Math 100b Winter 2025 Homework 7

Due 3/7/2025 at 5pm on Gradescope

Reading

Reading: Artin Chapter 3. Problems 3-6 below involve the basic definitions of vector space over a field, linear independence of a set of vectors, and basis of a vector space. We will cover all of the necessary material in class on Monday 3/3, or you can read it in Artin Sections 3.3-3.4.

Assigned Problems

1. Find $\gcd(2+4i, 5+5i)$ in the ring of Guassian integers $\mathbb{Z}[i]$, and justify your answer.

2. Let R be a UFD and let F be the field of fractions of R. Think of $R[x]$ as a subring of $F[x]$.

(a) Suppose that $f(x) = a_0 + a_1x + \cdots + a_nx^n \in R[x]$ and that (i) $\gcd(a_0, a_1, \ldots a_n) = 1 \in R$ and (ii) $f(x)$ is irreducible as an element of $F[x]$. Show that $f(x)$ is irreducible in $R[x]$.

(b) Show that $yx + y^2 + 1$ is irreducible in the polynomial ring $\mathbb{Q}[x, y]$. (Hint: write $\mathbb{Q}[x, y] = (\mathbb{Q}[y])[x]$ and take $R = \mathbb{Q}[y]$, $F = \mathbb{Q}(y)$ the field of fractions of R in part (a).

3. Let V be a vector space over a field F, such that $\{v_1, v_2, \ldots, v_n\}$ is a basis for V.

(a) Suppose that $0 \neq w \in V$ is a nonzero vector, and write $w = a_1v_1 + \cdots + a_nv_n$ for some $a_i \in F$. Suppose that i is any index such that $a_i \neq 0$. Prove that $\{v_1, v_2, \ldots, v_{i-1}, w, v_{i+1}, \ldots, v_n\}$ is also a basis for V. (This result is known as the "replacement lemma" because can replace some element of the basis with w and get another basis.)

(b) Suppose that $\{w_1, w_2, \ldots, w_m\}$ is a linearly independent set of vectors in V with $m \leq n$. Show that, possibly after rearranging the basis vectors v_i, then $\{w_1, w_2, \ldots, w_i, v_{i+1}, \ldots, v_n\}$ is a basis of V for all $1 \leq i \leq m$. In other words, we can replace the elements of the (rearranged) basis $\{v_i\}$ one by one with the w_i and still have a basis.

4. Using problem 3, show that if V is a vector space over F that has a basis with n elements, then every basis of V has n elements.

5. Let V be the set of all functions $\mathbb{R} \to \mathbb{R}$, which is an abelian group with pointwise addition $[f+g](x) = f(x)+g(x)$. Make V into a vector space over $\mathbb{R}$, where for $a \in \mathbb{R}$ and $f \in V$ we define $[af](x) = af(x)$.

Show that the set of functions $\{x^2, \sin x, \cos x, e^x\}$ is linearly independent over $\mathbb{R}$.

6. Let F be a field and let $F(x)$ be the field of rational functions in one variable— that is, the field of fractions of $F[x]$. Consider $F(x)$ as a vector space over F, where for $a \in F$, $\frac{f(x)}{g(x)} \in F(x)$, the scalar product $a \cdot \frac{f(x)}{g(x)} = \frac{af(x)}{g(x)}$ is just the product in $F(x)$.

Show that the set of elements

$$\left\{\frac{1}{x-a} \,\middle|\, a \in F\right\},$$

that is, the set of all reciprocals of monic degree 1 polynomials in $F[x]$, is an F-linearly independent subset of $F(x)$.