Math 31AH: Fall 2021
Practice Final Exam
Friday, 12/10/2021

Instructions: This is a 3 hour closed notes, closed books exam. Consultation with other humans is prohibited, including humans acting via websites such as Chegg. You need to clearly prove your claims; unsupported claims will get little credit. Please upload your exam to Gradescope after you are finished. You will have 10 additional minutes to upload your solutions to Gradescope.

Problem 1: Find a vector space V and a linear transformation $T : V \to V$ such that $V \supset T(V) \supset T^2(V)$ (strict containments) but $T^k(V) = T^2(V)$ for all $k > 2$.

Problem 2: Let $\mathbb{F}_5$ be the field with 5 elements. Prove that the number of 3×3 invertible matrices with entries in $\mathbb{F}_5$ is $(5^3 - 1) \cdot (5^3 - 5) \cdot (5^3 - 5^2)$.

Problem 3: Is the following real matrix A invertible?

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Problem 4: Let V be an $\mathbb{F}$-vector space. Give an explicit linear injection $\varphi : V \to V^{**}$.

Problem 5: Let V_2 be the vector space of polynomials in t with real coefficients of degree ≤ 2. Endow V_2 with the inner product

$$\langle f(t), g(t) \rangle := \int_0^1 f(t)g(t)dt$$

Find an orthonormal basis of V_2 containing $f_1(t) := \sqrt{3}t$.

Problem 6: Give an example of a square complex matrix which is diagonalizable but not orthogonally diagonalizable.

Problem 7: Let A be a diagonalizable $n \times n$ matrix which characteristic polynomial

$$\chi_A(t) = \det(A - tI) = (-1)^n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$$

Prove that $a_0 = \lambda_1 \cdot \lambda_2 \cdots \lambda_n$ where $\lambda_1, \lambda, \ldots, \lambda_n$ are the eigenvalues of A.

Problem 8: True or false: For any $\mathbb{F}$-vector spaces V and W, we have a well-defined linear map

$$\varphi : V^* \otimes V \otimes W \longrightarrow W$$

characterized by $\varphi(\lambda \otimes \mathbf{v} \otimes \mathbf{w}) := \lambda(\mathbf{v}) \cdot \mathbf{w}$.

Problem 9: If $T : V \to V$ has eigenvalues $\lambda_1, \ldots, \lambda_n$ and $U : W \to W$ has eigenvalues $\mu_1, \ldots, \mu_m$, what are the eigenvalues of $T \otimes U : V \otimes W \longrightarrow V \otimes W$? You may assume that T and U are diagonalizable.

Problem 10: Let A be an $n \times m$ complex matrix and let B be an $r \times n$ complex matrix. Suppose that

$$AA^* - B^*B = I_n$$

Prove that the linear map $\mathbb{C}^m \to \mathbb{C}^n$ given by $\mathbf{v} \mapsto A\mathbf{v}$ is surjective.