Math 170B: Homework 2

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Fixed Point Method

Since $-x\sin^2(1/x)$ is continuous, we just need to prove that g is continuous at 0. Since $-1 \leq \sin^2(1/x) \leq 1$, by the squeeze theorem

$$\lim_{x\to 0} -x\sin^2(1/x) = 0.$$

Thus g is continuous everywhere including 0. If there did exist a point $x_0 \in (0,1]$ then it must be that $\sin^2(1/x_0) = -1$. However this is impossible since a square cannot be negative. Thus $x = 0$ is the only fixed point.

Interpolation 1

$$\begin{aligned}
p_3(x) &= \sum_{k=0}^{3} L_k(x) y_k \\
&= 10\frac{(x-7)(x-1)(x-2)}{-8} + 146\frac{(x-3)(x-1)(x-2)}{120} + 2\frac{(x-3)(x-7)(x-2)}{-12} + 1\frac{(x-3)(x-7)(x-1)}{5} \\
&= -\frac{5(x-7)(x-1)(x-2)}{4} + \frac{73(x-3)(x-1)(x-2)}{60} - \frac{(x-3)(x-7)(x-2)}{6} + \frac{(x-3)(x-7)(x-1)}{5}
\end{aligned}$$

Interpolation 2

When $x = x_0$ we have that $x_0 - x_0 = 0$ so

$$g(x_0) + \frac{x_0 - x_0}{x_n - x_0}[g(x) - h(x)] = g(x_0) = f(x_0)$$

When $x = x_i$ for $i = 1, \dots, n-1$ we have that $[g(x_i) - h(x_i)] = 0$ so

$$g(x_i) + \frac{x_0 - x_i}{x_n - x_0}[g(x_i) - h(x_i)] = g(x_i) = f(x_i)$$

When $x = x_n$

$$g(x_n) + \frac{x_0 - x_n}{x_n - x_0}[g(x_n) - h(x_n)] = g(x_i) - [g(x_n) - h(x_n)] = h(x_n) = f(x_n)$$

Thus the function interpolates f at $x_0, \dots, x_n$.

Interpolation 3

Since $f^{(n)}(\eta_x) = \sinh(\eta_x)$ or $f^{(n)}(\eta_x) = \cosh(\eta_x)$ and both functions are less than 2 in the interval $[-1, 1]$, we have that $f^{(n)}(\eta_x)x \leq 2$. The error term is

$$\begin{aligned}
E(x) &= \frac{f^{(n)}(\eta_x)}{n!}\prod_{j=0}^{n-1}(x - x_j) \\
&\leq \frac{f^{(n)}(\eta_x)}{n!}2^{n-1}(x - 0) \\
&\leq \frac{2^n}{n!}
\end{aligned}$$

Interpolation 4

The Lagrange form is

$$\begin{aligned}
p_2(x) &= \sum_{k=0}^{3} L_k(x) y_k \\
&= 0L_0(x) + 1\frac{(x+2)(x-1)}{-2} - 1\frac{(x+2)x}{3} \\
&= -\frac{(x+2)(x-1)}{2} - \frac{(x+2)x}{3} \\
&= -\frac{5}{6}x^2 - \frac{7}{6}x + 1
\end{aligned}$$

The Newton form is

$$p_0 = 0$$

$$C_1 = \frac{1}{x_1+2} = \frac{1}{2}$$

$$p_1 = 0 + C_1(x+2) = \frac{1}{2}(x+2)$$

$$C_2 = \frac{-1-\frac{1}{2}(x_2+2)}{(x_2+2)x_2} = -\frac{5}{6}$$

$$p_2 = \frac{1}{2}(x+2) + C_2(x+2)x = \frac{1}{2}(x+2) - \frac{5}{6}(x+2)x = -\frac{5}{6}x^2 - \frac{7}{6}x + 1$$

Interpolation 5

The left hand side is the Lagrange interpolant and the right hand side is the Newton interpolant written using the divided differences. By the uniqueness of the interpolating polynomial, both sides must be equal.

Interpolation 6

Since both polynomials are equal, they must have equal coefficients. The coefficient for x^n in the Netwon interpolant is $f[x_0, \dots, x_n]$. The coefficient for x^n in the Lagrange interpolant is $\sum_{i=0}^{n} f(x_i) \prod_{\substack{j=0 \\ j\neq i}}^{n} (x_i - x_j)^{-1}$. Thus the equality holds.

Interpolation 7

x_i	$f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
4	63	$\frac{11-63}{2-4} = 26$	$\frac{2-26}{0-4} = 6$	$\frac{5-6}{3-4} = 1$
2	11	$\frac{7-11}{0-2} = 2$	$\frac{7-2}{3-2} = 5$	
0	7	$\frac{28-7}{3-0} = 7$		
3	28			

The final polynomial is

$$63 + 26(x-4) + 6(x-4)(x-2) + (x-4)(x-2)x$$

Bisection Method

Applying this function to $f(x) = x - 2e^{-x}$ on $[0, 1]$ yields $x = 0.8526$

```
function Xs = BisectionRoot(Fun,a,b)
    if Fun(a)*Fun(b) >= 0
        error("Does not bracket root")
    end
    Xs = (b+a)/2;
    while (b-a)/2 >= 0.000001
        if Fun(a)*Fun(Xs) < 0
            b = Xs;
        elseif Fun(Xs)*Fun(b) < 0
            a = Xs;
        else
            return;
        end
        Xs = (b+a)/2;
    end
end
```