Math 31BH: Assignment 8

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1. Let C be a plane curve, i.e. the image of a continuous function $f : \mathbb{R} \to \mathbb{R}^2$. What are the boundary points of C? Is C open, closed, neither, both?

   Solution: The boundary points of C are the points, v, for which the ball $B_\epsilon(v)$ contains both points in the image and outside the image for every $\epsilon > 0$. Unless C is space filling, C will be a closed curve.

2. Maximize the function $f(x, y) = x^2e^{-(x^4+y^2)}$ over $\mathbb{R}^2$.

   Solution: For $v = (x, y)$

$$x^2e^{-(x^4+y^2)} \leq (x^2 + y^2)e^{-(x^2+y^2)} = \|v\|e^{-\|v\|}$$

The function is also always nonnegative since x^2 and $e^{-(x^4+y^2)}$ are nonnegative. Therefore

$$0 \leq \lim_{\|v\|\to\infty} x^2e^{-(x^4+y^2)} \leq \lim_{\|v\|\to\infty} \|v\|e^{-\|v\|} = 0$$

Since $\lim_{\|v\|\to\infty} x^2e^{-(x^4+y^2)} = 0$ and the function is smooth, the maximum of the function must be a critical point.

The partial derivative with respect to x is

$$\frac{\partial}{\partial x}x^2e^{-(x^4+y^2)} = x^2e^{-x^4-y^2}(-4x^3) + 2xe^{-x^4-y^2)} = 2xe^{-x^4-y^2}(1 - 2x^4)$$

The left-hand term is only zero when $x = 0$ and the right hand term is only zero when $x = \pm\sqrt[4]{\frac{1}{2}}$. Therefore, the partial derivative with respect to x is only zero for points in the form $(0, y)$ or $(\pm\sqrt[4]{\frac{1}{2}}, y)$. The partial derivative with respect to y is

$$\frac{\partial}{\partial y}x^2e^{-(x^4+y^2)} = x^2e^{-x^4-y^2}(-2y) = -2x^2ye^{-x^4-y^2}$$

This is only zero when $x = 0$ or $y = 0$. Therefore, the partial derivative with respect to y is only zero for points in the form $(0, y)$ or $(x, 0)$. The critical points are thus points in the form $(0, y)$ or $(\pm\sqrt[4]{\frac{1}{2}}, 0)$.

$$f(0,y) = 0$$
$$f(\sqrt[4]{\frac{1}{2}}, 0) = \sqrt{\frac{1}{2}}e^{-\frac{1}{2}}$$
$$f(-\sqrt[4]{\frac{1}{2}}, 0) = \sqrt{\frac{1}{2}}e^{-\frac{1}{2}}$$

Therefore the global maximizer of f are the points $(\pm\sqrt[4]{\frac{1}{2}}, 0)$ with a value of $\sqrt{\frac{1}{2}}e^{-\frac{1}{2}}$ each.

3. Find the maximum of the function $f(x,y) = x^3 + xy$ on the unit square, and on the square with vertices $(\pm 1, \pm 1)$ and $(\pm 1, \mp 1)$.

   Solution: I will find the maximum on the square with vertices $(\pm 1, \pm 1)$ and $(\pm 1, \mp 1)$ first. Since the square is a compact set, the maximizer is either a critical point or a point on the boundary. The partial derivative with respect to x is
   $$\frac{\partial}{\partial x}x^3 + xy = 3x^2 + y$$
   This is zero for points in the form $(x, -3x^2)$. The partial derivative with respect to y is
   $$\frac{\partial}{\partial y}x^3 + xy = x$$
   This is zero for points in the form $(0, y)$ The only critical point is therefore $(0,0)$ This critical point has a value of $f(0,0) = 0$.

   Points on the top edge take the form $(x, 1)$. The expression $x^3 + x$, corresponding to the top edge values, is maximized when $x = 1$ with a value of 2.

   Points on the bottom edge take the form $(x, -1)$. The expression $x^3 - x$, corresponding to the bottom edge values, is maximized when $x = -\frac{1}{\sqrt{3}}$ with a value of $\frac{1}{3}^{\frac{1}{2}} - \frac{1}{3}^{\frac{3}{2}}$.

   Points on the right edge take the form $(1, y)$. The expression $1 + y$, corresponding to the right edge values, is maximized when $y = 1$ with a value of 2

   Points on the left edge take the form $(-1, y)$. The expression $-1 - y$, corresponding to the left edge values, is maximized when $y = -1$ with a value of 0.

   From this, we can see that the maximizer is the point $(1,1)$ with a value of 2. If only the unit square is considered, $(1,1)$ is still the maximizer since the unit square is a subset of the square centered at the origin, and $(1,1)$ is in the unit square.

4. Find the terms of order at most two in the Taylor expansion of $f(x,y) = \log(1+xy)$ at the point $(0,0)$.

   Solution: The value at $(0,0)$ is $f(0,0) = 0$ The gradient is $G_f(x,y) = [\frac{y}{1+xy}, \frac{x}{1+xy}]$ from taking the partial derivatives. The gradient at $(0,0)$ is thus $G_f(0,0) = [0,0]$. The Hessian is the matrix of the partial derivatives of the partial derivatives.

$$\begin{aligned} H_f(x,y) &= \begin{bmatrix} \frac{\partial}{\partial x}\frac{\partial}{\partial x} f & \frac{\partial}{\partial x}\frac{\partial}{\partial y} f \\ \frac{\partial}{\partial y}\frac{\partial}{\partial x} f & \frac{\partial}{\partial y}\frac{\partial}{\partial y} f \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial}{\partial x}\frac{y}{1+xy} & \frac{\partial}{\partial x}\frac{x}{1+xy} \\ \frac{\partial}{\partial y}\frac{y}{1+xy} f & \frac{\partial}{\partial y}\frac{x}{1+xy} \end{bmatrix} \\ &= \begin{bmatrix} -\frac{y^2}{(1+xy)^2} & \frac{1}{(1+xy)^2} \\ \frac{1}{(1+xy)^2} & -\frac{x^2}{(1+xy)^2} \end{bmatrix} \end{aligned}$$

   Therefore,

$$H_f(0,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

   Therefore,

$$\begin{aligned} P_2(x,y) &= f(0,0) + \langle G_f(0,0), v\rangle + \frac{1}{2}\langle H_f(0,0)v, v\rangle \\ &= 0 + 0 + \frac{1}{2}\langle [y,x], [x,y]\rangle \\ &= xy \end{aligned}$$

5. Repeat the previous problem with $f(x,y) = e^{x+y}$.

   Solution: The value at $(0,0)$ is $f(0,0) = 1$ The gradient is $G_f(x,y) = [e^{x+y}, e^{x+y}]$ since the partial derivatives of f are the function itself. Therefore the gradient at at $(0,0)$ is $G_f(0,0) = [1,1]$. Because the partial derivatives are still f, the hessian of f is the matrix

$$H_f(x,y) = \begin{bmatrix} e^{x+y} & e^{x+y} \\ e^{x+y} & e^{x+y} \end{bmatrix}$$

   Therefore the hessian at $(0,0)$ is

$$H_f(0,0) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Therefore

$$\begin{aligned} P_2(x,y) &= f(0,0) + \langle G_f(0,0), v\rangle + \frac{1}{2}\langle H_f(0,0)v, v\rangle \\ &= 1 + (x+y) + \frac{1}{2}\langle [x+y, x+y], [x,y]\rangle \\ &= 1 + (x+y) + \frac{1}{2}(x^2 + 2xy + y^2) \end{aligned}$$