Math 140A: Homework 2

Merrick Qiu

A

Let $\epsilon > 0$ and $x > 0$ be real numbers. If $x \leq \epsilon$, we can choose a such that $0 < a < x$ from property (2). If $x > \epsilon$, we need to choose some $x - \epsilon < a < x$ for $0 < x - a < \epsilon$ to hold.

Choose $b \in A$ such that $0 < b < \epsilon$ using property (2). By property (1), we can repeatedly add b to itself to get another element in A, so $nb \in A$ for all positive integers n.

By the archimedean principle we know that there exists some $nb > x - \epsilon$, and if we choose the smallest such n we know that $x - \epsilon < nb < x$ since $b < \epsilon$. Therefore for every x there exists $a \in A$ where $0 < x - a < \epsilon$.

B

1. We can use the bijection $f : (a, b) \to (c, d)$, $f(x) = c + (x - a) \cdot \frac{d-c}{b-a}$.

2. We can use the bijection $g : [a, b] \to [c, d]$, $g(x) = c + (x - a) \cdot \frac{d-c}{b-a}$.

3. Let $g : [a, b] \to [0, 1]$ from part (2) and $f(x) : (0, 1) \to (c, d)$ from part (1). Let $h : [0, 1] \to (0, 1)$

$$f(x) = \begin{cases} \frac{1}{2} & x = 0 \\ \frac{1}{n+2} & x = \frac{1}{n} \\ x & \text{otherwise} \end{cases}$$

   We can use the bijection $k : [a, b] \to (c, d)$, $k(x) = f(h(g(x)))$.

4. Let $k : [a, b] \to (-\frac{\pi}{2}, \frac{\pi}{2})$ from (3). We can use $l : [a, b] \to \mathbb{R}, l(x) = \tan(h(x))$.

C

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1. For the base case $n = 1$, $b^1 - 1 \geq 1(b-1)$. Assume that $b^k - 1 \geq k(b-1)$. This implies that $b^{k+1} - 1 \geq (k+1)(b-1)$ since

$$\begin{aligned} b^{k+1} - 1 &= b \cdot b^k - 1 \\ &= b(b^k - 1) + (b - 1) \\ &\geq b(k(b-1)) + (b-1) \\ &= (b-1)(bk+1) \\ &\geq (b-1)(k+1). \end{aligned}$$

   Thus, $b^n - 1 \geq n(b-1)$ for all positive integers n.

2. Substituting in $b^{\frac{1}{n}} \to b$ into the previous step implies $b - 1 \geq n(b^{\frac{1}{n}} - 1)$ for all positive integers n.

3. $n > \frac{b-1}{t-1}$ implies $n(t-1) > b - 1$. Since $b - 1 \geq n(b^{\frac{1}{n}} - 1)$, we have that $n(t-1) > b - 1 \geq n(b^{\frac{1}{n}} - 1)$. This then implies $n(t-1) > n(b^{\frac{1}{n}} - 1)$ which implies $t > b^{\frac{1}{n}}$.

4. Applying part (c) with $t = yb^{-w}$ yields $b^{\frac{1}{n}} < yb^{-w}$. This then implies that $b^{w+\frac{1}{n}} < y$ when the conditions in part (c) are met. $b^w < y$ implies that $t > 1$, but since there needs to be $n > \frac{b-1}{t-1}$, this statement is only true for sufficiently large n.

5. If in (d) we used $t = y^{-1}b^w$, we would find that $b^{w-\frac{1}{n}} > y$.

6. Suppose that $b^x \neq y$. Then either $b^x < y$ or $b^x > y$. If $b^x < y$ then we can pick $x + \frac{1}{n}$ for a sufficiently large n such that $b^{w+\frac{1}{n}} < y$ by part (d). This would lead to a contradiction since it would imply that $x \neq \sup A$ since x is not an upper bound.

   Likewise if $b^x > y$ then we can pick $x - \frac{1}{n}$ for a sufficiently large n such that $b^{w-\frac{1}{n}} > y$ by part (e). This would lead to a contradiction since it would also imply that $x \neq \sup A$ since x is not the best upper bound. Therefore $b^x = y$.

7. Since the supremum is unique, x is unique.

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In the case where $|x| - |y| \geq 0$ then we need to prove that $|x| - |y| \leq |x - y|$.

$$|x| = |x - y + y| \leq |x - y| + |y| \implies |x| - |y| \leq |x - y|.$$

In the case where $|x| - |y| < 0$ then we need to prove that $|y| - |x| \leq |x - y|$.

$$|y| = |y - x + x| \leq |y - x| + |x| \implies |y| - |x| \leq |x - y|.$$

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Let $z = a + bi$. $z\bar{z} = 1$ implies $a^2 + b^2 = 1$.

$$\begin{aligned}
|1 + z|^2 + |1 - z|^2 &= (1 + a)^2 + b^2 + (1 - a)^2 + b^2 \\
&= (1 + 2a + a^2) + (1 - 2a + a^2) + 2b^2 \\
&= 2 + (2a^2 + 2b^2) \\
&= 4
\end{aligned}$$

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$$\begin{aligned}
|x + y|^2 + |x - y|^2 &= (x + y) \cdot (x + y) + (x - y) \cdot (x - y) \\
&= |x|^2 + 2x \cdot y + |y|^2 + |x|^2 - 2x \cdot y + |y|^2 \\
&= 2|x|^2 + 2|y|^2
\end{aligned}$$

The sum of the areas of all squares drawn on the sides of a parallelograms is equal to the squares formed from the diagonals of the parallelogram.

Extra Practice Problem

1.

$$\begin{aligned}
\lambda \sum z_j \bar{w}_j &= \lambda(z, w) \\
= \sum \lambda z_j \bar{w}_j &= (\lambda z, w) \\
= \sum z_j \overline{\overline{\lambda} w_j} &= (z, \overline{\lambda} w)
\end{aligned}$$

2. $z_j \overline{z_j}$ is always nonnegative so the sum is as well. Forward direction can be proved by contradiction. Backwards is trivial.

3.

$$\begin{aligned}
(z, w) &= \sum z_j \overline{w}_j \\
&= \sum \overline{w_j \overline{z}_j} \\
&= \overline{\sum w_j \overline{z}_j} \\
&= \overline{(w, z)}
\end{aligned}$$

4. Use pythagorean theorem since u and w are orthogonal.