MAT140B WINTER 2024: PROBLEM SET 2

Due: S 28/01/2024, by 11:59pm

Directions: You can collaborate, but must write up the solutions independently and in a good handwriting. **Consulting solutions to problem sets of previous semesters or internet solutions is not allowed.**

Problem 1. Assume that f is a continuous function on $[0, 1]$. Assume that f' is also continuous on $[0, 1]$. Prove that there exists an $M \geq 0$ so that

$$|f(x) - f(y)| \leq M|x - y|$$

for all $x, y \in [0, 1]$.

Problem 2. Let f be a continuous function on $[0, 1]$. Assume that f' is also continuous on $[0, 1]$. Define $M = \{y \in \mathbb{R} : y = f(x) \text{ and } f'(x) = 0\}$. In other words, $M = f(\{x \in [0, 1] : f'(x) = 0\})$. Prove that M cannot contain an interval. Show by an example that M may contain infinitely many points.

Hint: for the proof, you could use that f' is actually uniformly continuous on $[0, 1]$ since $[0, 1]$ is compact and think about the image under f of a small ball around a critical point.

Problem 3. Directly compute

$$\int_0^1 x^2 dx$$

using Riemann sums.

Problem 4. Chapter 5: 8, 11, 14, 15, 25