Math 100A: Homework 8

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Problem 1

Since the trace is the sum of the diagonal entires and each diagonal entry is the dot product of the associated row in X and column in Y, then

$$\begin{aligned}\operatorname{tr}(XY) &= \sum_{i=1}^{n}\sum_{j=1}^{n} X_{i,j}Y_{j,i} \\ &= \sum_{j=1}^{n}\sum_{i=1}^{n} Y_{j,i}X_{i,j} \\ &= \operatorname{tr}(YX)\end{aligned}$$

As a corollary, we have that

$$\operatorname{tr}\bigl(SXS^{-1}\bigr) = \operatorname{tr}\bigl(SS^{-1}X\bigr) = \operatorname{tr}(X)$$

Problem 2

($\Longrightarrow$) Suppose there existed nonzero $w \in V$ such that $\mathbf{C}[G](w) = W \neq V$. $W \neq \{0\}$ since $W = \{0\}$ would imply $\rho(g)(w) = 0$ for all g. Since $\rho(g)$ is invertible, that implies $w = 0$ which is a contradiction.

W is a nontrivial G-invariant subspace since it is the span of $g \cdot w$ for all $g \in G$. Applying a group action g to an element in that span will still yield an element in that span. Therefore if V is irreducible it must be that $\mathbf{C}[G](v) = V$ for all nonzero $v \in V$.

($\Longleftarrow$) If $W \subseteq V$ was a nontrivial G-invariant subspace, then $\rho(g)(w) \in W$ for all g, and so $\mathbf{C}[G](w) \subseteq W$. However this contradicts the fact that $\mathbf{C}[G](v) = V$ for all $v \neq 0$ in V. Therefore V does not contain any G-invariant subspaces other than $\{0\}$ and V.

Problem 3

Let $W = \{(v_1, \ldots, v_n) : v_1 + \cdots + v_n = 0\}$. Suppose $w = (w_1, \ldots, w_n) \in W$ is not zero. It is sufficient to show that $\mathbf{C}[S_n](w) = W$ to show that W is irreducible by the previous question. There exists $i < j$ so that $w_i \neq w_j$ since $w \neq 0$. Let τ be the permutation of S_n that only exchanges i with j.

Notice that $\tau \cdot w - w \in \mathbf{C}[S_n](w)$ (since it is a linear combination of permutations of w) and that $\tau \cdot w - w = \alpha(e_i - e_j)$ for some $\alpha \neq 0$. Let σ_k be the permutation that exchanges j and n and then exchanges i with k. Then $\sigma_k(\tau \cdot w - w) = \alpha(e_k - e_n) \in \mathbf{C}[S_n](w)$ too. Since any $w \in W$ can be written as a linear combination of the $e_k - e_n$ terms, and each $e_k - e_n$ is in $\mathbf{C}[S_n](w)$, we have that $\mathbf{C}[S_n](w) = W$.