Math 31AH: Fall 2021
Homework 4
Due 5:00pm on Friday 10/22/2021

Problem 1: Projections. Let V and W be $\mathbb{F}$-vector spaces. Recall the direct sum of V and W

$$V \oplus W = \{(\mathbf{v}, \mathbf{w}) \,:\, \mathbf{v} \in V \,,\, \mathbf{w} \in W\}$$

defined last time. Define a function $P_V : V \oplus W \to V$ by $P_V(\mathbf{v}, \mathbf{w}) = \mathbf{v}$. The function P_V is called 'projection onto V'.

Prove that P_V is linear. When is P_V surjective? When is P_V injective?

Problem 2: Linear maps and spanning. Let $T : V \to W$ be a linear transformation of $\mathbb{F}$-vector spaces. If $S \subseteq V$ spans V, prove that

$$T(S) := \{T(\mathbf{v}) \,:\, \mathbf{v} \in S\}$$

spans Image(T).

Problem 3: Linear maps and independence. Let $T : V \to W$ be an injective linear transformation between $\mathbb{F}$-vector spaces. If $I \subseteq V$ is linearly independent in V, prove that

$$T(I) := \{T(\mathbf{v}) \,:\, \mathbf{v} \in I\}$$

is linearly independent in W. Does this necessarily hold if we remove the assumption that T is injective?

Problem 4: Representing matrices. Let V_n denote the $\mathbb{R}$-vector space of polynomials in x of degree $\leq n$ with real coefficients. The derivative map

$$T : V_3 \to V_2 \qquad T(f(x)) = f'(x)$$

is a linear transformation. Consider the ordered bases $\mathcal{B} := (x^3, x^2, x, 1)$ of V_3 and $\mathcal{C} := ((x+1)^2, x+1, 1)$ of V_2. Calculate $[T]_{\mathcal{C}}^{\mathcal{B}}$, the representing matrix of T with respect to these bases.

Problem 5: A matrix map. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Find a basis of Ker(T) and a basis of Image(T).

Problem 6: Inverses of linear maps. Let $T : V \to W$ be an invertible linear transformation between $\mathbb{F}$-vector spaces. Prove that the inverse $T^{-1} : W \to V$ is also linear.

Problem 7: Polynomial change of basis. Let V_3 be the vector space of polynomials in x of degree ≤ 3 with real coefficients. Consider the two bases $\mathcal{B}$ and $\mathcal{C}$ of V given by

$$\mathcal{B} := (1, x+1, x^2+x+1, x^3+x^2+x+1)$$
$$\mathcal{C} := (1, x-1, x^2-x+1, x^3-x^2+x-1)$$

Calculate the transition matrix from $\mathcal{B}$ to $\mathcal{C}$ and the transition matrix from $\mathcal{C}$ to $\mathcal{B}$.

Problem 8: Invariant subspaces and block matrices. Let V be an n-dimensional $\mathbb{F}$-vector space and let $W \subseteq V$ be a subspace of dimension m. Show that there exists an ordered basis

$$\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m, \mathbf{v}_{m+1}, \mathbf{v}_{m+2}, \dots, \mathbf{v}_n)$$

of V such that $(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m)$ is an ordered basis of W. (Hint: A problem on Homework 3 should make this a one-liner!)

Let $T : V \to V$ be a linear transformation. The subspace $W \subseteq V$ is called *invariant under* T if $T(\mathbf{w}) \in W$ for all $\mathbf{w} \in W$. If $\mathcal{B}$ is a basis of V as above, prove that W is invariant under T if and only if the representing matrix $[T]_{\mathcal{B}}^{\mathcal{B}}$ for T with respect to B has the block form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where A is $m \times m$ and C is $(n-m) \times (n-m)$.

Problem 9: (Optional; not to be handed in.) Let $\mathbb{K}$ be a finite field of characteristic p. You proved on a previous homework that p is prime. Show that $\mathbb{F} := \{0, 1, 2, \dots, p-1\} \subseteq \mathbb{K}$ is a subfield of $\mathbb{K}$ (i.e. is a subset which is also a field), so that $\mathbb{K}$ is a $\mathbb{F}$-vector space. Conclude that

> *Any finite field has size equal to some power p^r of a prime number p, where $r > 0$ is an integer.*

For example, there is no field with 6 elements.