Math 140C: Homework 6

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Rudin 11.1

Let E_n be the subset of E on which $f(x) > \frac{1}{n}$. Write $A = \bigcup E_n$. If $\mu(A) = 0$ then $\mu(E_n) = 0$ since $A \supset E_n$. If $\mu(E_n) = 0$ then $\mu(A) = 0$ since for the disjoint sets $E'_n = E_n \setminus \bigcup_1^{n-1} E_i$, $\mu(E') = 0$ and $\bigcup E' = \bigcup E = A$ so $\mu(A) = 0$ by countable additivity.

We know that $\mu(E_n) = 0$ since $0 \le \frac{1}{n}\mu(E_n) \le \int_{E_n} f\,du \le \int_E f\,du = 0$. Since A is the set where $f(x) > 0$ and $f(x) \ge 0$, $\mu(E_n) = 0$ implies that $\mu(A) = 0$, which then implies that $f(x) = 0$ almost everywhere.

Rudin 11.2

We can apply the conclusion of the previous problem twice on the set of non-negative values and non-positive values to show that $f^+(x) = 0$ almost everywhere and $f^-(x) = 0$. Thus $f(x) = 0$ almost everywhere on E since the set of values where $f(x) \neq 0$ is the union of the set of values where $f(x) < 0$ and $f(x) > 0$.

Rudin 11.5

For $0 \leq x \leq \frac{1}{2}$, the lim inf can be achieved by the subsequence of even elements, f_{2k}. For $\frac{1}{2} < x \leq 1$ the lim inf can be achieved by taking the subsequence of odd elements, f_{2k+1}. Thus we have that

$$f(x) = \liminf_{n\to\infty} f_n(x) = 0.$$

However each f_n is a simple function with integral $\int_0^1 f_n(x)\,dx = \frac{1}{2}$. This problem is in agreement with (77) since

$$0 = \int_E f\,d\mu \leq \liminf_{n\to\infty} \int_E f_n\,d\mu = \frac{1}{2}.$$

Rudin 11.6

Since $|f_n(x) - 0| < \frac{1}{n}$, $f_n(x) \to 0$ uniformly as $n \to \infty$.

However all f_n cannot be bounded by a function $g \in \mathscr{L}$, so Theorem 11.32 fails, as given by the fact that $\int_{-\infty}^{\infty} f_n \, dx = \int_{-n}^{n} \frac{1}{n} \, dx = 2$.