Math 120A: Homework 4

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Problem 1: Page 61 Question 8(a)

$$f(z) = \operatorname{Re}(z) = \frac{z + \overline{z}}{2}$$

$$\begin{aligned}
\frac{\Delta w}{\Delta z} &= \frac{\frac{z+\Delta z+\overline{z+\Delta z}}{2} - \frac{z+\overline{z}}{2}}{\Delta z} \\
&= \frac{\Delta z + \overline{\Delta z}}{2\Delta z} \\
&= \frac{1}{2} + \frac{\overline{\Delta z}}{2\Delta z}
\end{aligned}$$

Approaching $\Delta z \to 0$ horizontally yields 1 but approaching vertically yields 0, so the limit does not exist at any point. Therefore $f'(z)$ does not exist at any point.

Problem 2 : Page 62 Question 9

When $z = 0$,

$$\frac{\Delta w}{\Delta z} = \frac{f(\Delta z) - f(0)}{\Delta z} = \frac{\overline{\Delta z}^2}{\Delta z^2}$$

Along the x axis for points $(\Delta x, 0)$,

$$\frac{\overline{\Delta z}^2}{\Delta z^2} = \frac{\Delta x^2}{\Delta x^2} = 1$$

Along the y axis for points $(0, \Delta y)$

$$\frac{\overline{\Delta z}^2}{\Delta z^2} = \frac{\Delta y^2}{\Delta y^2} = 1$$

Along the diagonal for points $(\Delta x, \Delta x)$

$$\begin{aligned}
\frac{\overline{\Delta z}^2}{\Delta z^2} &= \frac{\overline{\Delta z}^2}{\Delta z^2} \cdot \frac{\overline{\Delta z}^2}{\overline{\Delta z}^2} \\
&= \frac{(\Delta x - i\Delta x)^4}{(\Delta x^2 + \Delta x^2)^2} \\
&= \frac{\Delta x^4 - 4i\Delta x^4 - 6\Delta x^4 + 4i\Delta x^4 + \Delta x^4}{(2\Delta x^2)^2} \\
&= \frac{-4\Delta x}{4\Delta x} \\
&= -1
\end{aligned}$$

Since the limit is not equal from all directions, the derivative does not exist for $z = 0$.

Problem 3: Page 70 Question 1(c)(d)

$$f(z) = 2x + ixy^2, \qquad u(x,y) = 2x, \qquad v(x,y) = xy^2$$

$$u_x = 2, \quad u_y = 1$$

$$v_x = y^2, \quad v_y = 2xy$$

$$u_x = v_y \implies 2 = 2xy$$

$$u_y = -v_x \implies 1 = -y^2$$

Since there does not exist real y such that $1 = -y^2$, $f'(z)$ does not exist at any point.

$$f(z) = e^x e^{-iy} = e^x(\cos(y) - i\sin(y))$$

$$u(x,y) = e^x \cos(y), \qquad v(x,y) = -e^x \sin(y)$$

$$u_x = e^x \cos(y), \quad u_y = -e^x \sin(y)$$

$$v_x = -e^x \sin(y), \quad v_y = -e^x \cos(y)$$

$$u_x = v_y \implies e^x \cos(y) = -e^x \cos(y)$$

$$u_y = -v_x \implies e^x \sin(y) = -e^x \sin(y)$$

These two set of equations has no solution since e^x is never zero and cos and sin are never simulteanously zero. Therefore, $f'(z)$ does not exist at any point.

Problem 4: Page 70 Question 2(b)

$$f(z) = e^{-x}e^{-iy} = e^{-x}(\cos(y) - i\sin(y))$$
$$u(x,y) = e^{-x}\cos(y), \qquad v(x,y) = -e^{-x}\sin(y)$$
$$u_x = -e^{-x}\cos(y), \quad u_y = -e^{-x}\sin(y)$$
$$v_x = e^{-x}\sin(y), \qquad v_y = -e^{-x}\cos(y)$$

Since u, v always exist, their partial derivatives always exist and are continuous, and the Cauchy-Riemann conditions hold, $f'(z)$ exists everywhere.

Problem 5: Page 71 Question 3(b)(c)

$$f(z) = x^2 + iy^2$$
$$u(x,y) = x^2, \qquad v(x,y) = y^2$$
$$u_x = 2x, \quad u_y = 0$$
$$v_x = 0, \qquad v_y = 2y$$

Therefore the derivative exists when $x = y$ and it is equal to

$$f'(x + ix) = u_x + iv_x = 2x$$

$$f(z) = z \operatorname{Im} z = (x + iy)y$$
$$u(x,y) = xy, \qquad v(x,y) = y^2$$
$$u_x = y, \quad u_y = x$$
$$v_x = 0, \quad v_y = 2y$$

Therefore the derivative exists when $z = 0$ and it is equal to

$$f'(0) = u_x + iv_x = 0$$