Math 31AH: Fall 2021
Homework 7
Due 5:00pm on Friday 11/19/2021

Problem 1: General Projections. Let $\mathbb{F}$ be an arbitrary field and let V be an $\mathbb{F}$-vector space. A linear projection $P : V \to V$ is a *projection* if $P \circ P = P$. If $P : V \to V$ is a projection, prove that $V = \text{Image}\, P \oplus \text{Ker}\, P$.

Problem 2: Trace. The *trace* of an $n \times n$ matrix $A = (a_{ij})$ is

$$\text{tr}(A) := \sum_{i=1}^{n} a_{ii}$$

If A, B are $n \times n$, prove that $\text{tr}(AB) = \text{tr}(BA)$. Deduce that

(1) Similar matrices have the same trace.
(2) If A is diagonalizable with eigenvalues $\lambda_1, \dots, \lambda_r$, then

$$\text{tr}(A) = \dim E_{\lambda_1} \cdot \lambda_1 + \cdots + \dim E_{\lambda_r} \cdot \lambda_r$$

You will learn the full importance of trace when you take representation theory.

Problem 3: Projections and trace. Let $P : V \to V$ be a projection, where V is a finite-dimensional vector space over an arbitrary field $\mathbb{F}$. We define $\text{tr}(P)$, the trace of P, as follows. Given an ordered basis $\mathcal{B}$ of V we set

$$\text{tr}(P) := \text{tr}[P]_{\mathcal{B}}^{\mathcal{B}}.$$

(1) Show that $\text{tr}(P)$ is well-defined (i.e. independent of the choice of $\mathcal{B}$). Your proof should work for any linear map $V \to V$, not just a projection.
(2) Prove that $\text{tr}(P) = \dim \text{Image}(P)$.

Problem 4: Polynomial Gram-Schmidt. Let V_3 be the vector space of polynomials of degree ≤ 3 and coefficients in $\mathbb{R}$. Endow V_3 with the inner product

$$\langle f(t), g(t) \rangle := \int_{-1}^{1} f(t) g(t) dt$$

The basis $\mathcal{B} = (1, t, t^2, t^3)$ of V_3 is not orthonormal. Apply Gram-Schmidt orthogonalization to $\mathcal{B}$ to get an orthonormal basis $\mathcal{C} = (f_0, f_1, f_2, f_3)$ of V_3.

Problem 5: Legendre Polynomials. Let V be the infinite-dimensional vector space of polynomials with real coefficients. Endow V with the inner product

$$\langle f(t), g(t)\rangle := \int_{-1}^{1} f(t)g(t)dt$$

For $n \geq 0$, let $P_n(t)$ be the *Legendre polynomial* defined by

$$P_n(t) := \frac{1}{2^n \cdot n!}\frac{d^n}{dt^n}(t^2-1)^n$$

Prove that $\langle P_n(t), P_m(t)\rangle = 0$ for $n \neq m$. (Hint: Integration by parts.)

Problem 6: Inner products and dual spaces. Let V be a finite-dimensional inner product space over $\mathbb{F}$. Given a fixed vector $\mathbf{v} \in V$, we have two functions

$$\varphi_{\mathbf{v}} : V \to \mathbb{F} \qquad \psi_{\mathbf{v}} : V \to \mathbb{F}$$

given by $\varphi_{\mathbf{v}}(\mathbf{w}) := \langle \mathbf{v}, \mathbf{w}\rangle$ and $\psi_{\mathbf{v}}(\mathbf{w}) := \langle \mathbf{w}, \mathbf{v}\rangle$.

When $\mathbb{F} = \mathbb{R}$, prove that that we have well-defined maps

$$\varphi : V \to V^* \qquad \psi : V \to V^*$$

given by $\varphi(\mathbf{v}) := \varphi_{\mathbf{v}}$ and $\psi(\mathbf{v}) := \psi_{\mathbf{v}}$ which are both isomorphisms. What happens when $\mathbb{F} = \mathbb{C}$?

Problem 7: Matrices and Inner Products. Let $V = \mathbb{R}^n$ and let A be an $n \times n$ real matrix. Define a function

$$\langle -, -\rangle : V \times V \to \mathbb{R}$$

by the rule

$$\langle \mathbf{v}, \mathbf{w}\rangle := \mathbf{v}^T A^T A\mathbf{w}$$

Prove that $\langle -, -\rangle$ is an inner product on $\mathbb{R}^n$ if and only if A is invertible.

Problem 8: Symmetric and Orthogonal. Consider the $\mathbb{R}$-vector space M_n of $n \times n$ real matrices. We have two subsets

$$S \subseteq M_n \quad \text{and} \quad O \subseteq M_n$$

consisting of symmetric (i.e. $A = A^T$) and orthogonal (i.e. $AA^T = I_n$) operators.

Which (if either) of S and O forms an $\mathbb{R}$-vector space? If either of them do, find their dimension.

Problem 9: (Optional; not to be handed in.) Let V be the infinite-dimensional vector space of continuous functions $[-\pi, \pi] \to \mathbb{R}$ which are **even**, i.e. $f(x) = f(-x)$ for all x. For any integer $n \geq 0$, define a function

$$f_n(x) := c_n \cos(nx)$$

where $c_n \in \mathbb{R}$ is a to-be-determined constant. Endow V with the inner product

$$\langle f, g \rangle := \int_{-\pi}^{\pi} f(x)g(x)dx$$

Prove that $\langle f_n, f_m \rangle = 0$ for $n \neq m$. Find values of c_n so that $\{f_0, f_1, f_2, \dots\}$ is orthonormal.