Math 188: Homework 7

Merrick Qiu

1 Necklaces in the Dihedral Group

The Dihedral group is composed of rotational symmetries and reflection symmetries. The number of cycles of a rotation is $\gcd(n, i)$ if we rotate by i places, and the number of cycles of a reflection is $\lceil \frac{n}{2} \rceil$. Thus,

$$|Y^X/G| = \frac{1}{|G|} \sum_{g \in G} |Y|^{c_X(g)} = \frac{1}{2n} \left(nk^{\lceil \frac{n}{2} \rceil} + \sum_{i=1}^{n} k^{\gcd(n,i)} \right)$$

2 Coloring a Matrix

There are four elements in the group of rotations of the matrix. The identity element has 9 1-cycles. A rotation clockwise and a rotation counter-clockwise have 1 1-cycle and 2 4-cycles. A 180 degree rotation has 1 1-cycle and 4 2-cycles. Thus the number of colorings is,

$$|Y^X/G| = \frac{1}{|G|}\sum_{g\in G}|Y|^{c_X(g)} = \frac{1}{4}(k^9 + k^5 + 2k^3)$$

. The cyclic indicator is

$$Z_X(G;t_1,\ldots,t_9) = \frac{1}{4}(t_1^9 + t_1t_4^2 + t_1t_2^4)$$

. The number of ways to color the matrix with three colors for three entries each is

$$\begin{aligned}
&[y_1^3y_2^3y_3^3]\frac{1}{4}\left(\left(\sum_{i=1}^k y_i\right)^9 + \left(\sum_{i=1}^k y_i\right)\left(\sum_{i=1}^k y_i^4\right)^2 + \left(\sum_{i=1}^k y_i\right)\left(\sum_{i=1}^k y_i^2\right)^4\right)\\
=&\frac{1}{4}\left(\binom{9}{3,3,3,0,\ldots} + 0 + 0\right)\\
=&420
\end{aligned}$$

3 Theorem 7.9

For each orbit $\alpha \in Y^X/G$, let $f \in \alpha$ be a representative. A bijection between these orbits and weak compositions of n with d parts exists where $a_i = |f^{-1}[\{i\}]|$ for a weak composition $(a_1, \ldots, a_d)$. The number of orbits is therefore $\binom{d+n-1}{n}$. Applying Theorem 7.9 yields

$$\begin{aligned}\binom{d+n-1}{n} = \frac{1}{n!}\sum_{g\in G} d^{c_X(g)} &\implies (d+n-1)_n = \sum_{g\in G} d^{c_X(g)} \\ &\implies (d+n-1)_n = \sum_{k=0}^{n} c(n,k)d^k\end{aligned}$$