MATH 31AH - Homework 7

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1 General Projections

Proof. If P is a projection, then $V = \text{image } P \oplus \ker P$.

If the intersection of image P and $\ker P$ is zero, and any vector in V can be written as a sum of a vector in image P and a vector in $\ker P$, then V is the direct sum of image P and $\ker P$. In other words, $\text{image } P \cap \ker P = 0$ and $V = \text{image } P + \ker P$ implies $V = \text{image } P \oplus \ker P$.

Assume that there exists $v \in \text{image } P \cap \ker P$ with $v \neq 0$. Since $v \in \text{image } P$, $P(x) = v$ for some $x \in V$. Since $P \circ P = P$, then $P(v) = P(P(x)) = P(x) = v \neq 0$. Since $v \in \ker P$, then $P(v) = 0$. This is a contradiction, therefore v cannot be nonzero and $\text{image } P \cap \ker P = 0$.

Every $v \in V$ can be written as $v = P(v) + (v - P(v))$. $P(v)$ is in the image of P. Since $P(v - P(v)) = P(v) - P(P(v)) = P(v) - P(v) = 0$, $(v - P(v))$ is in the kernal of P. Therefore, $V = \text{image } P + \ker P$. Since $\text{image } P \cap \ker P = 0$ and $V = \text{image } P + \ker P$, $V = \text{image } P \oplus \ker P$. □

2 Trace

Proof. $\text{tr}(AB) = \text{tr}(BA)$.

Let A and B be $n \times n$ matrices. The trace of a matrix is the sum of the diagonal entries, so

$$\text{tr}(AB) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} b_{ji} \text{ and } \text{tr}(BA) = \sum_{i=1}^{n} \sum_{j=1}^{n} b_{ij} a_{ji}$$

Since both traces sum up all term of the form $a_{ij} b_{ji}$, $\text{tr}(AB) = \text{tr}(BA)$. □

Proof. Similar matricies have the same trace.

Let A and B be similar matricies with $A = PBP^{-1}$. Since $\text{tr}(AB) = \text{tr}(BA)$, we have that

$$\begin{aligned} \text{tr}(A) &= \text{tr}(P(BP^{-1})) \\ &= \text{tr}((BP^{-1})P) \\ &= \text{tr}(BI) \\ &= \text{tr}(B) \end{aligned}$$

Therefore $\text{tr}(A) = \text{tr}(B)$ if $A \cong B$. □

Proof. $\mathrm{tr}(A) = \dim E_{\lambda_1} \cdot \lambda_1 + ... + \dim E_{\lambda_r} \cdot \lambda_r$

Since A is diagonalizable with eigenvalues $\lambda_1, ..., \lambda_r$, A is similar to a diagonal matrix, D, with the eigenvalues of A on the diagonal. Similar matricies have the same trace, so the trace of A is the trace of D, which is the sum of the eigenvalues on the diagonal matrix.

Since the dimension of an eigenspace is equal to the number of eigenvectors associated with the eigenvalue of the eigenspace, the dimension of an eigenspace is equal to the number of times that the eigenvalue appears on the diagonal of the diagonal matrix. The trace of D is equal to the sum of each eigenvalue multiplied by how many times it appears on the diagonal, so therefore, $\mathrm{tr}(A) = \mathrm{tr}(D) = \dim E_{\lambda_1} \cdot \lambda_1 + ... + \dim E_{\lambda_r} \cdot \lambda_r$.

□

3 Projections and trace

Proof. $\mathrm{tr}(P)$ is well defined.

The trace of all similar matrices are the same, so $\mathrm{tr}(P)$ is the same for all choices of $\mathcal{B}$. Therefore, $\mathrm{tr}(P)$ is well defined when V is finite dimensional. □

Proof. $\mathrm{tr}(P) = \dim \mathrm{Image}(P)$

Let $\mathcal{B} = \{e_1, ..., e_s\}$ be an ordered basis for $\mathrm{Image}(P)$. Since $\mathrm{Image}(P) \subseteq V$, $\mathcal{B}$ can be completed to form an ordered basis for V, $\mathcal{C} = \{e_1, ..., e_s, e_{s+1}, ..., e_n\}$. Let $[P]_{\mathcal{C}}^{\mathcal{C}}$ be the matrix representation of P in basis $\mathcal{C}$. A vector is in $\mathrm{Image}(P)$ iff the associated vector in $\mathbb{F}^n$ has zero in entries from $s+1$ through n; otherwise, $\mathcal{B}$ would no longer be a basis for $\mathrm{Image}(P)$.

Since $P \circ P = P$, if a vector $v \in \mathrm{Image}(P)$, then $P(v) = v$. Therefore the top-left $s \times s$ block of $[P]_{\mathcal{C}}^{\mathcal{C}}$ would be the identity matrix. Since all $v \in \mathrm{Image}(P)$ have zeros in entries from $s+1$ through n, the bottom $n-s$ rows of $[P]_{\mathcal{C}}^{\mathcal{C}}$ are zeros. Therefore the block matrix form of $[P]_{\mathcal{C}}^{\mathcal{C}}$ is

$$[P]_{\mathcal{C}}^{\mathcal{C}} = \begin{bmatrix} I_{s\times s} & B \\ 0 & 0 \end{bmatrix}$$

The trace of $[P]_{\mathcal{C}}^{\mathcal{C}}$ would therefore be s, which is simply the dimension of the image. Since the choice of basis does not change the trace of a transformation, we have that $\mathrm{tr}(P) = \dim \mathrm{Image}(P)$.

□

4 Polynomial Gram-Schmidt

For f_0, we have that

$$f_0 = \frac{1}{||1||} = \frac{1}{\sqrt{\int_{-1}^{1} 1\, dt}} = \frac{1}{\sqrt{2}}$$

For f_1, we have that

$$t - \langle t, \frac{1}{\sqrt{2}} \rangle \cdot \frac{1}{\sqrt{2}} = t - \int_{-1}^{1} \frac{1}{2} t\, dt = t - \left[\frac{1}{4}t^2\right]_{-1}^{1} = t$$

$$||t - \langle t, \frac{1}{\sqrt{2}} \rangle \cdot \frac{1}{\sqrt{2}}|| = ||t|| = \sqrt{\int_{-1}^{1} t^2\, dt} = \sqrt{\left[\frac{1}{3}t^3\right]_{-1}^{1}} = \sqrt{\frac{2}{3}}$$

$$f_1 = \frac{t - \langle t, \frac{1}{2} \rangle \cdot \frac{1}{2}}{||t - \langle t, \frac{1}{2} \rangle \cdot \frac{1}{2}||} = \frac{t}{\sqrt{\frac{2}{3}}} = \sqrt{\frac{3}{2}} t$$

For f_2 we have that

$$\begin{aligned} t^2 - \langle t^2, \sqrt{\frac{3}{2}} t \rangle \cdot \sqrt{\frac{3}{2}} t - \langle t^2, \frac{1}{\sqrt{2}} \rangle \cdot \frac{1}{\sqrt{2}} &= t^2 - \int_{-1}^{1} \frac{3}{2} t^3 \, dt \cdot t - \int_{-1}^{1} \frac{1}{2} t^2 \, dt \\ &= t^2 - \left[\frac{3}{8} t^4 \right]_{-1}^{1} \cdot t - \left[\frac{1}{6} t^3 \right]_{-1}^{1} \\ &= t^2 - \frac{1}{3} \end{aligned}$$

$$\begin{aligned} ||t^2 - \langle t^2, \sqrt{\frac{3}{2}} t \rangle \cdot \sqrt{\frac{3}{2}} t - \langle t^2, \frac{1}{\sqrt{2}} \rangle \cdot \frac{1}{\sqrt{2}}|| &= ||t^2 - \frac{1}{3}|| \\ &= \sqrt{\int_{-1}^{1} t^4 - \frac{2}{3} t^2 + \frac{1}{9} \, dt} \\ &= \sqrt{\left[\frac{1}{5} t^5 - \frac{2}{9} t^3 + \frac{1}{9} t \right]_{-1}^{1}} \\ &= \sqrt{\frac{8}{45}} \end{aligned}$$

$$f_2 = \frac{t^2 - \langle t^2, \sqrt{\frac{3}{2}} t \rangle \cdot \sqrt{\frac{3}{2}} t - \langle t^2, \frac{1}{\sqrt{2}} \rangle \cdot \frac{1}{\sqrt{2}}}{||t^2 - \langle t^2, \sqrt{\frac{3}{2}} t \rangle \cdot \sqrt{\frac{3}{2}} t - \langle t^2, \frac{1}{\sqrt{2}} \rangle \cdot \frac{1}{\sqrt{2}}||} = \frac{t^2 - \frac{1}{3}}{\sqrt{\frac{8}{45}}} = \sqrt{\frac{45}{8}} (t^2 - \frac{1}{3})$$

For f_3 we have that

$$\begin{aligned} & t^3 - \langle t^3, \sqrt{\frac{45}{8}} (t^2 - \frac{1}{3}) \rangle \cdot \sqrt{\frac{45}{8}} (t^2 - \frac{1}{3}) - \langle t^3, \sqrt{\frac{3}{2}} t \rangle \cdot \sqrt{\frac{3}{2}} t - \langle t^3, \frac{1}{\sqrt{2}} \rangle \cdot \frac{1}{\sqrt{2}} \\ &= t^3 - \int_{-1}^{1} \frac{45}{8} (t^5 - \frac{1}{3} t^3) \, dt \cdot (t^2 - \frac{1}{3}) - \int_{-1}^{1} \frac{3}{2} t^4 \, dt \cdot t - \int_{-1}^{1} \frac{1}{2} t^3 \, dt \\ &= t^3 - \frac{45}{8} \left[\frac{1}{6} t^6 - \frac{1}{12} t^4 \right]_{-1}^{1} \cdot (t^2 - \frac{1}{3}) - \left[\frac{3}{10} t^5 \right]_{-1}^{1} \cdot t - \left[\frac{1}{8} t^4 \right]_{-1}^{1} \\ &= t^3 - \frac{3}{5} t \end{aligned}$$

$$\begin{aligned}
||t^3 - \langle t^3, \sqrt{\frac{45}{8}}(t^2 - \frac{1}{3})\rangle \cdot \sqrt{\frac{45}{8}}(t^2 - \frac{1}{3}) - \langle t^3, \sqrt{\frac{3}{2}}t\rangle \cdot \sqrt{\frac{3}{2}}t - \langle t^3, \frac{1}{\sqrt{2}}\rangle \cdot \frac{1}{\sqrt{2}}|| \\
= ||t^3 - \frac{3}{5}t|| \\
\\
= \sqrt{\int_{-1}^{1} t^6 - \frac{6}{5}t^4 + \frac{9}{25}t^2\,dt} \\
= \sqrt{\left[\frac{1}{7}t^7 - \frac{6}{25}t^5 + \frac{3}{25}t^3\right]_{-1}^{1}} \\
= \sqrt{\frac{8}{175}}
\end{aligned}$$

$$\begin{aligned}
f_3 &= \frac{t^3 - \langle t^3, \sqrt{\frac{45}{8}}(t^2 - \frac{1}{3})\rangle \cdot \sqrt{\frac{45}{8}}(t^2 - \frac{1}{3}) - \langle t^3, \sqrt{\frac{3}{2}}t\rangle \cdot \sqrt{\frac{3}{2}}t - \langle t^3, \frac{1}{\sqrt{2}}\rangle \cdot \frac{1}{\sqrt{2}}}{||t^3 - \langle t^3, \sqrt{\frac{45}{8}}(t^2 - \frac{1}{3})\rangle \cdot \sqrt{\frac{45}{8}}(t^2 - \frac{1}{3}) - \langle t^3, \sqrt{\frac{3}{2}}t\rangle \cdot \sqrt{\frac{3}{2}}t - \langle t^3, \frac{1}{\sqrt{2}}\rangle \cdot \frac{1}{\sqrt{2}}||} \\
&= \frac{t^3 - \frac{3}{5}t}{\sqrt{\frac{8}{175}}} \\
&= \sqrt{\frac{175}{8}}(t^3 - \frac{3}{5}t)
\end{aligned}$$

Therefore, a orthonormal basis for V_3 is $(\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}t, \sqrt{\frac{45}{8}}(t^2 - \frac{1}{3}), \sqrt{\frac{175}{8}}(t^3 - \frac{3}{5}t))$

5 Legendre Polynomial

Proof. $\langle P_n(t), P_m(t)\rangle = 0$ for $n \neq m$

Let $n < m$ let

$$u = \frac{d^n}{dt^n}(t^2 - 1)^n \quad du = \frac{d^{n+1}}{dt^{n+1}}(t^2 - 1)^n dt$$

$$v = \frac{d^{m-1}}{dt^{m-1}}(t^2 - 1)^m \quad dv = \frac{d^m}{dt^m}(t^2 - 1)^m dt$$

Because of the chain rule, any derivative of $(t^2 - 1)^m$ up to the $m - 1$th derivative will have $t^2 - 1$ as a term and equal zero when $t = -1, 1$. Since the largest power in $(t^2 - 1)^n$ is t^{2n} and

$n+m > 2n$, the $m+n$th derivative of $(t^2-1)^n$ must equal zero. We have that

$$\begin{aligned}
\int_{-1}^{1} \frac{d^n}{dt^n}(t^2-1)^n \frac{d^m}{dt^m}(t^2-1)^m \, dt &= \left[\frac{d^n}{dt^n}(t^2-1)^n \frac{d^{m-1}}{dt^{m-1}}(t^2-1)^m\right]_{-1}^{1} - \int_{-1}^{1} \frac{d^{m-1}}{dt^{m-1}}(t^2-1)^m \frac{d^{n+1}}{dt^{n+1}}(t^2-1)^n \, dt \\
&= -\int_{-1}^{1} \frac{d^{n+1}}{dt^{n+1}}(t^2-1)^n \frac{d^{m-1}}{dt^{m-1}}(t^2-1)^m \, dt \\
&= \int_{-1}^{1} \frac{d^{n+2}}{dt^{n+2}}(t^2-1)^n \frac{d^{m-2}}{dt^{m-2}}(t^2-1)^m \, dt \\
&\vdots \\
&= \int_{-1}^{1} \frac{d^{n+m}}{dt^{n+m}}(t^2-1)^n \frac{d^0}{dt^0}(t^2-1)^m \, dt \\
&= \int_{-1}^{1} (t^2-1)^m \frac{d^{n+m}}{dt^{n+m}}(t^2-1)^n \, dt \\
&= 0
\end{aligned}$$

Since the inner product is commutative, and the inner product equals

$$\langle P_n(t), P_m(t)\rangle = \frac{1}{2^{n+m} n! m!} \int_{-1}^{1} \frac{d^n}{dt^n}(t^2-1)^n \frac{d^m}{dt^m}(t^2-1)^m \, dt = 0$$

we have that $\langle P_n(t), P_m(t)\rangle = 0$ for $n \neq m$.

$\square$

6 Inner products and dual spaces

Proof. φ is an isomorphism over $\mathbb{R}$ and $\mathbb{C}$. ψ is an isomorphism only over $\mathbb{R}$.

Let $\mathcal{B} = \{e_1, ..., e_n\}$ be an orthonormal basis for V, and let $\mathcal{B}^* = \{\lambda_1, ..., \lambda_n\}$ be the corresponding dual basis for V^*. Let i and j be arbitrary indexing variables.
When $i = j$,

$$\begin{aligned}
\varphi(e_i)(e_i) &= \langle e_i, e_i\rangle \\
&= 1 \\
&= \lambda_i(e_i)
\end{aligned}$$

When $i \neq j$,

$$\begin{aligned}
\varphi(e_i)(e_j) &= \langle e_i, e_j\rangle \\
&= 0 \\
&= \lambda_i(e_j)
\end{aligned}$$

Therefore $\varphi(e_i) = \lambda_i$ for all i. Let $v \in V$ be a vector such that $v = c_1e_1 + ... + c_ne_n$, let $v' \in V$ be a different vector such that $v' = d_1e_1 + ... + d_ne_n$ and let $w \in V$ be an arbitrary vector.

$$\begin{aligned}
\varphi(v)(w) &= \langle v, w\rangle \\
&= \langle c_1e_1 + ... + c_ne_n, w\rangle \\
&= c_1\langle e_1, w\rangle + ... + c_n\langle e_n, w\rangle \\
&= c_1\varphi(e_1)(w) + ... + c_n\varphi(e_n)(w) \\
&= c_1\lambda_1(w) + ... + c_n\lambda_n(w)
\end{aligned}$$

By a similar logic,

$$\varphi(v')(w) = d_1\lambda_1(w) + ... + d_n\lambda_n(w)$$

Since $\varphi(v)(w) \neq \varphi(v')(w)$, φ is injective. Since V is finite dimensional, $\dim V = \dim V^*$, φ is bijective. Since $\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$ and $\langle cu, w\rangle = c\langle u, w\rangle$, φ preserves addition and scalar multiplication. Since φ is bijective and preserves algebra, it is an isomorphism. When $\mathbb{F} = \mathbb{R}$, we have that $\langle v, w\rangle = \langle w, v\rangle$ so $\varphi = \psi$. Therefore, φ and ψ are both isomorphisms in $\mathbb{R}$. When $\mathbb{F} = \mathbb{C}$,

$$\begin{aligned}
\psi(cv)(w) &= \langle w, cv\rangle \\
&= \bar{c}\langle w, v\rangle \\
&= \bar{c}\psi(v)(w)
\end{aligned}$$

Thus, ψ is no longer an isomorphism in $\mathbb{C}$ since scalar multiplication is no longer preserved; however, φ remains an isomorphism in $\mathbb{C}$.

$\square$

7 Matrices and Inner Products

Proof. $\langle v, w\rangle = v^TA^TAw$ iff A is invertible

v^TA^TAw is an inner product iff it satisfies the properties of inner products.
For addition

$$\begin{aligned}
\langle u + v, w\rangle &= (u + v)^TA^TAw \\
&= (u^T + v^T)A^TAw \\
&= u^TA^TAw + v^TA^TAw \\
&= \langle u, w\rangle + \langle v, w\rangle
\end{aligned}$$

For scalar multiplication

$$\begin{aligned}
\langle cv, w\rangle &= (cv)^TA^TAw \\
&= cv^TA^TAw \\
&= c\langle v, w\rangle
\end{aligned}$$

For symmetry, the transpose of a 1×1 matrix is itself.

$$\begin{aligned}
\langle v, w \rangle &= v^T A^T A w \\
&= ((v^T A^T A w)^T)^T \\
&= (w^T A^T A v)^T \\
&= w^T A^T A v \\
&= \langle w, v \rangle
\end{aligned}$$

For positive-definiteness

$$\begin{aligned}
\langle v, v \rangle &= v^T A^T A v \\
&= (Av)^T A v \\
&= Av \cdot Av \\
&\geq 0
\end{aligned}$$

For $\langle v, v \rangle = 0$ iff $v = 0$

$$\begin{aligned}
\langle v, v \rangle = 0 &\iff v^T A^T A v = 0 \\
&\iff Av \cdot Av = 0 \\
&\iff Av = 0
\end{aligned}$$

If A is invertible, then $Av = 0$ iff $v = 0$. If A is not invertible, then there exists $v \neq 0$ with $Av = 0$. Therefore, $\langle v, w \rangle = v^T A^T A w$ iff A is invertible. □

8 Symmetric and Orthogonal

Proof. Symmetric matrices are an -vector space while orthogonal matrices are not.

Symmetric matrices are an -vector space since the zero matrix is symmetric and

$$A + B = A^T + B^T = (A + B)^T$$

$$cA = cA^T = (cA)^T$$

Symmetric matricies have dimension equal to the number of upper triangular entries.

$$\dim A = 1 + ... + n = \frac{n(n+1)}{2}$$

Orthogonal matricies do not form a vector space since the zero matrix is not orthogonal. Addition is also not closed since the identity matrix is orthogonal, but the sum of the identity matrix with itself is not orthogonal. □