Math 181A: Homework 9

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Problem 1

1. The likelihood ratio is
$$\frac{L(1)}{L(2)} = \frac{1}{2x}$$
This is a monotonically decreasing function of x. Thus we need to find $x \geq c'$ with probability 0.05. We have that
$$0.05 = \int_{c'}^{1} 1\,dx = 1 - c'.$$
This implies that $c' = 0.95$ so $x \geq 0.95$ is the critical region.

2. Assuming that $\theta = 2$, the probability that $x \geq 0.95$ is
$$\int_{0.95}^{1} 2x\,dx = 1 - 0.95^2 = 0.0975.$$

Problem 2

1. The likelihood ratio is

$$\frac{L(4)}{L(\lambda_1)} = \frac{4e^{-4x}}{\lambda_1 e^{-\lambda_1 x}} = \frac{4}{\lambda_1} e^{(\lambda_1 - 4)x}$$

   This is a monotonically increasing function of x. Thus we need to find $x \leq c'$ with probability 0.05. We have that

$$0.05 = \int_0^{c'} 4e^{-4x}\, dx = -e^{-4c'} + 1.$$

   This implies that $c' = -\frac{\ln(0.95)}{4} = 0.0128$. The critical region is therefore $x \leq 0.0128$.

2. The likelihood ratio is the same, except it is now a monotonically decreasing function of x. Thus we need to find $x \geq c'$ with probability 0.05. We have that

$$0.05 = \int_{c'}^{\infty} 4e^{-4x}\, dx = 0 + e^{-4c'}$$

   This implies that $c' = -\frac{\ln(0.05)}{4} = 0.749$. The critical region is therefore $x \geq 0.749$.

Problem 3

The likelihood ratio is

$$\frac{L(\lambda_0)}{L(\lambda_1)} = \frac{\prod_{i=1}^{n} \frac{\lambda_0^{x_i} e^{\lambda_0}}{x_i!}}{\prod_{i=1}^{n} \frac{\lambda_1^{x_i} e^{\lambda_1}}{x_i!}} = \prod_{i=1}^{n} \frac{\lambda_0^{x_i} e^{\lambda_0}}{\lambda_1^{x_i} e^{\lambda_1}} = e^{\lambda_0 - \lambda_1} \frac{\lambda_0}{\lambda_1}^{n\bar{x}}$$

Since $\lambda_1 > \lambda_0$, the likelihood ratio is a monotonically decreasing function of $\bar{x}$ and so the likelihood ratio test rejects the null hypothesis when $\bar{X} \geq c'$.

Problem 4

Since $\Omega = \mathbb{R}$, the generalized likelihood ratio is

$$\frac{L(\lambda_0)}{L(\hat{\lambda}_{MLE})} = \frac{\prod_{i=1}^{n} \lambda_0 e^{-\lambda_0 x_i}}{\prod_{i=1}^{n} \frac{1}{\bar{x}} e^{-\frac{1}{\bar{x}} x_i}} = \lambda_0^n e^{-\lambda_0 n \bar{x}} \bar{x}^n e^n = (\lambda_0 e \bar{x} e^{-\lambda_0 \bar{x}})^n$$

Since the likelihood ratio is a monotonically increasing function of $\bar{X} e^{-\lambda_0 \bar{X}}$, it can be rewritten as $\bar{X} e^{-\lambda_0 \bar{X}} \leq c'$.

Problem 5: 6.5.1

If k is the sum of all the values of k_i, the generalized likelihood ratio is

$$\begin{aligned}\frac{L(p_0)}{L(\hat{p}_{MLE})} &= \frac{\prod_{i=1}^{n}(1-p)^{k_i-1}p}{\prod_{i=1}^{n}(1-n/k)^{k_i-1}n/k} \\ &= \frac{p^n(1-p)^{k-n}}{(n/k)^n(1-n/k)^{k-n}} \\ &= \left(\frac{p}{nk}\right)^n\left(\frac{1-p}{1-n/k}\right)^{n-k}\end{aligned}$$

Problem 6

1. The ratio is

$$\frac{L(0.5)}{L(\hat{p}_{MLE})} = \frac{\binom{n}{x}0.5^n}{\binom{n}{x}(x/n)^x(1-x/n)^{n-x}}$$
$$= 0.5^n\binom{n}{x}(x/n)^{-x}(1-x/n)^{x-n}$$

2. The critical region is $x = \{0, 1, 11, 12\}$ The significance level is therefore

$$\binom{12}{0}0.5^{12} + \binom{12}{1}0.5^{12} + \binom{12}{11}0.5^{12} + \binom{12}{12}0.5^{12} = 26 \cdot 0.5^{12} = 0.00635$$

R Problem

My code yielded a confidence interval for the mean of $(202.3892, 215.7476)$ and a confidence interval for the median of $(235, 250)$. The estimated probability of being 5 units away from the mean is 0.236.