Math 31CH HW 5
Due May 10 at 11:59 pm by Gradescope Submission
6.2.1, 6.2.2, 6.2.3, 6.3.1, 6.3.3, 6.3.4, 6.3.5, 6.3.6, 6.3.7, 6.3.8, 6.3.11, 6.3.12

Professor Bennett Chow

EXERCISES FOR SECTION 6.2

Exercise 6.2.1: Set up each of the following integrals of form fields over parametrized domains as an ordinary multiple integral, and compute it.

a. $\int_{[\gamma(I)]} x\,dy + y\,dz$, where $I = [-1, 1]$, and $\gamma(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \\ t \end{pmatrix}$.

b. $\int_{[\gamma(U)]} x_1\,dx_2 \wedge dx_3 + x_2\,dx_3 \wedge dx_4$, where $U = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \middle| \; 0 \le u, v\,;\; u + v \le 2 \right\}$, $\gamma\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} uv \\ u^2 + v^2 \\ u - v \\ \ln(u + v + 1) \end{pmatrix}$.

Solution to 6.2.1: For part a

$$\begin{aligned}
\int_{[\gamma(U)]} x\,dy + y\,dz &= \int_{-1}^{1} x\,dy + y\,dz \left(P_{\begin{pmatrix} \sin t \\ \cos t \\ t \end{pmatrix}} \begin{bmatrix} \cos t \\ -\sin t \\ 1 \end{bmatrix} \right) dt \\
&= \int_{-1}^{1} \sin t(-\sin t) + \cos t(1)\,dt \\
&= \int_{-1}^{1} \frac{1}{2}\cos 2t - \frac{1}{2} + \cos t\,dt \\
&= \left[\frac{\sin 2t}{4} - \frac{1}{2}t + \sin t \right]_{-1}^{1} \\
&= \frac{\sin 2}{2} + 2\sin 1 - 1
\end{aligned}$$

For part b

$$\int_{[\gamma(U)]} x_1\, dx_2 \wedge\, dx_3 + x_2\, dx_3 \wedge\, dx_4$$

$$= \int_0^2 \int_0^{2-u} x_1\, dx_2 \wedge\, dx_3 + x_2\, dx_3 \wedge\, dx_4 \left(P_{\begin{pmatrix} uv \\ u^2+v^2 \\ u-v \\ \ln(u+v+1) \end{pmatrix}} \begin{bmatrix} v & u \\ 2u & 2v \\ 1 & -1 \\ \frac{1}{u+v+1} & \frac{1}{u+v+1} \end{bmatrix} \right) dv\, du$$

$$= \int_0^2 \int_0^{2-u} uv(-2u-2v) + (u^2+v^2)\frac{2}{u+v+1}\, dv\, du$$

$$= \frac{64}{45} - \frac{4}{3}\ln 3$$

The last step was calculated using Matlab.

```
>> syms u v
>> f = -2*u*v*(u+v) + 2*(u^2+v^2)/(1+u+v)
>> int(int(f,v,0,2-u),u,0,2)
   ans = 64/45 - (4*log(3))/3
```

Exercise 6.2.2: Repeat Exercise 6.2.1, for the following.

a. $\displaystyle\int_{[\gamma(U)]} x\, dy \wedge dz$, where $U = [-1,1] \times [-1,1]$, and $\gamma\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u^2 \\ u+v \\ v^3 \end{pmatrix}$.

b. $\displaystyle\int_{[\gamma(U)]} x_2\, dx_1 \wedge dx_3 \wedge dx_4$, where $U = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} \middle| \; 0 \le u, v, w\,;\; u+v+w \le 3 \right\}$, and $\gamma\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} uv \\ u^2+w^2 \\ u-v \\ w \end{pmatrix}$.

Solution to 6.2.2: For part a

$$\begin{aligned}
\int_{[\gamma(U)]} x\, dy \wedge dz &= \int_{-1}^{1}\int_{-1}^{1} x\, dy \wedge dz \left(P_{\begin{pmatrix} u^2 \\ u+v \\ v^3 \end{pmatrix}} \begin{bmatrix} 2u & 0 \\ 1 & 1 \\ 0 & 3v^2 \end{bmatrix} \right) dv\, du \\
&= \int_{-1}^{1}\int_{-1}^{1} 3u^2v^2\, dv\, du \\
&= \int_{-1}^{1} 2u^2\, du \\
&= \frac{4}{3}
\end{aligned}$$

For part b

$$\begin{aligned}
&\int_{[\gamma(U)]} x_2\, dx_1 \wedge dx_3 \wedge dx_4 \\
&= \int_0^3 \int_0^{3-u} \int_0^{3-u-v} x_2\, dx_1 \wedge dx_3 \wedge dx_4 \left(P_{\begin{pmatrix} uv \\ u^2+w^2 \\ u-v \\ w \end{pmatrix}} \begin{bmatrix} v & u & 0 \\ 2u & 0 & 2w \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) dw\, dv\, du \\
&= \int_0^3 \int_0^{3-u} \int_0^{3-u-v} (u^2+w^2)(-v-u)\, dw\, dv\, du \\
&= -\, 12.15
\end{aligned}$$

The last step was solved using Wolfram Alpha.

Exercise 6.2.3: Set up each of the following integrals of form fields over parametrized domains as an ordinary multiple integral.

a. $\int_{[\gamma(U)]} (x_1 + x_4)\, dx_2 \wedge dx_3$, where $U = \left\{ \begin{pmatrix} u \\ v \end{pmatrix} \middle| \; |v| \le u \le 1 \right\}$, and where $\gamma \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} e^u \\ e^{-v} \\ \cos(u) \\ \sin(v) \end{pmatrix}$.

b. $\int_{[\gamma(U)]} x_2\, x_4\, dx_1 \wedge dx_3 \wedge dx_4$, where $U = \left\{ \begin{pmatrix} u \\ v \\ w \end{pmatrix} \middle| \; (w-1)^2 \ge u^2 + v^2,\ 0 \le w \le 1 \right\}$, and where $\gamma \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} u+v \\ u-v \\ w+v \\ w-v \end{pmatrix}$.

Solution to 6.2.3: For part a

$$\int_{[\gamma(U)]} (x_1 + x_4)\, dx_2 \wedge dx_3 = \int_0^1 \int_{-u}^{u} (x_1 + x_4)\, dx_2 \wedge dx_3 \left(P_{\begin{pmatrix} e^u \\ e^{-v} \\ \cos u \\ \sin v \end{pmatrix}} \begin{bmatrix} e^u & 0 \\ 0 & -e^{-v} \\ -\sin u & 0 \\ 0 & \cos v \end{bmatrix} \right) du\, dv$$

$$= -\int_0^1 \int_{-u}^{u} (e^u + \sin v)(e^{-v} \sin u)\, du\, dv$$

For part b

$$\int_{[\gamma(U)]} x_2\, x_4\, dx_1 \wedge dx_3 \wedge dx_4$$

$$= \int_0^1 \int_{w-1}^{1-w} \int_{-\sqrt{(w-1)^2 - v^2}}^{-\sqrt{(w-1)^2 - v^2}} x_2\, x_4\, dx_1 \wedge dx_3 \wedge dx_4 \left(P_{\begin{pmatrix} u+v \\ u-v \\ w+v \\ w-v \end{pmatrix}} \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \right) du\, dv\, dw$$

$$= \int_0^1 \int_{w-1}^{1-w} \int_{-\sqrt{(w-1)^2 - v^2}}^{-\sqrt{(w-1)^2 - v^2}} 2(u-v)(w-v) dx_1\, du\, dv\, dw$$

Exercise 6.3.1: Is the constant vector field $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ a tangent vector field defining an orientation of the line of equation $x+y=0$? How about the line of equation $x-y=0$?

Solution to 6.3.1: The line of equation $x+y=0$ has a tangent vector of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ so the constant vector field $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ does not define an orientation.

The line of equation $x-y=0$ has a tangent vector of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ so the constant vector field $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ does define an orientation.

Exercise 6.3.3: Does any constant vector field define an orientation of the unit sphere in $\mathbb{R}^3$?

Solution to 6.3.3: No constant vector fields defines an orientation of the unit sphere since a constant vector field is not transversal to the unit sphere. This is because there exists a point in the unit sphere such that $x \in v^{\perp}$ for all v.

Exercise 6.3.4:
Find a vector that orients the curve given by $x + x^2 + y^2 = 2$.

Solution to 6.3.4: Using implicit differentiation,

$$\begin{aligned} x + x^2 + y^2 = 2 &\implies dx + 2x\,dx + 2y\,dy = 0 \\ &\implies 2y\,dy = (1 + 2x)\,dx \\ &\implies \frac{dy}{dx} = -\frac{1 + 2x}{2y} \end{aligned}$$

Therefore the following orients the curve in the counterclockwise direction.

$$t(x, y) = \begin{bmatrix} -2y \\ 1 + 2x \end{bmatrix}$$

Exercise 6.3.5:
Which of the vector fields

$$\begin{bmatrix}1\\1\\1\end{bmatrix}, \quad \begin{bmatrix}-1\\1\\1\end{bmatrix}, \quad \begin{bmatrix}-1\\-1\\1\end{bmatrix}, \quad \begin{bmatrix}-1\\-1\\-1\end{bmatrix}$$

define an orientation of the plane $P \subset \mathbb{R}^3$ of equation $x + y + z = 0$, and among these, which pairs define the same orientation?

Solution to 6.3.5: None of the four vectors solve the equation, so all of the vector fields orient the plane. Choosing the basis vectors we can calculate the orientation of the vector fields.

$$\begin{bmatrix}0\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}$$

$$\det \begin{bmatrix}1 & 0 & 1\\1 & 1 & 0\\1 & -1 & -1\end{bmatrix} = (-1) + (-2) = -3$$

$$\det \begin{bmatrix}-1 & 0 & 1\\1 & 1 & 0\\1 & -1 & -1\end{bmatrix} = (1) + (-2) = -1$$

$$\det \begin{bmatrix}-1 & 0 & 1\\-1 & 1 & 0\\1 & -1 & -1\end{bmatrix} = (1) + (0) = 1$$

$$\det \begin{bmatrix}-1 & 0 & 1\\-1 & 1 & 0\\-1 & -1 & -1\end{bmatrix} = (1) + (2) = 3$$

So $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}$ are the same orientation and $\begin{bmatrix}-1\\-1\\1\end{bmatrix}, \begin{bmatrix}-1\\-1\\-1\end{bmatrix}$ are another orientation.

Exercise 6.3.6: Find a vector field that orients the surface $S \subset \mathbb{R}^3$ given by $x^2 + y^3 + z = 1$.

Solution to 6.3.6: The gradient of the locus orients the surface

$$\nabla f(x, y, z) = \begin{bmatrix} 2x \\ 3y^2 \\ 1 \end{bmatrix}$$

Exercise 6.3.7: Let V be the plane of the equation $x + 2y - z = 0$. Show that the bases

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \vec{w}_1 = \begin{bmatrix} 2 \\ -3 \\ -4 \end{bmatrix}, \ \vec{w}_2 = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

give the same orientation

Solution to 6.3.7: The change of basis matrix is

$$P_{w \to v} = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$$

This matrix has a determinant of 7, so both basis have the same orientation.

Exercise 6.3.8: Let P be the plane of equation $x+y+z=0$.

a. Of the three bases

$$\begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}, \qquad \begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}, \qquad \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\-1\\1\end{bmatrix}$$

which gives a different orientation than the other two?

b. Find a normal vector to P that gives the same orientation as that basis.

Solution to 6.3.8:

The change of base matrix from the first to the second base is

$$P_{1\to 2} = \begin{bmatrix}-1 & -1\\ 0 & 1\end{bmatrix}$$

The change of base matrix from the second to the third base is

$$P_{2\to 3} = \begin{bmatrix}-1 & -1\\ 1 & 0\end{bmatrix}$$

$\det P_{1\to 2} = -1$ and $\det P_{2\to 3} = 1$, so the first basis has a different orientation than the other two. The orientation $\begin{bmatrix}1\\1\\1\end{bmatrix}$ has the same orientation as the first base since

$$\det \begin{bmatrix}1 & 1 & 0\\ 1 & 0 & 1\\ 1 & -1 & -1\end{bmatrix} = (1) - (-2) = 3 > 0$$

Exercise 6.3.11: Let $S \subset \mathbb{R}^4$ be the locus given by the equations $x_1^2 - x_2^2 = x_3$ and $2x_1x_2 = x_4$.

a. Show that S is a surface.

b. Find a basis for the tangent space to S at the origin that is direct for the orientation given by Proposition 6.3.9.

Solution to 6.3.11:

a. The locus of the surface is

$$f(x_1, x_2, x_3, x_4) = \begin{bmatrix} x_1^2 - x_2^2 - x_3 \\ 2x_1x_2 - x_4 \end{bmatrix}$$

. The derivative is

$$Df(x_1, x_2, x_3, x_4) = \begin{bmatrix} 2x_1 & -2x_2 & -1 & 0 \\ 2x_2 & 2x_1 & 0 & -1 \end{bmatrix}$$

From Theorem 3.1.10, S is a smooth manifold since the derivative is onto.

b. The tangent space consists of points where the derivative is zero, so the following work as basis vectors .

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

At the origin,

$$Df(0) = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Using the orientation from Proposition 6.3.9,

$$\det \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} = 1$$

Thus this basis is direct at the origin.

Exercise 6.3.12: Consider the manifold $M \subset \mathbb{R}^4$ of equation $x_1^2 + x_2^2 + x_3^2 - x_4 = 0$. Find a basis for the tangent space to M at the point $\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$ that is direct for the orientation given by Proposition 6.3.9.

Solution to 6.3.12: The derivative of the locus is

$$Df(x_1, x_2, x_3, x_4) = \begin{bmatrix}2x_1 & 2x_2 & 2x_3 & -1\end{bmatrix}$$

At the point it is

$$Df(1, 0, 0, 1) = \begin{bmatrix}2 & 0 & 0 & -1\end{bmatrix}$$

Thus the tangent space has a basis of

$$\begin{bmatrix}-1\\0\\0\\2\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\end{bmatrix}$$

The basis is direct since

$$\det\begin{bmatrix}2 & -1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\\-1 & 2 & 0 & 0\end{bmatrix} = 2(2) - (1) = 3$$