Math 140A: Homework 7

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A

1. $\{x_n\}$ is bounded so s_N and r_n exists for all $N \in \mathbb{N}$. For all $N \in \mathbb{N}$, s_N cannot exceed s_1 since the sup of a subsequence cannot be larger than the sup of the whole sequence(since the existence of the larger sup contradicts the fact that the sup is supposed to be the best upper bound), and s_N is bounded below by r_1 since the sup of a subsequence cannot be smaller than the inf of the whole sequence. Likewise, r_N is bounded below by r_1 and bounded above by s_1 so both $\{s_N\}$ and $\{r_n\}$ are bounded.

2. The sup of a sequence cannot be smaller than the sup of a subsequence and the inf of a sequence cannot be larger than the inf of a subsequence. This means that $s_K \geq s_{K+1}$ and $r_K \leq r_{K+1}$ for all $K \in \mathbb{N}$ so $\{s_N\}$ is non-increasing and $\{r_n\}$ is non-decreasing.

3. It cannot be that $s < \limsup x_n$. There must exist a subsequence $\{x_{n_k}\}$ with $s + c < \lim\{x_{n_k}\}$ for all $c > 0$. However, this subsequence must have infinitely many elements that are greater than $s + c$, but this contradicts the fact that there exists somes_N with $s < s_N < s+c$ and $x_n < s_N < s+c$ for all $n > N$.

   It also cannot be that $s > \limsup x_n$. Suppose that $s - \limsup x_n = c$. Choose a subsequence x_{n_k} so that $|s_k - x_{n_k}| < \frac{c}{4}$ for all k. Then choose K such that for all $k > K$, $|s - s_k| < \frac{c}{4}$. Thus $|s - x_{n_k}| = |s - s_k + s_k - x_{n_k}| \leq |s - s_k| + |s_k - x_{n_k}| < \frac{c}{2}$ for all $k > K$, so this subsequence is bounded by $\limsup x_n + \frac{c}{2}$ from below. Thus there must exist a subsequence of this subsequence that converges to something $\geq \limsup x_n + \frac{c}{2}$ which contradicts the definition of $\lim\sup$.

B

Let $\{G_{\alpha,\epsilon}\}$ be an open cover of K of open neighborhoods of radius $< \epsilon$ and let $\{G_{i,\epsilon}\}$ be a finite subcover of K chosen from $\{G_{\alpha,\epsilon}\}$. Create a sequence by appending all the centers of the neighborhoods of $\{G_{i,\epsilon}\}$ into the sequence for all $\epsilon = \frac{1}{n}$ where $n = 1, 2, 3, \ldots$. Thus for every point in $p \in K$, we are able to choose a subsequence that contains one point from each of the open covers whose neighborhood contains p and get a subsequence that converges to p.

C

$a_n \to a$ implies that for all $\epsilon > 0$, there exists some N such that for all $n > N$, $|a - a_n| < \epsilon$. We need to prove that for all $\epsilon' > 0$, there exists some N' such that for all $n > N'$, $|\sqrt{a} - \sqrt{a_n}| < \epsilon'$. Since square roots are always positive,

$$|\sqrt{a} - \sqrt{a_n}| = |\frac{a - a_n}{\sqrt{a} + \sqrt{a_n}}| = \frac{1}{\sqrt{a} + \sqrt{a_n}}|a - a_n| \leq \frac{1}{\sqrt{a}}|a - a_n| < \frac{1}{\sqrt{a}}\epsilon$$

Thus we can choose N' such that for all $n > N'$, $|a - a_n| < \sqrt{a}\epsilon'$, and this implies that $|\sqrt{a} - \sqrt{a_n}| < \epsilon'$, so we have shown that $\sqrt{a_n} \to \sqrt{a}$.

Rudin Question 5

If $\limsup a_n + b_n = -\infty$, then the statement trivially true. If $\limsup a_n + b_n = \infty$, then $\limsup a_n = \infty$ or $\limsup b_n = \infty$ since if they were both not infinity, then both sequences would be bounded above and so $\limsup a_n + b_n = \infty$ couldn't be true.

Thus consider the case where all the terms are finite. Let $\{a_{n_k} + b_{n_k}\}$ be a subsequence such that converges to $\limsup a_n + b_n$. Choose $\{a_{n_{k_m}}\}$ such that it converges to $\limsup a_{n_k}$ which means that $\{a_{n_{k_m}} + b_{n_{k_m}}\}$ converges to $\limsup a_n + b_n$. Since $\limsup a_n$ is greater than or equal to what $\{a_{n_{k_m}}\}$ converges to and likewise for $\limsup a_n$ for $\{b_{n_{k_m}}\}$, the inequality holds.

Rudin Question 20

Since $p_{n_i} \to p$, then for some K, for all $\epsilon > 0$, and for $i > K$, $d(p_{n_i}, p) < \epsilon$. Since $\{p_n\}$ is cauchy, there exists N such that for all $m, n > N$, $d(p_n, p_m) < \epsilon$. Choose $M \geq N$ large enough so that $k > K$ and $n_k > N$ for all $k > M$. Then $d(p_n, p) \leq d(p_n, p_{n_{K+1}}) + d(p_{n_{K+1}}, p) < 2\epsilon$, so the original sequence also converges to p.

Rudin Question 21

Suppose the intersection contained more than one point. Pick two points p_1 and p_2 from the interesection. Since $\lim \operatorname{diam} E_n = 0$, there exists E_n with $\operatorname{diam} E_n < d(p_1, p_2)$. However both these points cannot exist inside of E_n anymore by the definition of diameter, which is a contradiction.