Math 31AH: Spring 2021
Homework 3
Due 5:00pm on Friday 10/15/2021

Problem 1: Direct sum. Let V and W be an $\mathbb{F}$-vector spaces. The *direct sum* of V and W, denoted $V \oplus W$, is the $\mathbb{F}$-vector space

$$V \oplus W := \{(\mathbf{v}, \mathbf{w}) \,:\, \mathbf{v} \in V,\ \mathbf{w} \in W\}$$

with addition and scalar multiplication defined by

$$(\mathbf{v}, \mathbf{w}) + (\mathbf{v}', \mathbf{w}') := (\mathbf{v} + \mathbf{v}', \mathbf{w} + \mathbf{w}') \qquad c \cdot (\mathbf{v}, \mathbf{w}) := (c \cdot \mathbf{v}, c \cdot \mathbf{w})$$

where $\mathbf{v}, \mathbf{v}' \in V, \mathbf{w}, \mathbf{w}' \in W$, and $c \in \mathbb{F}$. Suppose $\mathcal{B}$ is a basis of V and $\mathcal{C}$ is a basis of W. Prove that

$$\mathcal{B} \oplus \mathcal{C} := \{(\mathbf{v}, \mathbf{0}) \,:\, \mathbf{v} \in \mathcal{B}\} \cup \{(\mathbf{0}, \mathbf{w}) \,:\, \mathbf{w} \in \mathcal{C}\}$$

is a basis of $V \oplus W$. In particular, we have the useful formula

$$\dim\, V \oplus W = \dim\, V + \dim\, W$$

whenever V and W are finite-dimensional.

Problem 2: Real sequences. Let V be the $\mathbb{R}$-vector space of all infinite sequences $(a_1, a_2, \dots)$ of real numbers, under the operations described in Homework 2. For any $i \geq 1$, let $\mathbf{e}_i \in V$ be the sequence

$$\mathbf{e}_i := (0, 0, \dots, 0, 1, 0, 0, \dots)$$

with a unique 1 in the i^{th} position and 0's elsewhere. Let

$$S = \{\mathbf{e}_1, \mathbf{e}_2, \dots\} \subseteq V$$

Does S span V? If not, describe the subspace $\mathrm{span}(S)$ of V. Prove your claims.

Problem 3: A basis for polynomials. Let V be the $\mathbb{R}$-vector space of polynomials $f(t)$ in a variable t with degree $\leq n$. Show that the set

$$\mathcal{B} := \{1, (t+1), (t+1)^2, \dots, (t+1)^n\}$$

is a basis of V.

Problem 4: Real-valued functions. Let V be the $\mathbb{R}$-vector space of differentiable functions $f : \mathbb{R} \to \mathbb{R}$. Find an infinite linearly independent subset $I \subseteq V$. (Hint: You may use knowledge from calculus about growth rates.)

Problem 5: Homogeneous systems. Let $\mathbb{F}$ be a field, let A be an $m \times n$ matrix over $\mathbb{F}$, and let $A\mathbf{x} = \mathbf{0}$ be the associated homogeneous system of linear equations. Prove that the solution set of this system is a subspace of $\mathbb{F}^n$.

Problem 6: Particular solutions. Let $\mathbb{F}$ be a field, let A be an $m \times n$ matrix over $\mathbb{F}$, let $\mathbf{b} \in \mathbb{F}^m$, and let $A\mathbf{x} = \mathbf{b}$ be the associated not-necessarily-homogeneous system of linear equations. Let $\mathbf{x}_0 \in \mathbb{F}^n$ be a vector such that

$$A\mathbf{x}_0 = \mathbf{b}$$

(The vector $\mathbf{x}_0$ is sometimes called a 'particular solution' of $A\mathbf{x} = \mathbf{b}$.) Let $W \subseteq \mathbb{F}^n$ be the solution set of the associated homogeneous system, i.e.

$$W := \{\mathbf{w} \in \mathbb{F}^n \,:\, A\mathbf{w} = \mathbf{0}\}$$

Prove that the full solution set to the original system $A\mathbf{x} = \mathbf{b}$ is

$$\{\mathbf{x} \in \mathbb{F}^n \,:\, A\mathbf{x} = \mathbf{b}\} = \{\mathbf{x}_0 + \mathbf{w} \,:\, \mathbf{w} \in W\}$$

Problem 7: Completing a basis. Let $n \geq 1$ be a positive integer and let V be an n-dimensional vector space over a field $\mathbb{F}$. Let $\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_s\}$ be a finite linearly independent subset of V. Prove that there exist $n - s$ vectors $\mathbf{v}_{s+1}, \mathbf{v}_{s+2}, \dots, \mathbf{v}_n \in V$ such that the set $\{\mathbf{v}_1, \dots, \mathbf{v}_s, \mathbf{v}_{s+1}, \dots, \mathbf{v}_n\}$ is a basis of V.

Problem 8: Trimming down to a basis. Let V be an $\mathbb{F}$-vector space where $\mathbb{F}$ is a field and $\{\mathbf{v}_1, \dots, \mathbf{v}_m\} \subseteq V$ be a finite subset which spans V. Prove that there exist $1 \leq i_1 < \cdots < i_n \leq m$ such that $\{\mathbf{v}_{i_1}, \dots, \mathbf{v}_{i_n}\}$ is a basis of V.

Problem 9: (Optional; not to be handed in.) Let $n \geq 0$ and let

$$V = \{S \,:\, S \subseteq [n]\}$$

Define addition on V by

$$S + T := S \Delta T$$

where $S \Delta T := (S - T) \cup (T - S)$ is the *symmetric difference* of S and T. Give a definition of scalar multiplication which turns V into an $\mathbb{F}_2$-vector space.