# COMP 5320/6320
# Design and Analysis of Computer Networks
# Homework Assignment 2
# Due on Monday, Nov. 8, on Canvas

**Instruction: Every student should finish the following questions independently. Please give justification for the results (i.e., show the calculation process and the corresponding diagram). Submit a scan of your answer sheet as a pdf file to Canvas.**

1. (5 points) Suppose that a rare disease has an average incidence of 1 in every 1000 persons. Assume that members of the population are affected independently and the number of affected follows Poisson distribution. Find the probability of k cases in a population of 10,000 for k=0,1,2 respectively.

**Solution:**

The expected value (mean) $=\lambda = .001*10{,}000 = 10$

10 new cases expected in this population →

$$P(X=0)=\frac{(10)^0 e^{-(10)}}{0!}=.0000454$$

$$P(X=1)=\frac{(10)^1 e^{-(10)}}{1!}=.000454$$

$$P(X=2)=\frac{(10)^2 e^{-(10)}}{2!}=.00227$$

2. (5 points) If new cases of West Nile in New England are occurring at a rate of 2 per month, then

(a) What's the probability that exactly 4 cases will occur in the next 3 months?

(b) What's the probability that exactly 6 cases will occur in the next 3 months?

**Solution:**

X ~ Poisson ($\lambda$=2/month)

$$\text{P(X = 4 in 3 months)} = \frac{(2*3)^4 e^{-(2*3)}}{4!} = \frac{6^4 e^{-(6)}}{4!} = 13.4\%$$

Exactly 6 cases?

$$\text{P(X = 6 in 3 months)} = \frac{(2*3)^6 e^{-(2*3)}}{6!} = \frac{6^6 e^{-(6)}}{6!} = 16\%$$

3. (10 points) Customers arrive at a restaurant according to a Poisson process with rate 10 customers/hour. The restaurant opens daily at 9:00 am. Calculate the following:

(a) When the restaurant opens at 9:00 am, the workers need 30 min to arrange the tables and chairs. What is the probability that they will finish the arrangement before the arrival of the first customer?
(b) Given that a new customer arrived at 9:13 am, what is the expected arrival time of the next customer?
(c) If a customer arrive at restaurant at 2:00 pm, what is the probability that the next customer will arrive before 2:10 pm.

Solution

(a) Prob ( finish the arrangement before the arrival of 1st customer)

= Prob ( no arrival during the 30-min interval )

$= Pr ( N(\frac{1}{2}) = 0)$

$= \frac{(\lambda t)^n}{n!} e^{-\lambda t} \Big|_{t=\frac{1}{2}, n=0} = e^{-10(\frac{1}{2})} = \boxed{0.00674.}$

(b) expected arrival time = arrival time of the current customer + ~~Eexpected~~ ~~interval~~ inter-arrival time

$= 9:13 + \frac{1}{\lambda} = 9:13 + \frac{1}{10}$ hour $= 9:13 + 6 \text{ min} = \boxed{9:19 \text{ am}}$

(c) $Pr\{\text{next arrival is before } 2:10 \mid \text{current arrival is at } 2:00\}$

$= Pr\{\text{inter-arrival time is} < 10 \text{ min}\}$

$= 1 - e^{-10(\frac{10}{60})} = \boxed{0.811}$

4. (20 points) Consider a small bank with one teller. Customers arrive to the bank according to a Poisson process with rate 8 customers per hour. The teller provides all kinds services for the

customers. Each customer takes on average 5 minutes to service. Assume that the service time is exponentially distributed. In steady-state, calculate the following:

(a) What is the probability that the teller is idle?
(b) What is the average number of customers waiting for service?
(c) On average, how long will a customer spend in the bank to complete his service?
(d) What is the probability that there are more than 5 customers in the bank?

Solution:

(a) M/M/1 with $\lambda = 8$, $\mu = \frac{60}{5} = 12$, $\rho = \frac{\lambda}{\mu} = \frac{2}{3}$.

$Pr\{\text{the teller is idle}\} = \pi_0 = 1-\rho = \frac{1}{3}$.

(b) $\bar{L}_q = \frac{\rho^2}{1-\rho} = \frac{4}{3}$ customers

(c) $\bar{S} = \frac{1}{\mu(1-\rho)} = \frac{1}{12 \times \frac{1}{3}} = 0.25$ hours

(d) $Pr\{n > 5\} = 1 - Pr\{n \le 5\} = 1 - \pi_0 - \pi_1 - \pi_2 - \pi_3 - \pi_4 - \pi_5$

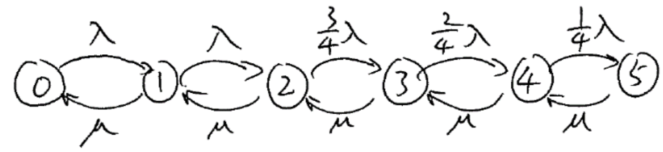
$= 0.088$.

5. (20 points) Consider a continuous-time Markovian system with discouraged job arrivals. Jobs arrive to a server according to a Poisson process, with an arrival rate of one job per 7 seconds. The jobs observe the queue. They do NOT join the queue with probability $l_k$ if they observe that there are $k$ jobs in the queue (This only refers to the number of jobs in the queue. The job being serviced, if any, is not included in this number.). $l_k$ = k/4 if k < 4, or 1, otherwise. The service time is exponentially distributed with mean time of 6 seconds.

(a) Please draw the state transition diagram for this queueing system;
(b) Write the Balance Equation for each state. If this is a birth-death process, please only write the Detailed Balance Equations;
(c) Determine the stationary distribution of the number of jobs in the system, and also calculate the mean number of jobs in the system;
(d) When the system becomes stationary, in an interval of 100 seconds, on average how many jobs enter the system (hint: when the system is stationary, the average number of jobs entering the system equals to the average number of jobs finishing their service then leaving the system)?

Solution:

(a)



(b) DBE:

$$\pi_0 \lambda = \pi_1 \mu \qquad \text{where } \lambda = \tfrac{1}{7} \text{ jobs / second}$$
$$\pi_1 \lambda = \pi_2 \mu \qquad \mu = \tfrac{1}{6} \text{ jobs / second}$$
$$\pi_2 \tfrac{3}{4}\lambda = \pi_3 \mu$$
$$\pi_3 \tfrac{2}{4}\lambda = \pi_4 \mu$$
$$\pi_4 \tfrac{1}{4}\lambda = \pi_5 \mu$$
$$\sum_{i=0}^{5} \pi_i = 1$$

(c) We should get $\boxed{\pi_0 = 0.3}$ and $\boxed{E[N] = 1.43}$

(d) $\pi_0 = 0.3$, so for the duration of 100 seconds, on average $\pi_0 \times 100$ seconds amount of time the server is idle, or equivalently $(1-\pi_0)\times 100$ seconds of time the server is busy. When the server is busy, it can service $\mu$ jobs in a second, so the # of jobs serviced in $(1-\pi_0)\times 100$ seconds is $\mu(1-\pi_0)\times 100 = \frac{1}{6} \times 0.7 \times 100 = \boxed{11.67 \text{ jobs}}$

In 100 seconds, on average 11.67 jobs enters the system.

6. (20 points) Consider a gas station located on a highway with five pumps. Cars arrive at the gas station according to a Poisson process at rate 50 cars/hour. Any car able to enter the gas station stops by one of the available pumps. If all pumps are occupied, the car will not enter the gas station and will just leave. Each car takes an exponential amount of time to refill, and the average refill time is 5 minutes.

(a) Draw the state transition diagram for the gas station.
(b) Determine the stationary distribution of the number of cars in the system.
(c) What is the probability that an arriving car will NOT be able to enter the gas station to refill? In 24 hours, on average how many cars cannot enter the gas station and thus have to leave (If you are the owner of the gas station, this is the business you will lose)?
(d) Now consider that you have bought a small parking lot right beside the gas station, so that a car can stop there and wait for any pump becomes available. Suppose that there are in total 2 spaces

in the parking lot. An arriving car will not enter the gas station and instead leave immediately if all parking spaces are occupied. How many business will be lost in 24 hours in this case?

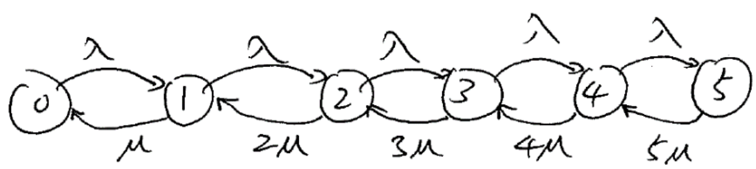
Solution:

(a) This is a M/M/5/0 queueing system.

Arrival rate $\lambda = 50$

service rate per pump $\mu = \frac{60}{5} = 12$

state-transition diagram




(b) To calculate the stationary distribution of # of cars, we write the detailed balance equation, because it's a birth-death process:

$$\pi_0 \lambda = \pi_1 \mu \Rightarrow \pi_1 = \frac{\lambda}{\mu}\pi_0 = \rho\pi_0$$

$$\pi_1 \lambda = 2\pi_2 \mu \Rightarrow \pi_2 = \frac{\lambda}{2\mu}\pi_1 = \frac{\lambda}{2\mu}\frac{\lambda}{\mu}\pi_0 = \frac{1}{2}\rho^2\pi_0$$

$$\pi_2 \lambda = 3\pi_3 \mu \Rightarrow \pi_3 = \frac{\lambda}{3\mu}\pi_2 = \frac{\lambda}{3\mu}\cdot\frac{\lambda}{2\mu}\cdot\frac{\lambda}{\mu}\pi_0 = \frac{1}{3}\times\frac{1}{2}\rho^3\pi_0$$

$$\pi_3 \lambda = 4\pi_4 \mu \Rightarrow \pi_4 = \frac{1}{4}\times\frac{1}{3}\times\frac{1}{2}\times\rho^4\pi_0$$

$$\pi_4 \lambda = 5\pi_5 \mu \Rightarrow \pi_5 = \frac{1}{5}\times\frac{1}{4}\times\frac{1}{3}\times\frac{1}{2}\rho^5\pi_0$$

$$\pi_0 + \pi_1 + \cdots + \pi_5 = 1$$

where $\rho = \frac{\lambda}{\mu} = \frac{50}{12} = 4.17$

$$\pi_0\left(1 + \rho + \frac{1}{2}\rho^2 + \frac{1}{6}\rho^3 + \frac{1}{24}\rho^4 + \frac{1}{120}\rho^5\right) = 1$$

So $\pi_0 = \frac{1}{49.05} = 0.0204$

$\pi_1 = 4.17 \times \pi_0 = \cancel{0.0834}\ 0.0851$

$\pi_2 = 8.69\pi_0 = 0.1774$

$\pi_3 = 12.09\pi_0 = \cancel{0.2418}\ 0.2465$

$\pi_4 = 12.59\pi_0 = \cancel{0.2518}\ 0.2570$

$\pi_5 = 10.51\pi_0 = 0.2144$

(c) Pr{ car can not enter }

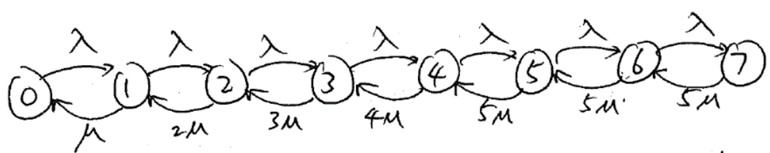
= Pr{ sees 5 cars in the system when the new car arrives}

$= \pi_5$

$= 0.2144.$

In 24 hours, on average $50 \times 24 = 1200$ car arrivals.
Among them 21.44% percent cannot enter the gas station, so
# of lost cars = $1200 \times 21.44\% = 257.28$

(d) when there are 2 parking spaces for waiting:




$\pi_5 \lambda = 5\mu\pi_6 \Rightarrow \pi_6 = \frac{\lambda}{5\mu}\pi_5 = \frac{1}{5} \times \frac{1}{120}\rho^6\pi_0$

$\pi_6 \lambda = 5\mu\pi_7 \Rightarrow \pi_7 = \frac{\lambda}{5\mu}\pi_6 = \frac{1}{5} \times \frac{1}{5} \times \frac{1}{120}\rho^7\pi_0$

$\sum_i \pi_i = 1$

$\pi_0 = 0.0154$

$\pi_7 = \frac{1}{3000}\rho^7\pi_0 = 0.1126$

So # of lost cars = $1200 \times 11.26\% = 135.12$ $\boxed{\text{almost cut by half!}}$

7. (20 points) Consider a communication link with a constant rate of 4.8kbit/s. Over the link we transmit two types of messages, both of exponentially distributed size. Messages arrive in a Poisson fashion with rate 10 messages/second. With probability 0.5 (independent from previous arrivals) the arriving message is of type 1 and has a mean length of 300 bits. Otherwise a message of type 2 arrives with a mean length of 150 bits. The buffer at the link can at most hold one message of type 1 or two messages of type 2. A message being transmitted still takes a place in the buffer.

(a) Draw the state transition diagram for the system. Note that in this case the state cannot simply be defined as the number of messages in the system, as messages are of different types.
(b) Determine the average times in the system for **accepted** messages of type 1 and 2, respectively.
(c) Determine the message loss probabilities for messages of type 1 and 2.

Solution:

(a) mean service time for type 1 message

$$E[T_1] = \frac{300}{4800} = \frac{1}{16} \text{ seconds.} \Rightarrow \mu_1 = 16$$

for type 2 message

$$E[T_2] = \frac{150}{4800} = \frac{1}{32} \text{ seconds.} \Rightarrow \mu_2 = 32.$$

Define states in the following way

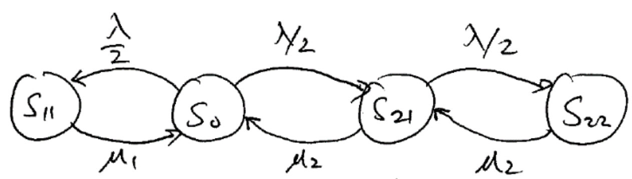
$S_0$: empty buffer

$S_{11}$: 1 Type-1 packet

$S_{21}$: 1 Type-2 packet

$S_{22}$: 2 Type-2 packet

state transition diagram:




DBE:

$$\begin{cases} \pi_{11}\mu_1 = \pi_0 \frac{\lambda}{2} \\ \pi_0 \frac{\lambda}{2} = \pi_{21}\mu_2 \\ \pi_{21}\frac{\lambda}{2} = \pi_{22}\mu_2 \\ \pi_{11} + \pi_0 + \pi_{21} + \pi_{22} = 1 \end{cases} \Rightarrow \begin{matrix} \pi_0 = 0.67, & \pi_{21} = 0.105 \\ \pi_{22} = 0.016, & \pi_{11} = 0.209 \end{matrix}$$

(b) Given a type 1 message is admitted, then it must be in state $S_{11}$, so average time in system is the average service time for type 1 message: $\boxed{\frac{1}{16}s}$

Given a type 2 message is admitted, there are 2 cases:

case 1: the admitted packet is in $S_{21}$, in that case time in system is $E[T_2] = \frac{1}{32}s$.

case 2: the admitted packet is in $S_{22}$, so time in system is $2E[T_2]$.

Given the packet is admitted $Prob(\text{case}1) = \frac{\pi_{21}}{\pi_{21}+\pi_{22}}$, $Prob(\text{Case } 2) = \frac{\pi_{22}}{\pi_{21}+\pi_{22}}$.

Combine them: $\frac{\pi_{21}}{\pi_{21}+\pi_{22}} \times E[T_2] + \frac{\pi_{22}}{\pi_{21}+\pi_{22}} \times 2E[T_2]$

(c) When a type 1 packet will get lost when it arrives at the queue?

~~When~~ This happens when the system is in states $S_{11}$, $S_{21}$, $S_{22}$.

so $\boxed{P_{Loss-type1} = \pi_{11} + \pi_{21} + \pi_{22}}$.

When will a type 2 packet get lost when it arrives at the queue?

This happens when the system is in states $S_{11}$ and $S_{22}$

so $\boxed{P_{Loss-type2} = \pi_{11} + \pi_{22}}$