Math 31AH: Spring 2021
Homework 2
Due 5:00pm on Friday 10/8/2021

Problem 1: A vector space? Let $\{a, b\}$ be a two-element set and let $V = \{a, b\} \times \mathbb{R}$. Define addition on V by

$$(a, x)+(a, y) := (a, x+y) \quad (a, x)+(b, y) := (a, x+y) \quad (b, x)+(b, y) := (b, x+y)$$

(so that a 'takes precedence over' b). Define scalar multiplication by

$$\lambda \cdot (a, x) := (a, \lambda x) \quad d\lambda \cdot (b, x) := (b, \lambda x)$$

for $\lambda \in \mathbb{R}$. Do these operations turn V into a real vector space? Prove your claim.

Problem 2: Working with vector space axioms. Let $\mathbb{F}$ be a field and let V be an $\mathbb{F}$-vector space. Suppose $a \in \mathbb{F}$ and $\mathbf{v} \in V$. If $a\mathbf{v} = \mathbf{0}$, prove that $a = 0$ or $\mathbf{v} = \mathbf{0}$.

Problem 3: Differentiable functions. Let V be the $\mathbb{R}$-vector space of all differentiable functions $f : \mathbb{R} \to \mathbb{R}$. Define two subsets $U, W \subseteq V$ as follows:

$$U := \{f \in V \,:\, f(3) = 0\} \qquad W := \{f \in V \,:\, f(3) = 7\}$$

Which (if either) of U or W are subspaces of V? Prove your claim.

Problem 4: Lines in the complex plane. For any real number c, define a subset $W_c \subseteq \mathbb{C}$ by

$$W_c := \{x + ic \,:\, x \in \mathbb{R}\}$$

That is, W_c is the set of complex numbers with imaginary part equal to c. For which values of $c \in \mathbb{R}$ is W_c a **real** vector space (under multiplication by real scalars and ordinary addition)? Prove your claim.

Problem 6: Eventually zero sequences. An infinite sequence $(a_1, a_2, \dots)$ of real numbers is *eventually zero* if there exists $N \in \mathbb{Z}_{\geq 0}$ such that $a_n = 0$ for all $n > N$.

It can be shown (and you do not have to prove) that the set V of all real sequences $(a_1, a_2, \dots)$ is an $\mathbb{R}$-vector space with addition

$$(a_1, a_2, \dots) + (b_1, b_2, \dots) := (a_1 + b_1, a_2 + b_2, \dots)$$

and scalar multiplication

$$\lambda \cdot (a_1, a_2, \dots) := (\lambda a_1, \lambda a_2, \dots)$$

If $W \subseteq V$ is the subset of eventually zero sequences, is W a subspace of V? Prove your claim.

Problem 6: A linear system over $\mathbb{R}$. Solve the following system of linear equations over the real numbers.

$$\begin{cases} 1\cdot x + 2\cdot y + 3\cdot z & = 1 \\ 4\cdot x + 5\cdot y + 6\cdot z & = 1 \\ 7\cdot x + 8\cdot y + 9\cdot z & = 1 \end{cases}$$

Problem 7: A linear system over $\mathbb{C}$. Solve the following system of linear equations over the complex numbers.

$$\begin{cases} x + iy & = 1 \\ x \qquad + z & = 1 \\ \qquad y - iz & = 2 \end{cases}$$

Problem 8: A linear system over $\mathbb{F}_2$. Solve the following system of linear equations over the field $\mathbb{F}_2$ with two elements.

$$\begin{cases} x + y & = 1 \\ x \qquad + z & = 1 \\ \qquad y + z & = 1 \end{cases}$$

(Here the 1's on the right-hand sides are regarded as $1 \in \mathbb{F}_2 = \{0, 1\}$.)

Problem 9: (Optional; not to be handed in.) Prove that the number of solutions to **any** finite system of linear equations over $\mathbb{F}_2$ is either zero, or else a power 2^a of 2.