Math 140A: Homework 5

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A

($\implies$) **Prove compactness implies finiteness.** Assume by contradiction that A is infinite. Choose the open cover of A that is made up of the open neighborhoods around all the points in A with radius $r < 1$. Since each of these neighborhoods only covers the point that is is centered on, a finite subcover does not exist since removing any one of these open neighborhoods would result in A not being covered. This is a contradiction so compact sets must be finite for metric spaces with the discrete metric.

($\impliedby$) **Prove finiteness implies compactness.** Suppose there is an open cover of A. For each point in A, choose a open set from the open cover that covers that point. Since there is a finite number of points in A, these open sets constitue a finite subcover of A.

Rudin 12

Let $\{G_\alpha\}$ be an open covering of K. Let G_{α_0} be the open set that covers 0. Include this set in the finite subcover. Since G_{α_0} is open, 0 is an interior point of G_{α_0} so there exists a radius $r > 0$ where $N_r(0) \subset G_{\alpha_0}$.

The set K only contains the limit point 0, so there is a finite number of points not within the neighborhood of 0 and so there is a finite number of points not covered by G_{α_0}. For each point not covered by G_{α_0}, pick an open set in the open covering that covers that point and add it into the finite subcover. We have constructed a finite subcover so K is compact.

Rudin 23

Since X is separable, let $Y \subset X$ be a countable dense subset of X. Let $\{V_\alpha\}$ be a collection of all the open neighborhoods centered around points in Y with rational radiuses. Since Y is countable and rational numbers are countable, $\{V_\alpha\}$ is also countable.

For every $x \in X$ and open set $G \subset X$ with $x \in G$, we can find an open neighborhood around x with rational radius such that $x \in N_r(x) \subset G$, so $\{V_\alpha\}$ acts as a countable base.

Rudin 24

Let $\delta > 0$ and pick $x_1 \in X$. Choose $x_1, \cdots x_j \in X$ such that $d(x_i, x_{i+1}) \geq \delta$ if possible. This process must stop after a finite number of steps, because if it was infinite, then this sequence would have a limit point and all the points could not be δ apart from each other.

Thus, X can therefore be covered by finitely many neighborhoods of radius δ since if it couldn't, we could add another point x_{j+1} to the sequence. We can take the union of all $\delta = \frac{1}{n}$ for all $n \in \mathbb{N}$ in order to generate a set of countable points that are dense in X. Thus X is separable.

Rudin 25

We can choose a cover of K composing of open neighborhoods of radius $\frac{1}{n}$, and since K is compact, we can find a finite subcover of this cover. If we union together the finite subcovers we get for all $n = 1, 2, 3, \cdots$, we get a set that also covers K, and the centers of all these neighborhoods forms a countable base for K. This is a countable base because for every $x \in K$ and every open set $G \subset K$ with $x \in G$, x is an interior point of G and so we can find an open neighborhood that is from one of the finite subcovers where $x \in N_{\frac{1}{n}}(x) \subset G$.

Since every point in K is arbitrarily close to some point in this countable base, K is also separable.

Rudin 26

Exercise 24 implies X has a separable base, and exercise 23 implies that since X has a separable base, it has a countable base. It follows that every open cover of X has a countable subcover $\{G_n\}$, which we can obtain as a union of a subcollection of the countable base. Assume by contradiction that there did not exist a finite subcollection of $\{G_n\}$ that covered X. Then the complement, F_n of $G_1 \cup \cdots \cup G_n$, must be nonempty for each n since no set of $G_1 \cup \cdots \cup G_n$ completely covers X.

However $\cap F_n$ is empty because $\cup G_n$ covers X. Let E is a set which contains a point from each F_n. E has a limit point, $p \in E$, since every infinite subset of X has a limit point. p must also belong to one of the sets G_n, and since G_n is open, there is a neighborhood around p that is contained by G_n. But this neighborhood around p cannot contain any point in F_m where $m > n$ which is a contradiction.