Math 170A, Fall 2023 ***HOMEWORK #1*** ***due Friday, October 13***

Homework problems that will be graded (Q1 - Q4, 30pts in total).

The first two problems, **Q1** and **Q2**, are meant to help you see why the result of Gaussian elimination can be obtained by multiplication of A by L^{-1}, where L is the matrix of multipliers.

Q1. (7pts) Let n be an integer, and A be a $n \times n$ matrix. Let a be a constant.

For a pair of indices (j, i) with $1 \leq i < j \leq n$, we let B_{ji}^a be the matrix whose entries are all 0, except for the (j, i) entry, which is a.

Now let L_{ji}^a be the $n \times n$ matrix

$$L_{ji}^a = I_n + B_{ji}^a,$$

where I_n is the identity matrix of dimension n.

Prove that

(a) Let $X = L_{ji}^a A$. Then X is the result of the row operation $R_j + aR_i \to R_j$ (adding a times R_i to R_j).

(b) The inverse of L_{ji}^a is

$$(L_{ji}^a)^{-1} = I_n - B_{ji}^a.$$

Hint: Think about which row operation you would apply to the matrix X to transform it back into A.

Q2. (8pts) For this problem, we will use the notations of **Q1**.

Let $1 \leq i < j < n$ and $1 \leq i < l < k \leq n$. Show that

$$L_{ji}^a L_{kl}^b = I + D \ ,$$

where D is a matrix that has 0 everywhere but in positions (j, i), for which $D(j, i) = a$, and (k, l), for which $D(k, l) = b$.

As an illustration, note that if $n = 4$, $i = 1$, $j = 3$, $k = 3$, $l = 2$,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & b & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & b & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Hint: L_{ji}^a effects the elementary row operation $R_j + a * R_i \to R_j$. What does that do to the rows of L_{kl}^b?

Q3. (8pts) Let A be an invertible upper triangular $n \times n$ matrix.

a) Let b a vector of length n with $b_i \neq 0$, and $b_{i+1} = b_{i+2} = \ldots = b_n = 0$, for some $1 \leq i \leq n$. Let x be the unique solution to the system $Ax = b$.

By using backward substitution, show that x has the same pattern of zeros as b, that is, $x_i \neq 0$, and $x_{i+1} = x_{i+2} = \ldots = x_n = 0$.

Hint: Start at the end and use "reverse" induction to work your way backward.

b) Prove that A^{-1} is also upper triangular.

Hint: One possibility is to note that the j-th column x_j of A^{-1} satisfies

$$A\,x_j = e_j, \quad j = 1, \ldots, n$$

where n is the size of the matrix and e_j is the j-th unit vector (jth column of the identity), with a 1 in position j and 0s everywhere else. Then, use a).

Q4. (7pts) Using basic programming ("for" loops and "if" statements), write two MATLAB functions that both take as an input

- $n \times n$ matrix A,
- $n \times n$ matrix B,
- $n \times 1$ vector x.

Check that the inputs are square matrices of the same size. Then, have the first function compute ABx through $(AB)x$ and the second through $A(Bx)$. Then

(a) Take a screenshot of your first function.

(b) Take a screenshot of your second function.

(c) Calculate the number of flops for both approaches, theoretically (in terms of n) and compare: which one is better to use?