Math 181A: Homework 2

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Problem 1: 5.2.17

For the MOM estimator, we have that

$$\begin{aligned}
\mu_1 &= E[y_i] \\
&= \int_0^\theta y \frac{2y}{\theta^2}\, dy \\
&= \left[\frac{2y^3}{3\theta^2}\right]_0^\theta \\
&= \frac{2}{3}\theta.
\end{aligned}$$

Solving out for the parameter yields

$$\theta = \frac{3}{2}\mu_1.$$

. Replacing the mean by the sample mean yields

$$\hat{\theta} = \frac{3}{2}\hat{\mu_1} = \frac{3}{2}\bar{Y}.$$

For the given random sample, so the mean is 50 $\theta_e = 75$.

For the MLE,

$$L(\theta) = \prod_{i=1}^{n} \frac{2y_i}{\theta^2} = \frac{2^n}{\theta^{2n}} \prod_{i=1}^{n} y_i$$

$L(\theta)$ is maximized when θ is minimized, so $\theta_e = y_max = 92$. Compared to the MOM estimator, the MLE overestimates the parameter.

Problem 2: 5.2.18

We have that

$$\begin{aligned}
\mu_1 &= E[y_i] \\
&= \int_0^1 y(\theta^2+\theta)y^{\theta-1}(1-y)\,dy \\
&= (\theta^2+\theta)\int_0^1 y^{\theta} - y^{\theta+1}\,dy \\
&= (\theta^2+\theta)\left[\frac{y^{\theta+1}}{\theta+1} - \frac{y^{\theta+2}}{\theta+2}\right]_0^1 dy \\
&= (\theta^2+\theta)\left(\frac{1}{\theta+1} - \frac{1}{\theta+2}\right) \\
&= \frac{\theta}{\theta+2}.
\end{aligned}$$

Then we can solve out for θ_e in terms of $\bar{y}$.

$$\begin{aligned}
\bar{y} = \frac{\theta_e}{\theta_e+2} &\implies \theta_e\bar{y} + 2\bar{y} = \theta_e \\
&\implies \theta_e(\bar{y}-1) = -2\bar{y} \\
&\implies \theta_e = \frac{2\bar{y}}{1-\bar{y}}.
\end{aligned}$$

Problem 3: 5.2.21

We have that

$$\begin{aligned}
\mu_1 &= E[y_i] \\
&= \int_{\theta_1-\theta_2}^{\theta_1+\theta_2} \frac{y}{2\theta_2}\, dy \\
&= \left[\frac{y^2}{4\theta_2}\right]_{\theta_1-\theta_2}^{\theta_1+\theta_2} \\
&= \theta_1.
\end{aligned}$$

$$\begin{aligned}
\mu_2 &= E[y_i^2] \\
&= \int_{\theta_1-\theta_2}^{\theta_1+\theta_2} \frac{y^2}{2\theta_2}\, dy \\
&= \left[\frac{y^3}{6\theta_2}\right]_{\theta_1-\theta_2}^{\theta_1+\theta_2} \\
&= \frac{1}{3}(3\theta_1^2 + \theta_2^2).
\end{aligned}$$

Thus we have that

$$\theta_{1e} = \bar{y}$$

$$\theta_{2e} = \sqrt{3\hat{\mu}_2 - 3\bar{y}^2} = \sqrt{\frac{3}{n}\sum_{i=1}^{n} y_i^2 - 3\bar{y}^2}$$

Problem 4: 5.2.4

We have that

$$L(\theta) = \prod_{i=1}^{n} \frac{\theta^{2k_i} e^{-\theta^2}}{k_i!} = \frac{\theta^{2\sum_{i=1}^{n} k_i}}{e^{\theta^2 n}} \prod_{i=1}^{n} \frac{1}{k_i!}$$

$$\ln L(\theta) = \left(2\sum_{i=1}^{n} k_i\right) \ln\theta - n\theta^2 + \ln \prod_{i=1}^{n} \frac{1}{k_i!}$$

$$\frac{d}{d\theta} \ln L(\theta) = \frac{1}{\theta}\left(2\sum_{i=1}^{n} k_i\right) - 2n\theta = \frac{2\sum_{i=1}^{n} k_i - 2n\theta^2}{\theta} = 0$$

This implies that

$$\theta_e = \sqrt{\frac{1}{n}\left(\sum_{i=1}^{n} k_i\right)}$$

Problem 5: 5.2.6

We have that

$$L(\theta) = \prod_{i=1}^{n} \frac{\theta}{2\sqrt{y_i}} e^{-\theta\sqrt{y_i}} = \frac{\theta^n e^{-\theta(\sum_{i=1}^{n} \sqrt{y_i})}}{2^n \prod_{i=1}^{n} \sqrt{y_i}}.$$

$$\ln L(\theta) = n \ln \frac{\theta}{2} - \theta \sum_{i=1}^{n} \sqrt{y_i} - \ln \prod_{i=1}^{n} \sqrt{y_i}.$$

$$\frac{d}{d\theta} \ln L(\theta) = \frac{n}{\theta} - \sum_{i=1}^{n} \sqrt{y_i} = 0.$$

This implies that

$$\theta_e = \frac{n}{\sum_{i=1}^{n} \sqrt{y_i}} \approx \frac{4}{8.77} \approx 0.456.$$

Problem 6: Uniform Distribution

1. We have that

$$\mu_1 = E[X_i] = \frac{a+b}{2}$$

$$\begin{aligned}\mu_2 &= E[X_i^2] \\ &= \int_a^b \frac{x^2}{b-a} \\ &= \left[\frac{x^3}{3(b-a)}\right]_a^b \\ &= \frac{b^3-a^3}{3(b-a)} \\ &= \frac{a^2+ab+b^2}{3}\end{aligned}$$

This implies that

$$a+b = 2\bar{x} \implies b = 2\bar{x} - a$$

$$\begin{aligned}\mu_2 = \frac{a^2+ab+b^2}{3} &\implies \mu_2 = \frac{a^2+a(2\bar{x}-a)+(2\bar{x}-a)^2}{3} \\ &\implies a^2 - 2\bar{x}a + 4\bar{x}^2 - 3\mu_2 = 0 \\ &\implies a = \frac{2\bar{x} \pm \sqrt{4\bar{x}^2 - (16\bar{x}^2 - 12\mu_2)}}{2} \\ &\implies a = \bar{x} \pm \sqrt{3\mu_2 - 3\bar{x}}\end{aligned}$$

Since a is the lower bound and b is the upper bound, we have that

$$\hat{a} = \bar{x} - \sqrt{\frac{3}{n}\sum_{i=1}^{n} x_i^2 - 3\bar{x}}$$

$$\hat{b} = \bar{x} + \sqrt{\frac{3}{n}\sum_{i=1}^{n} x_i^2 - 3\bar{x}}$$

2. The likelyhood function is

$$L(a,b) = \prod_{i=1}^{n} f_X(x_i; a, b) = \begin{cases} \frac{1}{(b-a)^n} & a \le x_i \le b \text{ for all i} \\ 0 & \text{otherwise} \end{cases}$$

The likelyhood function is maximized when $b - a$ is minimized. Thus $\hat{b} = \max(x_1, \ldots, x_n)$ and $\hat{a} = \min(x_1, \ldots, x_n)$.