MATH 31AH - Homework 6

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1 Eigenvalues and Eigenvectors

A has eigenvalues when $\det(A - \lambda I) = 0$.

$$A - \lambda I = \begin{bmatrix} 7-\lambda & -8 & 6 \\ 8 & -9-\lambda & 6 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

The characteristic polynomial of A is thus

$$\begin{aligned} (7-\lambda)(-9-\lambda)(1-\lambda) - (-8)(8)(1-\lambda) &= (-63 + 2\lambda + \lambda^2)(1-\lambda) + 64(1-\lambda) && (1) \\ &= (1 + 2\lambda + \lambda^2)(1-\lambda) && (2) \\ &= (1+\lambda)^2(1-\lambda) && (3) \end{aligned}$$

$(1+\lambda)^2(1-\lambda) = 0$ when $\lambda = -1, 1$, so those are our eigenvalues. Using these eigenvalues, we can calculate the matrices for the eigenspaces.

$$E_{-1} = \begin{bmatrix} 8 & -8 & 6 \\ 8 & -8 & 6 \\ 0 & 0 & 2 \end{bmatrix} \text{ and } E_1 = \begin{bmatrix} 6 & -8 & 6 \\ 8 & -10 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

Using RREF, we have that

$$\operatorname{rref}(E_{-1}) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \operatorname{rref}(E_1) = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore a basis for E_{-1} is $\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\}$ and a basis for E_1 is $\{\begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}\}$. Since there are not three eigenvalues, A is not diagonalizable.

2 More Eigenvalues and Eigenvectors

B has eigenvalues when $\det(B - \lambda I) = 0$.

$$B - \lambda I = \begin{bmatrix} 1-\lambda & \frac{1}{4} & 0 \\ 0 & \frac{1}{2}-\lambda & 0 \\ 0 & \frac{1}{4} & -1-\lambda \end{bmatrix}$$

The characteristic polynomial of B is thus $(1-\lambda)(\frac{1}{2}-\lambda)(-1-\lambda)$. The eigenvalues are thus $\lambda = -1, \frac{1}{2}, 1$ Using these eigenvalues, we can calculate the matrices for the eigenspaces.

$$E_{-1} = \begin{bmatrix} 2 & \frac{1}{4} & 0 \\ 0 & \frac{3}{2} & 0 \\ 0 & \frac{1}{4} & 0 \end{bmatrix} \text{ and } E_{\frac{1}{2}} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{4} & -\frac{3}{2} \end{bmatrix} \text{ and } E_1 = \begin{bmatrix} 0 & \frac{1}{4} & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{4} & -2 \end{bmatrix}$$

Using RREF, we have that

$$\text{rref}(E_{-1}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \text{rref}(E_{\frac{1}{2}}) = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -6 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \text{rref}(E_1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Therefore a basis for E_{-1} is $\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$, a basis for $E_{\frac{1}{2}}$ is $\{ \begin{bmatrix} -3 \\ 6 \\ 1 \end{bmatrix} \}$, and a basis for E_1 is $\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \}$ Since there are three eigenvalues, B is diagonalizable.

3 $\mathbb{R}$ and $\mathbb{C}$ and diagonalizability

The matrix has eigenvalues when $\det(R_\theta - \lambda I) = 0$. We have that the characteristic polynomial is

$$\begin{aligned} (\cos(\theta)-\lambda)^2 + \sin^2(\theta) &= \cos^2(\theta) - 2\lambda + \lambda^2 + \sin^2(\theta) && (4) \\ &= 1 - 2\lambda\cos(\theta) + \lambda^2 && (5) \end{aligned}$$

Using the quadratic formula, we get that our eigenvalues are

$$\begin{aligned} \lambda &= \frac{2\cos(\theta) \pm \sqrt{4\cos^2(\theta) - 4}}{2} && (6) \\ &= \cos(\theta) \pm \sqrt{\cos^2(\theta) - 1} && (7) \\ &= \cos(\theta) \pm i\sin(\theta) && (8) \\ &= e^{\pm i\theta} && (9) \end{aligned}$$

Our eigenspace matrices are

$$E_{e^{-i\theta}} = \begin{bmatrix} i\sin(\theta) & -\sin(\theta) \\ \sin(\theta) & i\sin(\theta) \end{bmatrix}$$

$$E_{e^{i\theta}} = \begin{bmatrix} -i\sin(\theta) & -\sin(\theta) \\ \sin(\theta) & -i\sin(\theta) \end{bmatrix}$$

So the corresponding eigenvectors are $v_{e^{-i\theta}} = (-i, 1)$ and $v_{e^{i\theta}} = (i, 1)$ Over $\mathbb{R}$, R_θ is only diagonalizable when $\theta = 0, \pi$ since these are the only two angles that result in real eigenvalues. Over $\mathbb{C}$, R_θ is diagonalizable for all θ because there are always two eigenvalues.

4 Polynomials and diagonalizability

Proof. $f(A)$ is diagonalizable when A is diagonalizable.

Base Case: Since A is diagonalizable, let P be a matrix such that $A = PDP^{-1}$ for some D. A^n can thus be written as $A = PD^nP^{-1}$.[1] Therefore $c_nA^n = P(c_nD^n)P^{-1}$, so each of the terms in $f(A)$ is diagonalizable, including when $n = 0$.

Inductive Step: For $n > 0$, assume that the sum of the first $n-1$ terms is diagonalizable with $f_{n-1}(A) = PBP^{-1}$ for some matrix B. Since the nth term can be written as $c_nA^n = P(c_nD^n)P^{-1}$, the sum of the first n terms is

$$\begin{aligned} f_{n-1}(A) + c_nA^n &= PBP^{-1} + P(c_nD^n)P^{-1} && (10)\\ &= P(BP^{-1} + c_nD^nP^{-1}) && (11)\\ &= P(B + c_nD^n)P^{-1} && (12) \end{aligned}$$

This completes the inductive step, and so $f(A)$ is diagonalizable when A is diagonalizable. ☐

5 Direct sums and diagonalizability

Proof. If λ is an eigenvalue of T or U, then λ is an eigenvalue of $T \oplus U$

If λ is an eigenvalue of T and $v \in T$ is the corresponding eigenvector, then the corresponding eigenvector in $T \oplus U$ with eigenvalue λ would be $(v, 0)$ since

$$\begin{aligned} (T \oplus U)(v, 0) &= (T(v), U(0)) && (13)\\ &= (\lambda v, 0) && (14)\\ &= \lambda(v, 0) && (15) \end{aligned}$$

Similarly, if λ is an eigenvalue of U and $w \in U$ is the corresponding eigenvector, then the corresponding eigenvector in $T \oplus U$ with eigenvalue λ would be $(0, w)$ since

$$\begin{aligned} (T \oplus U)(0, w) &= (T(0), U(w)) && (16)\\ &= (0, \lambda w) && (17)\\ &= \lambda(0, w) && (18) \end{aligned}$$

Therefore, if λ is an eigenvalue of T or U, then λ is an eigenvalue of $T \oplus U$ ☐

Proof. If T and U are diagonalizable, then $T \oplus U$ is diagonalizable.

Let V be m dimensional and let W be n dimensional. Since T and U are both diagonalizable, they have m and n eigenvalues respectivly.

Since every eigenvalue of T or U is an eigenvalue of $T \oplus U$, $T \oplus U$ has $m + n$ eigenvalues. Since $T \oplus U$ has $m + n$ dimensions and $m + n$ eigenvalues, $T \oplus U$ is diagonalizable. ☐

[1] See problem 7 for proof

6 A 6x6 example

We have that

$$A - \lambda I = \begin{bmatrix} -\lambda & 0 & 0 & 0 & 0 & 1 \\ 1 & -\lambda & 0 & 0 & 0 & 0 \\ 0 & 1 & -\lambda & 0 & 0 & 0 \\ 0 & 0 & 1 & -\lambda & 0 & 0 \\ 0 & 0 & 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 0 & 1 & -\lambda \end{bmatrix}$$

There are only two possible permutations, or "rook placements" that result in non-negative terms, so

$$\det(A - \lambda I) = (-\lambda)^6 - 1^6 \tag{19}$$
$$= \lambda^6 - 1 \tag{20}$$

$\lambda^6 - 1 = 0$ when $\lambda = 1, e^{i\frac{\pi}{3}}, e^{i\frac{2\pi}{3}}, -1, e^{i\frac{4\pi}{3}}, e^{i\frac{5\pi}{3}}$, the 6 roots of unity. The row reduced matrix of $A - \lambda I$ is

$$\begin{bmatrix} -\lambda & 0 & 0 & 0 & 0 & 1 \\ 0 & -\lambda & 0 & 0 & 0 & \frac{1}{\lambda} \\ 0 & 0 & -\lambda & 0 & 0 & \frac{1}{\lambda^2} \\ 0 & 0 & 0 & -\lambda & 0 & \frac{1}{\lambda^3} \\ 0 & 0 & 0 & 0 & -\lambda & \frac{1}{\lambda^4} \\ 0 & 0 & 0 & 0 & 0 & \frac{-\lambda^6+1}{\lambda^5} \end{bmatrix}$$

Since $\lambda^6 - 1 = 0$, we have that $\frac{-\lambda^6+1}{\lambda^5} = 0$. The rref of $A - \lambda I$ is thus

$$\text{rref}(A - \lambda I) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & -\frac{1}{\lambda} \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{\lambda^2} \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{\lambda^3} \\ 0 & 0 & 0 & 1 & 0 & -\frac{1}{\lambda^4} \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{\lambda^5} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Using the rref, we can determine the eigenvectors of the eigenspace.

$$v_1 = \left\{\left[\frac{1}{1}, \frac{1}{(1)^2}, \frac{1}{(1)^3}, \frac{1}{(1)^4}, \frac{1}{(1)^5}, 1,\right]\right\} \tag{21}$$
$$v_{e^{i\frac{\pi}{3}}} = \left\{\left[\frac{1}{e^{i\frac{\pi}{3}}}, \frac{1}{(e^{i\frac{\pi}{3}})^2}, \frac{1}{(e^{i\frac{\pi}{3}})^3}, \frac{1}{(e^{i\frac{\pi}{3}})^4}, \frac{1}{(e^{i\frac{\pi}{3}})^5}, 1,\right]\right\} \tag{22}$$
$$v_{e^{i\frac{2\pi}{3}}} = \left\{\left[\frac{1}{e^{i\frac{2\pi}{3}}}, \frac{1}{(e^{i\frac{2\pi}{3}})^2}, \frac{1}{(e^{i\frac{2\pi}{3}})^3}, \frac{1}{(e^{i\frac{2\pi}{3}})^4}, \frac{1}{(e^{i\frac{2\pi}{3}})^5}, 1,\right]\right\} \tag{23}$$
$$v_{-1} = \left\{\left[\frac{1}{-1}, \frac{1}{(-1)^2}, \frac{1}{(-1)^3}, \frac{1}{(-1)^4}, \frac{1}{(-1)^5}, 1,\right]\right\} \tag{24}$$
$$v_{e^{i\frac{4\pi}{3}}} = \left\{\left[\frac{1}{e^{i\frac{4\pi}{3}}}, \frac{1}{(e^{i\frac{4\pi}{3}})^2}, \frac{1}{(e^{i\frac{4\pi}{3}})^3}, \frac{1}{(e^{i\frac{4\pi}{3}})^4}, \frac{1}{(e^{i\frac{4\pi}{3}})^5}, 1,\right]\right\} \tag{25}$$
$$v_{e^{i\frac{5\pi}{3}}} = \left\{\left[\frac{1}{e^{i\frac{5\pi}{3}}}, \frac{1}{(e^{i\frac{5\pi}{3}})^2}, \frac{1}{(e^{i\frac{5\pi}{3}})^3}, \frac{1}{(e^{i\frac{5\pi}{3}})^4}, \frac{1}{(e^{i\frac{5\pi}{3}})^5}, 1,\right]\right\} \tag{26}$$

7 A real sequence

Proof. $A^n = PD^nP^{-1}$ if $A = PDP^{-1}$.

We will induct on n. For $n = 1$, we have that $A^1 = PD^1P^{-1}$. For $n > 1$, assume that $A^{n-1} = PD^{n-1}P^{-1}$. We have that

$$\begin{aligned} A^n &= A^{n-1}A && (27) \\ &= (PD^{n-1}P^{-1})(PDP^{-1}) && (28) \\ &= PD^nP^{-1} && (29) \end{aligned}$$

This completes the inductive step, so $A^n = PD^nP^{-1}$. $\square$

Proof. When $a_1 = a_2 = 1$ and $a_n = 4a_{n-1} - 2a_{n-2}$, $a_n = \frac{(2+\sqrt{2})^{n-2}+(2-\sqrt{2})^{n-2}}{2}$ for $n > 2$.

The nth number in the sequence can be found using the following equation:

$$\begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 1 & 0 \end{bmatrix}^{n-2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The matrix has characteristic polynomial of $(4-\lambda)(-\lambda)+2 = \lambda^2 - 4\lambda + 2$. Using the quadratic formula, we can find that the matrix has eigenvalues of $\lambda_1 = 2+\sqrt{2}$, $\lambda_2 = 2-\sqrt{2}$. The eigenvectors for these eigenvalues are $v_1 = (2+\sqrt{2}, 1)$ and $v_2 = (2-\sqrt{2}, 1)$ since they solve the corresponding homologous system of equations for the eigenspace.

Since there are two eigenvectors, we can diagonalize the matrix and simplify the expression to

$$\begin{aligned} \begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix} &= \begin{bmatrix} 2+\sqrt{2} & 2-\sqrt{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2+\sqrt{2} & 0 \\ 0 & 2-\sqrt{2} \end{bmatrix}^{n-2} \begin{bmatrix} \frac{1}{2\sqrt{2}} & \frac{\sqrt{2}-2}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} & \frac{\sqrt{2}+2}{2\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} && (30) \\ &= \frac{1}{2\sqrt{2}} \begin{bmatrix} 2+\sqrt{2} & 2-\sqrt{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (2+\sqrt{2})^{n-2} & 0 \\ 0 & (2-\sqrt{2})^{n-2} \end{bmatrix} \begin{bmatrix} \sqrt{2}-1 \\ \sqrt{2}+1 \end{bmatrix} && (31) \\ &= \frac{1}{2\sqrt{2}} \begin{bmatrix} 2+\sqrt{2} & 2-\sqrt{2} \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (\sqrt{2}-1)(2+\sqrt{2})^{n-2} \\ (\sqrt{2}+1)(2-\sqrt{2})^{n-2} \end{bmatrix} && (32) \\ &= \frac{1}{2\sqrt{2}} \begin{bmatrix} \sqrt{2}(2+\sqrt{2})^{n-2} + \sqrt{2}(2-\sqrt{2})^{n-2} \\ (\sqrt{2}-1)(2+\sqrt{2})^{n-2} + (\sqrt{2}+1)(2-\sqrt{2})^{n-2} \end{bmatrix} && (33) \\ & && (34) \end{aligned}$$

Therefore, $a_n = \frac{(2+\sqrt{2})^{n-2}+(2-\sqrt{2})^{n-2}}{2}$ for $n > 2$.

$\square$

8 Commuting operators and eigenspaces

Proof. E_λ is U-invariant when T and U are commutative.

Let $T, U : V \to V$ be two linear transformations that commute. Let $v \in E_\lambda$ be an eigenvector. Since it is a T-eigenvector, we have that $T(v) = \lambda(v)$. Since T and U commute, we have that

$$\begin{aligned} U(T(v)) &= U(\lambda v) && (35) \\ &= \lambda U(v) && (36) \\ &= T(U(v)) && (37) \end{aligned}$$

Since $T(U(v)) = \lambda U(v)$, we know that $U(v) \in E_\lambda$, and so E_λ is U-invariant. $\square$

Proof. E_λ is not necessarily U-invariant when T and U are not commutative.

Let $T, U : \mathbb{R}^2 \to \mathbb{R}^2$ such that T projects the vector onto the x-axis and U rotates the vector by $\pi/2$. E_λ would clearly not be U-invariant since the only eigenspace of T would be the x-axis, but U rotates vectors off of the x-axis. $\square$