Math 100b Winter 2025 Homework 2

Due 1/24/25 at 5pm on Gradescope

Reading

All references will be to Artin Algebra, 2nd edition.
finish reading Sections 11.1-11.3 and read Sections 11.4-11.5.

Assigned Problems

Write up neat and complete solutions to these problems. “Ring” will always mean commutative ring unless otherwise noted.

1. In the ring $\mathbb{Q}[x]$, find the remainder when $x^{20} + 2x^{19} + 5x - 7$ is divided by $(x + 2)$.

2. (Artin 11.3.3(b)) Let $\phi : \mathbb{R}[x] \to \mathbb{C}$ be the evaluation homomorphism defined by $f(x) \mapsto f(2 + i)$. Find a polynomial $g \in \mathbb{R}[x]$ such that $\ker \phi = (g)$, and justify your answer.

3. Let R and S be rings. Suppose that $\phi : R \to S$ is a ring homomorphism. Show that for any $s \in S$ there is a unique ring homomorphism $\widetilde{\phi} : R[x] \to S$ such that $\widetilde{\phi}(x) = s$ and $\widetilde{\phi}(r) = \phi(r)$ for all $r \in R$ (thinking of elements of R as constant polynomials in $R[x]$). [This is called the “freeness” of a polynomial ring, because it says that to define a homomorphism from $R[x]$ to another ring, once we say where the coefficients R go, we are free to send x anywhere we want, and this determines the homomorphism.]

4. The *Gaussian integers* is the subring $\mathbb{Z}[i]$ of the complex numbers given by $\{a + bi | a, b \in \mathbb{Z}\}$. Consider the principal ideal $I = (1 + i)$ in $\mathbb{Z}[i]$. Show that the factor ring $\mathbb{Z}[i]/(1 + i)$ has precisely 2 elements.

The following definition will be useful in the remaining problems.

Definition 0.1 *If R is a ring with element $a \in R$ and $n \in \mathbb{Z}$ is an integer, we define $n \cdot a$, the "nth multiple of a", in the same way as in any additive group:*

$$n \cdot a = \begin{cases} \overbrace{a + a + \cdots + a}^{n} & n \geq 1 \\ 0 & n = 0 \\ \overbrace{(-a) + (-a) + \cdots + (-a)}^{|n|} & n \leq -1. \end{cases}$$

(The $\cdot$ is often omitted.)

For any ring R, the characteristic *of R is defined to be the smallest positive integer n, if any, such that $n \cdot 1 = 0$, where 1 is the multiplicative identity of R. If none exists we define the characteristic of R to be 0.*

5. (a) Give an example of a ring of characteristic n, for each integer $n \geq 0$.
(b) Prove that if R is a ring of characteristic n, then $n \cdot a = 0$ for all $a \in R$.

6. Prove that the binomial formula holds in any (commutative) ring R, that is, if $a, b \in R$ and $n \geq 1$, then $(a + b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i}$. Here $\binom{n}{i}$ is the binomial coefficient $\binom{n}{i} = \dfrac{n!}{i!(n-i)!}$ in $\mathbb{Z}$ and $\binom{n}{i} a^i b^{n-i}$ means the $\binom{n}{i}$th multiple of $a^i b^{n-i}$.

7. (Artin 11.3.8) Let p be a prime number. Prove that if R is a ring of characteristic p, then the function $\phi : R \to R$ given by $\phi(a) = a^p$ is a ring homomorphism.