Math 140C: Homework 5

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Problem 1

1. If $A \in \mathcal{E}$ and $B \in \mathcal{E}$ then $A = \bigcup_{i=1}^N A_i$ and $B = \bigcup_{i=1}^M B_i$ for some intervals A_i, B_i and for some N, M. $A \cup B \in \mathcal{E}$ since the union of two finite unions of intervals is another finite union of intervals. $A - B \in \mathcal{E}$ since the set difference is right distributive over the union and subtracting a finite number of intervals from an interval results in a union of intervals.
$$\bigcup_{i=1}^N A_i - \bigcup_{i=1}^M B_i = \bigcup_{i=1}^N \left(A_i - \bigcup_{i=1}^M B_i \right) \in \mathcal{E}$$
$\mathcal{E}$ is not a σ-algebra since $I_n = [n, n+1] \in \mathcal{E}$ for all $n \in \mathbb{Z}$ but
$$\bigcup_{n \in \mathbb{Z}} I_n = \mathbb{R} \notin \mathcal{E}.$$

2. Let $A = \bigcup_{i=1}^n A_i$ for some intervals A_i. Then choose $I_j = A_j - \bigcup_{i=1}^{j-1} A_i$ so that I_j is pairwise disjoint and
$$A = \bigcup_{i=1}^N A_i = \bigcup_{j=1}^N I_j$$

3. Suppose there exist two different decompositions into pairwise disjoint intervals, $A = \bigcup_{j=1}^N A_j$ and $A = \bigcup_{j=1}^M B_j$. Let P_A and P_B be the partition representation of these decompositions such that if $A_j = [a, b]$ then $a, b \in P_A$ and likewise if $B_j = [a, b]$ then $a, b \in P_B$. Notice that the measure of a refinement of either partition will have the same measure, so we can simply take the common refinement of P_A and P_B to show that all decompositions have the same measure.

4. If A and B are disjoint elementary sets where $A = \bigcup_{i=1}^N A_i$ and $B = \bigcup_{i=1}^M B_i$ then m is additive since the all the A_i and B_i are pairwise disjoint.
$$\begin{aligned} m(A \cup B) &= m\left(\left(\bigcup_{i=1}^N A_i \right) \cup \left(\bigcup_{i=1}^M B_i \right) \right) \\ &= \sum_{i=1}^N m(A_i) + \sum_{i=1}^M m(B_i) \end{aligned}$$

Problem 2

The distance function, $d(A, B) = \mu^*(S(A, B))$, is the measure of the symmetric difference of A and B.

Property (27) is true since

$$
\begin{aligned}
d(A, B) &= \mu^*((A - B) \cup (B - A)) \\
&= \mu^*((B - A) \cup (A - B)) \\
&= d(B, A) \\
d(A, A) &= \mu^*(\emptyset) = 0
\end{aligned}
$$

Property (28) is true since

$$
\begin{aligned}
d(A, B) &= \mu^*(A - B) + \mu^*(B - A) \\
&= \mu^*(A - B - C) + \mu^*(A \cap C - B) + \mu^*(B - A - C) + \mu^*(B \cap C - A) \\
&\leq \mu^*(A - C) + \mu^*(B - C) + \mu^*(A \cap C - B) + \mu^*(B \cap C - A) \\
&\leq \mu^*(A - C) + \mu^*(C - A) + \mu^*(C - B) + \mu^*(B - C) \\
&= \mu^*((A - C) \cup (C - A)) + \mu^*((C - B) \cup (B - C)) \\
&= d(A, C) + d(C, B)
\end{aligned}
$$

For clarity, I will use A_1B_2 to represent $\mu^*((A_1 \cup B_2) - (A_2 \cup B_1))$, the measure of the set that is only in sets A_1 and B_2. Expanding out the values of each distance, its clear that property (29) holds.

$$
\begin{aligned}
d(A_1, B_1) + d(A_2, B_2) =& \mu^*(A_1 - B_1) + \mu^*(B_1 - A_1) + \mu^*(A_2 - B_2) + \mu^*(B_2 - A_2) \\
=& (A_1 + A_1A_2 + A_1B_2 + A_1A_2B_2) + (B_1 + A_2B_1 + B_1B_2 + A_2B_1B_2) + \\
& (A_2 + A_1A_2 + A_2B_1 + A_1A_2B_1) + (B_2 + A_1B_2 + B_1B_2 + A_1B_1B_2)
\end{aligned}
$$

$$
\begin{aligned}
d(A_1 \cup A_2, B_1 \cup B_2) &= \mu^*(A_1 \cup A_2 - B_1 \cup B_2) + \mu^*(B_1 \cup B_2 - A_1 \cup A_2) \\
&= (A_1 + A_2 + A_1A_2) + (B_1 + B_2 + B_1B_2)
\end{aligned}
$$

$$
\begin{aligned}
d(A_1 \cap A_2, B_1 \cap B_2) &= \mu^*(A_1 \cap A_2 - B_1 \cap B_2) + \mu^*(B_1 \cap B_2 - A_1 \cap A_2) \\
&= (A_1A_2 + A_1A_2B_1 + A_1A_2B_2) + (B_1B_2 + A_1B_1B_2 + A_2B_1B_2)
\end{aligned}
$$

$$
\begin{aligned}
d(A_1 - A_2, B_1 - B_2) &= \mu^*((A_1 - A_2) - (B_1 - B_2)) + \mu^*((B_1 - B_2) - (A_1 - A_2)) \\
&= (A_1 + A_1B_2 + A_1B_1B_2) + (B_1 + A_2B_1 + A_1A_2B_2)
\end{aligned}
$$

Problem 3

Assume $(X, \mathcal{M})$ is a countably infinite σ-algebra. For $x \in X$ define $A_x = \bigcap_{x \in E \in \mathcal{M}} E$. Since $\mathcal{M}$ is countable, A_x is the intersection of at most countably many sets and so $A_x \in \mathcal{M}$.

Now suppose that $A_x \cap A_y \neq \emptyset$. If $x \notin A_x \cap A_y$, then $x \in A_x - A_x \cap A_y \in \mathcal{M}$. This implies that $A_x \cap A_y = \emptyset$ since any element that could be in $A_x \cap A_y$ would not be in A_x. Thus $x \in A_x \cap A_y$. Similarly we know that $y \in A_x \cap A_y$. Since every set with x contains y and vice-versa, $A_x = A_y$ so we have that $A_x \cap A_y \neq \emptyset \implies A_x = A_y$.

Thus for a countably infinite set $x_i \in X$, A_{x_i} forms a disjoint partition of X. The function from $\mathcal{P}(\mathbb{N}) \to \mathcal{M}$ that takes $I \in \mathcal{P}(\mathbb{N})$ and gives $\bigcup_{i \in I} A_{x_i}$, is injective since each A_{x_i} is pairwise disjoint, meaning that $\mathcal{M}$ is uncountably infinite, a countradiction. Thus every σ-algebra is either finite or uncountably infinite.

Problem 4

1. Let $A = \bigcup_{i=1}^{N} A_i$ be the union of finitely many intervals. Since for any interval $A_i = [a, b]$, $m(A_i + t) = (b + t) - (a + t) = m(A_i)$ we have that

$$m(A + t) = \bigcup_{i=1}^{N} m(A_i + t) = \bigcup_{i=1}^{N} m(A_i) = m(A)$$

2. $A \subset \bigcup_{n=1}^{\infty} A_n$ is an open covering of A using elementary sets, then $\bigcup_{n=1}^{\infty} A_n + t$ is an open covering for $A + t$ and vice versa. Using this bijection of open coverings,

$$\mu^*(A + t) = \inf \sum_{n=1}^{\infty} \mu(A_n + t) = \inf \sum_{n=1}^{\infty} \mu(A_n) = \mu^*(A)$$