Math 31BH: Assignment 3

Due 01/23 at 23:59

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1. Prove that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(t) = |t|$ is continuous but not differentiable at $t = 0$.

   Solution: Since
   $$\lim_{t\to 0^-} \|t\| = \lim_{t\to 0^+} \|t\| = \|0\| = 0$$
   $f(t)$ is continuous at $t = 0$.

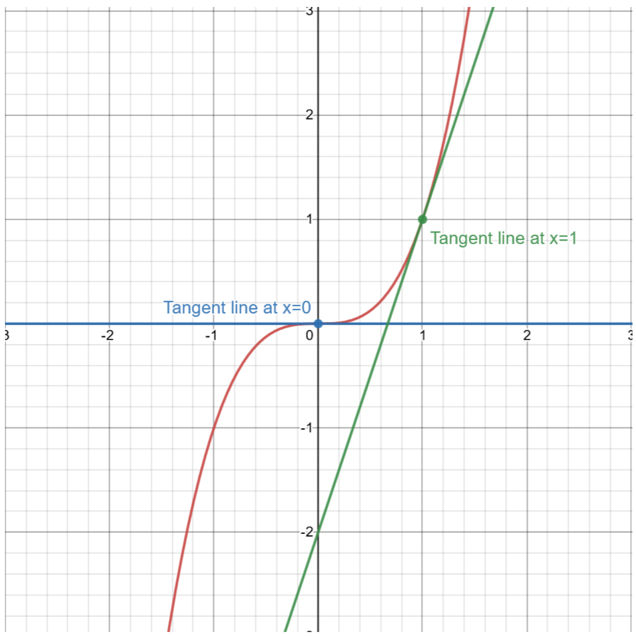
   The left hand limit for the derivative is
   $$\lim_{h\to 0^-} \frac{f(0+h)+f(h)}{h} = \frac{2|h|}{h} = -1$$
   and the right hand limit for the derivative is
   $$\lim_{h\to 0^+} \frac{f(0+h)+f(h)}{h} = \frac{2|h|}{h} = 1$$
   Since the left and right hand limits do not agree, the function is not differentiable at $t = 0$.

2. Consider the differentiable function $g\colon \mathbb{R} \to \mathbb{R}^2$ given by $g(t) = (t, t^3)$.

   (a) Sketch the tangent vector and the tangent line at $t = 0$ and $t = 1$.

   (b) Construct a function $h\colon \mathbb{R} \to \mathbb{R}$ with the same image as g such that $g(0) = h(0)$ but h is not differentiable at $t = 0$.

   Solution:



The function $h(t) = (t^{\frac{1}{3}}, t)$ has the same image as g but it is not differentiable at $t = 0$ because the derivative of $t^{\frac{1}{3}}$, which is $\frac{1}{3}t^{-\frac{2}{3}}$, is undefined at $t = 0$.

3. Consider the differentiable function $f\colon \mathbb{R} \to \mathbb{R}^2$ defined by $f(t) = (e^{kt}\cos t, e^{kt}\sin t)$ where k is a constant.

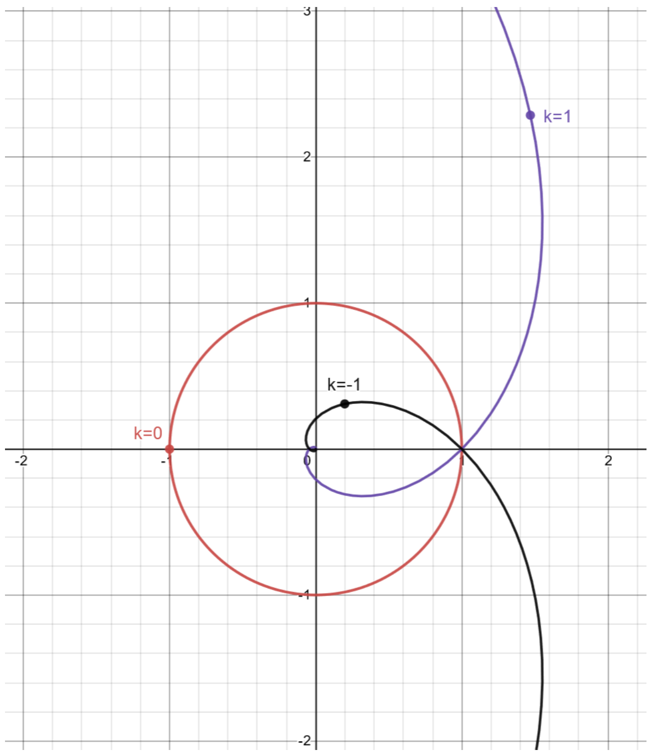
   (a) Sketch the image of f.

   (b) Prove that

   $$\frac{f'(t) \cdot f(t)}{\|f'(t)\|\|f(t)\|} = \frac{k}{\sqrt{1+k^2}}.$$

   (c) Prove that the angle between the tangent vector $f'(t)$ and the line joining $f(t)$ to $(0,0)$ is the same for all $t \in \mathbb{R}$.

Solution:



The derivative of f is $f'(t) = (ke^{kt}\cos(t) - e^{kt}\sin(t), ke^{kt}\sin(t) + e^{kt}\cos(t))$ so

$$\begin{aligned}
f'(t)\cdot f(t) &= e^{kt}\cos(t)(ke^{kt}\cos(t) - e^{kt}\sin(t)) + e^{kt}\sin(t)(ke^{kt}\sin(t) + e^{kt}\cos(t)) \\
&= ke^{2kt}\cos^2(t) - e^{2kt}\sin(t)\cos(t) + ke^{2kt}\sin^2(t) + e^{2kt}\sin(t)\cos(t) \\
&= ke^{2kt}(\cos^2(t) + \sin^2(t)) + (e^{2kt}\sin(t)\cos(t) - e^{2kt}\sin(t)\cos(t)) \\
&= ke^{2kt}
\end{aligned}$$

The norm of $f(t)$ is

$$\|f(t)\| = \sqrt{e^{2kt}\cos^2(t) + e^{2kt}\sin^2(t)} = \sqrt{e^{2kt}(\cos^2(t) + \sin^2(t))} = e^{kt}$$

and the norm of $f'(t)$ is

$$\|f'(t)\| = \sqrt{(ke^{kt}\cos(t) - e^{kt}\sin(t))^2 + (ke^{kt}\sin(t) + e^{kt}\cos(t))^2}$$

$$= \sqrt{k^2e^{2kt}\cos^2(t) - 2ke^{2kt}\sin(t)\cos(t) + e^{2kt}\sin^2(t) + k^2e^{2kt}\sin^2(t) + 2ke^{2kt}\sin(t)\cos(t) + e^{2kt}\cos^2(t)}$$

$$= \sqrt{k^2e^{2kt} + e^{2kt}}$$

$$= e^{kt}\sqrt{k+1}$$

Therefore,

$$\frac{f'(t) \cdot f(t)}{\|f'(t)\|\|f(t)\|} = \frac{ke^{2kt}}{e^{2kt}\sqrt{k+1}} = \frac{k}{\sqrt{1+k}}$$

Since $\cos(\theta) = \frac{v \cdot w}{\|v\|\|w\|}$ for any two vectors v, w, the equation from part b states that the cosine between $f(t)$ and $f'(t)$ is constant for all values of t. Therefore, the angle between the tangent vector and the line joining $f(t)$ to the origin is the same for all $t \in \mathbb{R}$.