Homework due Friday, November 17, at 11:00 pm Pacific Time.

A. Let $\{x_n\}$ be a bounded sequence in $\mathbb{R}$. For every $N \in \mathbb{N}$ define

$$s_N = \sup\{x_n : n \geq N\} \quad \text{and} \quad r_N = \inf\{x_n : n \geq N\}.$$

Prove that

(1) $\{s_N\}$ and $\{r_N\}$ are bounded.
(2) $\{s_N\}$ is non-increasing, i.e., $s_1 \geq s_2 \geq \cdots$, and $\{r_N\}$ is nondecreasing, i.e., $r_1 \leq r_2 \leq \cdots$.
(3) Let $s = \lim s_N = \inf\{s_N : N \in \mathbb{N}\}$, and $r = \lim r_N = \sup\{r_N : N \in \mathbb{N}\}$. Prove that $s = \limsup x_n$ and $r = \liminf x_n$.

B. Let K be a compact metric space. Show that there exists a sequence $\{p_n\}$ in K so that the set of subsequential limits of $\{p_n\}$ *equals* K.

C. Let $\{a_n\}$ be a sequence of positive real numbers. Assume that $a_n \to a$. Prove that $\sqrt{a_n} \to \sqrt{a}$.

D. Rudin, Chapter 3 (page 78), problems # 5, 20, 21.

The following problems are for your practice, and will not be graded.

(1) Let (X, d) be a metric space. Let $E, F \subset X$ be two connected subsets of X, and assume that $E \cap F \neq \emptyset$. Prove that $E \cup F$ is connected.

(2) Let (X_1, d_1) and (X_2, d_2) be two complete metric spaces. Let $X = X_1 \times X_2$, and define

$$d\big((x_1, x_2), (y_1, y_2)\big) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}.$$

Prove that (X, d) is a complete metric space.

(3) Let (X, d) be a metric space. Let $K \subset X$ be a nonempty compact set, and fix a point $p \in X \setminus K$.
  (a) Show that $\{d(p, q) : q \in K\}$ is a bounded subset of $\mathbb{R}$.
  (b) Set $\alpha = \sup\{d(p, q) : q \in K\}$. Prove that there is $q \in K$ satisfying $d(p, q) = \alpha$.