Math 100A: Homework 5

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Problem 1

Since $yxy^{-1} = x^{-1}$ and m is odd,

$$x = y^m x y^{-m} = x^{-1}$$

Therefore $x^2 = 1$.

Problem 2

Let $n = 2k+1$. Since $x^2 = 1$ and $x^n = x^{2k+1} = 1$, this implies that $x^{2k+1}(x^2)^{-k} = x = 1$. Substituting $x = 1$, we have the relations $y^m = 1$ and $yy^{-1} = 1$, which is a cyclic group of order m with generator y.

Problem 3

The last relation implies that $yx = x^{-1}y$, meaning we can conjugate all the y to the right and write all elements in $G_{n,m}$ in the form $x^a y^b$ for $a \in \{0, \dots, n-1\}$ and $b \in \{0, \dots, m-1\}$. Thus there are at most nm elements.

Problem 4

The cyclic group $N = \langle x \rangle$ is of order n so it is isomorphic to $\mathbb{Z}/n\mathbb{Z}$. It is a normal subgroup of $G_{n,m}$ because of the relation $yxy^{-1} = x^{-1}$.

The cyclic group $M = \langle y \rangle$ is of order m so it is isomorphic to $\mathbb{Z}/m\mathbb{Z}$. $N \cap M = \{1\}$ since the set of symbols x^k is distinct from the set of symbols y^k apart from identity.

Since every element in $G_{n,m}$ can be written as $x^a y^b$ for integers a, b we have that $G_{n,m} = NM$. Let the map $\Psi : G'_{n,m} \to G_{n,m}$ be given by $\Psi(x^a, y^b) = x^a y^b$ for integers a, b.

By the third relation $(\varphi \circ \pi)(y^m)$ is an automorphism that is conjugation by y^m

$$(\varphi \circ \pi)(y^m)(x^n) = \begin{cases} x^n & \text{if m is even} \\ x^{-n} & \text{if m is odd} \end{cases} = y^m x^n y^{-m}.$$

Ψ is an homomorphism since

$$\begin{aligned} \Psi((x^a, y^b)(x^c, y^d)) &= \Psi((x^a \varphi(\pi(y^b))(x^c), y^b y^d)) \\ &= x^a \varphi(\pi(y^b))(x^c) y^b y^d \\ &= x^a y^b x^c y^{-b} y^b y^d \\ &= x^a y^b x^c y^d \\ &= \Psi(x^a, y^b)\Psi(x^c, y^d) \end{aligned}$$

Ψ is injective since $N \cap M = \{1\}$ and it is surjective since $G_{n,m} = NM$. Therefore Ψ is an isomorphism between $\Psi : G'_{n,m}$ and $G_{n,m}$.