Homework due Friday, October 13, at 11:00 pm Pacific Time.

A. Let $A \subset \mathbb{R}$ be a non-empty subset which satisfies the following two properties

(1) $A = A + A$, and
(2) For every $\epsilon > 0$, there exists some $a \in A$ so that $0 < a < \epsilon$.

Prove that for every $x \in \mathbb{R}^{>0}$, there exists some $a \in A$ so that

$$0 < x - a < \epsilon$$

B. Let $a, b, c, d \in \mathbb{R}$ and assume $a < b$ and $c < d$. Give an explicit one-to-one correspondence between

(1) The points of the two open intervals (a, b) and (c, d).
(2) The points of the two closed intervals $[a, b]$ and $[c, d]$.
(3) The points of the closed interval $[a, b]$ and the open interval (c, d).
(4) The points of the closed interval $[a, b]$ and $\mathbb{R}$.

C. Rudin, Chapter 1 (page 21), problems 7, 13, 14, 17.

Section B: Extra practice problems: Problems in section B are for your practice; please do not hand them in. However, it is extremely important that you feel comfortable with these problems as some of them may appear on the exams.

For any positive integer n, define

$$\mathbb{C}^n = \{(z_1, \ldots, z_n) : z_j \in \mathbb{C}\}.$$

For any two elements $\mathbf{z} = (z_1, \ldots, z_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$ in $\mathbb{C}^n$, define

$$\mathbf{z} + \mathbf{w} = (z_1 + w_1, \ldots, z_n + w_n).$$

For any $\mathbf{z} = (z_1, \ldots, z_n)$ in $\mathbb{C}^n$ and any $\lambda \in \mathbb{C}$, define

$$\lambda \mathbf{z} = (\lambda z_1, \ldots, \lambda z_n).$$

Define a scalar product on $\mathbb{C}^n$ as follows. Let $\mathbf{z} = (z_1, \ldots, z_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$ be in $\mathbb{C}^n$, define

$$(\mathbf{z}, \mathbf{w}) = \sum_{j=1}^{n} z_j \overline{w}_j.$$

Let $\mathbf{z}, \mathbf{w} \in \mathbb{C}^n$ and let $\lambda \in \mathbb{C}$.

(1) Show that $(\lambda \mathbf{z}, \mathbf{w}) = \lambda(\mathbf{z}, \mathbf{w}) = (\mathbf{z}, \bar{\lambda} \mathbf{w})$.

(2) Show that $(\mathbf{z}, \mathbf{z})$ is a non-negative real number. Moreover, $(\mathbf{z}, \mathbf{z}) = 0$ if and only if $\mathbf{z} = (0, \ldots, 0)$.

(3) Show that $(\mathbf{z}, \mathbf{w}) = \overline{(\mathbf{w}, \mathbf{z})}$.

(4) Assume $\mathbf{w} \neq (0, \ldots, 0)$ and let $\mathbf{u} = \mathbf{z} - \frac{(\mathbf{z},\mathbf{w})}{(\mathbf{w},\mathbf{w})}\mathbf{w}$. Show that $(\mathbf{u}, \mathbf{w}) = 0$.

(5) Compute $(\mathbf{u}, \mathbf{u})$. Then use part (2) to prove the Cauchy-Schwarz inequality.