Math 31BH: Assignment 2

Due 01/16 at 23:59

1. Let $\mathbf{S}$ be the space of 2×2 symmetric matrices, and let $f \colon \mathbf{S} \to \mathbb{R}^2$ be the function which sends each $S \in \mathbf{S}$ to $(s_1, s_2) \in \mathbb{R}^2$, where s_1, s_2 are the eigenvalues of S and $s_1 \geq s_2$.

   (a) Write down an explicit formula for the function $g \colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by

   $$g(x_1, x_2, x_3) = f\left(\begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}\right).$$

   (b) Suppose $(a, b, c) \in \mathbb{R}^3$ is such that $g(a, b, c) = (2, 1)$. Prove that $g(a + 1, b, c) \neq (3.1, 1)$.

2. *Solution:* The eigenvalues $s_1 \geq s_2$ of the symmetric matrix $S = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}$ are the solutions of the characteristic equation $\det(sI - S) = 0$, which is the quadratic equation

   $$s^2 - (x_1 + x_3)s + (x_1x_3 - x_2^2) = 0.$$

   Applying the quadratic formula, we find that the map $g \colon \mathbb{R}^3 \to \mathbb{R}^2$ is given explicitly by

   $$g(x_1, x_2, x_3) = \frac{1}{2}\left(x_1 + x_3 + \sqrt{(x_1 - x_3)^2 + 4x_2^2}, x_1 + x_3 - \sqrt{(x_1 - x_3)^2 + 4x_2^2}\right).$$

   The second part of the question is optional (correct solutions receive extra credit). The reason for this is that I botched the formulation of the problem. I *should* have been asking you to compare the eigenvalues of a pair of symmetric matrices of the formu

   $$\begin{bmatrix} a & b \\ b & c \end{bmatrix} \text{ and } \begin{bmatrix} a & b+1 \\ b & c \end{bmatrix}$$

   using the Hoffman-Wielandt inequality, but I put the 1 in the wrong place.

3. Let $\mathbf{V}$ and $\mathbf{W}$ be Euclidean spaces. Prove that every linear function $f\colon \mathbf{V} \to \mathbf{W}$ is continuous.

   Solution: Let $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_m\}$ be an orthonormal basis of $\mathbf{W}$, and let $f_1, \dots, f_m \colon \mathbf{V} \to \mathbb{R}$ be the corresponding component functions of $f\colon \mathbf{V} \to \mathbf{W}$, i.e.

$$f(\mathbf{v}) = \sum_{i=1}^{m} f_i(\mathbf{v})\mathbf{c}_i \quad \forall \mathbf{v} \in \mathbf{V}.$$

   Then, as proved in Lecture 3, f is continuous if and only if each of the component functions f_i is continuous. We will show that this is the case.

   Since f is linear, so is each component function, meaning that each $f_i \mathbf{V} \to \mathbb{R}$ is a linear functional on $\mathbf{V}$. Thus, by the Riesz Representation Theorem, for each $1 \leq i \leq m$ there exists a vector $\mathbf{v}_i \in \mathbf{V}$ such that

$$f_i(\mathbf{v}) = \langle \mathbf{v}_i, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in \mathbf{V}.$$

   If $\mathbf{v}_i = \mathbf{0}_\mathbf{V}$, then the function f_i is identically zero (i.e. sends every vector in $\mathbf{V}$ to the number zero), and since a constant function is continuous we are done. Assuming $\mathbf{v}_i$ is nonzero, we argue as follows. For any two vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbf{V}$, we have

$$|f_i(\mathbf{v}_1) - f_i(\mathbf{v}_2)| = |\langle \mathbf{v}_i, \mathbf{v}_1 \rangle - \langle \mathbf{v}_i, \mathbf{v}_2 \rangle| = |\langle \mathbf{v}_i, \mathbf{v}_1 - \mathbf{v}_2 \rangle| \leq \|\mathbf{v}_i\| \|\mathbf{v}_1 - \mathbf{v}_2\|,$$

   by the Cauchy-Schwarz inequality. In particular, for any $\mathbf{v} \in \mathbf{V}$ and any $\varepsilon > 0$, the open ball of radius $\delta = \frac{\varepsilon}{2}$ centered at $\mathbf{v}$ is contained in the open ball of radius ε centered at $f_i(\mathbf{v})$, so f_i is continuous at $\mathbf{v}$.

4. Let $\mathbf{v}_1, \dots, \mathbf{v}_r$ be vectors in a Euclidean space $\mathbf{V}$.

   (a) Prove that the convex hull $\mathrm{Conv}(\mathbf{v}_1, \dots, \mathbf{v}_r)$ is a compact set.

   Solution: To prove that $C = \mathrm{Conv}(\mathbf{v}_1, \dots, \mathbf{v}_r)$ is a compact subset of $\mathbf{V}$, we need to prove that it is both bounded and closed.

   It is easy to verify that C is bounded. Every vector $\mathbf{v} \in C$ is of the form $\mathbf{v} = t_1\mathbf{v}_1 + \dots + t_r\mathbf{v}_r$ with the scalars $t_1, \dots, t_r$ nonnegative and summing to 1. Thus,

$$\|\mathbf{v}\| = \|t_1\mathbf{v}_1 + \dots + t_r\mathbf{v}_r\| \leq t_1\|\mathbf{v}_1\| + t_r\|\mathbf{v}_r\|.$$

   Thus, if $M = \max(\|\|\mathbf{v}_1\|, \dots, \|\mathbf{v}_r\|$, we have

$$\|\mathbf{v}\| \leq t_1 M + \dots + t_r M = (t_1 + \dots + t_r)M = M$$

for all $\mathbf{v} \in C$, which shows that C is bounded.

The proof that C is closed is actually a harder than I was anticipating, and the argument I found uses the fact that the continuous image of a compact set is compact together with the observation that C is the image of

$$B = \{(t_1, \ldots, t_r) \in \mathbb{R}^r : t_i \geq 0, \ \sum t_i = 1\}.$$

Since we did not cover the results which would allow you to write down a proof along these lines, **proving that C is closed is optional** (correct solutions receive extra credit),

(b) Let $f \colon \mathbf{V} \to \mathbb{R}$ be a linear function. Prove that f has both a maximum and a minimum value on $\mathrm{Conv}(\mathbf{v}_1, \ldots, \mathbf{v}_r)$, and show that it has both a maximizer and a minimizer in $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$.

Solution: Let $M = \max(f(\mathbf{v}_1), \ldots, f(\mathbf{v}_r))$ and let $m = \min(f(\mathbf{v}_1), \ldots, f(\mathbf{v}_r))$. For any $t_1, \ldots, t_r \in [0, 1]$ with $t_1 + \cdots + t_r = 1$, we have

$$m = t_1 m + \cdots + t_r m \leq t_1 f(\mathbf{v}_1) + \cdots + t_r f(\mathbf{v}_r) = f(t_1 \mathbf{v}_1 + \cdots + t_r \mathbf{v}_r),$$

and similarly

$$M = t_1 M + \cdots + t_r M \geq t_1 f(\mathbf{v}_1) + \cdots + t_r f(\mathbf{v}_r) = f(t_1 \mathbf{v}_1 + \cdots + t_r \mathbf{v}_r),$$

where in each case we used the linearity of f to get the final equality.