Math 120A: Homework 6

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Problem 1

(a) $u_x = -2x$, $u_{xx} = -2$ and $u_y = 2y + 2$, $u_{yy} = 2$ so u is harmonic since $u_{xx} + u_{yy} = 0$.

(b) We need $v_y = -2x$ and $v_x = -2y - 2$ for Cauchy-Riemann to hold. Integrating v_y with respect to y gives us that

$$v(x, y) = -2xy + g(x)$$

Differentiating by x gives us

$$v_x = -2y + g'(x) = -2y - 2$$

which implies $g(x) = -2x + C$. Therefore the harmonic conjugate is $v = -2xy - 2x + C$.

(c)

$$\begin{aligned} f(z) &= y^2 + 2y - x^2 + i(-2xy - 2x + C) \\ &= -(x + iy)^2 - 2i(x + iy) + iC \\ &= -z^2 - 2iz + iC \end{aligned}$$

Problem 2

$$
\begin{aligned}
\log(1-i) &= \log\left(\sqrt{2}e^{i\left(-\frac{\pi}{4}+2\pi n\right)}\right) \\
&= \frac{\ln(2)}{2} + i\left(-\frac{\pi}{4} + 2\pi n\right)
\end{aligned}
$$

The principal argument is $\frac{\pi}{4}$.

$$
\operatorname{Log}(1-i) = \frac{\ln(2)}{2} - i\frac{\pi}{4}
$$

Problem 3

$$\log(-i) = \log\left(e^{i\left(-\frac{\pi}{2}+2\pi n\right)}\right) = i\left(-\frac{\pi}{2} + 2\pi n\right)$$

$$(-i)^{-i} = e^{-i\log(-i)} = e^{-\frac{\pi}{2}+2\pi n}$$

The principal value is

$$(-i)^{-i} = e^{-\frac{\pi}{2}}$$

Problem 4

$$\log(i) = \log\left(e^{i\left(\frac{\pi}{2}+2\pi n\right)}\right) = i\left(\frac{\pi}{2}+2\pi n\right)$$

$$(i)^{1+i} = e^{(1+i)\log(i)} = e^{(i-1)\left(\frac{\pi}{2}+2\pi n\right)} = e^{-\left(\frac{\pi}{2}+2\pi n\right)} e^{i\left(\frac{\pi}{2}+2\pi n\right)} = e^{-\left(\frac{\pi}{2}+2\pi n\right)} i$$

The principal value is

$$(i)^{1+i} = e^{-\frac{\pi}{2}} i$$

Problem 5

$$\begin{aligned}
\sin\left(z+\frac{\pi}{2}\right) &= \frac{e^{i\left(z+\frac{\pi}{2}\right)}-e^{-i\left(z+\frac{\pi}{2}\right)}}{2i} \\
&= \frac{e^{i\frac{\pi}{2}}e^{iz}-e^{-i\frac{\pi}{2}}e^{-iz}}{2i} \\
&= \frac{e^{iz}+e^{-iz}}{2} \\
&= \cos(z)
\end{aligned}$$