Math 20D HW2

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Section 2.3

Problem 3
Both
Problem 4
Linear
Problem 12
We have that $P(x) = 4$ and $Q(x) = x^2e^{-4x}$.
The integrating factor is

$$\mu(x) = e^{\int 4\,dx} = e^{4x}$$

Multiplying by the integrating factor gives

$$\begin{aligned}\frac{dy}{dx} + 4y = x^2e^{-4x} &\implies \frac{d}{dx}e^{4x}y = x^2 \\ &\implies e^{4x}y = \frac{1}{3}x^3 + C \\ &\implies y = \frac{1}{3}x^3e^{-4x} + Ce^{-4x}\end{aligned}$$

Problem 14
We have that $P(x) = \frac{3}{x}$ and $Q(x) = \frac{\sin x}{x^2} - 3x$.
The integrating factor is

$$\mu(x) = e^{\int \frac{3}{x}\,dx} = e^{3\ln x} = x^3$$

Multiplying by the integrating factor gives

$$\begin{aligned}\frac{dy}{dx} + \frac{3}{x}y = \frac{\sin x}{x^2} - 3x &\implies \frac{d}{dx}x^3y = x\sin x - 3x^4 \\ &\implies x^3y = \sin x - x\cos x - \frac{3}{5}x^5 + C \\ &\implies y = \frac{\sin x}{x^3} - \frac{\cos x}{x^2} - \frac{3}{5}x^2 + \frac{C}{x^3}\end{aligned}$$

Problem 17
We have that $P(x) = -\frac{1}{x}$ and $Q(x) = xe^x$.
The integrating factor is

$$\mu(x) = e^{\int -\frac{1}{x}\,dx} = \frac{1}{x}$$

Multiplying by the integrating factor gives

$$\begin{aligned}\frac{dy}{dx} - \frac{y}{x} = e^x &\implies \frac{d}{dx}\frac{y}{x} = e^x \\ &\implies \frac{y}{x} = e^x + C \\ &\implies y = xe^x + Cx\end{aligned}$$

At $y(1) = e - 1$,

$$y = xe^x + Cx \implies e - 1 = e + C \implies C = -1$$

Therefore the final solution is

$$y = xe^x - x$$

Problem 20
We have that $P(x) = \frac{3}{x}$ and $Q(x) = 3x - 2$
The integrating factor is

$$\mu(x) = e^{\int \frac{3}{x}\,dx} = x^3$$

Multiplying by the integrating factor gives

$$\begin{aligned}\frac{dy}{dx}+\frac{3}{x}y=3x-2 &\implies \frac{d}{dx}x^3y=3x^4-2x^3\\ &\implies x^3y=\frac{3}{5}x^5-\frac{1}{2}x^4+C\\ &\implies y=\frac{3}{5}x^2-\frac{1}{2}x+\frac{C}{x^3}\end{aligned}$$

At $y(1)=1$,

$$y=\frac{3}{5}x^2-\frac{1}{2}x+\frac{C}{x^3} \implies 1=\frac{3}{5}-\frac{1}{2}+C \implies C=\frac{9}{10}$$

Therefore the final solution is

$$y=\frac{3}{5}x^2-\frac{1}{2}x+\frac{9}{10x^3}$$

Problem 29 Taking the reciprocal of both sides,

$$\frac{dy}{dx}=\frac{1}{e^{4y}+2x} \implies \frac{dx}{dy}=e^{4y}+2x \implies \frac{dx}{dy}-2x=e^{4y}$$

We have "switched" the roles of the variables, so $P(y)=-2$ and $Q(y)=e^{4y}$. The integrating factor is

$$\mu(y)=e^{\int -2\,dy}=e^{-2y}$$

Multiplying by the integrating factor gives

$$\begin{aligned}\frac{dx}{dy}-2x=e^{4y} &\implies \frac{d}{dy}e^{-2y}x=e^{2y}\\ &\implies e^{-2y}x=\frac{1}{2}e^{2y}+C\\ &\implies x=\frac{1}{2}e^{4y}+Ce^{2y}\end{aligned}$$

Section 2.4

Problem 4
Exact since both derivatives equal $xye^{xy} + x^{xy}$
Problem 5
Separable and linear
Problem 6
Exact since both derivatives equal $2y$
Problem 10
The equation is exact since both derivatives equal 1. Thus,

$$F(x,y) = \int 2x + y\,dx + g(y) = x^2 + xy + g(y)$$

Taking the derivative with respect to y,

$$x - 2y = x + g'(y) \implies g'(y) = -2y \implies g(y) = -y^2$$

Thus the general solution is

$$F(x,y) = x^2 + xy - y^2 = C$$

Problem 22 The equation is exact so,

$$F(x,y) = \int ye^{xy} - \frac{1}{y}\,dx = e^{xy} - \frac{x}{y} + g(y)$$

Taking the derivative with respect to y,

$$xe^{xy} + \frac{x}{y^2} = xe^{xy} + \frac{x}{y^2} + g'(y) \implies g'(y) = 0 \implies g(y) = C$$

Thus the general solution is in form

$$F(x,y) = e^{xy} - \frac{x}{y} = C$$

$C = e - 1$ at $y(1) = 1$ so the solution is

$$e^{xy} - \frac{x}{y} = e - 1$$

Problem 23 The equation is exact so,

$$F(x,y) = \int e^t y + te^t y\,dt = te^t y + g(y)$$

Taking the derivative with respect to y,

$$te^t + 2 = te^t + g'(y) \implies g'(y) = 2 \implies g(y) = 2y + C$$

Thus the general solution is in form

$$te^t y + 2y = C$$

$C = -2$ at $y(0) = -1$ so the solution is

$$te^t y + 2y = -2 \implies y = -\frac{2}{te^t + 2}$$

Problem 26 The equation is exact so,

$$F(x,y) = \int \tan y - 2\,dx = x\tan y - 2x + g(y)$$

Taking the derivative with respect to y,

$$x\sec^2 y + \frac{1}{y} = x\sec^2 y + g'(y) \implies g'(y) = \frac{1}{y} \implies g(y) = \ln y + C$$

Thus the general solution is in form

$$x \tan y - 2x + \ln y = C$$

$C = 0$ at $y(0) = 1$ so the solution is

$$x \tan y - 2x + \ln y = 0$$

Problem 29 The equation is not exact since

$$\frac{\partial}{\partial y}(y^2 + 2xy) = 2y + 2x$$

$$\frac{\partial}{\partial x} - x^2 = -2x$$

Multiplying by y^{-2} yields an exact solution since

$$\frac{\partial}{\partial y}(1 + 2\frac{x}{y}) = -2\frac{x}{y^2}$$

$$\frac{\partial}{\partial x}(-\frac{x^2}{y^2}) = -2\frac{x}{y^2}$$

Therefore,

$$F(x, y) = \int (1 + 2\frac{x}{y})\, dx = x + \frac{x^2}{y} + g(y)$$

Taking the derivative relative to y,

$$-\frac{x^2}{y^2} = -\frac{x^2}{y^2} + g'(y) \implies g'(y) = 0 \implies g(y) = C$$

The general solution is thus

$$x + \frac{x^2}{y} = C \implies y = \frac{x^2}{C - x}$$

The solution, $y = 0$, was lost.

Section 4.2

Problem 5 The root of the characteristic polynomial is $r = -4$

$$r^2 + 8y + 16 = (r+4)^2$$

The general solution is in the form

$$y(t) = c_1 e^{-4t} + c_2 t e^{-4t}$$

Problem 10 The roots of the characteristic polynomial are

$$r^2 - r - 11 = 0 \implies r = \frac{1 \pm \sqrt{1+44}}{2} \implies r = \frac{1 \pm 3\sqrt{5}}{2}$$

The general solution is in the form

$$y(t) = c_1 e^{\frac{1+3\sqrt{5}}{2}t} + c_2 e^{\frac{1-3\sqrt{5}}{2}t}$$

Problem 16 The roots of the characteristic polynomial are $r = -1, 5$

$$r^2 - 4r - 5 = (r-5)(r+1)$$

The general solution is in the form

$$y(t) = c_1 e^{-t} + c_2 e^{5t}$$

At $y(-1) = 3$ and $y'(-1) = 9$,

$$3 = ec_1 + e^{-5} c_2$$

$$9 = -ec_1 + 5e^{-5} c_2$$

Therefore $c_1 = e^{-1}$ and $c_2 = 2e^5$, so the final solution is

$$y(t) = e^{-t-1} + 2e^{5t+5}$$

Problem 18 The roots of the characteristic polynomial are

$$r^2 - 2r - 2 = 0 \implies r = \frac{2 \pm \sqrt{4+8}}{2} \implies r = 1 \pm 2\sqrt{3}$$

The general solution is in the form

$$z(t) = c_1 e^{t+2\sqrt{3}t} + c_2 e^{t-2\sqrt{3}t}$$

At $z(0) = 0$ and $z'(0) = 3$,

$$0 = c_1 + c_2$$

$$3 = (1+2\sqrt{3})c_1 + (1-2\sqrt{3})c_2$$

Therefore $c_1 = \frac{\sqrt{3}}{4}$ and $c_2 = -\frac{\sqrt{3}}{4}$, so the final solution is

$$z(t) = \frac{\sqrt{3}}{4} e^{t+2\sqrt{3}t} - \frac{\sqrt{3}}{4} e^{t-2\sqrt{3}t}$$

Problem 19 The root of the characteristic polynomial is $r = -1$

$$r^2 + 2r + 1 = (r+1)^2$$

The general solution is in the form

$$y(t) = c_1 e^{-t} + c_2 t e^{-t}$$

At $y(0) = 1$ and $y'(0) = 3$,

$$1 = c_1$$

$$-3 = -c_1 + c_2$$

Therefore, $c_1 = 1$ and $c_2 = -2$, so the final solution is

$$y(t) = e^{-t} - 2e^3 te^{-t}$$

Problem 27 The functions are dependent

$$\cos t \sin t = c \sin(2t) \implies \cos t \sin t = 2c \cos t \sin t \implies c = \frac{1}{2}$$

Problem 28 The functions are independent.

$$e^{3t} = ce^{-4t} \implies c = e^{7t}$$

Problem 29 The functions are independent since

$$te^{2t} = ce^{2t} \implies c = t$$

Problem 30 The functions are independent since

$$t^2 \cos(\ln t) = ct^2 \sin(\ln t) \implies c = \cot(\ln t)$$

Problem 31 The functions are dependent since

$$\tan^2 t - \sec^2 t = 3c \implies -1 = 3c \implies c = -\frac{1}{3}$$

Problem 32 The functions are dependent since

$$1 \cdot 0 + 0 \cdot e^t = 0$$