Math 31CH HW2
Due April 12 at 11:59 pm

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Exercise 4.8.1

(For the matrix A only.) Compute the determinant of the following matrix, using development by the first column or development by the first row.

$$\begin{pmatrix} 1 & -2 & 3 & 0 \\ 4 & 0 & 1 & 2 \\ 5 & -1 & 2 & 1 \\ 3 & 2 & 1 & 0 \end{pmatrix}$$

Solution. Using the cofactor formula,

$$\begin{aligned} \det A &= \det \begin{bmatrix} 0 & 1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix} + 2 \det \begin{bmatrix} 4 & 1 & 2 \\ 5 & 2 & 1 \\ 3 & 1 & 0 \end{bmatrix} + 3 \det \begin{bmatrix} 4 & 0 & 2 \\ 5 & -1 & 1 \\ 3 & 2 & 0 \end{bmatrix} \\ &= (-(-2) + 2(-5)) + 2(4(-1) - (-3) + 2(-1)) + 3(4(-2) + 2(13)) \\ &= -8 - 6 + 54 \\ &= 40 \end{aligned}$$

Exercise 4.8.5

Show by direction computation that if A, B are 2×2 matrices, then $tr(AB) = tr(BA)$.

Solution.

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

$$BA = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ae + cf & be + df \\ ag + ch & bg + dh \end{bmatrix}$$

$$\mathrm{tr}(AB) = (ae + bg) + (cf + dh) = (ae + cf) + (bg + dh) = \mathrm{tr}(BA)$$

Exercise 4.8.7

Part a. Use multilinearity to show that if a square matrix has a column of zeros, its determinant must be zero.

Solution. Let A be a $n \times n$ square matrix, let a_j denote the jth column of A, and let 0_n denote a column of zeros.

$$\begin{aligned}\det A &= \det[a_1 \ldots 0_n \ldots a_n] \\ &= \det[a_1 \ldots 0_n + 0_n \ldots a_n] \\ &= \det[a_1 \ldots 0_n \ldots a_n] + \det[a_1 \ldots 0_n \ldots a_n] \\ &= 2 \det A\end{aligned}$$

Therefore, $\det A = 0$ if A has a column of zeros.

Part b. Show that if two columns of a square matrix A are equal, $\det(A) = 0$.

Solution. Let columns a_i and a_j be equal. Using antisymmetry

$$\begin{aligned}\det A &= \det[\ldots a_i \ldots a_j \ldots] && (1) \\ &= -\det[\ldots a_j \ldots a_i \ldots] && (2) \\ &= -\det A && (3)\end{aligned}$$

Since $\det A = -\det A$, then $\det A = 0$.

Exercise 4.8.8

Let A and B be $n \times n$ matrices, with A invertible. Show that the function

$$f(B) = \frac{\det(AB)}{\det(A)}$$

satisfies multilinearity, antisymmetry, normalization, so $f(B) = \det(B)$.

Solution.
$f(B)$ is multilinear:

$$\begin{aligned}
f(B) &= f([\ldots c_1 b_1 + c_2 b_2 \ldots]) \\
&= \frac{\det[\ldots A(c_1 b_1 + c_2 b_2) \ldots]}{\det A} \\
&= \frac{c_1 \det[\ldots Ab_1 \ldots] + c_2 \det[\ldots Ab_2 \ldots]}{\det A} \\
&= c_1 f([\ldots b_1 \ldots]) + c_2 f([\ldots b_2 \ldots])
\end{aligned}$$

$f(B)$ is antisymmetric:

$$\begin{aligned}
f(B) &= f([\ldots b_i \ldots b_j \ldots]) \\
&= \frac{\det[\ldots Ab_i \ldots Ab_j \ldots]}{\det A} \\
&= -\frac{\det[\ldots Ab_j \ldots Ab_i \ldots]}{\det A} \\
&= -f([\ldots b_j \ldots b_i \ldots])
\end{aligned}$$

$f(B)$ is normal:

$$f(I) = \frac{\det AI}{\det A} = \frac{\det A}{\det A} = 1$$

Therefore, $f(B) = \det B$.

Exercise 4.8.11

Prove Theorem 4.8.10: If A is an $n \times n$ matrix, B is an $m \times m$ matrix, and C is an arbitrary $n \times m$ matrix, then

$$\det \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \det(A)\det(B)$$

Solution.

Let R_A be the matrix where $R_A A$ is in RREF, and let R_B be the matrix where $R_B B$ is in RREF. Then

$$\det \begin{bmatrix} R_A A & C \\ 0 & R_B B \end{bmatrix} = \det \begin{bmatrix} R_A & 0 \\ 0 & I \end{bmatrix} \det \begin{bmatrix} I & 0 \\ 0 & R_B \end{bmatrix} \det \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \det R_A \det R_B \det \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$$

If $\det A \neq 0$ and $\det B \neq 0$, then $R_A = A^{-1}$ and $R_B = B^{-1}$ so

$$\det \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = \frac{1}{\det R_A}\frac{1}{\det R_B} \det \begin{bmatrix} R_A A & C \\ 0 & R_B B \end{bmatrix} = \frac{1}{\det A^{-1}}\frac{1}{\det B^{-1}} \det \begin{bmatrix} I & C \\ 0 & I \end{bmatrix} = \det A \det B$$

If $\det A = 0$ or $\det B = 0$, then $\det \begin{bmatrix} R_A A & C \\ 0 & R_B B \end{bmatrix} = \det \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} = 0$ since $R_A A$ will have a column of zeros or $R_B B$ will have a row of zeros. Therefore the equation still holds when A or B are not invertible.

Exercise 4.8.13

Confirm that the six permutations of the number $1, 2, 3$ have the signatures listed in Example 4.8.13.

Solution.
The following have an even number of transpositions and so have positive signature.

$$\begin{aligned}
&123 \\
&231 \implies 132 \implies 123 \\
&312 \implies 213 \implies 123
\end{aligned}$$

The following have an odd number of transpositions and so have negative signature.

$$\begin{aligned}
132 &\implies 123 \\
213 &\implies 123 \\
321 &\implies 123
\end{aligned}$$

Exercise 4.8.14

Prove the Cayley-Hamilton theorem:

Part a. First prove it for diagonal matrices.

Solution. Let B be a $n \times n$ diagonal matrix with diagonal elements $\mu_1 \dots \mu_n$. Let $c_0 \dots c_{n-1}$ be constants dependent on $\mu_1 \dots \mu_n$. Then,

$$\chi_B(\lambda) = \det(\lambda I - B) = \prod_{i=1}^{n} (\lambda - \mu_i) = \lambda^n + c_{n-1}\lambda^{n-1} + \dots + c_0$$

Therefore,

$$\chi_B(B) = B^n + c_{n-1}B^{n-1} + \dots + c_0 I$$

Since B is a diagonal matrix, then

$$B^k = \begin{bmatrix} \mu_1^k & 0 & \dots & 0 \\ 0 & \mu_2^k & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \mu_n^k \end{bmatrix}$$

So to find the ith diagonal term in $\chi_B(B)$, each B^k term can be replaced with μ_i^k. The ith diagonal term in $\chi_B(B)$ ends up being $\mu_i^n + c_{n-1}\mu_i^{n-1} + \dots + c_0 = \chi_B(\mu_i)$. Since μ_i is an eigenvalue, $\chi_B(\mu_i) = 0$ for all i, so the Cayley-Hamilton theorem holds for all diagonal matrices.

$$\chi_B(B) = \begin{bmatrix} \chi_B(\mu_1) & 0 & \dots & 0 \\ 0 & \chi_B(\mu_2) & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & \chi_B(\mu_n) \end{bmatrix} = [0]$$

Part b. Next show that $\chi_{P^{-1}BP} = \chi_B$. Use this and part a to prove the theorem for diagonalizable matrices.

Solution. Substituting in $I = P^{-1}IP$,

$$\begin{aligned} \chi_{P^{-1}BP}(\lambda) &= \det(\lambda I - P^{-1}BP) \\ &= \det(\lambda P^{-1}IP - P^{-1}BP) \\ &= \det(P^{-1})\det(\lambda I - B)\det(P) \\ &= \det(\lambda I - B) \\ &= \chi_B(\lambda) \end{aligned}$$

Therefore, the Cayley-Hamilton theorem holds for diagonalizable matrices.

Part c. Finally, use Theorem 4.8.26 to prove it in general.

Solution. Theorem 4.8.26 says that there exists a sequence of complex diagonalizable matrices, A_i, that converges to any square matrix A. Since $\chi_{A_i}(A_i) = 0$ for all i, and the characteristic polynomial is continuous, $\chi_A(A) = 0$ for any square matrix A.