Math 140C: Homework 7

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Problem 1

By Theorem 11.24, we can treat $\phi(A)$ as a measure where

$$\phi(A) = \int_A x^\alpha \, dx.$$

Let $A_n = (\frac{1}{n}, 1)$ and $A = (0, 1)$. Since x^α is Riemann integrable on $(\frac{1}{n}, 1)$, we can write

$$\begin{aligned}
\int_0^1 x^\alpha &= \phi(A) \\
&= \lim_{n\to\infty} \phi(A_n) \\
&= \lim_{n\to\infty} \int_{\frac{1}{n}}^1 x^\alpha \\
&= \lim_{n\to\infty} \left[\frac{x^{\alpha+1}}{\alpha+1}\right]_{1/n}^1 \\
&= \frac{1}{\alpha+1}
\end{aligned}$$

Thus, the function is Lebesgue integrable when $\alpha > -1$. When $\alpha \leq -1$, the sequence of integrals diverges so the Lebesgue integral diverges as well.

Rudin 11.8

Theorem 6.20 says that $F'(x) = f(x)$ when f is continuous. Theorem 11.33 says that $f \in \mathcal{R}$ iff f is continuous almost everywhere. Therefore $F'(x) = f(x)$ almost everywhere on $[a, b]$.

Rudin 11.9

We can show that F is continuous at x if $F(x_n) \to F(x)$ for any sequence $x_n \to x$. To do so, we can apply the dominated convergence theorem. For any sequence $x_n \to x$, we can define $f_n \to f$ by

$$f_n(x) = \begin{cases} f(x) & a \leq x < x_n \\ 0 & \text{otherwise} \end{cases}.$$

Then choose $g = |f|$ to be the dominating function. Then we have that

$$\lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} \int_a^x f_n = \int_a^x f = F(x)$$

which implies that F is continuous since x was arbitrary.

Rudin 11.10

$f \in \mathscr{L}^2(\mu)$ on X implies

$$\int_X |f|^2 \, d\mu < \infty.$$

We can break X into two sets: let X_1 be the set where $|f(x)| > 1$ and X_2 be the set where $|f(x)| \leq 1$. Note that $\mu(X_2) < \infty$ since $\mu(X) < \infty$. Thus

$$\begin{aligned}
\int_X |f| \, d\mu &= \int_{X_1} |f| \, d\mu + \int_{X_2} |f| \, d\mu \\
&< \int_{X_1} |f|^2 \, d\mu + \mu(X_2) \\
&< \infty.
\end{aligned}$$

If we choose $X = \mathcal{R}$, then $f(x) = \frac{1}{1+|x|}$ is in $\mathscr{L}^2$ since

$$\begin{aligned}
\int_{\mathbb{R}} \left(\frac{1}{1+|x|}\right)^2 d\mu &= 2\int_0^\infty \frac{1}{(1+x)^2} \, d\mu \\
&< 2\int_0^\infty \frac{1}{x^2} \, d\mu \\
&< \infty
\end{aligned}$$

but $f \notin \mathscr{L}$ since

$$\begin{aligned}
\int_{\mathbb{R}} \frac{1}{1+|x|} \, d\mu &\geq \int_0^\infty \frac{1}{1+x} \\
&= [\ln(1+x)]_0^\infty \\
&\to \infty
\end{aligned}$$

Rudin 11.11

Suppose $\{f_n\}$ is a Cauchy sequence. Thus, we can find a sequence $\{n_k\}$ so that

$$||f_{n_k} - f_{n_{k+1}}|| < \frac{1}{2^k}$$

Let $g \in \mathscr{L}$. By the Schwarz inequality,

$$\begin{aligned}\int_X |g(f_{n_k} - f_{n_{k+1}})| \, d\mu &= \langle g, f_{n_k} - f_{n_{k+1}} \rangle \\ &\leq ||g|| ||f_{n_k} - f_{n_{k+1}}|| \\ &\leq \frac{||g||}{2^k}\end{aligned}$$

Therefore,

$$\sum_{k=1}^{\infty} \int_X |g(f_{n_k} - f_{n_{k+1}})| \, d\mu \leq ||g||$$

Then we interchange the summation and integration, which implies that

$$|g(x)| \sum_{k=1}^{\infty} |f_{n_k} - f_{n_{k+1}}| < \infty$$

which then implies that almost everywhere on X

$$\sum_{k=1}^{\infty} |f_{n_k} - f_{n_{k+1}}| < \infty.$$

Since the partial sum converges, it must be that the sequence converges.

Rudin 11.12

Yes its continuous on $[0, 1]$. If we have a sequence $x_n \to x$ we need to show that $g(x_n) \to g(x)$. Note that $f(x_n, y) \to f(x, y)$ because $f(x, y)$ is a continuous function of x for fixed y. Also since $|f(x_n, y)| \leq 1$, we can apply the dominated convergence theorem to say that $g(x_n) \to g(x)$.