Math 140A: Homework 8

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A

1. From theorem 3.37 we have that

$$\infty = \liminf n+1 = \liminf \frac{(n+1)!}{n!} \leq \liminf \sqrt[n]{n!}$$

Thus this sequence diverges and the limit is ∞.

2. The term can be rewritten as

$$\frac{\sqrt[n]{n!}}{n} = \sqrt[n]{\frac{n!}{n^n}}$$

From theorem 3.37,

$$\liminf \frac{c_{n+1}}{c_n} \leq \liminf \sqrt[n]{c_n} \leq \limsup \sqrt[n]{c_n} \leq \limsup \frac{c_{n+1}}{c_n}$$

Using $c_n = \frac{n!}{n}$,

$$\frac{c_{n+1}}{c_n} = \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = \frac{(n+1)n^n}{(n+1)^{n+1}} = \left(\frac{n}{n+1}\right)^n$$

$$\frac{1}{e} = \liminf \left(\frac{n}{n+1}\right)^n \leq \liminf \sqrt[n]{\frac{n!}{n^n}} \leq \limsup \sqrt[n]{\frac{n!}{n^n}} \leq \limsup \left(\frac{n}{n+1}\right)^n = \frac{1}{e}$$

Thus

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}$$

B

Let $\epsilon > 0$ and choose N such that $c^N d(x, f(x)) < \epsilon$. Then for all $n \geq N$,

$$\begin{aligned} d(f^n(x), f^{n+1}(x)) &= cd(f^{n-1}(x), f^n(x)) \\ &= c^2 d(f^{n-2}(x), f^{n-1}(x)) \\ &\vdots \\ &= c^n d(x, f(x)) \\ &< \epsilon \end{aligned}$$

By the triangle inequality, for all $m \geq n \geq N$

$$\begin{aligned} d(f^n(x), f^m(x)) &\leq d(f^n(x), f^{n+1}(x)) + d(f^{n+1}(x), f^{n+2}(x)) + \cdots + d(f^{m-1}(x), f^m(x)) \\ &= c^n d(x, f(x)) + c^{n+1} d(x, f(x)) + \cdots + c^{m-1} d(x, f(x)) \\ &< \epsilon + c\epsilon + \cdots + c^{m-n}\epsilon \\ &< \frac{\epsilon}{1-c} \end{aligned}$$

We can readjust our choice of N such that $c^N d(x, f(x)) < (1-c)\epsilon$, and then for all $n, m \geq N$

$$d(f^n(x), f^m(x)) < \epsilon$$

Thus the sequence is cauchy.

Rudin 7

Since $(\sqrt{a_n} - \frac{1}{n})^2 \geq 0$, this implies that $\frac{1}{2}(a_n + \frac{1}{n^2}) \geq \frac{\sqrt{a_n}}{n}$. Since $\sum a_n$ converges and $\sum \frac{1}{n^2}$ converges, $\frac{\sqrt{a_n}}{n}$ must also converge by the comparison test.

Rudin 8

Since $\{b_n\}$ is monotonic and bounded, there exists $\sup|b_n|$. Because $\sup|b_n|\sum a_n$ converges since it is just multiplying everything by a constant, and $|b_n| < \sup|b_n|$ for all n, $\sum a_n b_n$ converges as well.

Rudin 10

When $z > 1$ all the terms in the series are $\geq a_n$. Since all coefficients must be nonzero integers, $|a_n| \geq 1$ for all a_n. Thus $\lim_{n\to\infty} a_n z^n \neq 0$ so $\sum a_n z^n$ diverges. Thus the radius of convergence is at most 1.

Rudin 11

1. If $\sum a_n$ is unbounded then $\frac{a_n}{1+a_n}$ does not approach 0 so a_n diverges, and if a_n is bounded by M then $\frac{1}{1+M}a_n \leq \frac{a_n}{1+a_n}$, so by comparison, a_n diverges.

2. $$\begin{aligned}\frac{a_{N+1}}{S_{N+1}} + \cdots + \frac{a_{N+k}}{S_{N+k}} &\geq \frac{a_{N+1}}{S_{N+k}} + \cdots + \frac{a_{N+k}}{S_{N+k}} \\ &= \frac{S_{N+k} - S_N}{S_{N+k}} \\ &= 1 - \frac{S_N}{S_{N+k}}\end{aligned}$$

   Thus this series cannot be cauchy convergent since the RHS can be made arbitrarily close to 1 by taking sufficiently large k.

3. $$\frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{s_n - s_{n-1}}{s_{n-1}s_n} = \frac{a_n}{s_{n-1}s_n} \geq \frac{a_n}{s_n^2}$$

   Since $\sum_{n=2}^{N} \frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{1}{a_1} - \frac{1}{s_n}$, this sequence approaches $\frac{1}{a_1}$ and so by comparison, $\frac{a_n}{s_n^2}$ converges.

4. Since $\frac{a_n}{1+n^2a_n} < \frac{1}{n^2}$, $\sum \frac{a_n}{1+n^2a_n}$ converges. However $\sum \frac{a_n}{1+na_n}$ may converge or diverge.