Math 181A: Homework 8

Merrick Qiu

Problem 1: 7.4.12.

The mean of the sample is the center of the confidence interval.

$$\bar{y} = \frac{44.7 + 49.9}{2} = 47.3$$

The margin of error is

$$ME = \frac{49.9 - 44.7}{2} = 2.6 = t_{\alpha/2,n-1}\frac{s}{\sqrt{n}}$$

Since $t_{\alpha/2,n-1} = 2.1315$,

$$s = \frac{2.6}{2.1315}\sqrt{16} = 4.88$$

Problem 2: 7.4.20.

The critical value is $t_{\alpha/2,n-1} = 2.7333$. The margin of error is

$$ME = 2.7333 \cdot \frac{0.14139}{\sqrt{34}} = 0.066$$

The critical region is

$$(0.618 - 0.066, 0.618 + 0.066) = (0.552, 0.684)$$

Since the sample mean falls into this region, we fail to reject the null hypothesis.

The test statistic is

$$t = \frac{0.6373 - 0.618}{0.14139/\sqrt{34}} = 0.795$$

Since $t_{0.2,33} = 0.8527 > 0.795$, we know that the p-value is greater than 0.2, meaning we fail to reject the null hypothesis if $\alpha = 0.01$.

Problem 3: 7.4.21

We are testing $\mu = 0.0042$ for the null hypothesis and $\mu < 0.0042$ for the alternative hypothesis. The critical value is $t_{\alpha,n-1} = 1.8331$. The margin of error is

$$ME = 1.8331 \cdot \frac{0.000383}{\sqrt{10}} = 0.000222$$

The critical region is

$$(0.0042 - 0.000222, \infty) = (0.003978, \infty)$$

The sample mean of 0.0039 falls outside of this region, so we reject the null hypothesis.

The test statistic is

$$t = \frac{0.0039 - 0.0042}{0.000383/\sqrt{10}} = -2.48$$

Looking at the tample p-value is between 0.025 and 0.01, which means that the null hypothesis should be rejected for $\alpha = 0.05$.

Problem 4: 7.5.8

With $\chi^2_{\alpha/2,n-1} = 8.231$ and $\chi^2_{1-\alpha/2,n-1} = 31.526$, the interval for which S^2 has a 95 percent chance of falling into is

$$\left(8.231 \cdot \frac{12}{(19-1)}, 31.526 \cdot \frac{12}{(19-1)}\right) = (5.49, 21.01)$$

Problem 5: 7.3.4.

Since the variance of a chi square variable is twice its df,

$$\begin{aligned}\mathrm{Var}((n-1)S^2/\sigma^2) = 2n-2 &\implies \frac{(n-1)^2}{\sigma^4}\,\mathrm{Var}\,S^2 = 2n-2 \\ &\implies \mathrm{Var}\,S^2 = \frac{2\sigma^4}{n-1}\end{aligned}$$

Problem 6: Variance estimators

1.

$$\begin{aligned}
MSE &= \mathrm{Var}(c\sum_{i=1}^{n}(X_i-\bar{X})^2) + E[c\sum_{i=1}^{n}(X_i-\bar{X})^2-\sigma^2]^2 \\
&= c^2\sigma^4\,\mathrm{Var}(\sum_{i=1}^{n}\frac{1}{\sigma^2}(X_i-\bar{X})^2) + (c\sigma^2 E[\sum_{i=1}^{n}\frac{1}{\sigma^2}(X_i-\bar{X})^2]-\sigma^2)^2 \\
&= 2c^2\sigma^4(n-1) + (c\sigma^2(n-1)-\sigma^2)^2
\end{aligned}$$

2. The derivative of the MSE with respect to c is

$$\frac{\partial}{\partial c}(2c^2\sigma^4(n-1) + (c\sigma^2(n-1)-\sigma^2)^2) = 2(n-1)\sigma^4(cn+c-1)$$

This is equal to zero when $c = \frac{1}{n+1}$, so this is the value of c that minimizes the mean square error.