Math 100A: Homework 6

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Problem 1

For any two nonzero elements $(x, y) \in S$ and $(u, v) \in S$, we can multiply (x, y) by an invertible matrix to get to (u, v).

If $x, y \neq 0$ then

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \frac{u}{x} & 0 \\ 0 & \frac{v}{y} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

If $x = 0$ then

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & \frac{u}{y} \\ 0 & \frac{v}{y} \end{bmatrix} \begin{bmatrix} 0 \\ y \end{bmatrix}$$

If $y = 0$ then

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \frac{u}{x} & 1 \\ \frac{v}{x} & 0 \end{bmatrix} \begin{bmatrix} x \\ 0 \end{bmatrix}$$

Therefore $\mathrm{GL}_2(\mathbf{R})$ acts transitively on S.

Problem 2

A matrix is in the stabilizer if multiplying it with $[1,0]$ yields $[1,0]$. Notice that the second column of a stabilizing matrix can be arbitrary but the first column must be $[1,0]$ for the product to equal $[1,0]$.

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

However for the matrix to be invertible, we need $b \neq 0$ so the stabilizer is

$$\mathrm{GL}_2(\mathbf{R})_{[1,0]} = \left\{ \begin{bmatrix} 1 & a \\ 0 & b \end{bmatrix} : a, b \in \mathbb{R}, b \neq 0 \right\}.$$

Problem 3

(a) Since $g \in \mathrm{SL}_2(\mathbf{R})$ we have that $ad - bc = 1$. If $cz + d = cx + d + i(cy) = 0$, then $cx + d = 0$ and $cy = 0$. This implies that $c = 0$ since $y > 0$. This then implies that $d = 0$, however this means that $ad - bc = 0$ which is a contradiction, so $cz + d \neq 0$.

(b)

$$\begin{aligned}
\operatorname{Im} g \cdot z &= \operatorname{Im} \frac{az+b}{cz+d} \\
&= \operatorname{Im} \frac{(ax+b)+iay}{(cx+d)+icy} \cdot \frac{(cx+d)-icy}{(cx+d)-icy} \\
&= \operatorname{Im} \frac{(ax+b)(cx+d)+acy^2+i(ay(cx+d)-cy(ax+b))}{(cx+d)^2+(cy)^2} \\
&= \frac{acxy+ady-acxy-bcy}{|cz+d|^2} \\
&= \frac{(ad-bc)y}{|cz+d|^2} \\
&= \frac{y}{|cz+d|^2}
\end{aligned}$$

(c) First note that

$$1 \cdot z = \frac{1z+0}{0z+1} = z$$

If $G = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $H = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$ then

$$\begin{aligned}
(GH) \cdot z &= \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix} \cdot z \\
&= \frac{(ae+bg)z+af+bh}{(ce+dg)z+cf+dh} \\
&= \frac{\frac{aez+af+bgz+bh}{gz+h}}{\frac{cez+cf+dgz+dh}{gz+h}} \\
&= \frac{a\frac{ez+f}{gz+h}+b}{c\frac{ez+f}{gz+h}+d} \\
&= G \cdot \frac{ez+f}{gz+h} \\
&= G \cdot (H \cdot z)
\end{aligned}$$

Problem 4

1. For complex numbers $x + iy \in \mathcal{H}$ and $u + iv \in \mathcal{H}$, we need to show that there exists upper triangle matrix g such that $g \cdot (x + iy) = u + iv$. Let $g = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$.

$$\begin{aligned} g \cdot (x + iy) &= \frac{a(x + iy) + b}{d} \\ &= \left(\frac{ax + b}{d} \right) + i\frac{ay}{d} \\ &= u + iv \end{aligned}$$

   Since $v > 0$, we can choose $a = 1$, $d = \frac{y}{v}$, and $b = ud - x$ so that $g \cdot (x + iy) = u + iv$. Since it is possible to get from any element to any other element in $\mathcal{H}$ using group actions from B, there is only one orbit and B acts transitively on $\mathcal{H}$. Since $B \subseteq \mathrm{SL}_2(\mathbf{R})$, $\mathrm{SL}_2(\mathbf{R})$ also has just one orbit and so it also acts transitively on B.

2. Let $g \in \mathrm{SL}(2)$ where $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $ad - bc = 1$ since the determinant is 1.

$$\begin{aligned} g \cdot i &= \frac{ai + b}{ci + d} \\ &= \frac{b + ai}{d + ci} \cdot \frac{d - ci}{d - ci} \\ &= \frac{(bd + ac) + i(ad - bc)}{d^2 + c^2} \\ &= \frac{(bd + ac) + i}{d^2 + c^2} \end{aligned}$$

   We need $d^2 + c^2 = 1$ and $bd + ac = 0$ for g to be a stabilizer. This means that the inner product of the second row with itself is 1 and the inner product of the first and second row is 0. It also must be that $a^2 + b^2 = 1$ since the determinant is 1 and the rows are orthogonal. Algebraically we can see this by adding the two following equations to get the third

$$1 = (ad - bc)^2 = a^2d^2 - 2abcd + b^2c^2$$

$$0 = (ac + bd)^2 = a^2c^2 + 2abcd + b^2d^2$$

$$1 = a^2d^2 + a^2c^2 + b^2c^2 + b^2d^2 = a^2(c^d + d^2) + b^2(c^d + d^2) = a^2 + b^2$$

   Thus $gg^t = g^tg = I$, so the stabilizer is $\mathrm{SO}_2(\mathbf{R})$. Note that this means

3. If $g \in \mathrm{SL}_2(\mathbf{R})$ then there must exist $b \in B$ such that $b \cdot i = g \cdot i$ since $\mathrm{SL}_2(\mathbf{R})$ and B both act transitively on $\mathcal{H}$. Therefore $i = b^{-1}g \cdot i$ so $g^{-1}b$ is in the stabilizer of i, which is $\mathrm{SO}_2(\mathbf{R})$. Therefore $b^{-1}g = h$ for some $h \in \mathrm{SO}_2(\mathbf{R})$, ie $g = bh$.