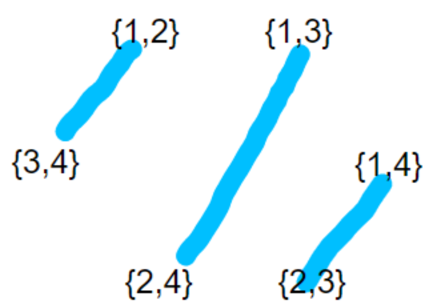
Math 158 - Homework 1

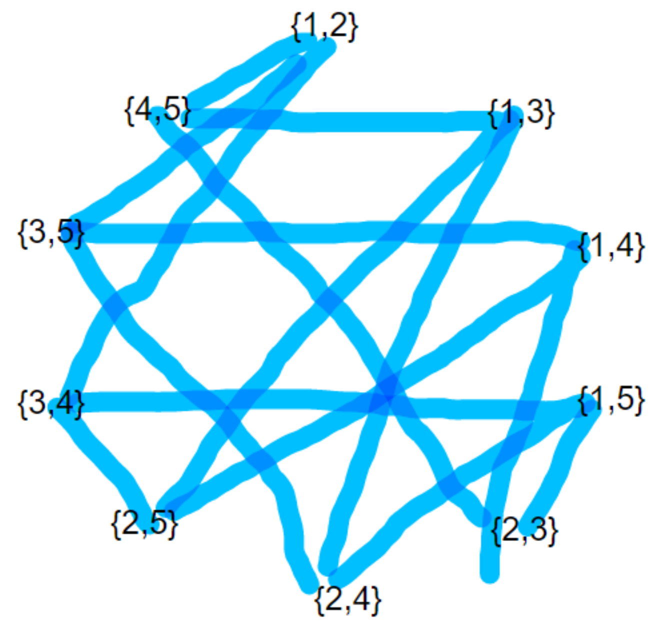
Question 1. Let $K_{n:r}$ denote the *Kneser graph*, whose vertex set is the set of r-element subsets of an n-element sets, and where two vertices form an edge if the corresponding sets are disjoint.

(a) Describe $K_{n:1}$ for $n \geq 1$.
(b) Draw $K_{4:2}$ and $K_{5:2}$.
(c) Determine $|E(K_{n:r})|$ for $n \geq 2r \geq 1$.

1. $V(K_{n:1})$ consists of all subsets of size 1 of [n], and all of these subsets are disjoint, so $K_{n:1}$ is isomorphic to the complete graph, K_n.

2. To the left is $K_{4:2}$ and to the right is $K_{5:2}$.





3. There are $\binom{n}{r}$ subsets and $\binom{n-r}{r}$ subsets that are disjoint with any given subset. Thus the sum of the degrees of $K_{n:r}$ is $\binom{n}{r}\binom{n-r}{r}$, and so there are $\frac{1}{2}\binom{n}{r}\binom{n-r}{r}$ edges.

Question 2. Let G be a digraph such that every vertex has positive in-degree. Prove that G contains a directed cycle.

Let P be a directed path from v_1 to v_r. Since every vertex has positive in-degree, there must always be a vertex, v_i, that connects to v_1. If v_i is on the path, then G contains the cycle, $v_1 v_2 \ldots v_i v_1$. If v_i is not on the path, then adding v_i to the path results in a longer path. Since G contains a finite number of vertices, repeatedly adding vertices to the path will eventually result in a cycle.

Question 3. Let G be an n-vertex graph with $n \geq 2$ and $\delta(G) \geq (n-1)/2$. Prove that G is connected and that the diameter of G is at most two.

Picking two arbitrary vertices, v, w, as the ends of a path. The neighborhoods of both vertices must overlap by the pigeonhole principle. There are $n-2$ verticies that are not v or w, yet each end of the path must be adjacent to at least $\frac{n-1}{2}$ other vertices. Thus v and w must either be directly adjacent to each other, or they must share a neighbor with which they are both adjacent to. Thus each vertex is connected by a path of at most size two, and the diameter of G is at most two. Since the diameter is finite, G must be connected.

Question 4.

(a) Let P and Q be longest paths in a connected graph G. Prove that

$$V(P) \cap V(Q) \neq \emptyset.$$

Assume that $V(P)$ and $V(Q)$ are disjoint. Since G is a connected graph, it must be possible to traverse from any vertex to any other vertex. Thus there must exist a path, R that connects some v_{pi} to some v_{qj} whose edge set is disjoint with the edge set of P and Q. Let P_{max} be the path from v_{p1} to v_{pi} if $i > k/2$ and v_{pk} to v_{pi} otherwise. Let Q_{max} be the path from v_{qj} to v_{q1} if $j > k/2$ and v_{qj} to v_{qk} otherwise.
The concatenation of P_{max}, R, and Q_{max} yields a path that is longer than P and Q. Thus $V(P)$ and $V(Q)$ cannot be disjoint.

Question 5. Prove that a graph of minimum degree at least $k \geq 2$ containing no triangles contains a cycle of length at least $2k$.

Let P be a longest path from v_1 to v_r for some graph G with minimum degree at least $k \geq 2$. Since this is already a longest path, the neighborhood of v_r must contain vertices on P.
The closest neighbor of v_r along the path is v_{r-1}. The second closest neighbor of v_r along the path must have index $i_2 \leq r-3$. to avoid a triangle. The third closest neighbor must have index $i_3 \leq r-5$ to avoid a triangle as well. Repeating this process, we see that the kth furthest neighbor along the path must have index $i_k \leq r-2k+1$. If v_{i_k} is the kth furthest neighbor, then $v_{i_k} v_{i_k+1} \ldots v_r v_{i_k}$ is a cycle of length at least $2k$.