Math 31BH: Assignment 7

Due 02/27 at 23:59

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1. Consider the function of three variables defined by $f(x, y, z) = x^2 y \sin(yz)$.

   (a) Show that f is differentiable on $\mathbb{R}^3$.

   (b) Calculate the gradient vector $\nabla f(1, -1, \pi)$.

   (c) Write down a formula for the derivative $f'((1, -1, \pi), (x, y, z))$.

   Solution:

   (a) f is differentiable on $\mathbb{R}^3$ since the partial derivatives exist.

   $$\frac{\partial}{\partial x} x^2 y \sin(yz) = 2xy \sin(yz)$$
   $$\frac{\partial}{\partial y} x^2 y \sin(yz) = x^2 yz \cos(yz) + x^2 \sin(yz)$$
   $$\frac{\partial}{\partial z} x^2 y \sin(yz) = x^2 y^2 \cos(yz)$$

   (b) The gradient is the vector of the partial derivatives.

   $$\nabla f(x, y, z)(2xy \sin(yz), x^2 yz \cos(yz) + x^2 \sin(yz), x^2 y^2 \cos(yz))$$
   $$\nabla f(1, -1, \pi) = (-2 \sin(-\pi), -\pi \cos(-\pi) + \sin(-\pi), \cos(-\pi)) = (0, \pi, -1)$$

   (c) The derivative can be represented as the scalar product between the gradient and (x, y, z).

   $$f'((1, -1, \pi), (x, y, z)) = \nabla f(1, -1, \pi) \cdot (x, y, z) = \pi y - z$$

2. Find the partial derivatives of $f(x, y) = x^y$.

   Solution: The partial derivative is computed by treating other variables as constants.

   $$\frac{\partial}{\partial x} x^y = y x^{y-1}$$
   $$\frac{\partial}{\partial y} x^y = \ln(x) x^y$$

3. Let $f(x, y) = x^2 + y^3$. Find the directional derivative of f at $\mathbf{v} = (-1, 3)$ in the direction of maximal increase of f.

Solution: The partial derivatives are

$$\frac{\partial}{\partial x} x^2 + y^3 = 2x$$
$$\frac{\partial}{\partial y} x^2 + y^3 = 3y^2$$

Therefore, the gradient is $w = (-2, 27)$ Therefore,

$$f'(v, w) = \nabla f(v) \cdot w = (-2, 27) \cdot (-2, 27) = 733$$

Since the question is finding the directional derivative, we divide by the norm of the gradient so, $f'(v, e) = \sqrt{733}$.

4. Let f be a differentiable function defined on an open set D in a Euclidean space $\mathbf{V}$. Suppose that $\mathbf{m} \in D$ is a maximum of f, i.e. $f(\mathbf{m}) \geq f(\mathbf{v})$ for all $\mathbf{v} \in D$. Prove that $\nabla f(\mathbf{m}) = \mathbf{0_V}$, the zero vector in $\mathbf{V}$.

Solution: Suppose that $\nabla f(m) \neq 0_v$. This means that

$$f'(m, \nabla f(m)) = \nabla f(m) \cdot \nabla f(m) > 0$$

Since

$$f'(m, \nabla f(m)) = \lim_{h \to 0} \frac{f(m + h\nabla f(m)) - f(m)}{h} = \lim_{h \to 0^+} \frac{f(m + h\nabla f(m)) - f(m)}{h} > 0$$

$$\begin{aligned} \lim_{h \to 0^+} \frac{f(m + h\nabla f(m)) - f(m)}{h} > 0 &\implies \lim_{h \to 0^+} f(m + h\nabla f(m)) - f(m) > 0 \\ &\implies \lim_{h \to 0^+} f(m + h\nabla f(m)) > f(m) \end{aligned}$$

there exists an $h > 0$ such that $f(m + h\nabla f(m)) > f(m)$. This contradicts the fact that $f(m)$ is the maximum, so $\nabla f(m)$ must be 0_v.