Math 31AH: Fall 2021
Homework 5 Solutions
Due 5:00pm on Friday 11/5/2021

Problem 1: Linear functionals and nonzero vectors. Let V be a finite-dimensional $\mathbb{F}$-vector space and let $\mathbf{v} \in V$ be nonzero. Prove there exists $\lambda \in V^*$ with $\lambda(\mathbf{v}) \neq 0$.

Solution: The set $\{\mathbf{v}\}$ is linearly independent, and so may be extended to a basis $\mathcal{B} = \{\mathbf{v}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ of V. Let $\mathcal{B}^* = \{\lambda, \lambda_1, \lambda_2, \dots, \lambda_n\}$ be the dual basis of V^*. Then $\lambda(\mathbf{v}) = 1 \neq 0$.

Problem 2: Induced maps. Let $T : V \to W$ be a linear transformation between finite-dimensional $\mathbb{F}$-vector spaces. Let $T^* : W^* \to V^*$ be the induced linear transformation between their dual spaces.

(1) If T is injective, prove that T^* is surjective.
(2) If T is surjective, prove that T^* is injective.

Solution: (1) Suppose T is injective. Let $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis of V and let $\mathcal{B}^* = \{\lambda_1, \dots, \lambda_n\}$ be its dual basis. It suffices to show that each $\lambda_i \in \mathcal{B}^*$ is in the image of T^*.

Since T is injective, the set $\{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n)\}$ is linearly independent in W. We may extend this set to a basis

$$\mathcal{C} = \{T(\mathbf{v}_1), \dots, T(\mathbf{v}_n), \mathbf{w}_{n+1}, \dots, \mathbf{w}_m\}$$

of W. Let $\{\mu_1, \dots, \mu_n, \mu_{n+1}, \dots, \mu_m\}$ be the corresponding dual basis. For $1 \leq i \leq n$ we claim $T^*(\mu_i) = \lambda_i$. Indeed, given $1 \leq i, j \leq n$ we have

$$T^*(\mu_i)(\mathbf{v}_j) = \mu_i(T(\mathbf{v}_j)) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

which agrees with $\lambda_i(\mathbf{v}_j)$. Since $T^*(\mu_i)$ and λ_i agree on the basis $\mathcal{B}$ we have $T^*(\mu_i) = \lambda_i$, and the map T^* is surjective as claimed.

(2) Suppose T is surjective. Let $\mu \in W^*$ be so that $T^*(\mu) = 0$. We show $\mu = 0$ as follows.

Let $\mathbf{w} \in W$. There exists $\mathbf{v} \in V$ so that $T(\mathbf{v}) = \mathbf{w}$. Then

$$\mu(\mathbf{w}) = \mu(T(\mathbf{v})) = (T^*(\mu))(\mathbf{v}) = 0$$

so that $\mu = 0$ as claimed and T^* is injective.

Problem 3: Infinite dimensionality and double duals. Let V be an infinite-dimensional $\mathbb{F}$-vector space with basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots\}$. Let $\varphi : V \to V^{**}$ be the linear map discussed in class given by

$$(\varphi(\mathbf{v}))(\lambda) := \lambda(\mathbf{v})$$

Is φ injective? Is φ surjective?

Solution: The map φ is injective, but not surjective. To see that φ is injective, let $\mathbf{v} \in V$ be so that $\varphi(\mathbf{v}) = 0$. We can write $\mathbf{v} = c_1\mathbf{e}_1 + \cdots + c_n\mathbf{e}_n$ for some n and some $c_i \in \mathbb{F}$. For any $1 \leq i \leq n$, we have a linear functional $\lambda_i \in V^*$ given by

$$\lambda_i(\mathbf{e}_j) := \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

Then

$$0 = \varphi(\mathbf{v})(\lambda_i) = \lambda_i(\mathbf{v}) = \lambda_i(c_1\mathbf{e}_1 + \cdots + c_n\mathbf{e}_n) = c_i$$

Since $c_i = 0$ for $i = 1, 2, \ldots, n$ this forces $\mathbf{v} = \mathbf{0}$ so that φ is injective.

To see that φ is not surjective, we argue as follows. For $i \geq 1$, let $\lambda_i \in V^*$ be the linear functional described above. The set $\{\lambda_1, \lambda_2, \ldots\}$ is linearly independent. Indeed, if $c_i \in \mathbb{F}$ are such that $c_1\lambda_1 + \cdots + c_n\lambda_n = 0$ then for $1 \leq i \leq n$ we have

$$0 = (c_1\lambda_1 + \cdots + c_n\lambda_n)(\mathbf{e}_i) = c_i$$

We may therefore extend the set $\{\lambda_1, \lambda_2, \ldots\}$ to a basis $\mathcal{B}$ of V^*. We have an element $f \in V^**$ defined by $f(\mu) = 1$ for all $\mu \in \mathcal{B}$. We claim that f is **not** in the image of φ. Indeed, let $\mathbf{v} \in V$ and write $\mathbf{v} = d_1\mathbf{e}_1 + \cdots + d_n\mathbf{e}_n$ for $d_i \in \mathbb{F}$. Then

$$0 = \lambda_{n+1}(d_1\mathbf{e}_1 + \cdots + d_n\mathbf{e}_n) = \lambda_{n+1}(\mathbf{v}) = \varphi(\mathbf{v})(\lambda_{n+1})$$

but $f(\lambda_{n+1}) = 1$. Thus $\varphi(\mathbf{v}) \neq f$.

Problem 4: Matrices, duals, and linear maps. Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{F})$ be an $n \times n$ matrix over $\mathbb{F}$. We use A to define a function

$$T_A : \mathbb{F}^n \to (\mathbb{F}^n)^*$$

by the rule $(T_A(\mathbf{v}))(\mathbf{w}) := \mathbf{v}^T A\mathbf{w}$. Prove that T_A is linear.

Solution: Let $c \in \mathbb{F}$, $\mathbf{v}, \mathbf{v}' \in V$ and $\mathbf{w} \in W$. We have

$$(T_A(\mathbf{v}+\mathbf{v}'))(\mathbf{w}) = (\mathbf{v}+\mathbf{v}')^T A\mathbf{w} = \mathbf{v}^T A\mathbf{w} + (\mathbf{v}')^T A\mathbf{w} = T_A(\mathbf{v})(\mathbf{w}) + T_A(\mathbf{v}')(\mathbf{w})$$

so that $T_A(\mathbf{v} + \mathbf{v}') = T_A(\mathbf{v}) + T_A(\mathbf{v}')$. Furthermore, we have

$$T_A(c\mathbf{v})(\mathbf{w}) = (c\mathbf{v})^T A\mathbf{w} = c(\mathbf{v}^T A\mathbf{w}) = cT_A(\mathbf{v})(\mathbf{w})$$

so that $T_A(c\mathbf{v}) = cT_A(\mathbf{v})$. We conclude that T_A is linear.

Problem 5: Internal Direct Sums. Let V be an $\mathbb{F}$-vector space and let $U, W \subseteq V$ be subspaces. Prove that the following are equivalent.

(1) Every vector $\mathbf{v} \in V$ can be written uniquely as a sum $\mathbf{v} = \mathbf{u} + \mathbf{w}$ for $\mathbf{u} \in U$ and $\mathbf{w} \in W$.

(2) The union $U \cup W$ spans V and we have $U \cap W = 0$.

In this case, we write $V = U \oplus W$.[1]

Solution: (1) $\Rightarrow$ (2) Since any $\mathbf{v} \in V$ may be written in the form $\mathbf{v} = \mathbf{u} + \mathbf{w}$ where $\mathbf{u} \in U$ and $\mathbf{w} \in W$, the set $U \cup W$ certainly spans V. If $\mathbf{v} \in U \cap W$ we have

$$\mathbf{0} = \mathbf{0} + \mathbf{0} = \mathbf{v} + (-\mathbf{v})$$

and since $\mathbf{v} \in U$ and $-\mathbf{v} \in W$ this forces $\mathbf{v} = \mathbf{0}$. Thus $U \cap W = 0$.

(2) $\Rightarrow$ (1) Let $\mathbf{v} \in V$. Since $U \cup W$ spans V, we may write $\mathbf{v} = \mathbf{u} + \mathbf{w}$ for some $\mathbf{u} \in U$ and $\mathbf{w} \in W$. Suppose $\mathbf{v} = \mathbf{u}' + \mathbf{w}'$ for some vectors $\mathbf{u}' \in U$ and $\mathbf{w}' \in W$. Then

$$\mathbf{0} = \mathbf{v} - \mathbf{v} = (\mathbf{u} - \mathbf{u}') + (\mathbf{w} - \mathbf{w}')$$

Since $\mathbf{0} = \mathbf{0} + \mathbf{0}$, this forces $\mathbf{u} - \mathbf{u}' = \mathbf{0}$ and $\mathbf{w} - \mathbf{w}' = \mathbf{0}$. That is, we have $\mathbf{u} = \mathbf{u}'$ and $\mathbf{w} = \mathbf{w}'$.

Problem 6: Determinants and transposition. Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{F})$ be a matrix and let A^T be its transpose. Prove that $\det A = \det A^T$.

Solution: First assume that A is invertible. We may write A is a product

$$A = E_1 E_2 \cdots E_r$$

of elementary matrices. This means that

$$A^T = (E_1 E_2 \cdots E_r)^T = E_r^T \cdots E_2^T E_1^T$$

It is certainly true that $\det E = \det E^T$ for any elementary matrix E. Thus

$$\begin{aligned} \det A = \det(E_1 \cdots E_r) &= \det E_1 \cdots \det E_r = \det E_r \cdots \det E_1 \\ &= \det E_r^T \cdots \det E_1^T = \det(E_r^T \cdots E_1^T) = \det A^T \end{aligned}$$

If A is not invertible, we have $\det A = 0$. We claim that A^T is not invertible, as well, so that $\det A^T = 0$. Indeed, the map A represents a linear map $T : \mathbb{F}^n \to \mathbb{F}^n$ whereas A^T represents the dual map $T^* : (\mathbb{F}^n)^* \to (\mathbb{F}_n)^*$. Problem 2 guarantees that T is invertible if and only if T^* is invertible.

[1] This is the 'internal' direct sum. Before, we saw 'external' direct sums. Starting with two little vector spaces U and W we contructed a new bigger vector space $U \oplus W = \{(\mathbf{u}, \mathbf{w}) : \mathbf{u} \in U, \mathbf{w} \in W\}$. In the 'internal' case we start with a big vector space V and decompose it as $V = U \oplus W$ where U, W are subspaces. If $V = U \oplus W$ is an internal direct sum, the map $\mathbf{u} + \mathbf{w} \mapsto (\mathbf{u}, \mathbf{w})$ is an isomophism to the external direct sum $U \oplus W$. Mathematicians use the same notation for, and don't make much distinction between, internal and external direct sums. This is a case of **notational abuse!**

Problem 7: Determinants and the plane. Consider the $\mathbb{R}$-vector space $V = \mathbb{R}^2$.

(1) If $T : V \to V$ is rotation counterclockwise by an angle θ, prove that $\det T = 1$.
(2) If $T : V \to V$ is reflection across some line L going through the origin, prove that $\det T = -1$.

Solution: (1) With respect to the standard (ordered) basis $\mathcal{B} = (\mathbf{e}_1, \mathbf{e}_2)$ of $\mathbb{R}^2$, the representing matrix for T is

$$[T]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

which has determinant $\det T = \cos^2\theta + \sin^2\theta = 1$.

(2) We make a clever choice of ordered basis $\mathcal{C} = (\mathbf{v}_1, \mathbf{v}_2)$ by letting $\mathbf{v}_1 \in L$ be a nonzero vector in L and letting $\mathbf{v}_2$ be a nonzero vector perpendicular to L. Then $T(\mathbf{v}_1) = \mathbf{v}_1$ and $T(\mathbf{v}_2) = -\mathbf{v}_2$ so that

$$[T]_{\mathcal{C}}^{\mathcal{C}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

which as determinant $\det T = -1$.

Problem 8: Determinants and block matrices. Let A be an $n \times n$ matrix, let B be an $n \times m$ matrix, and let C be an $m \times m$ matrix. Prove the identity

$$\det \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det(A) \cdot \det(C)$$

where 0 denotes a block of zeroes of size $m \times n$.

Solution: Let $E_1, \dots, E_r$ be elementary matrices such that $E_1 \cdots E_r A$ is in RREF. Let $E'_1, \dots, E'_s$ be elementary matrices such that $E'_1 \cdots E'_s B$ is in RREF. Then

$$\begin{pmatrix} E_1 & 0 \\ 0 & I_m \end{pmatrix} \cdots \begin{pmatrix} E_r & 0 \\ 0 & I_m \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & E'_1 \end{pmatrix} \cdots \begin{pmatrix} I_n & 0 \\ 0 & E'_s \end{pmatrix} \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$
$$= \begin{pmatrix} E_1 \cdots E_r A & B \\ 0 & E'_1 \cdots E'_s C \end{pmatrix}$$

is an upper triangular matrix, and the $r + s$ initial factors on the LHS are elementary matrices with

$$\det \begin{pmatrix} E_i & 0 \\ 0 & I_m \end{pmatrix} = \det E_i \qquad \det \begin{pmatrix} I_n & 0 \\ 0 & E'_j \end{pmatrix} = \det E'_j$$

Problem 9: (Optional; not to be handed in.) Let $x_1, x_2, \ldots, x_n$ be variables. Prove the *Vandermonde identity*

$$\det \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{pmatrix} = \prod_{1 \le i < j \le n} (x_i - x_j)$$