Math 20D HW9

Merrick Qiu

May 31, 2022

Section 7.6

Problem 2 The sketch is a box from 1 to 4.

$$\begin{aligned}\mathscr{L}\{u(t-1)-u(t-4)\} &= \mathscr{L}\{u(t-1)\}-\mathscr{L}\{u(t-4)\}\\ &= \frac{e^{-s}}{s}-\frac{e^{-4s}}{s}\end{aligned}$$

Problem 6

$$\begin{aligned}\mathscr{L}\{g(t)\} &= \mathscr{L}\{(t+1)u(t-2)\}\\ &= e^{-2s}\mathscr{L}\{t+3\}\\ &= e^{-2s}(\frac{1}{s^2}+\frac{3}{s})\end{aligned}$$

Problem 14

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{e^{-3s}}{s^2+9}\right\} &= u(t-3)\mathscr{L}^{-1}\left\{\frac{1}{s^2+9}\right\}(t-3)\\ &= \frac{\sin(3t-9)}{3}u(t-3)\end{aligned}$$

Problem 17

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{e^{-3s}(s-5)}{(s+1)(s+2)}\right\} &= u(t-3)\mathscr{L}^{-1}\left\{\frac{(s-5)}{(s+1)(s+2)}\right\}(t-3)\\ &= (7e^{6-2t}-6e^{3-t})u(t-3)\end{aligned}$$

Problem 22

$$\begin{aligned}\mathscr{L}\{w''+w\}=\mathscr{L}\{u(t-2)-u(t-4)\} &\implies s^2W-sw(0)-w'(0)+W=\frac{e^{-2s}-e^{-4s}}{s}\\ &\implies (s^2+1)W-s=\frac{e^{-2s}-e^{-4s}}{s}\\ &\implies W=\frac{e^{-2s}-e^{-4s}+1}{s(s^2+1)}\\ &\implies w=2u(t-2)\sin^2(1-\frac{t}{2})+u(t-4)(\cos(4-t)-1)-\cos(t)+1\end{aligned}$$

Problem 24

$$\begin{aligned}\mathscr{L}\{w''+w\}=\mathscr{L}\{3\sin 2t-3(\sin 2t)u(t-2\pi)\} &\implies s^2Y-sy(0)-y'(0)+Y=\frac{6-6e^{-2\pi s}}{s^2+4}\\ &\implies (s^2+1)Y-s-2=\frac{6-6e^{-2\pi s}}{s^2+4}\\ &\implies Y=\frac{1}{s^2+1}\left(\frac{6-6e^{-2\pi s}}{s^2+4}+s+2\right)\\ &\implies y=u(t-2\pi)(\sin(2t)-2\sin(2t))+\cos(t)-2\sin(t)(\cos(t)-2)\end{aligned}$$

Problem 31

$$\begin{aligned}\mathscr{L}\{y''+5y'+6y=g(t)\} &\implies s^2Y-sy(0)-y'(0)+5sY-5y(0)+6Y=\mathscr{L}\{t(u(t-1)-u(t-5))+u(t-5)\}\\ &\implies (s^2+6)Y-2=\frac{e^{-5s}(-4s+e^{4s}(s+1)-1)}{s^2}\\ &\implies Y=\frac{e^{-5s}(-4s+e^{4s}(s+1)-1)+2s^2}{s^2(s^2+6)}\\ &\implies y=2e^{-2t}-2e^{-3t}+\left(\frac{1}{36}+\frac{t-1}{6}-\frac{e^{-2(t-1)}+\frac{2e^{-3(t-1)}}{9}}{4}\right)u(t-1)\\ &-\left(\frac{19}{36}+\frac{t-5}{6}-\frac{7e^{-2(t-5)}}{4}+\frac{11e^{-3(t-5)}}{9}\right)u(t-5)\end{aligned}$$

Section 7.9

Problem 2

$$\int_{-\infty}^{\infty} e^{3t}\delta(t)\,dt = e^0 = 1$$

Problem 19

$$\begin{aligned}
\mathscr{L}\{w''+6w'+5w\} = \mathscr{L}\left\{e^t\delta(t-1)\right\} &\implies (s^2W-4)+6(sW)+5W = \mathscr{L}\{\delta(t-1)\}(s-1) \\
&\implies (s^2+6s+5)W-4 = e^{-(s-1)} \\
&\implies W = \frac{e^{-(s-1)}+4}{s^2+6s+5} \\
&\implies w = e^{-t}+e^{-5t}+\frac{e}{4}(e^{1-t}-e^{5-5t})u(t-1)
\end{aligned}$$

Problem 20

$$\begin{aligned}
\mathscr{L}\{y''+5y'+6y\} = \mathscr{L}\left\{e^{-t}\delta(t-2)\right\} &\implies (s^2Y-2s+5)+5(sY-2)+6Y = \mathscr{L}\{\delta(t-2)\}(s+1) \\
&\implies (s^2+5s+6)Y-2s-5 = e^{-2(s+1)} \\
&\implies Y = \frac{2s+5+e^{-2(s+1)}}{s^2+5s+6} \\
&\implies y = e^{-2t}+e^{-3t}+e^{-2}(e^{-2(t-2)}+e^{-3(t-2)})u(t-2)
\end{aligned}$$

Problem 22

$$\begin{aligned}
\mathscr{L}\{y''+y\} = \mathscr{L}\left\{\delta(t-\frac{\pi}{2})\right\} &\implies (s^2+1)Y-1 = e^{-\frac{\pi}{2}s} \\
&\implies Y = \frac{e^{-\frac{\pi}{2}s}+1}{s^2+1} \\
&\implies y = \sin t+(\cos t)u(t-\frac{\pi}{2})
\end{aligned}$$

Problem 23

$$\begin{aligned}
\mathscr{L}\{y''+y\} = \mathscr{L}\{-\delta(t-pi)+\delta(t-2\pi)\} &\implies (s^2+1)Y-1 = -e^{-\pi s}+e^{-2\pi s} \\
&\implies Y = \frac{-e^{-\pi s}+e^{-2\pi s}+1}{s^2+1} \\
&\implies y = (u(t-2\pi)+u(t-\pi)+1)\sin(t)
\end{aligned}$$

Problem 26

$$\begin{aligned}
\mathscr{L}\{y''-6y'+13\} = \mathscr{L}\{\delta(t)\} &\implies s^2Y-6sY+13Y = 1 \\
&\implies Y = \frac{1}{s^2-6s+13} \\
&\implies y = h = \frac{1}{2}e^{3t}\sin 2t
\end{aligned}$$

Section 8.2

Problem 2 Using the ratio test,

$$\lim_{n\to\infty}\left|\frac{\frac{3^n}{n!}}{\frac{3^{n+1}}{(n+1)!}}\right| = \lim_{n\to\infty}\left|\frac{n}{3}\right| = \infty$$

Thus the convergence set is $(-\infty,\infty)$.
Problem 3 Using the ratio test,

$$\lim_{n\to\infty}\left|\frac{\frac{n^2}{2^n}}{\frac{(n+1)^2}{2^{n+1}}}\right| = \lim_{n\to\infty}\left|\frac{2n^2}{(n+1)^2}\right| = \lim_{n\to\infty}\left|\frac{4n}{2n+2}\right| = 2$$

Thus the convergence set is $(-4,0)$.
Problem 5 Using the ratio test,

$$\lim_{n\to\infty}\left|\frac{\frac{3}{n^3}}{\frac{3}{(n+1)^3}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^3}{n^3}\right| = \lim_{n\to\infty}\left|\frac{6(n+1)}{6n}\right| = 1$$

Thus the convergence set is $(1,3)$.
Problem 6 Using the ratio test,

$$\lim_{n\to\infty}\left|\frac{\frac{(n+2)!}{n!}}{\frac{(n+3)!}{(n+1)!}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{n+3}\right| = 1$$

Thus the convergence set is $(-3,-1)$.
Problem 8

1. Using the ratio test and taking the square root,

$$\lim_{n\to\infty}\left|\frac{2^{2k}}{2^{2k+2}}\right| = \lim_{n\to\infty}\left|\frac{1}{4}\right| = \frac{1}{4}$$

   Thus the convergence set is $\left(-\frac{1}{2},\frac{1}{2}\right)$

2. Has the same limit as above, so convergence set is still $\left(-\frac{1}{2},\frac{1}{2}\right)$

3. Using the ratio test,

$$\lim_{n\to\infty}\left|\frac{\frac{(-1)^n}{(2n+1)!}}{\frac{(-1)^{n+1}}{(2n+3)!}}\right| = \lim_{n\to\infty}|-(2n+2)(2n+3)| = \infty$$

   Thus the convergence set is $(-\infty,\infty)$

4. Has the same limit as above, so convergence set is still $(-\infty,\infty)$

5. Has the same limit as above, so convergence set is still $(-\infty,\infty)$

6. Has same limit as part a but now taking the fourth root so convergence set is $\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$

Problem 9 We can rewrite $g(x)$ as

$$g(x) = \sum_{n=0}^{\infty} 2^{-n-1}x^n$$

Thus

$$f(x)+g(x) = \sum_{n=0}^{\infty}\left(\frac{1}{n+1}+2^{-n-1}\right)x^n$$

Problem 29 The derivative of cos is $-\sin$ so

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^(n+1)}{(2n)!}(x-\pi)^{2n}$$

Problem 30 The derivatives of x^{-1} cancel out with $n!$ so

$$\frac{1}{x} = \sum_{n=0}^{\infty}(-1)^n(x-1)^n$$

Problem 32 Similar to previous problem but one derivative behind.

$$\ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n}x^n$$

Section 8.3

Problem 2 The equation has a singular point at $x = 0$.
Problem 8 The euqation has no singular points.
Problem 14

$$\begin{aligned}
(x^2+1)y'' + y = 0 &\implies (x^2+1)\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0 \\
&\implies \sum_{n=2}^{\infty} n(n-1)a_n x^n + \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0 \\
&\implies \sum_{k=2}^{\infty} k(k-1)a_k x^k + \sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k + \sum_{k=0}^{\infty} a_k x^k = 0 \\
&\implies (2a_2 + a_0) + (6a_3 + a_1)x + \sum_{k=2}^{\infty}[(k(k-1)+1)a_k + (k+2)(k+1)a_{k+2}]x^k = 0
\end{aligned}$$

From this we get that

$$a_2 = -\frac{1}{2}a_0$$

$$a_3 = -\frac{1}{6}a_1$$

$$a_{k+2} = -\frac{k(k-1)+1}{(k+2)(k+1)}a_k$$

Writing out the first four terms gives

$$y = a_0\left(1 - \frac{x^2}{2} + \dots\right) + a_1\left(x - \frac{x^3}{6} + \dots\right)$$

Problem 19

$$\begin{aligned}
y' - 2xy = 0 &\implies \sum_{n=1}^{\infty} n a_n x^{n-1} - 2x\sum_{n=0}^{\infty} a_n x^n = 0 \\
&\implies \sum_{n=1}^{\infty} n a_n x^{n-1} - 2\sum_{n=0}^{\infty} a_n x^{n+1} = 0 \\
&\implies \sum_{k=0}^{\infty}(k+1)a_{k+1}x^k - 2\sum_{k=1}^{\infty} a_{k-1}x^k = 0 \\
&\implies a_1 + \sum_{k=1}^{\infty}[(k+1)a_{k+1} - 2a_{k-1}]x^k = 0
\end{aligned}$$

Thus we have that

$$a_1 = 0$$

$$a_{k+1} = \frac{2}{k+1}a_{k-1}$$

The solution is

$$a_0\sum_{n=0}^{\infty}\frac{1}{n!}x^{2n} = a_0 e^{x^2}$$

Problem 23

$$\begin{aligned}
z'' - x^2 z' - xz = 0 &\implies \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - x^2 \sum_{n=1}^{\infty} n a_n x^{n-1} - x \sum_{n=0}^{\infty} a_n x^n = 0 \\
&\implies \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=1}^{\infty} n a_n x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0 \\
&\implies \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k - \sum_{k=2}^{\infty} (k-1)a_{k-1} x^k - \sum_{k=1}^{\infty} a_{k-1} x^k = 0 \\
&\implies 2a_2 + (6a_3 - a_0)x + \sum_{k=2}^{\infty} [(k+2)(k+1)a_{k+2} - k a_{k-1}] x^k = 0
\end{aligned}$$

Thus we have that

$$a_2 = 0$$

$$a_3 = \frac{1}{6} a_0$$

$$a_{k+2} = \frac{k}{(k+2)(k+1)} a_{k-1}$$

The solution is

$$a_0 \left(1 + \sum_{k=1}^{\infty} \frac{\prod_1^k (3k-2)^2}{(3k)!} x^{3k} \right) + a_1 \left(x + \sum_{k=1}^{\infty} \frac{\prod_1^k (3k-1)^2}{(3k+1)!} x^{3k+1} \right)$$

Problem 25

$$\begin{aligned}
w'' + 3xw' - w = 0 &\implies \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 3x \sum_{n=1}^{\infty} n a_n x^{n-1} - \sum_{n=0}^{\infty} a_n x^n = 0 \\
&\implies \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + 3 \sum_{n=1}^{\infty} n a_n x^{n} - \sum_{n=0}^{\infty} a_n x^n = 0 \\
&\implies \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} x^k + 3 \sum_{k=1}^{\infty} k a_k x^k - \sum_{k=0}^{\infty} a_k x^k = 0 \\
&\implies (2a_2 - a_0) + \sum_{k=1}^{\infty} [(k+2)(k+1)a_{k+2} + (3k-1)a_k] x^k = 0
\end{aligned}$$

Thus we have that

$$a_2 = \frac{1}{2} a_0$$

$$a_{k+2} = -\frac{3k-1}{(k+2)(k+1)} a_k$$

$w(0) = 2$ implies $a_0 = 2$ and $w'(0) = 0$ implies $a_1 = 0$. The solution is

$$2 + x^2 - \frac{5}{12} x^4 + \frac{11}{72} x^6 + \ldots$$

Problem 26

$$
\begin{aligned}
&(x^2-x+1)y''-y'-y=0\\
\implies &(x^2-x+1)\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}-\sum_{n=1}^{\infty}na_nx^{n-1}-\sum_{n=0}^{\infty}a_nx^n=0=0\\
\implies &\sum_{n=2}^{\infty}n(n-1)a_nx^n-\sum_{n=2}^{\infty}n(n-1)a_nx^{n-1}+\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}-\sum_{n=1}^{\infty}na_nx^{n-1}-\sum_{n=0}^{\infty}a_nx^n=0\\
\implies &\sum_{k=2}^{\infty}k(k-1)a_kx^k-\sum_{k=1}^{\infty}(k+1)ka_{k+1}x^k+\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k-\sum_{k=0}^{\infty}(k+1)a_{k+1}x^k-\sum_{k=0}^{\infty}a_kx^k=0\\
\implies &(2a_2-a_1-a_0)+(-2a_2+6a_3-2a_2-a_1)x+\sum_{k=2}^{\infty}[(k(k-1)-1)a_k-(k+1)^2a_{k+1}+(k+2)(k+1)a_{k+2}]x^k=0
\end{aligned}
$$

Thus we have that

$$a_2=\frac{a_1+a_0}{2}$$

$$a_3=\frac{1}{6}(4a_2+a_1)$$

$$a_{k+2}=\frac{(k+1)^2a_{k+1}-(k(k-1)-1)a_k}{(k+2)(k+1)}$$

Since $y'(0)=1$ implies $a_1=1$, the first four terms are

$$x+\frac{x^2}{2}+\frac{x^3}{2}+\frac{x^4}{3}$$

Problem 27

$$
\begin{aligned}
(x+1)y''-y=0 \implies &(x+1)\sum_{n=1}^{\infty}n(n-1)a_nx^{n-2}-\sum_{n=0}^{\infty}a_nx^n=0\\
\implies &\sum_{n=2}^{\infty}n(n-1)a_nx^{n-1}+\sum_{n=2}^{\infty}n(n-1)a_nx^{n-2}-\sum_{n=0}^{\infty}a_nx^n=0\\
\implies &\sum_{k=1}^{\infty}(k+1)ka_{k+1}x^k+\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k-\sum_{k=0}^{\infty}a_kx^k=0\\
\implies &(2a_2-a_0)+\sum_{k=1}^{\infty}[(k+1)ka_{k+1}+(k+2)(k+1)a_{k+2}-a_k]x^k=0
\end{aligned}
$$

Thus we have that

$$a_2=\frac{1}{2}a_0$$

$$a_{k+2}=\frac{a_k-(k+1)ka_{k+1}}{(k+2)(k+1)}$$

The initial conditions imply $a_0=0$ and $a_1=1$. The first four terms are

$$x+\frac{1}{6}x^3-\frac{1}{12}x^4+\frac{11}{72}x^5+\dots$$