Math 140A: Homework 1

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A.

$$\sup E = \frac{13}{11}$$

$\frac{13}{11}$ is an upper bound since all elements in E are less than or equal to it. For any $c < \frac{13}{11}$, choose $b = \frac{13}{11}$ from E. Since $c < b \leq \frac{13}{11}$, $\frac{13}{11}$ is the supremum.

$$\inf E = \frac{5}{11}$$

$\frac{5}{11}$ is a lower bound since all elements in E are greater than it. Let $c > \frac{5}{11}$ where $c = \frac{5}{11} + \epsilon$ for some $\epsilon > 0$. Choose $b \in E$ where $b = \frac{5n+8}{11n}$ and $n > \frac{8}{11\epsilon}$. Since $\frac{5}{11} \leq b < c$, $\frac{5}{11}$ is the infimum.

B.

Let $a = \max(\sup T, \sup S)$. For all $x \in T \cup S$, $x \in T$ or $x \in S$. Thus $x < \sup T$ or $x < \sup S$ and so $x < \max(\sup T, \sup S)$. Therefore a is an upperbound.

Case 1 If $\sup T \leq \sup S$, then $a = \sup S$. For any c we choose, there exists some $b \in S$ such that $c < b \leq a$ by the definition of the supremum.

Case 2 Similarly if $\sup T > \sup S$, we can choose some $b \in T$ such that $c < b \leq a$. Therefore, a is the best upperbound and so it is the supremum.

C.

Let $a = (\sup S) * (\sup T)$. For all $x \in ST$, $x = st$ for some $s \in S$ and $t \in T$. Since $s \leq \sup S$, $t \leq \sup T$, and s and t are positive, $st \leq (\sup S)(\sup T)$. Thus, a is an upperbound for ST.

Let $c < a$ where $c = c_S c_T$ for some $c_S < \sup S$ and $c_T < \sup T$. Choose some b_S and b_T such that $c_S < b_S \leq \sup S$ and $c_T < b_T \leq \sup T$. Let $b = b_S b_T$. Since these are subsets of the positive real numbers, this implies that $c < b \leq (\sup S)(\sup T)$, meaning that a is the supremum of ST.

Let $a = \sup S + \sup T$. For all $x \in S + T$, $x = s + t$ for some $s \in S$ and $t \in T$. Since $s \leq \sup S$ and $t \leq \sup T$, $s + t \leq \sup S + \sup T$. Thus, a is an upperbound for $S + T$.

Let $c < a$ where $c = c_S + c_T$ for some $c_S < \sup S$ and $c_T < \sup T$. Choose some b_S and b_T such that $c_S < b_S \leq \sup S$ and $c_T < b_T \leq \sup T$. Let $b = b_S + b_T$. This implies that $c < b \leq \sup S + \sup T$, meaning that a is the supremum of $S + T$.

D

1. The addition of two rational functions is rational since the functions can be rewritten with a common denominator and then added to yield another rational function. Since addition of functions is commutative and associative, so is the addition of rational functions. 0 is a rational function and adding 0 to any rational function yields that rational function. The inverse of every rational function can be found by negating the coefficients of the numerator and adding a rational function to its inverse yields 0.

   The multiplication of two rational functions is rational since the multiplication of the numerators is a polynomial and the multiplication of the denominators is a polynomial. Since multiplication of functions is commutative and associative, so is the multiplication of rational functions. 1 is a rational function and multiplying 1 to any rational function yields that rational function. The inverse of every rational function can be found by swapping the numerator and the denominator, and multiplying a rational function to its inverse yields 1.

   Since the distributive law holds for functions, it also holds for rational functions.

2. For any two rational functions $\frac{p}{q}$ and $\frac{f}{g}$, $\frac{p}{q} - \frac{f}{g}$ either has $a_n b_m > 0$, $a_n b_m = 0$, or $a_n b_m < 0$. This means that either $\frac{p}{q} > \frac{f}{g}$, $\frac{p}{q} = \frac{f}{g}$, or $\frac{p}{q} < \frac{f}{g}$. Also for any three rational functions $\frac{p}{q}$, $\frac{f}{g}$, and $\frac{a}{b}$, if $\frac{p}{q} > \frac{f}{g}$ and $\frac{f}{g} > \frac{a}{b}$, then $\frac{p}{q} - \frac{f}{g} > 0$ and $\frac{f}{g} - \frac{a}{b} > 0$. This would imply that $\frac{p}{q} - \frac{a}{b} > 0$ and $\frac{p}{q} > \frac{a}{b}$. Thus F is an ordered set.

   Let $\frac{p}{q}$, $\frac{f}{g}$, and $\frac{a}{b}$ be three rational functions. $\frac{p}{q} > \frac{f}{g}$ implies that $\frac{p}{q} - \frac{f}{g} > 0$, which implies $(\frac{p}{q} + \frac{a}{b}) - (\frac{f}{g} + \frac{a}{b}) > 0$, which is equivalent to $\frac{p}{q} + \frac{a}{b} > \frac{f}{g} + \frac{a}{b}$.

   For rational functions $\frac{p}{q} > 0$ and $\frac{f}{g} > 0$, a_n of $\frac{p}{q} \cdot \frac{f}{q}$ is the product of a_n of $\frac{p}{q}$ and a_n of $\frac{f}{q}$. Similarly, b_m of $\frac{p}{q} \cdot \frac{f}{q}$ is the product of b_m of $\frac{p}{q}$ and b_m of $\frac{f}{q}$. Since the sign a_n and b_m match for $\frac{p}{q}$ and $\frac{f}{g}$ the sign of a_n and b_m in the product must match, so $\frac{p}{q} \cdot \frac{f}{g} > 0$. Since all the axioms of an ordered field have been met, F is an ordered field.

3. The order defined in (ii) is equivalent to the dictionary ordering since we only check the sign of the most significant terms in the numerator and denominator.

$$-x^5, 3 - 2x, 2, x + 6, x^2$$

4. For all $a \in \mathbb{R}$, $x - a > 0$ since $a_n b_m = 1 > 0$, so $x > a$.

E.

Exercise 1 Since r is rational it can be written as $r = \frac{f}{g}$. Assume that $r+x$ is rational. Then there exists integers p and q such that $\frac{p}{q} = r + x$. Then x could be written as $\frac{p}{q} - \frac{f}{g} = \frac{pg-fq}{qg}$, which is a contradiction. Therefore, $r + x$ must be irrational.

Also assume that rx is rational. Then there exists integers p and q such that $\frac{p}{q} = rx$. Then x could be written as $\frac{\frac{p}{q}}{\frac{f}{g}} = \frac{pg}{qf}$, which is a contradiction. Therefore, rx must be irrational.

Exercise 2 Assume that there exists a rational number $\frac{p}{q}$ in simplified form such that $\frac{p^2}{q^2} = 12$. $p^2 = 12q^2$ implies that p^2 and p are a multiple of 3, so $p = 3k$ for some k. Substituting this in for p yields $3k^2 = 4q^2$, which implies that q^2 and q are a multiple of 3. This is a contradiction since we assumed that $\frac{p}{q}$ was already simplified, so 12 has no rational root.

Exercise 5 Let $a = -\sup(-A)$. For all $x \in A$, $-x$ is an element of $-A$. By the definition of the supremum $-a \geq -x$ for all $-x$, which implies $a \leq x$ for all x and so a is a lower bound for A.

From the definition of the supremum, for all $c < -a$, there exists b such that $c < b \leq -a$. Therefore for all $-c > a$, you can choose $-b$ such that $a \leq -b < -c$, so a is the infimum.

Exercise 8 The square of a number is always nonnegative in an ordered field, but $i^2 = -1$, so the complex numbers cannot be an ordered field.