Math 188: Homework 3

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1 Another Catalan Interpretation

Let T_n be the number of ways of triangulating a polygon with $(n+2)$ vertices. Consider an arbitrary edge of this polygon. The number of ways of triangulating that polygon can be divided into disjoint subsets based on which triangle that edge is a part of.

Moving clockwise from the chosen edge, number the other vertexes from 0 to $n-1$. The triangle consisting of the chosen edge and the ith vertex divides the polygon into a $(i+2)$-gon and a $(n-i+1)$-gon. Thus the number of ways of triangulating the polygon with the chosen edge connected to the ith vertex is T_iT_{n-i-1}. Summing over all possible values of i yields

$$T_n = \sum_{i=0}^{n-1} T_iT_{n-i-1}.$$

Since $T_0 = 1$ and T_n has the same recurrence relation as the Catalan numbers, it follows that $T_n = C_n$.

2 Balanced Parenthesis with Stars

There are a_{n-1} strings of length n that start with "*". For strings that begin with a "(", consider the ")" that pairs with it. If there is a string of length i inside the parentheses, then there is a string of length $n-i-2$ to the right of the parenthesis. Since both of these strings must contain balanced parenthesis as well, there are $a_i a_{n-i-2}$ strings of length n that start with "(", so

$$a_n = a_{n-1} + \sum_{i=0}^{n-2} a_i a_{n-i-2}$$

The right term is the coefficient of x^{n-2} in $A(x)^2$, so

$$\begin{aligned}
A(x) &= 1 + x + \sum_{n\geq 2} a_n x^n \\
&= 1 + x + \sum_{n\geq 2} \left(a_{n-1} + \sum_{i=0}^{n-2} a_i a_{n-i-2} \right) x^n \\
&= 1 + x + \sum_{n\geq 2} a_{n-1} x^n + \sum_{n\geq 2} \left(\sum_{i=0}^{n-2} a_i a_{n-i-2} \right) x^n \\
&= 1 + x + x \sum_{n\geq 2} a_{n-1} x^{n-1} + x^2 \sum_{n\geq 2} \left(\sum_{i=0}^{n-2} a_i a_{n-i-2} \right) x^{n-2} \\
&= 1 + x + x(A(x) - 1) + x^2 A(x)^2 \\
&= 1 + xA(x) + x^2 A(x)^2.
\end{aligned}$$

Rearranging the above equation yields

$$x^2 A(x)^2 + (x-1)A(x) + 1 = 0.$$

Thus,

$$\begin{aligned}
a(x) &= x^2 \\
b(x) &= x - 1 \\
c(x) &= 1.
\end{aligned}$$

3 Compositions of 2n into 8 Parts

1. Each x_i can be written as $x_i = 2k_i$ for some $k_i \geq 0$, so

$$\begin{aligned} x_1 + x_2 + \ldots + x_8 = 2n &\implies 2k_1 + 2k_2 + \ldots + 2k_8 = 2n \\ &\implies k_1 + k_2 + \ldots + k_8 = n. \end{aligned}$$

   This is a weak composition of n with 8 parts so

$$\binom{8+n-1}{n} = \binom{n+7}{n}.$$

2. Each x_i can be written as $x_i = 2k_i + 1$ for some $k_i \geq 0$, so

$$\begin{aligned} x_1 + x_2 + \ldots + x_8 = 2n &\implies (2k_1 + 1) + (2k_2 + 1) + \ldots + (2k_8 + 1) = 2n \\ &\implies k_1 + k_2 + \ldots + k_8 = n - 4 \end{aligned}$$

   This is a weak composition of $n - 4$ with 8 parts so

$$\binom{8 + (n-4) - 1}{n-4} = \binom{n+3}{n-4}.$$

3. For a given x_8, the ways of choosing the values for the other x_i is a weak composition of $2n - x_8$ with 7 parts. Summing over the possible values of x_8 yields

$$\sum_{i=0}^{9} \binom{7 + (2n-i) - 1}{2n-i} = \sum_{i=0}^{9} \binom{2n-i+6}{2n-i}.$$

4 Sums over Compositions

1. Let D be the derivative. Consider the product

$$\begin{aligned}
P(x) &= \left(\sum_{a_1\geq 1} a_1 x^{a_1}\right)\left(\sum_{a_2\geq 2} a_2 x^{a_2}\right)\dots\left(\sum_{a_n\geq 1} a_n x^{a_n}\right) \\
&= \left(xD\left(\frac{1}{1-x}\right)\right)^n = \left(\frac{x}{(1-x)^2}\right)^n
\end{aligned}$$

Note that the $[x^k]$ term encodes the sum of products of compositions of k into n parts, so

$$[x^k]P(x) = \sum_{(a_1,\dots,a_n)} a_1 a_2 \dots a_n.$$

Using the binomial theorem yields and the fact that $\binom{-n}{k} = (-1)^k\binom{n+k-1}{k}$,

$$\begin{aligned}
P(x) = x^n(1-x)^{-2n} &= x^n \sum_{k\geq 0}(-1)^k\binom{-2n}{k}x^k \\
&= \sum_{k\geq n}(-1)^{k-n}\binom{-2n}{k-n}x^k = \sum_{k\geq n}(-1)^{k-n}\left((-1)^{k-n}\binom{2n+(k-n)-1}{k-n}\right)x^k \\
&= \sum_{k\geq n}\binom{n+k-1}{k-n}x^k.
\end{aligned}$$

Thus,

$$\sum_{(a_1,\dots,a_n)} a_1 a_2 \dots a_n = [x^k]P(x) = \binom{n+k-1}{k-n}$$

2. Using the same idea as part a, consider the product

$$\begin{aligned}
P(x) &= \left(\sum_{a_1\geq 1} 1^{a_1-1}x^{a_1}\right)\left(\sum_{a_2\geq 2} 2^{a_2-1}x^{a_2}\right)\dots\left(\sum_{a_n\geq 1} n^{a_n-n}x^{a_n}\right) \\
&= \left(\frac{x}{1-x}\right)\left(\frac{x}{1-2x}\right)\dots\left(\frac{x}{1-nx}\right) \\
&= \frac{x^k}{(1-x)(1-2x)\dots(1-kx)}.
\end{aligned}$$

This is the generating function for the stirling numbers, so

$$\sum_{(a_1,\dots,a_n)} 2^{a_2-1}3^{a_3-1}\dots n^{a_n-1} = [x^k]P(x) = S(k,n).$$

5 Tuples of subsets

1. Each element of $[n]$ can either be in S and T, be in T but not S, or not be in T or S, so there are 3^n pairs of subset such that $S \subseteq T$. However there are 2^n ways in which $S = T$ so there are $3^n - 2^n$ pairs of subsets such that $S \subsetneq T$.

2. Each element of $[n]$ has 2^k ways of appearing in k subsets, but since the element must appear in at least one subset, there are $2^k - 1$ ways for the element to appear in the subsets $(S_1, \ldots, S_k)$. Thus, there are $(2^k - 1)^n$ k-tuples of subsets of $[n]$.