# Math 170C: Homework 6

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## Problem 1

We are attempting to solve the system of equations $X' = AX$ where

$$A = \begin{bmatrix} 3 & -5 \\ 2 & 1 \end{bmatrix}.$$

This has solution $X(t) = e^{\lambda t}V$ where $\lambda$ and $V$ are the eigenvalue and eigenvector of $A$. The characteristic polynomial and its eigenvalues are

$$\begin{vmatrix} \lambda - 3 & 5 \\ -2 & \lambda - 1 \end{vmatrix} = (\lambda - 3)(\lambda - 1) + 10 = \lambda^2 - 4\lambda + 13$$

$$\lambda = \frac{4 \pm \sqrt{16 - 52}}{2} = 2 \pm 3i.$$

The eigenvectors for $\lambda_1 = 2 + 3i$ and $\lambda_2 = 2 - 3i$ are

$$\begin{bmatrix} -1 + 3i & 5 \\ -2 & 1 + 3i \end{bmatrix} V_1 = 0 \implies V_1 = \begin{bmatrix} 1 + 3i \\ 2 \end{bmatrix},$$

$$\begin{bmatrix} -1 - 3i & 5 \\ -2 & 1 - 3i \end{bmatrix} V_2 = 0 \implies V_1 = \begin{bmatrix} 1 - 3i \\ 2 \end{bmatrix}.$$

Thus for a constants $C_1$ and $C_2$, the solutions are

$$X(t) = C_1 e^{(2+3i)t} \begin{bmatrix} 1 + 3i \\ 2 \end{bmatrix} + C_2 e^{(2-3i)t} \begin{bmatrix} 1 - 3i \\ 2 \end{bmatrix}$$

## Problem 2

If $x_1 = x$ and $x_2 = x'$ then we can rewrite the problem as

$$\begin{cases} x_1' = x_2 \\ x_2' = -19x_1 - 20x_2 \end{cases}$$

with $X(0) = [2, -20]^T$. Note that

$$A = \begin{bmatrix} 0 & 1 \\ -19 & -20 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 19 & 1 \end{bmatrix} \begin{bmatrix} -19 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{18} & \frac{1}{18} \\ -\frac{19}{18} & -\frac{1}{18} \end{bmatrix}$$

It has solution

$$\begin{aligned} X(t) &= e^{At}X(0) \\ &= \begin{bmatrix} 0 & 1 \\ -19 & -20 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 19 & 1 \end{bmatrix} \begin{bmatrix} e^{-19t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} \frac{1}{18} & \frac{1}{18} \\ -\frac{19}{18} & -\frac{1}{18} \end{bmatrix} \begin{bmatrix} 2 \\ -20 \end{bmatrix} \\ &= \begin{bmatrix} e^{-19t} + e^{-t} \\ -19e^{-19t} - e^{-t} \end{bmatrix}. \end{aligned}$$

Looking at the eigenvalues and the solution, we can see that the problem is stiff. $-19$ is much larger in magnitude than $-1$ and so there is a "wide disparity in the time scales of the components in the vector solution". $e^{-19}$ is transient and we need very small time steps to account for its effects.

## Problem 3

The relevant polynomials are

$$p(z) = z^2 + \alpha z - (1+\alpha)$$

$$q(z) = -\frac{\alpha}{2}z^2 + \frac{4+3\alpha}{2}z$$

$$\phi(z) = \left(1 + \lambda h\frac{\alpha}{2}\right)z^2 + \left(\alpha - \lambda h\frac{4+3\alpha}{2}\right)z - (1+\alpha)$$

For stability we need that all the roots of $p$ lie in the disk $|z| \leq 1$ and roots of modulus 1 are simple. We need $-2 < \alpha < 0$ since the roots of $p$ are

$$z = \frac{-\alpha \pm \sqrt{\alpha^2 + 4(1+\alpha)}}{2} = \frac{-\alpha \pm (\alpha+2)}{2} = 1, -\alpha - 1$$

Our method is already consistent since $p(1) = 0$ and $p'(1) = q(1)$ for all $\alpha$.

$$p(1) = 1 + \alpha - (1+\alpha) = 0$$

$$p'(1) = 2 + \alpha$$

$$q(1) = -\frac{\alpha}{2} + \frac{4+3\alpha}{2} = 2 + \alpha$$

Convergence directly follows from stability and consistency.
The method is second order since

$$\begin{aligned} d_0 &= a_2 + a_1 + a_0 \\ &= 1 + \alpha - (1+\alpha) \\ &= 0 \end{aligned}$$

$$\begin{aligned} d_1 &= (2a_2 - b_2) + (a_1 - b_1) - b_0 \\ &= \left(2 + \frac{\alpha}{2}\right) + \left(\alpha - \frac{(4+3\alpha)}{2}\right) - 0 \\ &= 0 \end{aligned}$$

For A-stablility, we need the roots of $\phi$ ti lie in the unit disk $|z| < 1$. For this to happen, we can choose $\alpha = -1$ so that

$$\phi(z) = \left(1 - \frac{h\lambda}{2}\right)z^2 - \left(1 + \frac{h\lambda}{2}\right)z.$$

Then our roots lie inside of the unit circle when $\mathrm{Re}(h\lambda) < 0$.

$$z = 0 \quad z = \frac{2+h\lambda}{2-h\lambda}$$