# Math 140B: Homework 2

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## Problem 1

Let $x, y \in [0,1]$. By MVT, there exists $a$ such that $x \leq a \leq y$ and $f'(a)(x-y) = f(x) - f(y)$. Since a continuous function on a compact set attains a maximum, let $M = \sup_{x \in [0,1]} f'(x)$. Since $f'(a) \leq M$, we have that

$$|f(x) - f(y)| = |f'(a)(x-y)| \leq M|x-y|.$$

## Problem 2

Let $C = \{x \in [0,1] : f'(x)\}$ be the set of critical points. We want to show that the image $f(C)$ has no interval, $I = (a,b)$. For all $n \in \mathbb{N}$, there exists $I_1, \cdots, I_N$ such that

$$f(C) \subseteq \bigcup_{j=1}^{N} I_j$$

and

$$\sum_{j=1}^{N} |I_j| \leq \frac{K}{n}$$

for some $K > 0$. Pick $n_0$ such that $\frac{K}{n_0} < b - a$ for $(a,b) \subseteq f(C) \subseteq \bigcup_{j=1}^{N} I_j$. Thus

$$b - a = |(a,b)| \leq \left| \bigcup_{j=1}^{N} I_j \right| \leq \sum_{j=1}^{N} |I_j| \leq \frac{K}{n_0} < b - a$$

which is a contradiction, so $f(C)$ cannot contain an interval. A function like $f(x) = \frac{1}{x}\sin(\frac{1}{x})$ has infinite critical points.

## Problem 3

Since the function is monotonic and continuous, the Riemann integral exists.

$$\begin{aligned}
\int_0^1 x^2\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i}{n}\right)^2 \frac{1}{n} \\
&= \lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 \\
&= \lim_{n\to\infty} \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} \\
&= \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} \\
&= \lim_{n\to\infty} \frac{4n+3}{12n} \\
&= \frac{1}{3}
\end{aligned}$$

## Rudin 8

The definition of the derivative is

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}.$$

By the definition of the limit, for all $\epsilon > 0$ there exists a $\delta > 0$ such that when $0 < |t - x| < \delta$

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \epsilon.$$

This holds for vector valued functions of dimension $n$ as well since we can choose $\delta$ such that each component of $\frac{f(t)-f(x)}{t-x}$ is less than $\frac{\epsilon}{n}$ away from $f'(x)$.

## Rudin 11

Using L'Hospital's rule,

$$\begin{aligned}
\lim_{h\to 0}\frac{f(x+h)+f(x-h)-2f(x)}{h^2} &= \lim_{h\to 0}\frac{f'(x+h)+f'(x-h)}{2h}\\
&= \lim_{h\to 0}\frac{f'(x+h)+f'(x)}{2h}+\frac{f'(x)+f'(x-h)}{2h}\\
&= f''(x)
\end{aligned}$$

The following function has a limit that evaluates to 0 but has no second derivative

$$f(x)=\begin{cases} x^2 & x\ge 0\\ -x^2 & x<0\end{cases}$$

## Rudin 14

The definition of convex is that for all $x < y$,

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

If we let $t = \lambda x + (1-\lambda)y$ be the point that is the linear combination of $x$ and $y$, then $\lambda = \frac{y-t}{y-x}$, $1-\lambda = \frac{t-x}{y-x}$ where $x < t < y$. By the mean value theorem, there are values $c_1 < c_2$ such that

$$\frac{f(t)-f(x)}{t-x} = f'(c_1) \leq f'(c_2) = \frac{f(y)-f(t)}{y-t}.$$

Since $y-t = \lambda(y-x)$ and $t-x = (1-\lambda)(y-x)$,

$$\lambda(f(t)-f(x)) \leq (1-\lambda)(f(y)-f(t))$$

Isolating for $f(t)$ gives us the definition for convex

$$f(t) \leq \lambda f(x) + (1-\lambda)f(y)$$

The proof in the other direction is very similar. If $f''(x) \geq 0$ then $f'(x)$ must be monotonically increasing and vice versa.

# Rudin 15

For $h > 0$, Taylor's theorem says that

$$f(x+2h) = f(x) + f'(x) \cdot 2h + \frac{f''(\xi)}{2} \cdot (2h)^2$$

so if we take $h = \sqrt{\frac{M_0}{M_2}}$ then

$$\begin{aligned} |f'(x)| &= \left| \frac{1}{2h}(f(x+2h) - f(x)) - hf''(\xi) \right| \\ &\leq |\frac{2M_0}{2h} + hM_2| \\ &= |hM_2 + \frac{M_0}{h}| \\ &\leq 2\sqrt{M_0 M_2} \end{aligned}$$

This means that $M1 \leq 2\sqrt{M_0 M_2}$ which is equivalent to

$$M_1^2 \leq 4M_0 M_2$$

## Rudin 25

1. Newton's method estimates the root by finding where the tangent line at $x_k$ intersects the x-axis.

2. Since $x_1 \in (\xi, b)$ and $f$ is monotonically increasing, $f(x_1) > 0$. Since $f'(x_1) > 0$ as well, $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} < x_1$. By MVT, there exists $y$ such that $f'(y) = \frac{f(x_1)-f(x_2)}{x_1-x_2}$. Thus $f(x_2) = f(x_1) - f'(y)(x_1 - x_2) > f(x_1) - f'(x_1)(x_1 - x_2) = 0$ because $f'(y) < f'(x_1)$. Since $f(x_2) > 0$, we have that $\xi < x_2$. Using induction, we can show that $\xi < x_{n+1} < x_n$ for all $n$. Since the sequence of iterates is decreasing but bounded, it must converge to some value $l$, but since $l = l - \frac{f(l)}{f'(l)}$ , it must be that $l = \xi$.

3. Expanding the taylor series around $x_n$ yields

$$0 = f(\xi) = f(x_n) + f'(x_n)(\xi - x_n) + \frac{1}{2}f''(t_n)(\xi - x_n)^2$$

   Rearranging shows that

$$\frac{f(x_n)}{f'(x_n)} = -(\xi - x_n) - \frac{f''(t_n)}{2f'(x_n)}(\xi - x_n)^2$$

   Substituting this in to Newton's iteration yields our desired equation

$$x_{n+1} - \xi = x_n - \frac{f(x_n)}{f'(x_n)} - \xi = \frac{f''(t_n)}{2f'(x_n)}(\xi - x_n)^2$$

4. Since $A = M/2\delta$, $0 \le f''(t_n) \le M$ and $\delta < f'(x_n)$, we have that

$$0 \le x_{n+1}-\xi \le A(x_n-\xi)^2 = \frac{1}{A}[A(x_{n-1}-\xi)]^2 \le \frac{1}{A}[A^2(x_{n-2}-\xi)^2]^2 \cdots \le \frac{1}{A}[A(x_1-\xi)]^{2n}$$

5. When $f(x) = x$ then $f(x) = 0$ and vice versa. Since $g'(x) = \frac{f(x)f''(x)}{f'(x)^2}$, $g'(x)$ goes to zero as $x$ goes to $\xi$ as well.

6. The derivative of $x^{\frac{1}{3}}$ is $\frac{1}{3x^{\frac{2}{3}}}$ so

$$x_{n+1} = x_n - \frac{x_n^{\frac{1}{3}}}{\frac{1}{3x_n^{\frac{2}{3}}}} = -2x_n$$

   Newtons method thus diverges.