Math 170B: Homework 4

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Problem 1

Since $x_k = \frac{k}{3}$, we can scale up all the numbers by 3 using the substitution $x = \frac{1}{3}t$ to get rid of fractions

$$\begin{aligned}
W_0 &= \int_0^1 \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\,dx \\
&= \frac{1}{3}\int_0^3 \frac{(\frac{1}{3}t-\frac{1}{3})(\frac{1}{3}t-\frac{2}{3})(\frac{1}{3}t-1)}{(-\frac{1}{3})(-\frac{2}{3})(-1)}\,dt \\
&= -\frac{1}{18}\int_0^3 (t-1)(t-2)(t-3)\,dt \\
&= -\frac{1}{18}\left[\frac{1}{4}t^4 - 2t^3 + \frac{11}{2}t^2 - 6t\right]_{t=3} \\
&= \frac{1}{8}
\end{aligned}$$

$$\begin{aligned}
W_1 &= \frac{1}{6}\int_0^3 t(t-2)(t-3)\,dt \\
&= \frac{1}{6}\left[\frac{1}{4}t^4 - \frac{5}{3}t^3 + 3t^2\right]_{t=3} \\
&= \frac{3}{8}
\end{aligned}$$

$$\begin{aligned}
W_2 &= \frac{1}{6}\int_0^3 t(t-1)(t-3)\,dt \\
&= -\frac{1}{6}\left[\frac{1}{4}t^4 - \frac{4}{3}t^3 + \frac{3}{2}t^2\right]_{t=3} \\
&= \frac{3}{8}
\end{aligned}$$

$$\begin{aligned}
W_3 &= \frac{1}{18}\int_0^3 t(t-1)(t-2)\,dt \\
&= \frac{1}{18}\left[\frac{1}{4}t^4 - t^3 + t^2\right]_{t=3} \\
&= \frac{1}{8}
\end{aligned}$$

$$\begin{aligned}
\int_0^1 f(x)\,dx &\approx W_0 f(x_0) + W_1 f(x_1) + W_2 f(x_2) + W_3 f(x_3) \\
&= \frac{1}{8}f(x_0) + \frac{3}{8}f(x_1) + \frac{3}{8}f(x_2) + \frac{1}{8}f(x_3)
\end{aligned}$$

Problem 2

Since for polynomials of degree ≤ 4 the Lagrange interpolating polynomial exactly equals the polynomial, we just need to verify that the Newton-Cotes formula equals the given approximation when $a = 0$ and $b = 1$. Using the change of variables $x = a + th$ where $h = \frac{(b-2)}{2}$ and $x_k = a + kh$ for $k = 0, 1, 2, 3, 4$, we get that

$$
\begin{aligned}
W_0 &= \int_a^b \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} \\
&= \frac{h}{24}\int_0^4 (t-1)(t-2)(t-3)(t-4)\,dt \\
&= \frac{h}{24}\left[\frac{1}{5}t^5 - \frac{5}{2}t^4 + \frac{35}{3}t^3 - 25t^2 + 24t\right]_{t=4} \\
&= \frac{14}{45}h
\end{aligned}
$$

$$
\begin{aligned}
W_1 &= \frac{1}{4}\int_0^4 t(t-2)(t-3)(t-4)\,dt \\
&= -\frac{h}{6}\left[\frac{1}{5}t^5 - \frac{9}{4}t^4 + \frac{26}{3}t^3 - 12t^2\right]_{t=4} \\
&= \frac{64}{45}h
\end{aligned}
$$

$$
\begin{aligned}
W_2 &= \frac{1}{4}\int_0^4 t(t-1)(t-3)(t-4)\,dt \\
&= \frac{h}{4}\left[\frac{1}{5}t^5 - 2t^4 + \frac{19}{3}t^3 - 6t^2\right]_{t=4} \\
&= \frac{8}{15}h
\end{aligned}
$$

$$
\begin{aligned}
W_3 &= \frac{1}{4}\int_0^4 t(t-1)(t-2)(t-4)\,dt \\
&= -\frac{h}{6}\left[\frac{1}{5}t^5 - \frac{7}{4}t^4 + \frac{14}{3}t^3 - 4t^2\right]_{t=4} \\
&= \frac{64}{45}h
\end{aligned}
$$

$$
\begin{aligned}
W_4 &= \frac{1}{4}\int_0^4 t(t-1)(t-2)(t-3)\,dt \\
&= \frac{h}{24}\left[\frac{1}{5}t^5 - \frac{3}{2}t^4 + \frac{11}{3}t^3 - 3t^2\right]_{t=4} \\
&= \frac{14}{45}h
\end{aligned}
$$

Thus the general formula(which equals the given approximation when $h = \frac{1}{4}$) is

$$\int_a^b f(x)\,dx = \frac{14}{45}hf(x_0) + \frac{64}{45}hf(x_1) + \frac{8}{15}hf(x_2) + \frac{64}{45}hf(x_3) + \frac{14}{45}hf(x_4)$$

Problem 3

The problem can be solved as a system of equations.

$$f(x) = ae^x + b\cos\left(\frac{\pi}{2}x\right)$$

$$f(0) = a + b$$

$$f(1) = ea$$

$$\begin{aligned}
\int_0^1 ae^x + b\cos\left(\frac{\pi}{2}x\right) &= \left[ae^x + \frac{2b}{\pi}\sin\left(\frac{\pi}{2}x\right)\right]_{x=0}^{x=1} \\
&= \left(ae + \frac{2b}{\pi}\right) - a \\
&= (e-1)a + \frac{2}{\pi}b \\
&= \frac{2}{\pi}f(0) + \frac{(e-1)\pi - 2}{e\pi}f(1)
\end{aligned}$$

which we get from solving

$$\begin{bmatrix} 1 & e \\ 1 & 0 \end{bmatrix}\begin{bmatrix} A_0 \\ A_1 \end{bmatrix} = \begin{bmatrix} e-1 \\ \frac{2}{\pi} \end{bmatrix}$$

Problem 4

$$
\begin{aligned}
W_1 &= \int_a^b \frac{x - \frac{2}{3}}{-\frac{1}{3}} \\
&= -3\left[\frac{1}{2}x^2 - \frac{2}{3}x\right]_a^b \\
&= -\frac{1}{2}((3b^2 - 4b) - (3a^2 - 4a))
\end{aligned}
$$

$$
\begin{aligned}
W_2 &= \int_a^b \frac{x - \frac{1}{3}}{\frac{1}{3}} \\
&= 3\left[\frac{1}{2}x^2 - \frac{1}{3}x\right]_a^b \\
&= \frac{1}{2}((3b^2 - 2b) - (3a^2 - 2a))
\end{aligned}
$$

For generalize a, b with $h = \frac{b-a}{3}$ we have that

$$\int_a^b f(x)\,dx \approx W_1 f\,(a+h) + W_2 f\,(a+2h)$$

When $a = 0$ and $b = 1$ we have that

$$
\begin{aligned}
\int_0^1 f(x)\,dx &\approx W_1 f\left(\frac{1}{3}\right) + W_2 f\left(\frac{2}{3}\right) \\
&= \frac{1}{2} f\left(\frac{1}{3}\right) + \frac{1}{2} f\left(\frac{2}{3}\right)
\end{aligned}
$$

Matlab

```
function I = GaussQuad4(Fun)
    % over points -1, -1/3, 1/3, 1
    W0 = 0.3478548;
    W1 = 0.6521452;
    W2 = 0.6521452;
    W3 = 0.3478548;
    x1 = -0.86113631;
    x2 = -0.33998104;
    x3 = 0.33998104;
    x4 = 0.86113631;

    I = W0*Fun(x1) + W1*Fun(x2) + W2*Fun(x3) + W3*Fun(x4);
end
```

Using the change of variables $t = \frac{1}{3}$ we get that

$$\int_{-3}^{3} -\frac{5}{9}x^2 + 5\,dx = \int_{-1}^{1} -15x^2 + 15\,dx = 20$$

```
>> GaussQuad4(@(x) -15*x^2 + 15)

ans =

20.0000
```