Math 100B: Homework 8

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Problem 1

Let S be the set of all subsets $X' \subseteq V$ such that $X \subseteq X'$ and X' is linearly independent over F. Let $(S, \leq)$ be a poset where $X_1 \leq X_2$ iff $X_1 \subseteq X_2$. Note that S is nonempty since $X \in S$. Suppose T is a chain in S. Let $Y = \bigcup_{X_i \in T} X_i$ and we want to show that Y is an upper bound for T.

It's clear that if $X_i \in T$ then $X_i \leq Y$ so all we need to show is that $Y \in S$. First we have that $X \subseteq Y$ since $X \subseteq X_i$ for every $X_i \in T$. Now we need to show that $v_1, \ldots, v_n \in Y$ are linearly independent. Then $v_i \in X_i$ for some $X_i \in T$. Then $v_1, \ldots, v_n \in X_m$ where X_m is the largest among $X_1, \ldots, X_n$. Then since X_m consistents of linearly independent vectors, Y consists of linearly independent vectors and $Y \in S$.

Since each chain has a maximal element, S has a maximal element X' by Zorns lemma, which is a basis and contains X, which completes our proof. If X' was not a basis, then it would have a span smaller than V, which implies that X' is not maximal(since we can add a vector not in the span to X' while keeping it linearly independent) which is a contradiction.

Problem 2

(a) If A is invertible then we can choose $P = A$ since

$$A^{-1}ABA = BA$$

If B is invertible then we can choose $P = B^{-1}$ since

$$(B^{-1})^{-1}ABB^{-1} = BA$$

(b) In $\mathbb{R}^2$ we can choose

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

However AB and BA are not similar since

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Problem 3

(a) Let $v \in V$ be arbitrary and let $v' \in \operatorname{im}\phi$. such that $\phi(v) = v'$. Let $w = v - v'$ so we can write $\phi(v') + \phi(w) = v'$. Sine $v' \in \operatorname{im}\phi$ and $\phi^2 = \phi$ it must be that $\phi(v') = v'$. This implies that $\phi(w) = 0$ and $w \in \ker\phi$. Therefore $v = v' + w$ where $v' \in \operatorname{im}\phi$ and $w \in \ker\phi$, so $V = \operatorname{im}\phi \oplus \ker\phi$.

(b) Since the image and kernel are subspaces, they each have a basis. Their intersection only contains zero If the intersection had a nonzero vector, it would map to 0 since it was in the kernel but it would also need to map to itself since $\phi^2 = \phi$ which is a contradiction. Therefore the basis for the image and kernel are independent from each other and we can combine them to form a basis for the entire space V (since the dimension of V is the sum of the dimensions of the image and kernel by the rank-nullity theorem).

If the image has dimension m and V has dimension n, then $M_{\mathcal{B}}(\phi)$ would be a $n \times n$ matrix with zeros everywhere except for the first m entires of the diagonal.

Problem 4

(a) Since ϕ^2 is the identity map, $\phi^2(v) = v$ and $\phi(v - \phi(v)) = \phi(v) - v$. If $\phi(v) - v$ is zero then $v - \phi(v)$ has eigenvalue 0 and if $\phi(v) - v$ is not zero, then it has eigenvalue -1.

(b) Every vector $v \in V$ can be written as the sum $\frac{1}{2}(v + \phi(v)) + \frac{1}{2}(v - \phi(v))$. This is the sum of a vector in V_1 and a vector in V_{-1} since

$$\phi(\frac{1}{2}(v + \phi(v))) = \frac{1}{2}(\phi(v) + \phi(\phi(v))) = \frac{1}{2}(v + \phi(v))$$

$$\phi(\frac{1}{2}(v - \phi(v))) = \frac{1}{2}(\phi(v) - \phi(\phi(v))) = -\frac{1}{2}(v - \phi(v)).$$

(c) Since $V = V_1 \oplus V_{-1}$ and V_1 and V_{-1} both have eigenbasis, V also has an eigenbasis and so it is diagonal.

Problem 5

Over standard coordinates, the matrix

$$M_{\mathcal{B}}(\phi) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

is not diagonalizable because if $v = a_1 v_1 + a_2 v_2$ is an eigenvector, then $a_1 = a_2$, but there is only one such nonzero vector that satisfies this property(namely $v_1 + v_2$). Therefore there does not exist an eigenbasis for ϕ and so it is not diagonalizable.