Mathematics 100A Homework 4
Due: Tuesday October 29 2024

Instructions: Please write clearly and fully explain your solutions. It is OK to work with others to solve the problems, but if you do so, you should write your solutions up separately. Copying solutions from your peers or a solutions manual will be deemed academic misconduct. Chapter and problem numbers, if any, refer to *Algebra*, second edition, by Michael Artin. Please feel free to reach out to me or the TAs if you have any questions.

1. Suppose $r_1, r_2, \ldots, r_n$ are positive integers. Say that the r_j are *pairwise coprime* if $gcd(r_i, r_j) = 1$ if $i \neq j$. Prove the following generalization of the Chinese Remainder Theorem: Assume the positive integers $r_1, \ldots, r_n$ are pairwise coprime. Then the canonical map

$$\mathbf{Z}/(r_1 \cdots r_n\mathbf{Z}) \to (\mathbf{Z}/r_1\mathbf{Z}) \times \cdots \times (\mathbf{Z}/r_n\mathbf{Z})$$

   given by $a + r_1 \cdots r_n\mathbf{Z} \mapsto (a + r_1\mathbf{Z}, \ldots, a + r_n\mathbf{Z})$ is an isomorphism. **Note**: We proved the case $n = 2$ in class. You may want to induct on n.

2. Suppose p is a prime number. Prove *Wilson's Theorem*, which asserts that $(p-1)! \equiv -1 \pmod{p}$. **Hint**: If $x \in (\mathbf{Z}/p\mathbf{Z})^\times$, consider whether x and x^{-1} are distinct modulo p.

3. Suppose n is an integer and $n \equiv 3 \pmod{4}$. Prove that there does not exists integers x, y with $x^2 + y^2 = n$.

4. Suppose p is a prime number, and $p \equiv 1 \pmod{4}$. Prove that there exists $x \in \mathbf{Z}$ with $x^2 + 1 \equiv 0 \pmod{p}$. **Hint**: Use Wilson's theorem.