Math 31AH: Spring 2021
Homework 1
Due 5:00pm on Friday 10/1/2021

Problem 1: Arithmetic of sets. Determine whether the following three equalities hold for all sets A, B, and C. If equality does not hold, determine whether we have the containments $\subseteq$ or $\supseteq$. Prove your claims.

(1) $A \cap (B - C) = (A \cap B) - (A \cap C)$.
(2) $A \cup (B - C) = (A \cup B) - (A \cup C)$.
(3) $A \times (B - C) = (A \times B) - (A \times C)$.

Problem 2: Vectors on the circle. Let S be the unit circle in the plane $\mathbb{R}^2$ centered at the origin, i.e.

$$S = \{(x, y) \in \mathbb{R}^2 \,:\, x^2 + y^2 = 1\}.$$

True or false: there exist elements $\mathbf{v}, \mathbf{w} \in S$ such that $\mathbf{v} + \mathbf{w} \in S$. Prove your claim.

Problem 3: Ill-defined functions. Each of the following "functions" is not well-defined. Explain why they are not well-defined.

(1) $f : \mathbb{C} \to \mathbb{C}$, where $\mathbb{C} = \{x + iy \,:\, x, y \in \mathbb{R}\}$ is the set of complex numbers and $f(z) := \frac{1}{z^2+3}$.
(2) $g : \mathbb{Q} \to \mathbb{Z}$, where $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ is the set of integers, $\mathbb{Q} = \{\frac{a}{b} \,:\, a, b \in \mathbb{Z},\ b \neq 0\}$ is the set of rational numbers, and $g(\frac{a}{b}) = a - b$.
(3) $h : X \to \mathbb{R}_{>0}$, where $X := \{(x, y) \in \mathbb{R}^2 \,:\, y = x^2 - 1\}$ is a parabola in the plane, $\mathbb{R}_{>0} = \{x \in \mathbb{R} \,:\, x > 0\}$ are the positive reals, and $h(x, y) = y$.

Problem 4: Binary operations. Decide whether the given binary operations $\star$ on the given sets S are well-defined. Prove your claim.

(1) $S = \{(x, y) \in \mathbb{R}^2 \,:\, xy = 0\}$ and $(x, y) \star (x', y') := (x + x', y + y')$.
(2) $S = \mathbb{R}$ and $x \star y := \frac{x}{y^2+1}$.
(3) $S = \mathbb{C}$ and $x \star y := \frac{x}{y^2+1}$.

Problem 5: Multiplication in fields. Let $\mathbb{F}$ be a field and let $a, b \in \mathbb{F}$ be nonzero elements. Prove that $ab \neq 0$. (Hint: Use 'proof by contradiction'. Assume to the contrary that $ab = 0$ with $a, b \neq 0$. Prove that this forces one of a, b to be zero.)

Problem 6: Characteristic of a field. Let $\mathbb{F}$ be a field. The *characteristic* of $\mathbb{F}$, written $\mathrm{char}(\mathbb{F})$, is the minimum positive integer n such that we have

$$\overbrace{1 + 1 + \cdots + 1}^{n} = 0$$

inside $\mathbb{F}$. If no such n exists, the field $\mathbb{F}$ is said to have *characteristic zero* and we write $\mathrm{char}(\mathbb{F}) = 0$.

Let $\mathbb{F}$ be a field with $\mathrm{char}(\mathbb{F}) = n > 0$. Prove that n is prime. (Hint: Use Problem 5 in a clever way.)

Problem 7: A four-element field? Let $S = \{0, 1, 2, 3\}$ and define binary operations $+, \cdot$ on S to be addition and multiplication modulo 4.[1] Do these binary operations turn S into a field? Prove your claim.

Problem 8: A non-field. Let $\mathbb{F}$ be a field. Define binary operations $+$ and $\cdot$ on $\mathbb{F}^2 = \{(a, b) \,:\, a, b \in \mathbb{F}\}$ by the 'coordinatewise' rules

$$(a, b) + (a', b') := (a + a', b + b') \quad \text{and} \quad (a, b) \cdot (a', b') := (a \cdot a', b \cdot b')$$

Prove that these binary operations do **not** turn $\mathbb{F}^2$ into a field.

Problem 9: (Optional; not to be handed in.) When $\mathbb{F} = \mathbb{R}$ is the field of real numbers, we **can** endow $\mathbb{R}^2$ with the structure of a field via the alternative binary operations

$$(x, y)+(x', y') := (x+x', y+y') \quad \text{and} \quad (x, y)\cdot(x', y') := (xx'-yy', xy'+x'y)$$

Explain why this is the field $\mathbb{C}$ of complex numbers in disguise. Can these rules be used to define a field structure on $\mathbb{F}^2$ for any field $\mathbb{F}$? Why or why not?

[1] More precisely, given $x, y \in S$ we define $x + y \in S$ to be the remainder of the (usual) sum of x, y upon division by 4 and let $x \cdot y \in S$ be the remainder of the (usual) product of x, y upon division by 4.