Math 31BH: Assignment 5

Due 02/06 at 23:59

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1. Let $f, g\colon \mathbb{R} \to \mathbb{R}^2$ be differentiable functions, and define $D\colon \mathbb{R} \to \mathbb{R}$ by $D(t) = \det(f(t), g(t))$. Prove that

$$D'(t) = \det(f'(t), g(t)) + \det(f(t), g'(t)).$$

Solution: Since $D(t) = \det(f(t), g(t))$, we can write using component functions that

$$D(t) = \det(\begin{bmatrix} f_1(t) & g_1(t) \\ f_2(t) & g_2(t) \end{bmatrix}) = f_1(t)g_2(t) - f_2(t)g_1(t)$$

Taking the derivative we have that

$$\begin{aligned} D'(t) &= (f_1(t)g_2(t) - f_2(t)g_1(t))' \\ &= f_1(t)g_2'(t) + f_1'(t)g_2(t) - f_2(t)g_1'(t) - f_2'(t)g_1(t) \\ &= (f_1'(t)g_2(t) - f_2'(t)g_1(t)) + (f_1(t)g_2'(t) - f_2(t)g_1'(t)) \\ &= \det(f'(t), g(t)) + \det(f(t), g'(t)) \end{aligned}$$

2. Consider a particle moving in $\mathbb{R}^2$ such that its position at time time $t \in \mathbb{R}$ is given by $f(t) = (t^2, t^3)$.

   (a) Calculate the velocity and speed of the particle at time t.

   (b) Show that there is a unique time at which the particle has zero velocity, and calculate its acceleration vector at this time.

   (c) Write down an integral whose value is the distance traveled by the particle between time $t = -1$ and time $t = 1$. (You need not evaluate your integral, but I will be impressed if you do).

Solution:

(a) The velocity is $f'(t) = (2t, 3t^2)$ and the speed is

$$\|f'(t)\| = \sqrt{(2t)^2 + (3t^2)^2} = \sqrt{4t^2 + 9t^4}$$

(b) $2t = 0$ and $3t^2 = 0$ only have a solution at $t = 0$, so the velocity is only zero when $t = 0$. Since $f''(t) = (2, 6t)$, the acceleration is $f''(0) = (2, 0)$.

(c) The integral is

$$\int_{-1}^{1} \sqrt{4t^2 + 9t^4}\, dt$$

3. Consider a particle traveling along a helix in $\mathbb{R}^3$ such that its position at time $t \in \mathbb{R}$ is

$$f(t) = (at, b\cos\omega t, b\sin\omega t),$$

where a, b, ω are positive constants.

(a) Show that the particle is moving at constant speed.

(b) Show that for all times t the acceleration vector $f''(t)$ is a linear combination of the position vector $f(t)$ and the constant vector $(1, 0, 0)$.

(c) Show that the velocity vector $f'(t)$ and the acceleration vector $f''(t)$ are orthogonal for all times t.

Solution:

(a) The velocity of the particle is $f'(t) = (a, -b\omega\sin\omega t, b\omega\cos\omega t)$ so the speed is

$$\begin{aligned}
\|f'(t)\|^2 &= a^2 + b^2\omega^2\sin^2\omega t + b^2\omega^2\cos^2\omega t \\
&= a^2 + b^2\omega^2(\sin^2\omega t + \cos^2\omega t) \\
&= a^2 + b^2\omega^2
\end{aligned}$$

Since a, b, ω are constants, speed is constant.

(b) Since $f''(t) = (0, -b\omega^2\cos\omega t, -b\omega^2\sin\omega t)$, we can write

$$f''(t) = at\omega^2(1, 0, 0) - \omega^2 f(t)$$

(c) The scalar product of $f'(t)$ and $f''(t)$ is

$$f'(t)\cdot f''(t) = 0 + b^2\omega^3\sin\omega t\cos\omega t - b^2\omega^3\sin\omega t\cos\omega t = 0$$

Therefore, the velocity and acceleration are always orthogonal.