Math 188: Homework 4
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1 Stirling Numbers as Polynomials

1. **Lemma:** If a sequence is a polynomial of degree 0, its first difference is of degree 0. If a sequence is a polynomial of degree $d + 1 > 0$, the first difference is a polynomial of degree d since

$$\sum_{k=0}^{d+1} a_k(x+1)^k - \sum_{k=0}^{d+1} a_k x^k = \left(\sum_{k=0}^{d+1} a_k x^k + a_k \binom{k}{1} x^{k-1} + \ldots + a_k \right) - \sum_{k=0}^{d+1} a_k x^k$$

$$= \left(\sum_{k=0}^{d+1} a_k \binom{k}{1} x^{k-1} + a_k \binom{k}{2} x^{k-2} + \ldots + a_k \right).$$

This implies that if the first difference of a sequence is a polynomial of degree d where $d > 0$, then the sequence must be a polynomial of degree $d + 1$ since polynomials of other degrees cannot have a first difference of degree d.

Proof: We can induct on r to show that $S(n + r, n)$ is a polynomial of n of degree $2r$. For $r = 0$, $S(n, n) = 1$. Assume that $S(n + r - 1, n)$ is of degree $2r - 2$. For $r > 0$, applying the recursive formula yields

$$S(n + r, n) = S(n + r - 1, n - 1) + nS(n + r - 1, n)$$
$$\implies S(n + r, n) - S(n + r - 1, n - 1) = nS(n + r - 1, n).$$

By the inductive hypothesis, $nS(n+r-1, n)$ is of degree $2r-1$. Since the first difference of the sequence $(S(n + r, n))_{n \geq 0}$ is a polynomial of degree $2r - 1$ and $2r - 1 > 0$, $S(n + r, n)$ is a polynomial of degree $2r$. Similarly for $c(n + k, n)$, we have that $c(n, n) = 1$ for $r = 0$ and for $r > 0$,

$$c(n + r, n) = c(n + r - 1, n - 1) + (n + r - 1)c(n + r - 1, n)$$
$$\implies c(n + r, n) - c(n + r - 1, n - 1) = (n + r - 1)c(n + r - 1, n).$$

The first difference of $(c(n+r, n))_{n \geq 0}$ is a polynomial of degree $2r-1 > 0$, so $c(n + r, n)$ is a polynomial of degree $2r$.

2. Repeatedly applying the recursive formula for $S(n, k)$ yields

$$S(n + r, n) = S(n + r - 1, n - 1) + nS(n + r - 1, n)$$
$$= S(n + r - 2, n - 2) + (n - 1)S(n + r - 2, n - 1) + nS(n + r - 1, n)$$
$$\vdots$$
$$= \sum_{i=1}^{n} iS(i + r - 1, i).$$

For $r = 2$ we have that

$$S(n+2, n) = \sum_{i=1}^{n} iS(i+1, i) = \sum_{i=1}^{n} i\binom{i+1}{2}$$

$$= \frac{1}{2}\sum_{i=1}^{n} i^3 + \frac{1}{2}\sum_{i=1}^{n} i^2 = \frac{n^2(n+1)^2}{8} + \frac{n(n+1)(2n+1)}{12}$$

$$= \frac{1}{24}n(n+1)(n+2)(3n+1).$$

For $r = 3$ we have that

$$S(n+3, n) = \sum_{i=1}^{n} iS(i+2, i) = \sum_{i=1}^{n} \frac{1}{24}i^2(i+1)(i+2)(3i+1)$$

$$= \frac{1}{8}\sum_{i=1}^{n} i^5 + \frac{5}{12}\sum_{i=1}^{n} i^4 + \frac{3}{8}\sum_{i=1}^{n} i^3 + \frac{1}{12}\sum_{i=1}^{n} i^2$$

$$= \frac{1}{48}n^2(n+1)^2(n+2)(n+3)$$

Repeatedly applying the recursive formula for $c(n, k)$ yields

$$c(n+r, n) = c(n+r-1, n-1) + (n+r-1)c(n+r-1, n)$$
$$= c(n+r-2, n-2) + (n+r-2)c(n+r-2, n-1) + (n+r-1)c(n+r-1, n)$$

$$\vdots$$

$$= \sum_{i=1}^{n}(i+r-1)c(i+r-1, i)$$

For $r = 2$ we have that

$$c(n+2, n) = \sum_{i=1}^{n}(i+1)c(i+1, i)$$

$$= \sum_{i=1}^{n}(i+1)\binom{i+1}{2}$$

$$= \frac{1}{24}n(n+1)(n+2)(3n+5)$$

For $r = 3$ we have that

$$c(n+3, n) = \sum_{i=1}^{n}(i+2)c(i+2, i)$$

$$= \sum_{i=1}^{n} \frac{1}{24}i(i+1)(i+2)^2(3i+5)$$

$$= \frac{1}{48}n(n+1)(n+2)^2(n+3)^2$$

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2 Equality of Two Integer Partitions

The generating function for a_n is

$$\sum_{n \geq 0} a_n x^n = \prod_{i \geq 0} (1 + x^i + x^{2i}) = (1 + x + x^2)(1 + x^2 + x^4)(1 + x^3 + x^6)\ldots.$$

The generating function for b_n is

$$\sum_{n \geq 0} b_n x^n = \prod_{i \geq 0} \frac{1 - x^{3i}}{1 - x^i} = \frac{1}{(1 - x)(1 - x^2)(1 - x^4)(1 - x^5)\ldots}.$$

Since $1 + x^i + x^{2i} = \frac{1 - x^{3i}}{1 - x^i}$,

$$\sum_{n \geq 0} a_n x^n = \prod_{i \geq 0} (1 + x^i + x^{2i}) = \prod_{i \geq 0} \frac{1 - x^{3i}}{1 - x^i} = \sum_{n \geq 0} b_n x^n.$$

Since the generating functions for both sequences are equal, we have that $a_n = b_n$ for all n.

3 Generalization of Example 3.27

The coefficient of $y^k x^n$ in $\prod_{i \geq 0}(1 + x^{2i+1}y)$ counts the number of partitions of n that have k parts. Under the bijection, the number of parts in the odd partitions is equal to the number of "bends" (the number of pairs of columns and rows of the same index), in the Young Diagram. Since a partition with a size r Durfee square has r bends,

$$\prod_{i \geq 0}(1 + x^{2i+1}y) = \sum_{r \geq 0} x^{r^2} y^r \sum p_{\leq r}(n) x^2 n = \sum \frac{x^{r^2} y^r}{(1 - x^2)(1 - x^4) \dots (1 - x^{2r})}.$$

4 q-Binomial Constants

1. We can induct on d to show the identity. For $d = 0$ we have

$$\sum_{n \geq 0} \binom{n}{n}_q x^n = \sum_{n \geq 0} x^n = \frac{1}{1-x}$$

Assume that $\sum_{n \geq 0} \binom{n+d-1}{n}_q x^n = \prod_{i=0}^{d-1}(1-q^i x)^{-1}$. Using the symmetry of the q-binomial coefficient and the q-Pascal's identity yields

$$\sum_{n \geq 0} \binom{n+d}{n}_q x^n = \sum_{n \geq 0} \binom{n+d}{d}_q x^n$$

$$= q^d \sum_{n \geq 0} \binom{n+d-1}{d}_q x^n + \sum_{n \geq 0} \binom{n+d-1}{d-1}_q x^n$$

$$= q^d x \sum_{n \geq 0} \binom{n+d-1}{n-1}_q x^{n-1} + \sum_{n \geq 0} \binom{n+d-1}{n}_q x^n$$

$$= q^d x \sum_{n \geq 0} \binom{n+d}{n}_q x^n + \prod_{i=0}^{d-1}(1-q^i x)^{-1}.$$

Solving out for $\sum_{n \geq 0} \binom{n+d}{n}_q x^n$ shows that

$$\sum_{n \geq 0} \binom{n+d}{n}_q x^n = q^d x \sum_{n \geq 0} \binom{n+d}{n}_q x^n + \prod_{i=0}^{d-1}(1-q^i x)^{-1}$$

$$\implies (1 - q^d x) \sum_{n \geq 0} \binom{n+d}{n}_q x^n = \prod_{i=0}^{d-1}(1-q^i x)^{-1}$$

$$\implies \sum_{n \geq 0} \binom{n+d}{n}_q x^n = \prod_{i=0}^{d}(1-q^i x)^{-1}.$$

2. The generating function $\prod_{i=0}^{d}(1-q^i x)^{-1}$ represents making a choice of integers $m_1, \ldots, m_d$ and selecting the terms $x^{m_i} q^{i m_i}$, which corresponds to the partition where i gets used m_i times.

   Therefore a term of degree x^n corresponds to a partition with $\leq n$ parts, and a term of degree q^s corresponds to a partition of size s. Since all the parts in the partition have size $\leq d$, the coefficient of x^n is the sum $\sum_\lambda q^{|\lambda|}$ over all integer partitions, λ, whose Young diagram fits in a $n \times d$ rectangle.

5 Counting Linear Maps

1. A linear map $V \to M$ can be uniquely represented as a $m \times n$ matrix. Thus, there are q^{mn} linear maps.

2. The linear map is surjective if each of its m rows are linearly independent. Thus there are $\prod_{i=0}^{m-1}(q^n - q^i)$ surjective linear maps $V \to W$.

3. A rank k linear map $V \to W$ is equivalent to a surjective linear map onto a k-dimensional subspace of W. Thus there are $\prod_{i=0}^{k-1}(q^n - q^i)$ rank k linear maps.