Math 170A: Homework 3

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Question 1

$$||A|| = ||\alpha I|| = |\alpha|||I|| = |\alpha|$$
$$||A^{-1}|| = ||\frac{1}{\alpha}I|| = |\frac{1}{\alpha}|||I|| = |\frac{1}{\alpha}|$$
$$\det(A) = \det(\alpha I) = \alpha^n \det(I) = \alpha^n$$
$$\kappa_{||\cdot||}(A) = ||A||||A^{-1}|| = \alpha \cdot \frac{1}{\alpha} = 1$$

Question 2

(a)

$$\begin{aligned}\kappa(A) &= ||A|| \cdot ||A^{-1}|| \\ &= ||A^{-1}|| \cdot ||A|| \\ &= \kappa(A^{-1})\end{aligned}$$

(b) From class we know that for $Ax = b$

$$\frac{||\delta x||}{||x||} \leq \kappa(A) \cdot \frac{||\delta b||}{||b||}$$

Substituting in $A \to A^{-1}$, $x \to b$, and $b \to x$ and using the equality from part (a) yields

$$\begin{aligned}&\frac{||\delta b||}{||b||} \leq \kappa(A^{-1}) \cdot \frac{||\delta x||}{||x||} \\ \implies &\frac{||\delta b||}{||b||} \leq \kappa(A) \cdot \frac{||\delta x||}{||x||}\end{aligned}$$

Question 3

(a)

$$||A||_1 = \max_j \sum_i |a_{ij}| = \max\{5, 1, 5\} = 5$$
$$||A||_\infty = \max_i \sum_j |a_{ij}| = \max\{5, 4, 2\} = 5$$
$$||A||_F = \sqrt{3^2 + 3 \cdot 2^2 + 2 \cdot 1^2} = \sqrt{23}$$

(b) The inverse of A is

$$A^{-1} = \frac{1}{10}\begin{bmatrix} 2 & 2 & 0 \\ -2 & 3 & 10 \\ 2 & -3 & 0 \end{bmatrix}$$

Its norms are

$$||A^{-1}||_1 = \max_j \sum_i |a_{ij}| = \frac{1}{10}\max\{6, 8, 10\} = 1$$

$$||A^{-1}||_\infty = \max_i \sum_j |a_{ij}| = \frac{1}{10}\max\{4, 15, 5\} = \frac{3}{2}$$

$$||A^{-1}||_F = \frac{1}{10}\sqrt{10^2 + 2\cdot 3^2 + 4\cdot 2^2} = \sqrt{1.34}$$

The condition numbers are

$$\kappa_1(A) = 5 \cdot 1 = 5$$

$$\kappa_\infty(A) = 5 \cdot \frac{3}{2} = \frac{15}{2}$$

$$\kappa_F(A) = \sqrt{23}\cdot\sqrt{1.34} \approx 5.5516$$

(c) Dividing the companion inequality by $\kappa(A)$ yields

$$\frac{||\delta b||}{\kappa(A)||b||} \leq \frac{||\delta x||}{||x||}$$

Under all the norms, δb has norm ϵ, so the relative error can be bounded by

$$\frac{\epsilon}{\kappa(A)||b||} \leq \frac{||\delta x||}{||x||} \leq \kappa(A)\cdot\frac{\epsilon}{||b||}$$

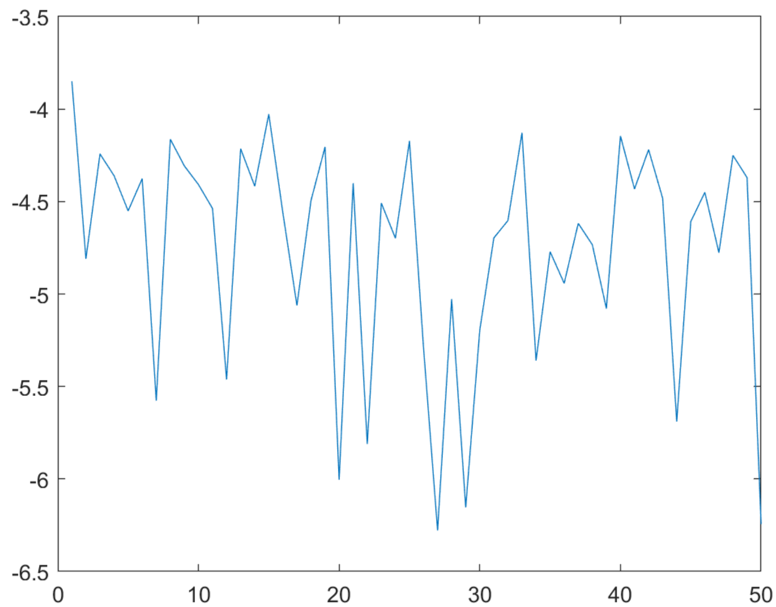
We can get the specific bounds by substituting in the condition numbers we got from (b).

Question 4

The reasoning is incorrect because it confuses the vector itself with its norm. Although it might be true that $0.01x = A^{-1}(0.01b)$, by submultiplicativity we have that $0.01||x|| \leq 0.01||A^{-1}||||b||$ so a 1 percent change in b can cause a greater than 1 percent change in x if $||A^{-1}||$ is large enough.

Question 5

```
function question5()
    q = zeros(50,1);
    for i = 1:50
        A = randn(500);
        b = randn(500,1);
        x = A\b;
        delta_b = 10^(-1)*randn(500,1);
        x_tilde = A\(b + delta_b);
        rel_error_sol = norm(x_tilde-x,1)/norm(x,1);
        rel_error_b = norm(delta_b,1)/norm(b,1);
        q(i) = rel_error_sol/(cond(A,1)*rel_error_b);
    end
    y = log10(q);
    plot(y);
end
```



Since $\log(q) < 0$ for all the points, we know that $q < 1$ for all the points, which agrees with the theory we learned in class.