Math 20D HW6

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Section 9.5

Problem 3 The characteristic polynomial is

$$\det(A-\lambda I) = (1-\lambda)(4-\lambda)+2 = 6-5\lambda+\lambda^2 = (\lambda-2)(\lambda-3)$$

The eigenvalues values are $\lambda_1 = 2$, $\lambda_2 = 3$.
The eigenvector for λ_1 is

$$(A-\lambda_1 I)x_1 = 0 \implies \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} x_1 = 0 \implies x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The eigenvector for λ_2 is

$$(A-\lambda_2 I)x_2 = 0 \implies \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} x_2 = 0 \implies x_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Problem 9 The characteristic polynomial is

$$\det(A-\lambda I) = \lambda^2+1$$

The eigenvalues are $\lambda_1 = -i$ and $\lambda_2 = i$.
The eigenvector for λ_1 is

$$(A-\lambda_1 I)x_1 = 0 \implies \begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix} x_1 = 0 \implies x_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

The eigenvector for λ_2 is

$$(A-\lambda_2 I)x_2 = 0 \implies \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} x_2 = 0 \implies x_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Problem 11 The characteristic polynomial is

$$\det(A-\lambda) = (-1-\lambda)(3-\lambda)+\frac{15}{4} = \frac{3}{4}-2\lambda+\lambda^2 = (\lambda-\frac{1}{2})(\lambda-\frac{3}{2})$$

The eigenvalues are $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \frac{3}{2}$
The eigenvector for λ_1 is

$$(A-\lambda_1 I)x_1 = 0 \implies \begin{bmatrix} -\frac{3}{2} & \frac{3}{4} \\ -5 & \frac{5}{2} \end{bmatrix} x_1 = 0 \implies x_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The eigenvector for λ_2 is

$$(A-\lambda_2 I)x_2 = 0 \implies \begin{bmatrix} -\frac{5}{2} & \frac{3}{4} \\ -5 & \frac{3}{2} \end{bmatrix} x_2 = 0 \implies x_2 = \begin{bmatrix} 3 \\ 10 \end{bmatrix}$$

The general solution is

$$y = c_1 e^{\frac{t}{2}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{\frac{3t}{2}} \begin{bmatrix} 3 \\ 10 \end{bmatrix}$$

Problem 12 The characteristic polynomial is

$$\det(A-\lambda I) = (1-\lambda)^2-36 = 35-2\lambda+\lambda^2 = (\lambda+5)(\lambda-7)$$

The eigenvalues are $\lambda_1 = -5$ and 7
The eigenvector for λ_1 is

$$(A-\lambda_1 I)x_1 = 0 \implies \begin{bmatrix} 6 & 3 \\ 12 & 6 \end{bmatrix} x_1 = 0 \implies x_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

The eigenvector for λ_2 is

$$(A-\lambda_2 I)x_2 = 0 \implies \begin{bmatrix} -6 & 3 \\ 12 & -6 \end{bmatrix} x_2 = 0 \implies x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The general solution is

$$y = c_1 e^{-5t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{7t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Problem 18

1. The matrix multiplies u_1 by -1
$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$
The matrix multiplies u_2 by -3
$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \end{bmatrix}$$
2. The equation is
$$y = e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
The equation starts at $(1, 1)$ at $t = 0$ and approaches the origin as $t \to \infty$ in a straight line.
3. The equation is
$$y = e^{-3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
The equation starts at $(-1, 1)$ at $t = 0$ and approaches the origin as $t \to \infty$ in a straight line.
4. The equation is
$$y = e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{-3t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
The equation starts at $(0, 2)$ at $t = 0$ and approaches the origin as $t \to \infty$ in a curve that goes rightward.

Problem 19 The characteristic polynomial is
$$\det(A - \lambda I) = (-1 - \lambda)(1 - \lambda) - 8 = -9 + \lambda^2 = (\lambda + 3)(\lambda - 3)$$
The eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = 3$.
The eigenvector for λ_1 is
$$(A - \lambda_1 I)x_1 = 0 \implies \begin{bmatrix} 2 & 1 \\ 8 & 4 \end{bmatrix} x_1 = 0 \implies x_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$
The eigenvector for λ_2 is
$$(A - \lambda_2 I)x_2 = 0 \implies \begin{bmatrix} -4 & 1 \\ 8 & -2 \end{bmatrix} x_2 = 0 \implies x_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$
The fundamental matrix is
$$\begin{bmatrix} e^{-3t} & e^{3t} \\ -2e^{-3t} & 4e^{3t} \end{bmatrix}$$

Problem 20 The characteristic polynomial is
$$\det(A - \lambda I) = (5 - \lambda)(-\lambda) + 4 = \lambda^2 - 5\lambda + 4 = (\lambda - 1)(\lambda - 4)$$
The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 4$.
The eigenvector for λ_1 is
$$(A - \lambda_1 I)x_1 = 0 \implies \begin{bmatrix} 4 & 4 \\ -1 & -1 \end{bmatrix} x_1 = 0 \implies x_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
The eigenvector for λ_2 is
$$(A - \lambda_2 I)x_2 = 0 \implies \begin{bmatrix} 1 & 4 \\ -1 & -4 \end{bmatrix} x_2 = 0 \implies x_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$
The fundamental matrix is
$$\begin{bmatrix} e^{t} & 4e^{4t} \\ -e^{t} & -e^{4t} \end{bmatrix}$$

Section 9.6

Problem 1 The characteristic equation is

$$\det(A - \lambda I) = (2-\lambda)(-2-\lambda) + 8 = 4 + \lambda^2$$

The eigenvalues are $\lambda_1 = -2i$ and $\lambda_2 = 2i$.
The eigenvector of λ_2 is

$$(A - \lambda_2 I)x = 0 \implies \begin{bmatrix} 2-2i & -4 \\ 2 & -2-2i \end{bmatrix} x = 0 \implies x = \begin{bmatrix} 1+i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The general solution is

$$y = c_1 \left(\cos 2t \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sin 2t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) + c_2 \left(\sin 2t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \cos 2t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

This can be rewritten as

$$y = c_1 \begin{bmatrix} \cos 2t - \sin 2t \\ \cos 2t \end{bmatrix} + c_2 \begin{bmatrix} \sin 2t + \cos 2t \\ \sin 2t \end{bmatrix}$$

Problem 2 The characteristic polynomial is

$$(-2-\lambda)(2-\lambda) + 5 = 1 + \lambda^2$$

The eigenvalues are $\pm i$ The eigenvector is

$$(A - \lambda I)x = 0 \implies \begin{bmatrix} -2-i & -5 \\ 1 & 2-i \end{bmatrix} x = 0 \implies x = \begin{bmatrix} i-2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The general solution is

$$y = c_1 \left(\cos t \begin{bmatrix} -2 \\ 1 \end{bmatrix} - \sin t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) + c_2 \left(\sin t \begin{bmatrix} -2 \\ 1 \end{bmatrix} + \cos t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

This can be rewritten as

$$y = c_1 \begin{bmatrix} -2\cos t - \sin t \\ \cos t \end{bmatrix} + c_2 \begin{bmatrix} -2\sin t + \cos t \\ \sin t \end{bmatrix}$$

Problem 5 The characteristic polynomial is

$$(-1-\lambda)^2 + 16 = 17 + 2\lambda + \lambda^2 \implies \lambda = \frac{-2 \pm \sqrt{4-68}}{2} = -1 \pm 4i$$

The eigenvector is

$$(A - \lambda I)x = 0 \implies \begin{bmatrix} -4i & -2 \\ 8 & -4i \end{bmatrix} x = 0 \implies x = \begin{bmatrix} 1 \\ -2i \end{bmatrix} =$$

The general solution is

$$y = c_1 \left(e^{-t} \cos 4t \begin{bmatrix} 1 \\ 0 \end{bmatrix} - e^{-t} \sin 4t \begin{bmatrix} 0 \\ -2 \end{bmatrix} \right) + c_2 \left(e^{-t} \sin 4t \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{-t} \cos 4t \begin{bmatrix} 0 \\ -2 \end{bmatrix} \right)$$

The fundamental matrix is

$$\begin{bmatrix} e^{-t}\cos 4t & e^{-t}\sin 4t \\ 2e^{-t}\sin 4t & -2e^{-t}\cos 4t \end{bmatrix}$$

Problem 6 The characteristic polynomial is

$$(-2-\lambda)(2-\lambda) + 8 = 4 + \lambda^2$$

The eigenvalues are $\pm 2i$ The eigenvector is

$$(A-\lambda I)x = 0 \implies \begin{bmatrix} -2-2i & -2 \\ 4 & 2-2i \end{bmatrix} x = 0 \implies x = \begin{bmatrix} 1 \\ -1-i \end{bmatrix}$$

The general solution is

$$y = c_1 \left(\cos 2t \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \sin 2t \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) + c_2 \left(\sin 2t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \cos 2t \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$$

The fundamental matrix is

$$\begin{bmatrix} \cos 2t & \sin 2t \\ -\cos 2t + \sin 2t & -\sin 2t - \cos 2t \end{bmatrix}$$

Problem 13 The characteristic equation is

$$(-3-\lambda)(-1-\lambda) + 2 = 5 + 4\lambda + \lambda^2 \implies \lambda = \frac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$$

The eigenvalues are $-2 \pm i$
The eigenvector is

$$\begin{bmatrix} -1-i & -1 \\ 2 & 1-i \end{bmatrix} x = 0 \implies x = \begin{bmatrix} 1 \\ -1-i \end{bmatrix}$$

The general solution is

$$y = c_1 \left(e^{-2t} \cos t \begin{bmatrix} 1 \\ -1 \end{bmatrix} - e^{-2t} \sin t \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right) + c_2 \left(e^{-2t} \sin t \begin{bmatrix} 1 \\ -1 \end{bmatrix} + e^{-2t} \cos t \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$$

This can be rewritten as

$$y = c_1 \begin{bmatrix} e^{-2t} \cos t \\ -e^{-2t} \cos t + e^{-2t} \sin t \end{bmatrix} + c_2 \begin{bmatrix} e^{-2t} \sin t \\ -e^{-2t} \sin t - e^{-2t} \cos t \end{bmatrix}$$

Solving out for the constants yields

1.
$$x(0) = \begin{bmatrix} c_1 \\ -c_1 - c_2 \end{bmatrix}$$
This means that $c_1 = -1$ and $c_2 = 1$. The solution is
$$\begin{bmatrix} e^{-2t} \sin t - e^{-2t} \cos t \\ -2e^{-2t} \sin t \end{bmatrix}$$

2.
$$x(\pi) = e^{-2\pi} \begin{bmatrix} -c_1 \\ c_1 - c_2 \end{bmatrix}$$
This means that $c_1 = -e^{2\pi}$ and $c_2 = 0$. The solution is
$$\begin{bmatrix} e^{-2t+2\pi} \cos t \\ -e^{-2t+2\pi} \cos t + e^{-2t+2\pi} \sin t \end{bmatrix}$$

3.
$$x(-2\pi) = e^{4\pi} \begin{bmatrix} c_1 \\ -c_1 - c_2 \end{bmatrix}$$
This means that $c_1 = 2e^{-4\pi}$ and $c_2 = -3e^{-4\pi}$. The solution is
$$\begin{bmatrix} e^{-2t+4\pi}(2\cos t - 3\sin t) \\ e^{-2t+4\pi}(\cos t + 5\sin t) \end{bmatrix}$$

4.
$$x(\pi/2) = e^{-\pi} \begin{bmatrix} c_2 \\ c_1 - c_2 \end{bmatrix}$$
This means that $c_1 = e^{\pi}$ and $c_2 = 0$ The solution is
$$\begin{bmatrix} e^{\pi-2t} \cos t \\ e^{\pi-2t}(\sin t - \cos t) \end{bmatrix}$$

Section 9.7

Problem 1 The characteristic polynomial is

$$(6-\lambda)(3-\lambda)-4 = 14-9\lambda+\lambda^2 = (\lambda-2)(\lambda-7)$$

The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 7$
The eigenvector for λ_1 is

$$\begin{bmatrix} 4 & 1 \\ 4 & 1 \end{bmatrix} x = 0 \implies x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The eigenvector for λ_2 is

$$\begin{bmatrix} -1 & 1 \\ 4 & -4 \end{bmatrix} x = 0 \implies x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The homogenous solution is in the form

$$y = c_1 e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{7t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Solving the system of equations,

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} -11 \\ -5 \end{bmatrix}$$

This has solution

$$a = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

The final solution is

$$y = c_1 e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{7t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Problem 2 The characteristic polynomial is

$$(1-\lambda)^2 - 4 = -3 - 2\lambda + \lambda^2 = (\lambda+1)(\lambda-3)$$

The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 3$.
The eigenvector for λ_1 is

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} x = 0 \implies x = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

The eigenvector for λ_2 is

$$\begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} x = 0 \implies x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The homogenous solution is in the form

$$y = c_1 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Solving the system of equations for the particular solution,

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} a_1 t + b_1 \\ a_2 t + b_2 \end{bmatrix} + \begin{bmatrix} -t-1 \\ -4t-2 \end{bmatrix}$$

This can be rewritten as

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = t \begin{bmatrix} a_1 + a_2 - 1 \\ 4a_1 + a_2 - 4 \end{bmatrix} + \begin{bmatrix} b_1 + b_2 - a_1 - 1 \\ 4b_1 + b_2 - a_2 - 2 \end{bmatrix}$$

Having $a_1 = 1$ and $a_2 = 0$ solves the first system and having $b_1 = 0$ and $b_2 = 2$ solves the second system. The final solution is

$$y = c_1 e^{-t} \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} t \\ 2 \end{bmatrix}$$

Problem 8 The particular solution has form

$$t^2 a + tb + c$$