Math 170A: Homework 6

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Q1

1.

$$\begin{aligned}||UA||_F^2 &= trace((UA)^TUA)\\ &= trace(A^TU^TUA)\\ &= trace(A^TA)\\ &= ||A||_F^2\end{aligned}$$

2.

$$\begin{aligned}||AV||_F^2 &= ||(AV)^T||_F^2\\ &= trace(AV(AV)^T)\\ &= trace(AVV^TA)\\ &= trace(AA^T)\\ &= ||A^T||_F^2\\ &= ||A||_F^2\end{aligned}$$

3. Since we can write any $n \times n$ matrix as $A = U\Sigma V$ using singular value decomposition, and since U and V are orthonormal, A has the same frobenius norm as Σ, which has frobenius norm $\sqrt{\sum_{i=1}^n \sigma_i^2}$.

Q2

Normalizing q_0 yields $[1, \frac{b}{a}]^T$, but repeatedly applying A simply switches the coordinates back and forth

$$\tilde{q_0} = [1, \frac{b}{a}]^T$$

$$A\tilde{q_0} = [\frac{b}{a}, 1]^T$$

$$A^2\tilde{q_0} = [1, \frac{b}{a}]^T$$

$$A^3\tilde{q_0} = [\frac{b}{a}, 1]^T$$

$$\vdots$$

The power method relies on there existing a largest eigenvalue, but the two eigenvalues of this matrix, 1 and -1 have the same magnitude.

Q3

The power method will converge to v_2. Since q does not depend on v_1, the power method will not be able to amplify v_1 so it will instead amplify v_2, the eigenvector with the next largest eigenvalue. This is why a random initial q needs to be chosen for the power method.

Q4

1. The characteristic polynomial is $(\lambda-1)^2 = \lambda^2 - 2\lambda + 1$, eigenvalue $\lambda = 1$ which has algebraic multiplicity 2 from its factorization.

2. Using the quadratic equation,

$$\hat{\lambda} = \frac{2 \pm \sqrt{4 - 4(1-\epsilon)}}{2} = 1 \pm \sqrt{\epsilon}$$

3.
$$|\hat{\lambda} - \lambda| = |1 \pm \sqrt{10^{-12}}2 - 1| = \times 10^{-6}$$
$$\frac{10^{-6}}{10^{-12}} = 10^6 \text{ times bigger}$$

4. Since a very small change in the coefficients can lead to an arbitrarily large relative change in the eigenvalues(by choosing a sufficiently small epsilon), the computation is numerically unstable.

Q5

1. MATLAB says the rank of A is 3
2. The pseudoinverse is

$$\begin{bmatrix} 0.5 & 0 & 0 & -1 \\ 1 & -0.4 & -0.2 & -1 \\ -1 & 0.4 & 0.2 & 2 \end{bmatrix}$$

3. Yes they agree, $A^\dagger A = I$ and

$$AA^\dagger = U_r U_r^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.8 & 0.4 & 0 \\ 0 & 0.4 & 0.2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$