Math 100b Winter 2025 Homework 4

Due 2/7/2025 at 5pm on Gradescope

Reading

All references will be to Artin Algebra, 2nd edition.

Reading: Sections 12.1-12.2.

Assigned Problems

Write up neat and complete solutions to these problems.

1. Let F be a field. Prove that $F[x]$ has infinitely many monic irreducible polynomials, and conclude that $F[x]$ has infinitely many distinct maximal ideals. (Hint: recall Euclid's proof that $\mathbb{Z}$ has infinitely many primes: If $p_1, \ldots, p_n$ are distinct prime numbers, then any prime factor of $(p_1 p_2 \ldots p_n) + 1$ is different from all of the p_i.)

2. Let p be a prime number, and let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ be the field with p elements. Consider the polynomial $f(x) = x^{p-1} - 1$ in the ring $\mathbb{F}_p[x]$.

(a) Show that every nonzero element of $\mathbb{F}_p$ is a root of $f(x)$.

(b) Write $f(x)$ as a product of irreducible polynomials in $\mathbb{F}_p[x]$ and justify your answer.

(c) By comparing the constant term of $f(x)$ with the constant term of its factorization into irreducibles in (b), prove Wilson's theorem, which states that when p is a prime number then $(p-1)! \equiv -1 \pmod{p}$.

3. Let F be a field and let $g(x) \in F[x]$ be a polynomial of degree $n \geq 1$. Consider the factor ring $R = F[x]/(g(x))$.

(a) Show that each element of R can be written as a coset $r(x) + (g(x))$ for a *unique* polynomial $r(x)$ such that $\deg r < \deg g$ or $r = 0$.

(b). Suppose that $F = \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is the field with p elements. Prove that the ring $\mathbb{F}_p[x]/(g(x))$ has exactly p^n elements.

4. Let $\mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z}$ be the field with 3 elements.

(a). Show that $E = \mathbb{F}_3[x]/(x^2+1)$ is a field with 9 elements.

(b). Consider the set of nonzero elements in the field E from (a), that is $E^\times = E - \{0\}$. $E^\times$ is a an abelian group under multiplication with 8 elements. Show that it is a cyclic group.

(c). Find a field with 27 elements.

5. Let F be a field. Let R be the following subring of $F[x]$:

$$R = \{f(x) = a_0 + a_2x^2 + a_3x^3 + \cdots + a_nx^n | a_i \in F\}.$$

In other words, R consists of all polynomials with no x term. Think about why R is a subring but you don't have to prove it. Note that R is an integral domain because it is a subring of $F[x]$.

(a). Show that x^2 and x^3 are irreducible elements of R.

(b). By considering the factorization of x^6 in R, show that R is not a unique factorization domain (UFD).

6. Let R be an integral domain. Suppose that S is a subset of R such that $0 \notin S$, $1 \in S$, and if $s, t \in S$ then $st \in S$. Such a subset is called a *multiplicative system*. Let F be the field of fractions of R.

(a). Let $RS^{-1} = \left\{\frac{r}{s} \,|\, r \in R, s \in S\right\}$, considered as a subset of F. Prove this is a subring of F.

(b). Show that RS^{-1} has the following universal property: If T is another ring and there is a homomorphism $\phi : R \to T$ such that $\phi(s)$ is a unit in T for all $s \in S$, then there is a unique homomorphism $\widehat{\phi} : RS^{-1} \to T$ such that $\widehat{\phi}\left(\frac{r}{1}\right) = \phi(r)$ for all $r \in R$.

7. Let $R = \mathbb{Z}$, let p be a prime number, and let $S = \{1, p, p^2, p^3, \dots\}$ be the set of powers of p. Fix an integer $n \geq 2$. Let RS^{-1} be the ring from problem 6.

How many ring homomorphisms $\phi : RS^{-1} \to \mathbb{Z}/n\mathbb{Z}$ are there? (The answer may depend on p and n).