Math 170A, Fall 2022 ***HOMEWORK #4*** ***due Friday, Nov 10***

Homework problems that will be graded (Q1 - Q5, 30pts in total):

Q1. Let

$$w_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, w_2 = \begin{bmatrix} 3 \\ 3 \\ -1 \\ -1 \end{bmatrix}, \text{ and } w_3 = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

be vectors in $\mathbb{R}^4$. Apply the classical Gram-Schmidt process to find an orthonormal basis for the subspace spanned by w_1, w_2, and w_3.

Q2. For $A = [w_1, w_2]$ with w_1 and w_2 as above, find the minimizer to the least squares problem $\min_{x \in \mathbb{R}^2} ||b - Ax||_2$ for $b = [1,\ 1,\ 0,\ 1]^T$, using the QR method discussed in class.

Q3. Let Q be an orthogonal $n \times n$ matrix ($QQ^T = Q^TQ = I$), and consider the induced 2-norm on $n \times n$ matrices A,

$$||A||_2 = \max_{x \neq 0} \frac{||Ax||_2}{||x||_2} ,$$

with $||x||_2 = \sqrt{\sum_{i=1}^{n} x_i^2}$ the usual 2-norm on vectors of length n.

a) Calculate $||Q||_2$ and $\kappa_2(Q) = \kappa_{||\cdot||_2}(Q)$.

b) Consider the perturbed system $(Q + \delta Q)\hat{x} = b$, with $\hat{x} = x + \delta x$. What can we say about the relative error $||\delta x||_2/||\hat{x}||_2$ of the solution, in terms of the induced 2-norm $||\delta Q||_2$ of the perturbation matrix δQ?

Q4. Let M be an $n \times m$ matrix, $n \geq m$, and A be an $m \times m$ matrix such that $A = M^TM$. Also, suppose M is full-rank.

If $M = QR$ is the reduced QR factorization of M, write down the relationship between R and A. Specifically, given the properties of M and R, what is R to A? Justify your answer.

Q5. In a class announcement from before the midterm, you were given scripts for the MATLAB functions **gs.m** and **mgs.m**, which implement the codes given in class for classical and, respectively, modified Gram-Schmidt.

Copy them in your MATLAB directory, then go back to the Command Window and type

$$A = hilb(8);$$

This makes A a particularly ill-conditioned 8×8 matrix form MATLAB's matrix library.

Then, call

$$\begin{aligned} [Q1, R1] &= gs(A); \\ [Q2, R2] &= mgs(A); \end{aligned}$$

to get the computed factors $Q1$ and $R1$, respectively, $Q2$ and $R2$, with the classical, respectively, modified Gram-Schmidt algorithms.

Check how close $Q1$ and $Q2$, respectively, R_1 and R_2 are by typing

$$\begin{aligned} \text{checkQ} &= \text{norm}(Q1 - Q2) \\ \text{checkR} &= \text{norm}(R1 - R2) \end{aligned}$$

Finally, let us now test the quality of the $Q1$ and $Q2$ factors. At the prompt, type

$$\text{orthoQ1} = \text{norm}(Q1' * Q1 - eye(8)),$$

followed by

$$\text{orthoQ2} = \text{norm}(Q2' * Q2 - eye(8))$$

These commands let you know how close to orthogonal the matrices $Q1$, respectively $Q2$, are.

By examining checkQ, checkR, orthoQ1 and orthoQ2, explain how classical Gram-Schmidt and modified Gram-Schmidt perform on this ill-conditioned matrix. Take screenshots of the Command Window, making sure the outputs to the last 4 commands are visible.