Math 100b Winter 2025 Homework 1

Due 1/17/25 at 5pm on Gradescope

Reading

All references will be to Artin Algebra, 2nd edition.

Read Sections 11.1, 11.2, 11.3.

Assigned Problems

Write up neat and complete solutions to these problems. "Ring" will always mean commutative ring unless otherwise noted.

1. (Artin 11.1.7(a)) Let U be an arbitrary set, and let R be the set of all subsets of U (that is, R is the power set of U). Define addition and multiplication in R by $A + B = (A \cup B) - (A \cap B)$ and $A \cdot B = A \cap B$. Is R a ring under these operations?

2. (Artin 11.1.6(a)) Let S be the set of all rational numbers a/b where b is not divisible by 3. Is S a subring of $\mathbb{Q}$?

3. (Artin 11.1.9) Let R be a set with two laws of composition satisfying all of the ring axioms except that addition is not necessarily assumed to be a commutative. Using the distributive law, prove that the commutativity of addition is automatic, so R is a ring.

4. An element a of a ring R is *nilpotent* if there is a positive integer n such that $a^n = 0$. Show that if $a \in R$ is nilpotent, then $1 - a$ is a unit in R. (Hint: think about the geometric series).

5. Let R be the set of all functions $f : \mathbb{R} \to \mathbb{R}$. R becomes a ring if we define the following "pointwise" operations: given $f, g \in R$, $f + g$ is the function defined by $[f + g](x) = f(x) + g(x)$, and fg is the function defined by $[fg](x) = f(x)g(x)$. It is not hard to show this is a commutative ring; think through this, but don't submit a proof of it. In particular, you should think about what the 0 and 1 elements are in this ring.

Describe the units, nilpotent elements, and zero-divisors of the ring R. (recall that a zero-divisor in a commutative ring S is an element $a \in S$ such that there exists a *nonzero* $b \in S$ with $ab = 0$.) In particular, prove that every element of R is either a unit or a zero-divisor.

6. Determine the units, nilpotent elements, and zero-divisors of the ring $R = \mathbb{Z}/n\mathbb{Z}$ for a positive integer $n \geq 2$. In particular, prove that every element of R is either a unit or a zero-divisor.

7. Let $r \in \mathbb{Q}$ be any rational number such that $\sqrt{r} \notin \mathbb{Q}$, and define

$$\mathbb{Q}[\sqrt{r}] = \{a + b\sqrt{r} | a, b \in \mathbb{Q}\}$$

as a subset of $\mathbb{C}$.

(a). Show that if $a_1 + a_2\sqrt{r} = b_1 + b_2\sqrt{r}$ in $\mathbb{Q}[\sqrt{r}]$, then $a_1 = b_1$ and $a_2 = b_2$.

(b). Prove that $\mathbb{Q}[\sqrt{r}]$ is a subfield of $\mathbb{C}$ (that is, a subring which is also a field). (Hint: Given $a + b\sqrt{r} \in \mathbb{Q}[\sqrt{r}]$, it will help to consider the product $(a + b\sqrt{r})(a - b\sqrt{r})$.)

8. Let $f(x) = x^5 + 4x^4 + 2x^3 + 3x^2$ and $g(x) = x^2 + 3$, considered as polynomials in $R = (\mathbb{Z}/n\mathbb{Z})[x]$ for some positive integer $n \geq 2$.

Find $q(x), r(x) \in R$ such that $f = qg + r$ with $r = 0$ or $\deg r < \deg g$. For which integers n is $r(x) = 0$ in R?