Math 140B: Homework 1

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Problem 1

A function $f : A \to \mathbb{R}$ is Holder continuous if there exists $C \geq 0$, $\alpha > 0$ such that for all $x, y \in A$,

$$d(f(x), f(y)) \leq C|x - y|^\alpha.$$

For all $\epsilon > 0$, if we choose $\delta = \left(\frac{\epsilon}{C}\right)^{\frac{1}{\alpha}}$ then f is uniformly continuous since

$$d(f(x), f(y)) \leq C|x - y|^\alpha < C\delta^\alpha = \epsilon.$$

Lipschitz continuous functions are Holder continuous with $\alpha = 1$, so they are also uniformly continuous.

Problem 2

Let $\epsilon > 0$ and $x \in \mathbb{R}$. Since $\alpha > 1$, $\frac{1}{n^{\alpha-1}}$ converges to 0 and it is possible to choose $n \in \mathbb{N}$ such that $n\left(\frac{1}{n}\right)^{\alpha} < \epsilon$. Since f is holder continuous, $|f(\frac{b}{n}) - f(0)| < \frac{\epsilon}{n}$ and likewise $|f(\frac{(k+1)b}{n}) - f(\frac{kb}{n})| < \frac{\epsilon}{n}$ for all $0 \leq k < n$. Thus, $|f(x) - f(0)| < \epsilon$ by the triangle inequality, and since ϵ was arbitrary, $f(x) = f(0)$ for all x.

Problem 3

Since $ff'' \geq 0$ and $(f')^2 \geq 0$,

$$(ff')' = ff'' + (f')^2 \geq 0.$$

Note that the derivative of the square of the function is nonnegative

$$(f^2)' = 2ff' \geq 0.$$

Therefore $|f|$ must monotonically increase, meaning that f increases when $f(x) > 0$, f decreases when $f(x) < 0$ and f can increase or decrease when $f(x) = 0$. Since $f(0) = 0$, if $f'(0) > 0$ it must remain monotonically increasing since $f > 0$ for all other points, and if $f'(0) < 0$ then f will be monotonically decreasing. If $f(0) \neq 0$, then a function like $f = (x - 0.5)^2$ shows that $f(0) = 0$ is necessary for the monotonicity to hold.

Rudin 2

For all intervals $[c, d] \subset (a, b)$, there exists a point $x \in [c, d]$ such that $f'(x)(d - c) = f(d) - f(c)$ by the mean value theorem. This implies that $f(d) > f(c)$ since $f'(x) > 0$. Since the choices of points c and d were arbitrary, f is strictly increasing in (a, b).

Let $t \to x$. Since f is continuous, the derivative exists and has the value of

$$\begin{aligned} g'(f(x)) &= \lim_{f(t) \to f(x)} \frac{g(f(t)) - g(f(x))}{f(t) - f(x)} \\ &= \lim_{t \to x} \frac{t - x}{f(t) - f(x)} \\ &= \lim_{t \to x} \frac{1}{\frac{f(t) - f(x)}{t - x}} \\ &= \frac{1}{f'(x)}. \end{aligned}$$

Rudin 3

$f(x)$ is one to one if it is monotonic. The derivative is

$$f'(x) = 1 + \epsilon g'(x)$$

If we choose $\epsilon < \frac{1}{M}$, then $f'(x) > 0$ since $|g'(x)| < M$. Thus, f is monotonically increasing and therefore it is one-to-one.

Rudin 4

Let $f(x)$ be

$$f(x) = C_0 x + \frac{C_1}{2}x^2 + \cdots + \frac{C_n}{n+1}x^{n+1}$$

The derivative is

$$f'(x) = C_0 + C_1 x + \cdots + C_n x^n.$$

Since $f(0) = 0$ and $f(1) = C_0 + \frac{C_1}{2} + \cdots + \frac{C_n}{n+1} = 0$, by the mean value theorem there is a point where the derivative of f is zero as well. Since the derivative is equal to the equation in question, the equation has a root between 0 and 1.

Rudin 5

Let $\epsilon > 0$ Since $f'(x) \to 0$, there exists M such that if $x > M$ then $|f'(x)| < \epsilon$. For all $x > M$, there is a point such that

$$f'(x_1) = f(x+1) - f(x)$$

by the mean value theorem. Since $|f'(x_1)| < \epsilon$, $g(x) = f(x+1) - f(x) < \epsilon$ as well so $g(x) \to 0$.

Rudin 6

Choose a, b where $0 < a < b$. Applying MVT on f for $[0, a]$ and $[a, b]$ implies that there exists c and d such that

$$f'(c) = \frac{f(a) - f(0)}{a - 0} = \frac{f(a)}{a}$$

$$f'(d) = \frac{f(b) - f(a)}{b - a}$$

Since f' is monotonically increasing, $f'(c) < f'(d)$ so

$$\begin{aligned} f'(c) < f'(d) &\implies \frac{f(a)}{a} < \frac{f(b) - f(a)}{b - a} \\ &\implies bf(a) - af(a) < af(b) - af(a) \\ &\implies bf(a) < af(b) \\ &\implies \frac{f(a)}{a} < \frac{f(b)}{b} \\ &\implies g(a) < g(b). \end{aligned}$$

Thus g is also monotonically increasing.