Math 170A: Homework 4

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Question 1

$$q_1 = \frac{w_1}{2} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$\tilde{q}_2 = \begin{bmatrix} 3 \\ 3 \\ -1 \\ -1 \end{bmatrix} - 2 \cdot \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix}$$

$$q_2 = \frac{\tilde{q}_2}{4} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\tilde{q}_3 = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 0 \end{bmatrix} - 4 \cdot \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} - 2 \cdot \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 1 \\ -1 \end{bmatrix}$$

$$q_3 = \frac{\tilde{q}_3}{\sqrt{20}} = \begin{bmatrix} \frac{3}{\sqrt{20}} \\ -\frac{3}{20} \\ \frac{1}{20} \\ -\frac{1}{20} \end{bmatrix}$$

Question 2

One can take q_1 and q_2 and append two more vectors that are orthonormal to them to get a Full QR decomposition of A

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 & 2 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The minimizer is

$$x = R^{-1}Q^T b = \begin{bmatrix} \frac{1}{2} & -\frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{8} \\ \frac{1}{8} \end{bmatrix}$$

Question 3

1. Since for an orthogonal matrix, $||Qx||_2 = ||x||_2$ for all x, we have that $||Q||_2 = 1$, $||Q^{-1}||_2 = 1$, and $\kappa(Q) = 1$.

2. We can rearrange the equation as

$$Q\hat{x} = b - \delta Q\hat{x}$$

which is equvalent to perturbing b with $\delta b = -\delta Q\hat{x}$. Thus we can say that

$$\frac{||\delta x||_2}{||\hat{x}||_2} \leq \frac{||\delta Q\hat{x}||_2}{||b||_2}$$

Question 4

We can write

$$A = M^T M = (QR)^T QR = R^T (Q^T Q) R = R^T R$$

We see that M is a $n \times m$ full rank matrix and that R is a $m \times m$ upper triangular matrix that happens to be the Cholesky decomposition of A since R times its transpose equals to A, which is positive definite.

Question 5

We can see that the orthogonal matrix Q differs signficantly between the two methods but the matrix R is mostly the same between the two matricies.

orthoQ1 demonstrates that Q_1 from classical Gram-Schmidt is very far from being orthogonal, but orthoQ1 demonstrates that Q_2 from modified Gram-Schmidth is much closer to being orthogonal.

Command Window

New to MATLAB? See resources for Getting Started.

```
>> A = hilb(8);
>> [Q1,R1] = gs(A);
>> [Q2,R2] = mgs(A);
>> checkQ = norm(Q1-Q2)

checkQ =

    1.4138

>> checkR = norm(R1-R2)

checkR =

   2.0354e-06

>> orthoQ1 = norm(Q1'*Q1-eye(8))

orthoQ1 =

    1.0325

>> orthoQ2 = norm(Q2'*Q2-eye(8))

orthoQ2 =

   4.3754e-07

fx >>
```