Math 100B: Homework 7

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Problem 1

We can first uniquely factorize each number and check if they share any primes in common

$$2 + 4i = 2(1 + 2i) = (1 + i)(1 - i)(1 + 2i)$$
$$5 + 5i = 5(1 + i) = (1 + 2i)(1 - 2i)(1 + i)$$

Therefore $\gcd(2 + 4i, 5 + 5i) = (1 + i)(1 + 2i) = -1 + 3i$.

Problem 2

(a) Suppose that $f(x)$ was reducible in $R[x]$. This would imply that $f(x) = g(x)h(x)$ where $g(x)$ and $h(x)$ are not units in $R[x]$ and they have positive degree (if they were constants then $f(x)$ would no longer have gcd $= 1$). We can write $g(x) = cg_0(x)$ and $h(x) = c'h_0(x)$ where $g_0(x)$ and $h_0(x)$ are primitive and $c, c' \in R$. Therefore $f(x) = cc'g_0(x)h_0(x)$ but by Gauss' lemma, $g_0(x)h_0(x)$ is primitive and so $c = c' = 1$. Therefore $g(x)$ and $h(x)$ are primitive.

However this is a factorization of $f(x)$ into two positive degree polynomials, which are not units in $F[x]$, which is a contradiction. Therefore $f(x)$ is irreducible in $R[x]$.

(b) Note that $\mathbb{Q}[x, y] = (\mathbb{Q}[y])[x]$ and take $R = \mathbb{Q}[y]$. Let $F = \mathbb{Q}(y)$ to be the field of fractions of R. Since $(y)x + (y^2 + 1)$ has $\gcd(y, y^2 + 1) = 1$, we can simply show that $yx + y^2 + 1$ is irreducible in $F[x]$ and apply part (a) to prove that it is irreducible in $R[x] = \mathbb{Q}[x, y]$.

Since $yx + y^2 + 1$ is a degree 1 polynomial in terms of x, it can only be written as the product of a degree 1 polynomial and a degree 0 polynomial, and all degree 0 polynomials in $F[x]$ are units since its a field of fractions, so $yx + y^2 + 1$ is therefore irreducible in $F[x]$ and $R[x]$.

Problem 3

(a) Since $B = \{v_1, \ldots, v_n\}$ is a basis, we can write $u = b_1v_1 + \ldots + b_nv_n$ for all $u \in V$. Let B' be B but with v_i replaced with w. B' is still linearly independent since if there was a way to form a nontrivial linear combination of its elements to get zero, that would immediately also give us a nontrivial linear combination that equals zero in B by simply substituting in $w = a_1v_1 + \ldots + a_nv_n$ into that linear combination.

In order to show that B' spans, we need to show it is possible to write $u = c_1v_1 + \ldots + c_{i-1}v_{i-1} + c_iw + c_{i+1}v_{i+1} + \ldots + c_nv_n$, which is u written in terms of the basis vectors in B'. First let us solve out for v_i in terms of the vectors in B'.

$$v_i = \frac{w - \sum_{j\neq i} a_jv_j}{a_i}$$

Next we can substitute to get the representation we need, which shows B' is a basis as well.

$$\begin{aligned}
u &= b_1v_1 + \ldots + b_iv_i + \ldots + b_nv_n \\
&= b_1v_1 + \ldots + b_i\left(\frac{w - \sum_{j\neq i} a_jv_j}{a_i}\right) + \ldots + b_nv_n \\
&= \left(b_1 - \frac{b_ia_1}{a_i}\right)v_1 + \ldots + \left(\frac{b_i}{a_i}\right)v_i + \ldots + \left(b_n - \frac{b_ia_n}{a_i}\right)v_n
\end{aligned}$$

(b) We can prove this by induction. Let $B_i = \{w_1, \ldots, w_i, v_{i+1}, \ldots, v_n\}$. In the base case, B_0 is the original basis we started with. Asumme that B_i is a basis. There exists some vector v_j that can be written as the linear combination of vectors in B_i that is not already in B_i. There must be some w_k used in this linear combination with nonzero coefficient, so we can replace w_k with v_j and then rearrange the vectors to get a new basis B_{i+1} by part (a).

Problem 4

If we have two bases, $B = \{v_1, \ldots, v_n\}$ and $B' = \{w_1, \ldots, w_m\}$ where $m > n$, then we can replace the elements of B' one by one with elements by B to create a basis that is a superset of B, which is a contradiction. Likewise if $m < n$ we can replace elements of B with elements of B' to create a basis that is a super set of B', which is also a contradiction. Therefore all basis have the same number of elements in V.

Problem 5

Suppose that there did exist a linear combination of these functions that equaled zero.

$$ax^2 + b\sin x + c\cos x + de^x = 0$$

At $x = 0$ we have that $c + d = 0$. At $x = 2\pi$ we have that $c + de^{2\pi} = 0$, which implies $de^{2\pi} - d = 0$, $d = 0$ and $c = 0$. At $x = \pi$ we have that $a\pi^2 = 0$ which implies $a = 0$. We are left with $b\sin x = 0$ which implies $b = 0$. Therefore only the trivial linear combination gives zero meaning these functions are independent.

Problem 6

If we have a linear combination of a finite number of reciprocals of monic degree 1 polynomials equal to 0, we can show that it implies all the coefficients are 0.

$$0 = \sum_{i=1}^{n} \frac{c_i}{x - a_i} = \frac{\sum_{i=1}^{n} c_i \prod_{j \neq i}(x - a_j)}{\prod_{i=1}^{n}(x - a_i)}$$

In the numerator, if $c_i \neq 0$ then $x - a_i$ does not divide the numerator and a_i is not a root, so it must be that all the $c_i = 0$. Therefore the set of reciprocals of monic degree 1 polynomials ia linearly independent.