Math 140B: Homework 6

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Problem 1

Note that $C_M^\alpha([0,1])$ is the set of all Holder continuous functions on $[0,1]$ with exponent α. Thus, $C_M^\alpha([0,1])$ is equicontinuous since for all $\epsilon > 0$, we can choose $\delta < \left(\frac{\epsilon}{M}\right)^{\frac{1}{\alpha}}$ to bound $|f(x) - f(y)| < \epsilon$ for all $f \in C_M^\alpha([0,1])$ and $x, y \in [0,1]$. Subsets of $C_M^\alpha([0,1])$ will also be equicontinuous, and if these subsets are also closed and bounded, they are compact due to Arzela-Ascoli theorem.

Rudin 7.20

Since the integral is linear operator and $\int_0^1 f(x)x^n\,dx = 0$ for all $n \in \mathbb{Z}$, $\int_0^1 f(x)P(x)\,dx = 0$ for any polynomial $P(x)$ as well. By the Weierstrass theorem, there exist a sequence of polynomials P_n such that

$$\lim_{n\to\infty} P_n(x) = f(x).$$

If we take the sequence of the integrals of the product of f with these polynomials, then by Theorem 7.16,

$$\lim_{n\to\infty} \int_0^1 f(x)P_n(x)\,dx = \int_0^1 f^2(x)\,dx = 0.$$

Thus, $f(x) = 0$ on $[0, 1]$.

Rudin 7.25

Fix n. For $i = 0, \cdots, n$ put $x_i = i/n$. Let f_n be a continuous function on $[0,1]$ such that $f_n(0) = c$,

$$f_n'(t) = \phi(x_i, f_n(x_i)) \qquad \text{if } x_i < t < x_{i+1} \tag{1}$$

and put

$$\Delta_n(t) = f_n'(t) - \phi(t, f_n(t)) \tag{2}$$

except at points x_i where $\delta_n(t) = 0$. Then

$$f_n(x) = c \int_0^x [\phi(t, f_n(t)) + \Delta_n(t)]\, dt \tag{3}$$

(a) Choose $M < \infty$ so that $|\phi| \leq M$, which implies that $|f_n'| \leq M$ from (1) and $|\Delta_n| \leq 2M$ from (2). $\Delta_n \in \mathscr{R}$ is Riemann integrable because f_n' is a step function and ϕ is continuous, so from (2) Δ_n has finitely many discontinuities. Since Δ_n is Riemann-integrable over $[0,1]$, $|f_n| \leq |c| + M = M_1$.

(b) Since $|f_n'| \leq M$ and $|f_n(x) - f_n(y)| =\leq \int_x^y |f_n'(t)|\, dt \leq M|x-y|$, (f_n) is equicontinuous with $\delta = \frac{\epsilon}{M}$.

(c) Since (f_n) is equicontinuous and bounded uniformly by M_1, by Arzela-Ascoli theorem there exists a subsequence that converges to some f uniformly.

(d) Since ϕ is uniformly continuous, the sequence $f_{n_k} \to f$ converging uniformly implies that

$$\phi(t, f_{n_k}(t)) \to \phi(t, f(t))$$

converges uniformly as well since $f_{n_k} - f$ can be arbitrarily small.

(e) Since ϕ converges uniformly and

$$\Delta_n(t) = \phi(x_i, f_n(x_i)) - \phi(t, f_n(t))$$

$\Delta_n(t) \to 0$ since $f_{n_k} \to f$ uniformly and $x_i \to t$ as well.

(f) Thus the solution to the problem exists and is

$$f(x) = c + \int_0^x \phi(t, f(t))\, dt$$

Rudin 8.1

Note that from induction,

$$\frac{d^n}{dx^n} e^{-1/x^2} = \frac{P_n(x)}{Q_n(x)} e^{-1/x^2}.$$

In the base case where $n = 1$,

$$\frac{d}{dx} e^{-1/x^2} = \frac{2}{x^3} e^{-1/x^2}$$

When the statement is true for $n = k$ then for $n = k + 1$

$$\begin{aligned}
\frac{d^{k+1}}{dx^{k+1}} e^{-1/x^2} &= \frac{d}{dx}\left(\frac{P_k(x)}{Q_k(x)} e^{-1/x^2}\right) \\
&= \frac{P_k(x)}{Q_k(x)} \frac{2}{x^3} e^{-1/x^2} + \frac{P_k'(x)Q_k(x) - P_k(x)Q_k'(x)}{Q_k(x)^2} e^{-1/x^2} \\
&= \left(\frac{P_k(x)}{Q_k(x)} \frac{2}{x^3} + \frac{P_k'(x)Q_k(x) - P_k(x)Q_k'(x)}{Q_k(x)^2}\right) e^{-1/x^2}
\end{aligned}$$

For all $n \in \mathbb{Z}$ if $y = \frac{1}{x}$ then from Theorem 8.6(f)

$$\lim_{x^+ \to 0} \frac{e^{-1/x^2}}{x^n} = \lim_{y \to \infty} y^n e^{-y^2} = 0$$

$$\lim_{x^- \to 0} \frac{e^{-1/x^2}}{x^n} = \lim_{y \to -\infty} y^n e^{-y^2} = 0$$

Thus, the derivative evaluated at zero exists and is

$$f^(n)(0) = \lim_{x \to 0} \frac{P_n(x)}{Q_n(x)} e^{-1/x^2} = 0$$

Rudin 8.4

(a)
$$\lim_{x\to 0}\frac{b^x-1}{x}=\frac{d}{dx}_{x=0}e^{x\log b}=[\log(b)e^{x\log b}]_{x=0}=\log b$$

(b)
$$\lim_{x\to 0}\frac{\log(1+x)}{x}=\frac{d}{dx}_{x=0}\log(1+x)=\left[\frac{1}{1+x}\right]_{x=0}=1$$

(c)
$$\lim_{x\to 0}(1+x)^{1/x}=\lim_{x\to 0}e^{\frac{\log(1+x)}{x}}=e$$

(d)
$$\lim_{x\to 0}\left(1+\frac{x}{n}\right)^n=\left(\left(1+\frac{x}{n}\right)^{1/(x/n)}\right)^n=e^x$$

Rudin 8.5

(a)

$$\lim_{x\to 0}\frac{e-(1+x)^{1/x}}{x}=\frac{d}{dx}_{x=0}(1{+}x)^{1/x}=\lim_{x\to 0}(1{+}x)^{1/x}\left(\frac{(1+x)\log(1+x)-x}{x^2(x+1)}\right)=\frac{e}{2}$$

(b)

$$\lim_{n\to\infty}\frac{n}{\log n}[n^{1/n}-1]=\lim_{n\to\infty}\frac{e^{\frac{\log n}{n}}-1}{\frac{\log n}{n}}=\frac{d}{dx}_{x=0}e^x=1$$

(c)

$$\lim_{x\to 0}\frac{\tan x-x}{x(1-\cos x)}=\lim_{x\to 0}\frac{\sin x-x\cos x}{x\cos x(1-\cos x)}=\lim_{x\to 0}\frac{x\sin x}{x\sin x-\cos x+1}=\lim_{x\to 0}\frac{\sin x+x\cos x}{2\sin x+x\cos x}=\frac{2}{3}$$

(d)

$$\lim_{x\to 0}\frac{x-\sin x}{\tan x-x}=\lim_{x\to 0}\frac{(x-\sin x)\cos x}{\sin x-x\cos x}=\lim_{x\to 0}\frac{1-\cos x}{x\sin x}=\lim_{x\to 0}\frac{\sin x}{x\cos x+\sin x}=\lim_{x\to 0}\frac{\cos x}{2\cos x}=\frac{1}{2}$$

1 Rudin 8.6

For $x \neq 0$ and $y = 0$, $f(x)f(0) = f(x+0)$ implies $f(0) = 1$. We have that

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x)f(h) - f(x)}{h} = f(x)f'(0)$$

If $c = f'(0)$ then the function $g(x) = e^{cx}$ satisfies the given conditions. $f(x) = g(x)$ since both these functions have $f(0) = g(0) = 1$ and $\frac{d}{dx}\frac{f}{g} = 0$.

Rudin 8.7

$\frac{\sin x}{x} < 1$ since $\sin x$ achieves a supremum of 1 over the interval $[0, \frac{\pi}{2}]$. $\frac{2}{\pi} < \frac{\sin x}{x}$ since over the interval $[0, \frac{\pi}{2}]$,

$$\frac{d}{dx}\frac{\sin x}{x} = \frac{x \cos x - \sin x}{x^2} < 0$$

so $\frac{\sin x}{x}$ is strictly decreasing. Since $\frac{\sin(\pi/2)}{\pi/2} = \frac{2}{\pi}$, the inequality holds.