Homework due Friday, October 6, at 11:00 pm Pacific Time.

A. Let
$$E = \left\{ \frac{5n+8}{11n} : n \in \mathbb{N} \right\}.$$
Compute $\sup E$ and $\inf E$. Justify your answer.

B. Let S and T be two bounded subsets of the real numbers. Prove that
$$\sup\bigl(T \cup S\bigr) = \max\bigl\{\sup T, \sup S\bigr\}.$$

C. Let S and T be two bounded, nonempty, subsets of the set of positive real numbers. Define $ST := \{st : s \in S, t \in T\}$ and $S + T := \{s + t : s \in S, t \in T\}$. Prove that
$$\sup(ST) = (\sup S) \cdot (\sup T) \quad \text{and} \quad \sup(S + T) = \sup S + \sup T.$$

D. Let F be the set of all rational functions
$$\frac{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0} \tag{1}$$
where the coefficients are real numbers and $b_m \neq 0$.

(i) Define addition and multiplication of two elements in F to be the usual addition and multiplication of functions. Show that with this addition and multiplication, F is a field.

(ii) We can define an order on F as follows. A rational function like (1) is positive if and only if a_n and b_m have the same sign, i.e. $a_n b_m > 0$. Now given two rational functions $\frac{p}{q}$ and $\frac{f}{g}$ we define:
$$\frac{p}{q} > \frac{f}{g} \text{ if and only if } \frac{p}{q} - \frac{f}{g} > 0.$$
Show with this ordering and the operations in part (i), F is an ordered field.

(iii) Write the following polynomials in order of increasing size using the order define in (ii): $x^2,\ -x^5,\ 2,\ x+6,\ 3-2x$.

(iv) Show that $x > a$ for all $a \in \mathbb{R}$.

E. Rudin, Chapter 1 (page 21), problems 1, 2, 5, 8.