Homework due Friday, November 24, at 11:00 pm Pacific Time.

A. Evaluate the following limits

(1) $\lim_{n\to\infty} \sqrt[n]{n!}$.

(2) $\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n}$. (You may use, without a proof: $\lim_{n\to\infty}(\frac{n+1}{n})^n = e$.)

(Hint: Theorem 3.37 maybe helpful.)

B. Let (X, d) be a compact metric space and let $f : X \to X$ be a function with the following property. There exists some $0 < c < 1$ so that

$$d(f(x), f(y)) \leq c\, d(x, y) \quad \text{for all } x, y \in X.$$

Let $x \in X$, and consider the sequence

$$f(x), f(f(x)), \ldots$$

Prove that this sequence is Cauchy.

D. Rudin, Chapter 3 (page 78), problems # 7, 8, 10, 11.

The following problems are for your practice, and will not be graded.

(1) Let $\sum b_n$ be a convergent series of real numbers, and let $\{a_n\}$ be a sequence of real numbers which is bounded below. Assume further that

$$a_{n+1} \leq a_n + b_n.$$

Prove that $\{a_n\}$ converges.

(2) Let $\{x_n\}$ be a bounded sequence of real numbers. Show that $\{x_n\}$ either has a monotonic subsequence.
(Hint: Call x_n a *peak point* if $x_m \leq x_n$ for all $m \geq n$. Consider two cases separately: there are infinitely many n so that x_n is a peak point, and there are only finitely many peak points.)

(3) Use problem 2 above to give an alternative proof of Heine-Borel theorem.

(4) Let $a_n \geq 0$ be a sequence of non-negative real numbers. Assume that

$$a_{n+m} \leq a_n + a_m \qquad \text{for all } m, n \in \mathbb{N}.$$

Prove that $\{\frac{a_n}{n}\}$ converges.
(Hint: Let $m \in \mathbb{N}$, show that $\limsup \frac{a_n}{n} \leq \frac{a_m}{m}$. Thus, $\limsup \frac{a_n}{n} \leq \inf \frac{a_m}{m}$. Use this to show that $\lim \frac{a_n}{n} = \inf \frac{a_n}{n}$.)