Mathematics 100A Homework 6
Due: Tuesday November 12 2024

Instructions: Please write clearly and fully explain your solutions. It is OK to work with others to solve the problems, but if you do so, you should write your solutions up separately. Copying solutions from your peers or a solutions manual will be deemed academic misconduct. Chapter and problem numbers, if any, refer to *Algebra*, second edition, by Michael Artin. Please feel free to reach out to me or the TAs if you have any questions.

1. Let $S = \mathbf{R}^2 \setminus \{(0,0)\}$ be the set of nonzero elements of $\mathbf{R}^2$. Prove that $\mathrm{GL}_2(\mathbf{R})$ operates transitively on S.

2. For $S = \mathbf{R}^2 \setminus \{(0,0)\}$, what is the stabilizer of the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for the action of $\mathrm{GL}_2(\mathbf{R})$ on S?

3. Suppose $z = x + iy \in \mathbf{C}$ has $y > 0$ and $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbf{R})$.

   (a) Check that $cz + d \neq 0$.

   (b) Let $g \cdot z = \frac{az+b}{cz+d}$, which is defined because $cz + d \neq 0$. Prove that $Im(g \cdot z) = \frac{y}{|cz+d|^2}$, where $Im(w)$ denotes the imaginary part of the complex number w.

   (c) Let $\mathcal{H} = \{z \in \mathbf{C} : Im(z) > 0\}$. If $g \in \mathrm{SL}_2(\mathbf{R})$ and $z \in \mathcal{H}$, the previous part shows $g \cdot z \in \mathcal{H}$. Prove that $(g, z) \mapsto g \cdot z$ gives an action of $\mathrm{SL}_2(\mathbf{R})$ on $\mathcal{H}$.

4. Let $K = \mathrm{SO}(2) \subseteq \mathrm{SL}_2(\mathbf{R})$ and set $B \subseteq \mathrm{SL}_2(\mathbf{R})$ the subgroup of upper triangular matrices.

   (a) Prove that B acts transitively on $\mathcal{H}$, and consequently $\mathrm{SL}_2(\mathbf{R})$ does as well.

   (b) Prove that the stabilizer of $i \in \mathcal{H}$ is $\mathrm{SO}(2)$.

   (c) Deduce that if $g \in \mathrm{SL}_2(\mathbf{R})$, there exists $b \in B$ and $k \in \mathrm{SO}(2)$ so that $g = bk$.