Math 140C: Homework 1

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Problem 1

1. ($\Longrightarrow$) Since $\boldsymbol{y} \in \operatorname{span}(E)$, we can write $\boldsymbol{y} = c_1\boldsymbol{v_1} + \cdots + c_r\boldsymbol{v_r}$ for some $c_i \in \mathbb{R}$. Thus we have that

   $$c_1\boldsymbol{v_1} + \cdots + c_r\boldsymbol{v_r} - \boldsymbol{y} = 0$$

   which implies that $E \cup \{\boldsymbol{y}\}$ is linearly dependent.

   ($\Longleftarrow$) Since $E \cup \{\boldsymbol{y}\}$ is linearly dependent, we can write $c_1\boldsymbol{v_1} + \cdots + c_r\boldsymbol{v_r} + c_{r+1}\boldsymbol{y} = 0$ for some $c_i \in \mathbb{R}$. Thus

   $$\boldsymbol{y} = -\frac{c_1}{c_{r+1}}\boldsymbol{v_1} - \cdots - \frac{c_r}{c_{r+1}}\boldsymbol{v_r}$$

   which implies that $\boldsymbol{y} \in \operatorname{span}(E)$.

2. If $\boldsymbol{x} \in \operatorname{span}(E)$ then $\boldsymbol{x} = a_1\boldsymbol{v_1} + \cdots + a_r\boldsymbol{v_r}$ for some $a_i \in \mathbb{R}$. It is also true that $\boldsymbol{x} = a_1\boldsymbol{v_1} + \cdots + a_r\boldsymbol{v_r} + 0\boldsymbol{y}$ so $\boldsymbol{x} \in \operatorname{span}(E \cup \{\boldsymbol{y}\})$.

   If $\boldsymbol{x} \in \operatorname{span}(E \cup \{\boldsymbol{y}\})$ then $\boldsymbol{x} = a_1\boldsymbol{v_1} + \cdots + a_r\boldsymbol{v_r} + a_{r+1}\boldsymbol{y}$ for some $a_i \in \mathbb{R}$. Since $E \cup \{\boldsymbol{y}\}$ is linearly dependent, $\boldsymbol{y} \in \operatorname{span}(E)$ so $\boldsymbol{y} = c_1\boldsymbol{v_1} + \cdots + c_r\boldsymbol{v_r}$ for some $c_i \in \mathbb{R}$. Thus

   $$\boldsymbol{x} = (a_1 + a_{r+1}c_1)\boldsymbol{v_1} + \cdots + (a_r + a_{r+1}c_r)\boldsymbol{v_r}$$

   so $\boldsymbol{x} \in \operatorname{span}(E)$

Rudin 9.1

If $\boldsymbol{x} \in \operatorname{span}(S)$ and $\boldsymbol{y} \in \operatorname{span} S$ then for some set of $\boldsymbol{v_1}, \cdots, \boldsymbol{v_n} \in S$ and constants $a_i, b_i \in \mathbb{R}$, we can write

$$\boldsymbol{x} = \sum_{i=1}^{n} a_i \boldsymbol{v_i} \qquad \boldsymbol{y} = \sum_{i=1}^{n} b_i \boldsymbol{v_i}$$

$\operatorname{span}(S)$ is a vector space since for all $c \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y} \in \operatorname{span}(S)$,

$$c\boldsymbol{x} = \sum_{i=1}^{n} c a_i \boldsymbol{v_i} \in \operatorname{span}(S)$$

$$\boldsymbol{x} + \boldsymbol{y} = \sum_{i=1}^{n} (a_i + b_i) \boldsymbol{v_i} \in \operatorname{span}(S).$$

Rudin 9.2

If A and B are linear transformations in X then for all $\boldsymbol{x}, \boldsymbol{v_1}, \boldsymbol{v_2} \in X$

$$\begin{aligned} BA(\boldsymbol{v_1} + \boldsymbol{v_2}) &= B(A(\boldsymbol{v_1} + \boldsymbol{v_2})) \\ &= B(A\boldsymbol{v_1} + A\boldsymbol{v_2}) \\ &= BA\boldsymbol{v_1} + BA\boldsymbol{v_2} \end{aligned}$$

$$BA(c\boldsymbol{x}) = B(cA\boldsymbol{x}) = cBA\boldsymbol{x}$$

Thus BA is also a linear transformation.
If A is one-to-one from X onto X then for all $\boldsymbol{x}, \boldsymbol{v_1}, \boldsymbol{v_2} \in X$ we can write

$$\boldsymbol{x} = A\boldsymbol{y} \qquad \boldsymbol{v_1} = A\boldsymbol{v_1} \qquad \boldsymbol{v_2} = A\boldsymbol{v_2}$$

for some vectors $\boldsymbol{y}, \boldsymbol{v_1}, \boldsymbol{v_2} \in X$. A^{-1} is a linear operator since

$$\begin{aligned} A^{-1}(\boldsymbol{v_1} + \boldsymbol{v_1}) &= A^{-1}(A\boldsymbol{v_1} + A\boldsymbol{v_2}) \\ &= A^{-1}A(\boldsymbol{v_1} + \boldsymbol{v_2}) \\ &= \boldsymbol{v_1} + \boldsymbol{v_2} \\ &= A^{-1}\boldsymbol{v_1} + A^{-1}\boldsymbol{v_1} \end{aligned}$$

$$A^{-1}(c\boldsymbol{x}) = A^{-1}(cA\boldsymbol{y}) = A^{-1}A(c\boldsymbol{y}) = c\boldsymbol{y} = cA^{-1}\boldsymbol{x}.$$

The inverse of A^{-1} is A since

$$A(A^{-1}\boldsymbol{x}) = A(A^{-1}A\boldsymbol{y}) = A\boldsymbol{y} = \boldsymbol{x}$$

Rudin 9.3

Suppose A is not 1-1. Then for some $\boldsymbol{y} \in Y$, there exists distinct $\boldsymbol{v}, \boldsymbol{w} \in X$ such that $A\boldsymbol{v} = \boldsymbol{y}$ and $A\boldsymbol{w} = \boldsymbol{y}$. Subtracting these two equations implies that

$$A(\boldsymbol{v} - \boldsymbol{w}) = 0$$

which contradicts our assumption that $A\boldsymbol{x} = 0$ only when $\boldsymbol{x} = 0$.

Rudin 9.4

Let $A \in L(X, Y)$ be a linear transformation Let $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{N}(A)$. $\mathcal{N}(A)$ is a vector space since

$$A(\boldsymbol{x} + \boldsymbol{y}) = A\boldsymbol{x} + A\boldsymbol{y} = \mathbf{0}$$

$$A(c\boldsymbol{x}) = cA\boldsymbol{x} = \mathbf{0}$$

Let $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{R}(A)$. We can write $\boldsymbol{x} = A\boldsymbol{p}$ and $\boldsymbol{y} = A\boldsymbol{q}$ for some $\boldsymbol{p}, \boldsymbol{q} \in X$. By the linearity of A,

$$\boldsymbol{x} + \boldsymbol{y} = A\boldsymbol{p} + A\boldsymbol{q} = A(\boldsymbol{p} + \boldsymbol{q}) \in \mathcal{R}(A)$$

$$c\boldsymbol{x} = cA\boldsymbol{p} = A(c\boldsymbol{p}) \in \mathcal{R}(A)$$

Rudin 9.5

Let $\boldsymbol{x} = \boldsymbol{x}_1\boldsymbol{e_1} + \cdots \boldsymbol{x}_n\boldsymbol{e_n}$ for the standard basis vectors $\boldsymbol{e}_i$. If we let $\boldsymbol{y} \in \mathbb{R}^n$ with $\boldsymbol{y}_i = A\boldsymbol{e_i}$ then

$$A\boldsymbol{x} = A(\boldsymbol{x}_1\boldsymbol{e_1} + \cdots \boldsymbol{x}_n\boldsymbol{e_n}) = c_1\boldsymbol{y_1} + \cdots + c_n\boldsymbol{y}_n = \boldsymbol{x} \cdot \boldsymbol{y}.$$

It is unique since if there was $\boldsymbol{z}$ such that $A\boldsymbol{x} = \boldsymbol{x} \cdot \boldsymbol{z}$, then

$$\boldsymbol{y} - \boldsymbol{z}|^2 = \boldsymbol{y} \cdot \boldsymbol{y} - \boldsymbol{y} \cdot \boldsymbol{z} - \boldsymbol{z} \cdot \boldsymbol{y} - \boldsymbol{z} \cdot \boldsymbol{z} = A(\boldsymbol{y}) - A(\boldsymbol{y}) - A(\boldsymbol{z}) + A(\boldsymbol{z}).$$

By the Schwarz inequality,

$$||A|| = \sup |A\boldsymbol{x}| = \sup |\boldsymbol{x} \cdot \boldsymbol{y}| \leq \sup |\boldsymbol{x}||\boldsymbol{y}|.$$

which implies that $||A|| \leq |y|$. Also note that $A\left(\frac{\boldsymbol{y}}{|\boldsymbol{y}|}\right) = \frac{\boldsymbol{y}}{|\boldsymbol{y}|} \cdot \boldsymbol{y} = |\boldsymbol{y}|$ so $||A|| \geq |y|$.