Math 31AH: Fall 2021
Homework 8
Due 5:00pm on Monday 11/29/2021

Problem 1: Quotients and matrices. Let V be a finite-dimensional $\mathbb{F}$-vector space and let $W \subseteq V$ be a subspace. Let $\mathcal{B} = (\mathbf{w}_1, \ldots, \mathbf{w}_m)$ be an ordered basis for W and extend to an ordered basis

$$\mathcal{C} = (\mathbf{w}_1, \ldots, \mathbf{w}_m, \mathbf{v}_{m+1}, \ldots, \mathbf{v}_n)$$

of V.

Let $T : V \to V$ be a linear operator. If W is T-invariant, prove that we have a well-defined linear transformation

$$\overline{T} : V/W \to V/W$$

given by $\overline{T}(\mathbf{v} + W) := T(\mathbf{v}) + W$. In this case, the matrix for T with respect to $\mathcal{C}$ has the block matrix form

$$[T]_{\mathcal{C}}^{\mathcal{C}} = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

Interpret the blocks A and C in terms of T and $\overline{T}$.

Problem 2: Quotients and Direct Sums. Let V and W be $\mathbb{F}$-vector spaces and form their direct sum $V \oplus W$. By common notational **abuse** we consider $W \subseteq V \oplus W$ as a subspace by means of

$$\{(\mathbf{0}, \mathbf{w}) \,:\, \mathbf{w} \in W\} \subseteq V \oplus W$$

Prove that $(V \oplus W)/W \cong V$.

Problem 3: Quotients and Duals. Let V be an $\mathbb{F}$-vector space and let $W \subseteq V$ be a subspace. Consider the subset $U \subseteq V^*$ given by

$$U := \{\lambda \in V^* \,:\, \lambda(\mathbf{w}) = 0 \text{ for all } \mathbf{w} \in W\}$$

(1) Prove that U is a subspace of V^*.
(2) Prove that W^* and V^*/U are isomorphic.
(3) Prove that $(V/W)^*$ and U are isomorphic.

Problem 4: Matrix Direct Sum. If A and B are matrices over a field $\mathbb{F}$, their *direct sum* is the block matrix

$$A \oplus B := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where the zero blocks have appropriate sizes.

If $T : V \to W$ and $T' : V' \to W'$ are linear transformations between $\mathbb{F}$-vector spaces, their *direct sum* $T \oplus T' : V \oplus V' \to W \oplus W'$ is defined by

$$(T \oplus T')(\mathbf{v}, \mathbf{v}') = (T(\mathbf{v}), T(\mathbf{v}'))$$

Explain the relationship between matrix direct sum and linear transformation direct sum.

Problem 5: Matrix Tensor Product. If A and B are matrices with A $m \times n$, their *tensor product* is the block matrix

$$A \otimes B := \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ & & \ddots & \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}$$

Prove that (whenever these products are defined) we have

$$(A \otimes B) \cdot (C \otimes D) = (AC) \otimes (BD)$$

Problem 6: Representing Tensor Transformations. Define two linear maps $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $U : V_2 \to V_1$ by

$$T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ -x \end{pmatrix} \qquad U(f(t)) = f'(t)$$

where V_n is the vector space of polynomials in t with coefficients in $\mathbb{R}$ of degree $\leq n$. Find a matrix representation of the tensor transformation

$$(T \otimes U) : (\mathbb{R}^2 \otimes V_2) \to (\mathbb{R}^2 \otimes V_1)$$

Problem 7: Tensors and Duals. Let V be an $\mathbb{F}$-vector space. Prove that we have a well-defined linear map

$$\varphi : V \otimes V^* \to \mathbb{F}$$

given by $\varphi(\mathbf{v})(\lambda) := \lambda(\mathbf{v})$.

Problem 8: Determinants and Tensors. Let $\mathbb{F}$ be a field. Prove that we have a well-defined linear map

$$\psi : (\mathbb{F}^n \otimes \cdots \otimes \mathbb{F}^n) \to \mathbb{F}$$

(where there are n factors of $\mathbb{F}^n$) given by

$$\psi : \mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_n \mapsto \det \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{pmatrix}$$

Problem 9: (Optional; not to be handed in.) Let $V, W,$ and U be $\mathbb{F}$-vector space. Prove the *tensor-hom adjunction isomorphism*

$$\mathrm{Hom}(V \otimes W, U) \cong \mathrm{Hom}(V, \mathrm{Hom}(W, U))$$

of $\mathbb{F}$-vector spaces.