Math 140A: Homework 4

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A

1. The statement is false. In $\mathbb{R}$, the intervals $(0, 1)$ and $(1, 2)$ are disjoint but their distance is 0.

2. The statement is false. The intervals $(0, 1)$ and $(1, 2)$ are separate since $[0, 1] \cap (1, 2) = \emptyset$ and $(0, 1) \cap [1, 2] = \emptyset$ but their distance is 0.

3. The statement is false. The intervals $(0, 1)$ and $(1, 2)$ are disjoint open sets but their distance is 0.

4. The statement is false. Let $A = \mathbb{N}$ and let $B = \{n + \frac{1}{n+1} : n \in \mathbb{N}\}$. A and B both only have isolated points so they are both closed. They are also both disjoint. However $d(A, B) = 0$, so this statement is false.

B

1. We need to show that all points in the closed neighborhood are limit points and all points not in the closed neighborhood are not limit points. If $y \in \overline{N}_r(x)$, then we need to show that y is a limit point by finding a point in $N_{r'}(y)$ for an arbitrary r'. If $r' > r - d(x,y)$, then choose $r' \leq r - d(x,y)$ since a point in a smaller neighborhood will also be in the larger neighborhood. Any point $z \in N_{r'}(y)$ will also be in $\overline{N}_r(x)$ since by the triangle inequality,

   $$d(x,z) \leq d(x,y) + d(y,z) \leq r$$

   If $y \notin \overline{N}_r(x)$, then we need to show that y is not a limit point by finding a r' such that $\overline{N}_r(x) \cap N_{r'}(y) = \emptyset$. Choose $r' < d(x,y) - r$. For all points $z \in N_{r'}(y)$

   $$r < d(x,y) - d(y,z) \leq d(x,z)$$

   by the triangle inequality(with $d(y,z)$ subtracted from both sides). Thus the closed neighborhood is a closed set.

2. We need to show that points inside the closed neighborhood are limit points of the open neighborhood and points outside the closed neighborhood are not limit points. If $y \in \overline{N}_r(x)$ then $d(x,y) \leq r$. For all r' we need to find a point $z \in N_{r'}(y)$ such that $z \in N_r(x)$ as well. Choose c such that $\frac{r-r'}{r} < c < 1$ and let $z = cx + (1-c)y$. This point is both in the neighborhood of y as well as in neighborhood x so y is a limit point.

   If $y \notin \overline{N}_r(x)$ then $d(x,y) > r$. We need to find a r' such that $N_{r'}(y) \cap N_r(x) = \emptyset$. Choose $r' < d(x,y) - r$. For all points $z \in N_{r'}(y)$

   $$r < d(x,y) - d(y,z) \leq d(x,z)$$

   by the triangle inequality, so y is not a limit point.

3. It is not true in general. For the discrete metric, a $N_1(x)$ only contains x and its closure also just contains x. However, $\overline{N}_1(x)$ contains all the points in the metric space.

C

$(\implies)$All non-empty open subsets O must contain a point $x \in O$, and this point is an interior point because the set is open. For some neighborhood $N_r(x)$, $N_r(x) \subset O$ since x is an interior point, but since A is dense, x is also a limit point of A and $a \in N_r(x)$ for some $a \in A$. Thus $A \cap O \neq \emptyset$ for all O.

$(\impliedby)$ Since all open neighborhoods are open and $A \cap O \neq \emptyset$ for all open sets, we know that every open neighborhood contains a point in A, which is the definition of A being dense.

Rudin Question 9

(a) For all $x \in E^{\circ}$, we know that there exists r such that $N_r(x) \subset E$ by the definition of an interior point. It is sufficient to show that x is an interior point of E° by showing that all points $y \in N_r(x)$ are also in E°. If we choose $r' < r - d(x,y)$, then $N_{r'}(y) \subset N_r(x) \subset E$ (since by the triangle inequality, a point $z \in N_{r'}(y)$ will have $d(x,z) \leq d(x,y) + d(y,z)$) so we know that y is also an interior point of E. Thus E° is always open.

(b) If $E^{\circ} = E$ then every point of E is an interior point so $E^{\circ} = E$ implies that E is open. If E is open, then every point of E is an interior point of E so $E \subset E^{\circ}$. Likewise, if a point is an interior point, it must be in E. So a set being open implies $E = E^{\circ}$.

(c) This is true since $G \subset E = E^{\circ}$.

(d) Since E is open, its complement is closed. This means that $E^c = \overline{E^c}$. Also since $E = E^{\circ}$ we have that

$$\begin{aligned} & E^{\circ} = E \\ \implies\ & E^{\circ c} = E^c \\ \implies\ & E^{\circ c} = \overline{E^c} \end{aligned}$$

(e) No, the set $\mathbb{R} \setminus \{0\}$ does not have an interior point at 0 but its closure does.

(f) No the rational numbers have a closure that is the real numbers, but the rational numbers have no interior points.

Rudin Question 22

The set of points which have only rational coordinates are dense in R^k. Let $(x_1, x_2, \cdots, x_k) \in \mathbb{R}^k$, with a neighborhood around that point of radius r.

We can choose $(y_1, y_2, \cdots, y_k) \in \mathbb{Q}^k$ such that $|x_i - y_i|^2 < \frac{r^2}{k}$ for all i by the density of the rationals in the reals,. Under the standard metric this point in $\mathbb{Q}^k$ will be at most r distance away from the original point in $\mathbb{R}^k$, thus $\mathbb{R}^k$ is separable.