**Math 31AH: Fall 2021**
**Midterm Solutions**
**Wednesday, 10/27/2021**

**Instructions:** This is a 50 minute closed notes, closed books exam. Consultation with other humans is prohibited, including humans acting via websites such as Chegg. You need to clearly prove your claims; unsupported claims will get little credit. Please upload your exam to Gradescope after you are finished. You will have 10 additional minutes to upload your solutions to Gradescope.

**Problem 1:** [20] Let $\mathbb{F}$ be a field. Endow $\mathbb{F}^2$ with binary operations

$$(a, b) + (a', b') := (a + a', b + b') \qquad (a, b) \cdot (a', b') := (a \cdot a', b \cdot b')$$

Prove that these operations do **not** turn $\mathbb{F}^2$ into a field.

**Solution:** Suppose $\mathbb{F}^2$ were a field under these operations. Since $(0, 0) + (a, b) = (a, b)$ for all $(a, b) \in \mathbb{F}^2$, the element $(0, 0)$ must be the 0 (i.e. additive identity) of $\mathbb{F}^2$. However, we have

$$(1, 0) \cdot (0, 1) = (0, 0)$$

so that a product of two nonzero elements of $\mathbb{F}^2$ is zero. We proved in class that this cannot happen in any field.

**Comments:** This was a problem on the homework. Students did well here in general. Some students showed that elements of the form $(a, 0)$ do not have multiplicative inverses. There was some slight issue with not specifying that $a \neq 0$ using this approach.

**Problem 2:** [10+10] Consider the two bases of $\mathbb{R}^2$ given by $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$, $\mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_2\}$ where

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection across the line $y = x$ and let $A, B \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$ be the matrices

$$A := [T]_{\mathcal{B}}^{\mathcal{B}} \qquad B := [T]_{\mathcal{C}}^{\mathcal{C}}$$

(1) Calculate $A$.
(2) Find an invertible matrix $P$ such that $B = PAP^{-1}$.

**Solution:** (1) We calculate

$$T(\mathbf{e}_1) = T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \mathbf{e}_2$$

and

$$T(\mathbf{e}_2) = T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \mathbf{e}_1$$

and conclude

$$A = [T]_{\mathcal{B}}^{\mathcal{B}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(2) Let $i : \mathbb{R}^2 \to \mathbb{R}^2$ be the identity map. We have

$$B = [T]_{\mathcal{C}}^{\mathcal{C}} = [i \circ T \circ i]_{\mathcal{C}}^{\mathcal{C}} = [i]_{\mathcal{C}}^{\mathcal{B}}[T]_{\mathcal{B}}^{\mathcal{B}}[i]_{\mathcal{B}}^{\mathcal{C}} = PAP^{-1}$$

where $P = [i]_{\mathcal{C}}^{\mathcal{B}}$. Since

$$\mathbf{e}_1 = \frac{1}{2}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2 \quad \text{and} \quad \mathbf{e}_2 = \frac{1}{2}\mathbf{v}_1 - \frac{1}{2}\mathbf{v}_2$$

we have

$$P = [i]_{\mathcal{C}}^{\mathcal{B}} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

**Comments:** This was an example from class. Students also did well here overall. Points lost tended to come from not having enough detail. There was some confusion here regarding $P$ vs. $P^{-1}$. In this particular case, $PAP^{-1} = P^{-1}AP$, but this is **not always true** in general. So, some people 'got lucky'.

**Problem 3:** [15] For a positive integer $n$, let $V_n$ be the $\mathbb{R}$-vector space of polynomials $f(t)$ of degree $\leq n$ in the variable $t$ with real coefficients. What is the dimension of the $\mathbb{R}$-vector space $\mathrm{Hom}(V_5, V_3) = \{T : V_5 \to V_3 \, : \, T \text{ is a linear transformation}\}$?

**Solution:** The vector space $V_n$ has basis $\{1, t, t^2, \ldots, t^n\}$ and so $\dim\, V_n = n + 1$. Furthermore, we proved in class that $\mathrm{Hom}(V, W) \cong \mathrm{Mat}_{r \times s}(\mathbb{R})$ whenever $V$ and $W$ are $\mathbb{R}$-vector spaces of dimensions $\dim V = s$ and $\dim W = r$ which implies $\dim \mathrm{Hom}(V, W) = \dim V \cdot \dim W$. We conclude that

$$\dim\, \mathrm{Hom}(V_5, V_3) = \dim V_5 \cdot \dim V_3 = 6 \cdot 4 = 24.$$

**Comments:** There was a fair amount of confusion about how to calculate $\dim \mathrm{Hom}(V, W)$ for finite-dimensional vector spaces $V$ and $W$. Some students felt that it was $\dim V$ or $\dim W$. Be sure you understand why it is neither of these. There were also some points lost for thinking that $\dim V_n = n$ rather than $n + 1$.

**Problem 4:** [25] If $T : V \to V$ is a linear transformation, a subspace $W \subseteq V$ is *$T$-invariant* if $T(\mathbf{w}) \in W$ for all $\mathbf{w} \in W$. If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + y + z \\ 2x + 2y + 2z \\ -x - y - z \end{pmatrix}$$

find **all** $T$-invariant subspaces $W \subseteq \mathbb{R}^3$.

**Solution:** The linear transformation $T$ has image

$$\mathrm{Image}(T) = \left\{ \begin{pmatrix} x + y + z \\ 2x + 2y + 2z \\ -x - y - z \end{pmatrix} \, : \, x, y, z \in \mathbb{R} \right\} = \mathrm{span} \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\}$$

which is a line in $\mathbb{R}^3$ through the origin. Any subspace $W \subseteq \mathbb{R}^3$ containing this line will be $T$-invariant. This includes

- $W = \text{Image}(T)$,
- $W =$ any plane containing the line $\text{Image}(T)$, and
- $W = \mathbb{R}^3$.

Furthermore, any subspace of $\text{Ker}(T)$ is $W$-invariant. We calculate

$$\text{Ker}(T) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \,:\, x + y + z = 0 \right\}$$

Any other subspace $W$ is *not* $T$-invariant. Indeed, given such a $W$ choose $\mathbf{w} \in W$ such that $T(\mathbf{w}) \neq \mathbf{0}$. Since $W$ does not contain $\text{Image}(T)$ and $\text{Image}(T)$ is 1-dimensional, we have $T(\mathbf{w}) \notin W$.

**Comments:** This was the most difficult problem on the exam and people struggled in general. The main idea is to realize that $T$ is sending all of $\mathbb{R}^3$ to a certain line $L$. Quite a few people got a bit lost in formulas, matrices, row reduction, etc. I deducted a couple points for students who forgot about the 'silly' invariant spaces 0 and $\mathbb{R}^3$.

**Problem 5:** [20] Give an example of a vector space $V$ and a linear map $T : V \to V$ which is injective but not surjective.

**Solution:** Let $V$ be the vector space of all polynomials $f(t)$ in the variable $t$ with real coefficients. Define $T : V \to V$ by $T(f(t)) = t \cdot f(t)$. Then $T$ is linear. $T$ is not surjective since $1 \notin \text{Image}(T)$. However, $T$ is injective since $t \cdot f(t) = t \cdot g(t)$ implies $f(t) = g(t)$.

**Comments:** The main issues here were caused by students who assumed that we were working with $\mathbb{F}^n$. Indeed, there is no such example when $V$ is finite-dimensional. So, saying things about pivot 1's in RREF isn't relevant here. The hope was that you would recall the infinite-dimensional examples we have been working with. (In fact, we can find such a $T$ whenever $V$ is an infinite-dimensional vector space.)