Math 140C: Homework 2

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Rudin 9.6

When $(x, y) = \mathbf{0}$, the partial derivatives are 0 since $f(0,0) = 0$ and

$$(D_1 f)(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

$$(D_2 f)(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = 0$$

When $(x, y) \neq \mathbf{0}$ the partial derivative can be taken by holding the other variable constant.

$$f(x,y) = \frac{xy}{x^2 + y^2}$$

$$(D_1 f)(x,y) = \frac{(x^2 + y^2)y - 2x^2 y}{(x^2 + y^2)^2}$$

$$(D_2 f)(x,y) = \frac{(x^2 + y^2)x - 2xy^2}{(x^2 + y^2)^2}$$

However, $f(x,x) = \frac{1}{2}$ for all x, but $f(0,0) = 0$, so f is not continuous at $(0,0)$.

Rudin 9.7

Fix $\boldsymbol{x} \in E$ and $\epsilon > 0$. Since E is open, there is an open ball $S \subset E$ centered at $\boldsymbol{x}$ with radius r. Since the partial derivatives are bounded, there exists M such that

$$|(D_j f)(\boldsymbol{y}) - (D_j f)(\boldsymbol{x})| < M \quad (\boldsymbol{y} \in S,\, 1 \leq j \leq n).$$

Suppose $\boldsymbol{h} = \sum h_j e_j$, $|\boldsymbol{h}| < r$ and put $\boldsymbol{v_k} = \sum_{i=1}^{k} h_j \boldsymbol{e_i}$. Since $\boldsymbol{v_0} = 0$ and $\boldsymbol{v_n} = \boldsymbol{h}$, we can rewrite

$$f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x}) = \sum_{j=1}^{n} [f(\boldsymbol{x}+\boldsymbol{v_j}) - f(\boldsymbol{x}+\boldsymbol{v_{j-1}})]$$

Since $|\boldsymbol{v_k}| < r$ and S is convex, the segments with end points $\boldsymbol{x}+\boldsymbol{v_{j-1}}$ and $\boldsymbol{x}+\boldsymbol{v_j}$ lie in S. By the mean value theorem,

$$\begin{aligned}
|f(\boldsymbol{x}+\boldsymbol{v_j}) - f(\boldsymbol{x}+\boldsymbol{v_{j-1}})| &= |f(\boldsymbol{x}+\boldsymbol{v_{j-1}}+h_j\boldsymbol{e_j}) - f(\boldsymbol{x}+\boldsymbol{v_{j-1}})| \\
&= |h_j (D_j f)(\boldsymbol{x}+\boldsymbol{v_{j-1}}+\theta_j h_j \boldsymbol{e_j})| \\
&< |\boldsymbol{h}|M
\end{aligned}$$

for some $\theta_j \in (0,1)$ since $\boldsymbol{x}+\boldsymbol{v_{j-1}}+\theta_j h_j \boldsymbol{e_j}$ is in the line segment between $\boldsymbol{v_{j-1}}$ and $\boldsymbol{v_j}$.

Thus,

$$\begin{aligned}
|f(\boldsymbol{x}+\boldsymbol{h}) - f(\boldsymbol{x})| &= \left| \sum_{j=1}^{n} [f(\boldsymbol{x}+\boldsymbol{v_j}) - f(\boldsymbol{x}+\boldsymbol{v_{j-1}})] \right| \\
&< \left| \sum_{j=1}^{n} |\boldsymbol{h}|M \right| \\
&= nM|\boldsymbol{h}| \\
&= nM|(\boldsymbol{x}+\boldsymbol{h}) - (\boldsymbol{x})|
\end{aligned}$$

so f is Lipschitz continuous.

Rudin 9.8

Choose a direction $\boldsymbol{u} \in \mathbb{R}^n$ and let $\varphi(t) = f(\boldsymbol{x}+t\boldsymbol{u})$. Since $\varphi'(t) = f'(\boldsymbol{x}+t\boldsymbol{u})(\boldsymbol{u})$ and $\varphi'(0) = 0$, we have that $f'(\boldsymbol{x})(\boldsymbol{u}) = 0$. Thus, $f'(\boldsymbol{x}) = 0$ since $\boldsymbol{u}$ was arbitrary.

Rudin 9.9

Let $\boldsymbol{x} \in E$. By the Corollary to theorem 9.19, there is an open ball containing $\boldsymbol{x}$ where f is constant. If $f(\boldsymbol{y}) = f(\boldsymbol{x})$, then there is also an open ball containing $\boldsymbol{y}$ which is constant. Therefore the set $\{\boldsymbol{y} | f(\boldsymbol{y}) = f(\boldsymbol{x})\}$ is open since its the union of open balls.

Similarly the complement of this set is also open since for any point $f(\boldsymbol{y}) \neq f(\boldsymbol{x})$, we can draw an open ball containing $\boldsymbol{y}$ that is constant.

Since E is connected and $\{\boldsymbol{y} | f(\boldsymbol{y}) = f(\boldsymbol{x})\}$ is clopen, it must be that $E = \{\boldsymbol{y} | f(\boldsymbol{y}) = f(\boldsymbol{x})\}$, meaning function is constant.

Rudin 9.10

Let $\boldsymbol{x} = [x_1, x_2, \cdots, x_n]$ and $\boldsymbol{y} = [x_1 + h, x_2, \cdots, x_n]$ for arbitrary $h \in \mathbb{R}$. Let $g(t) = f(\boldsymbol{x} + t\boldsymbol{e_1})$, which exists for all $t \in [0, h]$ since E is convex. By MVT, we know that $f(\boldsymbol{x}) = f(\boldsymbol{y})$ since

$$f(\boldsymbol{x}) - f(\boldsymbol{y}) = g(h) - g(0) = h(D_1 f)(\boldsymbol{x}) = 0.$$

Since $f(\boldsymbol{x}) = f(\boldsymbol{y})$ for arbitrary $\boldsymbol{x}$ and $\boldsymbol{y}$, $f(\boldsymbol{x})$ can only depend on $x_2, \cdots, x_n$.

The set only has to be convex in the first dimension, but if no condition is placed, then a function can "jump" in value across the gap of, say, a horseshoe since the derivative is a local property.

Rudin 9.11

$$\nabla(fg) = \begin{bmatrix} (D_1(fg))(\boldsymbol{x}) \\ (D_2(fg))(\boldsymbol{x}) \\ \vdots \\ (D_n(fg))(\boldsymbol{x}) \end{bmatrix} = \begin{bmatrix} f(\boldsymbol{x})(D_1 g)(\boldsymbol{x}) \\ f(\boldsymbol{x})(D_2 g)(\boldsymbol{x}) \\ \vdots \\ f(\boldsymbol{x})(D_n g)(\boldsymbol{x}) \end{bmatrix} + \begin{bmatrix} g(\boldsymbol{x})(D_1 f)(\boldsymbol{x}) \\ g(\boldsymbol{x})(D_2 f)(\boldsymbol{x}) \\ \vdots \\ g(\boldsymbol{x})(D_n f)(\boldsymbol{x}) \end{bmatrix} = f\nabla g + g\nabla f$$

$$\nabla(1/f) = \begin{bmatrix} (D_1(1/f))(\boldsymbol{x}) \\ (D_2(1/f))(\boldsymbol{x}) \\ \vdots \\ (D_n(1/f))(\boldsymbol{x}) \end{bmatrix} = \begin{bmatrix} (-f^{-2}(\boldsymbol{x}D_1(f))(\boldsymbol{x}) \\ (-f^{-2}\boldsymbol{x}D_2(f))(\boldsymbol{x}) \\ \vdots \\ (-f^{-2}\boldsymbol{x}D_n(f))(\boldsymbol{x}) \end{bmatrix} = -f^{-2}\nabla f$$