Math 31BH: Final Exam

1. **[10 points]** Let $\mathbf{V}$ and $\mathbf{W}$ be Euclidean spaces, and let $f\colon \mathbf{V} \to \mathbf{W}$ be a function such that $\|f(\mathbf{v})\| \leq C\|\mathbf{v}\|$ for all $\mathbf{v} \in \mathbf{V}$, where C is a positive constant. Give a delta/epsilon proof that f is continuous.

2. **[10 points]** Find a parametric equation for the tangent line to the curve $f(t) = (\cos t, \sin t)$ at $t = \frac{\pi}{3}$.

3. **[10 points]** A particle is moving on a helix H in $\mathbb{R}^3$ such that its position at time t is $f(t) = (at, b\cos \omega t, b \sin \omega t)$, where $b, \omega > 0$ are positive constants.

   (a) Calculate the velocity vector of the particle at time t.

   (b) Calculate the speed of the particle at time t.

   (c) Calculate the acceleration vector at time t and show that it is orthogonal to the velocity vector at time t.

4. **[10 points]** Let $g_1, g_2\colon \mathbb{R} \to \mathbb{R}$ be differentiable functions, and define $f_1, f_2\colon \mathbb{R}^2 \to \mathbb{R}$ by $f_1(x,y) = g_1(x+y)$ and $f_2(x,y) = g_2(x-y)$. Prove that the gradient of f_1 is orthogonal to the gradient of f_2 at every point of $\mathbb{R}^2$.

5. **[10 points]** Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be defined by $f(\mathbf{v}) = (\mathbf{v}\cdot\mathbf{v})^a$, where $a \in \mathbb{R}$ is a constant. Show that $\nabla f(\mathbf{v}) = 2a(\mathbf{v}\cdot\mathbf{v})^{a-1}\mathbf{v}$.

6. **[10 points]** Let $f(x,y) = x^2 + xy$, and let $S \subset \mathbb{R}^2$ be the convex hull of the points $(0,0), (1,0), (1,1), (0,1)$.

   (a) Prove that $f(x,y) = x^3 + xy$ has a maximum value on S.

   (b) Calculate the maximum of f on S.

7. **[10 points]** Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x,y) = (e^x \cos y, e^x \sin y)$.

   (a) Prove that f is not an invertible function on $\mathbb{R}^2$.

   (b) Calculate the Jacobian matrix $J_f(x,y)$.

   (c) Prove that f is locally invertible at any point $\mathbf{v} \in \mathbb{R}^2$.