Math 31CH HW3 SOLUTIONS
Due April 19 at 11:59 pm by Gradescope Submission

Professor Bennett Chow

EXERCISES FOR SECTION 4.9

Exercise 4.9.1: Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be given by the matrix

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 2 & 2 & 0 & \cdots & 0 \\ 3 & 3 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ n & n & n & \cdots & n \end{bmatrix},$$

and let $A \subset \mathbb{R}^n$ be the region given by

$$|x_1| + |x_2|^2 + |x_3|^3 + \cdots + |x_n|^n \leq 1.$$

What is $\operatorname{vol}_n T(A)/\operatorname{vol}_n A$?

Solution to 4.9.1: The fraction can be calculated using the determinant of T, which ends up being the product of the diagonal entries.

$$\frac{\operatorname{vol}_n T(A)}{\operatorname{vol}_n A} = \det T = n!$$

Exercise 4.9.4: What is the n-dimensional volume of the region

$$\{\mathbf{x} \in \mathbb{R}^n \,|\, x_i \geq 0 \text{ for all } i = 1, \ldots, n \text{ and } x_1 + \cdots + x_n \leq 1\}\,?$$

Solution to 4.9.4: Let $V_n(t)$ represent the volume of the region $S_n(t) = \{\mathbf{x} \in \mathbb{R}^n \,|\, x_i \geq 0 \text{ for all } i = 1, \ldots, n \text{ and } x_1 + \cdots + x_n \leq t\}$. Therefore,

$$\begin{aligned}
V_n(1) &= \int_{S_n(1)} |d^n x| \\
&= \int_0^1 V_{n-1}(1 - x_n)\, dx_n \\
&= \int_0^1 (1 - x_n)^{n-1} V_{n-1}(1)\, dx_n \\
&= V_{n-1}(1) \left[-\frac{1}{n}(1 - x_n)^n \right]_0^1 \\
&= \frac{1}{n} V_{n-1}(1)
\end{aligned}$$

Since $V_1(1) = 1$, by induction we have that $V_n(1) = \frac{1}{n!}$.

Exercise 4.9.6: Compute the volume of the three k-parallelograms ($k = 3, 4, 5$) spanned by the vectors:
(a)

$$\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix}.$$

(b)

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

(c)

$$\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

Solution to 4.9.6: The volumes of the parallelograms can be calculated by taking the absolute value of the determinant.

1. The volume is 18.

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 3 & -1 & 6 \\ -1 & -1 & 0 \end{bmatrix} = 1(6) - 2(6) + 3(-4) = -18$$

2. The volume is 3.

$$\det \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix} = 1(-1) - 1(1) + 1(-1) = -3$$

3. The volume is 32.

$$\det \begin{bmatrix} 4 & 3 & 2 & 1 & 0 \\ 3 & 2 & 1 & 0 & 1 \\ 2 & 1 & 0 & 1 & 2 \\ 1 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix} = 4(4) - 3(0) + 2(0) - 1(-16) = 32$$

EXERCISES FOR SECTION 4.10

Exercise 4.10.4: Use the change of variables formula to compute the volume of the region

$$\frac{x^2}{(z^3-1)^2} + \frac{y^2}{(z^3+1)^2} \le 1, \quad -1 \le z \le 1,$$

shown in the Figure on p. 497.

Solution to 4.10.4: Horizontal slices of the figure are ellipses, so we can use the map

$$\Phi \begin{pmatrix} r \\ \theta \\ z \end{pmatrix} \to \begin{pmatrix} (z^3-1)r\cos\theta \\ (z^3+1)r\sin\theta \\ z \end{pmatrix}$$

The determinant of the Lipschitzs derivative is

$$\begin{aligned} \det \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{bmatrix} &= \det \begin{bmatrix} (z^3-1)\cos\theta & -(z^3-1)r\sin\theta & 3z^2 r\cos\theta \\ (z^3+1)\sin\theta & (z^3+1)r\cos\theta & 3z^2 r\sin\theta \\ 0 & 0 & 1 \end{bmatrix} \\ &= 0 - 0 + 1((z^6-1)r\cos^2\theta + (z^6-1)r\sin^2\theta) \\ &= (z^6-1)r \end{aligned}$$

Therefore the integral is given by

$$\begin{aligned} \int_0^{2\pi}\int_0^1\int_{-1}^1 |(z^6-1)r|\,dz\,dr\,d\theta &= \int_0^{2\pi}\int_0^1 \frac{12}{7} r\,dr\,d\theta \\ &= \int_0^{2\pi} \frac{6}{7}\,d\theta \\ &= \frac{12}{7}\pi \end{aligned}$$

Exercise 4.10.5: (a) What is the area of the ellipse $x^2/a^2 + y^2/b^2 \leq 1$? *Hint*: Use the change of variables $u = x/a$, $v = y/b$.
(b) What is the volume of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 \leq 1$?

Solution to 4.10.5:

1. The following parameterization maps from the unit circle to the ellipse.

$$\Phi \begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} au \\ bv \end{pmatrix}$$

   The determinant of the Lipschitzs derivative is ab, meaning it streches the area of the unit circle by ab to get the area of the ellipse. Therefore the area of the ellipse is πab.

2. The following parameterization maps from the unit sphere to the ellipsoid.

$$\Phi \begin{pmatrix} u \\ v \\ w \end{pmatrix} \to \begin{pmatrix} au \\ bv \\ cw \end{pmatrix}$$

   The determinant of the Lipschitzs derivative is abc, so the volume of the ellipsoid is $\frac{4}{3}\pi abc$.

Exercise 4.10.8: (a) For fixed $a, b > 1$, let $U_{a,b}$ be the plane region in the first quadrant defined by the inequalities $1 \le xy \le a$, $x \le y \le bx$. Sketch $U_{2,4}$.

(b) Compute $\int_{U_{a,b}} x^2y^2\,|dx\,dy|$.

Solution to 4.10.8:

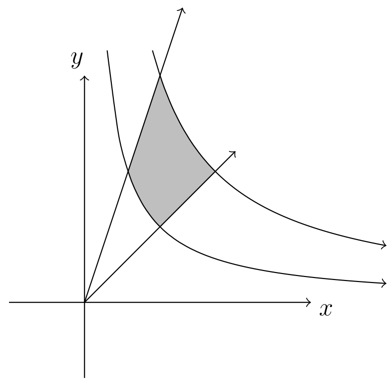

Figure 1: Exercise 4.10.8.a

Using the change of variables $u = xy$ and $v = \frac{y}{x}$, the parameterization is

$$\Phi \begin{pmatrix} u \\ v \end{pmatrix} \to \begin{pmatrix} \sqrt{\frac{u}{v}} \\ \sqrt{uv} \end{pmatrix}$$

The determinant of the Lipschitzs derivative is

$$\begin{aligned} \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} &= \det \begin{bmatrix} \frac{1}{2\sqrt{uv}} & -\frac{1}{2}\sqrt{\frac{u}{v^3}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} \end{bmatrix} \\ &= \frac{1}{4v} + \frac{1}{4v} \\ &= \frac{1}{2v} \end{aligned}$$

The integral is

$$\int_1^a \int_1^b \frac{u^2}{2v}\,dv\,du = \int_1^a \frac{u^2}{2} \ln b\,du = \frac{\ln b}{6}(a^3 - 1)$$

Exercise 4.10.12: Evaluate the iterated integral

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}}(x^2+y^2+z^2)^{3/2}dz\,dy\,dx.$$

Solution to 4.10.12: Using spherical coordinates, the integral over the quarter-sphere becomes

$$\int_0^{\pi}\int_0^{2}\int_0^{\frac{\pi}{2}} r^5\cos\varphi|\,d\varphi\,dr\,d\theta| = \pi\int_0^2 r^5\,dr = \frac{32}{3}\pi$$

Exercise 4.10.17: Let $Q_a = [0,a] \times [0,a] \subset \mathbb{R}^2$ be the square of side length a in the first quadrant, with two sides on the axes, and let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$\Phi \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u - v \\ e^u + e^v \end{pmatrix}.$$

Set $A = \Phi(Q_a)$.
(a) Sketch A, by computing the image of each of the sides of Q_a. It might help to being by drawing carefully the curves of the equations $y = e^x + 1$ and $y = e^{-x} + 1$.
(b) Show that $\Phi : Q_a \to A$ is one-to-one.
(c) What is $\displaystyle\int_A y\,|dx\,dy|$?

Solution to 4.10.17:

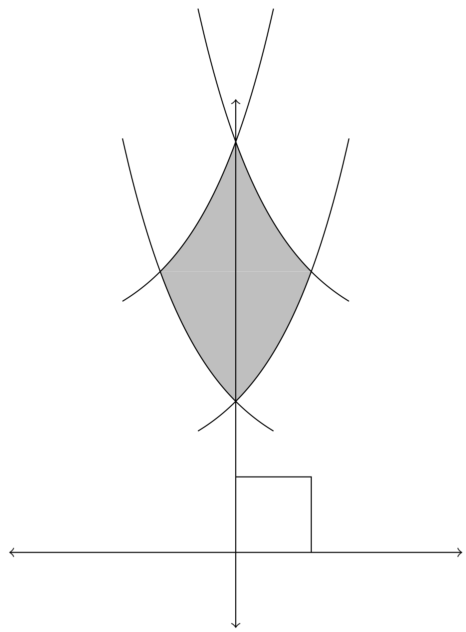

Figure 2: Exercise 4.10.17.a

Assume that Φ is not one-to-one. Therefore there exists different (u_1, v_1) and (u_2, v_2) such that $u_1 - v_1 = u_2 - v_2$. So either $u_1 > u_2$ and $v_1 > v_2$ or $u_2 > u_1$ and $v_2 > v_1$. However since e^x is a strictly increasing function, $e^{u_1} + e^{v_1} = e^{u_2} + e^{v_2}$ cannot be true. Therefore Φ must be one-to-one.
The determinant of the Lipschitzs derivative is

$$\det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det \begin{bmatrix} 1 & -1 \\ e^u & e^v \end{bmatrix} = e^u + e^v$$

Therefore the integral over A is

$$\int_A y\,|dx\,dy| = \int_0^a \int_0^a (e^u + e^v)^2\,|du\,dv| = a(e^{2a} - 1) + 2(e^a - 1)^2$$