Math 31AH: Fall 2021
Homework 6
Due 5:00pm on Friday 11/12/2021

Problem 1: Eigenvalues and Eigenvectors. Let A be the 3×3 real matrix

$$A = \begin{pmatrix} 7 & -8 & 6 \\ 8 & -9 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

Find the eigenvalues of A and a basis for each eigenspace of A. Is A diagonalizable?

Problem 2: More Eigenvalues and Eigenvectors. Let B be the 3×3 real matrix

$$B = \begin{pmatrix} 1 & 1/4 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1/4 & -1 \end{pmatrix}$$

Find the eigenvalues of B and a basis for each eigenspace of B. Is B diagonalizable?

Problem 3: $\mathbb{R}$ and $\mathbb{C}$ and diagonalizability. Let $0 \leq \theta < 2\pi$ be an angle expressed in radians. Consider the matrix

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

For which values of θ is R_θ diagonalizable over $\mathbb{R}$? What about over $\mathbb{C}$?

Problem 4: Polynomials and diagonalizability. Let $\mathbb{F}$ be a field and consider a polynomial $f(t) := c_n t^n + \cdots + c_1 t + c_0$ in t with coefficients $c_i \in \mathbb{F}$. Let A be an $n \times n$ matrix over $\mathbb{F}$ and define a new matrix $f(A)$ by

$$f(A) := c_n A^n + \cdots + c_1 A + c_0$$

If A is diagonalizable, prove that $f(A)$ is also diagonalizable.

Problem 5: Direct sums and diagonalizability. Let V and W be finite-dimensional $\mathbb{F}$-vector spaces and consider two linear transformations $T : V \to V$ and $U : W \to W$. We define the *direct sum transformation*

$$T \oplus U : V \oplus W \to V \oplus W$$

by $(T \oplus U)(\mathbf{v}, \mathbf{w}) := (T(\mathbf{v}), U(\mathbf{w}))$ for any $(\mathbf{v}, \mathbf{w}) \in V \oplus W$.

(1) If λ is an eigenvalue of T or an eigenvalue of U, prove that λ is also an eigenvalue of $T \oplus U$.
(2) If T and U are both diagonalizable, prove that $T \oplus U$ is also diagonalizable.

Problem 6: A 6×6 example. Let A be the 6×6 matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Find the eigenvalues of A over the complex numbers, and a basis for each eigenspace of A.

Problem 7: A real sequence. Suppose $A = PDP^{-1}$ for some $n \times n$ matrices A, P, D with P invertible and D diagonal. Prove that $A^n = PD^nP^{-1}$ for any positive integer n.

Define a sequence $(a_1, a_2, a_3, \dots)$ of real numbers recursively by the rule $a_1 = a_2 = 1$ and $a_n = 4a_{n-1} - 2a_{n-2}$ for $n \geq 3$. Find an explicit formula for a_n. *Hint: Consider the matrix equation*

$$\begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_{n-1} \\ a_{n-2} \end{pmatrix}$$

Problem 8: Commuting operators and eigenspaces. Let V be an $\mathbb{F}$-vector space and let $T, U : V \to V$ be two linear transformations. Assume that T and U *commute*, that is $T \circ U = U \circ T$. If $\lambda \in \mathbb{F}$ and $E_\lambda \subseteq V$ is the T-eigenspace corresponding to λ, prove that E_λ is U-invariant.

Does this conclusion necessarily hold if we drop the assumption that T and U commute?

Problem 9: (Optional; not to be handed in.) Let $\mathbb{F}$ be a field and let $\mathcal{A}$ be a collection of $n \times n$ matrices over $\mathbb{F}$. The collection $\mathcal{A}$ is called *simultaneously diagonalizable* if there exists a single invertible $n \times n$ matrix P over $\mathbb{F}$ such that PAP^{-1} is diagonal for all $A \in \mathcal{A}$.

Prove that $\mathcal{A}$ is simultaneously diagonalizable if and only if each matrix A is diagonalizable and $AB = BA$ for all $A, B \in \mathcal{A}$.