Math 31CH HW1 SOLUTIONS
Due April 5 at 11:59 pm by Gradescope Submission

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In-depth answers. Although you are not expect to present solutions at quite this level of detail, they are presented for increase understanding.

EXERCISES FOR SECTION 4.1

Exercise 4.1.9

Let $Q \subset \mathbb{R}^2$ be the unit square $0 \leq x, y < 1$.[1] Show that the function

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \sin(x-y)\,\mathbf{1}_Q\begin{pmatrix} x \\ y \end{pmatrix}$$

is integrable by providing an explicit bound for $U_N(f) - L_N(f)$ that tends to 0 as $N \to \infty$.

Solution. In essence, there reason we can get an explicit bound easily is that f is a Lipschitz continuous function. Consider a dyadic cube (square actually) C of level N. Each of its two dimensions (side lengths) is equal to $\frac{1}{2^N}$. A key fact is that the sine function satisfies the Lipschitz constant 1 estimate

$$|\sin(u) - \sin(v)| \leq |u - v|.$$

If you want to see this rigorously, just observe that $|\sin'(w)| = |\cos(w)| \leq 1$, and apply the mean value theorem.

Let (x_1, y_1) and (x_2, y_2) be points in the same dyadic square C of level N. Then

$$|x_1 - x_2| < \frac{1}{2^N}, \qquad |y_1 - y_2| < \frac{1}{2^N}.$$

We will find this estimate very useful. The dyadic squares of level N are

$$C_{\mathbf{k},N} = \left\{(x,y) \;\middle|\; \tfrac{k_1}{2^N} \leq x < \tfrac{k_1+1}{2^N},\ \tfrac{k_2}{2^N} \leq y < \tfrac{k_2+1}{2^N}\right\}$$

for $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2$. The unit square Q is the disjoint union of the dyadic squares $C_{\mathbf{k},N}$ of level N for

$$0 \leq k_1 < 2^N, \qquad 0 \leq k_2 < 2^N.$$

[1] Note that we slightly changed the exercise from the book's version, which actually makes it simpler.

Let (x_1, y_1) and (x_2, y_2) be points in the same dyadic square $C = C_{\mathbf{k},N}$ of level N and assume that $C \subset Q$. Then

$$\begin{aligned}
|f(x_1, y_1) - f(x_2, y_2)| &= |\sin(x_1 - y_1) - \sin(x_2 - y_2)| \\
&\leq |(x_1 - y_1) - (x_2 - y_2)| \\
&= |(x_1 - x_2) - (y_1 - y_2)| \\
&\leq |x_1 - x_2| - |y_1 - y_2| \\
&< \frac{1}{2^N} + \frac{1}{2^N} \\
&= \frac{1}{2^{N-1}}.
\end{aligned}$$

Thus, for each dyadic square $C_{\mathbf{k},N}$ intersecting Q we have that

$$\operatorname{osc}_{C_{\mathbf{k},N}}(f) \leq \frac{1}{2^{N-1}}.$$

From this we obtain that

$$\begin{aligned}
U_N(f) - L_N(f) &= \sum_{\mathbf{k} \in \mathbb{Z}^2} \operatorname{osc}_{C_{\mathbf{k},N}}(f) \operatorname{Area}(C_{\mathbf{k},N}) \\
&= \sum_{C_{\mathbf{k},N} \cap Q \neq \emptyset} \operatorname{osc}_{C_{\mathbf{k},N}}(f) \operatorname{Area}(C_{\mathbf{k},N}) \\
&\leq \frac{1}{2^{N-1}} \sum_{C_{\mathbf{k},N} \cap Q \neq \emptyset} \operatorname{Area}(C_{\mathbf{k},N}) \\
&= \frac{1}{2^{N-1}} \operatorname{Area}(Q) \\
&= \frac{1}{2^{N-1}}.
\end{aligned}$$

This gives an explicit bound for $U_N(f) - L_N(f)$. Since $\frac{1}{2^{N-1}} \to 0$ as $N \to \infty$, we conclude that f is integerable.

Exercise 4.1.10

a. What are the upper and lower sums $U_1(f)$ and $L_1(f)$ for the function

$$f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} x^2 + y^2 & \text{if } 0 < x, y < 1, \\ 0 & \text{otherwise,} \end{cases}$$

i.e., the upper and lower sums for the partition $\mathcal{D}_1(\mathbb{R}^2)$, shown in the figure at left (below actually)?

Solution to (a). Define $g : \mathbb{R}^2 \to \mathbb{R}$ by $g(\mathbf{x}) = x^2 + y^2$. Then $f = g\,\mathbf{1}_Q$, where

$$Q := (0, 1) \times (0, 1) = \{(x, y) \in \mathbb{R}^2 \,|\, 0 < x, y < 1\}.$$

Notice that g has the following monotonicity property:

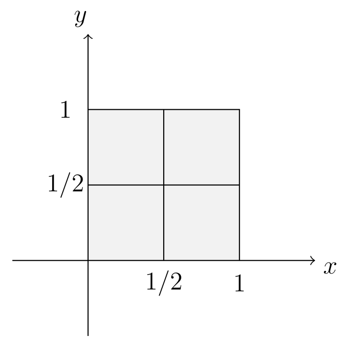

Figure 1: Figure for Exercise 4.1.10.

(1) If $x_1 \le x_2$, then $g(x_1, y) \le g(x_2, y)$.

(2) If $y_1 \le y_2$, then $g(x, y_1) \le g(x, y_2)$.

It is more concise to formulate this as:

If $x_1 \le x_2$ and $y_1 \le y_2$, then $g(x_1, y_1) \le g(x_2, y_2)$.

Recall that if $\mathbf{k} = (k_1, k_2) \in \mathbb{Z}^2$, then the corresponding dyadic cube (square actually since in dimension 2) of level 1 is given by

$$C_{\mathbf{k}} := C_{\mathbf{k},1} = \left\{(x, y) \,\middle|\, \tfrac{k_1}{2} \le x < \tfrac{k_1+1}{2},\ \tfrac{k_2}{2} \le y < \tfrac{k_2+1}{2}\right\}.$$

We have suppressed the "1" in the subscript for simplicity.

Note that $f > 0$ on Q, and $f = 0$ outside of Q. We have

$$\operatorname{supp}(f) = \bar{Q} = [0, 1] \times [0, 1] = \{(x, y) \in \mathbb{R}^2 \,|\, 0 \le x, y \le 1\}.$$

Thus there are exactly 4 dyadic squares of level 1 that intersect Q and each of these 4 dyadic squares are contained in $\bar{Q} = \operatorname{supp}(f)$. The 4 dyadic squares correspond to $\mathbf{k} \in \mathbb{Z}^2$ equal to

$$(0, 0), \quad (1, 0), \quad (0, 1), \quad (1, 1).$$

Also, the area of each dyadic square is equal to $\frac{1}{2^2} = \frac{1}{4}$. This means that

$$\begin{aligned}
U_1(f) &= \sum_{C \in \mathcal{D}_1(\mathbb{R}^2)} M_C(f) \operatorname{Area}(C) \\
&= \sum_{\mathbf{k} \in \mathbb{Z}^2} M_{C_{\mathbf{k}}}(f) \operatorname{Area}(C_{\mathbf{k}}) \\
&= M_{C_{(0,0)}}(f)\frac{1}{4} + M_{C_{(1,0)}}(f)\frac{1}{4} + M_{C_{(0,1)}}(f)\frac{1}{4} + M_{C_{(1,1)}}(f)\frac{1}{4}.
\end{aligned}$$

Now,

$$\begin{aligned}
M_{C_{(0,0)}}(f) &= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}, \\
M_{C_{(1,0)}}(f) &= (1)^2 + \left(\frac{1}{2}\right)^2 = \frac{5}{4}, \\
M_{C_{(0,1)}}(f) &= \left(\frac{1}{2}\right)^2 + (1)^2 = \frac{5}{4}, \\
M_{C_{(1,1)}}(f) &= (1)^2 + (1)^2 = 2.
\end{aligned}$$

Thus

$$\begin{aligned}
U_1(f) &= \sum_{C \in \mathcal{D}_1(\mathbb{R}^2)} M_C(f)\,\text{Area}(C) \\
&= \frac{1}{4}\left(\frac{1}{2} + \frac{5}{4} + \frac{5}{4} + 2\right) \\
&= \frac{5}{4}.
\end{aligned}$$

Similarly, one computes that

$$\begin{aligned}
m_{C_{(0,0)}}(f) &= (0)^2 + (0)^2 = 0, \\
m_{C_{(1,0)}}(f) &= \left(\frac{1}{2}\right)^2 + (0)^2 = \frac{1}{4}, \\
m_{C_{(0,1)}}(f) &= (0)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{4}, \\
m_{C_{(1,1)}}(f) &= \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}.
\end{aligned}$$

Thus

$$\begin{aligned}
L_1(f) &= \sum_{C \in \mathcal{D}_1(\mathbb{R}^2)} m_C(f)\,\text{Area}(C) \\
&= \frac{1}{4}\left(0 + \frac{1}{4} + \frac{1}{4} + \frac{1}{2}\right) \\
&= \frac{1}{4}.
\end{aligned}$$

b. Compute the integral of f and show that it is between the upper and lower sums.

Solution to (b). In this part you are allowed to use Fubini's theorem. Let g be defined as in the solution to part (a). We calculate that

$$\begin{aligned}
\int_{\mathbb{R}^2} f(\mathbf{x})\,|d^2\mathbf{x}| &= \int_{\mathbb{R}^2} g(\mathbf{x})\,\mathbf{1}_Q(\mathbf{x})\,|d^2\mathbf{x}| \\
&= \int_0^1 \int_0^1 (x^2 + y^2)\,dx\,dy \\
&= \int_0^1 \int_0^1 x^2\,dx\,dy + \int_0^1 \int_0^1 y^2\,dy\,dx \\
&= \int_0^1 \frac{1}{3}\,dy + \int_0^1 \frac{1}{3}\,dx \\
&= \frac{2}{3}.
\end{aligned}$$

Finally we observe that

$$L_1(f) = \frac{1}{4} < \int_{\mathbb{R}^2} f(\mathbf{x})\,|d^2\mathbf{x}| = \frac{2}{3} < \frac{5}{4} = U_1(f). \tag{1}$$

Exercise 4.1.14, Part a.

Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x \notin [0,1] \text{, or } x \text{ is rational,} \\ 1 & \text{if } x \in [0,1] \text{, and } x \text{ is irrational.} \end{cases}$$

What value do you get for the "left Riemann sum", where for the interval $C_{k,N} = \left\{ x \middle| \frac{k}{2^N} < x < \frac{k+1}{2^N} \right\}$ you choose the left endpomt $\frac{k}{2^N}$? For the sum you get when you choose the right endpoint $\frac{k+1}{2^N}$? The midpoint Riemann sum?

Solution. This is an easy question since the left endpoint $\frac{k}{2^N}$, the right endpoint $\frac{k+1}{2^N}$, and the midpoint $\frac{k+0.5}{2^N}$ are all rational numbers for any integer k. This means the output of f for any of these inputs is equal to 0. Therefore the Riemann sum corresponding to any one of these 3 choices is equal to zero.

Exercise 4.5.6

Part a. Show that as n increases, the volume of the n-dimensional unit *ball* becomes a smaller and smaller proportion of the smallest n-dimensional cube that contains it.[2]

Solution. Let β_n equal the volume of the n-dimensional unit ball $B_1(0)$ in $\mathbb{R}^n$. We have

$$\beta_n = \beta_{n-1} c_n,$$

where $c_n = \int_{-1}^{1} (1-t^2)^{\frac{n-1}{2}} dt$, and satisfies

$$c_0 = \pi, \quad c_1 = 2, \quad c_n = \frac{n-1}{n} c_{n-2}.$$

The cube that contains the n-dimensional unit ball is $[-1,1]^n$, which has volume 2^n. So we want to show that the function of n defined by

$$\frac{\beta_n}{2^n}$$

is monotone decreasing in n. That is, for any integer $n \geq 2$,

$$\frac{\beta_n}{2^n} < \frac{\beta_{n-1}}{2^{n-1}}.$$

This inequality, which we *want to prove*, is *equivalent to*

$$c_n = \frac{\beta_n}{\beta_{n-1}} < \frac{2^n}{2^{n-1}} = 2.$$

Now, $c_n < c_{n-2}$ for all $n \geq 2$. We also have $c_1 = 2 \leq 2$ and $c_2 = \pi/2 < 2$. This implies that $c_n < 2$ for all $n \geq 2$. (A rigorous proof would require mathematical induction, but you are NOT expected to do this.) Thus, we have proved the desired inequality.

Part b. What is the first n for which the ratio of volumes is smaller than 10^{-2}?

Solution. This and part (c) probably require a calculator. The answer is $n = 9$. Note that (see the table on p. 444 of Hubbard and Hubbard)

$$\begin{aligned}
\frac{\beta_1}{2^1} &= 1, \\
\frac{\beta_2}{2^2} &= \frac{\pi}{4}, \\
\frac{\beta_3}{2^3} &= \frac{4\pi}{3\cdot 8} = \frac{\pi}{6}, \\
\frac{\beta_4}{2^4} &= \frac{\pi^2}{2\cdot 16} = \frac{\pi^2}{32}, \\
\frac{\beta_5}{2^5} &= \frac{8\pi^2}{15\cdot 32} = \frac{\pi^2}{60}.
\end{aligned}$$

[2] The book uses the word "sphere" instead of "ball".

You can find a table of the volumes of balls here. For example, $\beta_9 = \frac{32\pi^4}{945}$. So we compute that

$$\frac{\beta_9}{2^9} = \frac{32\pi^4}{945 \cdot 512} = \frac{\pi^4}{945 \cdot 16} \approx 0.00644 < 10^{-2}.$$

(Check that $\frac{\beta_8}{2^8} > 10^{-2}$.)

Part c. What is the first n for which it is smaller than 10^{-6}?

Solution. The answer is $n = 18$. The same Wikipedia link as above gives β_n for $n \leq 15$. You can use that.

Exercise 4.5.7

Write as an iterated integral, and in six different ways, the triple integral of xyz over the region $x, y, z \geq 0,\ x + 2y + 3z \leq 1$. You need not compute the integrals.

Solution. We are going to present a more general solution. Let a, b, c be positive real numbers. Consider the region R defined by $x, y, z \geq 0,\ ax + by + cz \leq 1$. The first three inequalities says that R is contained in the first octant. The last inequality says that R is on one side of the plane $ax + by + cz = 1$. Which side? The side that contains the origin. The R is a tetrahedron. Its vertices are the four points

$$(0,0,0)\,, \quad \left(\frac{1}{a}, 0, 0\right), \quad \left(0, \frac{1}{b}, 0\right), \quad \left(0, 0, \frac{1}{c}\right).$$

We want to write the integral $\int_R f(\mathbf{x})\,|d^3\mathbf{x}|$ using Fubini's theorem as triple integrals in $6 = 3!$ different ways, ,where we need to determine the limits of integration:

$$\begin{aligned}
&\int_0^{1/c} \int_*^* \int_*^* f(x,y,z)\,dx\,dy\,dz, \\
&\int_0^{1/b} \int_*^* \int_*^* f(x,y,z)\,dx\,dz\,dy, \\
&\int_0^{1/c} \int_*^* \int_*^* f(x,y,z)\,dy\,dz\,dz, \\
&\int_0^{1/a} \int_*^* \int_*^* f(x,y,z)\,dy\,dz\,dx, \\
&\int_0^{1/b} \int_*^* \int_*^* f(x,y,z)\,dz\,dx\,dy, \\
&\int_0^{1/a} \int_*^* \int_*^* f(x,y,z)\,dz\,dy\,dx.
\end{aligned}$$

How did we figure out the limits 0 and $1/c$ for the variable z in the first integral? They are simply the smallest and largest possible values of z for which there exist x, y such that $(x,y,z) \in R$.

In the first integral, how do we figure out the limits for the variable y? We fix z and find the smallest and largest possible values of y for which there exist x such that

$(x, y, z) \in R$. Firstly, $y \geq 0$. Secondly, for fixed z, we want to find the largest y for which $ax + by + cz \leq 1$ for some x. This is given by taking $x = 0$ to get $1 = by + cz$ giving the largest value of y, which is $y = \frac{1-cz}{b}$. So the limits for y are 0 and $\frac{1-cz}{b}$.

Finally, the limits for the variable x are obtained as follows. Clearly 0 is the lower limit. Fixing z and y, the upper limit is given by the equation $ax+by+cz = 1$, so that it is $\frac{1-by-cz}{a}$. We conclude that the first integral is

$$\int_0^{1/c} \int_0^{\frac{1-cz}{b}} \int_0^{\frac{1-by-cz}{a}} f(x, y, z)\, dx\, dy\, dz.$$

All of the other five integrals are similarly derived.

In our example, $a = 1$, $b = 2$, and $c = 3$, and $f(x, y, z) = xyz$. So we get (as 1 of the 6 integrals)

$$\int_0^{1/3} \int_0^{(1-3z)/2} \int_0^{1-2y-3z} f(x, y, z)\, dx\, dy\, dz.$$

We do not evaluate.

Exercise 4.5.12

Part a. Represent the iterated integral $\displaystyle\int_0^a \left(\int_{x^2}^{a^2} \sqrt{y}\, e^{-y^2} dy \right) dx$ as the integral of $\sqrt{y}\, e^{-y^2}$ over a region of the plane. Sketch this region.

Solution. Firstly we examine the region R. It is given by the inequalities

$$0 \leq x \leq a, \qquad x^2 \leq y \leq a^2.$$

Part b. Use Fubini's theorem to make this integral into an iterated integral in the opposite order.

Solution. To reverse the order of integration, we find the minimum and maximum values of y. The maximum is still a. The minimum is when $x = 0$, which is $0^2 = 0$. Therefore when we rewrite the inequalities, the first inequality will be

$$0 \leq y \leq a^2.$$

To find the second inequality, given a fixed y we find the smallest and largest values of x. We have $x^2 \leq y$, that is, $x \leq \sqrt{y}$. So the lower limit for x is 0 and the upper limit is $\sqrt{y}$. Note that the region R is given by

$$0 \leq y \leq a^2, \qquad 0 \leq x \leq \sqrt{y}.$$

So, reversing the order of integration yields

$$\int_0^{a^2} \int_0^{\sqrt{y}} \sqrt{y} e^{-y^2} dx\, dy = \int_0^{a^2} y\, e^{-y^2} dy = -\, \frac{1}{2} e^{-y^2} \Big|_0^{a^2} = \frac{1}{2}(1 - e^{-a^4}).$$

Part c. Evaluate the integral.

Solution. This was answered above.

Exercise 4.5.15

Find the volume of the region

$$z \geq x^2 + y^2, \quad z \leq 10 - x^2 - y^2.$$

Solution. The inequalities defining the region imply

$$x^2 + y^2 \leq z \leq 10 - x^2 - y^2,$$

so that

$$2(x^2 + y^2) \leq 10.$$

So we can write the volume as a double integral (not a triple integral, which is also possible)

$$\int_R \big((10 - x^2 - y^2) - (x^2 + y^2)\big)|d^2\mathbf{x}|,$$

where $R = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}|^2 \leq 5\}$. In polar coordinates, we get

$$\begin{aligned}
\int_0^{2\pi} \int_0^{\sqrt{5}} (10 - r^2)\, r\, dr\, d\theta &= 2\pi \int_0^{\sqrt{5}} (10 - 2r^2)\, r\, dr \\
&= \pi \int_0^{\sqrt{5}} (20r - 4r^3)\, dr \\
&= \pi \left(10r^2 - r^4\right)\Big|_0^{\sqrt{5}} \\
&= \pi\,(50 - 25) \\
&= 25\pi.
\end{aligned}$$