Math 140B: Homework 3

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Rudin 1

Let P be a partition on $[a, b]$. If $x_0 \in [x_{i-1}, x_i]$ then $m_i = 0$ and $M_i = 1$ and $m_i = M_i = 0$ for all other intervals. Since Δx_i can be chosen to be arbitrarily small and α is continuous at x_0,

$$\underline{\int_a^b} f(x)\, d\alpha = \sup L(P, f, \alpha) = \sup 0 = 0$$

$$\overline{\int_a^b} f(x)\, d\alpha = \inf U(P, f, \alpha) = \inf \alpha(x_i) - \alpha(x_{i-1}) = 0$$

Thus $f \in \mathscr{R}(\alpha)$ and $\int_a^b f(x)\, d\alpha = 0$.

Rudin 2

Suppose $f(x_0) \neq 0$ for some x_0. Since f is continuous at x_0, there exists $\delta > 0$ such that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \frac{f(x_0)}{2}.$$

Since f is continuous, nonnegative, and bounded, it is Riemann integrable. Since all $x \in B_\delta(x_0)$ is positive,

$$\int_a^b f(x)\, dx \geq \int_{\max(a, x-\delta)}^{\min(b, x+\delta)} f(x)\, dx > 0$$

Rudin 4

Since the rationals and the irrationals are both dense in the reals, $M_i = 1$ and $m_i = 0$ for all intervals for all partitions. Thus every lower Riemann sum equals 0 and every upper Riemann sum equals $b - a$, so $f \notin \mathscr{R}$.

Rudin 5

The rational indicator function from problem 4 is a counterexample. It is not Riemann integrable but its square is just the constant function. However the integrability of f^3 does imply the integrability of f by theorem 6.11.

Rudin 6

Cover P with open intervals (u_j, v_j) where each interval has length $\alpha(v_j) - \alpha(u_j) < \epsilon$. If we remove these open intervals from $[0, 1]$ we get another compact set K, which f is uniformly continuous on, meaning $|s - t| < \delta \implies |f(s) - f(t)| < \epsilon$. If we form a partition where each u_j, v_j occurs in P, no point of any segument (u_j, v_j) occurs in P, and each $x_{i-1} \neq u_j$ has $\Delta x_i < \delta$ then

$$U(P, f, \alpha) - L(P, f, \alpha) \leq [\alpha(b) - \alpha(b)]\epsilon + 2M\epsilon$$

where $m \leq f(x) \leq M$ as bounds. Thus since ϵ was arbitrary we have that $f \in \mathscr{R}$ on $[0, 1]$.

Rudin 7

Let $\epsilon > 0$ and $M = \sup|f(x)|$. Let P be a partition that contains c when $0 < c \leq \frac{\epsilon}{4M}$ and the difference between the upper and lower Riemann sums is $< \frac{\epsilon}{4}$. Then the upper and lower Riemann sums on $[c, 1]$ using the points of P inside that intervals are also $< \frac{\epsilon}{4}$. Finally note that the value of the upper and lower Riemann sums in $[0, c]$ are also $< \frac{\epsilon}{4}$. Thus it must be that

$$\left|\int_0^1 f(x)\,dx - \int_c^1 f(x)\,dx\right| < \epsilon$$

The function $f(x) = (-1)^n(n+1)$ for $\frac{1}{n+1} < x \leq \frac{1}{n}$ has a limit

$$\int_c^1 f(x)\,dx = (-1)^N(N+1)(\frac{1}{N} - c) + \sum_{k=1}^{N-1} \frac{(-1)^k}{k}$$

where $\frac{1}{N+1} < c \leq \frac{1}{N}$. However this limit does not exist for $|f|$ since

$$\int_c^1 |f(x)|\,dx = (N+1)(\frac{1}{N} - c) + \sum_{k=1}^{N-1} \frac{1}{k}$$

Rudin 8

Note that for a partition of $[0, n]$ with the points $0, 1, 2, \cdots, n$, If the sum does not converge, then the integral does not converge since

$$\sum_{n=1}^{n} f(n) = L(f, P) \leq \int_0^n f(x)\,dx$$

Likewise if the integral does not converge then the sum does not converge since

$$\int_0^n f(x)\,dx \leq U(f, P) = \sum_{n=0}^{n} f(n)$$

Rudin 11

Define

$$P(\lambda) = \int_a^b (\lambda u(x)+v(x))^2 \, d\alpha = \lambda^2 \int_a^b u^2(x) \, d\alpha + 2\lambda \int_a^b u(x)v(x) \, d\alpha + \int_a^b v^2(x) \, d\alpha \geq 0$$

This implies a negative determinant so

$$\begin{aligned}
&\left(2\int_a^b u(x)v(x) \, d\alpha\right)^2 - 4\int_a^b u^2(x) \, d\alpha \int_a^b v^2(x) \, d\alpha \leq 0 \\
\implies & 4\left(\int_a^b u(x)v(x) \, d\alpha\right)^2 \leq 4\int_a^b u^2(x) \, d\alpha \int_a^b v^2(x) \, d\alpha \\
\implies & \int_a^b u(x)v(x) \, d\alpha \leq \left(\int_a^b u^2(x) \, d\alpha\right)^{\frac{1}{2}} \left(\int_a^b v^2(x) \, d\alpha\right)^{\frac{1}{2}} \\
\implies & \int_a^b |u(x)||v(x)| \, d\alpha \leq ||u||_2 ||v||_2
\end{aligned}$$

Expanding out the definition yields

$$\begin{aligned}
||f-h||_2^2 &= \int_a^b |f-h|^2 \, d\alpha \\
&= \int_a^b |(f-g)+(g-h)|^2 \, d\alpha \\
&= \int_a^b |f-g|^2 \, d\alpha + \int_a^b |f-g||g-h| \, d\alpha + \int_a^b |g-h|^2 \, d\alpha \\
&\leq ||f-g||_2^2 + 2||f-g||_2||g-h||_2 + ||g-h||_2^2 \\
&= (||f-g||_2 + ||g-h||_2)^2
\end{aligned}$$

Rudin 12

Define for a partition $P = \{x_0, \cdots, x_n\}$

$$g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)$$

Since g is continuous, and $|f-g|$ is bounded by $M_i - m_i$. We can choose a partition such that the upper riemann sum of f is bounded by $\frac{\epsilon^2}{2M}$ where M is the max of $|f(x)|$, which implies that

$$\sum (M_i - m_i)^2 [\alpha(x_i) - \alpha(x_{i-1})] < \epsilon^2$$

so $||g-f||_2 < \epsilon$.