Math 188: Homework 2

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Problem 1: Formal Power Series Composite Inverse

1. ($\Longrightarrow$) Let $F(x) = \sum_{i=1}^{\infty} a_n x^n$ and let $G(x) = \sum_{i=1}^{\infty} b_n x^n$ be some formal power series with $b_1 \neq 0$. If $F(G(x)) = x$, then $[x^1]F(G(x)) = a_1 b_1 = 1$ meaning $a_1 \neq 0$.
   ($\Longleftarrow$) If $[x^1]F(x) \neq 0$, then the constants b_i can be computed recursively given that $F(G(x)) = x$. This is possible because each b_i only depends on coefficients of smaller indexes.

$$\begin{aligned} a_1 b_1 &= 1 \\ a_1 b_2 + a_2 b_1^2 &= 0 \\ a_1 b_3 + 2a_2 b_1 b_2 + a_3 b_1^3 &= 0 \\ a_1 b_4 + a_2(b_2^2 + 2b_1 b_3) + 3a_3 b_1^2 b_2 + a_4 b_1^4 &= 0 \\ &\vdots \end{aligned}$$

   Note that the $G(x)$ constructed here has $G(0) = 0$, meaning that $G(x)$ satisfies $F(G(x)) \iff G(0) = 0$.

2. Let G^{-1} be the right composite inverse of G (it can be calculated in the same way that G was calculated from F). Let $I = x$ be the identity power series. Using the associativity of power series composition,

$$\begin{aligned} (G \circ F)(x) &= (G \circ (F \circ I))(x) \\ &= (G \circ (F \circ G \circ G^{-1}))(x) \\ &= (G \circ (F \circ G) \circ G^{-1})(x) \\ &= (G \circ G^{-1})(x) \\ &= x \end{aligned}$$

   Suppose that there exists some other power series G' such that $F(G'(x)) = x$.

$$\begin{aligned} (G \circ F \circ G')(x) &= (G \circ (F \circ G'))(x) = G(x) \\ &= ((G \circ F) \circ G')(x) = G'(x) \end{aligned}$$

   Thus, $G(x)$ is unique.

Problem 2: Binomial Theorem

1. Using the binomial theorem,

$$\sum_{i=0}^{n} \binom{n}{i} \frac{1}{2^i} = \left(1 + \frac{1}{2}\right)^n = \left(\frac{3}{2}\right)^n$$

2. Adding x^2 times the second derivative of the binomial theorem to x times the first derivative yields

$$\begin{aligned}(n(n-1)(1+x)^{n-2})x^2 + (n(1+x)^{n-1})x &= \sum_{i=0}^{n} i(i-1)\binom{n}{i}x^i + \sum_{i=0}^{n} i\binom{n}{i}x^i \\ &= \sum_{i=0}^{n} i^2 \binom{n}{i} x^i\end{aligned}$$

With $x = 3$,

$$\begin{aligned}\sum_{i=0}^{n} i^2 \binom{n}{i} 3^i &= (n(n-1)(4)^{n-2})3^2 + (n(4)^{n-1})3 \\ &= 3n(1+3n)4^{n-2}\end{aligned}$$

Problem 3: Choosing Cats and Dogs

1. $(1+x)^{a+b}$ can be expanded using the binomial theorem.

$$(1+x)^{a+b} = \sum_{n=0}^{\infty} \binom{a+b}{n} x^i$$

   $(1+x)^a(1+x)^b$ can be expanded using the binomial theorem and combined using the definition of products of power series.

$$\begin{aligned}(1+x)^a(1+x)^b &= \left(\sum_{n=0}^{\infty} \binom{a}{n} x^n\right)\left(\sum_{n=0}^{\infty} \binom{b}{n} x^n\right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} \binom{a}{i}\binom{b}{n-i}\right) x^n\end{aligned}$$

   Comparing the coefficients term-by-term yields

$$\binom{a+b}{n} = \sum_{i=0}^{n} \binom{a}{i}\binom{b}{n-i}$$

2. The number of ways of choosing n animals from a dogs and b cats is equivalent to the number of ways of choosing n animals with exactly 0 dogs plus choosing n animals with exactly 1 dog all the way to choosing n animals with exactly n dogs.

Problem 4: Arranging "MISSISSIPPI"

The letters count in "MISSISSIPPI" is one of 'M', two of 'P', four of 'I', and four of 'S' with eleven total letters.

$$\binom{11}{1,2,4,4} = \frac{11!}{1!2!4!4!} = 34650$$

Using the multinomial coefficient to count, there are 34650 total ways of arranging the letters in "MISSISSIPPI".

Problem 5: Rational Generating Functions

Using the binomial theorem, doing a change of variables, utilizing the fact that $k < d+1$ to set $n = 0$,

$$\begin{aligned}
\sum_{n\geq 0} f(n)x^n &= \frac{g_0 + g_1 x + \ldots + g_d x^d}{(1-x)^{d+1}} \\
&= \sum_{k=0}^{d} \frac{g_k x^k}{(1-x)^{d+1}} \\
&= \sum_{k=0}^{d}\sum_{n=0} g_k \binom{d+n}{n} x^{n+k} \\
&= \sum_{k=0}^{d}\sum_{n=k} g_k \binom{d+n-k}{n-k} x^{n} \\
&= \sum_{k=0}^{d}\sum_{n=0} g_k \binom{d+n-k}{n-k} x^{n}
\end{aligned}$$

For a given $n = t$,

$$f(t) = \sum_{k=0}^{d} g_k \binom{d+t-k}{t-k} = \sum_{k=0}^{d} g_k \binom{d+t-k}{d}$$

Plugging in $t = 0, \ldots, d$ yields the following system of equations in matrix form,

$$\begin{bmatrix} f(0) \\ f(1) \\ f(2) \\ \vdots \\ f(d) \end{bmatrix} = \begin{bmatrix} \binom{d}{d} & \binom{d-1}{d} & \binom{d-2}{d} & \cdots & \binom{0}{d} \\ \binom{d+1}{d} & \binom{d}{d} & \binom{d-1}{d} & \cdots & \binom{1}{d} \\ \binom{d+2}{d} & \binom{d+1}{d} & \binom{d}{d} & \cdots & \binom{2}{d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{2d}{d} & \binom{2d-1}{d} & \binom{2d-2}{d} & \cdots & \binom{d}{d} \end{bmatrix} \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \binom{d+1}{d} & 1 & 0 & \cdots & 0 \\ \binom{d+2}{d} & \binom{d+1}{d} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{2d}{d} & \binom{2d-1}{d} & \binom{2d-2}{d} & \cdots & 1 \end{bmatrix} \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ \vdots \\ g_d \end{bmatrix}$$

Working inductively, we can see that all g_k are integers. Since $f(0)$ is an integer and $f(0) = g_0$, g_0 is an integer. If $g_0, \ldots, g_k$ are integers then

$$f(k+1) = \sum_{i=0}^{k} \binom{d+k+1-i}{d} g_i + g_{k+1}$$

Since $f(k+1)$ is an integer and $\sum_{i=0}^{k} \binom{d+k+1-i}{d} g_i$ is a sum of products of integers, then g_{k+1} is also an integer. $f(a)$ is also an integer for all integer a since $f(t) = \sum_{k=0}^{d} g_k \binom{d+t-k}{d}$, and g_k and the binomial coefficients are integers for all $t \in \mathbb{Z}$.