Math 31BH: Final

Due 03/16 at 18:30

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1. Let $\epsilon > 0$ be arbitrary. Let $\delta = \frac{\epsilon}{C}$. For $v \in V$ with $\|v\| < \delta$, then $\|f(v)\| < \epsilon$ since

$$\|f(v)\| \leq C\|v\| < C\delta = C\frac{\epsilon}{C} = \epsilon$$

Therefore, f is continuous because for all $\epsilon > 0$, there exists a $\delta > 0$ such that $\|v\| < \delta$ implies $\|f(v)\| < \epsilon$.

2. The value of f at $\frac{\pi}{3}$ is

$$f(\frac{\pi}{3}) = (\frac{1}{2}, \frac{\sqrt{3}}{2})$$

The derivative of f at $\frac{\pi}{3}$ is

$$f'(t) = (-\sin(t), \cos(t))$$

$$f'(\frac{\pi}{3}) = (-\frac{\sqrt{3}}{2}, \frac{1}{2})$$

Using r as a parameter, a tangent line at t can be written in form $\{f(t) + rf'(t) : r \in \mathbb{R}\}$. Therefore the tangent line is $\{(\frac{1}{2}, \frac{\sqrt{3}}{2}) + r(-\frac{\sqrt{3}}{2}, \frac{1}{2}) : r \in \mathbb{R}\}$. Written as an equation it is $g(r) = (\frac{1}{2} - \frac{\sqrt{3}}{2}r, \frac{\sqrt{3}}{2} + \frac{1}{2}r)$.

3. (a) The velocity is the derivative of the position. Therefore the velocity at time t is

$$f'(t) = (a, -b\omega \sin \omega t, b\omega \cos \omega t)$$

(b) The speed of a particle is the magnitude of its veloicty. Therefore the speed at time t is

$$\|f'(t)\| = \sqrt{a^2 + b^2\omega^2 \sin^2 \omega t + b^2\omega^2 \cos^2 \omega t}$$

(c) The acceleration is the derivative of the velocity. Therefore the acceleration at time t is

$$f''(t) = (0, -b\omega^2 \cos \omega t, -b\omega^2 \sin \omega t)$$

The dot product of the acceleration and velocity is

$$\begin{aligned} f'(t) \cdot f''(t) &= (-b\omega \sin \omega t)(-b\omega^2 \cos \omega t) + (b\omega \cos \omega t)(-b\omega^2 \sin \omega t) \\ &= b^2\omega^3 \sin \omega t \cos \omega t - b^2\omega^3 \sin \omega t \cos \omega t \\ &= 0 \end{aligned}$$

Since the dot product of acceleration and velocity is zero for t, the acceleration and velocity are orthogonal at time t.

4. The gradient of f_1 is a vector of the partial derivatives of f_1. Using the chain rule to find the partial derivatives,

$$\begin{aligned}\nabla f_1(x,y) &= (\frac{\partial}{\partial x} g_1(x+y), \frac{\partial}{\partial y} g_1(x+y)) \\ &= (g_1'(x+y)\frac{\partial}{\partial x}(x+y), g_1'(x+y)\frac{\partial}{\partial y}(x+y)) \\ &= (g_1'(x+y), g_1'(x+y))\end{aligned}$$

Similarly for f_2,

$$\begin{aligned}\nabla f_2(x,y) &= (\frac{\partial}{\partial x} g_2(x-y), \frac{\partial}{\partial y} g_2(x-y)) \\ &= (g_2'(x-y)\frac{\partial}{\partial x}(x-y), g_2'(x+y)\frac{\partial}{\partial y}(x-y)) \\ &= (g_2'(x-y), -g_2'(x-y))\end{aligned}$$

The dot product of the gradients is

$$\nabla f_1(x,y) \cdot \nabla f_2(x,y) = g_1'(x+y)g_2'(x-y) - g_1'(x+y)g_2'(x-y) = 0$$

Since the dot product of the gradients is 0 for all x and y, the gradient of f_1 is orthogonal to the gradient of f_2 at every point in $\mathbb{R}^2$.

5. Writting v as its components, the function is

$$f(v_1, ..., v_n) = (v_1^2 + ... + v_n^2)^a$$

The partial derivative for an arbitrary component variable is

$$\begin{aligned}\frac{\partial}{\partial v_i} f(v_1, ..., v_i, ..., v_n) &= \frac{\partial}{\partial v_1}(v_1^2 + ... + v_i^2 + ... + v_n^2)^a \\ &= a(v_1^2 + ... + v_i^2 + ... + v_n^2)^{a-1} 2v_i \\ &= 2a(v \cdot v)^{a-1} v_i\end{aligned}$$

The gradient vector is therefore

$$\nabla f(v) = (2a(v \cdot v)^{a-1}v_1, ..., 2a(v \cdot v)^{a-1}v_n) = 2a(v \cdot v)^{a-1}v$$

6. (a) Since a convex hull is compact, the extreme value theorem says that a maximizer exists in S.

   (b) The gradient of f is

$$\nabla f(x,y) = (3x^2 + y, x)$$

   The gradient is only zero at the point $(0,0)$. Therefore $(0,0)$ is the only critical point, which is on the boundary. Therefore it is sufficient to only check the boundaries for a maximum.

For the left side $f(0, y) = 0$, which achieves a maximum of 0 for all points.
For the right side $f(1, y) = 1 + y$, which achieves a maximum of 2 at $y = 1$.
For the bottom side $f(x, 0) = x^3$, which achieves a maximum of 1 at $x = 1$.
For the top side $f(x, 1) = x^3 + 1$, which achieves a maximum of 2 at $x = 1$.
Therefore, f has a maximum value of 2 at the point $(1, 1)$ on S.

7. (a) Since $f(0, 0) = (1, 0)$ and $f(0, 2\pi) = (1, 0)$, the function is not injective, and therefore it is not invertible.

   (b) The Jacobian matrix is the matrix of the possible derivatives on the component functions.

$$J_f(x, y) = \begin{bmatrix} \frac{\partial}{\partial x} e^x \cos y & \frac{\partial}{\partial y} e^x \cos y \\ \frac{\partial}{\partial x} e^x \sin y & \frac{\partial}{\partial y} e^x \sin y \end{bmatrix} = \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix}$$

   (c) From the inverse function theorem, f is locally invertible if $\det(f'(v, \cdot)) \neq 0$ for a point $v \in \mathbb{R}^2$. The determinant is

$$\begin{aligned} \det(f'(v, \cdot)) &= \det(J_f(x, y)) \\ &= (e^x \cos y)(e^x \cos y) - (-e^x \sin y)(e^x \sin y) \\ &= e^{2x} \cos^2 y + e^{2x} \sin^2 y \\ &= e^{2x}(\cos^2 y + \sin^2 y) \\ &= e^{2x} \\ &\neq 0 \text{ for all } x \end{aligned}$$

   Since the determinant of the derivative is never zero, f is locally invertible for any point $v \in \mathbb{R}^2$.