**Math 31AH: Fall 2021**
**Midterm**
**Wednesday, 10/27/2021**

**Instructions:** This is a 50 minute closed notes, closed books exam. Consultation with other humans is prohibited, including humans acting via websites such as Chegg. You need to clearly prove your claims; unsupported claims will get little credit. Please upload your exam to Gradescope after you are finished. You will have 10 additional minutes to upload your solutions to Gradescope.

**Problem 1:** [20] Let $\mathbb{F}$ be a field. Endow $\mathbb{F}^2$ with binary operations

$$(a,b)+(a',b') := (a+a', b+b') \qquad (a,b)\cdot(a',b') := (a\cdot a', b\cdot b')$$

Prove that these operations do **not** turn $\mathbb{F}^2$ into a field.

**Problem 2:** [10+10] Consider the two bases of $\mathbb{R}^2$ given by $\mathcal{B} = \{\mathbf{e}_1, \mathbf{e}_2\}$, $\mathcal{C} = \{\mathbf{v}_1, \mathbf{v}_2\}$ where

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection across the line $y = x$ and let $A, B \in \mathrm{Mat}_{2\times 2}(\mathbb{R})$ be the matrices

$$A := [T]_{\mathcal{B}}^{\mathcal{B}} \qquad B := [T]_{\mathcal{C}}^{\mathcal{C}}$$

(1) Calculate $A$.
(2) Find an invertible matrix $P$ such that $B = PAP^{-1}$.

**Problem 3:** [15] For a positive integer $n$, let $V_n$ be the $\mathbb{R}$-vector space of polynomials $f(t)$ of degree $\leq n$ in the variable $t$ with real coefficients. What is the dimension of the $\mathbb{R}$-vector space $\mathrm{Hom}(V_5, V_3) = \{T : V_5 \to V_3 \,:\, T \text{ is a linear transformation}\}$?

**Problem 4:** [25] If $T : V \to V$ is a linear transformation, a subspace $W \subseteq V$ is *$T$-invariant* if $T(\mathbf{w}) \in W$ for all $\mathbf{w} \in W$. If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y+z \\ 2x+2y+2z \\ -x-y-z \end{pmatrix}$$

find **all** $T$-invariant subspaces $W \subseteq \mathbb{R}^3$.

**Problem 5:** [20] Give an example of a vector space $V$ and a linear map $T : V \to V$ which is injective but not surjective.