Math 120A: Homework 1

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Problem 1(c)

$$\begin{aligned}
\overline{(2+i)^2} &= \overline{(2+i)(2+i)} \\
&= \overline{(4-1)+4i} \\
&= \overline{3+4i} \\
&= 3-4i
\end{aligned}$$

Problem 1(d)

$$\begin{aligned}
\left|(2\overline{z}+5)(\sqrt{2}-i)\right| &= |2\overline{z}+5|\left|\sqrt{2}-i\right| \\
&= \left|\overline{2\overline{z}+5}\right|\sqrt{3} \\
&= \sqrt{3}|2z+5|
\end{aligned}$$

Problem 7

$$\begin{aligned}
\left|\mathrm{Re}(2+\overline{z}+z^3)\right| &\leq \left|2+\overline{z}+z^3\right| \\
&\leq |2|+|\overline{z}|+\left|z^3\right| \\
&\leq 2+1+1^3 \\
&= 4
\end{aligned}$$

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Problem 1(a)

$$\begin{aligned}
\frac{-2}{1+\sqrt{3}i} &= \frac{-2}{1+\sqrt{3}i}\frac{1-\sqrt{3}i}{1-\sqrt{3}i} \\
&= \frac{-2+2\sqrt{3}i}{1+3} \\
&= -\frac{1}{2}+\frac{\sqrt{3}}{2}i
\end{aligned}$$

Since the number is in the second quadrant and $\arctan\left(\sqrt{3}\right) = -\frac{\pi}{3}$, we have that $\operatorname{Arg}\frac{-2}{1+\sqrt{3}i} = \frac{2\pi}{3}$.

Problem 1(b)

$$\begin{aligned}
\arg(\sqrt{3}-i)^6 &= 6\cdot\arg(\sqrt{3}-i) \\
&= 6\cdot\arctan\left(-\frac{1}{\sqrt{3}}\right) \\
&= 6\cdot\left(-\frac{\pi}{6}\right) \\
&= -\pi
\end{aligned}$$

Thus the principal argument is π.

Problem 5(c)

$$\begin{aligned}
(\sqrt{3}+i)^6 &= (2e^{i\frac{\pi}{6}})^6 \\
&= 64e^{i\pi} \\
&= -64
\end{aligned}$$

Problem 5(d)

$$\begin{aligned}
(1+\sqrt{3}i)^{-10} &= (2e^{i\frac{\pi}{3}})^{-10} \\
&= 2^{-10}e^{i\frac{-10\pi}{3}} \\
&= -2^{-11}+2^{-11}\sqrt{3}i \\
&= 2^{-11}(-1+\sqrt{3}i)
\end{aligned}$$

Problem 6

Let $\theta_1 = \operatorname{Arg}(z_1)$ and $\theta_2 = \operatorname{Arg}(z_2)$. Since $\operatorname{Re} z_1, \operatorname{Re} z_2 > 0$, z_1 and z_4 must be in the first or fourth quadrant and so $\theta_1, \theta_2 \in (-\frac{\pi}{2}, \frac{\pi}{2}]$. Since $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$, we have that $\arg(z_1 z_2) = \theta_1 + \theta_2 + 2\pi n$ for some $n \in \mathbb{Z}$. However since $\theta_1, \theta_2 \in (-\frac{\pi}{2}, \frac{\pi}{2}]$, it must be that $\theta_1 + \theta_2 \in (-\pi, \pi]$. So $\operatorname{Arg}(z_1 z_2) = \arg(z_1 z_2)$ for $n = 0$ and so $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$.

Problem 10

$$
\begin{aligned}
& (\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta \\
\implies & \cos^3\theta + i3\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta = \cos 3\theta + i\sin 3\theta \\
\implies & (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta) = \cos 3\theta + i\sin 3\theta
\end{aligned}
$$

By equating the real and imaginary parts of both sides, we get the formulas

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$$