Math 181B: Homework 4

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Exercise 1

$$\begin{aligned}
\hat{y} = a + bx &\implies \hat{y} = a + b\bar{x} + cbs_x \\
&\implies \hat{y} = \bar{y} + cbs_x \\
&\implies \hat{y} = \bar{y} + c\left(\frac{rs_y}{s_x}\right)s_x \\
&\implies \hat{y} = \bar{y} + crs_y
\end{aligned}$$

Exercise 2

1. We can take the reciprocal to linearize the equation, and we see that $\frac{1}{y}$ is linear in terms of $\frac{1}{x}$ with slope a and intercept b instead of slope b and intercept a.

$$\frac{1}{y} = a\frac{1}{x} + b$$

Plugging in the reciprocals of x and y into the formulas for the slope and intercept yields

$$a = \frac{n\sum_{i=1}^{n}\frac{1}{x_i y_i} - \left(\sum_{i=1}^{n}\frac{1}{x_i}\right)\left(\sum_{i=1}^{n}\frac{1}{y_i}\right)}{n\left(\sum_{i=1}^{n}\frac{1}{x_i^2}\right) - \left(\sum_{i=1}^{n}\frac{1}{x_i}\right)^2}$$

$$b = \frac{\sum_{i=1}^{n}\frac{1}{y_i} - a\sum_{i=1}^{n}\frac{1}{x_i}}{n}$$

2. Here is the function that finds a and b

```
getAB = function(x, y) {
          n = length(x)
          aNumerator = n*sum(1/(x*y)) - sum(1/x)*sum(1/y)
          aDenominator = n*sum(1/x^2) - sum(1/x)^2
          a = aNumerator/aDenominator

          b = (sum(1/y) - a*sum(1/x))/n

          return(list(a=a, b=b))
}

x = 1:10
y = x/(2+3*x)

ab = getAB(x, y)
ab$a # [1] 2
ab$b # [1] 3
```

Exercise 3

We are using an estimator that is a linear combination of independent normals, so the estimator itself also has a normal distribution.

$$\hat{\beta}_1 = \frac{\sum_{i=1}^n (x_i - \bar{x})Y_i}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

The expected value of $\hat{\beta}_1$ is

$$\begin{aligned}
E[\hat{\beta}_1] &= \frac{\sum_{i=1}^n (x_i - \bar{x})E[Y_i]}{\sum_{i=1}^n (x_i - \bar{x})^2} \\
&= \frac{\sum_{i=1}^n (x_i - \bar{x})(\beta_0 + \beta_1 x_i)}{\sum_{i=1}^n (x_i - \bar{x})^2} \\
&= \beta_0 \frac{\sum_{i=1}^n (x_i - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2} + \beta_1 \frac{\sum_{i=1}^n (x_i - \bar{x}) \cdot x_i}{\sum_{i=1}^n (x_i - \bar{x})^2} \\
&= \beta_1
\end{aligned}$$

Since $\mathrm{Var}(Y_i) = \mathrm{Var}(\epsilon_i) = \sigma_i^2$, the variance of $\hat{\beta}_1$ is

$$\begin{aligned}
\mathrm{Var}(\hat{\beta}_1) &= \frac{\mathrm{Var}(\sum_{i=1}^n (x_i - \bar{x})Y_i)}{(\sum_{i=1}^n (x_i - \bar{x})^2)^2} \\
&= \frac{\sum_{i=1}^n (x_i - \bar{x})^2 \,\mathrm{Var}(Y_i)}{(\sum_{i=1}^n (x_i - \bar{x})^2)^2} \\
&= \frac{\sum_{i=1}^n (x_i - \bar{x})^2 \sigma_i^2}{(\sum_{i=1}^n (x_i - \bar{x})^2)^2}
\end{aligned}$$

Since $\hat{\beta}_1$ is a normal distribution with mean β_1 and variance $\frac{\sum_{i=1}^n (x_i - \bar{x})^2 \sigma_i^2}{(\sum_{i=1}^n (x_i - \bar{x})^2)^2}$, we conclude that $\hat{\beta}_1 \sim N(\beta_1, \frac{\sum_{i=1}^n (x_i - \bar{x})^2 \sigma_i^2}{(\sum_{i=1}^n (x_i - \bar{x})^2)^2})$.