Math 31BH: Assignment 2

Due 01/16 at 23:59

Merrick Qiu

1. Let $\mathbf{S}$ be the space of 2×2 symmetric matrices, and let $f\colon \mathbf{S} \to \mathbb{R}^2$ be the function which sends each $S \in \mathbf{S}$ to $(s_1, s_2) \in \mathbb{R}^2$, where s_1, s_2 are the eigenvalues of S and $s_1 \geq s_2$.

   (a) Write down an explicit formula for the function $g\colon \mathbb{R}^3 \to \mathbb{R}^2$ defined by

   $$g(x_1, x_2, x_3) = f\left(\begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}\right).$$

   (b) Suppose $(a, b, c) \in \mathbb{R}^3$ is such that $g(a, b, c) = (1, 2)$. Prove that $g(a + 1, b, c) \neq (2.1, 2)$.

   Solution:

   (a) The eigenvalues of S are the values λ with

   $$\det(A - \lambda I) = (x_1 - \lambda)(x_3 - \lambda) - x_2^2 = 0$$

   We can rearrange the expression and use the quadratic equal to solve for the roots.

   $$\begin{aligned}
(x_1 - \lambda)(x_3 - \lambda) - x_2^2 = 0 &\implies x_1x_3 - x_1\lambda - x_3\lambda + \lambda^2 - x_2^2 = 0 \\
&\implies \lambda^2 + (-x_1 - x_3)\lambda + (x_1x_3 - x_2^2) = 0 \\
&\implies \lambda = \frac{-(-x_1 - x_3) \pm \sqrt{(-x_1 - x_3)^2 - 4(x_1x_3 - x_2^2)}}{2} \\
&\implies \lambda = \frac{x_1 + x_3 \pm \sqrt{(x_1^2 + 2x_1x_3 + x_3^2) - (4x_1x_3 - 4x_2^2)}}{2} \\
&\implies \lambda = \frac{x_1 + x_3 \pm \sqrt{x_1^2 + 4x_2^2 + x_3^2 - 2x_1x_3}}{2}
\end{aligned}$$

   Therefore, we have that

   $$g(x_1, x_2, x_3) = \begin{bmatrix} \frac{x_1+x_3+\sqrt{x_1^2+4x_2^2+x_3^2-2x_1x_3}}{2} \\ \frac{x_1+x_3-\sqrt{x_1^2+4x_2^2+x_3^2-2x_1x_3}}{2} \end{bmatrix}$$

(b) From the equation for $g(a,b,c)$, we see that the sum of the eigenvalues is $a+c$. If $g(a,b,c) = (1,2)$ then the sum of the eigenvalues is

$$\lambda_1 + \lambda_2 = a + c = 3$$

For $g(a,b,c)$, the sum of the eigenvalues must be

$$\lambda_1 + \lambda_2 = (a+1) + c = 4$$

However, the sum $2.1 + 2 = 4.1$, so $g(a+1,b,c) \neq (2.1,2)$

2. Let $\mathbf{V}$ and $\mathbf{W}$ be Euclidean spaces. Prove that every linear function $f\colon \mathbf{V} \to \mathbf{W}$ is continuous.

Solution: Let $\delta =?$. Since $B_\delta(v)$ is bounded, there exists point p such that

3. Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be vectors in a Euclidean space $\mathbf{V}$.

(a) Prove that the convex hull $\mathrm{Conv}(\mathbf{v}_1, \ldots, \mathbf{v}_r)$ is a compact set.

(b) Let $f\colon \mathbf{V} \to \mathbb{R}$ be a linear function. Prove that f has both a maximum and a minimum value on $\mathrm{Conv}(\mathbf{v}_1, \ldots, \mathbf{v}_r)$, and show that it has both a maximizer and a minimizer in $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$.