Mathematics 100A Homework 7
Due: Tuesday November 19 2024

Instructions: Please write clearly and fully explain your solutions. It is OK to work with others to solve the problems, but if you do so, you should write your solutions up separately. Copying solutions from your peers or a solutions manual will be deemed academic misconduct. Chapter and problem numbers, if any, refer to *Algebra*, second edition, by Michael Artin. Please feel free to reach out to me or the TAs if you have any questions.

1. (Chapter 6, Problem 3.1) Let m be an orientation-reversing isometry. Prove algebraically that m^2 is a translation.

2. Suppose $D_{2n} = \langle x, y | x^n, y^2, yxy^{-1}x \rangle$ is the dihedral group of order $2n$. Write $x^2yxyx^3yx^4$ in the form x^iy^j.

3. (Chapter 6, Problem 5.3) Suppose L is a lattice in $\mathbf{R}^2$, so that $L = \mathbf{Z}v_1 + \mathbf{Z}v_2$ with $v_1, v_2 \in \mathbf{R}^2$ linearly independent. How many subgroups of index 3 does L have? **Hint**: Suppose $\Gamma \subseteq L$ has index three. First prove $\Gamma \supseteq 3L$.

4. (Chapter 6, Problem 5.4) Let $L \subseteq \mathbf{R}^2$ be a lattice, $L = \mathbf{Z}v_1 + \mathbf{Z}v_2$ with $v_1, v_2 \in \mathbf{R}^2$ linearly independent. Say that $w_1, w_2 \in \mathbf{R}^2$ is a *lattice basis* for L if $L = \mathbf{Z}w_1 + \mathbf{Z}w_2$. Prove that the lattice bases of L are exactly the set of pairs $(v_1, v_2)g$ where $g \in M_2(\mathbf{Z})$ has determinant ± 1. That is, w_1, w_2 is a lattice basis of L if and only if $w_1 = av_1 + cv_2$, $w_2 = bv_1 + dv_2$ where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbf{Z})$ has $ad - bc = \pm 1$.

5. (Chapter 6, Problem 5.10) Let f, g be rotations of the plane about distinct points, with angles of rotation θ and ϕ. Assume $f \neq 1$ and $g \neq 1$. Prove that the group generated by f and g contains a non-trivial translation.