Homework 2

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Most of the definitions, conventions and notations follow those in J. Hubbard and B. B. Hubbard's book. By abuse of notation, I may use 0 both for zero scalar and zero vector. Hopefully, this will cause no ambiguity.

Exercise 4.8.1(a)

Let's use development by the 1st row.

$$\begin{vmatrix} 1 & -2 & 3 & 0 \\ 4 & 0 & 1 & 2 \\ 5 & -1 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 0 \end{vmatrix} + 2\begin{vmatrix} 4 & 1 & 2 \\ 5 & 2 & 1 \\ 3 & 1 & 0 \end{vmatrix} + 3\begin{vmatrix} 4 & 0 & 2 \\ 5 & -1 & 1 \\ 3 & 2 & 0 \end{vmatrix}$$

$$= (-\begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix} + 2\begin{vmatrix} -1 & 2 \\ 2 & 1 \end{vmatrix}) + 2(4\begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} - \begin{vmatrix} 5 & 1 \\ 3 & 0 \end{vmatrix} + 2\begin{vmatrix} 5 & 2 \\ 3 & 1 \end{vmatrix})$$

$$+ 3(4\begin{vmatrix} -1 & 1 \\ 2 & 0 \end{vmatrix} + 2\begin{vmatrix} 5 & -1 \\ 3 & 2 \end{vmatrix})$$

$$= [-(-2) + 2 \times (-5)] + 2[4 \times (-1) - (-3) + 2 \times (-1)] + 3[4 \times (-2) + 2 \times 13]$$

$$= 40$$

Exercise 4.8.5

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. Then we have:

$$AB = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

$$BA = \begin{pmatrix} ea + fc & be + fd \\ ga + hc & gb + dh \end{pmatrix}$$

By direct computation, one sees

$$(ae + bg) + (cf + dh) = (ea + fc) + (gb + dh),$$

so that $tr(AB) = tr(BA)$.

Exercise 4.8.7

(a) $det(\mathbf{v}_1, \dots, \mathbf{0}, \dots \mathbf{v}_n) = det(\mathbf{v}_1, \dots, 0\mathbf{0}, \dots \mathbf{v}_n) = 0det(\mathbf{v}_1, \dots, \mathbf{0}, \dots \mathbf{v}_n) = 0$.
(b) Let $\mathbf{u} = \mathbf{v}$ and exchange them. Then

$$det(\dots \mathbf{u} \dots \mathbf{v} \dots) = -det(\dots \mathbf{v} \dots \mathbf{u} \dots) = -det(\dots \mathbf{u} \dots \mathbf{v} \dots)$$

The last equal is because $\mathbf{u} = \mathbf{v}$. This shows $det(\dots \mathbf{u} \dots \mathbf{v} \dots) = 0$.

Exercise 4.8.8

Write $B = (\mathbf{b}_1, \dots, \mathbf{b}_n)$ where $\mathbf{b}_i$ are column vectors. Regard f as a map $f : (\mathbb{R}^n)^n \to \mathbb{R}$. Then by block matrix multiplication, one has:

$$AB = A(\mathbf{b}_1, \dots, \mathbf{b}_n) = (A\mathbf{b}_1, \dots, A\mathbf{b}_n)$$

(Multilinearity)

$$\begin{aligned}
&f(\mathbf{b}_1, \dots, \mathbf{b}_{i-1}, \alpha\mathbf{u} + \beta\mathbf{v}, \mathbf{b}_{i+1}, \dots, \mathbf{b}_n) \\
&= \frac{det(AB)}{det(A)} \\
&= \frac{det(A\mathbf{b}_1, \dots, A\mathbf{b}_{i-1}, A(\alpha\mathbf{u} + \beta\mathbf{v}), A\mathbf{b}_{i+1}, \dots, A\mathbf{b}_n)}{det(B)} \\
&= \frac{det(A\mathbf{b}_1, \dots, A\mathbf{b}_{i-1}, \alpha A\mathbf{u} + \beta A\mathbf{v}, A\mathbf{b}_{i+1}, \dots, A\mathbf{b}_n)}{det(B)} \\
&= \frac{\alpha det(A\mathbf{b}_1, \dots, A\mathbf{u}, \dots, A\mathbf{b}_n) + \beta det(A\mathbf{b}_1, \dots, A\mathbf{v}, \dots, A\mathbf{b}_n)}{det(B)} \\
&= \alpha\frac{det(A\mathbf{b}_1, \dots, A\mathbf{u}, \dots, A\mathbf{b}_n)}{det(B)} + \beta\frac{det(A\mathbf{b}_1, \dots, A\mathbf{v}, \dots, A\mathbf{b}_n)}{det(B)} \\
&= \alpha f(A\mathbf{b}_1, \dots, A\mathbf{u}, \dots, A\mathbf{b}_n) + \beta f(A\mathbf{b}_1, \dots, A\mathbf{v}, \dots, A\mathbf{b}_n)
\end{aligned}$$

(Antisymmetry)

$$\begin{aligned}
&f(\mathbf{b}_1, \dots, \mathbf{b}_i, \dots, \mathbf{b}_j, \dots, \mathbf{b}_n) \\
&= \frac{det(A\mathbf{b}_1, \dots, A\mathbf{b}_i, \dots, A\mathbf{b}_j, \dots, A\mathbf{b}_n)}{detB} \\
&= -\frac{det(A\mathbf{b}_1, \dots, A\mathbf{b}_j, \dots, A\mathbf{b}_i, \dots, A\mathbf{b}_n)}{detB} \\
&= -f(\mathbf{b}_1, \dots, \mathbf{b}_j, \dots, \mathbf{b}_i, \dots, \mathbf{b}_n)
\end{aligned}$$

(Normalization)

$$f(\mathbf{e}_1, \dots, \mathbf{e}_n) = \frac{det(AI)}{det(A)} = \frac{det(A)}{det(A)} = 1$$

Exercise 4.8.11

Let's working over complex field. By proposition 4.8.24, there are invertible $n \times n$ matrix P and $m \times m$ matrix Q such that $P^{-1}AP$ and $Q^{-1}BQ$ are upper triangular. Let $\lambda_1, \dots, \lambda_n$ be diagonal of $P^{-1}AP$ and $\mu_1, \dots, \mu_m$ be diagonal of $Q^{-1}BQ$. Then we have:

$$\begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}^{-1} \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} = \begin{pmatrix} P^{-1}AP & P^{-1}CQ \\ 0 & Q^{-1}BQ \end{pmatrix}$$

The matrix $\begin{pmatrix} P^{-1}AP & P^{-1}CQ \\ 0 & Q^{-1}BQ \end{pmatrix}$ is a upper triangular matrix with diagonal $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_m$.

As a result, we have:

$$det \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = det \begin{pmatrix} P^{-1}AP & P^{-1}CQ \\ 0 & Q^{-1}BQ \end{pmatrix} = \lambda_1 \dots \lambda_n \mu_1 \dots \mu_m = det(A)det(B)$$

Here we use the property that for any triangular matrix, the determinant is the product of its diagonal.

Exercise 4.8.13

By theorem 4.8.11, only need to write σ_i as composition of transpositions and analyze the parity of the number of transpositions.

$\sigma_1 = (1\ 2)(1\ 2)$ has positive sign.

$\sigma_2 = (1\ 2\ 3) = (1\ 2)(2\ 3)$ has positive sign.

$\sigma_3 = (1\ 3\ 2) = (1\ 3)(3\ 2)$ has positive sign.

$\sigma_4 = (2\ 3)$ has negative sign.

$\sigma_5 = (1\ 2)$ has negative sign.

$\sigma_6 = (1\ 3)$ has negative sign.

Exercise 4.8.14

(a) Let A be a diagonal matrix whose diagonal entries are $a_1, \dots, a_n$. Then $\chi_A(t) = (t - a_1) \cdots (t - a_n)$. Notice that $A - a_i I$ is the diagonal matrix such that the i-th position on the diagonal is zero. As a result, $\chi_A(A) = (A - a_1 I) \dots (A - a_n I) = 0$.

(b) Notice that for any B, we have $det(P^{-1}BP) = det(B)$. And $P^{-1}(tI - A)P = tP^{-1}P - P^{-1}AP = tI - P^{-1}AP$. Then

$\chi_{P^{-1}AP}(t) = det(tI - P^{-1}AP) = det(P^{-1}(tI - A)P) = det(tI - A) = \chi_A(t)$.

Let's first prove a lemma.

Lemma 1. *Let $f(t)$ be a polynomial. Let A be a matrix and P be an invertible matrix. Then $f(P^{-1}AP) = P^{-1}f(A)P$.*

Proof. One can check the following property by direct computation:

$$(P^{-1}AP)^n = P^{-1}A^nP$$

(For example, $(P^{-1}AP)^2 = P^{-1}APP^{-1}AP = P^{-1}A^2P$.)

Then for general polynomial $f(t)$, use the property above and linearity. $\square$

Now, assume A is diagonalizable. Then there is invertible P and diagonal matrix D such that $A = P^{-1}DP$. As a result, we have:

$$\chi_A(A) = \chi_D(A) = \chi_D(P^{-1}DP) = P^{-1}\chi_D(D)P = 0$$

(c) We define a map $\phi : M_n \to M_n$ as $A \mapsto \chi_A(A)$ where M_n is the space of all $n \times n$ matrices. It is a continuous map since all components are polynomials.

As a result, $\phi^{-1}(0)$ is a closed set because under continuous map, inverse image of closed set is closed. From (b), we know all diagonalizable matrices are contained in $\phi^{-1}(0)$. Since diagonalizable matrices are dense in M_n by Theorem 4.8.26, we know $\phi^{-1}(0) = M_n$.