MATH 31AH - Homework 3

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1 Direct Sum

Proof. $\mathcal{B} \oplus \mathcal{C}$ is a basis for $V \oplus W$.

Let $(a, b) \in V \oplus W$. $(a, 0)$ can be written as a linear combination of vectors in $\{(v, 0) : v \in \mathcal{B}\}$ since $\mathcal{B}$ spans V. Simmilarly, $(0, b)$ can be written as a linear combinations of vectors in $\{(0, w) : w \in \mathcal{C}\}$ since $\mathcal{C}$ spans W. Since $(a, 0) + (0, b) = (a, b)$, (a, b) can be written as a linear combination of vectors in $\{(v, 0) : v \in \mathcal{B}\} \cup \{(0, w) : w \in \mathcal{C}\}$. Therefore, $\mathcal{B} \oplus \mathcal{C}$ spans $V \oplus W$.

The only way for a linear combination of $\mathcal{B} \oplus \mathcal{C}$ to be zero is if the linear combination of the vectors in $\{(v, 0) : v \in \mathcal{B}\}$ is zero and the linear combination of vectors in $\{(0, w) : w \in \mathcal{C}\}$ is zero. Since $\mathcal{B}$ is linearly independent, and $\mathcal{C}$ is linearly independent, The only linear combination of vectors in $\mathcal{B} \oplus \mathcal{C}$ that are zero is the trivial linear combination. Therefore, $\mathcal{B} \oplus \mathcal{C}$ is linearly independent.

Since $\mathcal{B} \oplus \mathcal{C}$ is linearly independent, and it spans $V \oplus W$, $\mathcal{B} \oplus \mathcal{C}$ is a basis for $V \oplus W$. $\square$

2 Real sequences

Proof. S spans V.

Any sequence $a = (a_1, a_2, ...)$ in V can be written as the linear combination $a_1e_1 + a_2e_2 + ...$ Therefore, S spans V. $\square$

3 A basis for polynomials

Proof. $\mathcal{B}$ is a basis for V.

Trivially, $\{1\}$ is a basis for polynomials of degree 0. Assume that $\{1, (t+1), (t+1)^2, ..., (t+1)^{n-1}\}$ is a basis for polynomials of degree $n-1$. Adding $(t+1)^n$ to this set makes this set span polynomials of degree n since a polynomial of degree n can be written as $a_n(t+1)^n + v$ where v is a polynomial of degree $n-1$. $(t+1)^n$ is also linearly independent from the other polynomials in the set since the degree of $(t+1)^n$ is higher than all the other polynomials.

Thus, $\{1, (t+1), (t+1)^2, ..., (t+1)^{n-1}, (t+1)^n\}$ is a basis for polynomials of degree n. This completes the inductive step, and so $\mathcal{B}$ is a basis for V. $\square$

4 Real-valued functions

Let I be the set of all functions $f(x) = x^n$ with $n \in \mathbb{Z}$. These functions are differentiable due to the power rule, and they are also independent since each of them have a different degree.

5 Homogenous systems

Proof. The solution set of a homogeneous system is a subspace of $\mathbb{F}^n$.

Let x and y be solutions to the homogeneous system $Ax = 0$. This implies that for every row i, $a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n = 0$ (or $Ax = 0$) and $a_{i1}y_1 + a_{i2}y_2 + ... + a_{in}y_n = 0$ (or $Ay = 0$). Adding these two equations together implies that $a_{i1}(x_1 + y_1) + a_{i2}(x_2 + y_2) + ... + a_{in}(x_n + y_n) = 0$ (or $A(x + y) = 0$) for every row i, implying that $x + y$ is also a solution to the homogeneous system. Therefore, the solution set of a homogeneous system is closed under addition.

Let x be a solution to the homogeneous system $Ax = 0$, and let $c \in \mathbb{F}$ be a scalar. This implies that for every row i, $a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n = 0$ (or $Ax = 0$). Multiplying this equation by c yields $a_{i1}(cx_1) + a_{i2}(cx_2) + ... + a_{in}(cx_n) = 0$ (or $A(cx) = 0$) for every row i, implying that cx is also a solution to the homogeneous ystem. Therefore, the solution set of a homogeneous system is closed under scalar multiplication.

Since solutions of $Ax = 0$ is a member of $\mathbb{F}^n$ and addition and scalar multiplication are closed, the solution set of a homogeneous system is a subspace of $\mathbb{F}^n$. $\square$

6 Particular solutions

Proof. $\{x \in \mathbb{F}^n : Ax = b\} = \{x_0 + w : w \in W\}$

Let x be an arbitrary solution to the system of linear equations, and let x_0 be a particular solution to the system of equations. This means that $Ax = b$ and $Ax_0 = b$. Subtracting the equations from each other yields $Ax - Ax_0 = 0$, which implies $A(x - x_0) = 0$. This means that $(x - x_0) \in W$ implying that $x - x_0 = w$ for some $w \in W$. Adding x_0 to both sides yields $x = x_0 + w$. Since x was arbitrary, all solutions to $Ax = b$ are in $\{x_0 + w : w \in W\}$

For all $w \in W$, $x_0 + w$ is a solution to $Ax = b$. Since $Ax_0 = b$ and $Aw = 0$, we have that $Ax_0 + Aw = b$. Factoring by A, we have that $A(x_0 + w) = b$. Thus all vectors in $\{x_0 + w : w \in W\}$ are solutions to $Ax = B$.

Since all solutions to $Ax = B$ can be expressed as $x_0 + w$, and all vectors expressed as $x_0 + w$ are solutions to $Ax = B$, we have that $\{x \in \mathbb{F}^n : Ax = b\} = \{x_0 + w : w \in W\}$. $\square$

7 Completing a basis

Proof. There exists vectors $v_{s+1}, v_{s+2}, ..., v_n$ such that $\{v_1, ..., v_s, v_{s+1}, ..., v_n\}$ is a basis of V.

Let $\mathcal{S} = \{v_1, v_2, ..., v_s\}$ be a linearly independent subset of the n-dimensional vector space V. From lecture 8, the basis for an n-dimensional vector space must be have n elements. Since the dimension of $\mathcal{S}$ is less than n, it cannot be a basis for V, and it cannot span V. Therefore there exists some vector $v_{s+1} \in V$ that is not in the span of $\mathcal{S}$, and is independent from all other vectors in $\mathcal{S}$. This vector can be added to $\mathcal{S}$ and it will still be linearly independent. A total of $n-s$ vectors can be added to $\mathcal{S}$ in a simmilar fashion until the size of $\mathcal{S}$ is n-dimensions. At this point, since $\mathcal{S}$ will have n vectors, and it is linearly independent, from the theorem of lecture 8, $\mathcal{S}$ will be a basis for V. Thus there exists vectors $v_{s+1}, v_{s+2}, ..., v_n$ such that $\{v_1, ..., v_s, v_{s+1}, ..., v_n\}$ is a basis of V. $\square$

8 Trimming down to a basis

Proof. A subset $\{v_1, ..., v_m\}$ can be "trimmed down" to a basis $\{v_{i_1}, ..., v_{i_n}\}$ of V.

Let $\mathcal{S} = \{v_1, ..., v_m\}$ be a subset of V that spans V. If $\mathcal{S}$ is linearly independent, then it is already a basis for V, and it must have n-elements as shown in lecture 8. This means that $\mathcal{S}$ is a basis $\{v_{i_1}, ..., v_{i_n}\}$ of V. If $\mathcal{S}$ is linearly dependent, then that means a nontrivial linear combination of vectors in $\mathcal{S}$ is zero, meaning $a_1v_1 + a_2v_2 + ... + a_mv_m = 0$. This means that some vector v_i with a nonzero coefficient a_i can be formed as a linear combination of the other vectors, i.e. $-\frac{a_1}{a_i}v_1 - \frac{a_2}{a_i}v_2 - ... - \frac{a_m}{a_i}v_m = v_i$.

If v_i is removed from $\mathcal{S}$, then it will still span V, since any linear combination with v_i can substituted in with the linear combination $-\frac{a_1}{a_i}v_1 - \frac{a_2}{a_i}v_2 - ... - \frac{a_m}{a_i}v_m$. Vectors can be removed from $\mathcal{S}$ until $\mathcal{S}$ is linearly independent. At this point, $\mathcal{S}$ will be a basis for V since it is linearly independent and it still spans V. $\mathcal{S}$ cannot have more than n elements at this point because that would imply the existence of a basis with more elements than the vector space, which was shown in lecture 8 to be impossible. Thus, $\mathcal{S}$ would have to have n elements, and so a basis $\{v_{i_1}, ..., v_{i_n}\}$ of V exists. $\square$

9 Making a $\mathbb{F}_2$-vector space(Optional)

V can be thought of as isomorphic to an $n+1$ dimension vector space of $\mathbb{F}_2$ where each vector-index indicates the presense or absense of the number of the index in the set. Addition can be thought of as "cancelling out" indexes where both vectors have a 1 since $1+1=0$ in $\mathbb{F}_2$. Scalar multiplication can be defined as returning the empty set if the scalar is 0, and returning the set if the scalar is 1 in order to complete the isomorphism.