Math 100A: Homework 3

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Problem 1

When $n = 1$, any element $g \in G$ distinct from the identity has order p. This is because $|\langle g \rangle|$ must divide p by Lagrange's theorem.

Assume that there exists an element with order p for all groups with order $p^1, p^2, \cdots, p^{k-1}$. Let G be a group with order p^k. Choose an element $g \in G$ distinct from the identity. The subgroup $\langle g \rangle$ must have order that divides p^k by Lagrange's theorem, meaning that it has order $p^1, p^2, \cdots, p^{k-1}, p^k$.

If the order is p^k, then g generates G and we can choose $g^{(p^{k-1})}$ as our element of order p.

Otherwise, there is an element $h \in \langle g \rangle$ with order p by our inductive hypothesis since $\langle g \rangle$ has order $p^1, p^2, \cdots, p^{k-1}, p^k$. Since $\langle g \rangle$ is a subgroup of G, the same element h is in G with order p as well.

Problem 2

Since G contains an element of order 10, 10 must divide $|G|$. Since G contains an element of order 6, 6 must divide $|G|$. Thus $|G|$ must be a multiple of 30 since 30 is the LCM of 10 and 6.

Problem 3

By corollary 11, $|G| = |\ker(\varphi)| \cdot |\operatorname{im}(\varphi)|$, $|\ker(\varphi)|$ divides $|G|$, and $|\operatorname{im}(\varphi)|$ divides $|G|$ and $|G'|$. Since ϕ is non-trivial we know that $|\operatorname{im}(\varphi)| \neq 1$. Thus $|\operatorname{im}(\varphi)| = 3$ since it must divide both 15 and 18. This then implies that $|\ker(\varphi)| = 6$ since $|G| = |\ker(\varphi)| \cdot |\operatorname{im}(\varphi)|$.

Problem 4

Since N is a subgroup of G of index 2, G is the disjoint union of two cosets, N and $G \setminus N$. Let $g \in G$. We want to show that $gN = Ng$.

In the case when $g \in N$, then these two sets are equal since $gN = N = Ng$.

Let $g \notin N$. This implies that $gN \neq N$ and so $gN = G \setminus N$. Similarly $Ng \neq N$ so $Ng = G \setminus N$. Therefore $gN = G \setminus N = Ng$.