Math 170A, Fall 2023 ***HOMEWORK #3*** ***due Friday, Oct. 27***

Homework problems that will be graded (Q1 - Q5, 30pts in total):

Q1. Let $A = \alpha I$ be a multiple of the $n \times n$ identity matrix, with $\alpha \in \mathbb{R}$, $\alpha \neq 0$, and consider $||\cdot||$ to be an induced matrix norm. Calculate $||A||$, $||A^{-1}||$, $\det(A)$, and $\kappa_{||\cdot||}(A)$.

Q2. Let A be a non-singular $n \times n$ matrix.

a) Show that, in any norm, $\kappa(A) = \kappa(A^{-1})$.

b) By rewriting $Ax = b$ as $A^{-1}b = x$, use the proof template we did in class to show the so-called **companion inequality**:

$$\frac{||\delta b||}{||b||} \leq \kappa(A)\frac{||\delta x||}{||x||} \ .$$

Q3. Let

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix},$$

and let $b = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

a) Calculate the norms $\|A\|_1$, $\|A\|_\infty$, $\|A\|_F$.

b) Compute the condition numbers $\kappa_1(A)$, $\kappa_\infty(A)$, $\kappa_F(A)$ with respect to the 1-norm, ∞-norm, Frobenius-norm respectively.

c) Let $\delta b = \begin{bmatrix} \epsilon \\ 0 \\ 0 \end{bmatrix}$. Consider the linear systems $Ax = b$, $A\hat{x} = b + \delta b$ and let $\delta x = \hat{x} - x$. Estimate the relative error from above and below through condition numbers, for norms $1, \infty$, and Frobenius. (see Q2 for the lower bound).

Q4. (Exercise 3, Page 94 in the textbook) What is wrong with the following reasoning?

Changing b by 1% means multiplying it by 1.01 . If $Ax = b$, then $A(1.01x) = 1.01b$, by linearity. So a 1% change of b, from b to $1.01b$, causes a 1% change in x, from x to $1.01x$. So forget the whole story about condition numbers—-a 1% change in b always causes a 1% change in x.

<u>Q5.</u> (similar to Exercise 4, Page 94 in the textbook) This is a MATLAB exercise.

Write a code that does the following.

- Pick a random 500 by 500 matrix A with the command $A = \text{randn}(500)$.

  Note the use of `randn`, not `rand`; this produces normally distributed, rather than uniformly distributed, variables. (If you don't know what that means, it won't matter for this problem.)
- Pick a random right-hand side $\text{b} = \text{randn}(500, 1)$.
- Solve $Ax = b$ using the backslash command $x = A\backslash b$.
- Then create a small perturbation, δb, of b using this command:

$$\text{delta\_b} = 10\hat{\ }(-3) * \text{randn}(500, 1)$$

- Solve $A\tilde{x} = b + delta\_b$ using the backslash. You can name your new variable $\tilde{x}$, x_tilde.
- Compute $\text{rel\_error\_sol} = ||x\_tilde - x||_1/||x||_1$; you can compute the 1 norm of a vector z by using the MATLAB command $norm(z, 1)$.
- Compute $\text{rel\_error\_b} = ||\text{delta\_b}||_1/||b||_1$.
- Finally, let $q = \frac{\text{rel\_error\_sol}}{cond(A,1)*\text{rel\_error\_b}}$; here $cond(A, 1)$ is using the command to compute the condition number of A in norm 1, in MATLAB.
- Now, repeat this 50 times, so you get $q(1), q(2), \ldots, q(50)$. Plot the vector $y = \log_{10}(q)$ by using the commands $y = log10(q)$ and plot(y).
- Explain how the picture agrees with the theory we learned in class (which predicts $q(i)$ will be less than 1.)

Please turn in a screenshot of the code, the picture, and your explanation.