Math 170C: Homework 2

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Problem 1

We need to solve the following integrals

$$a_{ij} = \int_0^{c_i} \ell(\tau)\,d\tau \qquad b_i = \int_0^1 \ell_i(\tau)\,d\tau.$$

The lagrange interpolating polynomials are

$$\ell_1(\tau) = \frac{(\tau-\frac{1}{2})(\tau-\frac{3}{4})}{(\frac{1}{4}-\frac{1}{2})(\frac{1}{4}-\frac{3}{4})} = 8\left(\tau-\frac{1}{2}\right)\left(\tau-\frac{3}{4}\right) = 8\tau^2-10\tau+3$$

$$\ell_2(\tau) = \frac{(\tau-\frac{1}{4})(\tau-\frac{3}{4})}{(\frac{1}{2}-\frac{1}{4})(\frac{1}{2}-\frac{3}{4})} = -16\left(\tau-\frac{1}{4}\right)\left(\tau-\frac{3}{4}\right) = -16\tau^2+16\tau-3$$

$$\ell_3(\tau) = \frac{(\tau-\frac{1}{4})(\tau-\frac{1}{2})}{(\frac{3}{4}-\frac{1}{4})(\frac{3}{4}-\frac{1}{2})} = 8\left(\tau-\frac{1}{4}\right)\left(\tau-\frac{1}{2}\right) = 8\tau^2-6\tau+1.$$

The indefinite integrals are

$$\int \ell_1(\tau)\,d\tau = \frac{8}{3}\tau^3-5\tau^2+3\tau+C$$

$$\int \ell_2(\tau)\,d\tau = -\frac{16}{3}\tau^3+8\tau^2-3\tau+C$$

$$\int \ell_3(\tau)\,d\tau = \frac{8}{3}\tau^3-3\tau^2+\tau+C$$

The weights are

$$b_1 = \int_0^1 \ell_1(\tau)\,d\tau = \frac{2}{3} \quad b_2 = \int_0^1 \ell_2(\tau)\,d\tau = -\frac{1}{3} \quad b_3 = \int_0^1 \ell_3(\tau)\,d\tau = \frac{2}{3}$$

$$a_{11} = \int_0^{\frac{1}{4}} \ell_1(\tau)\,d\tau = \frac{23}{48} \quad a_{12} = \int_0^{\frac{1}{4}} \ell_2(\tau)\,d\tau = -\frac{1}{3} \quad a_{13} = \int_0^{\frac{1}{4}} \ell_3(\tau)\,d\tau = \frac{5}{48}$$

$$a_{21} = \int_0^{\frac{1}{2}} \ell_1(\tau)\,d\tau = \frac{7}{12} \quad a_{22} = \int_0^{\frac{1}{2}} \ell_2(\tau)\,d\tau = -\frac{1}{6} \quad a_{23} = \int_0^{\frac{1}{2}} \ell_3(\tau)\,d\tau = \frac{1}{12}$$

$$a_{31} = \int_0^{\frac{3}{4}} \ell_1(\tau)\,d\tau = \frac{9}{16} \quad a_{32} = \int_0^{\frac{3}{4}} \ell_2(\tau)\,d\tau = 0 \quad a_{33} = \int_0^{\frac{3}{4}} \ell_3(\tau)\,d\tau = \frac{3}{16}.$$

The butcher table is

$$\begin{array}{c|ccc} 1/4 & 23/48 & -1/3 & 5/48 \\ 1/2 & 7/12 & -1/6 & 1/12 \\ 3/4 & 9/16 & 0 & 3/16 \\ \hline & 2/3 & -1/3 & 2/3 \end{array}$$

The order of accuracy of the method is $s + m = 3 + 1 = 4$ since the only inner product that equals 0 is

$$\langle q, \tau^0 \rangle = \int_0^1 \left(\tau - \frac{1}{4}\right)\left(\tau - \frac{1}{2}\right)\left(\tau - \frac{3}{4}\right)\tau^0 = 0.$$

Problem 2

The Adams-Bashford method estimates the integral with

$$\int_{t_n}^{t_n+h} f(x)\,dx \approx A_0 f_{n-3} + A_1 f_{n-2} + A_2 f_{n-1} + A_3 f_n.$$

For the Adams-Bashford method to be exact for polynomials of degree ≤ 3, the method must be exact for $1, x, x^2, x^3$ so

$$\begin{aligned}
\int_0^h 1\,dx &= h = A_0 + A_1 + A_2 + A_3 \\
\int_0^h x\,dx &= \frac{h^2}{2} = -3hA_0 - 2hA_1 - hA_2 \\
\int_0^h x^2\,dx &= \frac{h^3}{3} = 9h^2A_0 + 4h^2A_1 + h^2A_2 \\
\int_0^h x^3\,dx &= \frac{h^4}{4} = -27h^3A_0 - 8h^3A_1 - h^3A_2.
\end{aligned}$$

Solving this system of equations yields

$$A_0 = -\frac{3}{8}h \quad A_1 = \frac{37}{24}h \quad A_2 = -\frac{59}{24}h \quad A_3 = \frac{55}{24}h.$$

Plugging these values into the fundamental theorem of calculus gives us

$$\begin{aligned}
x_{n+1} &= x_n + \int_{t_n}^{t_n+h} f(x)\,dx \\
&= x_n - \frac{3}{8}hf_{n-3} + \frac{37}{24}hf_{n-2} - \frac{59}{24}hf_{n-1} + \frac{55}{24}hf_n \\
&= x_n + \frac{h}{24}[55f_n - 49f_{n-1} + 37f_{n-2} - 9f_{n-3}].
\end{aligned}$$

Problem 3

We have that

$$c_\ell = \int_{t_k}^{t_{k+1}} \left(\prod_{\substack{j=0 \\ j\neq \ell}}^{3} \frac{s - t_{k+1-j}}{t_{k+1-\ell} - t_{k+1-j}} \right) ds$$

$$c_0 = \int_0^h \frac{(s-0)(s+h)(s+2h)}{(h-0)(h+h)(h+2h)}\, ds = \frac{3}{8}h$$

$$c_1 = \int_0^h \frac{(s-h)(s+h)(s+2h)}{(0-h)(0+h)(0+2h)}\, ds = \frac{19}{24}h$$

$$c_2 = \int_0^h \frac{(s-h)(s-0)(s+2h)}{(-h-h)(-h-0)(-h+2h)}\, ds = -\frac{5}{24}h$$

$$c_3 = \int_0^h \frac{(s-h)(s-0)(s+h)}{(-2h-h)(-2h-0)(-2h+h)}\, ds = \frac{1}{24}h.$$

Putting these constants into the Adams-Moulton formula gives us

$$x_{n+1} = x_n + \frac{h}{24}[9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}]$$

Problem 4

We can do the change of variable $u = t_0 + hx$, $du = hdx$

$$\int_0^1 f(x)\,dx = \int_{t_0}^{t_0+h} f\left(\frac{u - t_0}{h}\right)\,\frac{du}{h} \approx \sum_{i=-n}^{n} A_i f(i).$$

Thus, we get that

$$\int_{t_0}^{t_0+h} f\left(\frac{u - t_0}{h}\right)\,du \approx h \sum_{i=-n}^{n} A_i f(i)$$

We can then evaluate f at points $i = t_0 + ih$ and likewise $u = t_0 + hx$ to get our final formula

$$\int_{t_0}^{t_0+h} f(x)\,dx \approx h \sum_{i=-n}^{n} A_i f(t_0 + ih)$$

Problem 5

If it is exactly for all polynomials of degree m, it will be exact for x^m for all choices of n, h, and A.
Choosing $n = 1$, $h = 1$, and $A = 0$ yields

$$\begin{aligned}
x_{n+1} &= x_n + \frac{5}{12}x'_{n+1} + \frac{8}{12}x'_n - x'_{n-1} \\
2^m &= 1 + \frac{5}{12}m2^{m-1} + \frac{8}{12}m \\
\left(2 - \frac{5}{12}m\right)2^{m-1} &= 1 + \frac{2}{3}m \\
\left(\frac{24-5m}{12}\right)2^{m-1} &= \frac{3+2m}{3} \\
2^{m-1} &= \frac{12+8m}{24-5m}
\end{aligned}$$

which has solutions at $m = 2, 3$.
Choosing $n = 1$, $h = 1$, and $A = 1$ yields

$$\begin{aligned}
x_{n+1} &= x_{n-1} + \frac{4}{12}x'_{n+1} + \frac{16}{12}x'_n + 4x'_{n-1} \\
2^m &= \frac{1}{3}m2^{m-1} + \frac{4}{3}m \\
\left(2 - \frac{1}{3}m\right)2^{m-1} &= \frac{4}{3}m \\
\left(\frac{6-m}{3}\right)2^{m-1} &= \frac{4m}{3} \\
2^{m-1} &= \frac{4m}{6-m}
\end{aligned}$$

which has solutions at $m = 2, 3, 4$. Thus, $m = 3$ and $A = 1$.