Math 181A: Homework 4

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Problem 1: 5.4.19

1. $\hat{\theta}_1 = Y_1$ is unbiased because

$$\begin{aligned} E[Y_1] &= \int_0^\infty \frac{1}{\theta} y e^{-y/\theta}\, dy \\ &= \left[-e^{-y/\theta}(y+\theta)\right]_0^\infty \\ &= 0 - (-\theta) \\ &= \theta. \end{aligned}$$

$\hat{\theta}_2 = \bar{Y}$ is unbiased because

$$E[\bar{Y}] = E[Y_1] = \theta.$$

The pdf of Y_{min} is

$$f_{Y_{min}}(y;\theta) = n f_Y(y)(1 - F_Y(y))^{n-1} = \frac{n}{\theta} e^{-ny/\theta}$$

We have that nY_{min} and Y_1 have the same pdf since

$$f_{nY_{min}} = \frac{1}{n} f_{Y_{min}}(\frac{y}{n}) = \frac{1}{\theta} e^{-y/\theta}.$$

Thus $E[nY_{min}] = \theta$ and it is also unbiased.

2. The variance of an exponential random variable is $\frac{1}{\lambda^2}$. In this case, $\lambda = \frac{1}{\theta}$ so $\text{Var}(Y_1) = \theta^2$. The variance of the sample is $\text{Var}(\bar{Y}) = \frac{\theta^2}{n}$. The variance of nY_{min} is the same as Y_1 so it is also $\text{Var}(nY_{min}) = \theta^2$.

3. Since $\hat{\theta}_1$ has the same variance as $\hat{\theta}_1$, the relative efficiency of $\hat{\theta}_1$ to $\hat{\theta}_3$ is 1. The relative efficiency of $\hat{\theta}_2$ to $\hat{\theta}_3$ is $\frac{\theta^2}{\theta^2/n} = n$.

Problem 2: 5.4.20

The expected value of a Poisson distribution is λ, so both $\hat{\lambda_1}$ and $\hat{\lambda_1}$ are unbiased. Since $\mathrm{Var}(\hat{\lambda_2}) = \frac{\mathrm{Var}(\hat{\lambda_1})}{n}$, the relative efficiency is $\frac{\mathrm{Var}(\hat{\lambda_1})/n}{\mathrm{Var}(\hat{\lambda_1})} = \frac{1}{n}$.

Problem 3: 5.4.22

The variance of the estimator is

$$\mathrm{Var}(cW_1 + (1-c)W_2) = c^2\sigma_1^2 + (1-c)^2\sigma_2^2.$$

To minimize the variance, we can take the derivative with respect to c yielding

$$\frac{\partial}{\partial c}(c^2\sigma_1^2 + (1-c)^2\sigma_2^2) = 2\sigma_1^2 c - 2\sigma_2^2(1-c) = 0.$$

$$\begin{aligned} 2\sigma_1^2 c - 2\sigma_2^2(1-c) = 0 &\implies 2c(\sigma_1^2 + \sigma_2^2) = 2\sigma_2^2 \\ &\implies c = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}. \end{aligned}$$

Problem 4

1. $\hat{\theta}$ has no bias because

$$\begin{aligned} E[c\hat{\theta_1} + (1-c)\hat{\theta_2}] &= cE[\hat{\theta_1}] + (1-c)E[\hat{\theta_2}] \\ &= c\theta + (1-c)\theta \\ &= \theta \end{aligned}$$

   The variance of $\hat{\theta}$ is

$$\begin{aligned} \operatorname{Var}(c\hat{\theta_1} + (1-c)\hat{\theta_2}) &= c^2 \operatorname{Var}(\hat{\theta_1}) + (1-c)^2 \operatorname{Var}(\hat{\theta_2}) + 2c(1-c) \operatorname{Cov}(\hat{\theta_1}, \hat{\theta_2}) \\ &= c^2\sigma^2 + (1-c)^2\frac{\sigma^2}{2} + \frac{2c(1-c)\sigma^2}{3} \end{aligned}$$

   For an unbiased variable, the mean squared error is the same as the variance.

2. Taking the partial derivative with respect to c yields

$$\begin{aligned} \frac{\partial}{\partial c}(c^2\sigma^2 + (1-c)^2\frac{\sigma^2}{2} + \frac{2c(1-c)\sigma^2}{3}) &= 2\sigma^2 c - \sigma^2(1-c) + \frac{2\sigma^2}{3}(1-2c) \\ &= 0 \end{aligned}$$

$$\begin{aligned} 2\sigma^2 c - \sigma^2(1-c) + \frac{2\sigma^2}{3}(1-2c) = 0 &\implies (3\sigma^2 - \frac{4\sigma^2}{3})c = \sigma^2 - \frac{2\sigma^2}{3} \\ &\implies c = \frac{\sigma^2/3}{5\sigma^2/3} = \frac{1}{5}. \end{aligned}$$

Problem 5: 5.5.1

The likelihood estimator yields $\hat{\theta} = \bar{Y}$ and therefore it has variance $\text{Var}(\hat{\theta}) = \frac{\theta^2}{n}$. We have that

$$\log f_Y(y;\theta) = -\log\theta - \frac{y}{\theta}$$

$$\frac{\partial}{\partial\theta}\log f_Y(y;\theta) = -\frac{1}{\theta} + \frac{y}{\theta^2}$$

$$\frac{\partial^2}{\partial\theta^2}\log f_Y(y;\theta) = \frac{1}{\theta^2} - \frac{2y}{\theta^3}$$

Since $E[Y] = \theta$,

$$I(\theta) = -E\left[\frac{1}{\theta^2} - \frac{2Y}{\theta^3}\right] = \frac{1}{\theta^2}$$

Therefore the Cramer-Rao lower bound is $\frac{\theta^2}{n}$ and so $\hat{\theta}$ is the best estimator.

Problem 6: 5.5.2

The variance of $\hat{\lambda}$ is $\frac{\lambda}{n}$ since it is a Poisson distribution. We have that

$$\log f_X(x;\lambda) = -\lambda + x\log\lambda - \log x!$$

$$\frac{\partial}{\partial\lambda}\log f_X(x;\theta) = -1 + x\frac{1}{\lambda}$$

$$\frac{\partial^2}{\partial\lambda^2}\log f_X(x;\theta) = -x\frac{1}{\lambda^2}$$

Since $E[X] = \lambda$,

$$I(\lambda) = -E[-X\frac{1}{\lambda^2}] = \frac{1}{\lambda}$$

Therefore the Cramer-Rao lower bound is $\frac{\lambda}{n}$ and so $\hat{\lambda}$ is the best estimator.