Math 31AH: Fall 2021
Homework 2 Solutions
Due 5:00pm on Friday 10/8/2021

Problem 1: A vector space? Let $\{a, b\}$ be a two-element set and let $V = \{a, b\} \times \mathbb{R}$. Define addition on V by

$$(a, x)+(a, y) := (a, x+y) \quad (a, x)+(b, y) := (a, x+y) \quad (b, x)+(b, y) := (b, x+y)$$

(so that a 'takes precedence over' b). Define scalar multiplication by

$$\lambda \cdot (a, x) := (a, \lambda x) \quad \lambda \cdot (b, x) := (b, \lambda x)$$

for $\lambda \in \mathbb{R}$. Do these operations turn V into a real vector space? Prove your claim.

Solution: No, these operations do not turn V into a real vector space. Working towards a contradiction, suppose that these operations **did** turn V into a real vector space. For any $x \in \mathbb{R}$ we have

$$(a, x) + (b, 0) = (a, x) \quad \text{and} \quad (b, x) + (b, 0) = (b, x)$$

so that the additive identity $\mathbf{0} \in V$ must be $(b, 0)$. However, the element $(a, 0) \in V$ does not have an additive inverse. Indeed, for any $x \in \mathbb{R}$ $(a, 0) + (a, x) = (a, x) \neq (b, 0)$ and $(a, 0) + (b, x) = (a, x) \neq (a, 0)$. We conclude that V is not a real vector space under these operations.

Problem 2: Working with vector space axioms. Let $\mathbb{F}$ be a field and let V be an $\mathbb{F}$-vector space. Suppose $a \in \mathbb{F}$ and $\mathbf{v} \in V$. If $a\mathbf{v} = \mathbf{0}$, prove that $a = 0$ or $\mathbf{v} = \mathbf{0}$.

Solution: Suppose $a \neq 0$. We multiply both sides of $a\mathbf{v} = \mathbf{0}$ by the scalar $a^{-1} \in \mathbb{F}$. The left-hand side becomes

$$a^{-1}(a\mathbf{v}) = (a^{-1}a)\mathbf{v} = 1 \cdot \mathbf{v} = \mathbf{v}$$

while the right-hand side becomes

$$a^{-1} \cdot \mathbf{0} = \mathbf{0}$$

yielding $\mathbf{v} = \mathbf{0}$.

Problem 3: Differentiable functions. Let V be the $\mathbb{R}$-vector space of all differentiable functions $f : \mathbb{R} \to \mathbb{R}$. Define two subsets $U, W \subseteq V$ as follows:

$$U := \{f \in V \,:\, f(3) = 0\} \qquad W := \{f \in V \,:\, f(3) = 7\}$$

Which (if either) of U or W are subspaces of V? Prove your claim.

Solution: We claim that U is a subspace of V, but W is not. Indeed, the zero function $f_0 : \mathbb{R} \to \mathbb{R}$ given by $f_0(x) = 0$ for all $x \in \mathbb{R}$ is the zero vector in V. Since $f_0(3) = 0$ we have $f_0 \in U$. Given $f, g \in U$ we have $(f + g)(3) = f(3) + g(3) = 0 + 0 = 0$ so that U is closed under addition. Given $c \in \mathbb{R}$ and $f \in U$ we have $(cf)(3) = c \cdot f(3) = c \cdot 0 = 0$ so that U is closed under scalar multiplication. We conclude that U is a subspace of V.

On the other hand, W is not a subspace of V. Indeed, the zero function $f_0 : \mathbb{R} \to \mathbb{R}$ satisfies $f_0(3) = 0 \neq 7$ so that $f_0 \notin W$ and W does not contain the zero vector of V.

Problem 4: Lines in the complex plane. For any real number c, define a subset $W_c \subseteq \mathbb{C}$ by

$$W_c := \{x + ic \,:\, x \in \mathbb{R}\}$$

That is, W_c is the set of complex numbers with imaginary part equal to c. For which values of $c \in \mathbb{R}$ is W_c a **real** vector space (under multiplication by real scalars and ordinary addition)? Prove your claim.

Solution: We claim that W_c is a real vector space if and only if $c = 0$. Indeed, if $c \neq 0$ we have $0 \notin W_c$, so that the zero vector is not a member of W_c. When $c = 0$, we have $W_0 = \{x + i \cdot 0 \,:\, x \in \mathbb{R}\} = \mathbb{R}$, so that W_0 **is equal to** the $\mathbb{R}$-vector space $\mathbb{R} = \mathbb{R}^1$.

Problem 5: Eventually zero sequences. An infinite sequence $(a_1, a_2, \dots)$ of real numbers is *eventually zero* if there exists $N \in \mathbb{Z}_{\geq 0}$ such that $a_n = 0$ for all $n > N$.

It can be shown (and you do not have to prove) that the set V of all real sequences $(a_1, a_2, \dots)$ is an $\mathbb{R}$-vector space with addition

$$(a_1, a_2, \dots) + (b_1, b_2, \dots) := (a_1 + b_1, a_2 + b_2, \dots)$$

and scalar multiplication

$$\lambda \cdot (a_1, a_2, \dots) := (\lambda a_1, \lambda a_2, \dots)$$

If $W \subseteq V$ is the subset of eventually zero sequences, is W a subspace of V? Prove your claim.

Solution: Yes, W is a subspace of V. Indeed, the sequence

$$\mathbf{0} := (0, 0, \dots) \in V$$

is the zero vector in V, and certainly eventually zero, so that $\mathbf{0} \in W$. If $(a_1, a_2, \dots) \in W$ and $\lambda \in \mathbb{R}$, choose N such that $n > N$ implies $a_n = 0$. Then $\lambda \cdot (a_1, a_2, \dots) = (\lambda a_1, \lambda a_2, \dots)$ and $n > N$ implies $\lambda a_n = \lambda \cdot 0 = 0$, so that $\lambda \cdot (a_1, a_2, \dots) \in W$. Finally, suppose $(a_1, a_2, \dots), (b_1, b_2, \dots)$ are two sequences in W. Then

$$(a_1, a_2, \dots) + (b_1, b_2, \dots) = (a_1 + b_1, a_2 + b_2, \dots)$$

so it suffices to show that $(a_1+b_1, a_2+b_2, \dots) \in W$. There exists $N_1 > 0$ such that $n > N_1$ implies $a_n = 0$. There also exists $N_2 > 0$ such that $n > N_2$ implies $b_n = 0$. In $N = \max(N_1, N_2)$ and $n > N$ then $a_n + b_n = 0 + 0 = 0$. We conclude that $(a_1+b_1, a_2+b_2, \dots) \in W$ and W is a subspace of V, as desired.

Problem 6: A linear system over $\mathbb{R}$. Solve the following system of linear equations over the real numbers.

$$\begin{cases} 1\cdot x + 2\cdot y + 3\cdot z &= 1 \\ 4\cdot x + 5\cdot y + 6\cdot z &= 1 \\ 7\cdot x + 8\cdot y + 9\cdot z &= 1 \end{cases}$$

Solution: We apply Gaussian Elimination to the system in question as follows:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \\ 7 & 8 & 9 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -6 & -3 \\ 0 & -6 & -12 & -6 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

From this we may read off the solution set

$$\{(x, y, z) = (t-1, -2t+1, t) \,:\, t \in \mathbb{R}\}$$

which forms a line in $\mathbb{R}^3$.

Problem 7: A linear system over $\mathbb{C}$. Solve the following system of linear equations over the complex numbers.

$$\begin{cases} x + iy & = 1 \\ x \qquad + z & = 1 \\ \qquad y - iz & = 2 \end{cases}$$

Solution: We apply Gaussian Elimination to the system in question as follows, recalling the arithmetic of complex numbers:

$$\left(\begin{array}{ccc|c} 1 & i & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & -i & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & i & 0 & 1 \\ 0 & -i & 1 & 0 \\ 0 & 1 & -i & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & i & 0 & 1 \\ 0 & 1 & i & 0 \\ 0 & 1 & -i & 2 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & i & 0 & 1 \\ 0 & 1 & i & 0 \\ 0 & 0 & -2i & 2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & i & 0 & 1 \\ 0 & 1 & i & 0 \\ 0 & 0 & 1 & i \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & i & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & i \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1-i \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & i \end{array}\right)$$

so this system has the unique solution $(x, y, z) = (1 - i, 1, i)$.

Problem 8: A linear system over $\mathbb{F}_2$. Solve the following system of linear equations over the field $\mathbb{F}_2$ with two elements.

$$\begin{cases} x + y \phantom{{}+z} &= 1 \\ x \phantom{{}+y} + z &= 1 \\ \phantom{x+{}} y + z &= 1 \end{cases}$$

(Here the 1's on the right-hand sides are regarded as $1 \in \mathbb{F}_2 = \{0, 1\}$.)

Solution: We apply Gaussian Elimination to the system in question as follows, using the arithmetic of $\mathbb{F}_2$:

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Since this final system contains the equation $0 = 1$, we conclude that the original system has no solutions.[1]

Problem 9: (Optional; not to be handed in.) Prove that the number of solutions to **any** finite system of linear equations over $\mathbb{F}_2$ is either zero, or else a power 2^a of 2.

[1]Over a finite field such as $\mathbb{F}_2$, it is also possible to solve systems by **checking all (finitely many) possible solutions.** In this case, the points $(x, y, z) \in \mathbb{F}_2^3$ are the eight triples $(0, 0, 0), (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)$ and you can directly show that none of them satisfy these three equations (symmetry can make this calculation faster).

It is almost always still faster to use Gaussian Elimination to solve linear systems over finite fields, though (unless the system has some 'special' form). Given a system in n variables over a field with q elements there would be q^n triples to check, and this grows quickly in n.