Math 31BH: Assignment 1

Due 01/09 at 23:59

1. Let $(\mathbf{V}, \langle \cdot, \cdot \rangle)$ be a Euclidean space, and let $\mathcal{B} = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be an orthonormal basis of $\mathbf{V}$. Show that the $n \times 1$ coordinate matrix of any $\mathbf{v} \in \mathbf{V}$ relative to $\mathcal{B}$ is given by

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} \langle \mathbf{e}_1, \mathbf{v} \rangle \\ \vdots \\ \langle \mathbf{e}_n, \mathbf{v} \rangle \end{bmatrix}.$$

Solution: Let $\mathbf{v} = \sum_{j=1}^{n} x_j \mathbf{e}_i$ be the unique representation of a given vector $\mathbf{v} \in \mathbf{V}$ as a linear combination of the basis vectors in $\mathcal{B}$. Then for any $1 \leq i \leq n$ we have

$$\langle \mathbf{e}_i, \mathbf{v} \rangle = \left\langle \mathbf{v}, \sum_{j=1}^{n} x_j \mathbf{e}_j \right\rangle = \sum_{j=1}^{n} x_i \langle \mathbf{e}_i, \mathbf{e}_j \rangle = x_i,$$

where the last equality follows from orthonormality of $\mathcal{B}$.

2. With the same notation as in the previous problem, let $A \in \mathrm{End}\mathbf{V}$ be a linear operator. Show that the $n \times n$ matrix of A relative to $\mathcal{B}$ is

$$[A]_{\mathcal{B}} = \begin{bmatrix} \langle \mathbf{e}_1, A\mathbf{e}_1 \rangle & \dots & \langle \mathbf{e}_1, A\mathbf{e}_n \rangle \\ \vdots & & \vdots \\ \langle \mathbf{e}_n, A\mathbf{e}_1 \rangle & \dots & \langle \mathbf{e}_n, A\mathbf{e}_n \rangle \end{bmatrix}.$$

Solution: Consider the image $A\mathbf{e}_j = \sum_{i=1}^{n} a_{ij} \mathbf{e}_i$ of the operator A applied to the jth vector of the basis $\mathcal{B}$, so that a_{ij} is the (i, j)-matrix element of $[A]_{\mathcal{B}}$. By the previous problem, $a_{ij} = \langle \mathbf{e}_i, A\mathbf{e}_j \rangle$.

3. With the same notation as in the previous problems, for each $1 \leq i, j \leq n$ let $E_{ij} \in \mathrm{End}\mathbf{V}$ be the linear operator defined by

$$E_{ij}\mathbf{e}_k = \langle \mathbf{e}_j, \mathbf{e}_k \rangle \mathbf{e}_i, \quad 1 \leq k \leq n.$$

What is the matrix of E_{ij} relative to $\mathcal{B}$?

Solution: By the previous problem, the (k, l)-matrix element of $[E_{ij}]_{\mathcal{B}}$ is

$$\langle \mathbf{e}_k, E_{ij}\mathbf{e}_l \rangle = \langle \mathbf{e}_k, \langle \mathbf{e}_j, \mathbf{e}_l \rangle \mathbf{e}_i \rangle = \langle \mathbf{e}_i, \mathbf{e}_k \rangle \langle \mathbf{e}_j, \mathbf{e}_l \rangle.$$

Thus, the matrix $[E_{ij}]_{\mathcal{B}}$ has a 1 in the (i,j)-position and all other matrix elements are 0. This is called the ijth "elementary matrix."

4. With the same notation as in the previous problems, prove that

$$E_{ij}E_{kl} = \langle \mathbf{e}_j, \mathbf{e}_k \rangle E_{il}.$$

Deduce from this that $\mathcal{E} = \{E_{ij} : 1 \le i,j \le n\}$ is an orthonormal basis of End$\mathbf{V}$, where by definition the scalar product of two operators $A, B \in$ End$\mathbf{V}$ is $\langle A, B\rangle = \operatorname{Tr} A^* B$, with Tr the trace and A^* the adjoint (aka transpose) of A. What is $\dim \operatorname{End}\mathbf{V}$?

Solution: The action of the operator $E_{ij}E_{kl}$ on a basis vector $\mathbf{e}_p$ is

$$E_{ij}E_{kl}\mathbf{e}_p = E_{ij}\langle \mathbf{e}_l, \mathbf{e}_p\rangle = \langle \mathbf{e}_l, \mathbf{e}_p\rangle E_{ij}\mathbf{e}_k = \langle \mathbf{e}_l, \mathbf{e}_p\rangle \langle \mathbf{e}_j, \mathbf{e}_k\rangle \mathbf{e}_i.$$

The action of the operator $\langle \mathbf{e}_j, \mathbf{e}_k\rangle E_{il}$ on the same basis vector $\mathbf{e}_p$ is the same:

$$\langle \mathbf{e}_j, \mathbf{e}_k\rangle E_{il}\mathbf{e}_p = \langle \mathbf{e}_j, \mathbf{e}_k\rangle \langle \mathbf{e}_l, \mathbf{e}_p\rangle \mathbf{e}_i.$$

Since the two operators have the same effect on every basis vector, they are the same operator.

Next, using the previous problem we have

$$\langle \mathbf{e}_k, E_{ij}\mathbf{e}_l\rangle = \langle \mathbf{e}_i, \mathbf{e}_k\rangle \langle \mathbf{e}_j, \mathbf{e}_l\rangle,$$

and also

$$\langle E_{ji}\mathbf{e}_k, \mathbf{e}_l\rangle = \langle \mathbf{e}_l, E_{ji}\mathbf{e}_k\rangle = \langle \mathbf{e}_i, \mathbf{e}_k\rangle \langle \mathbf{e}_j, \mathbf{e}_l\rangle.$$

Therefore, $E_{ij}^* = E_{ji}$, i.e. the transpose of the ijth elementary matrix is the jith elementary matrix. We thus have that

$$\langle E_{ij}, E_{kl}\rangle = \mathrm{Tr} E_{ij}^* E_{kl} = \mathrm{Tr} E_{ji}E_{kl} = \langle \mathbf{e}_i, \mathbf{e}_k\rangle \mathrm{Tr} E_{jl} = \langle \mathbf{e}_i, \mathbf{e}_k\rangle \langle \mathbf{e}_j, \mathbf{e}_l\rangle, \quad (1)$$

where we used the previous problem as well as the linearity of the trace. The resulting product is 0 unless $i = k$ and $j = l$, i.e. unless $E_{ij} = E_{kl}$, and in this case it is 1. Thus, $\mathcal{E}$ is an orthonormal set, and hence it is a linearly independent set.

It remains to show that $\mathcal{E}$ spans End$\mathbf{V}$, i.e. that any linear operator A on $\mathbf{V}$ is a linear combination of the operators E_{ij}. This is clear, since if $[A]_{\mathcal{B}} = [a_{ij}]_{i,j=1}^N$ is the matrix of A relative to the basis $\mathcal{B}$, then we have the equality of matrices

$$[A]_{\mathcal{B}} = \sum_{i,j=1}^{n} a_{ij}[E_{ij}]_{\mathcal{B}} = \left[\sum_{i,j=1}^{n} a_{ij}E_{ij}\right]_{\mathcal{B}},$$

which says that the matrix of A relative to $\mathcal{B}$ is equal to the matrix of $\sum_{i,j=1}^{n} a_{ij}E_{ij}$ relative to $\mathcal{B}$. Since any operator is uniquely determined by its matrix relative to $\mathcal{B}$, we have that

$$A = \sum_{i,j=1}^{n} a_{ij}E_{ij},$$

as required. We conclude that

$$\dim \mathrm{End}\mathbf{V} = |\{(i,j) \colon 1 \leq i,j \leq n\}| = n^2.$$

5. With the same notation as in the previous problems, prove that $\mathcal{S} = \{E_{ij} + E_{ji} \colon 1 \leq i \leq j \leq n\}$ is an orthogonal basis of the subspace $\mathrm{Sym}\mathbf{V}$ of $\mathrm{End}\mathbf{V}$ consisting of symmetric operators. What is $\dim \mathrm{Sym}\mathbf{V}$?

   Solution: For each $1 \leq i \leq j \leq n$, put $S_{ij} := E_{ij} + E_{ji}$. Let us check that S_{ij} is in fact a symmetric operator. We have

$$S_{ij}^* = E_{ij}^* + E_{ji}^* = E_{ji} + E_{ij} = S_{ij},$$

   so indeed $S_{ij} \in \mathrm{Sym}\mathbf{V}$. Now take $1 \leq i,j,k,l \leq n$ with $i \leq j$ and $k \leq l$. We then have

$$\langle S_{ij}, S_{kl}\rangle = \langle E_{ij}+E_{ji}, E_{kl}+E_{lk}\rangle = \langle E_{ij}, E_{kl}\rangle + \langle E_{ij}, E_{lk}\rangle + \langle E_{ji}, E_{kl}\rangle + \langle E_{ji}, E_{lk}\rangle.$$

   By the previous problem, the first and last scalar products in the sum on the right hand side are 0 unless $i = j$ and $k = l$, in which case they are 1, giving a total of 2. Now consider the middle terms. Again by the previous problem, the scalar product $\langle E_{ij}, E_{lk}\rangle$ is 0 unless $(i,j) = (l,k)$, but since $i \leq j$ and $l \geq k$ this is zero unless $i = j = k = l$, in which case it is 1. The same argument holds for $\langle E_{ji}, E_{kl}\rangle$. In summary, if $(i,j) \neq (k,l)$ then $\langle S_{ij}, S_{kl}\rangle = 0$, which gives orthogonality of the set $\mathcal{S}$, while

$$\langle S_{ij}, S_{ij}\rangle = \begin{cases} 2, & \text{if } i < j \\ 4, & \text{if } i = j \end{cases}.$$

   To show that $\mathcal{S}$ spans $\mathrm{Sym}\mathbf{V}$, let A by a symmetric operator. Then, the matrix of A relative to the basis $\mathcal{B}$ of $\mathbf{V}$ has the form $[A]_{\mathcal{B}} = [a_{ij}]_{i,j=1}^{n}$ with $a_{ij} = a_{ji}$, so that

$$A = \sum_{1\le i\le j\le n} a_{ij} S_{ij}$$

by the same argument as in the previous problem.

We conclude that

$$\dim \mathrm{Sym}\mathbf{V} = |\{(i,j)\colon 1 \le i \le j \le n\}| = n+(n-1)+\cdots+2+1 = \binom{n+1}{2}.$$