

## Homework 9, Math 181A Winter 2023

Due by Saturday noon, March 18 (pacific time).

Relevant section in textbook by Larsen and Marx: 6.5 (generalized likelihood ratio test). The content of likelihood ratio test is not covered in the textbook, apart from Lecture 22, Lecture 23 and Lecture 24, you can (but not necessary to) read Section 9.1 and 9.2 in “Mathematical Statistics and Data Analysis” by Rice as extra reference. R simulation problem is related to Lecture 25.

**Problem 1:** Suppose we observe a random variable  $X$  having the density

$$f_X(x; \theta) = \begin{cases} \theta x^{\theta-1} & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise,} \end{cases}$$

where  $\theta > 0$ .

- (a) What is the critical region for the likelihood ratio test of  $H_0 : \theta = 1$  against  $H_1 : \theta = 2$  at level of significance  $\alpha = 0.05$ ?
- (b) What is the power of the likelihood ratio test in part (a)?

**Problem 2:** Suppose we observe a random variable  $X$  having the exponential distribution with parameter  $\lambda$ , which means that  $X$  has the density

$$f_X(x; \lambda) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise,} \end{cases}$$

- (a) For  $\lambda_1 > 4$ , find the critical region of the likelihood ratio test for testing  $H_0 : \lambda = 4$  against  $H_1 : \lambda = \lambda_1$  at level of significance  $\alpha = 0.05$ .
- (b) For  $\lambda_1 < 4$ , find the critical region of the likelihood ratio test for testing  $H_0 : \lambda = 4$  against  $H_1 : \lambda = \lambda_1$  at level of significance  $\alpha = 0.05$ .

**Problem 3:** Suppose  $X_1, X_2, \dots, X_n$  is a sample of size  $n$  from a Poisson distribution with parameter  $\lambda > 0$ . Suppose we want to test  $H_0 : \lambda = \lambda_0$  against  $H_1 : \lambda = \lambda_1$ . Show that if  $\lambda_1 > \lambda_0$ , then the likelihood ratio test rejects  $H_0$  when  $\bar{X} \geq c'$ . (Note: Just show

the equivalence. You do not need to work out the value of  $c'$ .)

**Problem 4:** Suppose  $X_1, X_2, \dots, X_n$  is a sample of size  $n$  from the exponential distribution with parameter  $\lambda$ . Suppose that we want to test  $H_0 : \lambda = \lambda_0$  against  $H_1 : \lambda \neq \lambda_0$ . Show that the generalized likelihood ratio test rejects  $H_0$  when  $\bar{X}e^{-\lambda_0\bar{X}} \leq c'$ . (Note: Just show the equivalence. You do not need to work out the value of  $c'$ .)

**Problem 5:** Larsen and Marx question 6.5.1.

**Problem 6:** Suppose we observe a random variable  $X \sim \text{Binomial}(n, p)$ .

- (a) What is the generalized likelihood ratio for testing  $H_0 : p = 0.5$  against  $H_1 : p \neq 0.5$ ?
- (b) If we will reject  $H_0$  when  $|X - n/2| \geq c'$  (that is, we reject  $H_0$  when  $X$  is either too small or too large), what is the significance level of the test when  $n = 12$  and  $c' = 5$ ? Hint: use binomial distribution to compute the probability.

**R Simulation:** Old Faithful is a cone geyser located in Yellowstone National Park in Wyoming. Put the Old Faithful data set under your current working directory. You can use the code `read.table('oldfaithful.txt')` to read the data  $x_1, \dots, x_{272}$ . Take 1000 bootstrap samples to estimate a 95% confidence interval for each of (a) and (b) and give an estimate of (c):

- (a) mean  $\mu$ ,
- (b) median,
- (c)  $\mathbb{P}(|\bar{X} - \mu| > 5)$ .