Mathematics 100A Homework 5
Due: Tuesday November 5 2024

Instructions: Please write clearly and fully explain your solutions. It is OK to work with others to solve the problems, but if you do so, you should write your solutions up separately. Copying solutions from your peers or a solutions manual will be deemed academic misconduct. Chapter and problem numbers, if any, refer to *Algebra*, second edition, by Michael Artin. Please feel free to reach out to me or the TAs if you have any questions.

Suppose n, m are positive integers. Let

$$G_{n,m} = \langle x, y | x^n = 1, y^m = 1, yxy^{-1} = x^{-1} \rangle.$$

1. Suppose m is odd. Prove that $x^2 = 1$ in $G_{n,m}$.
2. Suppose m and n are odd. Prove that $G_{n,m}$ is isomorphic to the cyclic group of order m.
3. Prove that $G_{n,m}$ has at most nm elements.
4. Suppose m is even. Let $\pi : \mathbf{Z}/m\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \simeq \mu_2$ be the natural map, and $\varphi : \mu_2 \to \mathrm{Aut}(\mathbf{Z}/n\mathbf{Z})$ the map with $\varphi(-1)(a + n\mathbf{Z}) = -a + n\mathbf{Z}$. Let $G'_{n,m} = (\mathbf{Z}/n\mathbf{Z}) \rtimes_{\varphi\circ\pi} (\mathbf{Z}/m\mathbf{Z})$ be the semidirect product. Prove that $G_{n,m}$ is isomorphic to $G'_{n,m}$.