Math 188: Homework 5

Merrick Qiu

1 Gray Code Graph

1. Let us look at the scalar coordinate value of an arbitrary basis vector $i \in B_n$ of AE_x. We can sum over the coordinate values of the neighbors of i to get the coordinate value of i. Let y be a neighbor of i. Note that if y differs in the jth position and $x_j = 0$, then $x \cdot i = x \cdot y$. If $x_j = 1$, then $x \cdot i$ and $x \cdot y$ will differ by 1.

   Thus, there are $n - |x|$ neighbors of i such that the parity of $x \cdot i$ is the same as $x \cdot y$ and $|x|$ neighbors of i such that the parity of $x \cdot i$ is different than $x \cdot y$. It follows that

   $$\begin{aligned}(AE_x)_i &= \sum_{\substack{\text{y such that} \\ \text{(y,i) is an edge}}} (-1)^{x \cdot y} \\ &= (n - |x|)(E_x)_i - |x|(E_x)_i \\ &= (n - 2|x|)(E_x)_i.\end{aligned}$$

   Since this is true for all $i \in B_n$, E_x is a eigenvector of A with eigenvalue $n - 2|x|$.

2. Using the eigenvalues from part a, the number of closed walks of length d is

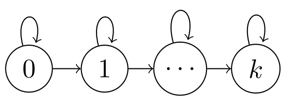
   $$\sum_{x \in B_n} (n - 2|x|)^d = \sum_{i=0}^{n} \binom{n}{i} (n - 2i)^d.$$

   The number of closed walks starting at each x is equal due to the symmetry of H_n, so the number of closed walks of length d beginning at x is

   $$\frac{1}{n} \sum_{i=0}^{n} \binom{n}{i} (n - 2i)^d.$$

2 Binary String Graph

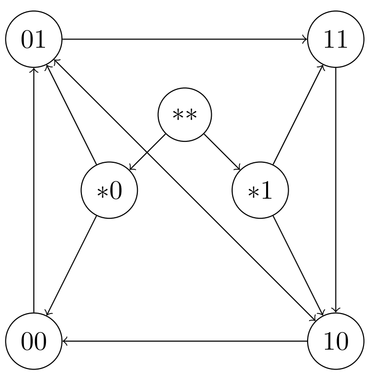
1. Construct a graph with verteces $0, \ldots, k$. Each vertex represents the number of zeros that have appeared. Each vertex has a self-loop that represents choosing a one. There are edges between consecutive vertices that represent choosing a zero. The length of a walk beginning at 0 is equal to the length of the associated string.

   Graph 1: Binary String with k Zeros.

2. Construct a graph with 7 verteces representing all the combinations that the last two characters can take on. The * character represents a missing character if the string length is less than 2.

   Each edge represents changing the string by adding a zero or one (The right symbol of the successor vertex of the edge is the added symbol). Note that 00 and 11 do not have self-loops to prevent three characters in a row. The length of a walk beginning at ** is equal to the length of the associated string

   Graph 2: Binary String Without Three Symbols in a Row.

3 Painting Tables

1. Let $\alpha(S)$ be the set of ways to paint tables red, let $\beta(S)$ be the set of ways to paint tables blue, and let $\gamma(S)$ be the set of ways to paint tables green. Since $|\alpha(S)| = 1$ when $|S|$ is odd and $|\alpha(S)| = 0$ when $|S|$ is even, the exponential generating function for α is

$$E_\alpha(x) = \sum_{n\geq 0} \frac{x^{2n+1}}{(2n+1)!} = \frac{e^x - e^{-x}}{2}.$$

Similarly for β,

$$E_\beta(x) = \sum_{n\geq 0} \frac{x^{2n}}{(2n)!} = \frac{e^x + e^{-x}}{2}.$$

We have that $E_\gamma(x) = e^x$ since $|\gamma(S)| = 1$. The exponential generating function for painting tables is

$$\begin{aligned} E_{\alpha\cdot\beta\cdot\gamma}(x) &= \frac{1}{4}e^x(e^x - e^{-x})(e^x + e^{-x}) \\ &= \frac{1}{4}(e^{3x} - e^{-x}) \\ &= \frac{1}{4}\left(\sum_{n\geq 0}\frac{(3x)^n}{n!} + \sum_{n\geq 0}\frac{(-x)^n}{n!}\right). \end{aligned}$$

Therefore, the number of ways to paint n tables according to the listed rules is $\frac{1}{4}(3^n + (-1)^n)$.

2. Let $\delta(S)$ be the set of ways to paint tables white or yellow. Since $|\delta(S)| = 2^{|S|}$ when $|S|$ is even and $|\delta(S)| = 0$ when $|S|$ is odd, the exponential generating function for δ is

$$E_\delta(x) = \sum_{n\geq 0} \frac{(2x)^{2n}}{(2n)!} = \frac{e^{2x} - e^{-2x}}{2}.$$

. The new exponential generating function for painting tables is

$$\begin{aligned} E_{\alpha\cdot\beta\cdot\gamma\cdot\delta}(x) &= \frac{1}{8}(e^{3x} - e^{-x})(e^{2x} - e^{-2x}) \\ &= \frac{1}{8}(e^{5x} - 2e^x + e^{-3x}). \end{aligned}$$

The number of ways to paint n tables is now $\frac{1}{8}(5^n + (-3)^n - 2)$.

4 EGF of set partitions and even blocks.

1. Let $\alpha(S)$ be the set of ways to have a set partition with a single even block. Note that $\alpha(\emptyset) = \emptyset$. The EGF of α is

$$E_\alpha(x) = \sum_{n \geq 1} \frac{x^{2n}}{(2n)!} = \frac{e^x + e^{-x}}{2} - 1 = \frac{e^x + e^{-x} - 2}{2}.$$

   We have that

$$B_k(x) = E_{\alpha^k}(x) = \left(\frac{e^x + e^{-x} - 2}{2} \right)^k.$$

2. If there is no restriction on the number of blocks, then

$$B(x) = E_{e^\alpha}(x) = \exp\left(\frac{e^x + e^{-x} - 2}{2} \right).$$

5 Idempotent Functions

The idempotent function $f : [n] \to [n]$ can be encoded as a directed graph with vertices $[n]$ and an edge $i \to j$ if $f(i) = j$. If there is an edge $i \to j$ then there must also be a self-loop $j \to j$. Let $\alpha(S)$ be the set of ways to have a connected graph with only one self-loop. Since there are $|S|$ possible choices for the vertex with the self-loop,

$$E_\alpha(x) = \sum_{n \geq 1} n \frac{x^n}{(n)!} = x \sum_{n \geq 1} \frac{x^{n-1}}{(n-1)!} = xe^x.$$

Thus,

$$A(x) = E_{e^\alpha}(x) = \exp(xe^x).$$