Math 181A: Homework 5

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Problem 1

The fisher information of X is

$$\log f_X(X;\theta) = \log(\theta+1) + \theta\log(X)$$

$$\frac{\partial}{\partial\theta}\log f_X(X;\theta) = \frac{1}{\theta+1} + \log(X)$$

$$\frac{\partial^2}{\partial\theta^2}\log f_X(X;\theta) = -\frac{1}{(\theta+1)^2}$$

$$I(\theta) = \frac{1}{(\theta+1)^2}$$

The asymptotic variance is

$$\frac{1}{nI(\theta)} = \frac{(\theta+1)^2}{n}$$

Problem 2

The fisher information of X is

$$\log f_X(X;p) = (X-1)\log(1-p) + \log(p)$$

$$\frac{\partial}{\partial p}\log f_X(X;p) = \frac{1-X}{1-p} + \frac{1}{p}$$

$$\frac{\partial^2}{\partial p^2}\log f_X(X;p) = \frac{1-X}{(1-p)^2} - \frac{1}{p^2}$$

$$I(p) = \frac{1}{p^2} - \frac{1-1/p}{(1-p)^2} = \frac{1}{p^2} + \frac{(1-p)/p}{(1-p)^2} = \frac{1}{p^2} + \frac{1}{p(1-p)} = \frac{1}{p^2(1-p)}$$

The asymptotic variance is

$$\frac{1}{nI(p)} = \frac{p^2(1-p)}{n}$$

The estimated value is

$$p_e = \frac{1}{\bar{x}} = 0.592$$

The marginal error is

$$Z_{\alpha/2}\sqrt{\frac{\hat{p}^2(1-\hat{p})}{n}} = 1.96\sqrt{\frac{0.592^2 \cdot 0.408}{100}} = 0.074$$

The confidence interval is therefore $(0.518, 0.666)$.

Problem 3: 5.7.2

We have that $\mu = 0$, $E[Y_i^2] = E[(Y_i - \mu)^2] = \sigma^2$. By the weak law of large numbers, S_n^2, which is the sample mean of Y^2, is a consistent estimator for $E[Y_i^2] = \sigma^2$.

Problem 4

1. $\hat{\lambda}_n$ is unbiased since $E[\hat{\lambda}_n] = E[X_n] = \lambda$.

2. $\hat{\lambda}_n$ is not consistent since we are only using the n-th random variable. The probability that the estimator is within some ϵ remains fixed at some constant value less than 1 and does not approach 1 as n goes to infinity.

Problem 5

The probability that a random person will support the mayor is $0.5 * 0.7 + 0.5 * 0.3 = 0.5$. The variance is $\frac{p(1-p)}{n} = \frac{0.5 \cdot 0.5}{500} = 0.0005$

The variance of the male population is $\frac{0.7 \cdot 0.3}{250} = 0.00084$. The variance of the female population is $\frac{0.3 \cdot 0.7}{250} = 0.00084$. The overall variance is $0.5^2 0.00084 + 0.5^2 0.00084 = 0.00042$. Thus the relative efficiency is $\frac{0.0005}{0.00042} = 1.19$.

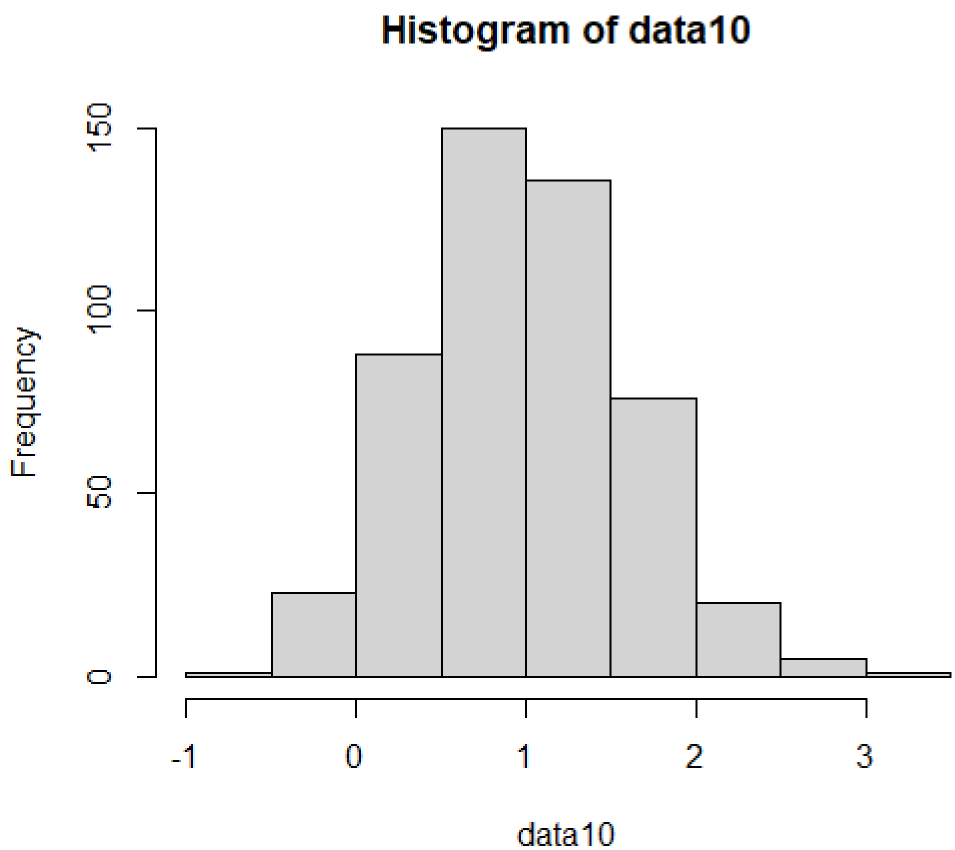
Problem 6: R Simulation

Here are the calculations for the critical values:

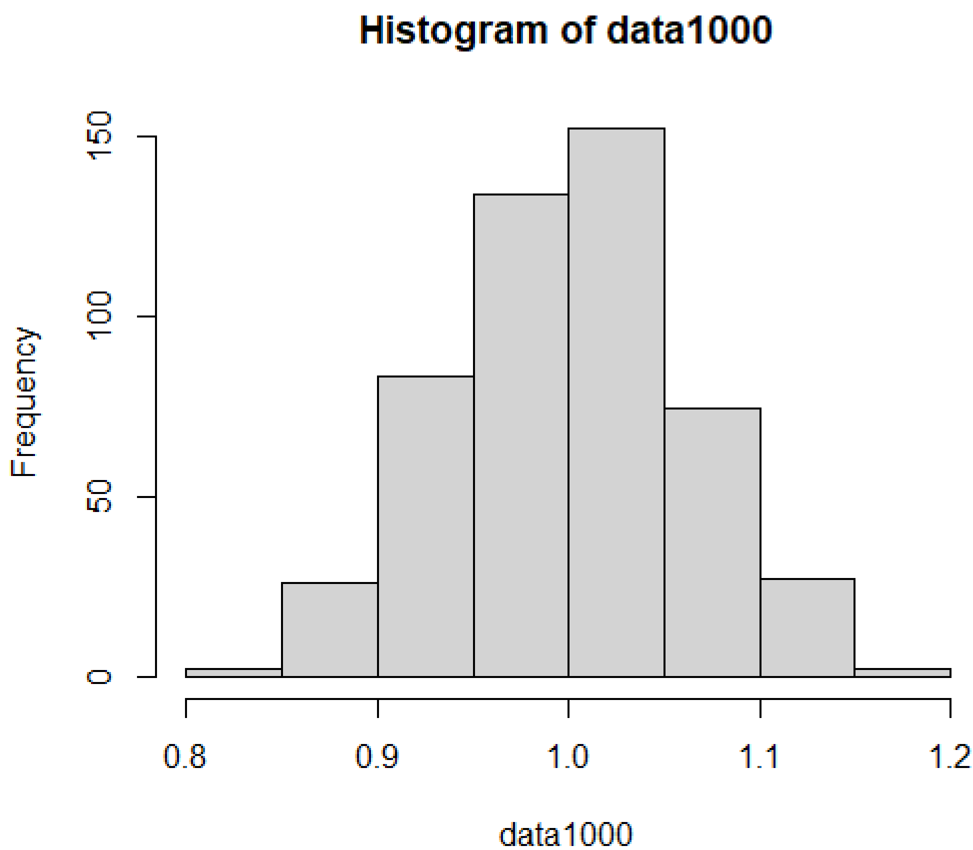
```
> quantile(sample1000, probs = 1-0.01/2)
   99.5%
2.606242

> quantile(sample1000, probs = 1-0.05/2)
   97.5%
2.037447

> quantile(sample1000, probs = 1-0.1/2)
     95%
1.744334

> sample10000 = rnorm(10000, 0, 1);

> quantile(sample10000, probs = 1-0.01/2)
   99.5%
2.667795

> quantile(sample10000, probs = 1-0.05/2)
   97.5%
1.987434

> quantile(sample10000, probs = 1-0.1/2)
     95%
1.622201
>
```

The values are relatively close to the theoretical values, with the $n = 10000$ being slightly closer.

Here is the histogram for $n = 10$



Here is the histogram for $n = 1000$



Since the sample mean is a consistent estimator for μ, its value will get closer and closer to μ for larger values of n, which is shown by the x axis of the $n = 1000$ histogram being tighter around 1.