Math 20D HW8

Merrick Qiu

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Section 7.5

Problem 5 Let $W = \mathscr{L}\{w\}(s)$

$$\begin{aligned}
\mathscr{L}\{w'' + w\} = \mathscr{L}\left\{t^2 + 2\right\} &\implies s^2W - sw(0) - w'(0) + W = \frac{2}{s^3} + \frac{2}{s} \\
&\implies (s^2+1)W - s + 1 = \frac{2}{s^3} + \frac{2}{s} \\
&\implies (s^2+1)W = \frac{2}{s^3} + \frac{2}{s} + s - 1 \\
&\implies W = \frac{2}{s^3(s^2+1)} + \frac{2}{s(s^2+1)} + \frac{s}{s^2+1} - \frac{1}{s^2+1} \\
&\implies W = \frac{2}{s^3} + \frac{s}{s^2+1} - \frac{1}{s^2+1} \\
&\implies w = t^2 + \cos t - \sin t
\end{aligned}$$

Problem 7 Let $Y = \mathscr{L}\{y\}(s)$

$$\begin{aligned}
\mathscr{L}\{y'' - 7y' + 10y\} = \mathscr{L}\{9\cos t + 7\sin t\} &\implies (s^2Y - sy(0) - y'(0)) - 7(sY - y(0)) + 10Y = \frac{9s}{s^2+1} + \frac{7}{s^2+1} \\
&\implies (s^2Y - 5s + 4) - (7sY - 35) + 10Y = \frac{9s+7}{s^2+1} \\
&\implies (s^2 - 7s + 10)Y - 5s + 39 = \frac{9s+7}{s^2+1} \\
&\implies Y = \frac{9s+7}{(s^2+1)(s-2)(s+5)} + \frac{5s-39}{(s-2)(s+5)} \\
&\implies Y = \frac{5s^3 - 39s^2 + 14s - 32}{(s^2+1)(s-2)(s+5)} \\
&\implies Y = \frac{s}{s^2+1} - \frac{4}{s-5} + \frac{8}{s-2} \\
&\implies y = \cos t - 4e^{5t} + 8e^{2t}
\end{aligned}$$

Problem 12 Let $y(t) = w(t-1)$ and $Y = \mathscr{L}\{y\}(s)$

$$\begin{aligned}
\mathscr{L}\{y'' - 2y' + y\} = \mathscr{L}\{6(t-1) - 2\} &\implies (s^2Y - sy(0) - y'(0)) - 2(sY - y(0)) + Y = \frac{6}{s^2} - \frac{8}{s} \\
&\implies (s^2Y - 3s - 7) - 2(sY - 3) + Y = \frac{6}{s^2} - \frac{8}{s} \\
&\implies (s^2 - 2s + 1)Y - 3s - 1 = \frac{6}{s^2} - \frac{8}{s} \\
&\implies (s^2 - 2s + 1)Y - 3s - 1 = \frac{6}{s^2(s-1)^2} - \frac{8}{s(s-1)^2} + \frac{3s+1}{(s-1)^2} \\
&\implies Y = \frac{3s^3 + s^2 - 8s + 6}{s^2(s-1)^2} \\
&\implies Y = \frac{6}{s^2} + \frac{4}{s} - \frac{1}{s-1} + \frac{2}{(s-1)^2} \\
&\implies y = 6t + 4 - e^t + 2te^t \\
&\implies w = 6(t+1) - e^{t+1} + 2(t+1)e^{t+1} \\
&\implies w = 6t + 10 + e^{t+1} + 2te^{t+1}
\end{aligned}$$

Problem 18

$$
\begin{aligned}
\mathscr{L}\left\{y''-2y'-y\right\}=\mathscr{L}\left\{e^{2t}-e^{t}\right\} &\implies (s^2Y-sy(0)-y'(0))-2(sY-y(0))-Y=\frac{1}{s-2}-\frac{1}{s-1}\\
&\implies (s^2Y-s-3)-2(sY-1)-Y=\frac{1}{s-2}-\frac{1}{s-1}\\
&\implies (s^2-2s-1)Y-s-1=\frac{1}{s-2}-\frac{1}{s-1}\\
&\implies Y=\frac{1}{s^2-2s-1}\left(\frac{1}{s-2}-\frac{1}{s-1}+s+1\right)
\end{aligned}
$$

Problem 23 Let $G=\mathscr{L}\left\{g(t)\right\}$

$$
\begin{aligned}
\mathscr{L}\left\{y''+4y\right\}=\mathscr{L}\left\{g(t)\right\} &\implies (s^2Y-sy(0)-y'(0))+4Y=G\\
&\implies (s^2+4)Y+s=G\\
&\implies Y=\frac{G-s}{s^2+4}
\end{aligned}
$$

The Laplace of g is

$$
\mathscr{L}\left\{g(t)\right\}=\int_0^2 te^{-st}\,dt+5\int_2^\infty e^{-st}\,dt=\frac{e^{-2s}(3s+e^{2s}-1)}{s^2}
$$

Thus

$$
Y=\frac{-s^3+1+3se^{-2s}-e^{-2s}}{s^2(s^2+4)}
$$

Problem 24 Let $G=\mathscr{L}\left\{g(t)\right\}$

$$
\begin{aligned}
\mathscr{L}\left\{y''-y\right\}=\mathscr{L}\left\{g(t)\right\} &\implies (s^2Y-sy(0)-y'(0))-Y=G\\
&\implies (s^2-1)Y-s-2=G\\
&\implies Y=\frac{G+s+2}{s^2-4}
\end{aligned}
$$

The Laplace of g is

$$
\begin{aligned}
\mathscr{L}\left\{g(t)\right\} &= \int_0^3 e^{-st}\,dt+\int_3^\infty te^{-st}\,dt\\
&= -\frac{e^{-3s}}{s}+\frac{1}{s}+\frac{3e^{-3s}}{s}+\frac{e^{-3s}}{s^2}\\
&= \frac{s+2se^{-3s}+e^{-3s}}{s^2}
\end{aligned}
$$

Thus

$$
Y=\frac{s^3+2s^2+s+2se^{-3s}+e^{-3s}}{s^3(s-1)(s+1)}
$$

Problem 33

$$
\begin{aligned}
\mathscr{L}\left\{t^2y'\right\} &= (-1)^2\frac{d^2}{ds^2}\mathscr{L}\left\{y'\right\}\\
&= \frac{d^2}{ds^2}sY(s)-y(0)\\
&= \frac{d}{ds}(sY'(s)+Y(s))\\
&= sY''(s)+2Y'(s)
\end{aligned}
$$

Review Problems Chapter 4

Problem 6 The auxillary equation has roots

$$\begin{aligned} r^2 + 8r - 14 &\implies r = \frac{-8 \pm \sqrt{64 + 56}}{2} \\ &\implies r = -4 \pm \sqrt{30} \end{aligned}$$

The general solution is

$$y = C_1 e^{-4-\sqrt{30}} + C_2 e^{-4+\sqrt{30}}$$

Problem 10 The auxillary equation has roots

$$r^2 + 11 = 0 \implies r = i\sqrt{11}$$

The general solution is

$$y = C_1 \cos\sqrt{11}t + C_2 \sin\sqrt{11}t$$

Problem 13 The auxillary equation has roots

$$r^2 + 16 = 0 \implies r = 4i$$

The homogenous solution is

$$y_h = c_1 \cos 4t + c_2 \sin 4t$$

The particular solution has form

$$y_p = (A_1 t + A_0)e^t$$

Thus,

$$\begin{aligned} A_1 e^t(t+2) + A_0 e^t + 16e^t(A_1 t + A_0) = te^t &\implies 17A_1 = 1, 2A_1 + 17A_0 = 0 \\ &\implies A_1 = \frac{1}{17}, A_0 = -\frac{2}{289} \end{aligned}$$

The general solution is

$$y = c_1 \cos 4t + c_2 \sin 4t + \frac{te^t}{17} - \frac{2e^t}{289}$$

Problem 20 The auxillary equation is

$$2r^2 - 1 = 0 \implies r = \pm\frac{1}{\sqrt{2}}$$

The homogeneous solution is

$$y_h = C_1 e^{\frac{t}{\sqrt{2}}} + C_2 e^{-\frac{t}{\sqrt{2}}}$$

The particular solution has form

$$y_p = (A_1 t + A_0)\cos t + (B_1 t + B_0)\sin t$$

Thus,

$$-2\cos(A_1 t + A_0 - 2B_0 + \sin t(2A_1 + B_1 t + B_0)) - ((A_1 t + A_0)\cos t + (B_1 t + B_0)\sin t) = t\sin t$$

This yields $A_1 = 0$, $A_0 = -\frac{4}{9}$, $B_1 = -\frac{1}{3}$, $B_0 = 0$. The general solution is

$$y = C_1 e^{\frac{t}{\sqrt{2}}} + C_2 e^{-\frac{t}{\sqrt{2}}} - \frac{4}{9}\cos t - \frac{1}{3}t\sin t$$

Problem 25 The auxillary equation has roots

$$4r^2 - 12r + 9 = (2r-3)^2 \implies r = \frac{3}{2}$$

The homogenous solution is

$$y_h = C_1 e^{\frac{3}{2}t} + C_2 t e^{\frac{3}{2}t}$$

The particular solution has the form

$$A_0 e^{3t} + B_0 e^{5t}$$

Plugging this into the equation and solving for the coefficients yields $A_0 = \frac{1}{9}$ and $B_0 = \frac{1}{49}$ The general solution is

$$y = C_1 e^{\frac{3}{2}t} + C_2 t e^{\frac{3}{2}t} + \frac{1}{9}e^{3t} + \frac{1}{49}e^{5t}$$

Problem 31 The auxillary equation has roots

$$r^2 - 2r + 10 \implies r = \frac{2 \pm \sqrt{4-40}}{2} \implies r = 1 \pm 3i$$

The homogenous solution has form

$$y_h = C_1 e^t \cos 3t + C_2 e^t \sin 3t$$

The particular solution has form

$$y_p = A_0 \cos 3t + B_0 \sin 3t$$

Solving out for the constants yields $A_0 = 0$ and $B_0 = -1$. The general solution is

$$y = C_1 e^t \cos 3t + C_2 e^t \sin 3t - \sin 3t$$

In order for the initial values to hold, $C_1 = 2$ and $C_2 = -\frac{7}{3}$. The solution is

$$y = 2e^t \cos 3t - \frac{7}{3}e^t \sin 3t - \sin 3t$$

0.1 Review Problems Chapter 9

Problem 1 The characteristic equation is

$$(6-\lambda)(1-\lambda)+6 = 12-7\lambda+\lambda^2 = (\lambda-3)(\lambda-4)$$

For $\lambda_1 = 3$

$$A-\lambda I = \begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix}$$

The eigenvector is

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda_2 = 4$

$$A-\lambda I = \begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix}$$

The eigenvector is

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

The general solution is thus

$$x(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Problem 2 The characteristic equation is

$$(3-\lambda)(1-\lambda)+10 = 13-4\lambda+\lambda^2 \implies \lambda = \frac{4 \pm \sqrt{16-52}}{2} \implies \lambda = 2 \pm 3i$$

The eigenvector of $\lambda_1 = 2+3i$ is

$$A-\lambda I = \begin{bmatrix} 1-3i & 2 \\ -5 & -1-3i \end{bmatrix} \implies \begin{bmatrix} 10 & 2+6i \\ -5 & -1-3i \end{bmatrix} \implies \implies \begin{bmatrix} 5 & 1+3i \\ 0 & 0 \end{bmatrix}$$

The eigenvector is

$$\begin{bmatrix} -1-3i \\ 5 \end{bmatrix}$$

The general solution is

$$x(t) = c_1 \left\{ e^{2t}\cos 3t \begin{bmatrix} -1 \\ 5 \end{bmatrix} - e^{2t}\sin 3t \begin{bmatrix} -3 \\ 0 \end{bmatrix} \right\} + c_2 \left\{ e^{2t}\sin 3t \begin{bmatrix} -1 \\ 5 \end{bmatrix} + e^{2t}\cos 3t \begin{bmatrix} -3 \\ 0 \end{bmatrix} \right\}$$

Problem 5 The characteristic equation is

$$(1-\lambda)(4-\lambda)+2 = 6-5\lambda+\lambda^2 = (\lambda-2)(\lambda-3)$$

For $\lambda = 2$,

$$A = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$$

The eigenvector is

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

For $\lambda = 3$,

$$A = \begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix}$$

The eigenvector is

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

The fundamental matrix is

$$\begin{bmatrix} e^{2t} & e^{3t} \\ -e^{2t} & -e^{3t} \end{bmatrix}$$

Problem 11 The characteristic polynomial is

$$-\lambda(3-\lambda)+2=\lambda^2-3\lambda+2=(\lambda-1)(\lambda-2)$$

For $\lambda = 1$

$$A-\lambda I=\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$$

The eigenvector is

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For $\lambda = 2$

$$A-\lambda I=\begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix}$$

The eigenvector is

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The general solution is

$$x(t)=c_1e^t\begin{bmatrix} 1 \\ 1 \end{bmatrix}+c_2e^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Plugging in the initial values yield

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix}=c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}+c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

So $c_1 = 3$ and $c_2 = -2$. The final solution is

$$x(t)=3e^t\begin{bmatrix} 1 \\ 1 \end{bmatrix}-2e^{2t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$