Math 100b Winter 2025 Homework 5

Due 2/21/2025 at 5pm on Gradescope

Reading

All references will be to Artin Algebra, 2nd edition.

Reading: Sections 12.5, Section 12.3-12.4.

Assigned Problems

1. Consider $R = \mathbb{Z}[\sqrt{-2}]$. Show that R is a Euclidean domain with respect to the norm function $N(a + b\sqrt{-2}) = a^2 + 2b^2$, hence R is a PID and a UFD. (Hint: follow a similar method as the one used in class to show that $\mathbb{Z}[i]$ is a UFD.)

2. Let $R = \mathbb{Z}[\sqrt{d}]$ with $d \leq -3$.

(a). Show that 2 is an irreducible element of R.

(b). Show that 2 is not a prime element of R, so R is not a UFD.

3. Consider $R = \mathbb{Z}[\sqrt{-2}]$ again.

(a) Show that for every integer prime number p, exactly one of the following happens:

(i). There do not exist integers a, b such that $p = a^2 + 2b^2$, and p is irreducible in R; or

(ii). There do exist integers a, b such that $p = a^2 + 2b^2$, and p factors as a product of two irreducibles in R.

(b). Show that $p = 2$ falls into case (ii), while if $p \equiv 5 \mod 8$ or $p \equiv 7 \mod 8$, then p falls into case (i).

Remark: It is known that the other possibilities for p, where $p \equiv 1 \mod 8$ or $p \equiv 3 \mod 8$, fall into case (ii).

4. Find, with justification, all possible pairs of integers a, b such that $a^2 + 2b^2 = 1122$. (Note: $1122 = (2)(3)(11)(17)$).

5. In this problem we consider factor rings of $R = \mathbb{Z}[\sqrt{-2}]$ which are fields.

(a). Show that if p is a prime number as in case (i) of #3(a), then $\mathbb{Z}[\sqrt{-2}]/(p)$ is a field with p^2 elements.

(b). Show that if p is a prime number as in case (ii) of #3(a), then writing $p = a^2 + 2b^2$ for $a, b \in \mathbb{Z}$, we have $\mathbb{Z}[\sqrt{-2}]/(a + b\sqrt{-2}) \cong \mathbb{Z}/p\mathbb{Z}$.

(Hint: similarly as we did in class with $\mathbb{Z}[i]$, first show that $\mathbb{Z}[\sqrt{-2}]/(p) \cong \mathbb{Z}[x]/(x^2 + 2, p)$.)