Math 140A: Homework 9

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A

Since the limit exists at a and the limit is equal to ℓ, we know that for all $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - \ell| < \epsilon$ for all point points $x \in [0, 1]$ with $0 < |x - a| < \delta$. Also note that a is the only point that is distance 0 from a, and since $f(a) = \ell$, we have that $|f(a) - \ell| = 0 < \epsilon$. Thus $|f(x) - \ell| < \epsilon$ for all point points $x \in [0, 1]$ and $|x - a| < \delta$, so g is continuous at a.

B

1. Since f is bounded, the diameter of the image of f is finite, and so the diameter of the image of f on a subset of the domain will be bounded.

   Thus $J_{f,x}(r)$ is bounded and it decreases monotonically as you decrease r, so any sequence of $J_{f,x}(r)$ values with smaller and smaller r values will converge, so the limit $J_f(x)$ exists.

2. ($\Longrightarrow$) Assume that $J_f(x) > 0$. Choose $0 < \epsilon < J_f(x)$. It is impossible to choose $\delta > 0$ in order to make f continuous at x since there must exist some point $p \in N_\delta(x)$ such that the distance between $f(x)$ and $f(p)$ will be $\epsilon < |f(x) - f(p)| \leq J_f(x)$. This is because $J_f(x) > 0$ and $J_f(x)$ is the limit of the diameters of the images of f for smaller and smaller neighborhoods around x. This is a contradiction, so $J_f(x) = 0$.

   ($\Longleftarrow$) For any $\epsilon > 0$, we want to show that there exists a $\delta > 0$ where $J_{f,x}(\delta) < \epsilon$. However since $J_f(x) = 0$, it is possible to find such a δ because we can find an r such that $J_{f,x}(r) < \epsilon$, and then set $\delta = r$.

3. If f is continuous, then this set is empty, which is vacuously closed. If f is not continuous, let p be a limit point of this set. Note that $p \in [0, 1]$ since $[0, 1]$ is closed. We want to show that $J_f(p) \geq \epsilon$, which we can show by showing that $J_{f,p}(r) \geq \epsilon$ for all r. Since p is a limit point of the set, we can find an x that is inside the r neighborhood and then choose r' such that $[x - r', x + r'] \subset [p - r, p + r]$. Then we can note that since $J_{f,x}(r') \geq \epsilon$, then $J_{f,p}(r) \geq \epsilon$. Thus $J_f(p) \geq \epsilon$ and p is in the set, so the set is closed.

C

(a) For all $\epsilon > 0$, we can choose $\delta = \frac{\epsilon}{c}$ which would mean that $d(f(x), f(y)) \leq cd(x, y) < \epsilon$. Thus f is continuous.

(b) For all $\epsilon > 0$, we need to choose N such that $d(f^n(x), f^m(x)) < \epsilon$ for all $m \geq n \geq N$. Let $y = f^{m-n}(x)$. If we choose N large enough such that $c^N d(x, y) < \epsilon$, then $d(f^n(x), f^n(y)) = d(^n(x), f^m(x)) < \epsilon$. Thus the sequence is cauchy.

(c) Since X is a compact metric space, and all compact metric spaces are complete, we know that the cauchy sequence from (b) converges to some x_0.

(d) Note that $X \supset f(X)$. $f(X) \supset f(f(X))$ as well since for all $x \in f(X)$, $x \in X$ as well, so $f(x) \in f(X)$ as well. By a similar argument we can construct a nested sequence of compact sets $X \supset f(X) \supset f(f(x))$ whose intersection is a single point, which is the fixed point x_0.

Rudin 1

No this does not. For the function

$$f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$$

The limit evaluates to 0 at $x = 0$ but it is not continuous at $x = 0$.

Rudin 3

Let p be a limit point of $Z(f)$. Since f is continuous, we know that at point p, there exists δ such that for all x with $d(x,p) < \delta$ implies that $d(f(x), f(p)) < \epsilon$.

Since p is a limit point, there exists some point q such that $d(p,q) < \delta$ and $f(q) = 0$. This would imply that $d(f(p), f(q)) < \epsilon$ because of continuity. If $f(p) \neq 0$ then there would not exist a δ for $0 < \epsilon < f(p)$ since $d(f(p), f(q)) = f(p)$, so it must be $f(p) = 0$. Since $f(p) = 0$, all limit points are in the zero set, so the zero set is closed.