Math 31BH: Assignment 3
Due 01/23 at 23:59

1. Prove that the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(t) = |t|$ is continuous but not differentiable at $t = 0$.

   Solution: This is a problem in single-variable calculus. To prove that a function $f(t)$ which contains 0 in its domain is continuous at 0, we must demonstrate that, given any $\varepsilon > 0$, there is a corresponding $\delta > 0$ such that

$$|t - 0| < \delta \implies |f(t) - f(0)| < \varepsilon.$$

   In the case at hand, the above reads

$$|t| < \delta \implies |t| < \varepsilon,$$

   so for this function any $\delta \in (0, \epsilon]$ will do.

   Now, if a function $f(t)$ which contains 0 in its domain is differentiable at 0, then there exists a number $f'(0) \in \mathbb{R}$ with the property that, given any $\varepsilon > 0$, there is a corresponding $\delta > 0$ such that

$$|h| < \delta \implies \left| f'(0) - \frac{f(h) - f(0)}{h} \right| < \varepsilon.$$

   Thus, if it were the case that $f(t) = |t|$ was differentiable at 0, with derivative $f'(0)$, there would exist a number $\delta > 0$ such that

$$|h| < \delta \implies \left| f'(0) - \frac{|h|}{h} \right| | < 1.$$

   For $h \in (0, \delta)$, the above gives

$$|f'(0) - 1| < 1,$$

   which means that the distance from $f'(0)$ to 1 is less than 1, so that $f'(0)$ must be positive. On the other hand, for $h \in (-\delta, 0)$, we instead get that

$$|f'(0) - (-1)| < 1,$$

   which means that the distance from $f'(0)$ to -1 is less than 1, so that $f'(0)$ must be negative. Since no number is simultaneously positive and negative, assuming the existence of $f'(0)$ has led to a contradiction, ergo $f'(0)$ does not exist.

2. Consider the differentiable function $g\colon \mathbb{R} \to \mathbb{R}^2$ given by $g(t) = (t, t^3)$.

   (a) Sketch the tangent vector and the tangent line at $t = 0$ and $t = 1$.

   (b) Construct a function $h\colon \mathbb{R} \to \mathbb{R}^2$ with the same image as g such that $g(0) = h(0)$ but h is not differentiable at $t = 0$.

   Solution: The image of g is the following curve in $\mathbb{R}^2$:

$$C = \{(x, y) \in \mathbb{R}^2 \colon x^3 - y = 0\} = \{(x, y) \in \mathbb{R}^2 \colon y = x^3\}.$$

   This might cause you to think about the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$, and this is a good place to point out the distinction between the image of a function, and the graph of a function: the curve C is the image of g, but it is the graph of f. Indeed, since $C \subset \mathbb{R}^2$, it cannot be the image of a function $f\colon \mathbb{R} \to \mathbb{R}$, which is a subset of $\mathbb{R}$. In particular, the image of $f(x) = x^3$ is the whole real line $\mathbb{R}$.

   The function $g\colon \mathbb{R} \to \mathbb{R}^2$ is one parameterization of the curve C, but there are (infinitely) many others, for example the function $h\colon \mathbb{R} \to \mathbb{R}^2$ defined by

$$h(t) = (t^{\frac{1}{3}}, t),$$

   whose component functions relative to the standard basis of $\mathbb{R}^2$ are

$$h_1(t) = t^{\frac{1}{3}} \quad \text{and} \quad h_2(t) = t.$$

   Clearly, $h(0) = g(0) = (0, 0)$; however, the function $h_1(t)$ is not differentiable at $t = 0$.

3. Consider the differentiable function $f\colon \mathbb{R} \to \mathbb{R}^2$ defined by $f(t) = (e^{kt}\cos t, e^{kt}\sin t)$ where k is a constant.

   (a) Sketch the image of f.

   (b) Prove that

$$\frac{f'(t) \cdot f(t)}{\|f'(t)\|\|f(t)\|} = \frac{k}{\sqrt{1 + k^2}}.$$

   (c) Prove that the angle between the tangent vector $f'(t)$ and the line joining $f(t)$ to $(0, 0)$ is the same for all $t \in \mathbb{R}$.

Solution: In order to sketch the image of f, it may help to write it in the form $f(t) = e^{kt}(\cos t, \sin t)$, from which you can see that $f(t)$ is a vector of norm

$$\|f(t)\| = e^{kt}$$

whose angle of inclination above the horizontal axis in $\mathbb{R}^2$ is t. In other words, the polar coordinates of $f(t)$ are (e^{kt}, t). Thus the image of $f(t)$ is a spiral emanating from the point $(0,0)$, which is *not* in the image of f because $e^{kt} > 0$ for all $t \in \mathbb{R}$. Actually, it may be better to think of the image of $f(t)$ as emanating from the point $f(0) = (1,0)$, and exploding counterclockwise outward from here through positive values of t, whilst spiraling clockwise towards $(0,0)$ through negative values of t.

In order to compute the derivative of $f(t)$, we differentiate its component functions f_1, f_2 relative to the standard basis of $\mathbb{R}^2$, which are

$$f_1(t) = e^{kt}\cos t \quad \text{and} \quad f_2(t) = e^{kt}\sin t.$$

By single-variable calculus (specifically, using the product rule and the chain rule), the derivatives of these functions are

$$f_1'(t) = ke^{kt}\cos t + e^{kt}\sin t \quad \text{and} \quad f_2'(t) = ke^{kt}\sin t - e^{kt}\cos t,$$

so that

$$\begin{aligned} f'(t) &= (ke^{kt}\cos t + e^{kt}\sin t, ke^{kt}\sin t - e^{kt}\cos t) \\ &= ke^{kt}(\cos t, \sin t) + e^{kt}(\sin t, -\cos t). \end{aligned}$$

Thus, writing

$$\mathbf{v}_1(t) = e^{kt}(\cos t, \sin t) \quad \text{and} \quad \mathbf{v}_2(t) = e^{kt}(\sin t, -\cos t),$$

and noting that $\mathbf{v}_1(t) \cdot \mathbf{v}_2(t) = 0$, we have

$$\begin{aligned} \|f'(t)\|^2 &= (k\mathbf{v}_1(t) + \mathbf{v}_2(t)) \cdot (k\mathbf{v}_1(t) + \mathbf{v}_2(t)) \\ &= k^2\|\mathbf{v}_1(t)\|^2 + 2k\mathbf{v}_1(t) \cdot \mathbf{v}_2(t) + \|\mathbf{v}_2(t)\|^2 \\ &= k^2 e^{2kt} + e^{2kt} \\ &= e^{2kt}(k^2+1), \end{aligned}$$

so that

$$\|f'(t)\| = e^{kt}\sqrt{k^2+1}.$$

Similarly,

$$f'(t) \cdot f(t) = \mathbf{v}_1(t) \cdot (k\mathbf{v}_1(t) + \mathbf{v}_2(t)) = ke^{2kt},$$

and we conclude that

$$\frac{f'(t) \cdot f(t)}{\|f'(t)\| \|f(t)\|} = \frac{k}{\sqrt{k^2+1}}.$$

Note that you can see the Cauchy-Schwarz inequality at work here: the fraction on the right hand side is a bit less than 1. Moreover, the angle θ between the vectors $f'(t)$ and $f(t)$ satisfies

$$\cos\theta = \frac{k}{\sqrt{k^2+1}},$$

i.e. it is the angle between the base and hypotenuse of a right triangle of base k and height 1, and this number does not depend on t. This says that the angle between the vector $f(t)$ and the tangent vector $f'(t)$ is the same for all times t, and for this reason the image of the function $f(t)$ is called an *equiangular spiral*. This family of curves (which depends on the parameter k) appears at various natural scales: the nautilus shell (small), the shape of a cyclone (medium), and the shape of galaxies (large).