Math 170B: Homework 5

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Problem 1

Let $x_n = x + nh$ and $f_n = f(x_n)$ for convenience. We can derive the first formula by interpolating $f(x)$ at x_0, x_1, x_2, x_3.

$$\begin{aligned}
p(x) &= f_0 + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3] \\
p'''(x) &= 6f[x_0, x_1, x_2, x_3] = 6\frac{f_3 - 3f_2 + 3f_1 - f_0}{6h^3} = \frac{1}{h^3}[f(x + 3h) - 3f(x + 2h) + 3f(x + h) - f(x)]
\end{aligned}$$

Using Taylor series, we have that

$$\begin{aligned}
f(x_0) &= f(x_0) \\
f(x_1) &= f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}h^2 + \frac{f'''(x_0)}{6}h^3 + \frac{f^{(4)}(x_0)}{24}h^4 + \frac{f^{(5)}(\xi(x_1))}{120}h^5 \\
f(x_2) &= f(x_0) + 2f'(x_0)h + 2f''(x_0)h^2 + \frac{4f'''(x_0)}{3}h^3 + \frac{2f^{(4)}(x_0)}{3}h^4 + \frac{4f^{(5)}(\xi(x_2))}{15}h^5 \\
f(x_3) &= f(x_0) + 3f'(x_0)h + \frac{9f''(x_0)}{2}h^2 + \frac{9f'''(x_0)}{2}h^3 + \frac{27f^{(4)}(x_0)}{8}h^4 + \frac{81f^{(5)}(\xi(x_1))}{40}h^5
\end{aligned}$$

Thus for some η,

$$f(x_3) - 3f(x_2) + 3f(x_1) - f(x_0) = h^3 f'''(x_0) + \frac{3}{2}f^{(4)}h^4 + \eta h^5$$

So the error term is $O(h)$.

Similarly when we interpolate at x_{-2}, x_{-1}, x_1, x_2,

$$\begin{aligned}
p(x) &= f_{-2} + (x - x_{-2})f[x_{-2}, x_{-1}] + (x - x_{-2})(x - x_{-1})f[x_{-2}, x_{-1}, x_1] + (x - x_{-2})(x - x_{-1})(x - x_1)f[x_{-2}, x_{-1}, x_1, x_2] \\
p'''(x) &= 6f[x_{-2}, x_{-1}, x_1, x_2] = 6\frac{f_2 - 2f_1 + 2f_{-1} - f_{-2}}{12h^3} = \frac{1}{2h^3}[f(x + 2h) - 2f(x + h) + 2f(x - h) - f(x - 2h)]
\end{aligned}$$

Using Taylor series, we have that

$$\begin{aligned}
f(x_{-2}) &= f(x_0) - 2f'(x_0)h + 2f''(x_0)h^2 - \frac{4f'''(x_0)}{3}h^3 + \frac{2f^{(4)}(x_0)}{3}h^4 - \frac{4f^{(5)}(\xi(x_{-2}))}{15}h^5 \\
f(x_{-1}) &= f(x_0) - f'(x_0)h + \frac{f''(x_0)}{2}h^2 - \frac{f'''(x_0)}{6}h^3 + \frac{f^{(4)}(x_0)}{24}h^4 - \frac{f^{(5)}(\xi(x_{-1}))}{120}h^5 \\
f(x_1) &= f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}h^2 + \frac{f'''(x_0)}{6}h^3 + \frac{f^{(4)}(x_0)}{24}h^4 + \frac{f^{(5)}(\xi(x_1))}{120}h^5 \\
f(x_2) &= f(x_0) + 2f'(x_0)h + 2f''(x_0)h^2 + \frac{4f'''(x_0)}{3}h^3 + \frac{2f^{(4)}(x_0)}{3}h^4 + \frac{4f^{(5)}(\xi(x_2))}{15}h^5
\end{aligned}$$

Thus for some η,

$$f(x + 2h) - 2f(x + h) + 2f(x - h) - f(x - 2h) = 2h^3 f'''(x_0) + \eta h^5$$

So the error term is $O(h^2)$. Since the error is better on the second formula, its more accurate

Problem 2

The approximation is

$$f'(x) \approx \frac{1}{2h}[-3f(x) + 4f(x+h) - f(x+2h)].$$

The taylor series

$$\begin{aligned}
f(x_0) &= f(x_0) \\
f(x_1) &= f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}h^2 + \frac{f'''(\xi(x_1))}{6}h^3 \\
f(x_2) &= f(x_0) + 2f'(x_0)h + 2f''(x_0)h^2 + \frac{4f'''(\xi(x_2))}{3}h^3
\end{aligned}$$

Thus

$$-3f(x) + 4f(x+h) - f(x+2h) = 2hf'(x_0) + \eta h^3$$

The error term is therefore $O(h^2)$.

1 Problem 3

Note that

$$\frac{\partial}{\partial x_i} a^t x = a_i \qquad \frac{\partial}{\partial x_i \partial x_j} a^t x = 0$$

Thus

$$\nabla f(x) = a \qquad \nabla^2 f(x) = 0$$

Problem 4

We have that for a positive semidefinite Q,

$$f(x) = \frac{1}{2}x^TQx + b^Tx.$$

Note that

$$\begin{aligned} f(x_k + \alpha p_k) &= \frac{1}{2}(x_k + \alpha p_k)^T Q(x_k + \alpha p_k) + b^T(x_k + \alpha p_k) \\ &= \frac{1}{2}x_k^T Q x_k + \alpha x_k^T Q p_k + \frac{1}{2}\alpha^2 p_k^T Q p_k + b^T x_k + \alpha b^T p_k. \end{aligned}$$

Thus

$$\frac{d}{d\alpha}(f(x_k + \alpha p)) = x_k^T Q p_k + \alpha p_k^T Q p_k + b^T p_k$$

Since

$$x_k^T Q + b^t = Q^t x_k + b = \nabla f_k.$$

Since

$$\frac{d}{d\alpha}(f(x_k + \alpha p_k)) = x_k^T Q p_k + \alpha p_k^T Q p_k + b^T p_k = 0.$$

We have that

$$\alpha = -\frac{\nabla f_k^T p_k}{p_k^T Q p_k}$$

Problem 5

Since for a matrix M

$$||Mx|| \leq ||M|| \cdot ||x||$$

We have that

$$||x|| = ||B^{-1}Bx|| \leq ||B^{-1}|| ||Bx|| \implies ||Bx|| \geq \frac{||x||}{||B^{-1}||}$$

From Cauchy Schwartz we have that

$$\cos\theta_k = -\frac{\nabla f_k^T p_k}{||\nabla f_k|| \cdot ||p_k||}$$

We can choose B_k to be a positive definite matrix matrix so that

$$\begin{aligned}
-\frac{\nabla f_k^T p_k}{||\nabla f_k|| \cdot ||p_k||} &= \frac{p_k^T B_k p_k}{||p_k^T B_k|| \cdot ||p_k||} \\
&\geq \frac{p_k^T B_k p_k}{||B_k|| \cdot ||p_k||^2} \\
&= \frac{p_k^T B_k^{1/2} B_k^{1/2} p_k}{||B_k|| \cdot ||p_k||^2} \\
&= \frac{||B_k^{1/2} p_k||^2}{||B_k|| \cdot ||p_k||^2} \\
&= \frac{||p_k||^2}{||B_k^{-1/2}||^2 \cdot ||B_k|| \cdot ||p_k||^2} \\
&= \frac{1}{||B_k^{-1}|| \cdot ||B_k||} \\
&\geq \frac{1}{M}
\end{aligned}$$

Matlab

```
function ydd = SecDeriv(x, y)
    h = x(2) - x(1);
    n = length(x);
    ydd = zeros(n,1);

    % forward difference approx of O(h^2)
    ydd(1) = (2*y(1)-5*y(2) + 4*y(3) - y(4))/h^3;
    % backward difference approx of O(h^2)
    ydd(n) = (2*y(n)-5*y(n-1) + 4*y(n-2) - y(n-3))/h^3;
    for i = 2:length(x)-1
        % center difference approx of O(h^2)
        ydd(i) = (y(i+1) - 2*y(i) + y(i-1))/h^2;
    end
```

```
>> SecDeriv(x, y)

ans =

  -15.5616
   -7.3800
   -6.9792
   -6.3176
   -5.2212
   -3.4280
   -0.4640
    4.4760
   12.4800
   25.7920
   47.7640
  139.4720
```