Math 170A: Homework 7

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Q1

We have that

$$I - \delta A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ 2 & -2 \end{bmatrix}$$

The characteristic polynomial is $(\frac{2}{3} - \lambda)(-2 - \lambda) + \frac{2}{3} = \lambda^2 + \frac{4}{3}\lambda - \frac{2}{3}$ The eigenvalues are $\frac{-\frac{4}{3} \pm \sqrt{\frac{16}{9} + \frac{8}{3}}}{2}$ which is $\lambda_1 = -\frac{2}{3} - \frac{\sqrt{10}}{3}$ and $\lambda_1 = -\frac{2}{3} + \frac{\sqrt{10}}{3}$. We can see that $|\lambda_1| > 1$ so the Jacobi iterative method will not always converge.

Q2

We have that

$$\begin{aligned}
x_1^{(1)} &= \frac{1-0}{1} = 1 \\
x_2^{(1)} &= \frac{1-1}{3} = 0 \\
x_1^{(2)} &= \frac{1-0}{1} = 1 \\
x_2^{(2)} &= \frac{1-1}{3} = 0 \\
&\vdots \\
x_1^{(5)} &= \frac{1-0}{1} = 1 \\
x_2^{(5)} &= \frac{1-1}{3} = 0
\end{aligned}$$

We see that Gauss-seidel converges to the answer after one iteration and remains at that value for each additional iteration.

Q3

Since $A^* = -A$, that means $A_{i,j} = -\overline{A_{j,i}}$, which is only possible if every entry has $\mathrm{Re}(A_{i,j}) = -\mathrm{Re}(A_{j,i})$ and $\mathrm{Im}(A_{i,j}) = \mathrm{Im}(A_{j,i})$. Thus $B = iA$ is a matrix with $\mathrm{Im}(A_{i,j}) = -\mathrm{Im}(A_{j,i})$ and $\mathrm{Re}(A_{i,j}) = \mathrm{Re}(A_{j,i})$, meaning B is a hermetian matrix. Thus B has all real eigenvalues, which are just the eigenvalues of A times i. Thus A must have all imaginary eigenvalues.

Q4

$$AA^T = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

The characteristic polynomial is

$$(2-\lambda)(1-\lambda)(1-\lambda) - (1-\lambda) - (1-\lambda) = -\lambda^3 + 4\lambda^2 - 3\lambda = -\lambda(\lambda-3)(\lambda-1)$$

The eigenvalues are $\lambda = 0, 1, 3$. By inspection, we can see that the vectors that satisfy $(A - \lambda I)v = 0$ are

$$v_0 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} v_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

The characteristic polynomial is

$$(2-\lambda)(2-\lambda) - 1 = \lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3)$$

The eigenvalues are $\lambda = 1, 3$. The eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} v_3 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We can put together the normalized eigenvectors of AA^T to form U, the normazliedeigenvectors of $A^T A$ to form V, and the square root of the eigenvalues from V to form the diagonals of Σ.

$$A = \begin{bmatrix} \frac{\sqrt{6}}{3} & 0 & \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & -\frac{\sqrt{3}}{3} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

Q5

1. The eigenvalues of B are the eigenvalues of A minus 0.25. The eigenvalues of C are the reciprocal of the eigenvalues of B.

2. The eigenvalue is 0.5, which corresponds to an eigenvalue of 0.25 in B, which corresponds to 4, which is the largest eigenvalue of C. Essentially we want to find the smallest eigenvalue in B because this is the largest eigenvalue in C, which will get amplified by the power method.

3. The eigenvalue is -0.25, which corresponds to 0.25 in B, which corresponds to 4, which is the largest eigenvalue of C.

4. It results in $3.5804e-04$, which is very small so q is a very good approximate.