Homework 5, Math 181A Winter 2023

Due by Saturday noon, February 18 (pacific time).

Relevant section in textbook by Larsen and Marx: 5.5, 5.7.

Relevant lecture notes: Lecture 11, Lecture 12 and Lecture 13.

Problem 1: Let $X_1, \ldots, X_n$ be a random sample from the density function $f_X(x;\theta) = (\theta+1)x^\theta$ for $0 < x < 1$, and $f_X(x;\theta) = 0$ otherwise, where $\theta > -1$. Find the asymptotic variance of $\hat{\theta}$, where $\hat{\theta}$ is the maximum likelihood estimator of θ. (You do not need to work out the MLE $\hat{\theta}$.)

Problem 2: Let $X_1, \ldots, X_n$ be i.i.d. random variables from a Geometric(p), $p \in (0,1)$, which means that $P_X(k;p) = (1-p)^{k-1}p$ for $k = 1, 2, \ldots$ and we have $E[X_i] = 1/p$.

(a) Find the asymptotic variance of the maximum likelihood estimator $\hat{p}$.

(b) Let $n = 100$ and you observe the following data:

Number	1	2	3	4	5
Frequency	60	21	12	4	3

which means that 60 of the random variables $X_1, \ldots, X_{100}$ equal 1, 21 of them equal 2, and so on. Find a 95% confidence interval for p.

Problem 3: Larsen and Marx question 5.7.2.

Hint: You may use the fact that if $Y \sim N(0, \sigma^2)$, $E(Y^4) = 3\sigma^4$.

Problem 4: Suppose $X_1, X_2, \ldots$ is a sequence of i.i.d. random variables having the Poisson distribution with mean λ. Let $\hat{\lambda}_n = X_n$ (the n-th random variable).

(a) Is $\hat{\lambda}_n$ an unbiased estimator of λ? Explain your answer.

(b) Is $\hat{\lambda}_n$ a consistent estimator of λ? Explain your answer.

Problem 5: Suppose that 70% of men and 30% of women in a state support the incumbent candidate of governor. Assume that the numbers of men and women in the state are equal.

(a) Suppose 500 people are surveyed at random, and $\hat{p}_1$ is the proportion of people surveyed who say they support the incumbent. What is the variance of $\hat{p}_1$?

(b) Suppose that 250 men and 250 women are surveyed at random, and $\hat{p}_2$ is the proportion of people surveyed who say they support the incumbent. What is the variance of $\hat{p}_2$?

(c) Calculate the relative efficiency of $\hat{p}_2$ with respect to $\hat{p}_1$.

R Simulation:

(a) For $n = 1000$, simulate a random sample of size n from $N(0, 1)$. Use the generated data to give an approximation to the critical values when $\alpha = 0.01, 0.05, 0.1$, and compare them with the theoretical values $z_{\alpha/2}$. Repeat with $n = 10,000$ to get a better approximation. Hint: Commands `quantile( )`, `qnorm( )` may be helpful.

(b) For $n = 10$, simulate a random sample of size n from $N(\mu, \sigma^2)$, where $\mu = 1$ and $\sigma^2 = 2$; compute the sample mean. Repeat the above simulation 500 times, plot the histogram of the 500 sample means. Now repeat the 500 simulations for $n = 1,000$. Compare these two sets of results with different sample sizes, and discuss it in the context of consistency.