MATH 31AH - Homework 5

Merrick Qiu

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1 Linear functionals and nonzero vectors

Proof. There exists some $\lambda \in V^*$ with $\lambda(v) \neq 0$.

Let $\mathcal{B} = \{v_1, v_2, ..., v_n\}$ be a basis for V, and let $\mathcal{B}^* = \{\lambda_1, \lambda_2, ..., \lambda_n\}$ be the dual basis of V^*. Since v is nonzero, there exists a linear combination $v = c_1v_1 + c_2v_2 + ...c_nv_n$ equal to v with some c_i nonzero. Using the definition of λ_i, we have that

$$\begin{aligned}
\lambda_i(v) &= \lambda_i(c_1v_1 + c_2v_2 + ...c_nv_n) && (1)\\
&= c_1\lambda_i(v_1) + c_2\lambda_i(v_2) + ... + c_i\lambda_i(v_i) + ... + c_n\lambda_i(v_n) && (2)\\
&= c_i\lambda_j(v_i) && (3)\\
&= c_i && (4)
\end{aligned}$$

. Since $\lambda_i(v) = c_i \neq 0$, there exists some $\lambda \in V^*$ with $\lambda(v) \neq 0$. □

2 Induced maps

Proof. If T is injective, then T^* is surjective.

The definition of T^* on some $\mu \in W^*$ and some $v \in V$ is $(T^*(\mu))(v) = \mu(T(v))$. T^* is surjective if and only if for every $\varphi \in V^*$, there exists some $\mu \in W^*$ such that $T^*(\mu) = \varphi$ or $(T^*(\mu))(v) = \mu(T(v)) = \varphi(v)$ for all $v \in V$.

Let $\varphi \in V^*$ be arbitrary. Let $\mathcal{B}$ be a basis for W. Since T is injective, there exists a left-inverse T^{-1} for it. We can uniquely define a linear transformation by where it maps the basis vectors in $\mathcal{B}$. For all $b \in \mathcal{B}$, let $\mu \in W^*$ such that if $b \in Image(T)$, then $\mu(b) = \varphi(T^{-1}(b))$. Otherwise, $\mu(b) = 0$. In other words,

$$\mu(b) = \begin{cases} \varphi(T^{-1}(b)) & b \in Image(T) \\ 0 & b \notin Image(T) \end{cases}$$

If $w \in Image(T)$, then $\mu(w) = \varphi(T^{-1}(w))$. Since $\mu(T(v)) = \varphi(T^{-1}(T(v))) = \varphi(v)$ for all $v \in V$, there exists a μ where $\mu(T(v)) = \varphi(v)$, and thus T^* is surjective. □

Proof. If T is surjective, then T^* is injective

The definition of T^* on some $\mu \in W^*$ and some $v \in V$ is $(T^*(\mu))(v) = \mu(T(v))$. T^* is injective if and only if for all $\varphi, \psi \in W^*$, $\varphi \neq \psi$ implies $T^*(\varphi) \neq T^*(\psi)$, or $\varphi(T(v)) \neq \psi(T(v))$ for some $v \in V$.

Let $\varphi, \psi \in W^*$ be arbitrary linear functionals such that $\varphi \neq \psi$. Let $\mathcal{B}$ be a basis for W. Since $\varphi \neq \psi$, there exists a $b \in \mathcal{B}$ such that $\varphi(b) \neq \psi(b)$. Since T is surjective, there exists a $v \in V$ such that $T(v) = b$.

Since $\varphi(T(v)) = \varphi(b) \neq \psi(b) = \psi(T(v))$, we know that $\varphi \neq \psi$ implies $T^*(\varphi) \neq T^*(\psi)$, so T^* is injective. $\square$

3 Infinite dimensionality and double duals

Proof. φ is still injective when V is infinite-dimensional.

The proof for the injectivity of φ when V is finite-dimensional still holds for infinite dimensions. φ is injective when $Ker(\varphi) = 0$. Let $\mathcal{B} = \{e_1, e_2, ...\}$ be a basis for V, and let $\mathcal{B}^* = \{\lambda_1, \lambda_2, ...\}$ be the dual set of V^* such that $\lambda_i(e_j) = 1$ when $i = j$, and $\lambda_i(e_j) = 0$ when $i \neq j$

Since all $v \in V$ can be writen as a finite linear combination, $v = c_1e_1 + ... + c_ne_n$, we have that for all $i \geq 1$,

$$\begin{aligned}
0 &= \varphi(v)(\lambda_i) && (5)\\
&= \lambda_i(v) && (6)\\
&= \lambda_i(c_1e_1 + ... + c_ne_n) && (7)\\
&= c_i && (8)\\
& && (9)
\end{aligned}$$

This forces $v = 0$. Therefore, $Ker(\varphi) = 0$ and φ is injective. $\square$

Proof. φ is not surjective when V is infinite-dimensional.

Let $\mathcal{B} = \{e_1, e_2, ...\}$ be a basis for V. Let $\mathcal{B}^* = \{\lambda_1, \lambda_2, ...\}$ be the dual set of $\mathcal{B}$ such that $\lambda_i(e_j) = 1$ when $i = j$ and $\lambda_i(e_j) = 0$ when $i \neq j$. Since $\mathcal{B}^*$ is a linearly independent subset of V^*, $\mathcal{B}^*$ is a subset of some basis of V^*.

φ is not surjective if there exists a $\mu \in V^{**}$ such that for all $v \in V$, $\varphi(v) \neq \mu$. In other words, for some $\mu \in V^{**}$, for all $v \in V$, for some $\lambda \in V^*$, $(\varphi(v))(\lambda) := \lambda(v) \neq \mu(\lambda)$. Let $\mu_i \in V^{**}$ be the double dual vector that sends $\lambda_i \in \mathcal{B}^*$ to 1 and all other basis vectors to 0. Let v be an arbitrary $v \in V$. We have that for $\lambda_i \in \mathcal{B}^*$,

$$\begin{aligned}
(\varphi(v))(\lambda_i) &= \lambda_i(v) && (10)\\
&= \lambda_i(c_1e_1 + ... + c_ie_i + ... + c_ne_n) && (11)\\
&= c_1\lambda_i(e_1) + ... + c_i\lambda_i(e_i) ... + c_n\lambda_i(e_n) && (12)\\
&= c_i && (13)
\end{aligned}$$

Since $(\mu_i)(\lambda_i) = 1$, we have that $c_i = 1$ for all i if $\lambda_i(v) = \mu_i(\lambda_i)$ for all v. Since v can only be a finite linear combination, this cannot be true. So, for some $\mu \in V^{**}$, for all $v \in V$, for some $\lambda \in V^*$, $(\varphi(v))(\lambda) := \lambda(v) \neq \mu(\lambda)$. Therefore, φ is not surjective.

$\square$

4 Matrices, duals, and linear maps

Proof. T_A is linear

For $A \in Mat_{n\times n}(\mathbb{F})$ and $v, v', w \in \mathbb{F}^n$,

$$(T_A(v+v'))(w) = (v+v')^T Aw \tag{14}$$
$$= (v^T + v'^T)Aw \tag{15}$$
$$= v^T Aw + v'^T Aw \tag{16}$$
$$= (T_A(v))(w) + (T_A(v'))(w) \tag{17}$$

For $A \in Mat_{n\times n}(\mathbb{F})$, $\lambda \in \mathbb{F}$, and $v, w \in \mathbb{F}^n$,

$$(T_A(\lambda v))(w) = (\lambda v)^T Aw \tag{18}$$
$$= \lambda v^T Aw \tag{19}$$
$$= (\lambda T_A(v))(w) \tag{20}$$

Since T_A preserves addition and scalar multiplication, T_A is linear. □

5 Internal Direct Sums

Proof. Being able to uniquely write $v = u + w$ is equivalent to $U \cup W$ spans V and $U \cap W = 0$.

If every vector $v \in V$ can be uniquely written as $v = u + w$ for $u \in U$ and $w \in W$, then $U \cup W$ spans V. Let $u \in U$ and $w \in W$. If some $x \in U, W$ with $x \neq 0$, then some $v \in V$ can be written as $(u + x) + w = u + (w + x) = v$. Since every vector v can only be uniquely written in one way, x cannot exist and so $U \cap W = 0$.

If the union $U \cup W$ spans V, then every vector $v \in V$ can be written as a linear combination of vectors in U and W; since U and W and subspaces, then some $v = u + w$ for $u \in U$ and $w \in W$. If $U \cap W = 0$, then the union of any basis of U and any basis of W is linearly independent since basis vectors cannot be the zero vector. This implies that there is only one way to write each v in terms of u and w.

Since uniquely writing $v = u + w$ implies $U \cup W$ spans V with $U \cap W = 0$ and $U \cup W$ spans V with $U \cap W = 0$ implies uniquely writing $v = u + w$, the two statements are equivalent. □

6 Determinants and transposition

Proof. $\det A = \det A^T$

If A is not invertible, then A^T is not invertible, and $\det A = \det A^T = 0$. If A is invertible, then it can be written as a product of elementary matrices, $A = E_1 E_2 ... E_n$. Since $(AB)^T = B^T A^T$, $A^T = (E_1 E_2 ... E_n)^T = E_n^T ... E_2^T E_1^T$. Since the transpose of an elementary matrix has the same determinant and $\det(AE) = \det(A)\det(E)$, we have that

$$\begin{aligned}
\det(A) &= \det(E_1 E_2 ... E_n) &(21)\\
&= \det(E_1)\det(E_2) ... \det(E_n) &(22)\\
&= \det(E_n) ... \det(E_2)\det(E_1) &(23)\\
&= \det(E_n^T) ... \det(E_2^T)\det(E_1^T) &(24)\\
&= \det(E_n^T ... E_2^T E_1^T) &(25)\\
&= \det(A^T) &(26)
\end{aligned}$$

Therefore, $\det A = \det A^T$. □

7 Determinants and the plane

Proof. A counterclockwise rotation has $\det(T) = 1$.

A counterclockwise rotation in $\mathbb{R}^2$ by θ takes the form $\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}$. The determinant of a 2D rotation matrix matrix is equal to

$$\begin{aligned}
\det(T) &= \cos(\theta)\cos(\theta) - (-\sin(\theta))(\sin(\theta)) &(27)\\
&= \cos^2(\theta) + \sin^2(\theta) &(28)\\
&= 1 &(29)
\end{aligned}$$

Therefore, $\det(T) = 1$. □

Proof. A reflection across a line has $\det(T) = -1$.

Reflecting a vector across a line that is θ counterclockwise of the x-axis is equivalent to rotating the vector by $-\theta$, reflecting across the x-axis, then rotating the vector by θ. The determinant for a rotation matrix is 1. The matrix for negating the y-component is $F = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$, so the determinant of reflecting across the x-axis is $1(-1) - 0(0) = -1$. If R is the matrix that rotates by θ, then the determinant of the matrix that reflection across a line is

$$\begin{aligned}
\det(T) &= \det(RTR^{-1}) &(30)\\
&= \det(R)\det(F)\det(R^{-1}) &(31)\\
&= 1(-1)(1) &(32)\\
&= -1 &(33)
\end{aligned}$$

Therefore, $\det T = -1$. □

8 Determinants and block matrices

Proof. If the bottom left $m \times n$ block is zeros, the determinant is equal to $\det(A) \cdot \det(C)$.

Base Case: When $\begin{bmatrix}A & B\\ 0 & C\end{bmatrix}$ is a 2×2 matrix, $\det(A) = AC - B \cdot 0 = AC = \det(A) \cdot \det(C)$.

Inductive Step: Assume that $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \det(A) \cdot \det(C)$ for when A is $m \times m$ and C is $n-1 \times n-1$. For any row index i, the cofactor formula says that

$$\det(M) = \sum_{j=1}^{m+n} (-1)^{i+j} m_{ij} \det(M_{ij})$$

Let $i = m+n$. In the cofactor formula, the first n terms are equal to zero since $m_{ij} = 0$. When $n < j \leq m+n$, the term is equal to $(-1)^{i+j} m_{ij} \det(M_{ij})$. By our assumption for when C is $n-1 \times n-1$, $\det(M_{ij}) = \det(A) \cdot \det(C_{ij})$. Therefore,

$$\det(M) = \sum_{j=n+1}^{m+n} (-1)^{i+j} m_{ij} \det(A) \cdot \det(C_{ij}) \quad = \det(A) \sum_{j=n+1}^{m+n} (-1)^{i+j} m_{ij} \det(C_{ij}) = \det(A) \cdot \det(C) \tag{34}$$

This completes the inductive step, so $\det(M) = \det(A) \cdot \det(C)$

$\square$