Math 140B: Homework 5

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Problem 1

Since F is equicontinuous, for all $\epsilon > 0$ there exists $\delta > 0$ such that for all points x, y where $|x - y| < \delta$

$$|f_c(x) - f_c(y)| < \epsilon$$

for all $c \in S$. Assume that f is not constant and choose x, y and $\epsilon > 0$ such that $|f_c(x) - f_c(y)| > \epsilon$ for some $c \in S$. Let $L < \frac{\delta}{|x-y|}$ and let $p = Lx$, $p = Ly$, and $k \in S$ such that $k \geq \frac{c}{L}$. Note that

$$|p - q| = L|x - y| < \delta$$

but that

$$|f(kp) - f(kq)| = |f(kLx) - f(kLy)| > |f(cx) - f(cy)| > \epsilon$$

which contradicts the equicontinuity of F. Thus f must be constant.

Rudin 12

For every fixed n,

$$\int_t^T f_n(x)\,dx \leq \left|\int_t^T f_n(x)\,dx\right| \leq \int_t^T g(x)\,dx \leq \int_0^\infty g(x)\,dx < \infty$$

so as $T \to \infty$ $\int_t^\infty f_n(x)\,dx$ converges.

If we define $h(t)$ to be

$$h_n(t) = \int_t^\infty f_n(x)\,dx \qquad h(t) = \int_t^\infty f(x)\,dx$$

then we just need to show that

$$\lim_{n\to\infty}\lim_{t\to 0} h_n(t) = \lim_{t\to 0}\lim_{n\to\infty} h_n(t) = \lim_{t\to 0} h(t)$$

which is true if $h_n \to h$ uniformly. Since $\int_0^\infty g(x)\,dx$ converges, we can choose $0 < a < b < \infty$ such that

$$\int_0^a g(x)\,dx < \frac{\epsilon}{2} \qquad \int_b^\infty g(x)\,dx < \frac{\epsilon}{2}$$

Since $f_n \to f$ uniformly on $[a, b]$, there exists N such that for $n > N$,

$$|f_n(x) - f(x)| < \frac{\epsilon}{b-a}$$

Thus $h_n \to h$ converges uniformly since

$$\begin{aligned}
|h_n(t) - h(t)| &= \int_t^\infty |f_n(x) - f(x)|\,dx \\
&\leq \int_t^a |f_n(x) - f(x)|\,dx + \int_a^b |f_n(x) - f(x)|\,dx + \int_b^\infty |f_n(x) - f(x)|\,dx \\
&\leq 2\int_0^a g(x)\,dx + \frac{\epsilon}{b-a}(b-a) + 2\int_b^\infty g(x)\,dx \\
&< 3\epsilon
\end{aligned}$$

Rudin 16

Since $\{f_n\}$ is equicontinuious, for all $\epsilon > 0$ there exists $\delta > 0$ such that for all points x, y where $|x - y| < \delta$

$$|f_n(x) - f_n(y)| < \epsilon$$

for all n. Since $\{f_n\}$ converges pointwise there exists N such that,

$$|f_n(x) - f(x)| < \epsilon.$$

for a fixed x and for $n > N$. By the triangle inequality

$$\begin{aligned}|f_n(x) - f(y)| &\leq |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &< 2\epsilon\end{aligned}$$

for all $|x - y| < \delta$. Since ϵ was arbitrary, $\{f_n\}$ is uniformly convergent.

Rudin 18

Since $\{f_n\}$ is uniformly bounded, there exists M such that $|f_n(x)| \leq M$ for all n and x. F_n is uniformly bounded as well since $|F_n(x)| \leq M(b-a)|$. For all $\epsilon > 0$, $|y-x| < \frac{\epsilon}{M}$ implies

$$|F_n(y) - F_n(x)| = \left| \int_x^y f_n(t) \right| < M|x-y| < \epsilon$$

so $\{f_n\}$ is equicontinuous. Thus by the Arzela-Ascoli theorem, $\{f_n\}$ is also sequentially compact.

Rudin 19

For the backwards direction, by Arzela-Ascoli we have that closed, pointwise bounded, and equicontinuous implies S is sequentially compact.

For the forwards direction, suppose that S was not equicontinuous. Thus for some $\epsilon > 0$ and for all $\delta > 0$, there exists x, y and $f \in S$ such that $d(x, y) < \delta$ and $|f(x) - f(y)| \geq \epsilon$. Let x_n and y_n be sequences of point in K such that $d(x_n, y_n) < \frac{1}{n}$ and $|f_n(x) - f_n(y)| \geq \epsilon$ for a sequence (f_n). No subsequence of (f_n) is equicontinuous, so no subsequence of (f_n) can converge uniformly, which contradicts the fact that S is compact.