Math 20D HW1

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April 4, 2022

Section 1.1

Classify the following as an ODE/PDE, give the order, and indicate the independent and dependent variables. If the equation is an ODE, indicate whether the equation is linear or nonlinear.

Problem 2

$$\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

1. Linear ODE with respect to x
2. Order 2
3. Independent variables: x
4. Dependent variables: y

Problem 3

$$\frac{dy}{dx} = \frac{y(2-3x)}{x(1-3y)}$$

1. Nonlinear ODE with respect to x
2. Order 1
3. Independent variables: x
4. Dependent variables: y

Problem 4

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

1. PDE
2. Order 2
3. Independent variables: x, y
4. Dependent variables: u

Problem 6

$$\frac{dx}{dt} = k(4-x)(1-x)$$

1. Nonlinear ODE with respect to t
2. Order 1
3. Independent variables: t
4. Dependent variables: x

Problem 8

$$\sqrt{1-y}\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0$$

1. Nonlinear ODE with respect to x
2. Order 2
3. Independent variables: x
4. Dependent variables: y

Problem 16 Write the differential equation that fits the description: The rate of change of A at time t is proportional to the square of A at time t.

Solution:

$$\frac{dA}{dt} = kA^2$$

Section 1.2

Problem 1

1. Show that $\phi(x) = x^2$ is a solution to
$$x\frac{dy}{dx} = 2y$$
2. Show that $\phi(x) = e^x - x$ is a solution to
$$\frac{dy}{dx} + y^2 = e^{2x} + (1-2x)e^x + x^2 - 1$$
3. Show that $\phi(x) = x^2 - x^{-1}$ is a solution to
$$x^2\frac{d^2y}{dx^2} = 2y$$

Solution:

1. Substituting $\phi(x)$ in for y yields an equation that is true for all $x \in (-\infty, \infty)$.
$$x\frac{dy}{dx} = 2y \implies x(2x) = 2(x^2) \implies 2x^2 = 2x^2$$
2. Substituting $\phi(x)$ in for y yields an equation that is true for all $x \in (-\infty, \infty)$.
$$\begin{aligned}\frac{dy}{dx} + y^2 = e^{2x} + (1-2x)e^x + x^2 - 1 &\implies (e^x - 1) + (e^x - x)^2 = e^{2x} + (1-2x)e^x + x^2 - 1\\ &\implies (e^x - 1) + (e^x - x)^2 = (e^x - x)^2 + e^x - 1\\ &\implies (e^x - 1) + (e^x - x)^2 = (e^x - 1) + (e^x - x)^2\end{aligned}$$
3. Substituting $\phi(x)$ in for y yields an equation that is true for all $x \in (0, \infty)$.
$$\begin{aligned}x^2\frac{d^2y}{dx^2} = 2y &\implies x^2(2 - 2x^{-3}) = 2(x^2 - x^{-1})\\ &\implies 2x^2 - 2x^{-1} = 2x^2 - 2x^{-1}\end{aligned}$$

Problem 4

Determine if $x = 2\cos t - 3\sin t$ is a solution to $x'' + x = 0$

Solution: It is a solution since
$$\begin{aligned}x'' + x = 0 &\implies (-2\cos t + 3\sin t) + (2\cos t - 3\sin t) = 0\\ &\implies 0 = 0\end{aligned}$$

Problem 10

Determine if $y - \ln(y) = x^2 + 1$ is an implicit solution to $\frac{dy}{dx} = \frac{2xy}{y-1}$

Solution: It is a solution since
$$\begin{aligned}y - \ln(y) = x^2 + 1 &\implies \frac{dy}{dx} - \frac{1}{y}\frac{dy}{dx} = 2x\\ &\implies \frac{dy}{dx}(1 - \frac{1}{y}) = 2x\\ &\implies \frac{dy}{dx}(\frac{y-1}{y}) = 2x\\ &\implies \frac{dy}{dx} = \frac{2xy}{y-1}\end{aligned}$$

Problem 22

Verify that $\phi(x) = c_1e^x + c_2e^{-2x}$ is a solution to
$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0$$
and find c_1 and c_2 that satisfies

1. $y(0) = 2$, $y'(0) = 1$
2. $y(1) = 1$, $y'(1) = 0$

Solution: Substituting in ϕ yields

$$\begin{aligned}\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0 &\implies (c_1e^x + 4c_2e^{-2x}) + (c_1e^x - 2c_2e^{-2x}) - 2(c_1e^x + c_2e^{-2x}) = 0 \\ &\implies 0 = 0\end{aligned}$$

1.
$$\begin{aligned}\phi(0) = 2 &\implies c_1e^0 + c_2e^0 = 2 \implies c_1 + c_2 = 2 \\ \phi'(0) = 1 &\implies c_1e^0 - 2c_2e^0 = 1 \implies c_1 - 2c_2 = 1\end{aligned}$$

   Solving this system of equations yields $c_1 = \frac{5}{3}$ and $c_2 = \frac{1}{3}$.

2.
$$\begin{aligned}\phi(1) = 1 &\implies c_1e^1 + c_2e^{-2} = 1 \implies ec_1 + e^{-2}c_2 = 1 \\ \phi'(1) = 0 &\implies c_1e^1 - 2c_2e^{-2} = 0 \implies ec_1 - 2e^{-2}c_2 = 0\end{aligned}$$

   Solving this system of equations yields $c_1 = \frac{2}{3e}$ and $c_2 = \frac{e^2}{3}$.

Problem 24

Does $\frac{dy}{dt} - ty = \sin^2 t$ with $y(\pi) = 5$ have a unique solution?

Solution: $f(t, y) = ty + \sin^2 t$ is always continuous and $\frac{\partial f}{\partial y} = t$ is also always continuous so the differential equation has a unique solution.

Problem 26

Does $\frac{dx}{dt} + \cos x = \sin t$ with $x(\pi) = 0$ have a unique solution?

Solution: $f(t, x) = \sin t - \cos x$ is always continuous and $\frac{\partial f}{\partial x} = \sin x$ is also always continuous so the differential equation has a unique solution.

Problem 28

Does $\frac{dy}{dx} = 3x - \sqrt[3]{y-1}$ with $y(2) = 1$ have a unique solution?

Solution: $f(x, y) = 3x - \sqrt[3]{y-1}$ is always continuous but $\frac{\partial f}{\partial y} = -\frac{1}{3}(y-1)^{-\frac{2}{3}}$ is not continuous for $y_0 = 1$ so the theorem does not apply.

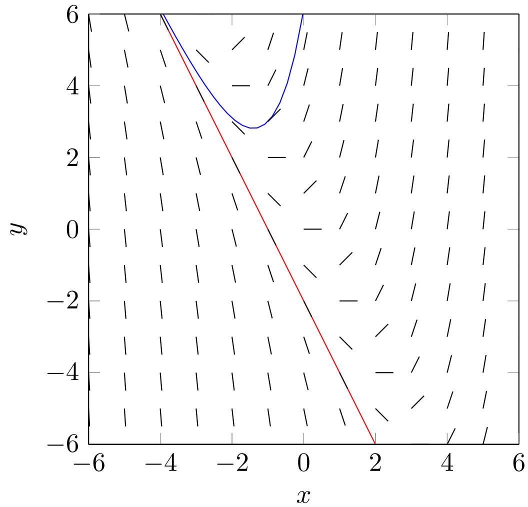
Section 1.3

Problem 2

For $\frac{dy}{dx} = 2x + y$,

1. Find and sketch the solution through $(0, -2)$
2. Sketch the solution through $(-1, 3)$
3. What can you say about the solution as $x \to \infty$ and $x \to -\infty$?

$\frac{dy}{dx} = 2x + y$



Solution:

1. The solution through $(0, -2)$ is shown by the red curve. The slope is

$$\frac{dy}{dx}|_{x=0} = 2(0) + y = y = -2$$

   In order for the point to pass through $(0, -2)$, the y-intercept must be -2. Therefore, the equation of the line is $y = -2x - 2$.

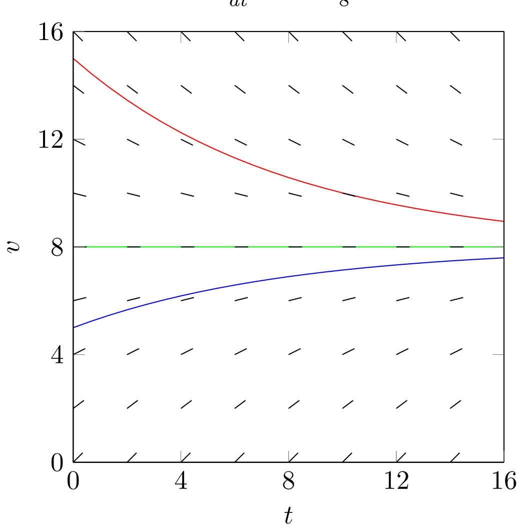
2. The solution through $(-1, 3)$ is shown by the blue curve.
3. The solution goes to ∞ for both $x \to \infty$ and $x \to -\infty$.

Problem 3

For $\frac{dv}{dt} = 1 - \frac{v}{8}$, sketch the solutions for $v(0) = 5, 8, 15$ Why is $v = 8$ the terminal velocity?

Solution:

$\frac{dv}{dt} = 1 - \frac{v}{8}$



The red curve is the solution for $y(0) = 15$, the green curve is the solution for $y(0) = 8$, and the blue curve is the solution for $y(0) = 5$. 8 is the terminal velocity because the velocity approaches 8 for all solutions.

Problem 6 Consider

$$\frac{dy}{dx} = x + \sin y$$

1. What is the slope at $(1, \frac{\pi}{2})$?

2. Argue that the solution curve increases for $x > 1$.

3. Show that all solutions satisfy

$$\frac{d^2y}{dx^2} = 1 + x\cos y + \frac{1}{2}\sin 2y$$

4. Prove that the curve through $(0, 0)$ has a relative minimum at $(0, 0)$.

Solution:

1. The slope is zero.

$$\frac{dy}{dx} = x + \sin y = 1 + \sin\frac{\pi}{2} = 2$$

2. $\sin y$ can be at minimum -1, so $x + \sin y$ must be positive for $x > 1$.

3. Using the chain rule

$$\begin{aligned}\frac{d^2y}{dx^2} &= 1 + \cos(y)\frac{dy}{dx} \\ &= 1 + \cos(y)(x + \sin y) \\ &= 1 + x\cos y + \sin y\cos y \\ &= 1 + x\cos y + \frac{1}{2}\sin 2y\end{aligned}$$

4. Since $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 1 > 0$ at $(0, 0)$, the curve has a relative minimum at this point.

Problem 8

$$\frac{dx}{dt} = t^3 - x^3$$

1. What is the velocity at $x = 1$ and $t = 2$?

2. Show that the acceleration is

$$\frac{d^2x}{dt^2} = 3t^2 - 3t^3x^2 + 3x^5$$

3. Can a particle at $x = 2$ and $t = 2.5$ reach $x = 1$ later?

Solution:

1.

$$\frac{dx}{dt} = 2^3 - 1^3 = 7$$

2. Using implicit differentiation,

$$\frac{d^2x}{dt^2} = 3t^2 - 3x^2\frac{dx}{dt} = 3t^2 - 3x^2(t^3 - x^3) = 3t^2 - 3t^3x^2 - x^5$$

3. Whenever $t > x$, $t^3 - x^3 > 0$ so x will increase. If $t < x$, x will decrease but only until $t = x$. Since the values t_0 and x_0 are already greater than 1, x cannot reach a value of 1 at a later time.

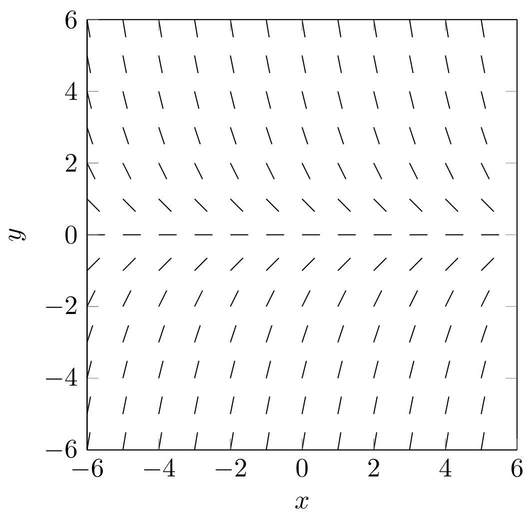
Problem 18

What is the behavior of a solution to the following equation as $x \to \infty$?

$$\frac{dy}{dx} = -y$$

Solution:

$\frac{dy}{dx} = -y$



$\frac{dy}{dx}$ is negative above the x-axis and positive below it, so solutions will tend towards $y = 0$ as $x \to \infty$.

Section 2.2

Problem 3 Is this equation separable?

$$\frac{ds}{dt} = t\ln(s^{2t}) + 8t^2$$

Solution: It is separable and can be written as

$$\frac{1}{\ln(s^2)+8}\,ds = t^2\,dt$$

Problem 5 Is this equation separable?

$$(xy^2 + 3y^2)dy - 2xdx = 0$$

Solution: It is separable and can be written as

$$y^2\,dy = \frac{2x}{x+3}\,dx$$

Problem 9 Solve the equation

$$\frac{dx}{dt} = \frac{t}{xe^{t+2x}}$$

Solution: Separating the equation yields

$$\begin{aligned}
\frac{dx}{dt} = \frac{t}{xe^{t+2x}} &\implies (xe^{t+2x})\frac{dx}{dt} = t\\
&\implies (xe^{2x})\,dx = te^{-t}\,dt\\
&\implies \int (xe^{2x})\,dx = \int te^{-t}\,dt\\
&\implies \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx = -te^{-t} - \int -e^{-t}\,dt\\
&\implies \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} = -te^{-t} - e^{-t} + C\\
&\implies e^{2x}(2x-1) + 4e^{-t}(t+1) = C
\end{aligned}$$

$\frac{1}{xe^{2x}} \neq 0$, so there are no constant solutions.

Problem 11 Solve the equation

$$x\frac{dv}{dx} = \frac{1-4v^2}{3v}$$

Solution: Separating the equation yields

$$\begin{aligned}
x\frac{dv}{dx} = \frac{1-4v^2}{3v} &\implies \frac{3v}{1-4v^2}\,dv = \frac{1}{x}\,dx\\
&\implies \int 3v(1-4v^2)^{-1}\,dv = \int \frac{1}{x}\,dx\\
&\implies -\frac{3}{8}\ln(1-4v^2) = \ln x + C\\
&\implies 1-4v^2 = Cx^{\frac{8}{3}}\\
&\implies v = \pm\frac{\sqrt{1-Cx^{\frac{8}{3}}}}{2}
\end{aligned}$$

$\frac{1-4v^2}{3v} = 0$ at $v = \frac{1}{2}$, which is a constant solution.

Problem 12 Solve the equation

$$\frac{dy}{dx} = \frac{\sec^2 y}{1+x^2}$$

Solution: Separating the equation yields

$$\begin{aligned}\frac{dy}{dx} = \frac{\sec^2 y}{1+x^2} &\implies \cos^2 y\, dy = \frac{1}{1+x^2}\, dx \\ &\implies \int \frac{1+\cos 2y}{2}\, dy = \int \frac{1}{1+x^2}\, dx \\ &\implies \frac{1}{2}y + \frac{1}{4}\sin 2y = \tan^{-1} x + C\end{aligned}$$

$\sec^2 y \neq 0$, so there are no constant solutions.
Problem 18 Solve $y' = x^3(1-y)$ with $y(0) = 3$
Solution: Separating the equation yields

$$\begin{aligned}y' = x^3(1-y) &\implies \frac{1}{1-y}\, dy = x^3\, dx \\ &\implies -\ln(y) = \frac{1}{4}x^4 + C \\ &\implies y = e^{-\frac{1}{4}x^4 + C}\end{aligned}$$

Pluggin in the initial value yields

$$3 = e^C \implies C = \ln(3)$$

Therefore, the solution is

$$y = e^{-\frac{1}{4}x^4 + \ln(3)}$$

Problem 20 Solve the equation at $y(1) = 1$

$$x^2 \frac{dy}{dx} = \frac{4x^2 - x - 2}{(x+1)(y+1)}$$

Solution: Separating the equation yields

$$\begin{aligned}x^2 \frac{dy}{dx} = \frac{4x^2 - x - 2}{(x+1)(y+1)} &\implies (y+1)\, dy = \frac{4x^2 - x - 2}{x^2(x+1)}\, dx \\ &\implies \int y + 1\, dy = \int -\frac{2}{x^2} + \frac{3}{x+1} + \frac{1}{x}\, dx \\ &\implies \frac{1}{2}y^2 + y = \frac{2}{x} + 3\ln(x+1) + \ln(x) + C\end{aligned}$$

Plugging in the initial value yields

$$\frac{3}{2} = 2 + 3\ln(2) + C \implies C = -\frac{1}{2} - 3\ln(2)$$

Therefore the implicit solution is

$$\frac{1}{2}y^2 + y = \frac{2}{x} + 3\ln(x+1) + \ln(x) - \frac{1}{2} - 3\ln(2)$$

Problem 22 Solve the equation at $y(0) = 2$

$$x^2 dx + 2y dy = 0$$

Solution: Separating the equation yields

$$\begin{aligned}x^2 dx + 2y dy = 0 &\implies \int 2y dy = \int -x^2 dx \\ &\implies y^2 = -\frac{1}{3}x^3 + C\end{aligned}$$

Plugging in the initial value gives

$$4 = 0 + C \implies C = 4$$

Therefore the implicit solution is

$$y^2 = -\frac{1}{3}x^3 + 4$$

Problem 26 Solve the equation at $y(0) = 1$

$$\sqrt{y}dx + (1+x)dy = 0$$

Solution: Separating the equation yields

$$\begin{aligned}
\sqrt{y}dx + (1+x)dy = 0 &\implies \int y^{-\frac{1}{2}}\,dy = \int -\frac{1}{1+x}dx \\
&\implies 2\sqrt{y} = -\ln(1+x) + C \\
&\implies y = \frac{(\ln(x+1)+C)^2}{4}
\end{aligned}$$

Plugging in the initial value gives

$$1 = \frac{C^2}{4} \implies C = 2$$

Therefore the solution is

$$y = \frac{(-\ln(x+1)+2)^2}{4}$$

Problem 30

1. Separate $\frac{dy}{dx} = (x-3)(y+1)^{\frac{2}{3}}$
2. Show that $y = -1$ satisfies the original equation
3. Show that there is no choice of C that will yield $y = -1$.

Solution:

1. Separating yields

$$\begin{aligned}
\frac{dy}{dx} = (x-3)(y+1)^{\frac{2}{3}} &\implies \int (y+1)^{-\frac{2}{3}}\,dy = \int (x-3)\,dx \\
&\implies 3(y+1)^{\frac{1}{3}} = \frac{1}{2}x^2 - 3x + C\,dx \\
&\implies y = -1 + (\frac{x^2}{6} - x + C)^3
\end{aligned}$$

2. At $y = -1$, $\frac{dy}{dx} = (x-3)\cdot 0 = 0$, which is true for constants.
3. For $y = 1$, $(\frac{x^2}{6} - x + C)^3 = 0$ has to be true for all x, but this is not the case.