31AH - Midterm 1

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1 Problem 1

$\mathbb{F}^2$ is not a field since not every non-zero number has a multiplicative inverse. For example, the element $(0, 1)$ is not the zero element, but it has no inverse since it has a zero in the first index and multiplication is element-wise.

2 Problem 2

Matrix A simply swaps the x and y values. Therefore the matrix is $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

The change of basis matrix P can be written by writing the vectors in $\mathcal{B}$ in terms of the vectors in $\mathcal{C}$.

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + -\frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Therefore the matrix is $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$.

3 Problem 3

$Hom(V_5, V_3)$ has dimension 24. A polynomial of degree n has $n + 1$ terms and has $n + 1$ dimensions. Therefore, V_5 has 6 dimensions and V_3 has 4 dimensions. Since the dimension of a linear transformation is the product of the dimension of the domain and the codomain, $Hom(V_5, V_3)$ has dimension $6 \cdot 4 = 24$.

4 Problem 4

The transformation can be represented as the matrix $T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{bmatrix}$.

Since all three columns are the same vector, the transformation simply maps to scalar multiples of $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

Therefore, the only three T-invariant subspaces $W \subseteq \mathbb{R}^3$ are, the zero subspace ($W = \{0\}$), the subspace containing all scalar multiples of $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ ($W = \{c \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} : c \in \mathbb{R}\}$), and $\mathbb{R}^3$.

5 Problem 5

Let V be the vector space $\mathbb{R}^1$, and let $T : V \to V$ with $T(x) = e^x$. Since no two real numbers can result in the same e^x, T is injective. Since the e^x is never negative, T is not surjective.