

Math 140B: Homework 4

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Rudin 19

If γ_1 is an arc, it is 1-1. Since ϕ is 1-1, γ_2 must also be 1-1 and so it is an arc as well. Similarly since ϕ^{-1} exists and is 1-1, γ_1 must be 1-1 if γ_2 is.

If γ_1 is a closed curve, then $\gamma_1(a) = \gamma_1(b)$. Since ϕ is 1-1, continuous, and $\phi(c) = a$, it must be that $\phi(d) = b$. Thus $\gamma_2(c) = \gamma_2(d)$ and γ_2 is also a closed curve. Similarly since ϕ^{-1} exists and is 1-1, $\gamma_2(c) = \gamma_2(d)$ implies $\gamma_1(a) = \gamma_1(b)$.

Since ϕ and ϕ^{-1} create corresponding partitions between $[a, b]$ and $[c, d]$, one curve must be rectifiable if the other is, and they must have the same length. This is clear since if $\{x_i\}$ is a partition of $[c, d]$ then

$$\sum |\gamma_1(\phi(x_i)) - \gamma_1(\phi(x_{i-1}))| = \sum |\gamma_2(x_i) - \gamma_2(x_{i-1})|$$

Rudin 1

By definition, a sequence of functions uniformly converges $f_n \rightarrow f$ if for all $\epsilon > 0$ there exists N such that $n \geq N$ implies

$$|f_n(x) - f(x)| \leq \epsilon$$

We can bound functions with $n \geq N$ below by $\inf f(x) - \epsilon$ and above by $\sup f(x) + \epsilon$. Since there is a finite number of functions where $n < N$, we can bound it below by $\inf_{n < N} \inf_{x \in \mathbb{R}} f_n(x)$ and above by $\sup_{n < N} \sup_{x \in \mathbb{R}} f_n(x)$. Thus every uniformly convergent sequence of bounded functions is uniformly bounded.

Rudin 2

If (f_n) converges uniformly then there exists N_1 such that $n \geq N_1$ implies

$$|f_n(x) - f(x)| < \frac{\epsilon}{2}.$$

Similarly there exists N_2 such that for $n \geq N_2$

$$|g_n(x) - g(x)| < \frac{\epsilon}{2}.$$

Thus for $n \geq \max(N_1, N_2)$,

$$|(f_n + g_n)(x) - (f + g)(x)| < \epsilon.$$

Since f_n and g_n are uniformly bounded, there exists M such that $|f_n(x)| < M$ and $|g_n(x)| < M$ for all n and x . Then choose N_1 so that for $n > N_1$

$$|f_n(x) - f(x)| < \frac{\epsilon}{2M}$$

and choose N_2 so that for $n > N_2$

$$|g_n(x) - g(x)| < \frac{\epsilon}{2M}.$$

Thus,

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &\leq |f_n(x)g_n(x) - f_n(x)g(x)| + |f_n(x)g(x) - f_n(x)g(x)| \\ &\leq M|g_n(x) - g(x)| + M|f_n(x) - f(x)| \\ &< M\frac{\epsilon}{2M} + M\frac{\epsilon}{2M} \\ &= \epsilon \end{aligned}$$

Rudin 3

$f_n(x) = x$ converges uniformly to $f(x) = x$ and $g_n(x) = \frac{1}{n}$ converges uniformly to $g(x) = 0$ but $f_n(x)g_n(x)$ does not converge because when $x = n$, $f_n(x)g_n(x) = 1$.

Rudin 4

The series diverges when $x = 0$ since all the terms are 1 and the series diverges when $x = -\frac{1}{n^2}$ for all natural numbers n since there is division by 0 on the n th term. For intervals $[\delta, \infty)$ for $\delta > 0$ the series converges uniformly since we can bound each term with

$$\frac{1}{n^2 x} \leq \frac{1}{n^2 \delta}.$$

For intervals $(-\infty, -\delta]$ and $x \neq -\frac{1}{n^2}$ the interval also converges uniformly since for a sufficiently large n such that $n^2 \geq \frac{2}{\delta}$

$$\left| \frac{1}{1 + n^2 x} \right| = \frac{1}{n^2} \frac{1}{\frac{1}{n^2} + x} \leq \frac{1}{n^2} \frac{1}{\frac{1}{n^2} + \delta} \leq \frac{2}{\delta n^2}$$

Rudin 5

f_n converges to 0 since all points $x \leq 0$ always evaluate to 0 and all positive points are $\frac{1}{n} < x$ for a sufficiently large n . However it does not converge uniformly since for every f_n , $f_n\left(\frac{1}{n+\frac{1}{2}}\right) = 1$.

Rudin 6

The series can be rewritten as

$$x^2 \sum \frac{(-1)^n}{n^2} + \sum \frac{(-1)^n}{n}$$

The left series converges since $x^2 \frac{(-1)^n}{n^2} \leq \frac{M}{n^2}$ if M is the sup over an interval for that term. The right sequence also converges, so the overall sequence converges uniformly. However the series does not converge absolutely since each term is greater than $\frac{1}{n}$, which diverges.

Rudin 8

Since each term is $< |c_n|$ and c_n converges, f_n converges uniformly. Since each individual term is continuous for $x \neq x_n$, the overall function is continuous as well.

Rudin 9

Since f_n converges uniformly, there exists N_1 such that

$$|f_n(x_n) - f_n(x)| < \frac{\epsilon}{2}.$$

Since f is continuous, there exists N_2 such that

$$|f(x_n) - f(x)| < \frac{\epsilon}{2}.$$

By triangle inequality,

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f_n(x)| + |f_n(x) - f(x)| < \epsilon$$

The converse is not true, as exemplified by $f_n = x^n$ for points approaching 1.