Math 31AH: Fall 2021
Homework 7
Due 5:00pm on Friday 11/19/2021

Problem 1: General Projections. Let $\mathbb{F}$ be an arbitrary field and let V be an $\mathbb{F}$-vector space. A linear projection $P : V \to V$ is a *projection* if $P \circ P = P$. If $P : V \to V$ is a projection, prove that $V = \operatorname{Image} P \oplus \operatorname{Ker} P$.

Solution: We show first that $\operatorname{Image} P \cap \operatorname{Ker} P = 0$. Indeed, let $\mathbf{v} \in V$ so that $P(\mathbf{v}) \in \operatorname{Image} P$. If $P(\mathbf{v}) \in \operatorname{Ker} P$ then $\mathbf{0} = P(P(\mathbf{v})) = P(\mathbf{v})$. This implies that $\operatorname{Image} P \cap \operatorname{Ker} P = 0$.

We now show that $\operatorname{Image} P + \operatorname{Ker} P = V$. Indeed, let $\mathbf{v} \in V$. Since $P(\mathbf{v} - P(\mathbf{v})) = P(\mathbf{v}) - P(P(\mathbf{v})) = P(\mathbf{v}) - P(\mathbf{v}) = \mathbf{0}$, we conclude that $\mathbf{v} - P(\mathbf{v}) \in \operatorname{Ker} P$. Then

$$\mathbf{v} = P(\mathbf{v}) + (\mathbf{v} - P(\mathbf{v})) \in \operatorname{Image} P + \operatorname{Ker} P$$

We conclude that $V = \operatorname{Image} P \oplus \operatorname{Ker} P$.

Problem 2: Trace. The *trace* of an $n \times n$ matrix $A = (a_{ij})$ is

$$\operatorname{tr}(A) := \sum_{i=1}^{n} a_{ii}$$

If A, B are $n \times n$, prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. Deduce that

(1) Similar matrices have the same trace.
(2) If A is diagonalizable with eigenvalues $\lambda_1, \dots, \lambda_r$, then

$$\operatorname{tr}(A) = \dim E_{\lambda_1} \cdot \lambda_1 + \cdots + \dim E_{\lambda_r} \cdot \lambda_r$$

You will learn the full importance of trace when you take representation theory.

Solution: We calculate

$$\operatorname{tr}(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i,j=1}^{n} a_{ij} b_{ji} = \sum_{j=1}^{n} (BA)_{jj} = \operatorname{tr}(BA)$$

If P, A are square with P invertible we see that

$$\operatorname{tr}(PAP^{-1}) = \operatorname{tr}(AP^{-1}P) = \operatorname{tr}(AI) = \operatorname{tr}(A)$$

so that similar matrices have the same trace. If A is diagonalizable, then A is similar to a diagonal matrix $\operatorname{diag}(\lambda_1, \dots, \lambda_1, \dots, \lambda_r, \dots, \lambda_r)$ where λ_i appears $\dim E_{\lambda_i}$ times. Since similar matrices have the same trace, we have

$$\operatorname{tr}(A) = \dim E_{\lambda_1} \cdot \lambda_1 + \cdots + \dim E_{\lambda_r} \cdot \lambda_r$$

Problem 3: Projections and trace. Let $P : V \to V$ be a projection, where V is a finite-dimensional vector space over an arbitrary field $\mathbb{F}$. We define $\operatorname{tr}(P)$, the trace of P, as follows. Given an ordered basis $\mathcal{B}$ of V we set

$$\operatorname{tr}(P) := \operatorname{tr}[P]_{\mathcal{B}}^{\mathcal{B}}.$$

(1) Show that $\operatorname{tr}(P)$ is well-defined (i.e. independent of the choice of $\mathcal{B}$). Your proof should work for any linear map $V \to V$, not just a projection.
(2) Prove that $\operatorname{tr}(P) = \dim \operatorname{Image}(P)$.

Solution: (1) If $\mathcal{C}$ is another ordered basis of V, then

$$[P]_{\mathcal{C}}^{\mathcal{C}} = [i]_{\mathcal{C}}^{\mathcal{B}}[P]_{\mathcal{B}}^{\mathcal{B}}[i]_{\mathcal{B}}^{\mathcal{C}} = [i]_{\mathcal{C}}^{\mathcal{B}}[P]_{\mathcal{B}}^{\mathcal{B}}([i]_{\mathcal{C}}^{\mathcal{B}})^{-1}$$

where $i : V \to V$ is the identity map. Applying Problem 2 we conclude that $\operatorname{tr}[P]_{\mathcal{B}}^{\mathcal{B}} = \operatorname{tr}[P]_{\mathcal{C}}^{\mathcal{C}}$.

(2) By Problem 1, we have a direct sum decomposition $V = \operatorname{Image}(P) \oplus \operatorname{Ker}(P)$. If $(\mathbf{v}_1, \ldots, \mathbf{v}_r)$ is a basis for $\operatorname{Image}(P)$ and $(\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n)$ is a basis for $\operatorname{Ker}(P)$ then $\mathcal{B} := (\mathbf{v}_1, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}, \ldots, \mathbf{v}_n)$ is a basis of V. We have

$$[P]_{\mathcal{B}}^{\mathcal{B}} = \operatorname{diag}(1, \ldots, 1, 0, \ldots, 0)$$

where there are r 1's and $n - r$ 0's. We conclude that

$$\operatorname{tr}(P) = r = \dim \operatorname{Image} P$$

Problem 4: Polynomial Gram-Schmidt. Let V_3 be the vector space of polynomials of degree ≤ 3 and coefficients in $\mathbb{R}$. Endow V_3 with the inner product

$$\langle f(t), g(t) \rangle := \int_{-1}^{1} f(t)g(t)dt$$

The basis $\mathcal{B} = (1, t, t^2, t^3)$ of V_3 is not orthonormal. Apply Gram-Schmidt orthogonalization to $\mathcal{B}$ to get an orthonormal basis $\mathcal{C} = (f_0, f_1, f_2, f_3)$ of V_3.

Solution: Let $(g_0, g_1, g_2, g_3) = (1, t, t^2, t^3)$. We have

$$\langle g_0, g_0 \rangle = \int_{-1}^{1} 1 \cdot 1dt = 2$$

so that $||g_0|| = \sqrt{2}$ and

$$f_0 = \frac{1}{\sqrt{2}}$$

We compute further that

$$g_1 - \langle g_1, f_0\rangle f_0 = t - \int_{-1}^{1} \frac{t}{\sqrt{2}} dt \cdot \frac{1}{\sqrt{2}} = t$$

and since

$$||t||^2 = \int_{t=-1}^{1} t^2 dt = \frac{2}{3}$$

we have

$$f_1 = \sqrt{\frac{3}{2}} \cdot t$$

Continuing, we find

$$g_2 - \langle g_2, f_0\rangle f_0 - \langle g_2, f_1\rangle f_1 = t^2 - \frac{1}{3}$$

and

$$||t^2 - \frac{1}{3}||^2 = \int_{-1}^{1} \left(t^2 - \frac{1}{3}\right)^2 dt = \frac{8}{45}$$

so that

$$f_2 = \sqrt{\frac{45}{8}} \cdot \left(t^2 - \frac{1}{3}\right)$$

Finally, we calcuate

$$g_3 - \langle g_3, f_0\rangle f_0 - \langle g_3, f_1\rangle f_1 - \langle g_3, f_2\rangle f_2 = t^3 - \frac{3}{5} \cdot t$$

and the norm of this expression is $\sqrt{\frac{8}{175}}$. We conclude that

$$f_3 = \sqrt{\frac{175}{8}} \cdot \left(t^3 - \frac{3}{5} \cdot t\right)$$

Problem 5: Legendre Polynomials. Let V be the infinite-dimensional vector space of polynomials with real coefficients. Endow V with the inner product

$$\langle f(t), g(t)\rangle := \int_{-1}^{1} f(t)g(t)dt$$

For $n \geq 0$, let $P_n(t)$ be the *Legendre polynomial* defined by

$$P_n(t) := \frac{1}{2^n \cdot n!} \frac{d^n}{dt^n}(t^2 - 1)^n$$

Prove that $\langle P_n(t), P_m(t)\rangle = 0$ for $n \neq m$. (Hint: Integration by parts.)

Solution: Suppose $m < n$. Applying integration by parts, we see that

$$\begin{aligned}\langle P_n(t), P_m(t)\rangle &= \int_{-1}^{1} P_n(t)\cdot P_m(t)dt \\ &= C\int_{-1}^{1} \frac{d^n}{dt^n}(t^2-1)^n P_m(t)dt \\ &= -C\int_{-1}^{1} \frac{d^{n-1}}{dt^{n-1}}(t^2-1)^n P_m'(t)dt + \left[C\frac{d^{n-1}}{dt^{n-1}}(t^2-1)^n P_m(t)\right]_{t=-1}^{t=1}\end{aligned}$$

where $C = \frac{1}{2^n n!}$ is a constant. The evaluation $[\cdots]_{t=-1}^{t=1}$ is zero since $\frac{d^{n-1}}{dt^{n-1}}(t^2-1)^n$ contains a factor of (t^2-1). Integrating by parts again yields

$$C\int_{-1}^{1} \frac{d^{n-2}}{dt^{n-2}}(t^2-1)^n P_m^{(2)}(t)dt - \left[C\frac{d^{n-2}}{dt^{n-2}}(t^2-1)^n P_m'(t)\right]_{t=-1}^{t=1}$$

where $P_m^{(2)}(t)$ is the second derivative of $P_m(t)$ and the evaluation $[\cdots]_{t=-1}^{t=1}$ again vanishes. Iterating this argument, we see that

$$\langle P_n(t), P_m(t)\rangle = \pm C\int_{-1}^{1} \frac{d^{n-m}}{dt^{n-m}}(t^2-1)^n P_m^{(m)}(t)dt$$

The polynomial $P_m(t)$ has degree m, so its m^{th} derivative $P_m^{(m)}(t)$ is a nonzero constant. Therefore (up to a nonzero constant) the inner product $\langle P_n(t), P_m(t)\rangle$ is

$$\int_{-1}^{1} \frac{d^{n-m}}{dt^{n-m}}(t^2-1)^n dt = \left[\frac{d^{n-m-1}}{dt^{n-m-1}}(t^2-1)^n\right]_{t=-1}^{t=1} = 0$$

where the integration is justified since $n-m>0$ and the evaluation is justified since the term in $[\cdots]$ contains a factor of (t^2-1).

Problem 6: Inner products and dual spaces. Let V be a finite-dimensional inner product space over $\mathbb{F}$. Given a fixed vector $\mathbf{v} \in V$, we have two functions

$$\varphi_{\mathbf{v}} : V \to \mathbb{F} \qquad \psi_{\mathbf{v}} : V \to \mathbb{F}$$

given by $\varphi_{\mathbf{v}}(\mathbf{w}) := \langle \mathbf{v}, \mathbf{w}\rangle$ and $\psi_{\mathbf{v}}(\mathbf{w}) := \langle \mathbf{w}, \mathbf{v}\rangle$.

When $\mathbb{F} = \mathbb{R}$, prove that that we have well-defined maps

$$\varphi : V \to V^* \qquad \psi : V \to V^*$$

given by $\varphi(\mathbf{v}) := \varphi_{\mathbf{v}}$ and $\psi(\mathbf{v}) := \psi_{\mathbf{v}}$ which are both isomorphisms. What happens when $\mathbb{F} = \mathbb{C}$?

Solution: When $\mathbb{F} = \mathbb{R}$, since $\langle \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{w}, \mathbf{v} \rangle$ we have $\varphi_{\mathbf{v}} = \psi_{\mathbf{v}}$ for all $\mathbf{v} \in V$ so that $\varphi = \psi$. These maps are well-defined because

$$\langle \mathbf{v}, c\mathbf{w} + c'\mathbf{w}' \rangle = c\langle \mathbf{v}, \mathbf{w} \rangle + c'\langle \mathbf{v}, \mathbf{w} \rangle$$

for all $\mathbf{v}, \mathbf{w}, \mathbf{w}' \in V$ and $c, c' \in \mathbb{R}$. These maps are linear because

$$\varphi_{c\mathbf{v}+c'\mathbf{v}'}(\mathbf{w}) = \langle c\mathbf{v} + c'\mathbf{v}', \mathbf{w} \rangle = c\langle \mathbf{v}, \mathbf{w} \rangle + c'\langle \mathbf{v}', \mathbf{w} \rangle = (c\varphi_{\mathbf{v}} + c'\varphi_{\mathbf{v}'})(\mathbf{w})$$

Since V is finite-dimensional, we have $\dim V = \dim V^*$ and it suffices to prove that φ is injective. Indeed, if $\varphi(\mathbf{v}) = \varphi_{\mathbf{v}} = 0$, then

$$0 = \varphi_{\mathbf{v}}(\mathbf{v}) = \langle \mathbf{v}, \mathbf{v} \rangle$$

which forces $\mathbf{v} = \mathbf{0}$. We conclude that φ (and hence ψ) is an isomorphism.

When $\mathbb{F} = \mathbb{C}$, the conclusion does not hold. Indeed, the map $\varphi : V \to V^*$ is not even a well-defined function. Since

$$\varphi_{\mathbf{v}}(i \cdot v) = \langle \mathbf{v}, i\mathbf{v} \rangle = -i\langle \mathbf{v}, \mathbf{v} \rangle = -i\varphi_{\mathbf{v}}(\mathbf{v})$$

we see that $\varphi_{\mathbf{v}} : V \to \mathbb{F}$ is not linear for any nonzero vector $\mathbf{v} \in V$. The function $\psi : V \to V^*$ is well-defined, but not linear. Indeed, we have

$$\psi_{i\mathbf{v}}(\mathbf{v}) = \langle \mathbf{v}, i\mathbf{v} \rangle = -i\langle \mathbf{v}, \mathbf{v} \rangle = -i\psi_{\mathbf{v}}(\mathbf{v})$$

so that $\psi_{i\mathbf{v}} \neq i\psi_{\mathbf{v}}$ for any nonzero vector $\mathbf{v}$.

Problem 7: Matrices and Inner Products. Let $V = \mathbb{R}^n$ and let A be an $n \times n$ real matrix. Define a function

$$\langle -, - \rangle : V \times V \to \mathbb{R}$$

by the rule

$$\langle \mathbf{v}, \mathbf{w} \rangle := \mathbf{v}^T A^T A \mathbf{w}$$

Prove that $\langle -, - \rangle$ is an inner product on $\mathbb{R}^n$ if and only if A is invertible.

Solution: Suppose A is not invertible. Since A is square, there is a nonzero vector $\mathbf{v} \in \mathbb{R}^n$ with $A\mathbf{v} = \mathbf{0}$. Then

$$\langle \mathbf{v}, \mathbf{v} \rangle = \mathbf{v}^T A^T A \mathbf{v} = (A\mathbf{v})^T (A\mathbf{v}) = \mathbf{0}^T \mathbf{0} = 0$$

so that $\langle -, - \rangle$ is not an inner product.

Now suppose A is invertible. Given $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ we have

$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T A^T A \mathbf{w} = (\mathbf{w}^T A^T A \mathbf{v})^T = \mathbf{w}^T A^T A \mathbf{v} = \langle \mathbf{w}, \mathbf{v} \rangle$$

Given $\mathbf{v} \in \mathbb{R}^n$, if $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ then $A\mathbf{v} \cdot A\mathbf{v} = -0$ under the standard dot product. This forces $A\mathbf{v} = \mathbf{0}$. Since A is invertible, this means $\mathbf{v} = 0$. Finally, we verify linearity in the first component (where (sesqui)linearity in the second component follows from symmetry). Indeed, we have

$$\langle c\mathbf{v}+c'\mathbf{v}', \mathbf{w}\rangle = (c\mathbf{v}+c'\mathbf{v}')^T A^T A\mathbf{w} = c\mathbf{v}^T A^T A\mathbf{w}+c'(\mathbf{v}')^T A^T A\mathbf{w} = c\langle \mathbf{v}, \mathbf{w}\rangle+c'\langle \mathbf{v}', \mathbf{w}\rangle$$

as desired.

Problem 8: Symmetric and Orthogonal. Consider the $\mathbb{R}$-vector space M_n of $n \times n$ real matrices. We have two subsets

$$S \subseteq M_n \quad \text{and} \quad O \subseteq M_n$$

consisting of symmetric (i.e. $A = A^T$) and orthogonal (i.e. $AA^T = I_n$) operators.

Which (if either) of S and O forms an $\mathbb{R}$-vector space? If either of them do, find their dimension.

Solution: The set O does not form an $\mathbb{R}$-vector space since the zero matrix is not orthogonal. On the other hand, the set S does form an $\mathbb{R}$-vector space. Indeed, the zero matrix is symmetric and the equations

$$(A + B)^T = A^T + B^T \qquad (cA)^T = cA^T$$

imply that symmetric matrices are closed under taking linear combinations.

We claim that the dimension of S is $1+2+\cdots+n = \frac{n(n+1)}{2}$. Indeed, if F_{ij} denotes the $n\times n$ matrix with a 1 in the (i, j)-position and zeroes elsewhere, a basis of S is given by

$$\{F_{ii} : 1 \leq i \leq n\} \cup \{F_{ij} + F_{ji} : 1 \leq i < j \leq n\}$$

and this set has $\frac{n(n+1)}{2}$ elements.

Problem 9: (Optional; not to be handed in.) Let V be the infinite-dimensional vector space of continuous functions $[-\pi, \pi] \to \mathbb{R}$ which are **even**, i.e. $f(x) = f(-x)$ for all x. For any integer $n \geq 0$, define a function

$$f_n(x) := c_n \cos(nx)$$

where $c_n \in \mathbb{R}$ is a to-be-determined constant. Endow V with the inner product

$$\langle f, g\rangle := \int_{-\pi}^{\pi} f(x)g(x)dx$$

Prove that $\langle f_n, f_m\rangle = 0$ for $n \neq m$. Find values of c_n so that $\{f_0, f_1, f_2, \ldots\}$ is orthonormal.