MAT140C SPRING 2024: PROBLEM SET 4

Directions: You can collaborate, but must write up the solutions independently and in a good handwriting. **Consulting solutions to problem sets of previous semesters or internet solutions is not allowed.**

Problem 1. Consider the space $\mathcal{E}$ of elementary subsets of $\mathbb{R}$, that is finite unions of intervals. Prove the following facts.

(1) $\mathcal{E}$ is an algebra, but not σ-algebra. (hint: reason on connected components)

(2) Given any set $A \in \mathcal{E}$ there exists a finite collection of pairwise disjoint intervals I_j such that $A = \bigcup_{j=1}^{N} I_j$.

(3) Let $m \colon \mathcal{E} \to [0, \infty)$ be defined using the decomposition of (2) as in class, then show that m is well defined. (hint: use partitions)

(4) Prove that m is additive.

Problem 2. Let $d(A, B)$ be as in Definition 11.9 of Rudin. Prove properties (27) to (29).

Problem 3. Show that a σ-algebra of subsets of a set X is either finite, or has uncountably many elements.

Problem 4. Let E be the family of elementary subsets of $\mathbb{R}$. Denote by $m \colon \mathcal{E} \to [0, \infty)$ the Lebesgue set function and by $\mu^*(A)$ the outer measure of a set $A \subset \mathbb{R}$. For a scalar $t \in \mathbb{R}$ let the set $A + t := \{x + t : x \in A\}$.

(1) Show that if $A \in \mathcal{E}$, then $m(A + t) = m(A)$ for all $t \in \mathbb{R}$.

(2) Show that for an arbitrary $A \subset \mathbb{R}$, $\mu^*(A + t) = \mu^*(A)$, for all $t \in \mathbb{R}$.