Math 100b Winter 2025 Homework 3

Due 1/31/2025 at 5pm on Gradescope

Reading

All references will be to Artin Algebra, 2nd edition.

Reading: I think about the material in 11.5 differently than Artin, so you can just skim that section (some of the same ideas will be covered in lecture in different form). Read 11.6-11.7 and the first half of 11.8 only (not Hilbert's Nullstellensatz and the material following it). We will not cover 11.9.

Assigned Problems

Write up neat and complete solutions to these problems. "Ring" will always mean commutative ring unless otherwise noted.

1. Recall that an element r in a ring R is nilpotent if there exists $n \geq 1$ such that $r^n = 0$ in R. Let R be a nonzero ring and let N be the set of all nilpotent elements of R. N is called the *nilradical* of R.

(a). Prove that N is an ideal of R.

(b). Prove that the ring R/N has no nonzero nilpotent elements.

(c). If P is any prime ideal of the ring R, show that $N \subseteq P$.

2. Recall the ring R of all functions $f : \mathbb{R} \to \mathbb{R}$ with pointwise addition and multiplication, which we considered in Homework 1 Exercise #5. For any subset X of $\mathbb{R}$, we define

$$I_X = \{f \in R \,|\, f(a) = 0 \text{ for all } a \in X\}.$$

(a). Prove that for any set X, I_X is an ideal of R and in fact is a principal ideal.

(b). For which sets X is I_X a maximal ideal? For which sets X is I_X a prime ideal?

3. Let R be a subring of a ring S.

(a). If I is an ideal of S, show that $I \cap R$ is an ideal of R.

(b). If I is a prime ideal of S, show that $I \cap R$ is prime in R.

(c). if I is a maximal ideal of S, must $I \cap R$ be maximal in R? Prove or give a counterexample.

4. Let $R = \mathbb{Z}[x]$ be the ring of polynomials with integer coefficients. Let $I = (2, x)$ be the ideal generated by 2 and x in R. Show that I is not a principal ideal of R, so R is not a principal ideal domain.

Remark 0.1 *The notion of a product of 2 rings is discussed in Section 11.6 of the book. The product of finitely many rings is defined in an entirely analogous way. Products appear in the next two problems. We will cover this notion in lecture on Monday 1/27.*

5. Let F be a field and consider the ring $F[x]$.

(a). Let $f, g \in F[x]$. Show that there is an inclusion of principal ideals $(f) \subseteq (g)$ if and only if g divides f, that is, $f = gh$ for some $h \in F[x]$.

(b). Suppose that $f(x) = (x - a)(x - b)$ for elements a, b of F with $a \neq b$. Consider the homomorphism $\phi : F[x] \to F \times F$ given by $\phi(f(x)) = (f(a), f(b))$. Using this homomorphism show that $F[x]/(f) \cong F \times F$.

(c). Suppose that $g(x) = (x - a)^2$ for some $a \in F$. Is $F[x]/(g(x))$ isomorphic to $F \times F$? (Hint: how many ideals does this ring have?)

6. Recall the example from Homework 1 Exercise #1: R is the set of subsets of some fixed set U, with operations $A + B = (A \cup B) - (A \cap B)$ and $A \cdot B = A \cap B$. In that exercise you proved that R is indeed a ring.

Let U be a finite set with n elements. Prove that that ring R defined above is isomorphic to the ring

$$\overbrace{\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z}}^{n},$$

that is, the product of n copies of the ring $\mathbb{Z}/2\mathbb{Z}$.