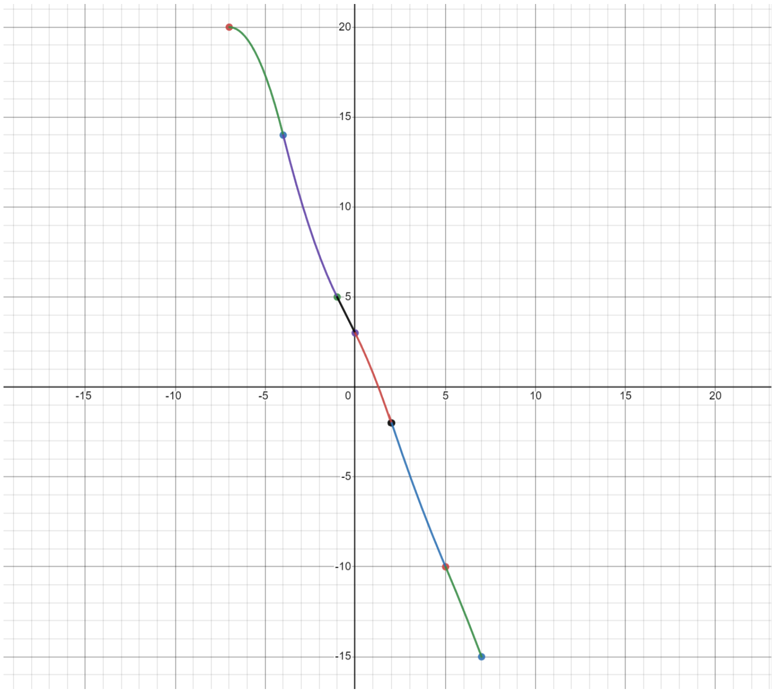
```
ans =

   -0.6667   -9.3333  -12.6667
    0.3333   -1.3333    3.3333
    0.0000   -2.0000    3.0000
   -0.2500   -2.0000    3.0000
    0.1111   -3.4444    4.4444
   -0.0833   -1.5000   -0.4167
```

From the matlab code, we can see that the spline for example 6.2 is

$$S(x) = \begin{cases} -\frac{2}{3}x^2 - \frac{28}{3}x - \frac{38}{3} & -7 \leq x \leq -4 \\ \frac{1}{3}x^2 - \frac{4}{3}x + \frac{10}{3} & -4 \leq x \leq -1 \\ -2x + 3 & -1 \leq x \leq 0 \\ -\frac{1}{4}x^2 - 2x + 3 & 0 \leq x \leq 2 \\ \frac{1}{9}x^2 - \frac{31}{9}x + \frac{40}{9} & 2 \leq x \leq 5 \\ -\frac{1}{12}x^2 - \frac{3}{2}x - \frac{5}{12} & 5 \leq x \leq 7 \end{cases}$$

Plotting this spline, we get



20
15
10
5
0
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