Math 140C: Homework 3

Merrick Qiu

Rudin 9.13

We have that $|f(t)|^2 = f(t) \cdot f(t) = 1$. Differentiating this yields

$$f(t) \cdot f'(t) + f'(t) \cdot f(t) = 2f'(t) \cdot f(t) = 0,$$

which implies that $f'(t) \cdot f(t) = 0$. Geometrically, the velocity of a partical on the surface of a sphere is perpenticular to the radius of that sphere.

Rudin 9.14

(a) At the origin

$$D_1 f(0,0) = \lim_{x\to 0} \frac{f(x,0)}{x} = \lim_{x\to 0} \frac{x}{x} = 1$$

$$D_1 f(0,0) = \lim_{y\to 0} \frac{f(0,y)}{y} = \lim_{y\to 0} \frac{0}{y} = 0$$

When $(x, y) \neq 0$ the partial derivatives can be bounded by

$$\begin{aligned} D_1 f &= \frac{3x^2(x^2+y^2) - 2x^4}{(x^2+y^2)^2} \\ &= \frac{x^2(x^2+3y^2)}{(x^2+y^2)^2} \\ &\leq \frac{3x^2(x^2+y^2)}{(x^2+y^2)^2} \\ &\leq \frac{3(x^2+y^2)^2}{(x^2+y^2)^2} \\ &= 3 \end{aligned}$$

$$\begin{aligned} D_2 f &= \frac{0 - 2x^3 y}{(x^2+y^2)^2} \\ &= -\frac{2x^3 y}{(x^2+y^2)^2} \\ &= \frac{(x^2+y^2-(x+y)^2)x^2}{(x^2+y^2)^2} \\ &\leq \frac{(x^2+y^2)^2}{(x^2+y^2)^2} \\ &= 1. \end{aligned}$$

(b) If $\boldsymbol{u} = (x, y)$ and $x^2 + y^2 = 1$ then

$$D_u f(0,0) = \lim_{t\to 0} \frac{f(tx,ty) - f(0,0)}{t} = x^3.$$

$$|x^3| \leq 1.$$

(c) Since the total derivative of f exists away form the origin, by the chain rule we have that $g'(t) = f'(\gamma(t))\gamma'(t)$ exists when $\gamma(t)$ is away from the origin.

If $\gamma(t) = (x(t), y(t))$ and $\gamma(t_0) = 0$, then

$$\begin{aligned}
g'(0) &= \lim_{t \to t_0} \frac{g(t) - g(t_0)}{t - t_0} \\
&= \frac{f(x(t), y(t)) - f(x(t_0), y(t_0))}{t - t_0} \\
&= \frac{\frac{x(t)^3}{x(t)^2 + y(t)^2}}{t - t_0} \\
&= \frac{\left(\frac{x(t) - x(t_0)}{t - t_0}\right)}{\left(\frac{x(t) - x(t_0)}{t - t_0}\right)^2 + \left(\frac{y(t) - y(t_0)}{t - t_0}\right)^2} \\
&= \frac{x'(t_0)^3}{x'(t_0)^2 + y'(t_0)^2}.
\end{aligned}$$

Thus g is differentiable everywhere. If $\gamma'(t)$ is continuous, then note that $g'(t)$ is also continuous away from the origin. $\gamma'(t)$ is continuous at the origin since

$$\begin{aligned}
\lim_{t \to t_0} g'(t) &= \lim_{t \to t_0} \frac{x(t)^4 x'(t) + 3x(t)^2 y(t)^2 x'(t) - 2x(t)^3 y(t) y'(t)}{(x(t)^2 + y(t)^2)^2} \\
&= \lim_{t \to t_0} \frac{((t - t_0)x'(t))^4 x'(t) + 3((t - t_0)x'(t))^2((t - t_0)y'(t))^2 x'(t) - 2x(t)^3 y(t) y'(t)}{(((t - t_0)x'(t))^2 + ((t - t_0)y'(t))^2)^2} \\
&= \frac{x'(t_0)^5 + x'(t_0)^3 y'(t_0)^2}{(x'(t_0)^2 + y'(t_0))^2} \\
&= \frac{x'(t_0)^3}{x'(t_0)^2 + y'(t_0)^2} \\
&= g'(t_0)
\end{aligned}$$

(d) The partial derivatives indicate that the derivative should be

$$\sum_{i=1}^{n} (D_i f)(\boldsymbol{x}) u_i = u_1.$$

However part (b) says that

$$(D_u f)(x) = u_1^3$$

which is a contradiction.

Rudin 9.15

(a) The inequality holds because

$$\begin{aligned}
4x^4y^2 &\le (x^4+y^2)^2 \\
4x^4y^2 &\le x^8 + 2x^4y^2 + y^4 \\
0 &\le x^8 - 2x^4y^2 + y^4 \\
0 &\le (x^4 - y^2)^2.
\end{aligned}$$

Since

$$\frac{4x^6y^2}{(x^4+y^2)^2} \le \frac{x^2(x^4+y^2)^2}{(x^4+y^2)^2} = x^2$$

and all the other terms tend to 0, we have that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$ so f is continuous.

(b)

$$g_\theta(0) = f(0,0) = 0$$

$$\begin{aligned}
g'_\theta(0) &= \lim_{t\to 0} \frac{f(t\cos\theta, t\sin\theta) - f(0,0)}{t} \\
&= \lim_{t\to 0} \frac{1}{t}\left[(t\cos\theta)^2 + (t\sin\theta)^2 - 2(t\cos\theta)(t\sin\theta) - \frac{4(t\cos\theta)^6(t\sin\theta)^2}{((t\cos\theta)^4 + (t\sin\theta)^2)^2}\right] \\
&= \lim_{t\to 0} t - 2t\cos\theta\sin\theta - 4t^5\frac{\cos^6\theta\sin^2\theta}{(t^2\cos^4\theta + \sin^2\theta)^2} \\
&= 0
\end{aligned}$$

When $t \ne 0$ we have that

$$g'_\theta(t) = 2t - 6t^2\cos^2\theta\sin\theta - 4\cos^6\theta\sin^2\theta\left(\frac{4t^3(t^2\cos^4\theta + \sin^2\theta)^2 - 4t^5\cos^4(t^2\cos^4\theta + \sin^2\theta)}{(t^2\cos^4\theta+\sin^2\theta)^4}\right)$$

Therefore,

$$g''_\theta(0) = \lim_{t\to 0}\frac{g'(t) - g'(0)}{t} = 2$$

(c) $(0,0)$ is not a local minimum for f since $f(x,x^2) = -x^4$ and $-x^4$ is strictly decreasing.

Rudin 9.16

If $f(0) = 0$ and

$$f(t) = t + 2t^2 \sin\left(\frac{1}{t}\right)$$

then

$$f'(0) = \lim_{t\to 0} \frac{f(t) - f(0)}{t} = \lim_{t\to 0} 1 + 2t \sin\left(\frac{1}{t}\right) = 1$$

and when $t \neq 0$

$$f'(t) = 1 - 2\cos\left(\frac{1}{t}\right) + 4t \sin\left(\frac{1}{t}\right)$$

so f' is not continuous at 0. $|f'| \leq 7$ is bounded since cos and sin have values from -1 to 1. For any neighborhood around 0, there always exists points at which f is increasing and points at which it is decreasing since $f'(\frac{1}{n\pi}) = 1 + 2(-1)^n$ which is positive at even n and negative otherwise, so f cannot be one to one.

Rudin 9.17

(a) The range is $\mathbb{R}^2$ except for $(0,0)$ since cosine and sine cannot both be zero.

(b) The Jacobian is always zero, so by the inverse function theorem every point has a neighborhood in which it is one-to-one. However it is not one-to-one in the whole space since cosine and sine are periodic.

$$\begin{aligned} J_f(x) &= \det \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} \\ &= \det \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix} \\ &= e^{2x} \cos^2 x + e^{2x} \sin^2 x = e^{2x} \end{aligned}$$

(c) We can take $g(x,y) = (\log \sqrt{x^2+y^2}, \arctan \frac{y}{x})$ as the inverse. Plugging in $\boldsymbol{a}$ into the derivative calculated in (b) gives

$$f'(\boldsymbol{a}) = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

The derivative of g is

$$g'(x,y) = \begin{bmatrix} \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{bmatrix}$$

Plugging in $\boldsymbol{b} = (1/2, \sqrt{3}/2)$ yields

$$g'(\boldsymbol{b}) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

Thus formula (52) is true since

$$g'(\boldsymbol{b})f'(\boldsymbol{a}) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(d) Since x determines magnitude and y determines phase, lines perpendicular to the x-axis become circles around the origin and lines perpendicular to the y-axis become lines radiating away from the origin.

Rudin 9.19

Subtracting the second and the third equation from the first yields

$$u^2 - 3u = 0$$

which implies that $u = 0, 3$ and so we cant solve for x, y, z in terms of u. However for the other 3 variables, the matrix of the first three variables only has 2 independent columns, so we can simply fix u and solve for the two remaining variables in terms of the variable we choose.