Math 140B: Homework 7

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Problem 8.11

Let $\epsilon > 0$. Since $f(x) \to 1$ as $x \to \infty$, we can choose A so that $1 - \epsilon < f(x) < 1 + \epsilon$ when $x \geq A$.

Since

$$(1-\epsilon)e^{-tx} = t\int_A^\infty e^{-tx}(1-\epsilon)\,dx \leq t\int_A^\infty e^{-tx}f(x)\,dx \leq t\int_A^\infty e^{-tx}(1+\epsilon)\,dx = (1+\epsilon)e^{-tx}$$

we have that

$$\lim_{t\to 0} t\int_A^\infty e^{-tx}f(x)\,dx = 1.$$

Since $e^{-tx} \leq 1$ when $x \geq 0$

$$\lim_{t\to 0}\left|t\int_0^A e^{-tx}f(x)\,dx\right| \leq \lim_{t\to 0} t\int_0^A |f(x)|\,dx = 0$$

Thus finally we have that

$$\lim_{t\to 0} t\int_0^\infty e^{-tx}f(x)\,dx = \lim_{t\to 0} t\int_0^A e^{-tx}f(x)\,dx + \lim_{t\to 0} t\int_A^\infty e^{-tx}f(x)\,dx = 1$$

Problem 8.12

1. If the Fourier series is
$$\frac{1}{2}a_0 + \sum_{n=1}^{N}(a_n \cos(nx) + b_n \sin(nx))$$
then
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) = \frac{2\delta}{\pi}$$
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx) = \frac{1}{\pi}\int_{-\delta}^{\delta}\cos(nx) = \frac{2\sin(n\delta)}{\pi n}$$
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx) = 0$$

2. When $f(x)$ is Lipschitzs continuous, we have that
$$f(x) = \frac{\delta}{\pi} + \frac{2}{\pi}\sum_{n=1}^{N}\frac{\sin(n\delta)}{n}\cos(nx).$$
Since $f(x)$ is Lipschitzs at $x = 0$ we get that,
$$\frac{\delta}{\pi} + \frac{2}{\pi}\sum_{n=1}^{N}\frac{\sin(n\delta)}{n} = 1.$$
Solving for the sum yields
$$\sum_{n=1}^{N}\frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2}$$

3. From Parseval's theorem,
$$\frac{2\delta}{\pi} = \frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx = \sum_{n=0}^{\infty}|a_n|^2 = \frac{2\delta^2}{\pi^2} + \sum_{n=1}^{\infty}\frac{4\sin^2(n\delta)}{\pi^2 n^2}$$
Solving for the sum yields
$$\sum_{n=1}^{\infty}\frac{\sin^2(n\delta)}{n^2\delta} = \frac{\pi - \delta}{2}$$

4. If we let $\delta \to 0$, then this is simply the Riemann integral of $\frac{\sin x}{x}$ with $\Delta x = \delta$. Thus
$$\int\left(\frac{\sin x}{x}\right)^2 dx = \frac{\pi}{2} = \frac{\pi}{2}$$

5. Putting $\delta = \frac{\pi}{2}$ yields
$$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin^2(n\pi/2)}{n^2} = \frac{\pi}{4} \implies \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

Rudin 8.13

Note that the fourier series of $f(x) = x$ is

$$f(x) = c_0 + \sum_{n=-\infty}^{\infty} c_n e^{inx} = \pi + \sum_{n=-\infty}^{\infty} \frac{i}{n} e^{inx}$$

since

$$c_0 = \frac{1}{2\pi} \int_0^{2\pi} x \, dx = \pi$$

$$\begin{aligned} c_n &= \frac{1}{2\pi} \int_0^{2\pi} x e^{-inx} \\ &= \frac{1}{2\pi} \left[-\frac{1}{in} x e^{-inx} + \frac{1}{n^2} e^{-inx} \right]_0^{2\pi} \\ &= \frac{1}{2\pi} \left[\left(-\frac{2\pi}{in} + \frac{1}{n^2} \right) - \left(\frac{1}{n^2} \right) \right] \\ &= -\frac{1}{in} = \frac{i}{n} \end{aligned}$$

From Parseval's theorem

$$\frac{4\pi^2}{3} = \frac{1}{2\pi} \int_0^{2\pi} |x|^2 \, dx = \sum_{-\infty}^{\infty} |c_n|^2 = \pi^2 + 2 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Solving for the sum of inverse squares yields

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Rudin 8.14

Note that the fourier series of $f(x) = (\pi - |x|)^2$ is

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{N}(a_n \cos(nx) + b_n \sin(nx)) = \frac{\pi^2}{3} + \sum_{n=1}^{N} \frac{4}{n^2} \cos nx$$

since

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - |x|)^2 \, dx = \frac{2}{\pi}\int_{0}^{\pi} (\pi - x)^2 \, dx = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} (\pi - x)^2 \cos(nx) \, dx = (-1)^n \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos(nx) \, dx = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - |x|)^2 \sin(nx) \, dx = 0$$

Since $f(x)$ is Lipschitzs continuous, we can evaluate at $x = 0$ to get

$$f(0) = \pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{N} \frac{4}{n^2} \implies \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

From Parseval's theorem,

$$\frac{2\pi^4}{5} = \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - |x|)^4 = \frac{2}{9}\pi^4 + 16\sum_{n=1}^{\infty} \frac{1}{n^4}$$

This implies that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$