Math 31AH: Fall 2021
Homework 8 Solutions
Due 5:00pm on Monday 11/29/2021

Problem 1: Quotients and matrices. Let V be a finite-dimensional $\mathbb{F}$-vector space and let $W \subseteq V$ be a subspace. Let $\mathcal{B} = (\mathbf{w}_1, \dots, \mathbf{w}_m)$ be an ordered basis for W and extend to an ordered basis

$$\mathcal{C} = (\mathbf{w}_1, \dots, \mathbf{w}_m, \mathbf{v}_{m+1}, \dots, \mathbf{v}_n)$$

of V.

Let $T : V \to V$ be a linear operator. If W is T-invariant, prove that we have a well-defined linear transformation

$$\overline{T} : V/W \to V/W$$

given by $\overline{T}(\mathbf{v} + W) := T(\mathbf{v}) + W$. In this case, the matrix for T with respect to $\mathcal{C}$ has the block matrix form

$$[T]_{\mathcal{C}}^{\mathcal{C}} = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

Interpret the blocks A and C in terms of T and $\overline{T}$.

Solution: We claim that the block A is the matrix for $T\mid_W : W \to W$ (the restriction of T to W) with respect to the ordered basis $\mathcal{B}$. Indeed, if $1 \leq j \leq m$ we have

$$T\mid_W (\mathbf{w}_j) = T(\mathbf{w}_j) = a_{1j}\mathbf{w}_1 + \cdots + a_{mj}\mathbf{w}_m$$

On the other hand, writing $\mathcal{C}'$ for the ordered basis of V/W given by $(\mathbf{v}_{m+1} + W, \dots, \mathbf{v}_n + W)$ we claim that C is the matrix for the quotient map $\overline{T} : V/W \to V/W$ with respect to $\mathcal{C}'$. Indeed, for $m + 1 \leq j \leq n$ we have

$$T(\mathbf{v}_j) = \sum_i b_{ij}\mathbf{w}_i + \sum_{i'} c_{i'j}\mathbf{v}_{i'}$$

so that

$$\overline{T}(\mathbf{v}_j + W) = \sum_i b_{ij}\mathbf{w}_i + \sum_{i'} c_{i'j}\mathbf{v}_{i'} + W = \sum_{i'} c_{i'j}\mathbf{v}_{i'} + W$$

Problem 2: Quotients and Direct Sums. Let V and W be $\mathbb{F}$-vector spaces and form their direct sum $V \oplus W$. By common notational **abuse** we consider $W \subseteq V \oplus W$ as a subspace by means of

$$\{(\mathbf{0}, \mathbf{w}) : \mathbf{w} \in W\} \subseteq V \oplus W$$

Prove that $(V \oplus W)/W \cong V$.

Solution: We have a linear map $\varphi : V \oplus W \to V$ given by the projection

$$\varphi(\mathbf{v}, \mathbf{w}) := \mathbf{v}$$

Since $\varphi(\mathbf{0}, \mathbf{w}) = \mathbf{0}$ for all $\mathbf{w} \in W$, the map φ induces a linear map $\Phi : (V \oplus W)/W \to V$ given by

$$\Phi : (\mathbf{v}, \mathbf{w}) + W \mapsto \mathbf{v}$$

On the other hand, we have a linear map $\Psi : V \to (V \oplus W)/W$ given by

$$\Psi(\mathbf{v}) = (\mathbf{v}, \mathbf{0}) + W$$

It is not difficult to check that Φ and Ψ are mutually inverse, so that $(V \oplus W)/W \cong V$.

Problem 3: Quotients and Duals. Let V be an $\mathbb{F}$-vector space and let $W \subseteq V$ be a subspace. Consider the subset $U \subseteq V^*$ given by

$$U := \{\lambda \in V^* \, : \, \lambda(\mathbf{w}) = 0 \text{ for all } \mathbf{w} \in W\}$$

(1) Prove that U is a subspace of V^*.
(2) Prove that W^* and V^*/U are isomorphic.
(3) Prove that $(V/W)^*$ and U are isomorphic.

Solution: (1) Certainly the zero functional is in U. If $\lambda, \lambda' \in U$ and $c, c' \in \mathbb{F}$ then

$$(c\lambda + c'\lambda')(\mathbf{w}) = c \cdot \lambda(\mathbf{w}) + c' \cdot \lambda'(\mathbf{w}) = 0 + 0 = 0$$

for any $\mathbf{w} \in W$ so that $c\lambda + c'\lambda' \in U$. Thus U is a subspace of V^*.

(2) We have a function $\varphi : V^* \to W^*$ given by letting $\varphi(\lambda) := \lambda \mid_W$, the restriction of λ to W. It is clear that φ is linear. Also, the kernel of φ is U by definition.

We claim that φ is surjective. Indeed, if $\mu \in W^*$ we may extend μ to a linear functional $\tilde{\mu} \in V^*$ as follows. Take a basis $\mathcal{B}$ of W and extend to a basis $\mathcal{C}$ of V. Define $\tilde{\mu}(\mathbf{w}) = \mu(\mathbf{w})$ for all $\mathbf{w} \in \mathcal{B}$ and let $\tilde{\mu}(\mathbf{v}) = 0$ for all $\mathbf{v} \in \mathcal{C} - \mathcal{B}$. It follows that φ is surjective. Thus, we have isomorphisms

$$W^* \cong V^*/\mathrm{Ker}\varphi = V^*/U$$

(3) A typical element $\lambda \in U$ is a linear map $\lambda : V \to \mathbb{F}$ for which $\lambda(\mathbf{w}) = 0$ for all $\mathbf{w} \in W$. We therefore have an induced linear map $\overline{\lambda} : V/W \to \mathbb{F}$ defined by

$$\overline{\lambda}(\mathbf{v} + W) = \lambda(\mathbf{v})$$

The function

$$\varphi : U \to (V/W)^*$$

given by $\lambda \mapsto \overline{\lambda}$ is easily seen to be linear. If $\varphi(\lambda) = 0$ then

$$0 = \overline{\lambda}(\mathbf{v} + W) = \lambda(\mathbf{v})$$

for all $\mathbf{v} \in V$ so that $\lambda = 0$ and φ is injective. Also, if $\mu : V/W \to \mathbb{F}$ is linear we have $\varphi(\mu \circ \pi) = \mu$ where $\pi : V \twoheadrightarrow V/W$ is the canonical projection. We conclude that φ is an isomorphism.

Problem 4: Matrix Direct Sum. If A and B are matrices over a field $\mathbb{F}$, their *direct sum* is the block matrix

$$A \oplus B := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where the zero blocks have appropriate sizes.

If $T : V \to W$ and $T' : V' \to W'$ are linear transformations between $\mathbb{F}$-vector spaces, their *direct sum* $T \oplus T' : V \oplus V' \to W \oplus W'$ is defined by

$$(T \oplus T')(\mathbf{v}, \mathbf{v}') = (T(\mathbf{v}), T'(\mathbf{v}'))$$

Explain the relationship between matrix direct sum and linear transformation direct sum.

Solution: Assume that the vector spaces in question are finite-dimensional. Let $\mathcal{B}, \mathcal{B}', \mathcal{C}, \mathcal{C}'$ be ordered bases of V, V', W, W' respectively. Then

$$[T \oplus T']_{\mathcal{C}\oplus\mathcal{C}'}^{\mathcal{B}\oplus\mathcal{B}'} = [T]_{\mathcal{C}}^{\mathcal{B}} \oplus [T']_{\mathcal{C}'}^{\mathcal{B}'}$$

where $\mathcal{B} \oplus \mathcal{B}'$ and $\mathcal{C} \oplus \mathcal{C}'$ are ordered by placing every unprimed basis vector before every primed basis vector.

Problem 5: Matrix Tensor Product. If A and B are matrices with A $m \times n$, their *tensor product* is the block matrix

$$A \otimes B := \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ & & \ddots & \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}$$

Prove that (whenever these products are defined) we have

$$(A \otimes B) \cdot (C \otimes D) = (AC) \otimes (BD)$$

Solution: Applying the arithmetic of block matrix multiplication, the (i, j)-block of $A \otimes B) \cdot (C \otimes D)$ is

$$\sum_k (a_{ik}B) \cdot (c_{kj}D) = \left(\sum_k a_{ik}c_{jk}\right) \cdot BD$$

Since $\sum_k a_{ik}c_{kj}$ is the (i, j)-entry of the product AC we are done.

Problem 6: Representing Tensor Transformations. Define two linear maps $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $U : V_2 \to V_1$ by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 2y \\ -x \end{pmatrix} \qquad U(f(t)) = f'(t)$$

where V_n is the vector space of polynomials in t with coefficients in $\mathbb{R}$ of degree $\leq n$. Find a matrix representation of the tensor transformation

$$(T \otimes U) : (\mathbb{R}^2 \otimes V_2) \to (\mathbb{R}^2 \otimes V_1)$$

Solution: Let $(\mathbf{e}_1, \mathbf{e}_2)$ be the standard basis of $\mathbb{R}^2$. The matrix for T in this basis is then

$$\begin{pmatrix} 1 & 2 \\ -1 & 0 \end{pmatrix}$$

Considering the bases $(t^2, t, 1)$ and $(t, 1)$ of V_2 and V_1, the matrix for U with respect to these bases is

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

A matrix representing $T \otimes U$ is the matrix tensor product

$$\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix} \otimes \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 4 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Problem 7: Tensors and Duals. Let V be an $\mathbb{F}$-vector space. Prove that we have a well-defined linear map

$$\varphi : V \otimes V^* \to \mathbb{F}$$

given by $\varphi(\mathbf{v} \otimes \lambda) := \lambda(\mathbf{v})$.

Solution: We check that the map $V \times V^* \mapsto \mathbb{F}$ given by $(\mathbf{v}, \lambda) \mapsto \lambda(\mathbf{v})$ is bilinear. Indeed, we have

$$(c\lambda + c'\lambda')(\mathbf{v}) = c\lambda(\mathbf{v}) + c'\lambda(\mathbf{v}')$$

and

$$\lambda(c\mathbf{v} + c'\mathbf{v}') = c\lambda(\mathbf{v}) + c'\lambda(\mathbf{v}')$$

for all $\lambda, \lambda' \in V^*, c, c' \in \mathbb{F}$, and $\mathbf{v}, \mathbf{v}' \in V$. By the Universal Property of the Tensor Product, we have the claimed well-defined linear map characterized by

$$\varphi : \mathbf{v} \otimes \lambda \mapsto \lambda(\mathbf{v})$$

Problem 8: Determinants and Tensors. Let $\mathbb{F}$ be a field. Prove that we have a well-defined linear map

$$\psi : (\mathbb{F}^n \otimes \cdots \otimes \mathbb{F}^n) \to \mathbb{F}$$

(where there are n factors of $\mathbb{F}^n$) given by

$$\psi : \mathbf{v}_1 \otimes \cdots \otimes \mathbf{v}_n \mapsto \det \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{pmatrix}$$

Solution: By the Universal Property of the Tensor Product, it is enough to show that for any fixed i we have

$$\det \begin{pmatrix} \mathbf{v}_1 & \cdots & c \cdot \mathbf{v}_i & \cdots & \mathbf{v}_n \end{pmatrix} = c \cdot \det \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_i & \cdots & \mathbf{v}_n \end{pmatrix}$$

and

$$\det \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_i + \mathbf{v}_i' & \cdots & \mathbf{v}_n \end{pmatrix} = \\ \det \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_i & \cdots & \mathbf{v}_n \end{pmatrix} + \det \begin{pmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_i' & \cdots & \mathbf{v}_n \end{pmatrix}$$

for all $c \in \mathbb{F}$ and $\mathbf{v}_1, \ldots, \mathbf{v}_i, \mathbf{v}_i', \ldots, \mathbf{v}_n \in \mathbb{F}^n$. But these are both properties of the determinant function.

Problem 9: (Optional; not to be handed in.) Let V, W, and U be $\mathbb{F}$-vector space. Prove the *tensor-hom adjunction isomorphism*

$$\mathrm{Hom}(V \otimes W, U) \cong \mathrm{Hom}(V, \mathrm{Hom}(W, U))$$

of $\mathbb{F}$-vector spaces.