Mathematics 100A Homework 9
Due: Tuesday December 3 2024

Instructions: Please write clearly and fully explain your solutions. It is OK to work with others to solve the problems, but if you do so, you should write your solutions up separately. Copying solutions from your peers or a solutions manual will be deemed academic misconduct. Chapter and problem numbers, if any, refer to *Algebra*, second edition, by Michael Artin. Please feel free to reach out to me or the TAs if you have any questions.

1. Let $D_{2n} = \langle x, y | x^n = 1, y^2 = 1, yxy^{-1} = x^{-1} \rangle$ be the dihedral group of order $2n$. For $\theta \in \mathbf{R}$, let $R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$. Define $\rho_1 : D_{2n} \to \mathrm{GL}_2(\mathbf{C})$ as $\rho_1(x) = R_{2\pi/n}$ and $\rho_2(y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Define $\rho_2 : D_{2n} \to \mathrm{GL}_2(\mathbf{C})$ as $\rho_2(x) = \begin{pmatrix} e^{2\pi i/n} & 0 \\ 0 & e^{-2\pi i/n} \end{pmatrix}$ and $\rho_2(y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Prove that ρ_1 and ρ_2 are isomorphic.

2. Let $C_n = \mathbf{Z}/n\mathbf{Z}$ denote a cyclic group of order n. This group has n conjugacy classes, so there are n irreducible representations, up to isomorphism. Write down explicitly the n maps $\rho : \mathbf{Z}/n\mathbf{Z} \to \mathrm{GL}_1(\mathbf{C})$.

3. Recall the dihedral group $D_{2n} = \langle x, y | x^n = 1, y^2 = 1, yxy^{-1} = x^{-1} \rangle$. Suppose $\rho : D_{2n} \to \mathrm{GL}(V)$ is an irreducible representation of the dihedral group of order $2n$. Prove $\dim(\rho) \leq 2$. **Hint**: Let $C_n \subseteq D_{2n}$ be the index two normal subgroup which is cyclic of size n. Restricting ρ to C_n, we can write V as a direct sum of irreducible representations of C_n. In particular, there is $v \in V$ with $gv = \lambda(g)v$ for every $g \in C_n$, $\lambda(g) \in \mathbf{C}^\times$. Let $W = \mathrm{Span}(v, y \cdot v)$. Prove that W is a D_{2n} invariant subspace of V.