Math 170A, Fall 2023 ***HOMEWORK #7*** ***due Friday, Dec 8***

Homework problems that will be graded (Q1 - Q5, 30pts in total):

Q1. Let $A = \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}$. Will the Jacobi iterative method converge on A, regardless of the starting vector $x^{(0)}$?

Q2. Let $B = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Take 5 steps of Gauss-Seidel to attempt to solve $Bx = b$, starting with $x^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$; write down $x^{(1)}, x^{(2)}, x^{(3)}, x^{(4)}, x^{(5)}$. Based on what you see, do you think Gauss-Seidel will converge?

Q3. A matrix $A \in \mathbb{C}^{n\times n}$ is called skew-Hermitian if $A^* = -A$.

Show that skew-Hermitian matrices have only purely imaginary eigenvalues, i.e., any eigenvalue of a skew-Hermitian matrix has the form $\lambda = ai$ with $a \in \mathbb{R}$.

Hint: Show that iA is Hermitian, then use what we know about Hermitian matrices.

Q4. Using one of the two methods of computing the SVD using eigendecompositions outlined in class, compute **by hand** the *reduced* SVD of the matrix

$$A = \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Q5. (MATLAB problem) Run the `sample_inverse_PM.m` code provided. Below is an explanation of what the code does.

The first line of the code resets the pseudo-random number generator to default.

Lines $2-4$ construct a random matrix A with prescribed eigenvalues given in line 2.

Line 5 constructs a matrix B which shifts the eigenvalues of A by the same amount, while leaving the eigenvectors unchanged.

Line 6 inverts B and names the inverse C.

The remainder of the code applies the power method to the matrix C, starting from the vector of all ones, and using 8 iterations.

Run the code and examine it closely, then answer the following questions.

a) What are the eigenvalues of B and C? Start with the eigenvalues of A and see how they get changed when we transform A into B and then C.

b) Use **reasoning**, not MATLAB, to answer this question, and show your reasoning. By the end of the code, q is a very good approximation for an eigenvector of A. What is the corresponding eigenvalue λ?

c) If you replaced 0.25 with -0.5 in line 5 of the script, which eigenvector of A would q approximate? What would be the corresponding eigenvalue of A? Like in b), use and show your reasoning.

d) After running the code with 0.25 in the line 5 of the script, show how good an approximation the vector q is to an eigenvector of A by typing in MATLAB

`norm((A-λ*eye(6))*q)` ,

where λ is the number you obtained in b). This computes the 2-norm of the vector in parentheses; if it is small, the vector q is a very good approximation to an eigenvector with eigenvalue λ.

sample_inverse_PM.m

```
rng('default')
[Q,R]= qr(randn(6));
D = diag([2, -1, -0.25, 0.5, 2, -5]);
A = Q*D*Q';
B = A - 0.25*eye(6);
C = inv(B);

q = ones(6,1); s = 1;
for j=1:12
        q_old = q;
        q_new = C*q_old;
        [~, ind] = max(abs(q_new));
        s = q_new(ind(1));
        q = q_new/s;
end;
```