Math 100B: Homework 3

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Problem 1

(a) If $r, s \in R$ and $r^n = 0$ for some n then $(rs)^n = r^n s^n = 0(s^n) = 0$ so $rs \in N$. If we have $s^m = 0$ for some m then $(r+s)^{nm} = r^{nm} + Prs + s^{nm} = Prs \in N$ since $rs \in N$ (P is some polynomial from the middle terms of the binomial expansion). Thus $(Prs)^p = 0$ for some p so $(r+s)^{nmp} = 0$ and $r + s \in N$. Therefore N is an ideal since it is closed under addition and closed under multiplication with an arbitrary element from the ring.

(b) If $r \in R$ was a nonzero nilpotent element then $r + N = 0 + N$ since $r \in N$ by definition.

(c) Let $r \in N$ arbitrary with $r^n = 0$. Notice that $0 \in P$ so either $r \in P$ or $r^{n-1} \in P$. If $r^{n-1} \in P$ then either $r \in P$ or $r^{n-2} \in P$. Inducting over the exponent, we have that $r \in P$, but since r was arbitrary, $N \subset R$.

Problem 2

(a) If $f \in I_X$ and $g \in R$ then $f(a) = 0$ for all $a \in X$. Thus $(fg)(a) = 0$ for all a so $fg \in I_X$. If $f, g \in I_X$ then $f(a) = 0$ and $g(a) = 0$ for all $a \in X$. Thus $(f + g)(a) = 0$ for all a so $f + g \in I_X$. Therefore R is a ring. The function

$$f(a) = \begin{cases} 0 & a \in X \\ 1 & a \notin X \end{cases}$$

is the generator of the principal ideal R since any function $h \in R$ can be written as the product of f and some function $g \in R$ that matches h for all values not in X.

(b) I_X is a maximal ideal when $\mathbb{R} - X$ only contains a single point since the only other ideal that contains it is the entire ring R. If $\mathbb{R} - X$ contains more than one point then $I_X \subset I_Y$ where Y is X with an additional missing point added so I_X is not maximal.

I_X is a prime ideal when X contains less than two points. If X is the empty set then $I_X = R$ which is trivally prime. If X contains a single point a and if $fg \in I_X$ then $fg(a) = 0$ implies either $f(a) = 0$ or $g(a) = 0$ so I_X is prime. If X contains two or more points $a, b \in X$ then it is possible for $fg \in I_x$ If $f(a) = 0$ and $g(b) = 0$ but $f(b) \neq 0$ and $g(a) \neq 0$.

Problem 3

(a) If $r \in I \cap R$ and $s \in R$ then $rs \in I$ since I is an ideal and $rs \in R$ because $r \in R$ and$s \in R$. Thus $rs \in I \cap R$. If $r, s \in I \cap R$ then $r + s \in I$ and $r + s \in R$ since I and R are both subgroups under addition. Thus $r + s \in I \cap R$ and $I \cap R$ is an ideal of R.

(b) If $a, b \in R$ and $ab \in I \cap R$ then either $a \in I$ or $b \in I$ because I is a prime ideal of S. However $a, b \in R$ so either $a \in I \cap R$ or $b \in I \cap R$, which means $I \cap R$ is a prime in R.

(c) No. If I is a subring and $R = I$, then $I \cap R = R$ is definitionally is not maximal. For example if $I = R = \mathbb{R}$ and $S = \mathbb{C}$ then $\mathbb{R}$ is not a maximal ideal of $\mathbb{R}$.

Problem 4

Suppose that $I = (p)$ was a principal ideal generated by $p \in R$. This means that $2 = pq$ for some $q \in R$, and so p must be a constant. It must also be that $x = pr$ for some $r \in R$ so q must be a linear polynomial and $p = -1, 1$ so that p divides 1(which is the coefficient of x). However neither -1 or 1 generate I so it cannot be that I is a principal ideal and R is not a principal ideal domain.

Problem 5

(a) ($\Longrightarrow$) Let $f, g \in F[x]$ and $(f) \subseteq (g)$. Since $f \in (f)$ it is also $f \in (g)$. Therefore $f = gh$ for some $h \in F[x]$.

($\Longleftarrow$) Let $f, g \in F[x]$ and $f = gh$ for some $h \in F[x]$. From the definition of the principal ideal,

$$\begin{aligned}(f) &= \{fr | \forall r \in F[x]\} \\ &= \{ghr | \forall r \in F[x]\} \\ &\subseteq \{gr | \forall r \in F[x]\} \\ &= (g).\end{aligned}$$

(b) The kernel of ϕ is $\ker \phi = \{f(x) \in F[x] \colon f(a) = f(b) = 0\}$. Therefore $\ker \phi = (x-a)(x-b)$ since it has roots a, b and no polynomial of degree ≤ 1 has a and b as roots.

The homomorphism ϕ is surjective since for arbitrary $(p, q) \in F \times F$, $\phi\left(\frac{(a-x)q+(x-b)p}{a-b}\right) = (p, q)$.

By the first isomorphism theorem, $F[x]/(f) \cong F \times F$.

(c) Since F is a field it only has the zero ideal and the unit ideal. So $F \times F$ only has the four ideals $\{0\} \times \{0\}$, $\{0\} \times F$, $F \times \{0\}$, $F \times F$. According to the correspondence theorem for quotient rings, the ideals of $F[x]/(g(x))$ are in correspondence with the ideals of $F[x]$ that contain $g(x)$, and since $F[x]$ is a principal ideal domain its only ideals that contain $g(x)$ are $F[x]$, $((x-a)^2)$, and $(x-a)$. Therefore $F \times F$ is not isomorphic to $F[x]/(g(x))$ since they have a different number of ideals.

Problem 6

If R is a ring with finitely many elements such that every element of R is idempotent, then R is ismomorphic to n copies of $\mathbb{Z}/2\mathbb{Z}$. However since $A \cap A = A$ for all $A \in R$, all n elements of R are idempotent.