SUMS 31AH - Midterm Review

Merrick Qiu

October 25, 2021

1 Find the change of basis matrix P, from β to α

$\alpha = \{\begin{bmatrix}1 & -1\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & -1\end{bmatrix}, \begin{bmatrix}0 & -1\\0 & -1\end{bmatrix}\}$

$\beta = \{\begin{bmatrix}1 & -2\\1 & -2\end{bmatrix}, \begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}, \begin{bmatrix}2 & -1\\1 & -2\end{bmatrix}, \begin{bmatrix}-2 & 0\\1 & -1\end{bmatrix}\}$

$$\begin{bmatrix}1 & -2\\1 & -2\end{bmatrix} = \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\1 & -1\end{bmatrix} + \begin{bmatrix}0 & -1\\0 & -1\end{bmatrix}$$
$$\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix} = \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\1 & -1\end{bmatrix}$$
$$\begin{bmatrix}2 & -1\\1 & -2\end{bmatrix} = \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix} + \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\1 & -1\end{bmatrix} + \begin{bmatrix}0 & 0\\1 & -1\end{bmatrix}$$
$$\begin{bmatrix}-2 & 0\\1 & -1\end{bmatrix} = -\begin{bmatrix}1 & -1\\0 & 0\end{bmatrix} - \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix} + \begin{bmatrix}0 & 0\\1 & -1\end{bmatrix}$$
$$P = \begin{bmatrix}1 & 0 & 1 & -1\\0 & 1 & 1 & -1\\1 & -1 & 1 & 1\\1 & 0 & 1 & 0\end{bmatrix}$$

2 Union and Intersection of two subspaces a subspace?

The union of two subspaces is not necessarily a subspace(for $dim(v) >= 2$). For example, the x and y axis. The intersection of two subspaces is a subspace. Zero is in S_1 and S_2, so zero is in $S_1 \cap S_2$. Let $c \in \mathcal{F}$ and $v \in S_1 \cap S_2$. Since $cv \in S_1$ and $cv \in S_2$, we know that $cv \in S_1 \cap S_2$. Let $v, w \in S_1 \cap S_2$. Since $v + w \in S_1$ and $v + w \in S_2$, $v + w \in S_1 \cap S_2$.

3 Linear transformation $T : \mathbb{R} \to \mathbb{R}$ from [-1, 1] to [1,3]?

Since [-1, 1] contains zero but [1,3] doesn't, and a linear transformation always maps zero to zero, the statement is false. Think of linear transformations from $\mathbb{R}$ to $\mathbb{R}$ as scalar multiplication.

4 If A and AB are invertible, then B is invertible.

Socks and shoes principle: If A and B are invertible, then $(AB)^{-1} = B^{-1}A^{-1}$ Hint: If T invertible, then exists U such that $I = TU = UT$.

$$\begin{aligned}(AB)^{-1}AB = I &\implies ((AB)^{-1}A)B = I && (1)\\ &\implies (AB)^{-1}A = B^{-1}(Left) && (2)\end{aligned}$$

$$\begin{aligned}AB(AB)^{-1} = I &\implies B(AB)^{-1} = A^{-1} && (3)\\ &\implies B(AB)^{-1}A = A^{-1}A = I && (4)\\ &\implies (AB)^{-1}A = B^{-1}(Right) && (5)\end{aligned}$$

5 Does there exist 2024 independent vectors in $\mathbb{R}^{2021}$?

No, a basis is the maximal linearly independent set, but since the basis has 2021 elements, there cannot be a linearly independent set with 2024 elements.