Math 120A: Homework 5

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Problem 1: Page 71 Question 4(a)

$$
\begin{aligned}
f(z) &= \frac{1}{z^4} \\
&= \frac{1}{(re^{i\theta})^4} \\
&= \frac{1}{r^4}e^{-i4\theta} \\
&= \frac{1}{r^4}(\cos 4\theta - i\sin 4\theta)
\end{aligned}
$$

$$u = \frac{\cos 4\theta}{r^4}, \quad v = -\frac{\sin 4\theta}{r^4}$$

$$ru_r = -\frac{4\cos 4\theta}{r^4} = v_\theta$$

$$u_\theta = -\frac{4\sin 4\theta}{r^4} = -rv_r$$

By the polar form of the Cauchy-Riemann equations, f is differentiable when $z \neq 0$.

$$
\begin{aligned}
f'(z) = e^{-i\theta}(u_r + iv_r) &= e^{-i\theta}\left(-\frac{4\cos 4\theta}{r^5} + i\frac{4\sin 4\theta}{r^5}\right) \\
&= -\frac{4}{r^5}e^{-i\theta}e^{-i4\theta} \\
&= -\frac{4}{(re^{i\theta})^5} \\
&= -\frac{4}{z^5}
\end{aligned}
$$

Problem 2: Page 76 Question 1(c)(d)

$$f(z) = e^{-y}\sin x - ie^{-y}\cos x$$
$$u = e^{-y}\sin x, \quad v = -e^{-y}\cos x$$
$$u_x = e^{-y}\cos x \quad u_y = -e^{-y}\sin x$$
$$v_x = e^{-y}\sin x \quad v_y = e^{-y}\cos x$$

Since $u_x = v_y$ and $u_y = -v_x$ everywhere, the function is entire.

$$\begin{aligned}
f(z) &= (z^2-2)e^{-x}e^{-iy} \\
&= ((x+iy)^2-2)e^{-x}(\cos y - i\sin y) \\
&= ((x^2-y^2-2)+2ixy)e^{-x}(\cos y - i\sin y) \\
&= e^{-x}((x^2-y^2-2)\cos y - i(x^2-y^2-2)\sin y + 2ixy\cos y + 2xy\sin y) \\
&= e^{-x}((x^2-y^2-2)\cos y + 2xy\sin y + i(2xy\cos y - (x^2-y^2-2)\sin y))
\end{aligned}$$

$$u = e^{-x}(x^2-y^2-2)\cos y + 2e^{-x}xy\sin y, \quad v = 2e^{-x}xy\cos y - e^{-x}(x^2-y^2-2)\sin y$$

$$\begin{aligned}
u_x &= 2e^{-x}x\cos y - e^{-x}(x^2-y^2-2)\cos y + 2e^{-x}y\sin y - 2e^{-x}xy\sin y \\
u_y &= -e^{-x}(x^2-y^2-2)\sin y - 2e^{-x}y\cos y + 2e^{-x}xy\cos y + 2e^{-x}x\sin y \\
v_x &= 2e^{-x}y\cos y - 2e^{-x}xy\cos y - 2e^{-x}x\sin y + e^{-x}(x^2-y^2-2)\sin y \\
v_y &= -2e^{-x}xy\sin y + 2e^{-x}x\cos y - e^{-x}(x^2-y^2-2)\cos y + 2e^{-x}y\sin y
\end{aligned}$$

Since $u_x = v_y$ and $u_y = -v_x$ everywhere, the function is entire.

Problem 3: Page 76 Question 2(a)(c)

$$f(z) = xy + iy$$

$$u = xy, \quad v = y$$

$$u_x = y \quad u_y = x$$
$$v_x = 0 \quad v_y = 1$$

By the Cauchy-Riemann equations, the function is only differentiable at the point $(0, 1)$. No point has a neighborhood of complex differentiable points so the function is nowhere analytic.

$$f(z) = e^y e^{ix} = e^y \cos(x) + ie^y \sin(x)$$

$$u = e^y \cos(x), \quad v = e^y \sin(x)$$

$$u_x = -e^y \sin(x) \quad u_y = e^y \cos(x)$$
$$v_x = e^y \cos(x) \quad v_y = e^y \sin(x)$$

By the Cauchy-Riemann equations, the function is only differentiable on the line $(0, y)$. No point has a neighborhood of complex differentiable points so the function is nowhere analytic.

Problem 4: Page 76 Question 4(c)

$$f(z) = \frac{z^2}{(z+2)(z^2+2z+2)}$$

Since f is the quotient of two polynomials, the function is analytic everywhere except for when the quotient is zero, which are the singular points. Factoring the quotient yields

$$(z+2)(z^2+2z+2) = (z+2)(z+1+i)(z+1-i).$$

Therefore the singular points are $z = -2, -1 \pm i$.

Problem 5: Page 77 Question 7

Since the function is analytic on a domain, it must follow the Cauchy-Riemann equations hold for each point. However since $v = 0$, this implies that $u_x = v_y = 0$ and $u_y = -v_x = 0$, which means that f must be constant.