Math 140C: Homework 4

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Rudin 9.26

Let $f(x, y) = g(x)$, where g is a function that is nowhere continuous. Thus, $D_2 f = 1$ and $D_{12} f = 0$ but $D_1 f$ doesn't exist.

Rudin 9.27

1. f is continuous away from $(0,0)$, so we just need to show it is continuous at the origin. Using polar coordinates shows us that f is continuous at the origin.

$$\begin{aligned} f(r\cos\theta, r\sin\theta) &= \frac{r^2\cos\theta\sin\theta(r^2\cos^2\theta - r^2\sin^2\theta)}{r^2} \\ &= \frac{r^2\cos 2\theta(\cos^2\theta - \sin^2\theta)}{2} \\ &= \frac{r^2\cos 2\theta\sin 2\theta}{2} \\ &= \frac{r^2\sin 4\theta}{4} \end{aligned}$$

$$\lim_{r\to 0}|f(x,y)| = \lim_{r\to 0}\frac{r^2\sin 4\theta}{4} \le \lim_{r\to 0}\frac{r^2}{4} = 0 = f(0,0)$$

At the origin,

$$D_1f(0,0)\lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0}\frac{0}{x} = 0$$

$$D_2f(0,0)\lim_{y\to 0}\frac{f(0,y)-f(0,0)}{y} = \lim_{y\to 0}\frac{0}{y} = 0$$

D_1f exists away from the origin and it is continuous since

$$\begin{aligned} D_1f(x,y) &= \frac{(x^2+y^2)(3x^2y-y^3)-(2x)(x^3y-xy^3)}{(x^2+y^2)^2} \\ &= \frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2} \end{aligned}$$

$$D_1f(r\cos\theta, r\sin\theta) = \frac{r^5(\cos^4\theta\sin\theta + 4\cos^2\theta\sin^3\theta - \sin^5\theta)}{r^4}$$
$$\lim_{r\to 0}|D_1f(r\cos\theta, r\sin\theta)| \le \lim_{r\to 0}6r = 0 = D_1f(0,0).$$

D_2f exists away from the origin and it is continuous since

$$\begin{aligned} D_2f(x,y) &= \frac{(x^2+y^2)(x^3-3xy^2)-(2y)(x^3y-xy^3)}{(x^2+y^2)^2} \\ &= \frac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2} \end{aligned}$$

$$D_2f(r\cos\theta, r\sin\theta) = \frac{r^5(\cos^5\theta - 4\cos^3\theta\sin^2\theta - \cos\theta\sin^4\theta)}{r^4}$$
$$\lim_{r\to 0}|D_2f(r\cos\theta, r\sin\theta)| \le \lim_{r\to 0}6r = 0 = D_2f(0,0).$$

2. Away from the origin, D_{12} is continuous and has value

$$\begin{aligned} D_{12}f(x,y) &= \frac{(x^2+y^2)(5x^4-12x^2y^2-y^4)-4x(x^5-4x^3y^2-xy^4)}{(x^2+y^2)^3} \\ &= \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3} \end{aligned}$$

$$D_{12}f(r\cos\theta, r\sin\theta) = \cos^6\theta + 9\cos^4\theta\sin^2\theta - 9\cos^2\theta\sin^4\theta - \sin^6\theta.$$

Since D_{12} is independent of r but has different values for different θ, D_{12} does not converge as $r \to 0$. Since we are in $\mathbb{R}^2$, $D_{12}f = D_{21}f$ so $D_{21}f$ is not continuous at the origin either.

3.

$$\begin{aligned} D_{12}f(0,0) &= \lim_{x\to 0} \frac{D_2f(x,0)-D_2f(0,0)}{x} \\ &= \lim_{x\to 0} \frac{x^5}{x^5} \\ &= 1 \end{aligned}$$

$$\begin{aligned} D_{21}f(0,0) &= \lim_{y\to 0} \frac{D_1f(0,y)-D_1f(0,0)}{y} \\ &= \lim_{y\to 0} -\frac{y^5}{y^5} \\ &= -1 \end{aligned}$$

Rudin 9.28

Since each piece is continuous, we just need to check that the pieces equal each other at the boundaries.

When $x = 0$,

$$0 = x$$

When $x = \sqrt{t}$,

$$x = -x + 2\sqrt{t}.$$

When $x = 2\sqrt{t}$

$$-x + 2\sqrt{t} = 0.$$

Since the same boundaries hold for $t < 0$, f is continuous. Note that $\varphi(x, t) = 0$ in the neighborhood $\frac{1}{4}x^2 < t < \frac{1}{4}x^2$, so $(D_2\varphi)(x, 0) = 0$.

When $|t| < \frac{1}{4}$,

$$\begin{aligned}
f(t) &= \int_{-1}^{1} \varphi(x, t)\, dx \\
&= \int_{-1}^{0} 0\, dx + \int_{0}^{\sqrt{t}} x\, dx + \int_{\sqrt{t}}^{2\sqrt{t}} -x + 2\sqrt{t}\, dx + \int_{2\sqrt{t}}^{1} 0\, dx \\
&= \left[\frac{1}{2}x^2\right]_0^{\sqrt{t}} + \left[-\frac{1}{2}x^2 + 2\sqrt{t}x\right]_{\sqrt{t}}^{2\sqrt{t}} \\
&= \frac{1}{2}t + (-2t + 4t) - \left(-\frac{1}{2}t + 2t\right) \\
&= t.
\end{aligned}$$

Thus,

$$f'(0) = 1 \neq 0 = \int_{-1}^{1} (D_2\varphi)(x, 0)\, dx.$$

Rudin 9.29

We want to show that for any permutation σ,

$$D_{i_1 i_2 \ldots i_k} f = D_{i_{\sigma(1)} i_{\sigma(2)} \ldots i_{\sigma(k)}} f.$$

Theorem 9.41 says that we can choose any two adjacent indices and switch them, $D_{i_{k-1} i_k} f = D_{i_k i_{k-1}} f$. Note that we can repeatedly apply theorem 9.41 in order to switch any two indices in a permutation. Also note that any permutation can be written as the composition index switches. Thus through the repeated application of theorem 9.41, our conclusion holds for all k and for all σ.

Problem 2

1. Each component of f is continuous, so f is continuous. The Jacobian matrix is
$$f'(x,y,z) = \begin{bmatrix} 2(x+z) & -1 & 2(x+z) \\ -2x & 1 & -1 \end{bmatrix}.$$
Since all the partial derivatives exist and are continuous, $f \in C^1(E)$.

2. Since $f \in C^1(E)$ is a mapping from $\mathbb{R}^{1+2}$ into $\mathbb{R}^1$ and $f(a,b) = f(1,1,0) = 0$, by the implicit function theorem, there exists $U \in \mathbb{R}^{1+2}$, $W \in \mathbb{R}^1$, and g such that $g(1) = (1,0)$ and $f(x, g(x)) = 0$ for all $x \in W$.

3.
$$\begin{aligned} g'(1) &= -(A_x)^{-1} A_y \\ &= -\begin{bmatrix} -1 & 2(x+z) \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 2(x+z) \\ -2x \end{bmatrix} \\ &= -\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -2 \end{bmatrix} \\ &= -\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 0 \end{bmatrix} \end{aligned}$$