Math 140A: Homework 10

Merrick Qiu

A

Choose $\epsilon > 0$. From the limits we know that there exist c_1, c_2 such that $x > c_1$ implies that $|x - \ell_1| < \frac{\epsilon}{2}$ and $x < c_2$ implies that $|x - \ell_2| < \frac{\epsilon}{2}$. For all $c_2 - 1 \leq x \leq c_1 + 1$, f is uniformly continuous since $[c_2 - 1, c_1 + 1]$ is a compact set. Assume that we choose a $\delta < 1$. $[c_1, \infty]$ and $[-\infty, c_2]$ are uniformly continuous for any δ since we chose c_1 and c_2 such that all points are within ϵ of each other. Thus, the function is uniformly continuous.

B

Choose $\epsilon > 0$. If we choose $\delta < \left(\frac{\epsilon}{c}\right)^{\frac{1}{\alpha}}$, Then

$$d(f(x), f(y)) \leq Cd(x, y)^{\alpha} < \epsilon$$

so f is uniformly continuous.

C

f is a homeomorphism iff the image of an open set is an open set as well. All connected sets in $\mathbb{R}$ are intervals, and the image of f on a connected set must also be connected, so $f((a, b))$ must also be an interval for all $a, b \in \mathbb{R}$. This interval must be open as well since continuous bijections map open intervals to open intervals. Since all open sets can be expressed as the countable union of open intervals, f maps open sets to open sets. Thus f is a homeomorphism.

Rudin 2

Let $y \in f(\overline{E})$ and $x \in \overline{E}$ with $f(x) = y$. Since $\overline{E}$ is a closure, there must exist a sequence $x_i \to x$ where $x_i \in E$ for all i. Thus all the points in the sequence $f(x_i)$ are in the image, $f(E)$, and $\lim f(x_i) = f(\lim x_i) = y$ due to the continuity of f, so $y \in \overline{f(E)}$. Since y was arbitrary, we have that $f(\overline{E}) \subset \overline{f(E)}$.

If $f : [1, \infty) \to \mathbb{R}$, $f(x) = \frac{1}{x}$ and $E = [1, \infty)$, then $f(\overline{E}) = (0, 1]$ but $\overline{f(E)} = [0, 1]$

Rudin 4

Since E is dense in X, every point $x \in X$ is the limit of some sequence of points, $x_i \to x$ where $x_i \in E$, for all i. Since f is continuous we have that $\lim f(x_i) = f(\lim x_i) = f(x)$, so every point $f(x)$ can also be represented as a sequence of points from $f(E)$. Thus $f(E)$ is dense in $f(X)$.

If $g(x_i) = f(x_i)$ for all $x_i \in E$, then $f(x) = \lim f(x_i) = \lim g(x_i) = g(x)$, so $f(x) = g(x)$.

Rudin 6

($\implies$) Take a sequence of points in the graph $(x_n, f(x_n))$. Since E is compact, there is a subsequence $x_{n_k} \to x$ for some x. Since f is continuous, we also have that $f(x_{n_k}) \to f(x)$, so any sequence of points in the graph has a convergent subsequence, so the graph is compact.

($\impliedby$) Let $x \in E$. Take a sequence of points in the graph $(x_n, f(x_n))$ such that $x_n \to x$. By compactness, there exists a convergent subsequence $(x_{n_k}, f(x_{n_k})) \to (x, f(x))$. It is the case that $f(x_{n_k}) \to f(x)$ because if it did not, then the graph would fail to contain the limit point $(x, f(x))$, which would contradict the fact that the graph is compact.

Rudin 8

Let $a = \inf E$ and $b = \sup E$ For some $\epsilon > 0$, there exists δ such that all points $p, q \in E$ with $|p - q| < \delta$ implies that $|f(p) - f(q)| < \epsilon$. We can "divide up" E into $N = \lceil \frac{b-a}{\delta} \rceil$ sections. We have that $\inf f(E) \geq f(a) - N * \epsilon$ and $\sup f(E) \leq f(a) + N * \epsilon$ since the maximum amount of change that can occur within a delta neighborhood of a point is ϵ. Thus, $f(E)$ is bounded.

Rudin 14

If we define $g(x) = f(x) - x$, $0 \leq g(0) \leq 1$ and $-1 \leq g(1) \leq 0$. By the intermediate value theorem, $g(x) = 0$ at some point, so there exists some point where $f(x) = x$.

Rudin 18

At every point p, the limit is 0. For any $\epsilon > 0$ we can find n such that $\frac{1}{n} < \epsilon$ by the archimedes principle. Then we can choose δ such that all rational numbers with denominator $< n$ are not in the delta neighborhood around p. This is possible because there is a finite number of rational numbers with denominators from 1 to n.

All the points x in this δ neighborhood have $|x - 0| < \epsilon$, so the function converges to 0 at all points. However only irrational points evaluate to 0 so the function is only continuous at irrational points, but it has simple discontinuity at every rational point.