Homework due Friday, December 1, at 11:00 pm Pacific Time.

A. Let $f : [0,1] \to \mathbb{R}$ be a function and let $a \in [0,1]$. Suppose there is some $\ell \in \mathbb{R}$ so that

$$\lim_{x \to a} f(x) = \ell.$$

Define

$$g(x) = \begin{cases} f(x) & x \neq a \\ \ell & x = a \end{cases}$$

Prove that g is continuous at a.

B. Let $f : [a,b] \to \mathbb{R}$ be a bounded function. For every $x \in [a,b]$ define the function $J_{f,x} : (0,,\infty) \to \mathbb{R}$ by

$$J_{f,x}(r) = \mathrm{diam}\Big(f\Big((x-r, x+r) \cap [a,b]\Big)\Big).$$

(1) Prove that $\lim_{r \to 0^+} J_{f,x}(r)$ exists for every $x \in [a,b]$. Denote this limit by $J_f(x)$.
(2) Prove that f is continuous at x if and only if $J_f(x) = 0$.
(3) Show that for every $\varepsilon > 0$ the set

$$\{x \in [a,b] : J_f(x) \geq \varepsilon\}$$

is a closed set.

C. Let (X,d) be a compact metric space and let $f : X \to X$ be a function with the following property. There exists some $0 < c < 1$ so that

$$d(f(x), f(y)) \leq c\, d(x,y) \quad \text{for all } x, y \in X.$$

(1) Prove that f is continuous.
(2) Let $x \in X$ and consider the sequence

$$f(x), f(f(x)), \ldots$$

Prove that this sequence is Cauchy.
(3) Use part (b) to conclude that there exists a unique $x_0 \in X$ so that $f(x_0) = x_0$.
(4) Prove part (c) using a theorem nested sequence of compact sets by considering $X \supset f(X) \supset f(f(X)) \supset \cdots$.

D. Rudin, Chapter 4 (page 98), problems # 1, 3.

The following problems are for your practice, and will not be graded.

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be a function which satisfies

$$f(x+y) = f(x) + f(y) \text{ for all } x, y \in \mathbb{R}$$

 (a) Prove that there exists some $\lambda \in \mathbb{R}$ such that $f(r) = \lambda r$ for all $r \in \mathbb{Q}$.
 (b) Prove that if f is continuous at 0, then it is continuous at every $x \in \mathbb{R}$.
 (c) Prove that if f is continuous at 0, then $f(x) = \lambda x$ for all $x \in \mathbb{R}$, where λ is the number given in (a).

(2) Let the notation be as in problem B.

 (a) Show that the set of discontinuity f is a union of (at most) countably many closed sets.
 (b) Construct (with justification) a function on $\mathbb{R}$ which is discontinuous on $\mathbb{Q}$ and continuous on $\mathbb{Q}^c$. (Hint: recall that $\mathbb{Q}$ is countable.)
 (c) (Bonus problem) Does there exist any function on $\mathbb{R}$ which is discontinuous on $\mathbb{Q}^c$ and continuous on $\mathbb{Q}$? (Hint: Use Rudin, Chapter 2, problem # 30.)