Math 31AH: Spring 2021
Homework 1
Due 5:00pm on Friday 10/1/2021

Problem 1: Arithmetic of sets. Determine whether the following three equalities hold for all sets A, B, and C. If equality does not hold, determine whether we have the containments $\subseteq$ or $\supseteq$. Prove your claims.

(1) $A \cap (B - C) = (A \cap B) - (A \cap C)$.
(2) $A \cup (B - C) = (A \cup B) - (A \cup C)$.
(3) $A \times (B - C) = (A \times B) - (A \times C)$.

Solution: (1) This equality is true. Indeed, let $x \in A \cap (B - C)$. Then $x \in A$ and $x \in B - C \subseteq B$, so that $x \in A \cap B$. Furthermore, since $x \in B - C$ we have $x \notin C$, so that $x \notin A \cap C$. We conclude that $x \in (A \cap B) - (A \cap C)$.

On the other hand, suppose $x \in (A \cap B) - (A \cap C)$. Then $x \in A \cap B$ so that $x \in A$ and $x \in B$. If $x \in C$ then (since $x \in A$) we have $x \in A \cap C$, which contradicts $x \in (A \cap B) - (A \cap C)$. We conclude that $x \in A \cap (B - C)$.

(2) We have the containment $A \cup (B - C) \supseteq (A \cup B) - (A \cup C)$. Indeed, suppose $x \in (A \cup B) - (A \cup C)$. If $x \in A$ then certainly $x \in A \cup (B - C)$. If $x \in B$ then since $x \notin A \cup C$ we have $x \notin C$ so that $x \in B - C$. We conclude that $x \in A \cup (B - C)$.

To see why equality does not hold in general, let $A = \{a\}, B = \{b\}$, and $C = \{c\}$ be singleton sets. Then $A \cup (B - C) = \{a\} \cup \{b\} = \{a, b\}$ whereas $(A \cup B) - (A \cup C) = \{a, b\} - \{a, c\} = \{b\}$.

(2) This equality is true. Indeed, let $(x, y) \in A \times (B - C)$. Then $x \in A$ and $y \in B - C \subseteq B$ so that $x \in A \times B$. Since $y \notin C$, we have $(x, y) \notin A \times C$ so that $(x, y) \in (A \times B) - (A \times C)$.

Now suppose $(x, y) \in (A \times B) - (A \times C)$. Since $(x, y) \in A \times B$ we have $x \in A$ and $y \in B$. If $y \in C$ then $(x, y) \in A \times C$, which is a contradiction. We conclude that $(x, y) \in A \times (B - C)$.

Problem 2: Vectors on the circle. Let S be the unit circle in the plane $\mathbb{R}^2$ centered at the origin, i.e.

$$S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

True or false: there exist elements $\mathbf{v}, \mathbf{w} \in S$ such that $\mathbf{v} + \mathbf{w} \in S$. Prove your claim.

Solution: This is true. Indeed, let $\mathbf{v} = (1/2, \sqrt{3}/2)$ and $\mathbf{w} = (1/2, -\sqrt{3}/2)$. We have $\mathbf{v}, \mathbf{w} \in S$ and $\mathbf{v} + \mathbf{w} = (1, 0) \in S$.

Problem 3: Ill-defined functions. Each of the following "functions" is not well-defined. Explain why they are not well-defined.

(1) $f : \mathbb{C} \to \mathbb{C}$, where $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$ is the set of complex numbers and $f(z) := \frac{1}{z^2+3}$.
(2) $g : \mathbb{Q} \to \mathbb{Z}$, where $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ is the set of integers, $\mathbb{Q} = \{\frac{a}{b} : a, b \in \mathbb{Z},\ b \neq 0\}$ is the set of rational numbers, and $g(\frac{a}{b}) = a - b$.
(3) $h : X \to \mathbb{R}_{>0}$, where $X := \{(x, y) \in \mathbb{R}^2 : y = x^2 - 1\}$ is a parabola in the plane, $\mathbb{R}_{>0} = \{x \in \mathbb{R} : x > 0\}$ are the positive reals, and $h(x, y) = y$.

Solution: (1) We have $\sqrt{3}i \in \mathbb{C}$ and trying to evaluate $f(z) = \frac{1}{z^2+3}$ at $z = \sqrt{3}i$ results in division by zero.

(2) The fractions $\frac{1}{2}$ and $\frac{2}{4}$ represent the same element of $\mathbb{Q}$. However $g(\frac{1}{2}) = 1 - 2 = -1$ and $g(\frac{2}{4}) = 2 - 4 = -2$. Since $-1 \neq -2$, the rule for $g(\frac{a}{b})$ does not assign an unambiguous value to $\frac{1}{2} = \frac{2}{4}$.

(3) We have $(0, -1) \in X$ and $h(0, -1) = -1$, which is not an element of the range $\mathbb{R}_{>0}$ of the rule for $h(x, y)$.

Problem 4: Binary operations. Decide whether the given binary operations $\star$ on the given sets S are well-defined. Prove your claim.

(1) $S = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$ and $(x, y) \star (x', y') := (x + x', y + y')$.
(2) $S = \mathbb{R}$ and $x \star y := \frac{x}{y^2+1}$.
(3) $S = \mathbb{C}$ and $x \star y := \frac{x}{y^2+1}$.

Solution: (1) This binary operation is not well-defined. Indeed, we have $(1, 0), (0, 1) \in S$ and $(1, 0) \star (0, 1) = (1, 1) \notin S$.

(2) This is a well-defined binary operation $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Indeed, the expression $\frac{x}{y^2+1}$ is well-defined for any $x, y \in \mathbb{R}$ since $y^2 + 1 \neq 0$ for all $y \in \mathbb{R}$.

(3) This binary operation is not well-defined. Indeed, attempting to compute $1 \star i$ involves division by zero since $i^2 + 1 = 0$.

Problem 5: Multiplication in fields. Let $\mathbb{F}$ be a field and let $a, b \in \mathbb{F}$ be nonzero elements. Prove that $ab \neq 0$. (Hint: Use 'proof by contradiction'. Assume to the contrary that $ab = 0$ with $a, b \neq 0$. Prove that this forces one of a, b to be zero.)

Solution: Assume that $ab = 0$ for $a, b \neq 0$. Since $a \neq 0$, we may multiply both sides of $ab = 0$ by a^{-1} to obtain

$$0 = a^{-1} \cdot 0 = a^{-1}ab = 1 \cdot b = b$$

so that $b = 0$. This contradicts the assumption that $b \neq 0$.

Problem 6: Characteristic of a field. Let $\mathbb{F}$ be a field. The *characteristic* of $\mathbb{F}$, written $\mathrm{char}(\mathbb{F})$, is the minimum positive integer n such that we have

$$\overbrace{1+1+\cdots+1}^{n} = 0$$

inside $\mathbb{F}$. If no such n exists, the field $\mathbb{F}$ is said to have *characteristic zero* and we write $\mathrm{char}(\mathbb{F}) = 0$.

Let $\mathbb{F}$ be a field with $\mathrm{char}(\mathbb{F}) = n > 0$. Prove that n is prime. (Hint: Use Problem 5 in a clever way.)

Solution: Suppose $\mathrm{char}(\mathbb{F}) = n > 0$ is not prime. Then we may factor $n = ab$ where $a, b < n$ are positive integers. By the distributive law, we have

$$\begin{aligned} 0 &= \overbrace{1+1+\cdots+1}^{n} \\ &= \overbrace{1\cdot(\overbrace{1+1+\cdots+1}^{b}) + \cdots + 1\cdot(\overbrace{1+1+\cdots+1}^{b})}^{a} \\ &= (\overbrace{1+1+\cdots+1}^{a}) \cdot (\overbrace{1+1+\cdots+1}^{b}) \end{aligned}$$

and Problem 5 implies that either $\overbrace{1+1+\cdots+1}^{a} = 0$ or $\overbrace{1+1+\cdots+1}^{b} = 0$. Since $a, b < n$ are positive integers this contradicts the definition of $\mathrm{char}(\mathbb{F})$.[1]

Problem 7: A four-element field? Let $S = \{0, 1, 2, 3\}$ and define binary operations $+, \cdot$ on S to be addition and multiplication modulo 4.[2] Do these binary operations turn S into a field? Prove your claim.

Solution: These binary operations do not turn S into a field. Indeed, in S we have $2 \cdot 2 = 0$ since $4 = 0$ modulo 4. Since $2 \neq 0$, Problem 5 shows that S is not a field under these operations.

Problem 8: A non-field. Let $\mathbb{F}$ be a field. Define binary operations $+$ and $\cdot$ on $\mathbb{F}^2 = \{(a, b) : a, b \in \mathbb{F}\}$ by the 'coordinatewise' rules

$$(a, b) + (a', b') := (a + a', b + b') \quad \text{and} \quad (a, b) \cdot (a', b') := (a \cdot a', b \cdot b')$$

Prove that these binary operations do **not** turn $\mathbb{F}^2$ into a field.

[1] Technically, since 1 is not prime, you would need to observe that $1 \neq 0$ to see that $\mathrm{char}(\mathbb{F}) \neq 1$. I am not expecting this here.

[2] More precisely, given $x, y \in S$ we define $x + y \in S$ to be the remainder of the (usual) sum of x, y upon division by 4 and let $x \cdot y \in S$ be the remainder of the (usual) product of x, y upon division by 4.

Solution: The elements $(1,0),(0,1) \in \mathbb{F}^2$ are nonzero and yet their product $(1,0)\cdot(0,1) = (0,0)$ is zero (i.e., the additive identity) in $\mathbb{F}^2$. By Problem 5, the set $\mathbb{F}^2$ is not a field under these operations.

Problem 9: (Optional; not to be handed in.) When $\mathbb{F} = \mathbb{R}$ is the field of real numbers, we **can** endow $\mathbb{R}^2$ with the structure of a field via the alternative binary operations

$$(x,y)+(x',y') := (x+x', y+y') \quad \text{and} \quad (x,y)\cdot(x',y') := (xx'-yy', xy'+x'y)$$

Explain why this is the field $\mathbb{C}$ of complex numbers in disguise. Can these rules be used to define a field structure on $\mathbb{F}^2$ for any field $\mathbb{F}$? Why or why not?