Math 120A: Homework 3

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Page 43 Problem 2(b)

$$
\begin{aligned}
f(z) &= \frac{\overline{z}^2}{z} \\
&= \frac{\overline{z}^3}{z\overline{z}} \\
&= \frac{(x-iy)^3}{x^2+y^2} \\
&= \frac{x^3 - 3ix^2y - 3xy^2 + iy^3}{x^2+y^2} \\
&= \frac{x^3 - 3xy^2}{x^2+y^2} + i\frac{y^3 - 3x^2y}{x^2+y^2}
\end{aligned}
$$

Page 43 Problem 3

$$
\begin{aligned}
f(z) &= x^2 - y^2 - 2y + i(2x - 2xy) \\
&= \left(\frac{z+\overline{z}}{2}\right)^2 - \left(\frac{z-\overline{z}}{2i}\right)^2 - \frac{z-\overline{z}}{i} + i\left(z + \overline{z} - \frac{(z+\overline{z})(z-\overline{z})}{2i}\right) \\
&= \frac{z^2 + 2z\overline{z} + \overline{z}^2}{4} - \frac{z^2 - 2z\overline{z} + \overline{z}^2}{-4} + i(z - \overline{z}) + i\left(z + \overline{z} - \frac{z^2 - \overline{z}^2}{2i}\right) \\
&= \frac{z^2 + \overline{z}^2}{2} + 2iz - \frac{z^2 - \overline{z}^2}{2} \\
&= \overline{z}^2 + 2iz
\end{aligned}
$$

Page 43-44 Problem 4

$$
\begin{aligned}
f(z) &= z + \frac{1}{z} \\
&= re^{i\theta} + \frac{1}{r}e^{-i\theta} \\
&= r\cos(\theta) + ir\sin(\theta) + \frac{1}{r}\cos(-\theta) + i\frac{1}{r}\sin(-\theta) \\
&= \left(r + \frac{1}{r}\right)\cos(\theta) + i\left(r - \frac{1}{r}\right)\sin(\theta)
\end{aligned}
$$

Page 44 Problem 5

The mapping $w = z^2$ is equivalent to the mapping

$$u = x^2 - y^2, \quad v = 2xy$$

For the image to be in the square domain, we want to find all x, y so that $1 \le u \le 2$ and $1 \le v \le 2$. Thus the preimage of the square is the following region

$$R = \{(x,y) | 1 \le x^2 - y^2 \le 2, 1 \le 2xy \le 2\}$$

This region includes two curved boxes in the first and third quadrant bound by the hyperbolas $x^2 - y^2 = 1$, $x^2 - y^2 = 2$, $y = \frac{1}{2x}$, and $y = \frac{1}{x}$.

Problem 4

1. The limit is $4i$. Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{3}$. If $|z - i| < \delta$, then

$$\begin{aligned} |f(z) - L| &= |(3z + i) - 4i| \\ &= 3|z - i| \\ &< 3\delta \\ &= \epsilon. \end{aligned}$$

2. The limit is 3. Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{3}$. If $|z - 1| < \delta$, then

$$\begin{aligned} |f(z) - L| &= |(2z + \overline{z}) - 3| \\ &= |(3x - 3) + iy| \\ &\leq |(3x - 3) + 3iy| \\ &= 3|(x - 1) + iy| \\ &= 3|z - 1| \\ &< 3\delta \\ &= \epsilon. \end{aligned}$$

3. The limit is 0. Let $\epsilon > 0$. Choose $\delta = \epsilon$. If $|z| < \delta$, then

$$\begin{aligned} |f(z) - L| &= \left|\frac{\overline{z}^2}{z}\right| \\ &= \frac{|\overline{z}|^2}{|z|} \\ &= \frac{|z|^2}{|z|} \\ &= |z| \\ &< \delta \\ &= \epsilon \end{aligned}$$