Homework due Friday, December 8, at 11:00 pm Pacific Time.

A. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function so that $\lim_{x\to\infty} f(x) = \ell_1$ and $\lim_{x\to-\infty} f(x) = \ell_2$, where $\ell_1, \ell_2 \in \mathbb{R}$. Prove that f is uniformly continuous.

B. Let (X, d) be a metric space and let $f : X \to X$ be a function. Suppose there exist $C > 0$ and $\alpha > 0$ so that

$$d(f(x), f(y)) \leq C d(x, y)^\alpha \quad \text{for all } x, y \in X.$$

Prove that f is uniformly continuous.

C. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous bijection, prove that f is a homeomorphism.

(Hint: Use the fact that connected subsets of $\mathbb{R}$ are intervals to determine what $f([a, b])$ can be for a closed interval $[a, b]$.)

D. Rudin, Chapter 4 (page 98), problems # 2, 4, 6, 8, 14, 18.

The following problems are for your practice, and will not be graded.

(1) Rudin, Chapter 4 (page 98), # 13, 15.

(2) Let $p \in [0, 1] \times [0, 1]$, and let $C = [0, 1] \times [0, 1] \setminus \{p\}$.
  - (a) Let $x, y \in C$. Show that there exists a continuous function $f : [0, 1] \to C$ so that $f(0) = x$ and $f(1) = y$.
  - (b) Use part (a) and an argument similar to Rudin problem 21, page 44, to show that C is connected.

(3) Let (X, d) be a compact metric space and let $f : X \to X$ be a function with the following property.

$$d(f(x), f(y)) < d(x, y) \quad \text{for all } x \neq y \in X.$$

Prove that there exists a unique $x_0 \in X$ so that $f(x_0) = x_0$.
(Hint: Consider $g : X \to \mathbb{R}$ where $g(x) = d(x, f(x))$.)

(4) Let $f : [0, 1] \to [0, 1]$ be a function which satisfies

$$\lim_{y\to x} f(y) \text{ exists for all } x \in [0, 1].$$

Define $g(x) = \lim_{y\to x} f(y)$. Prove that g is continuous on $[0, 1]$.

(5) (a) Suppose f is a continuous function on $\mathbb{R}$ and assume that

$$\lim_{x\to\infty} f(x) = \infty = \lim_{x\to-\infty} f(x).$$

Prove that there exists some $y \in \mathbb{R}$ such that $f(y) \leq f(x)$ for all $x \in \mathbb{R}$.

(b) Let n be an even natural number and let

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

be a polynomial of degree n. Prove that f has a minimum, i.e. there exists some $y \in \mathbb{R}$ such that $f(y) \leq f(x)$ for all $x \in \mathbb{R}$.

(6) Let n be an odd natural number and let

$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

be a polynomial of degree n.

(a) Prove that $\lim_{x\to\infty} f(x) = \infty$, and $\lim_{x\to-\infty} f(x) = -\infty$.

(b) Conclude that f has at least one real root, i.e. there exists some $x_0 \in \mathbb{R}$ such that $f(x_0) = 0$.

(7) (Bonus Problem) Let $f : [0,1] \to \mathbb{R}$ be a function such that

$$\lim_{y\to x} f(y) \text{ exists for all } x \in [0,1].$$

Prove that the set where f is discontinuous is at most countable.
(Hint: Prove that for every $\epsilon > 0$ there are only finitely many x such that

$$|\lim_{y\to x} f(y) - f(x)| > \epsilon.)$$

(8) (Bonus Problem) Rudin, Chapter 4 (page 98), problems 17, 22, 23, 24

(9) (Bonus Problem) Let p be a prime number. For any integer $m \in \mathbb{Z}$ define

$$\nu_p(m) = \begin{cases} \text{power of } p \text{ in the prime factorization of } m & \text{if } m \neq 0 \\ \infty & \text{if } m = 0 \end{cases}$$

and define the following **norm** on $\mathbb{Q}$

$$|\frac{m}{n}|_p = \begin{cases} p^{\nu_p(n)-\nu_p(m)} & \text{if } \frac{m}{n} \neq 0 \\ 0 & \text{if } \frac{m}{n} = 0 \end{cases}.$$

(a) Show that this definition is well defined, i.e. if $\frac{m}{n} = \frac{m'}{n'}$, then $|\frac{m}{n}|_p = |\frac{m'}{n'}|_p$.

(b) For any two $r, s \in \mathbb{Q}$ define $d(r,s) = |r-s|_p$. Show that $(\mathbb{Q}, d)$ is a metric space.

(c) Show that $\mathbb{Z}$ is a bounded subset of $\mathbb{Q}$ with respect to this metric.

(d) Does $\{p^n : n \in \mathbb{N}\}$ have a limit point in $\mathbb{Q}$?

(10) (Bonus Problem)* With the notation as in the previous problem. Let

$$\mathbb{Z}_p = \Big\{\sum_{n=0}^{\infty} a_n p^n : a_n = 0, 1, \cdots, p-1\Big\}.$$

Extend the definition of $|\ |_p$ to $\mathbb{Z}_p$ by

$$|x|_p = p^{-n}$$

where $x = a_np^n + a_{n+1}p^{n+1} + \cdots$, i.e., $n \geq 0$ is the smallest index so that $a_n \neq 0$ in the expansion of x. Define a metric d on $\mathbb{Z}_p$ using $|\ |_p$ similar to the above.

(a) We may identify $\mathbb{Z} \subset \mathbb{Z}_p$ by expanding an integer base p. Show that $\mathbb{Z}$ is dense in $\mathbb{Z}_p$.
(b) Show that $\mathbb{Z}_p$ equipped with the above metric is compact and hence a complete metric space.
(c) Let $A \subset \mathbb{Z}_p$ be a nonempty connected subset of $\mathbb{Z}_p$. Show that A has exactly one point.
(d) Let $f : [0, 1] \to \mathbb{Z}_p$ be a continuous function. Show that f is constant.