Math 188: Homework 6

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1 Another Proof of Cayley's Formula

1. Let T_n be all labeled trees with verticies $1, \dots, n$. The partial derivative of $\mathbf{C}_n$ is
$$\frac{\partial \mathbf{C}_n}{\partial x_n} = \sum_{T_n} d_n x_1^{d_1} \dots x_{n-1}^{d_{n-1}} x_n^{d_n - 1}.$$
Substituting in $x_n = 0$ leaves only trees in T_n such that $d_n = 1$. These trees can be generated by taking all trees in T_{n-1} and choosing a vertex to attach the nth vertex to. Since attaching the nth vertex to the ith vertex increases d_i by one, we have that
$$\begin{aligned}\mathbf{C}_n^{(n)} &= \sum_{i=1}^{n-1} \left(x_i \sum_{T_{n-1}} x_1^{d_1} \dots x_{n-1}^{d_{n-1}} \right) \\ &= (x_1 + x_2 + \dots + x_{n-1}) \mathbf{C}_{n-1}.\end{aligned}$$
The partial derivative of $\mathbf{D}_n$ is
$$\frac{\partial \mathbf{D}_n}{\partial x_n} = (n-2)(x_1 \dots x_n)(x_1 + \dots + x_n)^{n-3} + (x_1 \dots x_{n-1})(x_1 + \dots + x_n)^{n-2}.$$
Substituting in $x_n = 0$ yields
$$\begin{aligned}\mathbf{D}_n^{(n)} &= (x_1 \dots x_{n-1})(x_1 + \dots + x_{n-1})^{n-2} \\ &= (x_1 \dots x_{n-1}) \mathbf{D}_{n-1}\end{aligned}$$

2. Since the variables are symmetric with respect to each other, the proof from part (a) can be repeated to show that for all $i = 1, \dots, n$
$$\mathbf{C}_n^{(i)} = \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} x_j \right) \mathbf{C}_{n-1} \quad \text{and} \quad \mathbf{D}_n^{(i)} = \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} x_j \right) \mathbf{D}_{n-1}.$$
Assuming that $\mathbf{C}_{n-1} = \mathbf{D}_{n-1}$, we have that $\mathbf{C}_n^{(i)} = \mathbf{D}_n^{(i)}$ for all $i = 1, \dots, n$.

3. From part (b), $\mathbf{C}_{n-1} = \mathbf{D}_{n-1}$ implies $\mathbf{C}_n^{(i)} = \mathbf{D}_n^{(i)}$ for all $i = 1, \dots, n$, which then implies that the coefficients of all x_i are equal. This then implies that $\mathbf{C}_n = \mathbf{D}_n$ since they are both polynomials. Since $\mathbf{C}_1 = \mathbf{D}_1 = 1$ and $\mathbf{C}_2 = \mathbf{D}_2 = x_1 x_2$, we have that $\mathbf{C}_n = \mathbf{D}_n$ for all $n \geq 1$ from induction.

2 Ordering the letters of MATHEMATICS

Let U be the set of orderings of MATHEMATICS without restriction. Let M, A, and T be the sets of ways of ordering MATHEMATICS with a consecutive repeated 'M', 'A', and 'T' respectively. The number of orderings of MATHEMATICS is therefore $|U \setminus M \cup A \cup T|$.

There are $|U| = \frac{10!}{2!2!2!}$ ways to order MATHEMATICS with no restrictions. Grouping pairs of letters togther, there are $|M| = |A| = |T| = \frac{9!}{2!2!}$ ways to order MATHEMATICS such that a two characters appear consecutively. Similarly, there are $|M \cap A| = |M \cap T| = |A \cap T| = \frac{8!}{2!}$ ways for two characters to appear consecutively twice and $|M \cap A \cap T| = 7!$ ways for all three characters to appear consecutively twice. Using the inclusion-exclusion principle, the number of ways to order MATHEMATICS where each letter does not appear consecutively is

$$\begin{aligned}
&|U \setminus M \cup A \cup T| \\
=&|U| - |M| - |A| - |T| + |M \cap A| + |M \cap T| + |A \cap T| - |M \cap A \cap T| \\
=&\frac{10!}{2!2!2!} - 3\left(\frac{9!}{2!2!}\right) + 3\left(\frac{8!}{2!}\right) - 7! \\
=&236880.
\end{aligned}$$

3 Circle of Marriage

1. Let A_i be the set of lines where ith couple is standing next to each other. By grouping the couples together, the number of ways for j couples to stand next to each other is $2^j(2n-j)!$. Using the inclusion-exclusion principle, the number of ways to have everyone stand apart from their spouse is

$$\begin{aligned}
&(2n)! - |A_1 \cup \ldots \cup A_n| \\
=&(2n)! - \sum_{j=1}^{n} (-1)^{j-1} \sum_{1 \le i_1 < \ldots < i_j \le n} |A_{i_1} \cap \ldots \cap A_{i_h}| \\
=&(2n)! - \sum_{j=1}^{n} (-1)^{j-1} 2^j \binom{n}{j} (2n-j)! \\
=&(2n)! + \sum_{j=1}^{n} (-2)^j \binom{n}{j} (2n-j)!.
\end{aligned}$$

2. Let B_i be the set of circles where ith couple is standing next to each other. Note that the number of circular permutations of n things is $(n-1)!$. Using inclusion-exclusion yields

$$\begin{aligned}
&(2n-1)! - |B_1 \cup \ldots \cup B_n| \\
=&(2n-1)! - \sum_{j=1}^{n} (-1)^{j-1} \sum_{1 \le i_1 < \ldots < i_j \le n} |B_{i_1} \cap \ldots \cap B_{i_h}| \\
=&(2n-1)! - \sum_{j=1}^{n} (-1)^{j-1} 2^j \binom{n}{j} (2n-j-1)! \\
=&(2n-1)! + \sum_{j=1}^{n} (-2)^j \binom{n}{j} (2n-j-1)!.
\end{aligned}$$

4 Vector Subspace Morbin'

The set of r-dimensional subspaces Z where $X \subseteq Z \subseteq Y$ are in bijection with (r - dim X)-dimensional subspaces of Y/X. Using the q-binomial theorem with $t = -1$ yields that

$$\begin{aligned}
\sum_{Z \in [X,Y]} \mu(X, Z) &= \sum_{Z \in [X,Y]} (-1)^{\dim Z - \dim X} q^{\binom{\dim Z - \dim X}{2}} \\
&= \sum_{Q \subseteq Y/X} (-1)^{\dim Q} q^{\binom{\dim Q}{2}} \\
&= \sum_{k=0}^{d} \binom{d}{k}_q (-1)^k q^{\binom{k}{2}} \\
&= \prod_{k=0}^{d-1} (1 - q^k) \\
&= \delta_{0,d} = \delta_{X,Y}
\end{aligned}$$

Since this equality characterizes the Mobius function, we have that $\mu(X, Y) = (-1)^d q^{\binom{d}{2}}$.

5 Number of Connected Labeled Graphs

Let x be a set partitions of $[n]$, let x_i be the size of the ith block of x, and let $|x|$ be the number of blocks in x. Let $g(x)$ be the number of labeled graphs such that there no edges between vertices in different blocks. Let $f(x)$ be the number of labeled graphs such that there are no edges between blocks and each block is connected. Since $g(y) = \sum_{x \leq y} f(x)$ for all set partitions y, we have that

$$\begin{aligned} f(y) &= \sum_{x \leq y} g(x)\mu(x, y) \\ &= \sum_{x \leq y} 2^{\sum_{i=1}^{|x|} \binom{x_i}{2}} \mu(x, y). \end{aligned}$$

If y is the set partition with a single block, then $f(y)$ counts the number of connected labeled graphs.