Math 170B: Homework 1

Merrick Qiu

Bisection Method

Since the interval is of length 1, $\log_2(10^6) - 1 = 18.93$ so 19 steps are needed to get an accuracy of 10^{-6}. Since $|x_T| \geq 2$, we only need 18 steps to get a relative accuracy of 10^{-6}.

Newton Method

We are trying to find the roots of

$$f = \begin{bmatrix} 4x_1^2 - x_2^2 \\ 4x_1x_2^2 - x_1 - 1 \end{bmatrix}.$$

The Jacobian is

$$f' = \begin{bmatrix} 8x_1 & -2x_2 \\ 4x_2^2 - 1 & 8x_1x_2 \end{bmatrix}.$$

The first iteration is

$$\begin{aligned} x_1 &= x_0 - f'(x_0)^{-1} f(x_0) \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 & -2 \\ 3 & 0 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 & \frac{1}{3} \\ -\frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} \\ \frac{1}{2} \end{bmatrix} \end{aligned}$$

The second iteration is

$$\begin{aligned} x_2 &= x_1 - f'(x_1)^{-1} f(x_1) \\ &= \begin{bmatrix} \frac{1}{3} \\ \frac{1}{2} \end{bmatrix} - \begin{bmatrix} \frac{8}{3} & -1 \\ 0 & \frac{4}{3} \end{bmatrix}^{-1} \begin{bmatrix} \frac{7}{36} \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{3} \\ \frac{1}{2} \end{bmatrix} - \begin{bmatrix} \frac{3}{8} & \frac{9}{32} \\ 0 & \frac{3}{4} \end{bmatrix} \begin{bmatrix} \frac{7}{36} \\ -1 \end{bmatrix} \\ &= \begin{bmatrix} \frac{13}{24} \\ \frac{5}{4} \end{bmatrix} \end{aligned}$$

Problem 1.3

The derivatives are

$$f(x) = e^x - x - 2$$
$$f'(x) = e^x - 1$$
$$f''(x) = e^x$$

Note that $f'(x) > 0$ and $f''(x) > 0$ when $x > 0$ and $f'(x) < 0$ and $f''(x) > 0$ when $x < 0$. Thus by theorem 1.9, Newton's method converges to the positive root when $x > 0$ and Newton's method converges to the negative root when $x < 0$.

Problem 1.6

The taylor expansion around x_k is

$$\begin{aligned}
0 = f(\xi) = f(x_k) + f'(x_k)(\xi - x_k) + \frac{f''(\eta)}{2}(\xi - x_k)^2 &\implies \\
0 = \frac{f(x_k)}{f'(x_k)} + (\xi - x_k) + \frac{f''(\eta)}{2f'(x_k)}(\xi - x_k)^2 &\implies \\
\frac{f(x_k)}{f'(x_k)} = -(\xi - x_k) - \frac{f''(\eta)}{2f'(x_k)}(\xi - x_k)^2
\end{aligned}$$

By the mean value theorem there exists χ_k such that

$$f'(x_k) - f'(\xi) = (x_k - \xi)f''(\chi_k)$$

$$\xi - x_k = -\frac{f'(x_k) - f'(\xi)}{f''(\chi_k)} = -\frac{f'(x_k)}{f''(\chi_k)}$$

Substituting these in yields

$$\begin{aligned}
\xi - x_{k+1} &= \xi - x_k + \frac{f(x_k)}{f'(x_k)} \\
&= \xi - x_k - (\xi - x_k) - \frac{f''(\eta_k)}{2f'(x_k)}(\xi - x_k)^2 \\
&= -\frac{f''(\eta_k)}{2f'(x_k)}(\xi - x_k)^2 \\
&= \left(-\frac{f''(\eta_k)(\xi - x_k)}{2f'(x_k)}\right)\left(-\frac{f'(x_k)}{f''(\chi_k)}\right) \\
&= (\xi - x_k)\frac{f''(\eta_k)}{2f''(\chi_k)}
\end{aligned}$$

Note that $f''(\eta_k) < M$, $f''(\chi_k) > m$, and $M < 2m$ so $\frac{f''(\eta_k)}{2f''(\chi_k)} < 1$, so x_0 will converge. Additionally since $\eta_k \to \xi$ and $\chi_k \to \xi$, $f''(\eta_k) \to f''(\xi)$ and $f''(\chi_k) \to f''(\xi)$. Thus,

$$\frac{\xi - x_{k+1}}{\xi - x_k} = \frac{f''(\eta_k)}{2f''(\chi_k)} \to \frac{1}{2}$$

The asymptotic rate of convergence is thus $\rho = -\log\frac{1}{2} = \log 2$. Using newtons method we find that it converges to the correct root of 0 by the rate of convergence we found. $x_0 = 1$, $x_1 = 0.58$, $x_2 = 0.32$, $x_3 = 0.17$, $x_4 = 0.09$, $x_5 = 0.04$, ...

Problem 1.7

The taylor expansion around ξ is

$$f(x_k) = f(\xi) + f'(\xi)(x_k - \xi) + \frac{f''(x_k)}{2}(x_k - \xi)^2 + \frac{f'''(\eta_k)}{6}(x_k - \xi)^3 = \frac{f'''(\eta_k)}{6}(x_k - \xi)^3$$

$$f'(x_k) = f'(\xi) + f''(\xi)(x_k - \xi) + \frac{f'''(\chi_k)}{2}(x_k - \xi)^2 = \frac{f'''(\chi_k)}{2}(x_k - \xi)^2$$

Substituting yields

$$\begin{aligned}
\xi - x_{k+1} &= \xi - x_k + \frac{f(x_k)}{f'(x_k)} \\
&= \xi - x_k + \frac{\frac{f'''(\eta_k)}{6}(x_k - \xi)^3}{\frac{f'''(\chi_k)}{2}(x_k - \xi)^2} \\
&= \xi - x_k + \frac{f'''(\eta_k)(x_k - \xi)}{3f'''(\chi_k)} \\
&= (\xi - x_k)\left(1 - \frac{f'''(\eta_k)}{3f'''(\chi_k)}\right)
\end{aligned}$$

If we assume that η_k and χ_k both lie between ξ and x_k and that $0 < m < |f'''(x)| < M$ and $M < 3m$ then

$$\frac{\xi - x_{k+1}}{\xi - x_k} = 1 - \frac{f'''(\eta_k)}{3f'''(\chi_k)} < 1 - \frac{M}{m} < 1$$

$$\frac{\xi - x_{k+1}}{\xi - x_k} \to \frac{2}{3}$$

Problem 1.10

We have the secant method is

$$x_{k+1} = \frac{x_k f(x_{k-1}) - x_{k-1} f(x_k)}{f(x_{k-1}) - f(x_k)}$$

Substituting into the function yields

$$\begin{aligned}
\varphi(x_k, x_{k-1}) &= \frac{x_{k+1} - \xi}{(x_k - \xi)(x_{k-1} - \xi)} \\
&= \frac{\frac{x_k f(x_{k-1}) - x_{k-1} f(x_k)}{f(x_{k-1}) - f(x_k)} - \xi}{(x_k - \xi)(x_{k-1} - \xi)} \\
&= \frac{x_k f(x_{k-1}) - x_{k-1} f(x_k) - \xi f(x_{k-1}) + \xi f(x_k)}{(f(x_{k-1}) - f(x_k))(x_k - \xi)(x_{k-1} - \xi)}
\end{aligned}$$

By L'Hospitals rule on x_k,

$$\begin{aligned}
\lim_{x_k \to \xi} \frac{x_k f(x_{k-1}) - x_{k-1} f(x_k) - \xi f(x_{k-1}) + \xi f(x_k)}{(f(x_{k-1}) - f(x_k))(x_k - \xi)(x_{k-1} - \xi)} &= \lim_{x_k \to \xi} \frac{f(x_{k-1}) - x_{k-1} f'(x_k) + \xi f'(x_k)}{(x_{k-1} - \xi)(f(x_{k-1}) - f(x_k) - f'(x_k)(x_k - \xi))} \\
&= \frac{f(x_{k-1}) - x_{k-1} f'(\xi) + \xi f'(\xi)}{(x_{k-1} - \xi)(f(x_{k-1}) - f(\xi))}
\end{aligned}$$

By L'Hopitals rule on x_{k-1} twice,

$$\begin{aligned}
\lim_{x_{k-1} \to \xi} \frac{f(x_{k-1}) - x_{k-1} f'(\xi) + \xi f'(\xi)}{(x_{k-1} - \xi)(f(x_{k-1}) - f(\xi))} &= \lim_{x_{k-1} \to \xi} \frac{f'(x_{k-1}) - f'(\xi)}{(x_{k-1} - \xi) f'(x_{k-1}) + f(x_{k-1}) - f(\xi)} \\
&= \lim_{x_{k-1} \to \xi} \frac{f''(x_{k-1})}{(x_{k-1} - \xi) f''(x_{k-1}) + f'(x_{k-1}) + f'(x_{k-1})} \\
&= \frac{f''(\xi)}{2 f'(\xi)}
\end{aligned}$$

If we assume that

$$\lim_{k \to \infty} \frac{|x_{k+1} - \xi|}{|x_k - \xi|^q} = A$$

then

$$\lim_{k \to \infty} \frac{|x_k - \xi|^{\frac{1}{q}}}{|x_{k-1} - \xi|} = A^{\frac{1}{q}}$$

so

$$\lim_{k \to \infty} \frac{|x_{k+1} - \xi|}{|x_k - \xi|^{q - 1/q} |x_{k-1} - \xi|} = A^{1 + \frac{1}{q}}$$

When $q - 1/q = 1$ we have that $A^{1+\frac{1}{q}} = \frac{f''(\xi)}{2f'(\xi)}$ from our first limit. Solving out for $q^2 - q - 1$ yields positive root $q = \frac{1+\sqrt{5}}{2}$. Finally solving out for A yields

$$\lim_{k \to \infty} \frac{|x_{k+1} - \xi|}{|x_k - \xi|^q} = A = \left(\frac{f''(\xi)}{2 f'(\xi)} \right)^{q/(1+q)}$$