Homework due Friday, November 10, at 11:00 pm Pacific Time.

A. Prove that a metric space X is connected if and only if the following holds. The only subsets of X which are both closed and open are $\emptyset$ and X.

B. Let $\{a_n\}$ be a sequence of bounded real numbers. Prove that $\ell \in \mathbb{R}$ is a subsequential limit of $\{a_n\}$ if and only if for every $\varepsilon > 0$, the set $\{n \in \mathbb{N} : |a_n - \ell| < \varepsilon\}$ is infinite.

C. Rudin, Chapter 2 (page 43), problem # 19, 20, 21.

D. Rudin, Chapter 3 (page 78), problems # 1, 3, 5

The following problems are for your practice, and will not be graded.

(1) Let (X, d) be a metric space and let $E \subset X$ be a connected set. Prove that any subset $E \subset F \subset \overline{E}$ is connected. In particular, $\overline{E}$ is connected.

(2) Using the definition of the limit of a sequence prove the following ($\mathbb{R}$ is considered with the standard metric).
   (a) $\lim_{n\to\infty} \frac{2n+1}{3n+5} = \frac{2}{3}$.
   (b) $\lim\left(\frac{n}{n+1}\right)^2 = 1$.

(3) Let (X, d) be a metric space, and let $\{p_n\}$ be a sequence in X. Assume that all the subsequences $\{p_{n_i}\}$ where $\mathbb{N} \setminus \{n_i : i\}$ is infinite converge (e.g. we are including sequences like $p_1, p_3, p_5, \ldots$ but not something like $p_2, p_3, p_4, p_5, \ldots$).
   Prove that $\{p_n\}$ converges.
   (Note that we are not assuming all the subsequences converge to the same limit, this indeed is the first thing you need to show.)

(4) (a) Prove that $\lim_{n\to\infty} \frac{4^n}{n!} = 0$.
   (b) Let $a \in \mathbb{R}$, prove that $\lim_{n\to\infty} \frac{a^n}{n!} = 0$.

(5) Let $\{a_n\}$ be a sequence of non-negative real numbers. For every $n \geq 1$, define
$$\sigma_n = \frac{a_1 + \cdots + a_n}{n}$$
   Prove that
$$\liminf a_n \leq \liminf \sigma_n \leq \limsup \sigma_n \leq \limsup a_n.$$