MATH 31AH - Homework 2

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1 A vector space?

V is not a vector space because the additive inverse does not exist. The zero element is $(b, 0)$ since for any vector v, $v + (b, 0) = v$. Let v be a vector (a, x) with $x \in \mathbb{R}$. Assume $-v$ exists. This would mean $v + (-v) = (b, 0)$. However, since the first element of v is a, the first element of $v + (-v)$ must be a. Therefore, the additive inverse does not exist for every element in V, and V is not a vector space.

2 Working with vector space axioms.

If $av = 0$, then $a = 0$ or $v = 0$.

Proof. Assume $av = 0$ with $a \neq 0$ and $v \neq 0$. Since $a \neq 0$, there exists a multiplicative inverse a^-1. Multiplying $av = 0$ by a^{-1} results in $ava^{-1} = 0 \cdot a^{-1}$, which implies $v = 0$. This contradicts $v \neq 0$, Therefore $a = 0$ or $v = 0$ must be true.

Note: Multiplying the zero vector by a scalar gives the zero vector. $0 + 0 = 0 \implies a(0 + 0) = a \cdot 0 \implies a \cdot 0 + a \cdot 0 = a \cdot 0 \implies a \cdot 0 = 0$.

□

3 Differentiable functions.

3.1 U is a subspace of V.

Proof. The zero function, $f(x) = 0$) is in U. Addition in U is closed since $(f+g)(3) = 0 = 0+0 = f(3) + g(3)$. Scalar multiplication in U is closed since $(\lambda \cdot g)(3) = \lambda \cdot g(3) = \lambda \cdot 0 = 0$. Since U is a subset of V, U is a subspace of V. □

3.2 W is not a subspace of V.

Proof. Addition in W is not closed since $(f + g)(3) = 7 \neq 14 = f(3) + g(3)$. Therefore, W is not a subspace of V. □

4 Lines in the complex plane.

W_c is only a vector space if and only if $c = 0$.

Proof. Addition in W_0 is closed since $(x + 0i) + (y + 0i) = x + y + 0i \in W_0$. Scalar multiplication in W_0 is closed since $\lambda \cdot (y + 0i) = \lambda y + 0i \in W_0$. Let c be any nonzero real number. Addition in W_c is not closed since $(x + ic) + (y + ic) = x + y + i(2c) \notin W_c$ and $2c \neq c$.

Since W_c is a vector space when $c = 0$, and W_c is not a vector space otherwise, W_c is only a vector space if and only if $c = 0$. $\square$

5 Eventually zero sequences

W is a subspace of V

Proof. The zero sequence is in W. Addition in W is closed since for any $a, b \in W$, u+w is eventually zero at $max(\{N_a, N_b\})$. This is because for any $n > N_a$ and $n > N_b$, $a_n = 0$ and $b_n = 0$. This implies that $(a + b)_n = 0$, so $a + b \in W$.

Multiplication in W is closed since for any $a \in W$ and $\lambda \in \mathbb{R}$, λa is eventually zero at N_a. This is because for any $n > N_a$, $a_n = 0$. This implies that $(\lambda a)_n = 0$ so $\lambda a \in W$.

Since $W \subseteq V$, and addition and multiplication is closed under W, W is a subspace of V. $\square$

6 A linear system over $\mathbb{R}$.

The system of linear equations can be solved with an augmented matrix.

$$
\begin{aligned}
&\begin{pmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \\ 7 & 8 & 9 & 1 \end{pmatrix}
\begin{array}{l} \urcorner^{-4} \\ \leftarrow\lrcorner_{+} \\ \end{array} \\
\Rightarrow &\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -3 \\ 7 & 8 & 9 & 1 \end{pmatrix}
\begin{array}{l} \urcorner^{-7} \\ \\ \leftarrow\lrcorner_{+} \end{array} \\
\Rightarrow &\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -3 \\ 0 & -6 & -12 & -6 \end{pmatrix}
\begin{array}{l} \\ \urcorner^{-2} \\ \leftarrow\lrcorner_{+} \end{array} \\
\Rightarrow &\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -3 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\begin{array}{l} \\ \mid \cdot -\frac{1}{3} \\ \end{array} \\
\Rightarrow &\begin{pmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\begin{array}{l} \leftarrow\urcorner^{+} \\ \lrcorner_{-2} \\ \end{array} \\
\Rightarrow &\begin{pmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

The augmented matrix shows that $x = z - 1$ and $y = -2z + 1$.

7 A linear system over $\mathbb{C}$.

The system of linear equations can be solved with an augmented matrix.

$$
\begin{aligned}
&\begin{pmatrix} 1 & i & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & -i & 2 \end{pmatrix}
\begin{array}{l} -1 \\ \leftarrow + \\ \end{array} \\
\Rightarrow &\begin{pmatrix} 1 & i & 0 & 1 \\ 0 & -i & 1 & 0 \\ 0 & 1 & -i & 2 \end{pmatrix}
\begin{array}{l} \\ -i \\ \leftarrow + \end{array} \\
\Rightarrow &\begin{pmatrix} 1 & i & 0 & 1 \\ 0 & -i & 1 & 0 \\ 0 & 0 & -2i & 2 \end{pmatrix}
\begin{array}{l} \leftarrow + \\ 1 \\ \end{array} \\
\Rightarrow &\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & -i & 1 & 0 \\ 0 & 0 & -2i & 2 \end{pmatrix}
\begin{array}{l} \\ \leftarrow + \\ -\frac{1}{2}i \end{array} \\
\Rightarrow &\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & -i & 0 & -i \\ 0 & 0 & -2i & 2 \end{pmatrix}
\begin{array}{l} \leftarrow + \\ \\ -\frac{1}{2}i \end{array} \\
\Rightarrow &\begin{pmatrix} 1 & 0 & 0 & 1-i \\ 0 & -i & 0 & -i \\ 0 & 0 & -2i & 2 \end{pmatrix}
\begin{array}{l} \\ \mid \cdot i \\ \mid \cdot \frac{1}{2}i \end{array} \\
\Rightarrow &\begin{pmatrix} 1 & 0 & 0 & 1-i \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & i \end{pmatrix}
\end{aligned}
$$

The augmented matrix shows that $x = 1 - i$, $y = 1$, and $z = i$.

8 A linear system over $\mathbb{F}_2$.

The system of linear equations can be solved with an augmented matrix.

$$
\begin{aligned}
&\begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{array}{l} 1 \\ + \\ \end{array} \\
\Rightarrow &\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{array}{l} \\ 1 \\ + \end{array} \\
\Rightarrow &\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Since the third row can never be true, there are no solutions.

9 Number of solutions to $\mathbb{F}_2$ (Optional).

Let the degrees of freedom of the system of equations be a. For each degree of freedom, there is a variable that can either be 0 or 1. There are two possible elements for each variable, so the total number of possible solutions is thus 2^a.