Math 170A, Fall 2023 HOMEWORK #6 due Monday, Nov 27

Homework problems that will be graded (Q1 - Q5, 30pts in total):

Q1. Let A be an $n \times n$ real matrix and denote by $\|A\|_F$ the Frobenius norm of A. Recall that the Frobenius norm has the following equivalent definition:

$$||A||_F^2 = \sum_{i,j} |a_{ij}|^2 = trace(A^T A).$$

a) Show that $||A||_F = ||UA||_F$ for any real orthogonal $n \times n$ matrix U.

b) Show that $||A||_F = ||AV||_F$ for any real orthogonal $n \times n$ matrix V.

c) Conclude that $||A||_F = \sqrt{\sum_{i=1}^{n} \sigma_i^2}$, where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n \geq 0$ are the singular values of A.

Q2. **Work this problem out by hand.** Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Carry out the power method with starting vector $q_0 = [a, b]^T$, where $a, b \geq 0$ and $a > b$. Calculate the first few terms; what do you notice?

Explain why the sequence fails to converge. What is the problem with the convergence argument we had in the lecture? Which of the conditions is violated?

Q3. Let A be an $n \times n$ complex matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ ordered in decreasing order of magnitude, that is, $|\lambda_1| \geq |\lambda_2| \geq \ldots \geq |\lambda_n|$. Let $v_1, \ldots, v_n$ be the corresponding eigenvectors.

Assume that $\lambda_1 = 5$, $\lambda_2 = 3$, $\lambda_3 = 2$, and consider performing the power method on A starting with a vector q which *does not depend on the eigenvector* v_1:

$$\mathbf{q} = \mathbf{v_2} + \sum_{\mathbf{i=3}}^{\mathbf{n}} \mathbf{c_i v_i}, \quad \text{for some } c_i \in \mathbb{C}\ , \quad 3 \leq i \leq n\ .$$

If we start the power method on A with the vector $\mathbf{q}$, will the method converge? If so, to what? Explain.

Q4. This exercise is meant to show you why computing roots of polynomials is, in general, numerically unstable.

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, the 2×2 identity matrix.

(a) Calculate the characteristic polynomial of A and show it has one eigenvalue $\lambda = 1$ with algebraic multiplicity 2.

(b) We now perturb one coefficient of the characteristic polynomial slightly and consider the equation

$$\lambda^2 - 2\lambda + (1 - \varepsilon) = 0,$$

where $0 < \varepsilon \ll 1$. Solve the equation for the roots $\hat{\lambda}_1$ and $\hat{\lambda}_2$, in terms of ε.

(c) Show that if $\varepsilon = 10^{-12}$, $|\hat{\lambda}_1 - \lambda|$ and $|\hat{\lambda}_2 - \lambda|$ are one million times bigger than ε.

(d) Conclude that, given a relative change $\epsilon \ll 1$ in one of the coefficients of the matrix, the relative change in the eigenvalues can grow arbitrarily large; therefore, the computation is numerically unstable.

Q5. (MATLAB problem)

Let

$$A = \begin{bmatrix} 2 & 2 & 2 \\ 4 & 0 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

a) Use the MATLAB function rank to figure the rank of A.

b) MATLAB's pinv command returns the pseudoinverse of a matrix. Use pinv to compute the pseudoinverse $A^\dagger$ of A.

c) Use MATLAB's svd command to find the singular values of $AA^\dagger$ and $A^\dagger A$, and calculate $AA^\dagger$ and $A^\dagger A$.

In HW 5, **Q1**b, **Q1**c you were asked to compute $A^\dagger A$ and $AA^\dagger$ for a general matrix A. Does your answer for those questions agree with the results computed by MATLAB?