Math 31CH HW1
Due April 5 at 11:59 pm by Gradescope Submission

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Exercise 4.1.9

Let $Q \subset \mathbb{R}^2$ be the unit square $0 \leq x, y < 1$.[1] Show that the function

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \sin(x-y)\,\mathbf{1}_Q\begin{pmatrix} x \\ y \end{pmatrix}$$

is integrable by providing an explicit bound for $U_N(f) - L_N(f)$ that tends to 0 as $N \to \infty$.

Solution. The inequality $|\sin(x_2) - \sin(x_1)| \leq |x_2 - x_1|$ (i.e. Lipschitzs continuous with $C = 1$) provides a bound for the oscilation of any dyadic cube $C_{k,N}$. For some (x_2, y_2) and (x_1, y_1) in $\overline{C}$,

$$\begin{aligned}
\operatorname{osc}_C(f) &= |\sin(x_2 - y_2) - \sin(x_1 - y_1)| \\
&\leq |(x_2 - y_2) - (x_1 - y_1)| \\
&= |(x_2 - x_1) - (y_2 - y_1)| \\
&\leq |x_2 - x_1| + |y_2 - y_1| \\
&= \frac{1}{2^N} + \frac{1}{2^N} \\
&= \frac{1}{2^{N-1}}
\end{aligned}$$

In the dyadic paving $\mathcal{D}_N$, there are 2^{2N} dyadic cubes with a volume of $\frac{1}{2^{2N}}$. So $U_N(f) - L_N(f)$ is bounded by $\frac{1}{2^{N-1}}$ since

$$\begin{aligned}
U_N(f) - L_N(f) &= \sum_{C \in \mathcal{D}_N} \operatorname{osc}_C(f) \operatorname{vol}_2 C \\
&\leq \sum_{C \in \mathcal{D}_N} \frac{1}{2^{N-1}} \frac{1}{2^{2N}} \\
&= 2^{2N} \frac{1}{2^{N-1}} \frac{1}{2^{2N}} \\
&= \frac{1}{2^{N-1}}
\end{aligned}$$

Since $\frac{1}{2^{N-1}}$ tends to 0 as $N \to \infty$, $U_N(f) - L_N(f)$ tends to 0 as $N \to \infty$, and so f is integrable.

[1] Note that we slightly changed the exercise from the book's version, which actually makes it simpler.

Exercise 4.1.10

a. What are the upper and lower sums $U_1(f)$ and $L_1(f)$ for the function

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} x^2 + y^2 & \text{if } 0 < x, y < 1, \\ 0 & \text{otherwise,} \end{cases}$$

i.e., the upper and lower sums for the partition $\mathcal{D}_1(\mathbb{R}^2)$, shown in the figure at left (below actually)?

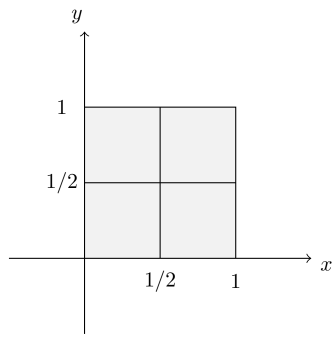

Figure 1: Figure for Exercise 4.1.10.

Solution to (a). Since f monotonically increases as x increases or y increases, the supremum of the dyadic cube occurs at the top right of the cube, and the infinum of the dyadic cubes occurs at the bottom left of the cube.

$$U_1(f) = \frac{1}{4}(\frac{1}{2} + \frac{5}{4} + \frac{5}{4} + 2) = \frac{5}{4}$$
$$L_1(f) = \frac{1}{4}(0 + \frac{1}{4} + \frac{1}{4} + \frac{1}{2}) = \frac{1}{4}$$

b. Compute the integral of f and show that it is between the upper and lower sums.

Solution to (b). Using Fubini's theorem, the integral can be written as

$$\int_{\mathbb{R}^2} f|dxdy| = \int_{-\infty}^{\infty} (\int_{-\infty}^{\infty} f(x,y)\, dx)\, dy = \int_0^1 (\int_0^1 x^2 + y^2\, dx)\, dy$$

This can then be solved using single-variable calculus

$$\begin{aligned} \int_0^1 (\int_0^1 x^2 + y^2\, dx)\, dy &= \int_0^1 \left[\frac{x^3}{3} + xy^2\right]_{x=0}^{x=1} dy \\ &= \int_0^1 \frac{1}{3} + y^2\, dy \\ &= \left[\frac{y}{3} + \frac{y^3}{3}\right]_{y=0}^{y=1} \\ &= \frac{2}{3} \end{aligned}$$

The integral is between the upper and lower sums since

$$\frac{1}{4} \le \frac{2}{3} \le \frac{5}{4}$$

Exercise 4.1.14, Part a.

Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x \notin [0,1], \text{ or } x \text{ is rational,} \\ 1 & \text{if } x \in [0,1], \text{ and } x \text{ is irrational.} \end{cases}$$

What value do you get for the "left Riemann sum", where for the interval $C_{k,N} = \left\{ x \middle| \frac{k}{2^N} < x < \frac{k+1}{2^N} \right\}$ you choose the left endpomt $\frac{k}{2^N}$? For the sum you get when you choose the right endpoint $\frac{k+1}{2^N}$? The midpoint Riemann sum?

Solution.

1. The inverval C_{k_N} will have a left Riemann sum of 0 since $k \in \mathbb{Z}$ implies the rationality of the left endpoint, $\frac{k}{2^N} \in \mathbb{Q}$.
2. The inverval C_{k_N} will have a right Riemann sum of 0 since $k \in \mathbb{Z}$ implies the rationality of the right endpoint, $\frac{k+1}{2^N} \in \mathbb{Q}$.
3. The inverval C_{k_N} will have a midpoint Riemann sum of 0 since $k \in \mathbb{Z}$ implies the rationality of the midpoint, $\frac{k+\frac{1}{2}}{2^N} \in \mathbb{Q}$.

Exercise 4.5.6

Part a. Show that as n increases, the volume of the n-dimensional unit *ball* becomes a smaller and smaller proportion of the smallest n-dimensional cube that contains it.[2]

Solution. The volume of the n-dimensional cube that contains the unit sphere is 2^n. Let β_n be the volumen of the n-dimensional unit sphere. Let c_n be defined such that $\beta_n = c_n \beta_{n-1}$. In order to show that the proportion of the sphere decreases, it is sufficient to show that $c_n < 2$ for all $n > 1$ since

$$\frac{\beta_n}{2^n} < \frac{\beta_{n-1}}{2^{n-1}} \iff \frac{c_n \beta_{n-1}}{2^n} < \frac{\beta_{n-1}}{2^{n-1}} \iff c_n < 2$$

From exercise 4.5.4,

$$c_n = \frac{n-1}{n} c_{n-2}$$

Since $c_0 = \pi$, $c_1 = 2$, and $\frac{n-1}{n} < 1$ for $n > 0$, $c_n < 2$ must be true for all $n > 1$. Therefore, the volume of a n-sphere decreases in proportion to the cube that contains it.

Part b. What is the first n for which the ratio of volumes is smaller than 10^{-2}?

Solution. From exercise 4.5.5, the volume of a unit sphere in n-dimensions if n is even is

$$\beta_n = \beta_{2k} = \frac{\pi^k}{k!}$$

If n is odd, the volume is

$$\beta_n = \beta_{2k+1} = \frac{\pi^k k! 2^{2k+1}}{(2k+1)!}$$

The inequality $\frac{\pi^k}{2^{2k} k!} < 10^{-2}$ is first true at $k = 5$ which corresponds to $n = 10$.
The inequality $\frac{\pi^k k!}{(2k+1)!} < 10^{-2}$ is first true at $k = 4$ which corresponds to $n = 9$.
Therefore, $n = 9$ is the first n for which the ratio is smaller than 10^{-2}.

Part c. What is the first n for which it is smaller than 10^{-6}?

Solution.
The inequality $\frac{\pi^k}{2^{2k} k!} < 10^{-6}$ is first true at $k = 9$ which corresponds to $n = 18$.
The inequality $\frac{\pi^k k!}{(2k+1)!} < 10^{-6}$ is first true at $k = 9$ which corresponds to $n = 19$.
Therefore, $n = 18$ is the first n for which the ratio is smaller than 10^{-6}.

[2] The book uses the word "sphere" instead of "ball".

Exercise 4.5.7

Write as an iterated integral, and in six different ways, the triple integral of xyz over the region $x, y, z \geq 0,\ x + 2y + 3z \leq 1$. You need not compute the integrals.

Solution.

1. $$\int_0^1 \int_0^{\frac{1}{2}(1-x)} \int_0^{\frac{1}{3}(1-x-2y)} dz\, dy\, dx$$

2. $$\int_0^{\frac{1}{2}} \int_0^{(1-2y)} \int_0^{\frac{1}{3}(1-x-2y)} dz\, dx\, dy$$

3. $$\int_0^1 \int_0^{\frac{1}{3}(1-x)} \int_0^{\frac{1}{2}(1-x-3z)} dy\, dz\, dx$$

4. $$\int_0^{\frac{1}{3}} \int_0^{1-3z} \int_0^{\frac{1}{2}(1-x-3z)} dy\, dx\, dz$$

5. $$\int_0^{\frac{1}{2}} \int_0^{\frac{1}{3}(1-2y)} \int_0^{1-2y-3z} dx\, dz\, dy$$

6. $$\int_0^{\frac{1}{3}} \int_0^{\frac{1}{2}(1-3z)} \int_0^{1-2y-3z} dx\, dy\, dz$$

Exercise 4.5.12

Part a. Represent the iterated integral $\int_0^a \left(\int_{x^2}^{a^2} \sqrt{y}\, e^{-y^2}\, dy \right) dx$ as the integral of $\sqrt{y}\, e^{-y^2}$ over a region of the plane. Sketch this region.

Solution.

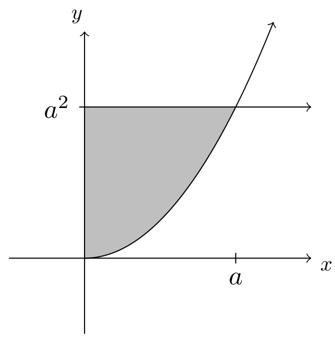

Figure 2: Figure for Exercise 4.5.12.a

Part b. Use Fubini's theorem to make this integral into an iterated integral in the opposite order.

Solution.

$$\int_0^{a^2} \int_0^{\sqrt{y}} \sqrt{y} e^{-y^2}\, dx\, dy$$

Part c. Evaluate the integral.

Solution.

$$\begin{aligned}
\int_0^{a^2} \int_0^{\sqrt{y}} \sqrt{y} e^{-y^2}\, dx\, dy &= \int_0^{a^2} \left[\sqrt{y} e^{-y^2} x \right]_{x=0}^{x=\sqrt{y}} dy \\
&= \int_0^{a^2} y e^{-y^2}\, dy \\
&= \left[-\frac{1}{2} e^{-y^2} \right]_{y=0}^{y=a^2} \\
&= \frac{1}{2} - \frac{1}{2} e^{-a^4}
\end{aligned}$$

Exercise 4.5.15

Find the volume of the region

$$z \geq x^2 + y^2, \quad z \leq 10 - x^2 - y^2.$$

Solution.

In cylindrical coordinates, $z = x^2 + y^2 = \sqrt{x^2 + y^2}^2 = r^2$. The volume can be found by multiplying half of the volume by 2. The integral can be written as

$$2\int_0^{2\pi}\int_0^{\sqrt{5}}\int_0^{r^2} r\,dz\,dr\,d\theta$$

$$\begin{aligned}
2\int_0^{2\pi}\int_0^{\sqrt{5}}\int_0^{r^2} r\,dz\,dr\,d\theta &= 2\int_0^{2\pi}\int_0^{\sqrt{5}} [rz]_{z=0}^{z=r^2}\,dr\,d\theta \\
&= 2\int_0^{2\pi}\int_0^{\sqrt{5}} r^3\,dr\,d\theta \\
&= 2\int_0^{2\pi}\left[\frac{r^4}{4}\right]_{r=0}^{r=\sqrt{5}}\,d\theta \\
&= 2\int_0^{2\pi}\frac{25}{4}\,d\theta \\
&= 2\frac{25\pi}{2} \\
&= 25\pi
\end{aligned}$$