Math 31AH: Fall 2021
Homework 4 Solution
Due 5:00pm on Friday 10/22/2021

Problem 1: Projections. Let V and W be $\mathbb{F}$-vector spaces. Recall the direct sum of V and W

$$V \oplus W = \{(\mathbf{v}, \mathbf{w}) \,:\, \mathbf{v} \in V \,,\, \mathbf{w} \in W\}$$

defined last time. Define a function $P_V : V \oplus W \to V$ by $P_V(\mathbf{v}, \mathbf{w}) = \mathbf{v}$. The function P_V is called 'projection onto V'.

Prove that P_V is linear. When is P_V surjective? When is P_V injective?

Solution: Let $(\mathbf{v}, \mathbf{w}), (\mathbf{v}', \mathbf{w}') \in V \oplus W$ and $c \in \mathbb{F}$. We have

$$P_V((\mathbf{v}, \mathbf{w}) + (\mathbf{v}', \mathbf{w}')) = P_V(\mathbf{v}+\mathbf{v}', \mathbf{w}+\mathbf{w}') = \mathbf{v}+\mathbf{v}' = P_V(\mathbf{v}, \mathbf{w}) + P_V(\mathbf{v}', \mathbf{w}')$$

and

$$P_V(c(\mathbf{v}, \mathbf{w})) = P_V(c\mathbf{v}, c\mathbf{w}) = c\mathbf{v} = cP_V(\mathbf{v}, \mathbf{w})$$

so that P_V is linear.

For any $\mathbf{v} \in V$ we have $P_V(\mathbf{v}, \mathbf{0}) = \mathbf{v}$ so that P_V is surjective.

We claim that P_V is injective if any only if $W = 0$. If $W \neq 0$, choose $\mathbf{w} \neq \mathbf{0}$ in W. Then $P_V(\mathbf{v}, \mathbf{w}) = \mathbf{v} = P_V(\mathbf{v}, \mathbf{0})$. Since $(\mathbf{v}, \mathbf{w}) \neq (\mathbf{v}, \mathbf{0})$, P_V is not injective. If $W = 0$, then $V \oplus W = \{(\mathbf{v}, \mathbf{0}) \,:\, \mathbf{v} \in V\}$ and

$$P_V(\mathbf{v}, \mathbf{0}) = P_V(\mathbf{v}', \mathbf{0}) \Leftrightarrow \mathbf{v} = \mathbf{v}' \Leftrightarrow (\mathbf{v}, \mathbf{0}) = (\mathbf{v}', \mathbf{0})$$

so that P_V is injective.

Problem 2: Linear maps and spanning. Let $T : V \to W$ be a linear transformation of $\mathbb{F}$-vector spaces. If $S \subseteq V$ spans V, prove that

$$T(S) := \{T(\mathbf{v}) \,:\, \mathbf{v} \in S\}$$

spans Image(T).

Solution: Let $\mathbf{w} \in \text{Image}(T)$. There exists $\mathbf{v} \in V$ so that $T(\mathbf{v}) = \mathbf{w}$. Since S spans V, there exist $\mathbf{v}_i \in S$ and $c_i \in \mathbb{F}$ so that

$$\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$$

Since T is linear we have

$$\mathbf{w} = T(\mathbf{v}) = T(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n) = c_1T(\mathbf{v}_1) + \cdots + c_nT(\mathbf{v}_n)$$

so that $\mathbf{w} \in \text{span}T(S)$. We conclude that $T(S)$ spans Image(T).

Problem 3: Linear maps and independence. Let $T : V \to W$ be an injective linear transformation between $\mathbb{F}$-vector spaces. If $I \subseteq V$ is linearly independent in V, prove that

$$T(I) := \{T(\mathbf{v}) \,:\, \mathbf{v} \in I\}$$

is linearly independent in W. Does this necessarily hold if we remove the assumption that T is injective?

Solution: Suppose $\mathbf{v}_i \in I$ and $c_i \in \mathbb{F}$ are such that

$$c_1 T(\mathbf{v}_1) + \cdots + c_n T(\mathbf{v}_n) = \mathbf{0}$$

in W. We claim that $c_1 = \cdots = c_n = 0$. Indeed, since T is linear

$$\mathbf{0} = c_1 T(\mathbf{v}_1) + \cdots + c_n T(\mathbf{v}_n) = T(c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n)$$

and since T is injective $\mathrm{Ker}(T) = 0$ so that

$$c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$$

in V. Since I is linearly independent in V, we have $c_1 = \cdots = c_n = 0$. We conclude that $T(I)$ is linearly independent in W.

This does **not** necessarily hold if we remove the assumption that T is injective. For example, let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the zero map, $T(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in \mathbb{R}^2$. The set $\{\mathbf{e}_1, \mathbf{e}_2\}$ consisting of the two standard basis vectors is linearly independent in $\mathbb{R}^2$, but $T(\{\mathbf{e}_1, \mathbf{e}_2\}) = \{\mathbf{0}\}$ is not linearly independent.

Problem 4: Representing matrices. Let V_n denote the $\mathbb{R}$-vector space of polynomials in x of degree $\leq n$ with real coefficients. The derivative map

$$T : V_3 \to V_2 \qquad T(f(x)) = f'(x)$$

is a linear transformation. Consider the ordered bases $\mathcal{B} := (x^3, x^2, x, 1)$ of V_3 and $\mathcal{C} := ((x+1)^2, x+1, 1)$ of V_2. Calculate $[T]_{\mathcal{C}}^{\mathcal{B}}$, the representing matrix of T with respect to these bases.

Solution: We take the derivative of the elements of $\mathcal{B}$ and expand the result in terms of the entries of $\mathcal{C}$:

$$\begin{aligned}
T(x^3) &= 3x^2 = 3(x+1)^2 - 6(x+1) + 3(1)\\
T(x^2) &= 2x = 0(x+1)^2 + 2(x+1) - 2(1)\\
T(x) &= 1 = 0(x+1)^2 + 0(x+1) + 1(1)\\
T(1) &= 0 = 0(x+1)^2 + 0(x+1) + 0(1)
\end{aligned}$$

Recording these coefficients in a matrix gives

$$[T]_{\mathcal{C}}^{\mathcal{B}} = \begin{pmatrix} 3 & 0 & 0 & 0 \\ -6 & 2 & 0 & 0 \\ 3 & -2 & 1 & 0 \end{pmatrix}$$

Problem 5: A matrix map. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Find a basis of $\operatorname{Ker}(T)$ and a basis of $\operatorname{Image}(T)$.

Solution: We have $\operatorname{Ker}(T) = \{\mathbf{x} \in \mathbb{R}^3 \,:\, A\mathbf{x} = \mathbf{0}\}$. The RREF of A is computed as in the previous homework

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

so that

$$\operatorname{Ker}(T) = \{\mathbf{x} \in \mathbb{R}^3 \,:\, A\mathbf{x} = \mathbf{0}\} = \left\{ \begin{pmatrix} z \\ -2z \\ z \end{pmatrix} \,:\, z \in \mathbb{R} \right\}$$

and a basis of $\operatorname{Ker}(T)$ is the one-element set

$$\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$$

The image of T is given by

$$\operatorname{Image}(T) = \{A\mathbf{x} \,:\, \mathbf{x} \in \mathbb{R}^3\} = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 \,:\, c_1, c_2, c_3 \in \mathbb{R}\}$$

where $A = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix}$ are the columns of A. By the previous paragraph, we have $\mathbf{v}_1 - 2\mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$ so that $\{\mathbf{v}_1, \mathbf{v}_2\}$ spans $\operatorname{Image}(A)$. In fact, the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is also linearly independent (indeed, the RREF of the 3×2 matrix $\begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix}$ has a pivot 1 in each column, so that there are no nontrivial solutions (c_1, c_2) to $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$). Therefore, the space $\operatorname{Image}(T)$ has basis

$$\{\mathbf{v}_1, \mathbf{v}_2\} = \left\{ \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 8 \end{pmatrix} \right\}$$

Problem 6: Inverses of linear maps. Let $T : V \to W$ be an invertible linear transformation between $\mathbb{F}$-vector spaces. Prove that the inverse $T^{-1} : W \to V$ is also linear.

Solution: Let $\mathbf{w}, \mathbf{w}' \in W$ and $c \in \mathbb{F}$. Since T is invertible, there exist unique $\mathbf{v}, \mathbf{v}' \in V$ with $T(\mathbf{v}) = \mathbf{w}$ and $T(\mathbf{v}') = \mathbf{w}'$. Then $T^{-1}(\mathbf{w}) = \mathbf{v}$ and $T^{-1}(\mathbf{w}) = \mathbf{v}'$. To calculate $T^{-1}(\mathbf{w} + \mathbf{w}')$ observe that

$$T(\mathbf{v} + \mathbf{v}') = T(\mathbf{v}) + T(\mathbf{v}') = \mathbf{w} + \mathbf{w}'$$

so that

$$T^{-1}(\mathbf{w} + \mathbf{w}') = \mathbf{v} + \mathbf{v}' = T^{-1}(\mathbf{w}) + T^{-1}(\mathbf{w}')$$

Furthermore, we have $T(c\mathbf{v}) = cT(\mathbf{v}) = c\mathbf{w}$ so that

$$T^{-1}(c\mathbf{w}) = c\mathbf{v} = cT^{-1}(\mathbf{w})$$

and we conclude that T^{-1} is linear.

Problem 7: Polynomial change of basis. Let V_3 be the vector space of polynomials in x of degree ≤ 3 with real coefficients. Consider the two bases $\mathcal{B}$ and $\mathcal{C}$ of V given by

$$\begin{aligned}\mathcal{B} &:= (1, x+1, x^2+x+1, x^3+x^2+x+1)\\ \mathcal{C} &:= (1, x-1, x^2-x+1, x^3-x^2+x-1)\end{aligned}$$

Calculate the transition matrix from $\mathcal{B}$ to $\mathcal{C}$ and the transition matrix from $\mathcal{C}$ to $\mathcal{B}$.

Solution: We express the elements of $\mathcal{B}$ in terms of those of $\mathcal{C}$ as follows

$$\begin{aligned}1 &= 1(1) + 0(x-1) + 0(x^2-x+1) + 0(x^3-x^2+x-1)\\ x+1 &= 2(1) + 1(x-1) + 0(x^2-x+1) + 0(x^3-x^2+x-1)\\ x^2+x+1 &= 2(1) + 2(x-1) + 1(x^2-x+1) + 0(x^3-x^2+x-1)\\ x^3+x^2+x+1 &= 2(1) + 2(x-1) + 2(x^2-x+1) + 1(x^3-x^2+x-1)\end{aligned}$$

so that the transition matrix from $\mathcal{B}$ to $\mathcal{C}$ is

$$[\mathrm{id}_{V_3}]_{\mathcal{C}}^{\mathcal{B}} = \begin{pmatrix} 1 & 2 & 2 & 2\\ 0 & 1 & 2 & 2\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 1\end{pmatrix}$$

Expressing the elements of $\mathcal{C}$ in terms of those of $\mathcal{B}$ yields

$$\begin{aligned}
1 &= 1(1) + 0(x+1) + 0(x^2+x+1) + 0(x^3+x^2+x+1)\\
x-1 &= -2(1) + 1(x+1) + 0(x^2+x+1) + 0(x^3+x^2+x+1)\\
x^2-x+1 &= 2(1) + (-2)(x+1) + 1(x^2+x+1) + 0(x^3+x^2+x+1)\\
x^3-x^2+x-1 &= (-2)(1) + 2(x+1) + (-2)(x^2+x+1) + 1(x^3+x^2+x+1)
\end{aligned}$$

so that the transition matrix from $\mathcal{C}$ to $\mathcal{B}$ is

$$[\mathrm{id}_{V_3}]_{\mathcal{B}}^{\mathcal{C}} = \begin{pmatrix} 1 & -2 & 2 & -2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This is the inverse of the transition matrix from $\mathcal{B}$ to $\mathcal{C}$.

Problem 8: Invariant subspaces and block matrices. Let V be an n-dimensional $\mathbb{F}$-vector space and let $W \subseteq V$ be a subspace of dimension m. Show that there exists an ordered basis

$$\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m, \mathbf{v}_{m+1}, \mathbf{v}_{m+2}, \dots, \mathbf{v}_n)$$

of V such that $(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m)$ is an ordered basis of W. (Hint: A problem on Homework 3 should make this a one-liner!)

Let $T : V \to V$ be a linear transformation. The subspace $W \subseteq V$ is called *invariant under* T if $T(\mathbf{w}) \in W$ for all $\mathbf{w} \in W$. If $\mathcal{B}$ is a basis of V as above, prove that W is invariant under T if and only if the representing matrix $[T]_{\mathcal{B}}^{\mathcal{B}}$ for T with respect to B has the block form

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

where A is $m \times m$ and C is $(n-m) \times (n-m)$.

Solution: Let $(\mathbf{v}_1, \dots, \mathbf{v}_m)$ be an ordered basis of W. The vectors $\mathbf{v}_1, \dots, \mathbf{v}_m$ are linearly independent in V so by Problem 7 on Homework 3, this list may be extended to an ordered basis

$$\mathcal{B} = (\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m, \mathbf{v}_{m+1}, \mathbf{v}_{m+2}, \dots, \mathbf{v}_n)$$

of V.

Suppose W is invariant under T. For $1 \le j \le m$ we have $\mathbf{v}_j \in W$ so that $T(\mathbf{v}_j) \in W$. Since $(\mathbf{v}_1, \dots, \mathbf{v}_m)$ is a basis of W we have

$$T(\mathbf{v}_j) = c_1\mathbf{v}_1 + \cdots + c_m\mathbf{v}_m = c_1\mathbf{v}_1 + \cdots + c_m\mathbf{v}_m + 0\mathbf{v}_{m+1} + \cdots + 0\mathbf{v}_n$$

for some $c_1, \ldots, c_m \in \mathbb{F}$. The j^{th} column of $[T]_{\mathcal{B}}^{\mathcal{B}}$ is therefore

$$\begin{pmatrix} c_1 \\ \vdots \\ c_m \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and the matrix $[T]_{\mathcal{B}}^{\mathcal{B}}$ has the required form.

On the other hand, suppose the matrix $[T]_{\mathcal{B}}^{\mathcal{B}}$ has the given form. This implies that for $1 \leq j \leq m$ there are scalars a_{ij} so that

$$T(\mathbf{v}_j) = a_{1j}\mathbf{v}_1 + \cdots + a_{mj}\mathbf{v}_m \in W$$

Since W is a subspace of V, we conclude that

$$T(c_1\mathbf{v}_1 + \cdots + c_m\mathbf{v}_m) = c_1T(\mathbf{v}_1) + \cdots + c_mT(\mathbf{v}_m) \in W$$

for any scalars $c_1, \ldots, c_m \in W$. Since $\mathbf{v}_1, \ldots, \mathbf{v}_m$ span W this shows that W is invariant under T.

Problem 9: (Optional; not to be handed in.) Let $\mathbb{K}$ be a finite field of characteristic p. You proved on a previous homework that p is prime. Show that $\mathbb{F} := \{0, 1, 2, \ldots, p-1\} \subseteq \mathbb{K}$ is a subfield of $\mathbb{K}$ (i.e. is a subset which is also a field), so that $\mathbb{K}$ is a $\mathbb{F}$-vector space. Conclude that

> *Any finite field has size equal to some power p^r of a prime number p, where $r > 0$ is an integer.*

For example, there is no field with 6 elements.