Math 100A: Homework 9

Merrick Qiu

Problem 1

We can show that the both representations have the same trace to show that they are isomorphic. Each element $g \in D_{2n}$ can be written in the form $g = x^a y^b$ for $0 \leq a < n$ and $0 \leq y < 2$.

If $b = 0$ then

$$\begin{aligned}
\chi_{\rho_1}(x^a) &= \operatorname{tr}\left(\begin{bmatrix} \cos(\frac{2\pi}{n}) & -\sin(\frac{2\pi}{n}) \\ \sin(\frac{2\pi}{n}) & \cos(\frac{2\pi}{n}) \end{bmatrix}\right) \\
&= 2\cos\left(\frac{2\pi}{n}\right)
\end{aligned}$$

$$\begin{aligned}
\chi_{\rho_1}(x^a) &= \operatorname{tr}\left(\begin{bmatrix} e^{2\pi i/n} & 0 \\ 0 & e^{-2\pi i/n} \end{bmatrix}\right) \\
&= \left(\cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)\right) + \left(\cos\left(\frac{2\pi}{n}\right) - i\sin\left(\frac{2\pi}{n}\right)\right) \\
&= 2\cos\left(\frac{2\pi}{n}\right)
\end{aligned}$$

If $b = 1$ then

$$\begin{aligned}
\chi_{\rho_1}(x^a y) &= \operatorname{tr}\left(\begin{bmatrix} \cos(\frac{2\pi}{n}) & \sin(\frac{2\pi}{n}) \\ \sin(\frac{2\pi}{n}) & -\cos(\frac{2\pi}{n}) \end{bmatrix}\right) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\chi_{\rho_1}(x^a) &= \operatorname{tr}\left(\begin{bmatrix} 0 & e^{2\pi i/n} \\ e^{-2\pi i/n} & 0 \end{bmatrix}\right) \\
&= 0
\end{aligned}$$

Since the traces of both representations are equal, the representations are isomorphic.

Problem 2

If $a \in \mathbb{Z}/n\mathbb{Z}$ then the kth representation is

$$\rho_k(a) = e^{ak(2\pi i/n)}$$

This is a homomorphism for all k since

$$\rho_k(a+b) = e^{(a+b)k(2\pi i/n)} = e^{ak(2\pi i/n)} \cdot e^{bk(2\pi i/n)} = \rho_k(a)\rho_k(b)$$

Problem 3

Let $C_n \subseteq D_{2n}$ be the cyclic normal subgroup of size n. Restricting ρ to C_n, we can write V as a direct sum of irreducible representations of C_n by Maschke's theorem. In particular, we have a nonzero eigenvector $v \in V$ such that $gv = \lambda(g)v$ for every $g \in C_n$ by Schur's lemma. Let $W = \text{Span}(v, y \cdot v)$. We will now prove that W is a D_{2n} invariant subspace of V.

Let $w \in W$ with $w = av + by \cdot v$ for $a, b \in \mathbb{C}$. Let $x^a \in D_{2n}$.

$$\begin{aligned}
\rho(x^a)(av + b\rho(y)v) &= a\rho(x^a)v + b\rho(x^a y)v \\
&= a\rho(x^a)v + b\rho(y)\rho(x^{-a})v \\
&= a\lambda(x^a)v + b\lambda(x^{-a})\rho(y)v
\end{aligned}$$

Similarly for $x^a y \in D_{2n}$,

$$\begin{aligned}
\rho(x^a y)(av + b\rho(y)v) &= a\rho(x^a y)v + b\rho(x^a)v \\
&= a\rho(y)\rho(x^{-a})v + b\rho(x^a)v \\
&= a\lambda(x^{-a})\rho(y)v + b\lambda(x^a)v
\end{aligned}$$

Therefore W is a D_{2n} invariant subspace of V. Since ρ is an irreducible representation, we have that $W = V$ and that $\dim(V) \leq 2$ since $\dim(W) = 2$.