Math 31AH: Fall 2021
Final Exam
Friday, 12/10/2021

Instructions: This is a 3 hour closed notes, closed books exam. Consultation with other humans is prohibited, including humans acting via websites such as Chegg. You need to clearly prove your claims; unsupported claims will get little credit. Please upload your exam to Gradescope after you are finished. You will have 10 additional minutes to upload your solutions to Gradescope.

Problem 1: Find a vector space V and a linear transformation $T : V \to V$ for which $\mathrm{Image}\, T = \mathrm{Ker}\, T$.

Problem 2: Let $\mathbb{F}_3$ be the field with three elements. How many linear transformations $T : \mathbb{F}_3^6 \to \mathbb{F}_3^5$ are there?

Problem 3: Find a basis of the kernel $\mathrm{Ker}\, A$ where A is the real matrix

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 1 & 3 & 5 & 7 \end{pmatrix}$$

Problem 4: Let $V = \mathbb{R}^n$ and let $\mathbf{v} \in V$ be nonzero. Prove there exists $\lambda \in V^*$ with $\lambda(\mathbf{v}) \neq 0$.

Problem 5: Define a function $\langle -, - \rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$\langle \mathbf{v}, \mathbf{w} \rangle := \mathbf{v}^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathbf{w}$$

Is $\langle -, - \rangle$ an inner product on $\mathbb{R}^2$?

Problem 6: Let C be the 5×5 matrix

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Find a basis for each eigenspace of C over the complex numbers.

Problem 7: Let A be an $n \times n$ matrix with columns $A = \begin{pmatrix} \mathbf{v}_1 \, \mathbf{v}_2 \, \cdots \, \mathbf{v}_{n-1} \, \mathbf{v}_n \end{pmatrix}$ and form $B = \begin{pmatrix} \mathbf{v}_n \, \mathbf{v}_1 \, \mathbf{v}_2 \cdots \mathbf{v}_{n-1} \end{pmatrix}$ by cycling the columns of A. Then $\det B = c \cdot \det A$ for some number c depending on n. Find c.

Problem 8: True or false: For any vector spaces V and W, every vector in $V \otimes W$ has the form $\mathbf{v} \otimes \mathbf{w}$ for $\mathbf{v} \in V$ and $\mathbf{w} \in W$. Justify your answer.

Problem 9: Let A and B be $n \times n$ real matrices. If A and B are diagonalizable, is AB necessarily diagonalizable?

Problem 10: For any vector spaces V and W, there is a linear map

$$\varphi : \mathrm{Hom}(V, W) \otimes V \to W$$

characterized by $\varphi(T \otimes \mathbf{v}) := T(\mathbf{v})$. If $\dim V = 4$ and $\dim W = 5$, what is the dimension of $\operatorname{Ker} \varphi$?