Math 100A: Homework 2

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Problem 1

$$\begin{aligned} ab &= a^{15}b \\ &= a^{12}(a^3b) \\ &= a^{12}ba^3 \\ &= ba^{15} \\ &= ba \end{aligned}$$

Problem 2

Suppose ab has order n. This means that $(ab)^n = 1$.

$$\begin{aligned} 1 &= (ab)^n \\ \iff 1 &= a(ba)^{n-1}b \\ \iff a^{-1}b^{-1} &= (ba)^{n-1} \\ \iff ba(a^{-1}b^{-1}) &= ba(ba)^{n-1} \\ \iff 1 &= (ba)^n \end{aligned}$$

Thus $(ab)^n = 1$ iff $(ba)^n = 1$.

Problem 3

If G has no proper subgroup, it must be cyclic. If it was not cyclic, then the group generated by an element of G would be a proper subgroup.

Since G is cyclic and it doesn't have any proper subgroups, it must be finite. If it was infinite, then it would have a proper subgroup. For example if g generates an infinite cyclic group G, then the group generated by g^2 would be a proper subgroup.

Let G be the finite cyclic group generated by g. G must either have order 1 or order p. Suppose the order, n, of G can be written as the product of two integers greater than 1, $n = pq$. Then the group generated by g^p would be a proper subgroup, which is a contradiction.

Problem 4

Suppose $a^m = 1$ and $b^n = 1$. Since G is abelian, we have that

$$(ab)^{mn} = a^{mn}b^{mn} = (a^m)^n(b^n)^m = 1$$

Thus ab has finite order.

Take $SL_2(\mathbb{R})$ to be an example of a non-abelian group.

$$a = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \qquad ab = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$b = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \qquad (ab)^n = \begin{bmatrix} 1 & -n \\ 0 & 1 \end{bmatrix}$$

$a^2 = b^2 = I$ but $(ab)^n$ is never I, so this proves ab need not have finite order in a non-abelian group.

Problem 5

Since G is cyclic, each element $g \in G$ can be written as $g = x^n$ for some integer n. Since φ is surjective, each element $h \in G'$ can be written as $h = \varphi(g)$ for some $g \in G$. Thus G' is a cyclic group generated by $\varphi(x)$ since each element h can be written as

$$h = \varphi(g) = \varphi(x^n) = \varphi(x)^n.$$

Suppose G is abelian. Let $h, h' \in G'$ with $h = \varphi(g)$ and $h' = \varphi(g')$ for some $g, g' \in G$. Then G' is abelian since

$$\begin{aligned} hh' &= \varphi(g)\varphi(g') \\ &= \varphi(gg') \\ &= \varphi(g'g) \\ &= \varphi(g')\varphi(g) \\ &= h'h. \end{aligned}$$