Math 100B: Homework 2

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Problem 1

We can write $x^{20} + 2x^{19} + 5x - 7 = (x+2)(x^{19} + 5) - 17$ so the remainder is -17.

Problem 2

The kernel is equal to the ideal $\ker\phi = (x^2 - 4x + 5)$ since $x^2 - 4x + 5$ is the monic polynomial of lowest degree in the kernel. If there was a polynomial of lower degree $ax + b \in \ker\phi$, then that would imply $2a + ia + b = 0$ which implies $a = 0$ and $b = 0$.

Problem 3

The homomorphism that sends $\tilde{\phi}(x) \to s$ and $\tilde{\phi}(r) \to \phi(r)$ is unique.

The action of this homomorphism on a element $\sum a_i x^i \in R[x]$ is uniquely determined by

$$\tilde{\phi}\left(\sum a_i x^i\right) = \sum \tilde{\phi}(a_i)\tilde{\phi}(x)^i = \sum \phi(a_i)s^i.$$

It is a homomorphism since it sends 1 to 1, and it respects addition and multiplication.

$$\tilde{\phi}(1) = \phi(1) = 1$$

$$\begin{aligned}
\tilde{\phi}(f+g) &= \tilde{\phi}\left(\sum (a_i + b_i)x^i\right) \\
&= \sum \tilde{\phi}((a_i + b_i)x^i) \\
&= \sum \phi(a_i + b_i)s^i \\
&= \sum \phi(a_i)s^i + \sum \phi(b_i)s^i \\
&= \tilde{\phi}(f) + \tilde{\phi}(g)
\end{aligned}$$

$$\begin{aligned}
\tilde{\phi}(fg) &= \tilde{\phi}\left(\sum a_i b_j x^{i+j}\right) \\
&= \sum \tilde{\phi}\left(\sum a_i b_j x^{i+j}\right) \\
&= \sum \phi(a_i b_j)s^{i+j} \\
&= \sum \phi(a_i)s^i + \sum \phi(b_j)s^j \\
&= \tilde{\phi}(f)\tilde{\phi}(g)
\end{aligned}$$

Problem 4

We have that $\mathbb{Z}/(x^2+1) = \mathbb{Z}[i]$ and the factor ring is defined as $\mathbb{Z}[i]/(i+1) = R$. We can obtain R by applying these relations in the opposite order by first applying the homomorphism $\mathbb{Z}[x] \to \mathbb{Z}$ that kills $x+1$(i.e. it sends $x \to -1$) and then kills $x^2+1 = 2$ (i.e. it takes modulo 2) to obtain that $R = \mathbb{F}_2$, which has two elements.

$$\begin{tikzcd}
\mathbb{Z}[x] \arrow[r, "\text{kill } x+1"] \arrow[d, "\text{kill } x^2+1"'] & \mathbb{Z} \arrow[d, "\text{kill } 2"] \\
\mathbb{Z}[i] \arrow[r, "\text{kill } i+1"'] & \mathbb{F}_2
\end{tikzcd}$$

Problem 5

(a) The ring $\mathbb{Z}/n\mathbb{Z}$ has characteristic n.

(b) We have that $n \cdot a = n \cdot (1 \cdot a) = (n \cdot 1) \cdot a = 0$.

Problem 6

Over the integers, Pascal's identity says

$$\begin{aligned}
\binom{n}{i-1} + \binom{n}{i} &= \frac{n!}{(i-1)!(n+1-i)!} + \frac{n!}{i!(n-i)!} \\
&= n!\left(\frac{i}{i!(n+1-i)!} + \frac{n+1-i}{i!(n+1-i)!}\right) \\
&= n!\left(\frac{n+1}{i!(n+1-i)!}\right) \\
&= \frac{(n+1)!}{i!(n+1-i)!} \\
&= \binom{n+1}{i}.
\end{aligned}$$

We can prove the binomial theorem by induction. In the base case where $n = 1$ then

$$(a+b) = a^0b^1 + a^1b^0$$

Assuming that the binomial theorem holds for n, we will now prove it holds for $n+1$ using the distributive property for rings and Pascal's identity for integers.

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)(a+b)^n \\
&= (a+b)\sum_{i=0}^{n}\binom{n}{i}a^ib^{n-i} \\
&= \sum_{i=0}^{n}\binom{n}{i}a^{i+1}b^{n-i} + \sum_{i=0}^{n}\binom{n}{i}a^ib^{n+1-i} \\
&= \sum_{i=1}^{n+1}\binom{n}{i-1}a^ib^{n+1-i} + \sum_{i=0}^{n}\binom{n}{i}a^ib^{n+1-i} \\
&= \sum_{i=0}^{n+1}\binom{n}{i-1}a^ib^{n+1-i} + \sum_{i=0}^{n+1}\binom{n}{i}a^ib^{n+1-i} \\
&= \sum_{i=0}^{n+1}\left(\binom{n}{i-1} + \binom{n}{i}\right)a^ib^{n+1-i} \\
&= \sum_{i=0}^{n+1}\binom{n+1}{i}a^ib^{n+1-i}
\end{aligned}$$

Notice that we rewrote the range of summation for convenience since the terms where $i = 0$ and $i = n+1$ are equal to zero.

Problem 7

The homomorphism sends 1 to 1.

$$\phi(1) = 1^p = 1.$$

The homomorphism respects multiplication

$$\phi(ab) = (ab)^p = a^p b^p = \phi(a)\phi(b)$$

Notice that

$$\binom{p}{i} = \frac{p!}{i!(p-i)!}$$

is a multiple of p for $0 < i < p$ since $i!$ and $(p-i)!$ are coprime with p when p is prime.

Using the binomial theorem, the homomorphism respects addition.

$$\begin{aligned}
\phi(a+b) &= (a+b)^p \\
&= \sum_{i=0}^{p} \binom{p}{i} a^i b^{p-i} \\
&= a^p + b^p \\
&= \phi(a) + \phi(b)
\end{aligned}$$