Math 20D HW7

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Section 7.2

Problem 4 For $s > 3$

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t)\,dt \\
&= \int_0^\infty te^{(3-s)t}\,dt \\
&= \lim_{N\to\infty} \left[\left(\frac{t}{3-s} - \frac{1}{(3-s)^2} \right) e^{(3-s)t} \right]_0^N \\
&= \lim_{N\to\infty} \left[\left(\frac{N}{3-s} - \frac{1}{(3-s)^2} \right) e^{(3-s)N} \right] + \frac{1}{(3-s)^2} \\
&= \lim_{N\to\infty} \left[\frac{1}{-(3-s)^2 e^{-(3-s)N}} \right] + \frac{1}{(3-s)^2} \\
&= \frac{1}{(3-s)^2}
\end{aligned}$$

Problem 6 For $s > 0$,

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t)\,dt \\
&= \int_0^\infty e^{-st} \cos bt\,dt \\
&= \lim_{N\to\infty} \left[\frac{e^{-st}(-s\cos bt + b\sin bt)}{s^2+b^2} \right]_0^N \\
&= \frac{s}{s^2+b^2}
\end{aligned}$$

Problem 11 For all s

$$\begin{aligned}
F(s) &= \int_0^\infty e^{-st} f(t)\,dt \\
&= \int_0^\pi e^{-st} \sin t\,dt + \int_\pi^\infty 0\,dt \\
&= \left[-\frac{e^{-st}(s\sin(t) + \cos(t))}{s^2+1} \right]_0^\pi \\
&= \frac{e^{-\pi s}}{s^2+1} + \frac{1}{s^2+1}
\end{aligned}$$

Problem 17 For $s > 3$

$$\begin{aligned}
\mathscr{L}(e^3 t\sin 6t - t^3 + e^t) &= \mathscr{L}(e^3 t \sin 6t) - \mathscr{L}(t^3) + \mathscr{L}(e^t) \\
&= \frac{6}{(s-3)^2+36} - \frac{6}{s^4} + \frac{1}{s-1}
\end{aligned}$$

Problem 18 For $s > 0$

$$\begin{aligned}
\mathscr{L}(t^4 - t^2 - t + \sin\sqrt{2}t) &= \mathscr{L}(t^4) - \mathscr{L}(t^2) - \mathscr{L}(t) + \mathscr{L}(\sin\sqrt{2}t) \\
&= \frac{24}{s^5} - \frac{2}{s^3} - \frac{1}{s^2} + \frac{\sqrt{2}}{s^2+2}
\end{aligned}$$

Problem 23 The graph is only piecewise continuous. **Problem 28** The graph is continuous since

$$\lim_{t\to0} \frac{\sin t}{t} = \lim_{t\to0} \frac{\cos t}{1} = 1$$

Section 7.3

Problem 3 For $s > 6$

$$\begin{aligned}\mathscr{L}(e^{-t}\cos 3t + e^{6t} - 1) &= \mathscr{L}(e^{-t}\cos 3t) + \mathscr{L}(e^{6t}) - \mathscr{L}(1) \\ &= \frac{s+1}{(s+1)^2+9} + \frac{1}{s-6} - \frac{1}{s}\end{aligned}$$

Problem 7 For $s > 0$

$$\begin{aligned}\mathscr{L}((t-1)^4) &= \mathscr{L}(t^4 - 4t^3 + 6t^2 - 4t + 1) \\ &= \frac{24}{s^5} - \frac{24}{s^4} + \frac{12}{s^3} - \frac{4}{s^2} + \frac{1}{s}\end{aligned}$$

Problem 12 For $s > 0$

$$\begin{aligned}\mathscr{L}(\sin 3t\cos 3t) &= \frac{1}{2}\mathscr{L}(\sin 6t) \\ &= \frac{3}{s^2+36}\end{aligned}$$

Problem 16 For $s > 0$

$$\begin{aligned}\mathscr{L}(t\sin^2 t) &= \frac{1}{2}\mathscr{L}(t - t\cos 2t) \\ &= \frac{1}{2s^2} + \frac{d}{ds}(\frac{s}{s^2+4}) \\ &= \frac{1}{2s^2} + \frac{4-s^2}{2(s^2+4)^2}\end{aligned}$$

Problem 17 For $s > 0$

$$\begin{aligned}\mathscr{L}(\sin 2t\sin 5t) &= \frac{1}{2}\mathscr{L}((\cos 3t - \cos 7t)) \\ &= \frac{s}{2(s^2+9)} - \frac{s}{2(s^2+49)}\end{aligned}$$

Problem 21 Substituting in $s - a$ yields

$$\mathscr{L}(e^{at}f(t))(s) = \frac{s-a}{((s-a)^2+b^2)}$$

Problem 26
For $t \geq T$, $|f(t)| \leq Me^{\alpha t}$. For $0 \leq t < T$, the function is bounded since since each piecewise function is continuous and the endpoints have limits. If the upperbound of $0 \leq t < T$ is N, then $|f(t)| \leq Ke^{\alpha t}$ where $k = \max\{M, N\}$.

Since this function is bounded by $Ke^{\alpha t}$

$$\begin{aligned}\lim_{s\to\infty} |\mathscr{L}\{f(t)\}(s)| &\leq \lim_{s\to\infty}\int_0^\infty Ke^{\alpha t}e^{-st} \\ &= \lim_{s\to\infty}\frac{K}{s-\alpha} \\ &= 0\end{aligned}$$

Problem 31 Using the definition,

$$\mathscr{L}\{g\}(s) = \int_c^\infty e^{-st}f(t-c)\,dt = \int_0^\infty e^{-st-cs}f(t)\,dt = e^{-cs}\mathscr{L}\{f\}(s)$$

Section 7.4

Problem 9 Completing the square

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{3s-15}{2s^2-4s+10}\right\} &= \mathscr{L}^{-1}\left\{\frac{\frac{3}{2}(s-1)}{(s-1)^2+4}\right\} - \mathscr{L}^{-1}\left\{\frac{6}{(s-1)^2+4}\right\} \\ &= \frac{3}{2}e^t\cos 2t - 3e^t\sin 2t\end{aligned}$$

Problem 12 Solving out for the constants

$$\begin{aligned}\frac{-s-7}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2} &\implies -s-7 = A(s-2)+B(s+1) \\ &\implies -s-7 = (A+B)s - 2A + B \\ &\implies A=2, B=-3\end{aligned}$$

The partial fraction expansion is

$$\frac{-s-7}{(s+1)(s-2)} = \frac{2}{s+1} - \frac{3}{s-2}$$

Problem 13 Solving out for the constants

$$\begin{aligned}\frac{-2s^2-3s-2}{s(s+1)^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2} &\implies -2s^2-3s-2 = A(s+1)^2 + Bs(s+1) + Cs \\ &\implies -2s^2-3s-2 = (A+B)s^2 + (2A+B+C)s + A \\ &\implies A=-2, B=0, C=1\end{aligned}$$

The partial fraction expansion is

$$\frac{-2s^2-3s-2}{s(s+1)^2} = \frac{1}{(s+1)^2} - \frac{2}{s}$$

Problem 22 Using partial fraction decomposition,

$$\begin{aligned}\frac{s+11}{(s-1)(s+3)} = \frac{A}{s-1} + \frac{B}{s+3} &\implies s+11 = A(s+3)+B(s-1) \\ &\implies s+11 = (A+B)s + 3A - B \\ &\implies A=3, B=-2\end{aligned}$$

The inverse laplacian is,

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{s+11}{(s-1)(s+3)}\right\} &= \mathscr{L}^{-1}\left\{\frac{3}{s-1} - \frac{2}{s+3}\right\} \\ &= 3e^t - 2e^{-3t}\end{aligned}$$

Problem 23 Using partial fraction decomposition,

$$\begin{aligned}\frac{5s^2+34s+53}{(s+3)^2(s+1)} = \frac{A}{s+1} + \frac{B}{s+3} + \frac{C}{(s+3)^2} &\implies 5s^2+34s+53 = A(s+3)^2 + B(s+1)(s+3) + C(s+1) \\ &\implies 5s^2+34s+53 = (A+B)s^2 + (6A+4B+C)s + 9A + 3B + C \\ &\implies A=6, B=-1, C=2\end{aligned}$$

The inverse laplacian is

$$\begin{aligned}\mathscr{L}^{-1}\left\{\frac{5s^2+34s+53}{(s+3)^2(s+1)}\right\} &= \mathscr{L}^{-1}\left\{\frac{6}{s+1} - \frac{1}{s+3} + \frac{2}{(s+3)^2}\right\} \\ &= 6e^{-t} - e^{-3t} + 2te^{-3t}\end{aligned}$$

Problem 27 Using partial fraction decomposition,

$$\frac{5}{(s+1)(s-2)(s+2)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{C}{s+2} \implies 5 = A(s-2)(s+2) + B(s+1)(s+2) + C(s+1)(s-2)$$
$$\implies 5 = (A+B+C)s^2 + (3B-C)s - 4A + 2B - 2C$$
$$\implies A = -\frac{5}{3}, B = \frac{5}{12}, C = \frac{5}{4}$$

The inverse laplacian is

$$\mathscr{L}^{-1}\left\{-\frac{5}{3}\frac{1}{s+1} + \frac{5}{12}\frac{1}{s-2} + \frac{5}{4}\frac{1}{s+2}\right\} = -\frac{5}{3}e^{-t} + \frac{5}{12}e^{2t} + \frac{5}{4}e^{-2t}$$

Problem 28 Using partical fraction decomposition,

$$\frac{s^2+4}{s(s+1)(s+3)(s-2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+3} + \frac{D}{s-2} \implies s^2 + 4 = A(s+1)(s+3)(s-2) + Bs(s+3)(s-2) + Cs(s+1)$$
$$\implies A = -\frac{2}{3}, B = \frac{5}{6}, C = -\frac{13}{3}, D = \frac{4}{15}$$

The inverse laplacian is

$$\mathscr{L}^{-1}\left\{-\frac{2}{3}\frac{1}{s} + \frac{5}{6}\frac{1}{s+1} - \frac{13}{3}\frac{1}{s+3} + \frac{4}{15}\frac{1}{s-2}\right\} = -\frac{2}{3} + \frac{5}{6}e^{-t} - \frac{13}{30}e^{-3t} + \frac{4}{15}e^{2t}$$