Math 100b Winter 2025 Homework 8

Due 3/14/2025 at 5pm on Gradescope

Reading

Reading: Artin Chapter 4

Assigned Problems

1. Let V be a vector space over the field F and let X be a set of vectors in V which is linearly independent over F. Show that X can be extended to a basis of V, that is, there is a basis of V which contains X.

(Hint: use Zorn's Lemma in a similar way as we did in class to prove that every vector space has a basis. How do you need to change the poset that you apply Zorn's Lemma to?)

2. (a). Suppose that A and B are $n \times n$ matrices over a field F and that at least one of them is invertible. Prove that AB and BA are similar matrices; that is, there is an invertible matrix P such that $P^{-1}(AB)P = BA$.

(b). Show that AB and BA are not necessarily similar if A and B are both singular.

In the next few problems we will use the following definition:

Definition 0.1 *For any vector space V over F, if V_1 and V_2 are subspaces of V then we write $V = V_1 \oplus V_2$, and say that V is the internal direct sum of the subpsaces V_1 and V_2, if every $v \in V$ can be written as $v = v_1 + v_2$ for* unique *vectors $v_1 \in V_1$ and $v_2 \in V_2$.*

3. Let $\phi : V \to V$ be a linear transformation of a vector space V of finite dimension n. Suppose that ϕ is idempotent; that is, $\phi^2 = \phi$ (where $\phi^2 = \phi \circ \phi : V \to V$).

(a). Show that $V = (\ker \phi) \oplus (\operatorname{im} \phi)$.

(b). Show that there is a basis $\mathcal{B} = \{v_1, \dots, v_n\}$ of V and a number $0 \leq k \leq n$ such that $\phi(v_i) = v_i$ for $i \leq k$ and $\phi(v_i) = 0$ for $i > k$. In other words, ϕ is just a projection of V onto the subspace spanned by the first k vectors in the basis. Calculate $M_{\mathcal{B}}(\phi)$, the matrix of ϕ with respect to this basis.

4. Let $\phi : V \to V$ be a linear transformation of a vector space of finite dimension n over a field F in which $2 = 1 + 1$ is a unit. Suppose that $\phi^2 = \phi \circ \phi$ is the identity map.

(a). Prove that for any vector $v \in V$, either $v - \phi(v)$ is an eigenvector with eigenvalue -1, or else it is 0.

(b). Show that $V = V_1 \oplus V_{-1}$, where $V_a = \{v \in V | \phi(v) = av\}$ means the eigenspace of V associated to the value $a \in F$.

(c). Show that ϕ is diagonalizable: that is, there is a basis $\mathcal{B}$ of V such that $M_{\mathcal{B}}(\phi)$ is diagonal.

5. Show that there is a linear transformation $\phi : V \to V$, where V is a vector space of dimension 2 over $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$, such that ϕ^2 is the identity map but ϕ is not diagonalizable. That is, there is no basis $\mathcal{B}$ of V such that $M_{\mathcal{B}}(\phi)$ is diagonal.