Math 170C: Homework 1

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Problem 1

We have that

$$
\begin{aligned}
& & x(0) &= 2 \\
x'(t) &= -tx^2 & x'(0) &= 0 \\
x''(t) &= -x^2 - 2txx' = -x^2 + 2t^2x^3 & x''(0) &= -4
\end{aligned}
$$

$$
\begin{aligned}
x(0.1) &\approx x(0) + 0.1x'(0) + \frac{0.1^2}{2}x''(0) \\
&= 2 + 0 - 0.02 \\
&= 1.98
\end{aligned}
$$

Problem 2

The second derivative is

$$\begin{aligned}
x''(t) &= 2xx' + xe^t + x'e^t \\
&= (2x^3 + 2x^2e^t) + (xe^t) + (x^2e^t + xe^{2t}) \\
&= 2x^3 + 3x^2e^t + xe^t + xe^{2t}.
\end{aligned}$$

The third derivative is

$$\begin{aligned}
x'''(t) &= (6x^2x') + (6xx'e^t + 3x^2e^t) + (x'e^t + xe^t) + (x'e^{2t} + 2xe^{2t}) \\
&= (6x^4 + 6x^3e^t) + (6x^3e^t + 6x^2e^{2t} + 3x^2e^t) + (x^2e^t + xe^{2t} + xe^t) + (x^2e^{2t} + xe^{3t} + 2xe^{2t}) \\
&= 6x^4 + 12x^3e^t + 7x^2e^{2t} + 4x^2e^t + 3xe^{2t} + xe^t + xe^{3t}
\end{aligned}$$

We have that

$$\begin{aligned}
& & x(0) &= 1 \\
x'(t) &= x^2 + xe^t & x'(0) &= 2 \\
x''(t) &= 2x^3 + 3x^2e^t + xe^t + xe^{2t} & x''(0) &= 7 \\
x'''(t) &= 6x^4 + 12x^3e^t + 7x^2e^{2t} + 4x^2e^t + 3xe^{2t} + xe^t + xe^{3t} & x'''(0) &= 34
\end{aligned}$$

$$\begin{aligned}
x(0.01) &\approx x(0) + 0.01x'(0) + \frac{0.01^2}{2}x''(0) + \frac{0.01^3}{6}x'''(0) \\
&= 1 + 0.02 + 0.00035 + 0.000005\overline{6} \\
&= 1.020355\overline{6}
\end{aligned}$$

Problem 3

$$x' = \cos(tx)$$

$$x'' = -\sin(tx)(x + tx')$$

$$x''' = -\sin(tx)(x'+x'+tx'')-\cos(tx)(x+tx')^2 = -\sin(tx)(2x'+tx'')-\cos(tx)(x+tx')^2$$

$$x^{(4)} = -\sin(tx)(3x''+tx''')-\cos(tx)(x+tx')(2x'+tx'')-2\cos(tx)(x+tx')(2x'+x'')+\sin(tx)(x+tx')^3$$

Problem 4

We need to impose

$$\begin{aligned} w_1 + w_2 &= 1 \\ w_2\alpha &= \frac{1}{2} \\ w_2\beta &= \frac{1}{2} \end{aligned}$$

Thus when $\alpha = \beta = \frac{2}{3}$, $w_1 = \frac{1}{4}$ and $w_2 = \frac{3}{4}$. Substituting these values in the form of Runge-Kutta yields

$$x(t+h) = x + \frac{1}{4}hf + \frac{3}{4}h\left[f + \frac{2}{3}hf_t + \frac{2}{3}hff_x\right] + \mathcal{O}(h^3)$$

Problem 5

Euler's formula for some error term Ch^2 is

$$x(t+h) = x(t) + hf(t, x(t)) + Ch^2$$

Using step size $\frac{h}{2}$ gives

$$x\left(t+\frac{h}{2}\right) = x(t) + \frac{h}{2}f(t, x(t)) + \frac{C}{4}h^2$$

$$\begin{aligned} x(t+h) &= x\left(t+\frac{h}{2}\right) + \frac{h}{2}f\left(t+\frac{h}{2}, x\left(t+\frac{h}{2}\right)\right) + \frac{C}{4}h^2 \\ &= x(t) + \frac{h}{2}f(t, x(t)) + \frac{h}{2}f\left(t+\frac{h}{2}, x(t) + \frac{h}{2}f(t, x(t))\right) + \frac{C}{4}h^2 \end{aligned}$$

$$4x(t+h) = 4x(t) + 2hf(t, x(t)) + 2hf\left(t+\frac{h}{2}, x(t) + \frac{h}{2}f(t, x(t))\right) + 2Ch^2$$

Subtracting the two equations to get rid of the local error term yields

$$3x(t+h) = 3x(t) + hf(t, x(t)) + 2hf\left(t+\frac{h}{2}, x(t) + \frac{h}{2}f(t, x(t))\right)$$

$$x(t+h) = x(t) + \frac{1}{3}hf(t, x(t)) + \frac{2}{3}hf\left(t+\frac{h}{2}, x(t) + \frac{h}{2}f(t, x(t))\right)$$