Math 31BH: Assignment 6

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1. Given an example of a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ which is differentiable at $(0,0)$ in the direction $(1,0)$, but not in the direction $(0,1)$.

   Solution: The function $f(x_1, x_2) = \|x_2\|$ is differentiable at $(0,0)$ in direction $(1,0)$ with derivative $\frac{\partial}{\partial x_1} = 0$, but it is not differentiable in direction $(0,1)$ since $\|x_2\|$ does not have a derivative relative to x_2 at $(0,0)$.

   This is because the left-hand limit of the newton quotient of $\|x_2\|$ equals -1, but the right-hand limit of the newton quotient equals 1 at $(0,0)$.

2. Let $D \subseteq \mathbf{V}$ be an open set in a Euclidean space, and suppose $f\colon D \to \mathbb{R}$ is differentiable at $\mathbf{v} \in D$ with respect to $\mathbf{w} \in \mathbf{V}$. Prove that f is $\mathbf{v}$ with respect to any scalar multiple of $\mathbf{w}$.

   Solution: Using the newton quotient, $f'(v, aw) = af'(v, w)$ for any scalar a since

$$\begin{aligned} f'(v, aw) &= \lim_{h \to 0} \frac{f(v + haw) - f(v)}{h} \\ &= a \lim_{ha \to 0} \frac{f(v + haw) - f(v)}{ha} \\ &= af'(v, w) \end{aligned}$$

   If $a = 0$, then the newton quotient trivially ends up equalling 0, so the above proof works for all scalars. Therefore f is differentiable with respect to any scalar multiple of w.

3. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be the function defined by $f(\mathbf{v}) = \mathbf{v} \cdot \mathbf{v}$.

   (a) Compute the partial derivatives of f at a given point $\mathbf{v} \in \mathbb{R}^n$.

   (b) Prove that f is continuously differentiable on $\mathbb{R}^n$.

   (c) Compute the gradient of f at a given $\mathbf{v} \in \mathbb{R}^n$.

   Solution:

   (a) Decomposing v into its n components,

$$\frac{\partial}{\partial x_i} f(v) = \frac{\partial}{\partial x_i}(v \cdot v) = \frac{\partial}{\partial x_i} \sum_{j=1}^{n} x_j^2 = 2x_i$$

(b) f is differentiable since partial derivative exist for all basis vectors, and the partial derivatives, $2x_i$, are all continuous.

(c) The gradient is a vector composed of all the partial derivatives so,

$$\nabla f(v) = (2x_1, ..., 2x_n) = 2v$$

4. Let A be a linear operator on a Euclidean space $\mathbf{V}$, and define a function $f\colon \mathbf{V} \to \mathbb{R}$ by $f(\mathbf{v}) = \langle \mathbf{v}, A\mathbf{v} \rangle$. Prove that f is continuously differentiable on $\mathbf{V}$, and compute the gradient $\nabla f(\mathbf{v})$ for each $\mathbf{v} \in \mathbf{V}$.

   Solution: Using the newton quotient and the bilinearity of the scalar product,

$$\begin{aligned}
f'(v,w) &= \lim_{h\to 0} \frac{f(v+hw) - f(v)}{h} \\
&= \lim_{h\to 0} \frac{\langle v+hw, A(v+hw)\rangle - \langle v, Av\rangle}{h} \\
&= \lim_{h\to 0} \frac{\langle v+hw, Av + Ahw)\rangle - \langle v, Av\rangle}{h} \\
&= \lim_{h\to 0} \frac{\langle v, Av\rangle + \langle v, Ahw\rangle + \langle hw, Av\rangle + \langle hw, Ahw\rangle - \langle v, Av\rangle}{h} \\
&= \lim_{h\to 0} \frac{h\langle v, Aw\rangle + h\langle Av, w\rangle + h^2\langle w, Aw\rangle}{h} \\
&= \lim_{h\to 0} \langle v, Aw\rangle + \langle Av, w\rangle + h\langle w, Aw\rangle \\
&= \langle A^* v, w\rangle + \langle Av, w\rangle \\
&= \langle (A + A^*)v, w\rangle
\end{aligned}$$

   Therefore f is always differentiable. Using the Cauchy-Schwartz inequality, the derivative is Lipschitz continuous since

$$\|f'(v,w)\| = \|\langle (A+A^*)v, w\rangle\| = \|\langle v, (A+A^*)w\rangle\| \leq \|v\|\|(A+A^*)w\|$$

   Since $f'(v,w) = \langle (A+A^*)v, w\rangle$, the gradient is $(A+A^*)v$.