MATH 31AH - Practice Final

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1 Strict containments

Let $T : \mathbb{R}_1[x] \to \mathbb{R}_1[x]$ be the derivative of polynomials of degree one. Since $\mathbb{R}_1[x] \supset \mathbb{R}_0[x] \supset 0$ and $T(0) = 0$, the linear transformation fufills the requirements.

2 Invertible matrices

The matrix is is invertible if all the rows are linearly independent. There are $5^3 - 1$ choices for the first row since the first row canot be zero. There are $5^3 - 5$ choices for the second row since the second row cannot be a scalar multiple(which there are five of) of the first row. There are $5^3 - 5^2$ choices for the third row since the third row cannot be a linear combination of the first two rows. Multiplying all these possible permutations yields $(5^3 - 1)(5^3 - 5)(5^3 - 5^2)$

3 Is this matrix invertible?

Yes it is invertible since the row eschelon is upper triangular.

$$
\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}
$$
$$
\Rightarrow \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}
$$
$$
\Rightarrow \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}
$$
$$
\Rightarrow \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \\ 1 & 1 & 1 & 0 \end{pmatrix}
$$
$$
\Rightarrow \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}
$$

4 Linear injection into double dual

Let $\varphi : V \to V^{**}$ with $\varphi(v)(\lambda) := \lambda(v)$. This is a linear injection. See 5.3 for proof.

5 Gram schmidt

We need a vector that is orthogonal with $\sqrt{3}t$ Do some calculus and $3\sqrt{3}t - 2\sqrt{3}$ will work.

$$
\int_0^1 9t^2 - 6t = \left[3t^3 - 3t^2\right]_0^1 = 0
$$

Normalize $3\sqrt{3}t - 2\sqrt{3}$ and do regular gram schmidt on t^2

6 Not diagonalizable square complex matrix

A matrix is orthogonally diagonalizable iff it is self-adjoint. This matrix is diagonalizable since it has eigenvalues $\lambda = 3, -1$

$$\begin{bmatrix} 1 & i \\ -4i & 1 \end{bmatrix}$$

7 Eigenvalues

Since it has n eigenvalues, the polynomial can be rewritten as $(\lambda_1 - t)(\lambda_2 - t)...(\lambda_n - t)$. Thus a_0 will be the product of the lambdas.

8 Well defined?

Use universal property of quotient spaces.

$$\begin{aligned}
\varphi(c_1\lambda_1 + c_2\lambda_2, v, w) &= (c_1\lambda_1 + c_2\lambda_2)(v)(w) \\
&= (c_1\lambda_1(v) + c_2\lambda_2(v))w \\
&= c_1\lambda_1(v)w + c_2\lambda_2(v)w \\
&= c_1\varphi(\lambda_1, v, w) + c_2\varphi(\lambda_2, v, w)
\end{aligned}$$

Do same thing for other three. Its multilinear so its well defined.

9 Eigenvalues of tensor

It will have eigenvalues that are all possible products of the eigenvalues of V and W.

$$\begin{aligned}
(T \otimes U)(v_i \otimes w_j) &= T(v_i) \otimes U(w_j) \\
&= \lambda_i v_i \otimes \mu_j w_j \\
&= \lambda_i \mu_j (v_i \otimes w_j)
\end{aligned}$$

10 Adjoint matrixes

Proof. Av is surjective.

Let $A = \{a_1, ..., a_n\}^T$ be an $n \times m$ complex matrix. Then the ijth entry of AA^* would be $a_i \cdot \overline{a_j}$. Let $B = \{b_1, ..., b_n\}$ be an $r \times n$ complex matrix. Then the ijth entry of B^*B would be $\overline{b_i} \cdot b_j$. Since $AA^* - B^*B = I_n$, we have that

$$(AA^*)_{ij} - (B^*B)_{ij} = a_i \cdot \overline{a_j} - b_j \cdot \overline{b_i} = \langle a_i, a_j \rangle - \langle b_j, b_i \rangle = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

Since the standard inner product of a vector to itself is real and nonnegative, we have that $(B^*B)_{ii} \geq 0$ and $(AA^*)_{ii} = (B^*B)_{ii} + 1 \geq 1$ on the diagonals.

Assume that there exists a nonzero $v \in \mathbb{C}^n$ such that $AA^*v = 0$. Therefore we have that

$$\begin{aligned} AA^*v - B^*Bv = I_n v &\implies -B^*Bv = I_n v \\ &\implies B^*Bv = -v \end{aligned}$$

This contradicts the fact that the diagonal entries of AA^* are greater than or equal to 1, so AA^* must have a kernel of zero. Since AA^* is square, AA^* is invertible and there exists an inverse $AA^*(AA^*)^{-1} = I$. Therefore, $A^*(AA^*)^{-1}$ is the right inverse of A and so A is surjective. $\square$