Math 31CH HW4

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Exercise 5.3.1

Part a. Let $(r(t), \theta(t))$ be a parametrization of a curve in polar coordinates. Show that the length of the piece of curve between $t = a$ and $t = b$ is given by the integral $\int_a^b \sqrt{(r'(t))^2 + (r(t))^2(\theta'(t))^2} dt$.

Solution. The derivative of the coordinates are

$$\gamma'(t) = \begin{bmatrix} r'(t)\cos(\theta(t)) - r(t)\theta'(t)\sin(\theta(t)) \\ r'(t)\sin(\theta(t)) + r(t)\theta'(t)\cos(\theta(t)) \end{bmatrix}$$

The squared sum of the x and y coordinates are

$$\begin{aligned}
\gamma'(t)_x^2 + \gamma'(t)_y^2 &= r'(t)^2\cos^2(\theta(t)) + r(t)^2\theta'(t)^2\sin^2(\theta(t)) + r'(t)^2\sin^2(\theta(t)) + r'(t)^2\theta'(t)^2\cos^2(\theta(t)) \\
&= r'(t)^2(\cos^2(\theta(t)) + \sin^2(\theta(t))) + r(t)^2\theta'(t)^2(\cos^2(\theta(t)) + \sin^2(\theta(t))) \\
&= r'(t)^2 + r(t)^2\theta'(t)^2
\end{aligned}$$

Therefore the path length is

$$\int_a^b |\gamma'(t)|\, dt = \int_a^b \sqrt{\gamma'(t)_x^2 + \gamma'(t)_y^2}\, dt = \int_a^b \sqrt{(r'(t))^2 + (r(t))^2(\theta'(t))^2}\, dt$$

Part b. Consider the spiral in polar coordinates when $r(t) = e^{-\alpha t}$ and $\theta(t) = t$, for $\alpha > 0$. What is its length between $t = 0$ and $t = b$? What is the limit of this length as $\alpha \to 0$.

Solution. The path length is

$$\begin{aligned}
\int_0^b \sqrt{(r'(t))^2 + (r(t))^2(\theta'(t))^2}\, dt &= \int_0^b \sqrt{\alpha^2 e^{-2\alpha t} + e^{-2\alpha t}}\, dt \\
&= \int_0^b \sqrt{\alpha^2 + 1} e^{-\alpha t}\, dt \\
&= -\frac{\sqrt{\alpha^2 + 1}}{\alpha}(e^{-\alpha b} - 1)
\end{aligned}$$

The limit of of the path length as $\alpha \to 0$ is

$$\begin{aligned}
\lim_{\alpha \to 0} -\frac{\sqrt{\alpha^2 + 1}}{\alpha}(e^{-\alpha b} - 1) &= \lim_{\alpha \to 0} -\frac{\sqrt{\alpha^2 + 1}}{\alpha}((1 - b\alpha + \frac{b^2\alpha^2}{2} \ldots) - 1) \\
&= \lim_{\alpha \to 0} -\sqrt{\alpha^2 + 1}(b(-1 + \frac{b\alpha}{2} \ldots)) \\
&= b
\end{aligned}$$

Part c. Show that the spiral turns infinitely many times around the origin as $t \to \infty$. Does the length tend to ∞ as $b \to \infty$?

Solution. θ grows without bound as t increases, so it turns infinitely many times.

$$\lim_{t \to \infty} \theta(t) = \lim_{t \to \infty} t = \infty$$

The length does not tend towards ∞ since

$$\lim_{b \to \infty} -\frac{\sqrt{\alpha^2 + 1}}{\alpha}(e^{-\alpha b} - 1) = \frac{\sqrt{\alpha^2 + 1}}{\alpha}$$

Exercise 5.3.3

Part a. Suppose that $t \mapsto (r(t), \theta(t), \phi(t))$ is a parametrization of a curve in $\mathbb{R}^3$, written in spherical coordinates. Find the formula analogous to the integral in Exercise 5.3.1, part a, for the length of the arc between $t = a$ and $t = b$.

Solution. The coordinates are given by

$$\gamma(t) = \begin{bmatrix} r(t)\cos(\theta(t))\cos(\varphi(t)) \\ r(t)\sin(\theta(t))\cos(\varphi(t)) \\ r(t)\sin(\varphi(t)) \end{bmatrix}$$

The derivative is

$$\gamma(t) = \begin{bmatrix} r'(t)\cos(\theta(t))\cos(\varphi(t)) - r(t)\theta'(t)\sin(\theta(t))\cos(\varphi(t)) - r(t)\varphi'(t)\cos(\theta(t))\sin(\varphi(t)) \\ r'(t)\sin(\theta(t))\cos(\varphi(t)) + r(t)\theta'(t)\cos(\theta(t))\cos(\varphi(t)) - r(t)\varphi'(t)\sin(\theta(t))\sin(\varphi(t)) \\ r'(t)\sin(\varphi(t)) + r(t)\varphi'(t)\cos(\varphi(t)) \end{bmatrix}$$

After a lot of ugly algebra to calculate the norm,

$$\int_a^b |\gamma'(t)|\, dt = \int_a^b \sqrt{\gamma'(t)_x^2 + \gamma'(t)_y^2 + \gamma'(t)_z^2}\, dt = \int_a^b \sqrt{r'(t)^2 + r^2\phi'(t)^2 + r^2\cos^2(\varphi(t))\theta'(t)^2}\, dt$$

Part b. What is the length of the curve parametrized by $r(t) = cos\ t,\ \theta(t) = tan\ t,\ \phi(t) = t$, between $t = 0$ and $t = a$, where $0 < a < \pi/2$?

Solution.

$$\begin{aligned}\int_0^a \sqrt{r'(t)^2 + r^2\phi'(t)^2 + r^2\cos^2(\varphi(t))\theta'(t)^2}\, dt &= \int_0^a \sqrt{\sin^2(t) + \cos^2(t) + \cos^2(t)\cos^2(t)\sec^4(t)}\, dt \\ &= \int_0^a \sqrt{2}\, dt \\ &= \sqrt{2}a\end{aligned}$$

Exercise 5.3.5

Part a. Set up (but do not compute) the integral giving the surface area of the part of the surface of equation $z = \frac{x^2}{4} + \frac{y^2}{9}$, where $z \le a^2$.

Solution. The manifold can be parameterized by

$$\gamma(r,\theta) = \begin{bmatrix} 2r\cos\theta \\ 3r\sin\theta \\ r^2 \end{bmatrix}$$

The Jacobian is

$$D\gamma = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} \end{bmatrix} = \begin{bmatrix} 2\cos\theta & -2r\sin\theta \\ 3\sin\theta & 3r\cos\theta \\ 2r & 0 \end{bmatrix}$$

Multiplying by the transpose yields

$$D\gamma^T D\gamma = \begin{bmatrix} 2\cos\theta & 3\sin\theta & 2r \\ -2r\sin\theta & 3r\cos\theta & 0 \end{bmatrix} \begin{bmatrix} 2\cos\theta & -2r\sin\theta \\ 3\sin\theta & 3r\cos\theta \\ 2r & 0 \end{bmatrix} = \begin{bmatrix} 4+5\sin^2\theta+4r^2 & 5r\sin\theta\cos\theta \\ 5r\sin\theta\cos\theta & 4r^2+5r^2\cos^2\theta \end{bmatrix}$$

The volume of the transformation is

$$\sqrt{\det[D\gamma^T D\gamma]} = \sqrt{(4+5\sin^2\theta+4r^2)(4r^2+5r^2\cos^2\theta) - (5r\sin\theta\cos\theta)^2} = 2r\sqrt{9+r^2(4+5\cos^2\theta)}$$

The integral is therefore,

$$\int_0^{2\pi}\int_0^a 2r\sqrt{9+r^2(4+5\cos^2\theta)}\,dr\,d\theta$$

Part b. What is the volume of the region $\frac{x^2}{4} + \frac{y^2}{9} \le z \le a^2$?

Solution. We can use cylindrical coordinates to calculate the volume.

$$6\int_0^{2\pi}\int_0^a\int_{r^2}^{a^2} r\,dz\,dr\,d\theta = 6\int_0^{2\pi}\int_0^a a^2r - r^3\,dr\,d\theta = 6\int_0^{2\pi} \frac{a^4}{2} - \frac{a^4}{4}\,d\theta = 3\pi a^4$$

I multply by 6 since the

$$\det\begin{bmatrix} 2\cos\theta & -2r\sin\theta \\ 2\sin\theta & 3r\cos\theta \end{bmatrix} = 6$$

Exercise 5.3.6

Let S be the part of the paraboloid of revolution $z = x^2 + y^2$ where $z \leq 9$. Compute the integral $\int_S (x^2 + y^2 + 3z^2)|d^2x|$.

Solution. Lets use the transformation

$$\gamma(r, \theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \\ r^2 \end{bmatrix}$$

The Jacobian of the transformation is

$$D\gamma = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \\ 2r & 0 \end{bmatrix}$$

The determinant of $D\gamma^T D\gamma$ is

$$\begin{aligned} \det D\gamma^T D\gamma &= \det \begin{bmatrix} \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \\ 2r & 0 \end{bmatrix} \\ &= \det \begin{bmatrix} 1 + 4r^2 & 0 \\ 0 & r^2 \end{bmatrix} \\ &= r^2 + 4r^4 \end{aligned}$$

Using the substitution $t = \sqrt{1 + 4r^2}$

$$\begin{aligned} \int_S (x^2 + y^2 + 3z^2)|d^2x| &= \int_0^{2\pi} \int_0^3 (r^3 + 3r^5)\sqrt{1 + 4r^2}\, dr\, d\theta \\ &= 2\pi \int_0^3 r^3\sqrt{1 + 4r^2}\, dr + 2\pi \int_0^3 3r^5\sqrt{1 + 4r^2}\, dr \\ &= \frac{\pi}{8} \int_1^{\sqrt{37}} t^4 - t^2\, dt + \frac{3\pi}{32} \int_1^{\sqrt{37}} (t^2 - 1)(t^4 - t^2)\, dt \\ &= \frac{7 + 277574\sqrt{37}}{420}\pi \\ &\approx 4020\pi \end{aligned}$$

Exercise 5.3.8

What is the surface area of the part of the paraboloid of revolution $z = x^2 + y^2$ where $z \leq 1$?

Solution. We can repurpose the integral from the previous problem

$$\begin{aligned}
\int_S |d^2x| &= \int_0^{2\pi} \int_0^1 r\sqrt{1+4r^2}\, dr\, d\theta \\
&= 2\pi \left[\frac{1}{12}(4r^2+1)^{\frac{3}{2}} \right]_0^1 \\
&= 2\pi \frac{5\sqrt{5}-1}{12} \\
&= \frac{5\sqrt{5}-1}{6}\pi
\end{aligned}$$

Exercise 5.3.13(a)

Let S^2 be the unit sphere and let S_1 be the part of the cylinder of equation $x^2+y^2=1$ with $-1 \le z \le 1$. Show that the horizontal radial projection $S_1 \to S^2$ preserves area.

Solution. The unit sphere is given by the equation $x^2+y^2+z^2=1$. The radial projection using cylindrical coordinates is

$$\gamma(z,\theta) = \begin{bmatrix} \sqrt{1-z^2}\cos\theta \\ \sqrt{1-z^2}\sin\theta \\ z \end{bmatrix}$$

The Jacobian is

$$D\gamma = \begin{bmatrix} -\frac{z}{\sqrt{1-z^2}}\cos\theta & -\sqrt{1-z^2}\sin\theta \\ -\frac{z}{\sqrt{1-z^2}}\sin\theta & \sqrt{1-z^2}\cos\theta \\ 1 & 0 \end{bmatrix}$$

The determinant of the Jacobian times its transpose is

$$\begin{aligned} \det D\gamma^T D\gamma &= \det \begin{bmatrix} -\frac{z}{\sqrt{1-z^2}}\cos\theta & -\frac{z}{\sqrt{1-z^2}}\sin\theta & 1 \\ -\sqrt{1-z^2}\sin\theta & \sqrt{1-z^2}\cos\theta & 0 \end{bmatrix} \begin{bmatrix} -\frac{z}{\sqrt{1-z^2}}\cos\theta & -\sqrt{1-z^2}\sin\theta \\ -\frac{z}{\sqrt{1-z^2}}\sin\theta & \sqrt{1-z^2}\cos\theta \\ 1 & 0 \end{bmatrix} \\ &= \det \begin{bmatrix} \frac{z^2}{1-z^2}+1 & 0 \\ 0 & 1-z^2 \end{bmatrix} \\ &= 1 \end{aligned}$$

Since $\sqrt{\det D\gamma^T D\gamma} = 1$, the horizontal radial projection preserves area.

Exercise 5.3.15

Part a. Show that when ϕ, ψ, θ satisfy

$$-\pi/2 \le \phi \le \pi/2, \quad -\pi/2 \le \psi \le \pi/2, \quad 0 \le \theta < 2\pi$$

the map $\gamma(\theta, \phi, \psi) = (\cos\psi\cos\phi\cos\theta, \cos\psi\cos\phi\sin\theta, \cos\psi\sin\phi, \sin\psi)$ parametrizes the unit sphere S^3 in $\mathbb{R}^4$.

Solution. For the unit sphere, $x^2 + y^2 + z^2 + w^2 = 1$. Plugging the values in to show that the image of γ is on the sphere,

$$\begin{aligned} x^2 + y^2 + z^2 + w^2 &= (\cos^2\psi\cos^2\phi\cos^2\theta + \cos^2\psi\cos^2\phi\sin^2\theta) + \cos^2\psi\sin^2\phi + \sin^2\psi \\ &= (\cos^2\psi\cos^2\phi + \cos^2\psi\sin^2\phi) + \sin^2\psi \\ &= (\cos^2\psi + \sin^2\psi) \\ &= 1 \end{aligned}$$

$w \in [-1, 1]$ since $-\pi/2 \le \psi \le \pi/2$, so all w coordinates of the sphere are covered. For a given w, the equation traces out the entire S^2 sphere of radius $\cos\psi$.

$$x^2 + y^2 + z^2 + \sin^2\psi = 1 \implies x^2 + y^2 + z^2 = \cos^2\psi$$

Thus all points on S^3 are mapped onto.

Part b. Use this parametrization to compute $vol_3(S^3)$.

Solution. The Jacobian of the transformation is

$$D\gamma = \begin{bmatrix} -\cos\psi\cos\phi\sin\theta & -\cos\psi\sin\phi\cos\theta & -\sin\psi\cos\phi\cos\theta \\ \cos\psi\cos\phi\cos\theta & -\cos\psi\sin\phi\sin\theta & -\sin\psi\cos\phi\sin\theta \\ 0 & \cos\psi\cos\phi & -\sin\psi\sin\phi \\ 0 & 0 & \cos\psi \end{bmatrix}$$

The determinant of the Jacobian times its transform is

$$\det D\gamma^T D\gamma = \det \begin{bmatrix} \cos^2\psi\cos^2\phi & 0 & 0 \\ 0 & \cos^2\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \cos^4\psi\cos^2\phi$$

The volume is thus

$$\begin{aligned} \int_0^{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2\psi\cos\phi \, d\phi \, d\psi \, d\theta &= 2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos^2\psi \, d\psi \\ &= \pi\left[(\sin(2x) + 2x)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \\ &= 2\pi^2 \end{aligned}$$

Exercise 6.1.3

Compute the following numbers:

$$a.\ dx_1 \wedge dx_4 \left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -1 \\ 2 \end{bmatrix} \right) \qquad b.\ (dx_1 \wedge dx_2 + 2dx_2 \wedge dx_3) \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right)$$

$$c.\ dx_4 \wedge dx_2 \left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -1 \\ 2 \end{bmatrix} \right) \qquad d.\ dx_1 \wedge dx_2 \wedge dx_2 \left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right)$$

Solution.

1. $$\det \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = 0$$

2. $$\det \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} + 2 \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 3 + 2(-1) = 1$$

3. $$\det \begin{bmatrix} 2 & 2 \\ 0 & -3 \end{bmatrix} = -6$$

4. $$\det \begin{bmatrix} 1 & -2 & 2 \\ 3 & 1 & 2 \\ 3 & 1 & 2 \end{bmatrix} = 1(0) + 2(0) + 2(0) = 0$$

Exercise 6.1.6

Which of the following expressions make sense? Evaluate those that do.

$$a.\ dx_1 \wedge dx_2 \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \right) \qquad b.\ dx_1 \wedge dx_3 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right)$$

$$c.\ dx_1 \wedge dx_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right) \qquad d.\ dx_1 \wedge dx_2 \wedge dx_4 \left(\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right)$$

$$e.\ dx_1 \wedge dx_2 \wedge dx_3 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right) \qquad f.\ dx_1 \wedge dx_2 \wedge dx_3 \left(\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 2 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right)$$

Solution.

1. $$\det \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = 3$$

2. Does not make sense

3. Does not make sense

4. Does not make sense

5. Does not make sense

6. $$\det \begin{bmatrix} 1 & 3 & 2 \\ 0 & 7 & 0 \\ 3 & 2 & 1 \end{bmatrix} = 1(7) - 0 + 3(-14) = -35$$

Exercise 6.1.8

Verify that the wedge product of two 1-forms does not commute, and that the wedge product of a 2-form and a 1-form does commute.

Solution. Two 1-forms do not commute

$$\begin{aligned}(\phi \wedge \omega)(v_1, v_2) &= \phi(v_1)\omega(v_2) - \phi(v_2)\omega(v_1) \\ &= -(\omega(v_1)\phi(v_2) - \omega(v_2)\phi(v_1)) \\ &= -(\omega \wedge \phi)(v_1, v_2)\end{aligned}$$

However, the wedge product of a 2-from and a 1-form does commute.

$$\begin{aligned}(\phi \wedge \omega)(v_1, v_2, v_3) &= \phi(v_1, v_2)\omega(v_3) - \phi(v_1, v_3)\omega(v_2) + \phi(v_2, v_3)\omega(v_1) \\ &= \omega(v_1)\phi(v_2, v_3) - \omega(v_2)\phi(v_1, v_3) + \omega(v_3)\phi(v_1, v_2) \\ &= (\omega \wedge \phi)(v_1, v_2, v_3)\end{aligned}$$

Exercise 6.1.10

Let $\mathbf{a}, \mathbf{v}, \mathbf{w}$ be vectors in $\mathbb{R}^3$ and let ϕ be the 2-form on $\mathbb{R}^3$ given by $\phi(\mathbf{v}, \mathbf{w}) = det(\mathbf{a}, \mathbf{v}, \mathbf{w})$. Write ϕ as a linear combination of elementary 2-forms on $\mathbb{R}^3$, in terms of the coordinates of $\mathbf{a}$.

Solution. Using the cofactor formula on the first column yields

$$\begin{aligned}\phi(v, w) &= \det(a, v, w)\\ &= a_1(dx_2 \wedge dx_3) - a_2(dx_1 \wedge dx_3) + a_3(dx_1 \wedge dx_2)\end{aligned}$$

Exercise 6.1.11

Let ϕ and ψ be 2-forms. Use definition 6.1.12 to write the wedge product $\phi \wedge \psi(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$ as a combination of values of ϕ and ψ evaluated on appropriate vectors (as in equations 6.1.28 and 6.1.32).

Solution. Simply sum up with the permutations, ensure the correct sign, and make sure that it follows the "shuffle" rule.

$$\begin{aligned}
(\phi \wedge \psi)(v_1, v_2, v_3, v_4) = {} & \phi(v_1, v_2)\psi(v_3, v_4) \\
& - \phi(v_1, v_3)\psi(v_2, v_4) \\
& + \phi(v_1, v_4)\psi(v_2, v_3) \\
& + \phi(v_2, v_3)\psi(v_1, v_4) \\
& - \phi(v_2, v_4)\psi(v_1, v_3) \\
& + \phi(v_3, v_4)\psi(v_1, v_2)
\end{aligned}$$

Exercise 6.1.6

Which of the following expressions make sense? Evaluate those that do.

$$a.\ (x_1 - x_4)dx_3 \wedge dx_2 \left(P_0 \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right) \right) \quad b.\ e^x dy \left(P_{\begin{bmatrix} 2 \\ 1 \end{bmatrix}} \left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}, \right) \right)$$

$$c.\ x_1^2 dx_3 \wedge dx_2 \wedge dx_1 \left(P_{\begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}} \left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 0 \end{bmatrix} \right) \right)$$

Solution.

1. Does not make sense
2. $e^2 \det[2] = 2e^2$
3. $$4 \det \begin{bmatrix} 3 & -1 & -1 \\ 2 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} = 4(7) = 28$$