Math 31BH: Assignment 4

Due 01/30 at 23:59

1. Verify that the following sets $C \subseteq \mathbb{R}^2$ are curves by giving an explicit paramterization, i.e. for each give a continuously differentiable function defined on a subset of $\mathbb{R}$ and taking values in $\mathbb{R}^2$ whose image is C.

   (a) The parabola $C = \{(x, y) \in \mathbb{R}^2 \colon y + 1 = (x - 2)^2\}$.

   Solution: Writing the defining equation of this parabola as $y = (x-2)^2 - 1$, we see immediately that a parameterization is given by $f \colon \mathbb{R} \to \mathbb{R}^2$ defined by $f(t) = (t, (t-2)^2 - 1)$.

   (b) The circle $C = \{(x, y) \in \mathbb{R}^2 \colon (x - 1)^2 + (y - 2)^2 = 4\}$.

   Solution: This is a circle with center $(1, 2)$ and radius 2. Consequently, every point on this circle can be written as

$$(1, 2) + (2\cos t, 2\sin t), \quad t \in [0, 2\pi]$$

   which corresponds to the parameterization

$$f(t) = (1 + 2\cos t, 2 + 2\sin t), \quad t \in [0, 2\pi].$$

   Since the component functions $f_1(t) = 1 + 2\cos t$ and $f_2(t) = 2 + 2\sin t$ of f are continuously differentiable, the function f itself is continuously differentiable, and its derivative is

$$f'(t) = (-2\sin t, 2\cos t).$$

   The parametric equation of the tangent line at the point $f(t) \in C$ is thus

$$\{f(t) + rf'(t) \colon r \in \mathbb{R}\} = \{1 + 2\cos t - 2r\sin t, 2 + 2\sin t + 2r\cos t) \colon r \in \mathbb{R}\}.$$

   (c) The ellipse $C = \{(x, y) \in \mathbb{R}^2 \colon 4x^2 + y^2 = 1\}$.

   Solution: The locus of points $(x, y) \in \mathbb{R}^2$ satisfying the relation

$$\left(\frac{x}{a}\right)^2 + \left(\frac{x}{b}\right)^2 = 1$$

   is the image of the unit circle $x^2 + y^2 = 1$ stretched by the factor $a > 0$ in the horizontal direction, and the factor $b > 0$ in the vertical direction. Equivalently, this is the image of the function $f\colon [0, 2\pi] \to \mathbb{R}^2$ defined by

$$f(t) = (a\cos t, b\sin t), \quad t \in [0, 2\pi].$$

In the particular case at hand, $a = \frac{1}{2}$ and $b = 1$.

(d) The set $C = \{(x, y) \in \mathbb{R}^2 \colon x = |y|\}$.

Solution: The most obvious parameterization $f\colon \mathbb{R} \to \mathbb{R}^2$ defined by

$$f(t) = (|t|, t)$$

doesn't work; the first component function $f_1(t) = t$ is fine, but the second one $f_2(t) = |t|$ is not differentiable at $t = 0$. However, the fact that this fails does not mean that there cannot exist a continuously differentiable parameterization of C. One such is given by the function $g\colon \mathbb{R} \to \mathbb{R}^2$ defined by

$$g(t) = (t|t|, t^2),$$

the component functions of which are $g_1(t) = t|t|$ and $g_2(t) = t^2$. The second component function is a polynomial, so differentiable for all t. The first component function is

$$g_1(t) = \begin{cases} t^2, \ t > 0 \\ 0, \ t = 0 \\ -t^2, t < 0 \end{cases},$$

so is clearly differentiable for all $t \neq 0$. The Newton quotient at $t = 0$ is

$$\frac{g_1(0 + h) - g_1(0)}{h} = \frac{h|h|}{h} = |h|,$$

which has limit 0 as $h \to 0$, so $g_1'(0)$ exists and equals the number 0, so that $g'(0)$ exists and equals the vector $(0, 0)$.

2. For each of the curves C in the previous problem, find the equation of the tangent line at each point on C where it exists, and specify those points at which it does not.

Solution: The tangent line at $f(t)$ exists whenever the tangent vector $f'(t)$ exists and is not the zero vector, and it is then the curve defined by

$$L_{f(t)}(s) = f(t) + sf'(t), \quad s \in \mathbb{R}.$$

For all functions in Problem 1 except the last, the tangent line always exists and it is straightforward to apply the above formula. For the final function, the tangent vector always exists, but it is the zero vector at $t = 0$, so the corresponding tangent line does not exist.