Mathematics 100A Homework 8
Due: Tuesday November 26 2024

Instructions: Please write clearly and fully explain your solutions. It is OK to work with others to solve the problems, but if you do so, you should write your solutions up separately. Copying solutions from your peers or a solutions manual will be deemed academic misconduct. Chapter and problem numbers, if any, refer to *Algebra*, second edition, by Michael Artin. Please feel free to reach out to me or the TAs if you have any questions.

1. Suppose $X, Y \in M_n(\mathbf{C})$ are $n \times n$ matrices and $S \in \mathrm{GL}_n(\mathbf{C})$.
   (a) Prove that $\mathrm{tr}(XY) = \mathrm{tr}(YX)$.
   (b) Deduce that $\mathrm{tr}(SXS^{-1}) = \mathrm{tr}(X)$.

2. Suppose G is a finite group, and V is a finite-dimensional $\mathbf{C}$ vector space, and $\rho : G \to \mathrm{GL}(V)$ is a representation. For $v \in V$, define

$$\mathbf{C}[G](v) = \mathrm{Span}_{g \in G}\{g \cdot v\} = \left\{ \sum_{g \in G} a_g \rho(g)(v) : a_g \in \mathbf{C} \right\},$$

   the span of all the elements $\rho(g)(v)$ as g ranges over elements of G. Prove that V is irreducible if and only if $\mathbf{C}[G](v) = V$ for all $v \neq 0$ in V.

3. The standard representation of the symmetric group S_n is defined as follows: Let S_n act on $\mathbf{C}^n$ by permutations, $\sigma(v_1, \ldots, v_n) = (v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(n)})$. Set $W \subseteq \mathbf{C}^n$ to be the subspace of elements whose coordinates sum to 0, $W = \{v = (v_1, \ldots, v_n) : v_1 + \cdots + v_n = 0\}$, so that W is $n-1$-dimensional. We let S_n act on W by the restriction of the S_n action on $\mathbf{C}^n$. The representation of S_n on W is called the standard representation. Prove that it is irreducible.
   Hint: We will apply the criterion of the previous problem. Suppose $w = (w_1, \ldots, w_n) \in W$ is not zero. Then there exists $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$ so that $w_i \neq w_j$. Let τ be the permutation of S_n that exchanges i with j and leaves the other indices fixed. Then $\tau \cdot w - w = \alpha(e_i - e_j)$ for some $\alpha \neq 0$. Here e_k is the k^{th} standard basis vector of $\mathbf{C}^n$. Now check that the span of the S_n translates of $e_i - e_j$ is all of W.