Math 100b Winter 2025 Homework 6

Due 2/28/2025 at 5pm on Gradescope

Reading

All references will be to Artin Algebra, 2nd edition.

Reading: Sections 12.3-12.4.

Assigned Problems

1. Find all irreducible polynomials of degree at most 3 in the polynomial ring $(\mathbb{Z}/3\mathbb{Z})[x]$.

2. Completely factor the polynomial $x^4 + 1$ as a product of irreducibles in the following rings, briefly indicating how you know the factors you find are irreducible.

(a). $\mathbb{C}[x]$. (Hint: De Moivre's theorem).

(b). $\mathbb{R}[x]$.

(c). $\mathbb{Q}[x]$.

(d). $(\mathbb{Z}/3\mathbb{Z})[x]$.

3. Let p be prime in $\mathbb{Z}$. Consider the reduction mod p homomorphism $\phi : \mathbb{Z}[x] \to (\mathbb{Z}/p\mathbb{Z})[x]$ which sends $h(x) = a_0 + a_1x + \cdots + a_nx^n$ to $\overline{h}(x) = \overline{a_0} + \overline{a_1}x + \cdots + \overline{a_n}x^n$.

(a). Suppose that $f(x) = a_0 + a_1x + \cdots + a_nx^n \in \mathbb{Z}[x]$ where $\overline{a_n} \neq 0$ in $\mathbb{Z}/p\mathbb{Z}$, and suppose that $\overline{f}(x)$ is irreducible as a polynomial in $\mathbb{Z}/p\mathbb{Z}[x]$. Prove that $f(x)$ is irreducible in $\mathbb{Q}[x]$. (This is called the "reduction mod p" method for showing irreducibility).

(b). Use the method of part (a) to show that $80x^3 - 8x + 100$ is irreducible in $\mathbb{Q}[x]$.

4. (a). Consider $x^6 + x^3 + 1 \in \mathbb{Z}[x]$. Use Eisenstein's criterion to show that this polynomial is irreducible in $\mathbb{Z}[x]$, by first making a substitution.

(b). Factor $x^9 - 1$ as a product of irreducibles in $\mathbb{Z}[x]$. (Hint: it is a difference of two cubes).

5. Decide if the following polynomials are irreducible in $\mathbb{Q}[x]$ or not. Use any method, but justify your answer.

(a) $2x^3 + x - 4$.

(b). $x^4 - 4x^3 + 6$.

(c). $x^4 + 10x^2 + 1$.

6. Suppose that $f(x)$ and $g(x)$ are polynomials with rational coefficients whose product $h(x) = f(x)g(x)$ has integer coefficients. Prove that the product of any coefficient of f with any coefficient of g is an integer.