COMP 5320/6320
Design and Analysis of Computer Networks
Homework Assignment 2
Due on Monday, Nov. 7, on Canvas

Instruction: Every student should finish the following questions independently. Please give justification for the results (i.e., show the calculation process and the corresponding diagram). Submit a scan of your answer sheet as a pdf file to Canvas.

1. Suppose that a rare disease has an average incidence of 1 in every 1000 persons. Assume that members of the population are affected independently and the number of affected follows Poisson distribution. Find the probability of k cases in a population of 10,000 for k=0,1,2 respectively.

Poisson distribution:

$E(X) = \lambda = \frac{1}{1000}$ person's infected $\times$ 10000 persons

$= 10$ persons expected to be infected in the year

$P(X = n) = \frac{\lambda^n}{n!} e^{-\lambda}, \quad n = 0, 1, 2, \ldots$

$P(X = 0) = \frac{10^0}{0!} e^{-10} = 0.0000454 = \boxed{4.54 \times 10^{-5}}$

$P(X = 1) = \frac{10^1}{1!} e^{-10} = 0.0004539 = \boxed{4.539 \times 10^{-4}}$

$P(X = 2) = \frac{10^2}{2!} e^{-10} = 0.002269 = \boxed{2.269 \times 10^{-3}}$

2. If new cases of West Nile in New England are occurring at a rate of 2 per month, then
(a) What's the probability that exactly 4 cases will occur in the next 3 months?
(b) What's the probability that exactly 6 cases will occur in the next 3 months?

Poisson process:

$E[N(t)] = \lambda t = \frac{2 \text{ cases}}{\text{month}} \times 3 \text{ months} = 6$ expected cases

$P\{N(t) = k\} = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad k = 0, 1, 2, \ldots$

$P\{N(t) = 4\} = \frac{6^4}{4!} e^{-6} = \boxed{0.134}$

$P\{N(t) = 6\} = \frac{6^6}{6!} e^{-6} = \boxed{0.161}$

3. Customers arrive at a restaurant according to a Poisson process with rate 10 customers/hour. The restaurant opens daily at 9:00 am. Calculate the following:

(a) When the restaurant opens at 9:00 am, the workers need 30 min to arrange the tables and chairs. What is the probability that they will finish the arrangement before the arrival of the first customer?
(b) Given that a new customer arrived at 9:13 am, what is the expected arrival time of the next customer?
(c) If a customer arrive at restaurant at 2:00 pm, what is the probability that the next customer will arrive before 2:10 pm.

Poisson process:

$E[N(t)] = \lambda t = \frac{10 \text{ customers}}{\text{hour}} \times 0.5 \text{ hours} = 5$ expected customers

$P\{N(t) = k\} = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad k = 0, 1, 2$

$P\{N(0.5) = 0\} = \frac{5^0}{0!} e^{-5} = \boxed{0.00674}$

inter-arrival times are iid exponential

$\Rightarrow E(X) = \frac{1}{\lambda} = \frac{1}{10 \text{ customers/hour}}$

9:13 am + $E(X)$ = 9:13 am + $1/10$ hour = $\boxed{9:19 \text{ am}}$

memoryless

$\Rightarrow P$(next customer arrives in less than 10 minutes)

= 1 - P(no customers arrive in the 10 minutes)

$= 1 - P\{N(10/60) = 0\}$

$= 1 - \frac{(10(10/60))^0}{0!} e^{-10(10/60)} = \boxed{0.811}$

4. Consider a small bank with one teller. Customers arrive to the bank according to a Poisson process with rate 8 customers per hour. The teller provides all kinds services for the customers. Each customer takes on average 5 minutes to service. Assume that the service time is exponentially distributed. In steady-state, calculate the following:

(a) What is the probability that the teller is idle?
(b) What is the average number of customers waiting for service?
(c) On average, how long will a customer spend in the bank to complete his service?
(d) What is the probability that there are more than 5 customers in the bank?

$\lambda = 8$ customers / hour

$\mu = 1$ customer / $5/60$ hours $= 12$ customers / hour

prob. teller is idle : $\pi_0 = 1 - \rho$

$= 1 - \frac{8}{12} = \boxed{0.333}$

avg. queue length : $E(L^q) = \frac{\rho^2}{1-\rho}$

$= \frac{(8/12)^2}{1 - (8/12)}$

$= \boxed{1.333}$

avg. sojourn time : $E(S) = \frac{E(L)}{\lambda}$

$= \frac{\frac{\rho}{1-\rho}}{\lambda} = \frac{\frac{8/12}{1-(8/12)}}{8}$

$= \boxed{0.25 \text{ hours} = 15 \text{ minutes}}$

prob. more than 5 custumers :

$P(X > 5) = 1 - P(X \leq 5) = 1 - \sum_{n=0}^{5} \rho^n (1-\rho)$

$* \pi_n = (1-\rho)\rho^n$

$= 1 - \sum_{n=0}^{5} \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$

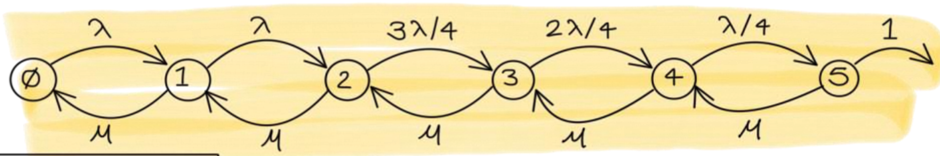
$= 1 - \sum_{n=0}^{5} \left(\frac{8}{12}\right)^n \left(1 - \frac{8}{12}\right)$

$= 1 - 0.91221$

$= \boxed{0.088}$

5. Consider a continuous-time Markovian system with discouraged job arrivals. Jobs arrive to a server according to a Poisson process, with an arrival rate of one job per 7 seconds. The jobs observe the queue. They do NOT join the queue with probability l_k if they observe that there are k jobs in the queue (This only refers to the number of jobs in the queue. The job being serviced, if any, is not included in this number.). l_k = k/4 if k < 4, or 1, otherwise. The service time is exponentially distributed with mean time of 6 seconds.

(a) Please draw the state transition diagram for this queueing system;
(b) Write the Balance Equation for each state. If this is a birth-death process, please only write the Detailed Balance Equations;
(c) Determine the stationary distribution of the number of jobs in the system, and also calculate the mean number of jobs in the system;
(d) When the system becomes stationary, in an interval of 100 seconds, on average how many jobs enter the system (hint: when the system is stationary, the average number of jobs entering the system equals to the average number of jobs finishing their service then leaving the system)?



$\pi_0 \lambda = \pi_1 \mu$ $\quad\Rightarrow \pi_1 = \pi_0 \lambda / \mu$

$\pi_1 \lambda = \pi_2 \mu$ $\quad\Rightarrow \pi_2 = \pi_1 \lambda/\mu = \pi_0 \lambda^2/\mu^2$

$\pi_2 \frac{3\lambda}{4} = \pi_3 \mu$ $\quad\Rightarrow \pi_3 = \pi_2 3\lambda/4\mu = \frac{3\pi_0 \lambda^3}{4\mu^3}$

$\pi_3 \frac{2\lambda}{4} = \pi_4 \mu$ $\quad\Rightarrow \pi_4 = \pi_3 2\lambda/4\mu = \frac{3\pi_0 \lambda^4}{8\mu^4}$

$\pi_4 \frac{\lambda}{4} = \pi_5 \mu$ $\quad\Rightarrow \pi_5 = \pi_4 \lambda/4\mu = \frac{3\pi_0 \lambda^5}{32\mu^5}$

$\sum_{n=0}^{5} \pi_n = 1$

Stationary distribution:

$\rho = \lambda/\mu = (1/7)/(1/6) = 6/7$

$\pi_0 = \dfrac{1}{1+\rho+\rho^2+\frac{3}{4}\rho^3+\frac{3}{8}\rho^4+\frac{3}{32}\rho^5} = \boxed{0.302}$

$\pi_1 = \pi_0 \rho = 0.302\,(6/7) = \boxed{0.259}$

$\pi_2 = \pi_0 \rho^2 = 0.302\,(6/7)^2 = \boxed{0.222}$

$\pi_3 = \frac{3}{4}\pi_0 \rho^3 = \frac{3}{4}(0.302)\left(\frac{6}{7}\right)^3 = \boxed{0.143}$

$\pi_4 = \frac{3}{8}\pi_0 \rho^4 = \frac{3}{8}(0.302)\left(\frac{6}{7}\right)^4 = \boxed{0.061}$

$\pi_5 = \frac{3}{32}\pi_0 \rho^5 = \frac{3}{32}(0.302)\left(\frac{6}{7}\right)^5 = \boxed{0.013}$

$E(X) = \sum_{n=0}^{5} n\pi_n = 0(0.302) + 1(0.259) + 2(0.222) + 3(0.143) + 4(0.061) + 5(0.013)$

mean jobs in system $\approx \boxed{1.44}$

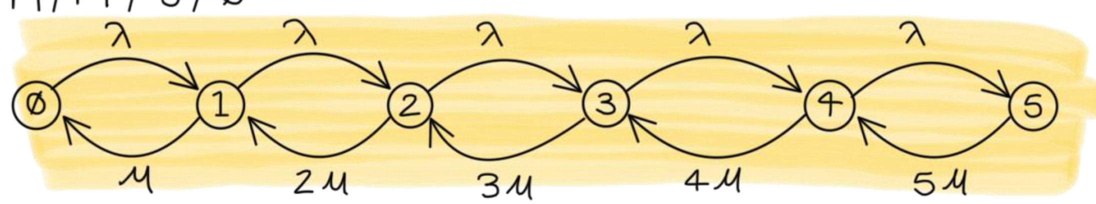
In an interval of 100s:

$\mu(1-\pi_0)100 = \frac{1}{6}(1-0.302)100 \approx \boxed{11.6}$

6. Consider a gas station located on a highway with five pumps. Cars arrive at the gas station according to a Poisson process at rate 50 cars/hour. Any car able to enter the gas station stops by one of the available pumps. If all pumps are occupied, the car will not enter the gas station and will just leave. Each car takes an exponential amount of time to refill, and the average refill time is 5 minutes.

(a) Draw the state transition diagram for the gas station.
(b) Determine the stationary distribution of the number of cars in the system.
(c) What is the probability that an arriving car will NOT be able to enter the gas station to refill? In 24 hours, on average how many cars cannot enter the gas station and thus have to leave (If you are the owner of the gas station, this is the business you will lose)?
(d) Now consider that you have bought a small parking lot right beside the gas station, so that a car can stop there and wait for any pump becomes available. Suppose that there are in total 2 spaces in the parking lot. An arriving car will not enter the gas station and instead leave immediately if all parking spaces are occupied. How many business will be lost in 24 hours in this case?

M/M/5/Ø



$\lambda = 50$ cars/hour, $\mu = 1$ car / (5/60 hours) $= 12$ cars/hour

From notes, stationary distribution:

$\boxed{\pi_n = \pi_0 \dfrac{(\lambda/\mu)^n}{n!}}$ for $n = 0, 1, 2, \ldots 5$

$$\pi_0 = \left[\sum_{n=0}^{5} \frac{(\lambda/\mu)^n}{n!} \right]^{-1} = \left[\sum_{n=0}^{5} \frac{(50/12)^n}{n!} \right]^{-1} \approx \boxed{0.0204}$$

$$\pi_1 = \pi_0 (50/12) = (0.0204)(50/12) \approx \boxed{0.0852}$$

$$\pi_2 = \pi_0 \frac{(50/12)^2}{2!} = (0.0204)\frac{(50/12)^2}{2!} \approx \boxed{0.177}$$

$$\pi_3 = \pi_0 \frac{(50/12)^3}{3!} = (0.0204)\frac{(50/12)^3}{3!} \approx \boxed{0.246}$$

$$\pi_4 = \pi_0 \frac{(50/12)^4}{4!} = (0.0204)\frac{(50/12)^4}{4!} \approx \boxed{0.257}$$

$$\pi_5 = \pi_0 \frac{(50/12)^5}{5!} = (0.0204)\frac{(50/12)^5}{5!} \approx \boxed{0.214}$$

Prob. car doesn't enter:

$P_{\text{not enter}} = \boxed{\pi_5 = 0.214}$ or 21.4%

Number of cars expected to be blocked in 24 hours:

$\lambda \pi_5 (24 \text{ hours}) = 50(0.214)(24) = \boxed{256.7}$

M/M/5/2 : 2 parking buffers

$$\begin{tikzcd}
0 \arrow[r, bend left, "\lambda"] & 1 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "\lambda"] & 2 \arrow[l, bend left, "2\mu"] \arrow[r, bend left, "\lambda"] & 3 \arrow[l, bend left, "3\mu"] \arrow[r, bend left, "\lambda"] & 4 \arrow[l, bend left, "4\mu"] \arrow[r, bend left, "\lambda"] & 5 \arrow[l, bend left, "5\mu"] \arrow[r, bend left, "\lambda"] & 6 \arrow[l, bend left, "5\mu"] \arrow[r, bend left, "\lambda"] & 7 \arrow[l, bend left, "5\mu"]
\end{tikzcd}$$

$\pi_n = \dfrac{(c\rho)^n}{n!}\pi_0$

$\rho = \lambda / c\mu = 5/6$

$\pi_{5+n} = \rho^n \pi_5$

$$\pi_0 = \left(\sum_{n=0}^{4} \frac{(25/6)^n}{n!} + \frac{(25/6)^5}{5!} \frac{1}{1 - 5/6} \right)^{-1} = 0.00284$$

$$\pi_5 = \frac{(25/6)^5}{5!}(0.00284) = 0.0297$$

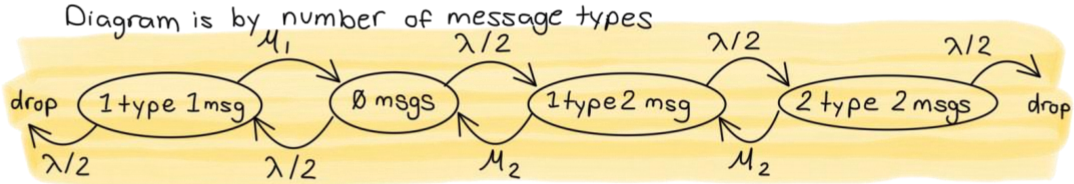
$$\pi_7 = (5/6)^2 \pi_5 = (5/6)^2 (0.0297) = 0.021$$

$\lambda \pi_7 (24 \text{ hours}) = 50(0.021)(24) = \boxed{24.75}$

7. Consider a communication link with a constant rate of 4.8kbit/s. Over the link we transmit two types of messages, both of exponentially distributed size. Messages arrive in a Poisson fashion with rate 10 messages/second. With probability 0.5 (independent from previous arrivals) the arriving message is of type 1 and has a mean length of 300 bits. Otherwise a message of type 2 arrives with a mean length of 150 bits. The buffer at the link can at most hold one message of type 1 or two messages of type 2. A message being transmitted still takes a place in the buffer.

(a) Draw the state transition diagram for the system. Note that in this case the state cannot simply be defined as the number of messages in the system, as messages are of different types.
(b) Determine the average times in the system for **accepted** messages of type 1 and 2, respectively.
(c) Determine the message loss probabilities for messages of type 1 and 2.

Diagram is by number of message types



* different service time since msgs are different sizes

$\frac{\lambda}{2} \pi_{\emptyset} = \mu_1 \pi_{1\,type\,1\,msg}$

$\frac{\lambda}{2} \pi_{\emptyset} = \mu_2 \pi_{1\,type\,2\,msg}$

$\frac{\lambda}{2} \pi_{1\,type\,2\,msg} = \mu_2 \pi_{2\,type\,2\,msg}$

$\pi_{\emptyset} + \pi_{1\,type\,1\,msg} + \pi_{1\,type\,2\,msg} + \pi_{2\,type\,2\,msg} = 1$

* $\mu_1 = \frac{300\ bits}{4800\ bits/sec} = \frac{1}{16}\ sec = 16\ msgs/sec$

$\mu_2 = \frac{150\ bits}{4800\ bits/sec} = \frac{1}{32}\ sec = 32\ msgs/sec$

$\lambda = 10\ msgs/sec$

$5\pi_{\emptyset} = 16\,\pi_{1\,type\,1\,msg}$

$5\pi_{\emptyset} = 32\,\pi_{1\,type\,2\,msg}$

$5\pi_{1\,type\,2\,msg} = 32\,\pi_{2\,type\,2\,msg}$

$\sum \pi_n = 1$

$\pi_{\emptyset} = \frac{1024}{1529}$, $\pi_{1\,type\,1\,msg} = \frac{320}{1529}$, $\pi_{1\,type\,2\,msg} = \frac{160}{1529}$

$\pi_{2\,type\,2\,msg} = \frac{25}{1529}$

average sojourn time for type 1 = $\boxed{1/16\ sec}$

average sojourn time for type 2 = $\dfrac{\frac{1}{32}\pi_{\emptyset} + \frac{2}{32}\pi_{1\,type\,2\,msg}}{\pi_{\emptyset} + \pi_{1\,type\,2\,msg}}$

$= \boxed{0.035\ sec}$

prob. type 1 is blocked $= \pi_{1\,type\,1\,msg} + \pi_{1\,type\,2\,msg} + \pi_{2\,type\,2\,msg}$

$= \boxed{0.33}$

prob. type 2 is blocked $= \pi_{1\,type\,1\,msg} + \pi_{2\,type\,2\,msg}$

$= \boxed{0.23}$