MATH 31AH - Homework 1

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1 Arithmetic of sets

1.1 $A \cap (B - C) = (A \cap B) - (A \cap C)$

The equality holds.

Proof. The equality can be demonstrated with set properties.

$$\begin{aligned} A \cap (B - C) &= A \cap (B \cap \overline{C}) \\ &= (A \cap B) \cap \overline{C} \\ &= \emptyset \cup ((A \cap B) \cap \overline{C})) \\ &= ((A \cap B) \cap \overline{A}) \cup ((A \cap B) \cap \overline{C}) \\ &= (A \cap B) \cap (\overline{A} \cup \overline{C}) \\ &= (A \cap B) \cap \overline{(A \cap C)} \\ &= (A \cap B) - (A \cap C) \end{aligned}$$

□

1.2 $A \cup (B - C) = (A \cup B) - (A \cup C)$

The equality does not hold, but $A \cup (B - C) \supseteq (A \cup B) - (A \cup C)$ is true.

Proof. The left hand side can be simplified:

$$\begin{aligned} A \cup (B - C) &= A \cup (B \cap \overline{C}) \\ &= (A \cup B) \cap (A \cup \overline{C}) \\ &= A \cup (B \cap \overline{C}) \end{aligned}$$

The right hand side can be simplified:

$$\begin{aligned} (A \cup B) - (A \cup C) &= (A \cup B) \cap \overline{(A \cup C)} \\ &= (A \cup B) \cap (\overline{A} \cap \overline{C}) \\ &= (A \cap (\overline{A} \cap \overline{C})) \cup (B \cap (\overline{A} \cap \overline{C})) \\ &= \emptyset \cup (B \cap (\overline{A} \cap \overline{C})) \\ &= (B \cap (\overline{A} \cap \overline{C})) \\ &= \overline{A} \cap (B \cap \overline{C}) \end{aligned}$$

Since $\overline{A} \cap (B \cap \overline{C}) \subseteq (B \cap \overline{C}) \subseteq A \cup (B \cap \overline{C})$, we know that $(A \cup B) - (A \cup C) \subseteq A \cup (B - C)$

Let $x \in A$, so $x \in A \cup (B - C)$. However, $x \in A$ implies that $x \notin \overline{A} \cap B \cap \overline{C}$, which implies that $x \notin (A \cup B) - (A \cup C)$. Therefore $A \cup (B - C) \nsubseteq (A \cup B) - (A \cup C)$, so there is no equality. □

1.3 $A \times (B - C) = (A \times B) - (A \times C)$

The equality holds.

Proof. Let $x \in A \times (B - C)$. Then $x = (x_1, x_2)$ where $x_1 \in A$ and $x_2 \in B - C$. The statement $x \in (A \times B) - (A \times C)$ is true if and only if $(x_1, x_2) \in A \times B$ and $(x_1, x_2) \notin A \times C$. Note, $(x_1, x_2) \in A \times B$ because $x_1 \in A$ and $x_2 \in B - C$. Because $x_2 \in B - C$, $x_2 \notin C$, so $(x_1, x_2) \notin A \times C$. Therefore, $x \in (A \times B) - (A \times C)$ and $A \times (B - C) \subseteq (A \times B) - (A \times C)$.

Let $x \in (A \times B) - (A \times C)$. Then $x = (x_1, x_2)$ where $(x_1, x_2) \in A \times B$ and $(x_1, x_2) \notin A \times C$. Because $(x_1, x_2) \notin A \times C$, $x_1 \notin A$ or $x_2 \notin C$. Because $x_1 \in A$ from $(x_1, x_2) \in A \times B$, $x_1 \notin A$ cannot be true. Therefore $x_2 \notin C$. Since $x_1 \in A$, $x_2 \in B$, and $x_2 \notin C$, it follows that $x \in A \times (B - C)$ and $A \times (B - C) \supseteq (A \times B) - (A \times C)$.

Since $A \times (B - C) \subseteq (A \times B) - (A \times C)$ and $A \times (B - C) \supseteq (A \times B) - (A \times C)$, it follows that $A \times (B - C) = (A \times B) - (A \times C)$. □

2 Vectors on a circle

There exists elements $v, s \in S$ such that $v + w \in S$ where S is the unit circle.

Proof. Let $v = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $w = (\frac{1}{2}, -\frac{\sqrt{3}}{2})$. $(\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2 = (\frac{1}{2})^2 + (-\frac{\sqrt{3}}{2})^2 = 1$ so v and w are in S. v+w = (1, 0), which is in S. Therefore, there exists vectors in the unit circle that add to another vector in the unit circle. □

3 Ill-defined functions

3.1 $f : \mathbb{C} \to \mathbb{C}$ and $f(z) = \frac{1}{z^2+3}$

The function is not well-definied when z = $\sqrt{3}i$ because of a divide by zero error.

3.2 $g : \mathbb{Q} \to \mathbb{Z}$ and $g(\frac{a}{b}) = a - b$

The function is not well-defined since each rational number can be expressed with many different combinations of numerators and denominators.

3.3 $h : X \to \mathbb{R}_{>0}$ and $h(x, y) = y$

The function is not well-defined since the value of $h(0, -1)$ is -1, which is negative and not in the codomain of $\mathbb{R}_{>0}$.

4 Binary operations

4.1 $S = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$ and $(x, y) \star (x', y') = (x + x', y + y')$

This function is not well defined since $x = (1, 0)$ and $y = (0, 1)$ does not have $xy = 0$.

4.2 $S = \mathbb{R}$ and $x \star y = \frac{x}{y^2+1}$

This function is well-defined since the square of a real number plus 1 cannot be zero.

4.3 $S = \mathbb{C}$ and $x \star y = \frac{x}{y^2+1}$

This function is not well-defined since for $y = i$, there will be a divide by zero error.

5 Multiplication in fields

If $a, b \in \mathbb{F}$ are nonzero elements, then $ab \neq 0$.

Proof. Assume $ab = 0$ with $a, b \neq 0$. Since b is not zero, it has multiplicative inverse b^-1. Multiplying both sides by b^-1 gives $a \cdot b \cdot b^-1 = 0 \cdot b^-1 \implies a \cdot 1 = 0 \implies a = 0$. This contradicts the initial assumption that a is nonzero. Therefore $a, b \neq 0$ implies $ab \neq 0$. □

6 Characteristic of a field

If $char(\mathbb{F}) = n > 0$ then n is prime.

Proof. Assume that n is not prime. This means that there exists numbers $1 < a \leq b < n$ such that $ab = n$. Since $\overbrace{1 + 1 + ... + 1}^{n} = 0$, and $ab = n$ we have that $\overbrace{1 + 1 + ... + 1}^{\text{ab}} = 0$ which implies $ab = 0$. Since $1 < a \leq b < n$, $ab \neq 0$ by problem 5, which contradicts $ab = 0$. Therefore n must be prime. □

7 A four-element field?

Proof. Field S, a four element field with arithmetic modulo 4, does not exist.

A field must have a multiplicative inverse for every element in it. The element 2 in set S does not have a multiplicative inverse since $0 \cdot 2 = 0$, $1 \cdot 2 = 2$, $2 \cdot 2 = 0$, and $3 \cdot 2 = 2$. Therefore S is not a field. □

8 A non-field

Proof. $\mathbb{F}^2$ with ”coordinate-wise” arithmetic is not a field.

A field must have a multiplicative inverse for every nonzero element. Let $x = (0, a)$ where a is an arbitrary nonzero element. Since the x is not the zero element, it must have a multiplicative inverse. Since the first element of x is 0, it does not have a multiplicative inverse, and so the multiplicative inverse of x does not have a valid first element. This means that x does not have a multiplicative inverse under "coordinate-wise" arithmetic". Therefore, $\mathbb{F}^2$ is not a field. $\square$

9 $\mathbb{R}^2$ and $\mathbb{C}$ (Optional)

x is analogous to the real component of a complex number, and y is analogous to the imaginary component of a complex number. When viewed in this way, the arithmetic operations for $\mathbb{R}^2$ and $\mathbb{C}$ are isomorphic. These rules can be used to define a field structure $\mathbb{F}^2$ for every $\mathbb{F}$:

1. Addition is commutative.
   (a) $(a, b) + (c, d)$
   (b) $(a + c, b + d)$
   (c) $(c + a, d + b)$
   (d) $(c, d) + (a, b)$
2. Addition is associative.
   (a) $((a, b) + (c, d)) + (e, f)$
   (b) $(a + c, b + d) + (e, f)$
   (c) $(a + c + e, b + d + f)$
   (d) $(a + (c + e), b + (d + f))$
   (e) $(a, b) + (c + e, d + f)$
   (f) $(a, b) + ((c, d) + (e, f))$
3. Multiplication is distributive.
   (a) $(a, b) \cdot ((c, d) + (e, f)) =$
   (b) $(a, b) \cdot (c + e, d + f) =$
   (c) $(ac + ae - bd - bf, ad + af + bc + be) =$
   (d) $(ac - bd, ad + bc) + (ae - bf, af + be) =$
   (e) $(a, b) \cdot (c, d) + (a, b) \cdot (e, f)$
4. Zero is $(0, 0)$.
5. One is $(1, 0)$.
6. Additive inverse: $-(x, y) = (-x, -y)$
7. Multiplicative inverse: $(x, y)^-1 = (\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2})$
8. $(0, 0) \neq (1, 0)$.