Math 31AH: Fall 2021
Homework 6 Solutions

Problem 1: The characteristic polynomial of A is

$$\det(A-\lambda I) = \det\begin{pmatrix} 7-\lambda & -8 & 6 \\ 8 & -9-\lambda & 6 \\ 0 & 0 & 1-\lambda \end{pmatrix} = ((7-\lambda)(-9-\lambda)+64)(1-\lambda)$$

which simplifies to $-(\lambda-1)(\lambda+1)^2$. This polynomial has real roots $\lambda = \pm 1$. The eigenspace E_1 is the solution set to $(A-I)\mathbf{x} = \mathbf{0}$. Row reduction yields

$$A - I = \begin{pmatrix} 6 & -8 & 6 \\ 8 & -10 & 6 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

so that E_1 has basis

$$\left\{ \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix} \right\}$$

Similarly, the eigenspace E_{-1} is the solution set to $(A+I) = 0$. Row reduction yields

$$A + I = \begin{pmatrix} 8 & -8 & 6 \\ 8 & -8 & 6 \\ 0 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

so that E_{-1} is 1-dimensional with basis

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Since $E_1 + E_{-1} \neq \mathbb{R}^3$, we conclude that A is not diagonalizable.

Problem 2 The characteristic polynomial of B is

$$\det(B-\lambda I) = \det\begin{pmatrix} 1-\lambda & 1/4 & 0 \\ 0 & 1/2-\lambda & 0 \\ 0 & 1/4 & -1-\lambda \end{pmatrix} = (1-\lambda)(\frac{1}{2}-\lambda)(-1-\lambda)$$

which has roots $\lambda = 1, 1/2, -1$. Since B has three distinct eigenvalues and is 3×3, we immediately know that B is diagonalizable.

To find the eigenspace E_1, we row reduce

$$A-I = \begin{pmatrix} 0 & 1/4 & 0 \\ 0 & -1/2 & 0 \\ 0 & 1/4 & -2 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -8 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -8 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

so that E_1 has basis $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.

To find $E_{1/2}$, we row reduce

$$A - \frac{1}{2}I = \begin{pmatrix} 1/2 & 1/4 & 0 \\ 0 & 0 & 0 \\ 0 & 1/4 & -3/2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -6 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -6 \\ 0 & 0 & 0 \end{pmatrix}$$

so that $E_{1/2}$ has basis $\left\{ \begin{pmatrix} -3 \\ 6 \\ 1 \end{pmatrix} \right\}$.

Finally, to find E_{-1}, we row reduce

$$A + I = \begin{pmatrix} 2 & 1/4 & 0 \\ 0 & 3/2 & 0 \\ 0 & 1/4 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1/8 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so that E_{-1} has basis $\left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Problem 3: Over $\mathbb{R}$, the matrix R_θ represents the map on $\mathbb{R}^2$ given by counterclockwise rotation by θ radians. This rotation has no eigenvectors (and hence is not diagonalizable) when $\theta \neq 0, \pi$. When $\theta = 0$ (resp. $\theta = \pi$) this matrix is I_2 (resp. $-I_2$), hence diagonalizable.

Over $\mathbb{C}$, this matrix is always diagonalizable. Indeed, its characteristic polynomial is

$$\det \begin{pmatrix} \cos(\theta) - \lambda & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) - \lambda \end{pmatrix} = (\cos(\theta) - \lambda)^2 + \sin^2(\theta) = \lambda^2 - 2\cos(\theta)\lambda + 1$$

which has roots

$$\lambda = \frac{2\cos(\theta) \pm \sqrt{4\cos^2(\theta) - 4}}{2} = \cos(\theta) \pm i\sin(\theta) = e^{\pm i\theta}$$

For $\theta \neq 0, \pi$, these roots are distinct, so we have two distinct eigenvalues and are automatically diagonalizable. When $\theta = 0$ or π, we are diagonalizable over $\mathbb{R}$, hence over $\mathbb{C}$.

Problem 4: If A is diagonalizable, we can write $A = PDP^{-1}$, where P is invertible and D is diagonal. Then

$$\begin{aligned} f(A) = f(PDP^{-1}) &= c_n(PDP^{-1})^n + \cdots + c_1(PDP^{-1}) + c_0 I \\ &= P(c_n D^n)R^{-1} + \cdots + P(c_1 D)P^{-1} + P(c_0 I)P^{-1} \\ &= P(c_n D^n + \cdots + c_1 D + c_0 I)P^{-1} \\ &= Pf(D)P^{-1} \end{aligned}$$

since $f(D)$ is diagonal, we conclude that $f(A)$ is diagonalizable.

Problem 5: (1) If λ is an eigenvalue of T, there exists $\mathbf{v} \in V$ nonzero so that $T(\mathbf{v}) = \lambda\mathbf{v}$. Then

$$(T \oplus U)(\mathbf{v}, \mathbf{0}) = (T(\mathbf{v}), U(\mathbf{0})) = \lambda\mathbf{v}, \mathbf{0}) = \lambda(\mathbf{v}, \mathbf{0})$$

so that λ is an eigenvalue of $T \oplus U$. The case where λ is an eigenvalue of U follows by symmetry.

(2) Since T and U are diagonalizable, there exists a basis $\mathcal{B}$ of V consisting of eigenvectors of T and a basis $\mathcal{C}$ of W consisting of eigenvectors of U. On a previous homework, we constructed a basis

$$\mathcal{B} \oplus \mathcal{C} = \{(\mathbf{v}, \mathbf{0}) \,:\, \mathbf{v} \in \mathcal{B}\} \cup \{(\mathbf{0}, \mathbf{w}) \,:\, \mathbf{w} \in \mathcal{C}\}$$

of $V \oplus W$. The calculation in (1) shows that every element in this basis is an eigenvector of $T \oplus U$.

Problem 6: We could calculate the eigenvalues and eigenvectors of A naÏvely, but the special form of A suggests a more clever strategy.

Let $\omega := \exp(\pi i/3)$. For $1 \leq i \leq 6$, define a vector $\mathbf{v}_i \in \mathbb{C}^6$ by

$$\mathbf{v}_i := \begin{pmatrix} \omega^0 \\ \omega^i \\ \omega^{2i} \\ \omega^{3i} \\ \omega^{4i} \\ \omega^{5i} \end{pmatrix}$$

Since we have

$$A \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \end{pmatrix} = \begin{pmatrix} z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_1 \end{pmatrix}$$

we see that $\mathbf{v}_i$ is an eigenvector of A with eigenvalue ω^i. Since $\omega^1, \omega^2, \ldots, \omega^6$ are six distinct eigenvalues of A and A is 6×6, these are all the eigenvalues of A. Each eigenspace E_{ω^i} is 1-dimensional with basis $\{\mathbf{v}_i\}$.

Problem 7: For any $n \geq 0$ we have

$$(PDP^{-1})^n = (PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1}) = PD^nP^{-1},$$

where the middle expression has n factors.

Let A be the 2×2 matrix appearing in the hint. We diagonalize A as follows. The characteristic polynomial is

$$\det \begin{pmatrix} 4-\lambda & -2 \\ 1 & -\lambda \end{pmatrix} = (4-\lambda)(-\lambda) + 2 = \lambda^2 - 4\lambda + 2$$

which has solutions

$$\lambda = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2}$$

For each eigenvalue $\lambda = 2 \pm \sqrt{2}$, we find an eigenvector. When $\lambda = 2 + \sqrt{2}$, we may choose any nontrivial solution $\mathbf{v}_+$ of

$$\begin{pmatrix} 2-\sqrt{2} & -2 \\ 1 & -2-\sqrt{2} \end{pmatrix} \mathbf{v}_+ = \mathbf{0}$$

We take

$$\mathbf{v}_+ = \begin{pmatrix} 2+\sqrt{2} \\ 1 \end{pmatrix}$$

When $\lambda = 2 - \sqrt{2}$, we may choose any nontrivial solution $\mathbf{v}_-$ of

$$\begin{pmatrix} 2+\sqrt{2} & -2 \\ 1 & -2+\sqrt{2} \end{pmatrix} \mathbf{v}_- = \mathbf{0}$$

We choose

$$\mathbf{v}_- = \begin{pmatrix} 2-\sqrt{2} \\ 1 \end{pmatrix}$$

Therefore, if

$$P = \begin{pmatrix} 2+\sqrt{2} & 2-\sqrt{2} \\ 1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} 2+\sqrt{2} & 0 \\ 0 & 2-\sqrt{2} \end{pmatrix}$$

so that

$$P^{-1} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & -2+\sqrt{2} \\ -1 & 2+\sqrt{2} \end{pmatrix}$$

we have

$$\begin{aligned}
A^n &= (PDP^{-1})^n \\
&= PD^nP^{-1} \\
&= \frac{1}{2\sqrt{2}} \begin{pmatrix} 2+\sqrt{2} & 2-\sqrt{2} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} (2+\sqrt{2})^n & 0 \\ 0 & (2-\sqrt{2})^n \end{pmatrix} \begin{pmatrix} 1 & -2+\sqrt{2} \\ -1 & 2+\sqrt{2} \end{pmatrix} \\
&= \frac{1}{2\sqrt{2}} \begin{pmatrix} (2+\sqrt{2})^{n+1} & (2-\sqrt{2})^{n+1} \\ (2+\sqrt{2})^n & (2-\sqrt{2})^n \end{pmatrix} \begin{pmatrix} 1 & -2+\sqrt{2} \\ -1 & 2+\sqrt{2} \end{pmatrix} \\
&= \frac{1}{2\sqrt{2}} \begin{pmatrix} (2+\sqrt{2})^{n+1} - (2-\sqrt{2})^{n+1} & (-2+\sqrt{2})(2+\sqrt{2})^{n+1} + (2+\sqrt{2})(2-\sqrt{2})^{n+1} \\ (2+\sqrt{2})^n - (2-\sqrt{2})^n & (-2+\sqrt{2})(2+\sqrt{2})^n + (2+\sqrt{2})(2-\sqrt{2})^n \end{pmatrix}
\end{aligned}$$

Since

$$\begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix} = A^{n-2} \begin{pmatrix} a_2 \\ a_1 \end{pmatrix} = A^{n-2} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

we see that

$$\begin{aligned}
a_n &= \frac{1}{2\sqrt{2}} \left[(2+\sqrt{2})^{n-1} - (2-\sqrt{2})^{n-1} + (-2+\sqrt{2})(2+\sqrt{2})^{n-1} + (2+\sqrt{2})(2-\sqrt{2})^{n-1} \right] \\
&= \frac{1}{2\sqrt{2}} \left[(-1+\sqrt{2})(2+\sqrt{2})^{n-1} + (1+\sqrt{2})(2-\sqrt{2})^{n-1} \right]
\end{aligned}$$

Problem 8: Suppose $\mathbf{v} \in E_\lambda$. We need to show that $U(\mathbf{v}) \in E_\lambda$. To do this, we calculate

$$T(U(\mathbf{v})) = U(T(\mathbf{v})) = U(\lambda\mathbf{v}) = \lambda U(\mathbf{v})$$

This conclusion does not necessarily hold if T and U do not commute. For example, if

$$T \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad U \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

as operators on $\mathbb{R}^2$ then the first standard basis vector $\mathbf{e}_1$ is in E_1 but

$$T(U(\mathbf{e}_1)) = T(\mathbf{e}_2) = -\mathbf{e}_2 = -U(\mathbf{e}_1)$$

so that $U(\mathbf{e}_1) \notin E_1$.