Math 31BH: Midterm Exam

1. **[10 points]** Define $f\colon \mathbb{R} \to \mathbb{R}^2$ by $f(t) = (2\cos t, \sin t)$.

   (a) Show that f is a 2π-periodic function, i.e. $f(t + 2\pi) = f(t)$ for all $t \in \mathbb{R}$.

   (b) Show that the image of f is $C = \{(x, y) \in \mathbb{R}^2 \colon \frac{1}{4}x^2 + y^2 = 1\}$. What kind of curve is this?

   (c) Construct a non-periodic parameterization of C, i.e. a function $g\colon \mathbb{R} \to \mathbb{R}^2$ whose image is C, but such that there does not exist a constant $p \in \mathbb{R}$ such that $g(t + p) = g(t)$ for all $t \in \mathbb{R}$.

2. **[10 points]** Let $\mathbb{R}^{2\times 2}$ denote the vector space of 2×2 matrices with the norm

$$\left\| \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \right\| = \sqrt{(a_{11})^2 + (a_{21})^2 + (a_{12})^2 + (a_{22})^2},$$

   and consider the function $f\colon \mathbb{R} \to \mathbb{R}^{2\times 2}$ defined by $f(t) = \begin{bmatrix} e^t & e^{2t} \\ e^{3t} & e^{4t} \end{bmatrix}$.

   (a) What are the component functions of f relative to the basis of elementary matrices?

   (b) Verify that $f(t)$ is a smooth curve, i.e. continuously differentiable with nonvanishing derivative.

   (c) Write down (but don't evaluate) an integral which gives the length of the arc $\{f(t) \colon t \in [0, 1]\}$ in $\mathbb{R}^{2\times 2}$.

3. **[10 points]** All functions referenced in this problem are differentiable.

   (a) Given $f\colon \mathbb{R} \to \mathbb{R}^m$, define $g(t) = f(t) \cdot f(t)$ for all $t \in \mathbb{R}$. Prove that $g'(t) = 2f(t) \cdot f'(t)$.

   (b) Prove that the position and velocity of a particle moving on a sphere are always orthogonal.

   (c) Prove that the velocity and acceleration of a particle moving at constant speed are always orthogonal.