Math 31AH: Fall 2021
Homework 4
Due 5:00pm on Friday 11/5/2021

Problem 1: Linear functionals and nonzero vectors. Let V be a finite-dimensional $\mathbb{F}$-vector space and let $\mathbf{v} \in V$ be nonzero. Prove there exists $\lambda \in V^*$ with $\lambda(\mathbf{v}) \neq 0$.

Problem 2: Induced maps. Let $T : V \to W$ be a linear transformation between finite-dimensional $\mathbb{F}$-vector spaces. Let $T^* : W^* \to V^*$ be the induced linear transformation between their dual spaces.

(1) If T is injective, prove that T^* is surjective.
(2) If T is surjective, prove that T^* is injective.

Problem 3: Infinite dimensionality and double duals. Let V be an infinite-dimensional $\mathbb{F}$-vector space with basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots\}$. Let $\varphi : V \to V^{**}$ be the linear map discussed in class given by

$$(\varphi(\mathbf{v}))(\lambda) := \lambda(\mathbf{v})$$

Is φ injective? Is φ surjective?

Problem 4: Matrices, duals, and linear maps. Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{F})$ be an $n \times n$ matrix over $\mathbb{F}$. We use A to define a function

$$T_A : \mathbb{F}^n \to (\mathbb{F}^n)^*$$

by the rule $(T_A(\mathbf{v}))(\mathbf{w}) := \mathbf{v}^T A\mathbf{w}$. Prove that T_A is linear.

Problem 5: Internal Direct Sums. Let V be an $\mathbb{F}$-vector space and let $U, W \subseteq V$ be subspaces. Prove that the following are equivalent.

(1) Every vector $\mathbf{v} \in V$ can be written uniquely as a sum $\mathbf{v} = \mathbf{u}+\mathbf{w}$ for $\mathbf{u} \in U$ and $\mathbf{w} \in W$.
(2) The union $U \cup W$ spans V and we have $U \cap W = 0$.

In this case, we write $V = U \oplus W$.[1]

Problem 6: Determinants and transposition. Let $A \in \mathrm{Mat}_{n\times n}(\mathbb{F})$ be a matrix and let A^T be its transpose. Prove that $\det A = \det A^T$.

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[1]This is the 'internal' direct sum. Before, we saw 'external' direct sums. Starting with two little vector spaces U and W we contructed a new bigger vector space $U \oplus W = \{(\mathbf{u}, \mathbf{w}) : \mathbf{u} \in U, \mathbf{w} \in W\}$. In the 'internal' case we start with a big vector space V and decompose it as $V = U \oplus W$ where U, W are subspaces. If $V = U \oplus W$ is an internal direct sum, the map $\mathbf{u} + \mathbf{w} \mapsto (\mathbf{u}, \mathbf{w})$ is an isomophism to the external direct sum $U \oplus W$. Mathematicians use the same notation for, and don't make much distinction between, internal and external direct sums. This is a case of **notational abuse!**

Problem 7: Determinants and the plane. Consider the $\mathbb{R}$-vector space $V = \mathbb{R}^2$.

(1) If $T : V \to V$ is rotation counterclockwise by an angle θ, prove that $\det T = 1$.
(2) If $T : V \to V$ is reflection across some line L going through the origin, prove that $\det T = -1$.

Problem 8: Determinants and block matrices. Let A be an $n \times n$ matrix, let B be an $n \times m$ matrix, and let C be an $m \times m$ matrix. Prove the identity

$$\det \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det(A) \cdot \det(C)$$

where 0 denotes a block of zeroes of size $m \times n$.

Problem 9: (Optional; not to be handed in.) Let $x_1, x_2, \ldots, x_n$ be variables. Prove the *Vandermonde identity*

$$\det \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ x_1 & x_2 & x_3 & \cdots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \cdots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \cdots & x_n^{n-1} \end{pmatrix} = \prod_{1 \le i < j \le n} (x_i - x_j)$$