Math 100B: Homework 6

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Problem 1

Polynomials of degree 0 are units so they aren't irreducible. Polynomials of degree 1 must be the product of a degree 0 polynomial and a degree 1 polynomial so they are irreducible. Polynomials of degree 2 or 3 that are reducible must have a root since their product contains at least one degree 1 polynomial. Checking the roots of all monic polynomials gives us the following table. Therefore all the irreducibles up to associates are, $x, x+1, x+2, x^2+1, x^2+x+2, x^2+2x+2, x^3+2x+1, x^3+2x+2, x^3+x^2+2, x^3+x^2+x+2, x^3+x^2+2x+1, x^3+2x^2+1, x^3+2x^2+x+1, x^3+2x^2+2x+2$.

	$f(0)$	$f(1)$	$f(2)$
x^2	0	1	1
x^2+1	1	2	2
x^2+2	2	0	0
x^2+x	0	2	0
x^2+x+1	1	0	1
x^2+x+2	2	1	2
x^2+2x	0	0	2
x^2+2x+1	1	1	0
x^2+2x+2	2	2	1
x^3	0	1	2
x^3+1	1	2	0
x^3+2	2	0	1
x^3+x	0	2	1
x^3+x+1	1	0	2
x^3+x+2	2	1	0
x^3+2x	0	0	0
x^3+2x+1	1	1	1
x^3+2x+2	2	2	2
x^3+x^2	0	2	0
x^3+x^2+1	1	0	1
x^3+x^2+2	2	1	2
x^3+x^2+x	0	0	2
x^3+x^2+x+1	1	1	0
x^3+x^2+x+2	2	2	1
x^3+x^2+2x	0	1	1
x^3+x^2+2x+1	1	2	2
x^3+x^2+2x+2	2	0	0
x^3+2x^2	0	0	1
x^3+2x^2+1	1	1	2
x^3+2x^2+2	2	2	0
x^3+2x^2+x	0	1	0
x^3+2x^2+x+1	1	2	1
x^3+2x^2+x+2	2	0	2
x^3+2x^2+2x	0	2	2
x^3+2x^2+2x+1	1	0	0
x^3+2x^2+2x+2	2	1	1

Problem 2

(a) We can use De Moivre's theorem to find the four roots of -1, where $m = 0, 1, 2, 3$.

$$\left(\cos\left(\frac{\pi + 2\pi m}{4}\right) + i\sin\left(\frac{\pi + 2\pi m}{4}\right)\right)^4 = \cos(\pi + 2\pi m) + i\sin(\pi + 2\pi m) = -1$$

Any degree 1 polynomial is irreducible since it can only be written as the product of a degree 1 polynomial with some constant.

$$x^4 + 1 = \left(x - \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\left(x + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\left(x + \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\left(x - \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$$

(b) Although none of the factors in $\mathbb{C}[x]$ are in $\mathbb{R}[x]$, we can combine the terms to get factors that are in $\mathbb{R}[x]$. These terms are irreducible because if they weren't, that would imply real roots.

$$\begin{aligned} x^4 + 1 &= \left(x - \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\left(x - \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)\left(x + \frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\left(x + \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) \\ &= (x^2 - \sqrt{2}x + 1)(x^2 + \sqrt{2}x + 1) \end{aligned}$$

(c) Over $\mathbb{Q}$, $x^4 + 1$ is already irreducible since $\mathbb{Q}[x] \subseteq \mathbb{R}[x]$ which are both UFDs, but the factors in $\mathbb{R}[x]$ are irrational and combining them just gives you $x^4 + 1$.

(d) Since $x^4 + 1$ is irreducible in $\mathbb{Q}$, it is also irreducible in $\mathbb{Z}$. Since the reduction mod p mapping $\mathbb{Z}[x] \to (\mathbb{Z}/3\mathbb{Z})[x]$ is an isomorphism, this means that it is also irreducible in $(\mathbb{Z}/3\mathbb{Z})[x]$. See problem 3(a) for more details.

Problem 3

1. Suppose that $f(x) = g(x)h(x)$ was reducible in $\mathbb{Z}[x]$, where degree of g and h are greater than 0. Then the homomorphism implies

$$\begin{aligned}\overline{f}(x) &= (\phi(f))(x) \\ &= (\phi(g)\phi(h))(x) \\ &= \overline{g}(x)\overline{h}(x)\end{aligned}$$

which is a contradiction since $\overline{f}(x)$ is irreducible but $\overline{g}(x)$ and $\overline{h}(x)$ are not units and nonzero since $\overline{a_n} \neq 0$. Therefore $f(x)$ must be irreducible.

2. $80x^3 - 8x + 100$ is irreducible in $\mathbb{Q}$ iff irreducible in $\mathbb{Z}$ if irreducible in $\mathbb{Z}/p\mathbb{Z}$ for some prime p. Choosing $p = 5$ works since $-3x$ is irreducible in $\mathbb{Z}/5\mathbb{Z}$.

Problem 4

We can write $x^9 - 1 = (x-1)(x^2+x+1)(x^6+x^3+1)$. $(x-1)$ is irreducible since it is degree 1, (x^2+x+1) is irreducible since it is a degree 2 polynomial with no roots. (x^6+x^3+1) is irreducible by Eisenstein's criterion.

Let $\Phi(x) = x^6 + x^3 + 1$. We can substitute $x \to x+1$ to get

$$\Phi(x+1) = (x+1)^6 + (x+1)^3 + 1 = x^6 + 6x^5 + 15x^4 + 21x^3 + 18x^2 + 9x + 3.$$

Notice that all ther coefficients except for the leading coefficient are divisible by 3 and the constant term is not divisible by 9, so it is irreducible.

Problem 5

(a) $2x^3 + x - 4$ is irreducible since it only has irrational roots that can be calculated through the cubic formula, but any factorization of $2x^3 + x - 4$ must have a degree 1 polynomial.

(b) By the Eisenstein criterion, all the coefficients except for the leading coefficient are divisble by 2 but the constant is not divisible by 4 so it is irreducible.

(c) First notice that $x^4 + 10x^2 + 1$ does not have rational roots so it must factor into two degree two polynomials.

$$(x^2 + ax + b)(x^2 + cx + d) = x^4 + (a + c)x^3 + (ac + b + d)x^2 + (ad + bc)x + bd$$

This gives up the following system of equations

$$\begin{aligned} a + c &= 0 \\ ac + b + d &= 10 \\ ad + bc &= 0 \\ bd &= 1 \end{aligned}$$

From the last equation, either $b = d = 1$ or $b = d = -1$. In the first case, the second equation gives $ac = 8$ but the third equation gives $a + c = 0$ which is a contradiction. Similarly, $ac = 12$ but $-a - c = 0$ is a contradiction. Therefore $x^4 + 10x^2 + 1$ is irreducible since it cannot be factorized into degree 2 polynomials.

Problem 6

As proved in class, there exist a rational number $r \in \mathbb{Q}$ such that $h(x) = f'(x)g'(x)$ where $f'(x) = rf(x)$ and $g'(x) = r^{-1}g(x)$ have integer coefficients. The product of two coefficients of $f'(x)$ and $g'(x)$ is clearly an integer and the product of two coefficients of $f(x)$ and $g(x)$ is equivalent to the product of the associated product of coefficients from $f'(x)$ and $g'(x)$ since $rr^{-1} = 1$. Therefore the product of any coefficient of f with any coefficient of g is an integer.