Homework due Friday, October 20, at 11:00 pm Pacific Time.

A. Determine all the limit points of the following sets in $\mathbb{R}$ and determine whether the sets are open or closed (or neither).

(1) All integers.
(2) The interval $(a, b]$.
(3) All rational numbers.
(4) All numbers of the form $(-1)^n + \frac{1}{m}$, where $m, n = 1, 2, \dots$
(5) All numbers of the form $\frac{1}{n} + \frac{1}{m}$ where $m, n = 1, 2, \dots$
(6) All numbers of the form $\frac{(-1)^n}{1+(1/n)}$, where $n = 1, 2, \dots$

B. In this problem you may use without a proof the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ with the usual metric.

(1) Show that $A = \{a + b\mathbf{i} : a, b \in \mathbb{Q}\}$ is dense in $\mathbb{C}$ with the usual metric on $\mathbb{C}$.
(2) Show that $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$ with the usual metric on $\mathbb{R}^n$.
(3) Let A be as in part (a). Show that A^n is dense in $\mathbb{C}^n$ with the usual metric on $\mathbb{C}^n$.

C. Let $A \subset \mathbb{R}$ be an additive subgroup (that is: $A \neq \emptyset$ and $A - A = A$). Prove that A is either discrete or dense in $\mathbb{R}$.

D. Rudin, Chapter 2 (page 43), problems # 2, 5, 6, 8.

Section B: Extra practice problems: Problems in section B are for your practice; please do not hand them in. However, it is extremely important that you feel comfortable with these problems as some of them may appear on the quizzes and/or exams.

(1) Rudin, Chapter 2 (page 43), problems 3, 7
(2) Let (X_1, d_1) and (X_2, d_2) be two metric spaces. Let $X = X_1 \times X_2$.
   (a) Define
   $$\rho_\infty((x_1, x_2), (y_1, y_2)) = \max\{\mathrm{d}_1(x_1, y_1), \mathrm{d}_2(x_2, y_2)\}.$$
   Prove that (X, ρ_∞) is a metric space.
   (b) Define
   $$\rho_1((x_1, x_2), (y_1, y_2)) = \mathrm{d}_1(x_1, y_1) + \mathrm{d}_2(x_2, y_2).$$
   Prove that (X, ρ_1) is a metric space.

(c) Define

$$\rho_2((x_1,x_2),(y_1,y_2)) = \Big(\mathrm{d}_1(x_1,y_1)^2 + \mathrm{d}_2(x_2,y_2)^2\Big)^{1/2}.$$

Prove that (X,ρ_2) is a metric space.

(d) Let (X_i,d_i) be $\mathbb{R}$ with respect to the standard metric for $i=1,2$. Describe $N_1(0)$ in $X=\mathbb{R}^2$ with respect the metrics ρ_∞, ρ_1, and ρ_2.

(3) Let

$$\ell^\infty = \Big\{(a_1,a_2,\dots) : a_n\in\mathbb{R} \ \text{ and } \ \sup\{|a_n| : n\in\mathbb{N}\}<\infty\Big\}.$$

We use componentwise addition and scalar multiplication on ℓ^∞. For any $(a_n)\in\ell^\infty$, define

$$\big\|(a_n)\big\|_\infty = \sup\{|a_n| : n\in\mathbb{N}\}.$$

Define

$$\mathrm{d}\big((a_n),(b_n)\big) = \big\|(a_n-b_n)\big\|_\infty$$

for any two sequences (a_n) and (b_n).

(a) Show that $\mathrm{d}\big((a_n),(b_n)\big)$ is a finite number for any two (a_n) and (b_n) in ℓ^∞.

(b) Show that $\big(\ell^\infty,\mathrm{d}\big)$ is a metric space.

(c) (Bonus Problem) Is $\big(\ell^\infty,\mathrm{d}\big)$ a separable space? (A metric space X is called *separable* if there is a countable and dense subset $A\subset X$, e.g., $\mathbb{R}$ with the usual metric is separable.)