MATH 31AH - Final

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Problem 1

If the image and kernel of a transformation are the same, then that means that taking the transformation twice will result in zero. If T is the derivative with V being polynomials of degree one or less, then $\text{Image}(T) = \text{Kernel}(T)$ since the second derivative of degree one polynomials is zero.

Problem 2

Since $\mathbb{F}_3^6$ has 6 dimensions and $\mathbb{F}_3^5$ has 5 dimensions, all linear transformations $T : \mathbb{F}_3^6 \to \mathbb{F}_3^5$ can be represented by 5×6 matricies with elements in $\mathbb{F}_3$. Therefore, there are $5 \times 6 \times 3 = 90$ different transformations.

Problem 3

We can use gaussian elimination to solve $Ax = 0$.

$$\begin{pmatrix}1 & 2 & 3 & 4\\ 0 & 1 & 2 & 3\\ 1 & 3 & 5 & 7\end{pmatrix} \begin{array}{l} -1 \\ \\ + \end{array}$$

$$\Rightarrow \begin{pmatrix}1 & 2 & 3 & 4\\ 0 & 1 & 2 & 3\\ 0 & 1 & 2 & 3\end{pmatrix} \begin{array}{l} \\ -1 \\ + \end{array}$$

$$\Rightarrow \begin{pmatrix}1 & 2 & 3 & 4\\ 0 & 1 & 2 & 3\\ 0 & 0 & 0 & 0\end{pmatrix} \begin{array}{l} + \\ -2 \\ \\ \end{array}$$

$$\Rightarrow \begin{pmatrix}1 & 0 & -1 & -2\\ 0 & 1 & 2 & 3\\ 0 & 0 & 0 & 0\end{pmatrix}$$

$Ax = 0$ therefore has basis vectors of

$$\mathcal{B} = \{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix} \}$$

Problem 4

Proof. There exists λ with $\lambda(v) \neq 0$ if $v \neq 0$.

Let $\mathcal{B} = \{e_1, ..., e_n\}$ be the canonical basis of $\mathbb{R}^n$. Let $\mathcal{B}^* = \{\lambda_1, ..., \lambda_n\}$ be the dual basis of $\mathbb{R}^n$. Since $v \neq 0$, there exists a linear combination of the basis vectors such that $v = c_1e_1 + ... + c_ie_i + ... + c_ne_n$ with some $c_i \neq 0$. We have that for λ_i,

$$\begin{aligned} \lambda_i(v) &= \lambda_i(c_1e_1 + ... + c_ie_i + ...c_ne_n) \\ &= c_i \\ &\neq 0 \end{aligned}$$

Therefore, there exists λ with $\lambda(v) \neq 0$ if $v \neq 0$. □

Problem 5

$\langle -, - \rangle$ is not an inner product since there exists a $v \neq 0$ such that $\langle v, v \rangle = 0$. Let $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then

$$\begin{aligned} \langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rangle &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= 0 \end{aligned}$$

Problem 6

We have that

$$A - \lambda I = \begin{bmatrix} -\lambda & 0 & 0 & 0 & 1 \\ 1 & -\lambda & 0 & 0 & 0 \\ 0 & 1 & -\lambda & 0 & 0 \\ 0 & 0 & 1 & -\lambda & 0 \\ 0 & 0 & 0 & 1 & -\lambda \end{bmatrix}$$

There are only two permutations that result in non-negative terms, so the characteristic polynomial is

$$\begin{aligned} \det(A - \lambda I) &= (-\lambda)^5 + 1^5 && (1) \\ &= -\lambda^5 + 1 && (2) \end{aligned}$$

so $\lambda = \sqrt[5]{1}$. Therefore the five eigenvalues are $\lambda = 0, e^{i\frac{2\pi}{5}}, e^{i\frac{4\pi}{5}}, e^{i\frac{7\pi}{5}}, e^{i\frac{8\pi}{5}}$. If we perform row reduction on $A - \lambda I$ we the following matrix

$$\begin{bmatrix} -\lambda & 0 & 0 & 0 & 1 \\ 0 & -\lambda & 0 & 0 & \frac{1}{\lambda} \\ 0 & 0 & -\lambda & 0 & \frac{1}{\lambda^2} \\ 0 & 0 & 0 & -\lambda & \frac{1}{\lambda^3} \\ 0 & 0 & 0 & 0 & \frac{1}{\lambda^4} - \lambda \end{bmatrix}$$

Since

$$\frac{1}{\lambda^4} - \lambda = \frac{1-\lambda^5}{\lambda^4} = \frac{0}{\lambda^4} = 0$$

we can set the bottom right entry to zero. The final rref matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{1}{\lambda} \\ 0 & 1 & 0 & 0 & -\frac{1}{\lambda^2} \\ 0 & 0 & 1 & 0 & -\frac{1}{\lambda^3} \\ 0 & 0 & 0 & 1 & -\frac{1}{\lambda^4} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This means that each eigenspace will have a one dimensional eigenbasis in the form $\{\left[\frac{1}{\lambda}, \frac{1}{\lambda^2}, \frac{1}{\lambda^3}, \frac{1}{\lambda^4}, 1\right]\}$. for the corresponding eigenvalue of the eigenspace $\lambda = 0, e^{i\frac{2\pi}{5}}, e^{i\frac{4\pi}{5}}, e^{i\frac{7\pi}{5}}, e^{i\frac{8\pi}{5}}$.

Problem 7

A matrix that has one column switched will have a determinant with the opposite sign of the determinant of the original matrix. B has switched columns $n-1$ times in order for the nth column to move to the beginning of the matrix. Therefore, $c = (-1)^{n-1}$.

Problem 8

This statement is false, since vectors in the form $v \otimes w$ only includes simple tensors. Let $t \in \mathbb{R}^2 \otimes \mathbb{R}^2$ be a tensor with $t = e_1 \otimes e_2 + e_2 \otimes e_1$, where e_1, e_2 are the standard basis vectors of $\mathbb{R}^2$. If it is expressable in the form $v \otimes w$, then

$$\begin{aligned} (e_1 \otimes e_2) + (e_2 \otimes e_1) &= (ae_1 + be_2) \otimes (ce_1 + de_2) \\ &= ac(e_1 \otimes e_1) + ad(e_1 \otimes e_2) + bc(e_2 \otimes e_1) + bd(e_2 \otimes e_2) \end{aligned}$$

Therefore

$$\begin{aligned} ac &= 0 \\ ad &= 1 \\ bc &= 1 \\ bd &= 0 \end{aligned}$$

This is impossible since the second equation implies a is nonzero and the third equation implies that c is nonzero, but then $ac = 0$ would be impossible. This is a contradiction, so there exists tensors that cannot be written in the form $v \otimes w$.

Problem 9

This statement is false.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

AB is not diagonalizable. Since there exists a counterexample, the statement is false.

Problem 10

Due to the rank-nullity theorem, $\dim(\hom(V, W) \otimes V) = \dim(\ker(\varphi)) + \dim(\text{image}(\varphi))$ We have that $\dim(\hom(V, W) \otimes V) = (4 \cdot 5) \cdot 4 = 80$. φ is also surjective because for every $w \in W$, for some $v \neq 0$, there exists a transformation T such that $T(v) = w$, so there exists some $T \otimes v$ with $\varphi(T \otimes v) = w$. Therefore $\dim(\text{image}(\varphi)) = \dim(W) = 5$. Therefore, $\dim(\ker(\varphi)) = 80 - 5 = 75$.