Divide-and-Conquer: Polynomial Multiplication

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Algorithmic Design and Techniques
Algorithms and Data Structures

Outline

1. Problem Overview
2. Naïve Algorithm
3. Naïve Divide and Conquer Algorithm
4. Faster Divide and Conquer

Uses of multiplying polynomials

- Error-correcting codes
- Large-integer multiplication
- Generating functions
- Convolution in signal processing

Multiplying Polynomials

Example

Multiplying Polynomials

Example

$$A(x) = 3x^2 + 2x + 5$$

Multiplying Polynomials

Example

$$A(x) = 3x^2 + 2x + 5$$
$$B(x) = 5x^2 + x + 2$$

Multiplying Polynomials

Example

$$
\begin{aligned}
A(x) &= 3x^2 + 2x + 5 \\
B(x) &= 5x^2 + x + 2 \\
A(x)B(x) &= 15x^4 + 13x^3 + 33x^2 + 9x + 10
\end{aligned}
$$

Multiplying polynomials

Input: Two $n-1$ degree polynomials:

$$a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

$$b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1 x + b_0$$

Output:

Multiplying polynomials

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Output: The product polynomial:

$$c_{2n-2}x^{2n-2} + c_{2n-3}x^{2n-3} + \cdots + c_1 x + c_0$$

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where:

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Multiplying Polynomials

Example

Input: $n = 3, A = (3, 2, 5), B = (5, 1, 2)$

Multiplying Polynomials

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Input: $n = 3, A = (3, 2, 5), B = (5, 1, 2)$

$$A(x) = 3x^2 + 2x + 5$$
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Multiplying Polynomials

Example

Input: $n = 3, A = (3, 2, 5), B = (5, 1, 2)$

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Multiplying Polynomials

Example

Input: $n = 3, A = (3, 2, 5), B = (5, 1, 2)$

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Output: $C = (15, 13, 33, 9, 10)$

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$\texttt{MultPoly}(A, B, n)$

$product \leftarrow Array[2n-1]$
$\texttt{for } i \texttt{ from } 0 \texttt{ to } 2n-2\texttt{:}$
$\quad product[i] \leftarrow 0$

MultPoly(A, B, n)

$product \leftarrow Array[2n-1]$
`for` i `from 0 to` $2n-2$`:`
  $product[i] \leftarrow 0$
`for` i `from 0 to` $n-1$`:`
  `for` j `from 0 to` $n-1$`:`
    $product[i+j] \leftarrow product[i+j] + A[i] \times B[j]$

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  `for` j `from` 0 `to` $n-1$`:`
    $product[i+j] \leftarrow product[i+j] + A[i] \times B[j]$
`return` $product$

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$\texttt{for } i \texttt{ from } 0 \texttt{ to } n-1\texttt{:}$
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Runtime: $O(n^2)$

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  $D_1(x) = a_{n-1}x^{\frac{n}{2}-1} + a_{n-2}x^{\frac{n}{2}-2} + ... + a_{\frac{n}{2}}$
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- Let $B(x) = E_1(x)x^{\frac{n}{2}} + E_0(x)$ where
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- $AB = (D_1x^{\frac{n}{2}} + D_0)(E_1x^{\frac{n}{2}} + E_0)$
  $= (D_1E_1)x^n + (D_1E_0 + D_0E_1)x^{\frac{n}{2}} + D_0E_0$

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- Calculate D_1E_1, D_1E_0, D_0E_1, and D_0E_0

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- Calculate D_1E_1, D_1E_0, D_0E_1, and D_0E_0

Recurrence: $T(n) = 4T(\frac{n}{2}) + kn$.

Polynomial Mult: Divide & Conquer

$A(x) = 4x^3 + 3x^2 + 2x + 1$

$B(x) = x^3 + 2x^2 + 3x + 4$

Polynomial Mult: Divide & Conquer

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

Polynomial Mult: Divide & Conquer

$A(x) = 4x^3 + 3x^2 + 2x + 1$

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$$D_1 E_1 = 4x^2 + 11x + 6$$

Polynomial Mult: Divide & Conquer

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$$D_1 E_1 = 4x^2 + 11x + 6 \qquad D_1 E_0 = 12x^2 + 25x + 12$$

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$$AB = (4x^2 + 11x + 6)x^4 + \\ (12x^2 + 25x + 12 + 2x^2 + 5x + 2)x^2 +$$

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$$\begin{aligned} AB = {} & (4x^2 + 11x + 6)x^4 + \\ & (12x^2 + 25x + 12 + 2x^2 + 5x + 2)x^2 + \\ & 6x^2 + 11x + 4 \end{aligned}$$

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$$\begin{aligned} AB = {} & (4x^2 + 11x + 6)x^4 + \\ & (12x^2 + 25x + 12 + 2x^2 + 5x + 2)x^2 + \\ & 6x^2 + 11x + 4 \\ = {} & 4x^6 + 11x^5 + 20x^4 + 30x^3 + 20x^2 + 11x + 4 \end{aligned}$$

Function $\mathtt{Mult2}(A, B, n, a_l, b_l)$

Function Mult2(A, B, n, a_l, b_l)

$R = \texttt{array}[0..2n-1]$

Function Mult2(A, B, n, a_l, b_l)

```
R = array[0..2n − 1]
if n = 1:
   R[0] = A[a_l] ∗ B[b_l] ; return R
```

Function Mult2(A, B, n, a_l, b_l)

```
R = array[0..2n-1]
if n = 1:
    R[0] = A[a_l] * B[b_l] ; return R
R[0..n-2] = Mult2(A, B, n/2, a_l, b_l)
```

Function Mult2(A, B, n, a_l, b_l)

$R =$ `array`$[0..2n-1]$

`if` $n = 1$:

  $R[0] = A[a_l] * B[b_l]$; `return` R

$R[0..n-2]$ = `Mult2`$(A, B, \frac{n}{2}, a_l, b_l)$

$R[n..2n-2]$ = `Mult2`$(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l + \frac{n}{2})$

Function Mult2(A, B, n, a_l, b_l)

$R =$ `array`$[0..2n-1]$

`if` $n = 1$:

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$R[0..n-2]$ = `Mult2`$(A, B, \frac{n}{2}, a_l, b_l)$

$R[n..2n-2]$ = `Mult2`$(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l + \frac{n}{2})$

D_0E_1 = `Mult2`$(A, B, \frac{n}{2}, a_l, b_l + \frac{n}{2})$

Function Mult2(A, B, n, a_l, b_l)

$R =$ `array`$[0..2n-1]$

`if` $n = 1$:

  $R[0] = A[a_l] * B[b_l]$; `return` R

$R[0..n-2]$ = `Mult2`$(A, B, \frac{n}{2}, a_l, b_l)$

$R[n..2n-2]$ = `Mult2`$(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l + \frac{n}{2})$

D_0E_1 = `Mult2`$(A, B, \frac{n}{2}, a_l, b_l + \frac{n}{2})$

D_1E_0 = `Mult2`$(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l)$

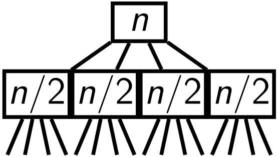
$R[\frac{n}{2} \ldots n + \frac{n}{2} - 2]$ += $D_1E0 + D_0E_1$

Function Mult2(A, B, n, a_l, b_l)

```
R = array[0..2n-1]
if n = 1:
   R[0] = A[a_l] * B[b_l] ; return R
R[0..n-2] = Mult2(A, B, n/2, a_l, b_l)
R[n..2n-2] = Mult2(A, B, n/2, a_l + n/2, b_l + n/2)
D_0E_1 = Mult2(A, B, n/2, a_l, b_l + n/2)
D_1E_0 = Mult2(A, B, n/2, a_l + n/2, b_l)
R[n/2 ... n + n/2 - 2] += D_1E0 + D_0E_1
return R
```

$\boxed{n}$

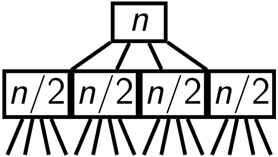
level

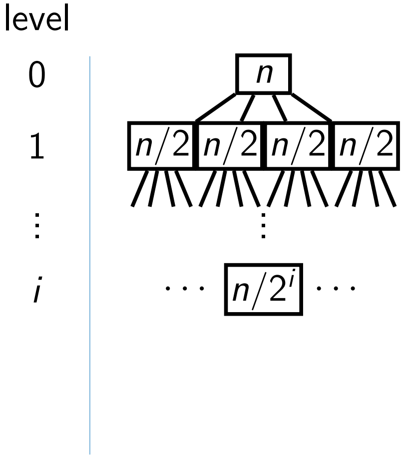


level

0

1




level
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1
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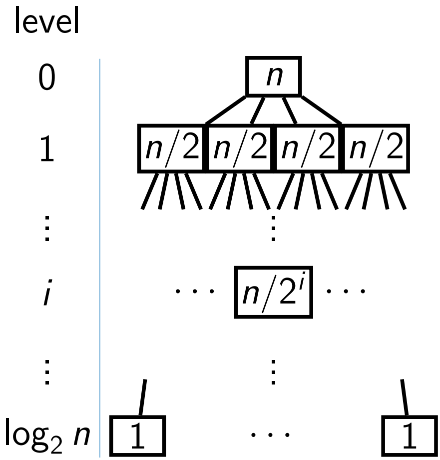
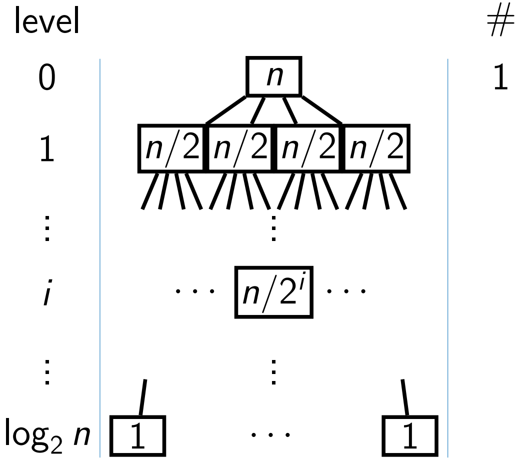
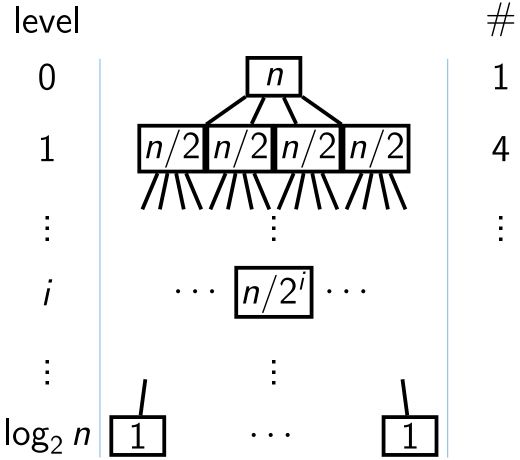
n/2
n/2
n/2
n/2
⋮
⋯ n/2^i ⋯


level
0
1
$\vdots$
i
$\vdots$
$\log_2 n$
n
$n/2$
$n/2$
$n/2$
$n/2$
$\cdots$
$n/2^i$
$\cdots$
1
$\cdots$
1


level
#
0
1
1
n
n/2
n/2
n/2
n/2
i
n/2^i
log_2 n
1
1


level
#
0
n
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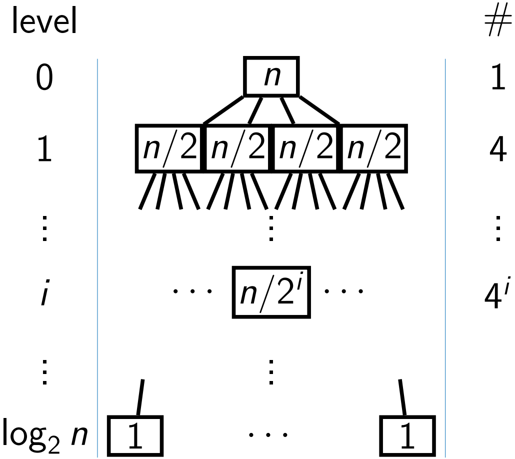
1
n/2
n/2
n/2
n/2
4
i
n/2^i
log_2 n
1
1


level
#
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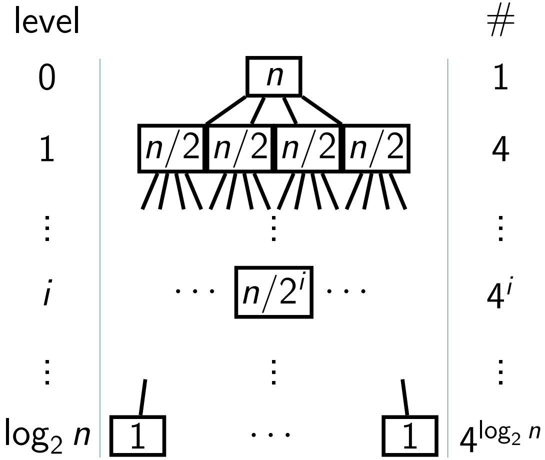
$n/2$ $n/2$ $n/2$ $n/2$
4
i
$\cdots$ $n/2^i$ $\cdots$
4^i
$\log_2 n$
1 $\cdots$ 1


level
#
0
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i
$\cdots$ $n/2^i$ $\cdots$
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$\log_2 n$
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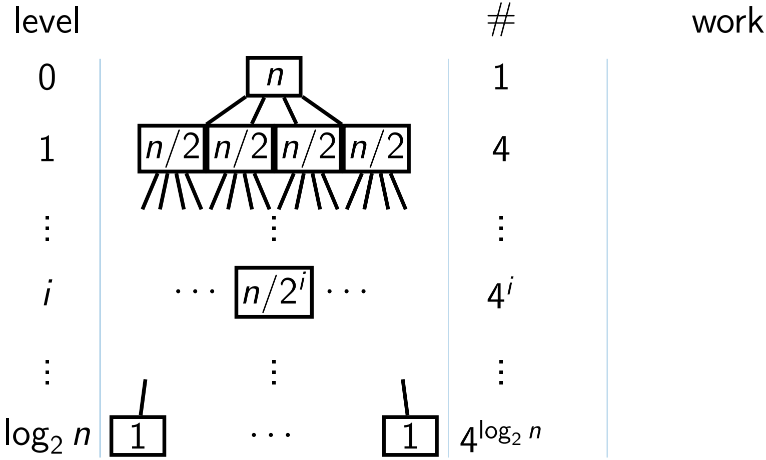
$4^{\log_2 n}$


level
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work
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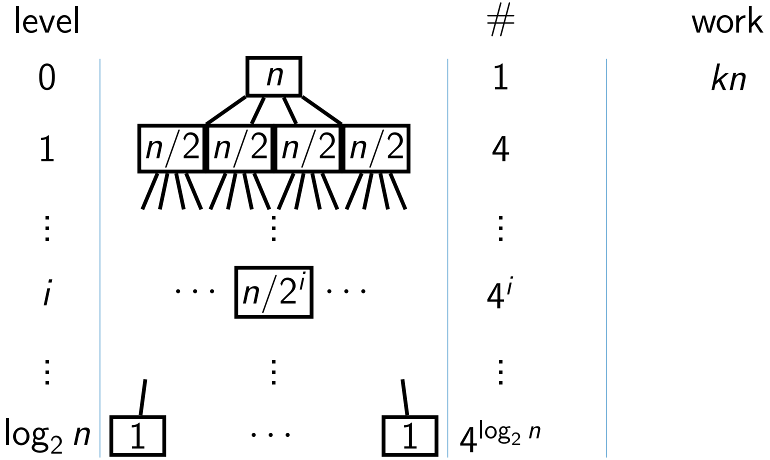
$\cdots$ $n/2^i$ $\cdots$
4^i
$\log_2 n$
1
$\cdots$
1
$4^{\log_2 n}$



level
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work
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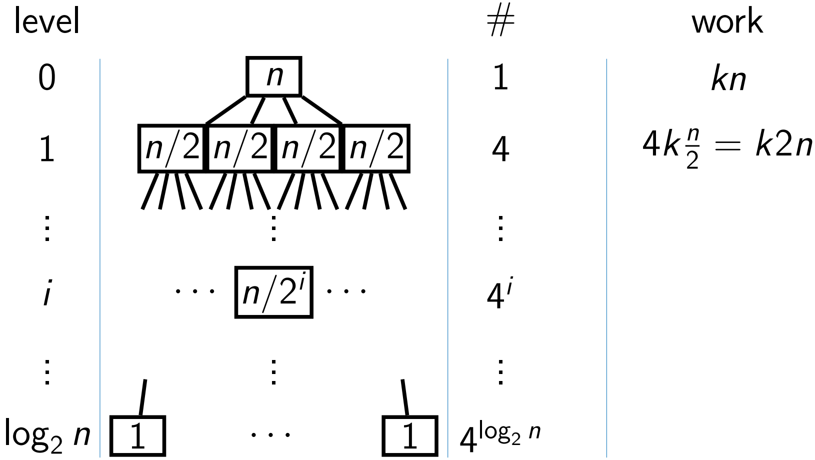
$n/2$ $n/2$ $n/2$ $n/2$
4
$\vdots$
$\vdots$
$\vdots$
i
$\cdots$ $n/2^i$ $\cdots$
4^i
$\vdots$
$\vdots$
$\vdots$
$\log_2 n$
1 $\cdots$ 1
$4^{\log_2 n}$


level
#
work
0
n
1
kn
1
$n/2$ $n/2$ $n/2$ $n/2$
4
$4k\frac{n}{2} = k2n$
$\vdots$
i
$\cdots$ $n/2^i$ $\cdots$
4^i
$\vdots$
$\log_2 n$
1 $\cdots$ 1
$4^{\log_2 n}$

level		#	work
0	$\boxed{n}$	1	kn
1	$\boxed{n/2}\boxed{n/2}\boxed{n/2}\boxed{n/2}$	4	$4k\frac{n}{2} = k2n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$\cdots \boxed{n/2^i} \cdots$	4^i	$4^i k\frac{n}{2^i} = k2^i n$
$\vdots$	$\vdots$	$\vdots$	
$\log_2 n$	$\boxed{1} \quad \cdots \quad \boxed{1}$	$4^{\log_2 n}$	

level		#	work
0	$\boxed{n}$	1	kn
1	$\boxed{n/2}\boxed{n/2}\boxed{n/2}\boxed{n/2}$	4	$4k\frac{n}{2} = k2n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$\cdots \boxed{n/2^i} \cdots$	4^i	$4^i k \frac{n}{2^i} = k2^i n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\log_2 n$	$\boxed{1} \quad \cdots \quad \boxed{1}$	$4^{\log_2 n}$	$k4^{\log_2 n} = kn^2$

level		#	work
0	$\boxed{n}$	1	kn
1	$\boxed{n/2}\boxed{n/2}\boxed{n/2}\boxed{n/2}$	4	$4k\frac{n}{2} = k2n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$\cdots \boxed{n/2^i} \cdots$	4^i	$4^i k \frac{n}{2^i} = k2^i n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\log_2 n$	$\boxed{1} \quad \cdots \quad \boxed{1}$	$4^{\log_2 n}$	$k4^{\log_2 n} = kn^2$

Total: $\sum_{i=0}^{\log_2 n} 4^i k \frac{n}{2^i} = \Theta(n^2)$

Outline

1. Problem Overview
2. Naïve Algorithm
3. Naïve Divide and Conquer Algorithm
4. Faster Divide and Conquer

Karatsuba approach

Karatsuba approach

$$A(x) = a_1 x + a_0$$

Karatsuba approach

$$A(x) = a_1 x + a_0$$

$$B(x) = b_1 x + b_0$$

Karatsuba approach

$$
\begin{aligned}
A(x) =& a_1 x + a_0 \\
B(x) =& b_1 x + b_0 \\
C(x) =& a_1 b_1 x^2 + (a_1 b_0 + a_0 b_1)x + a_0 b_0
\end{aligned}
$$

Karatsuba approach

$$A(x) = a_1 x + a_0$$
$$B(x) = b_1 x + b_0$$
$$C(x) = a_1 b_1 x^2 + (a_1 b_0 + a_0 b_1)x + a_0 b_0$$

Needs 4 multiplications

Karatsuba approach

$$A(x) = a_1 x + a_0$$

$$B(x) = b_1 x + b_0$$

$$C(x) = a_1 b_1 x^2 + (a_1 b_0 + a_0 b_1)x + a_0 b_0$$

Needs 4 multiplications

Rewrite as:

Karatsuba approach

$$A(x) = a_1 x + a_0$$

$$B(x) = b_1 x + b_0$$

$$C(x) = a_1 b_1 x^2 + (a_1 b_0 + a_0 b_1)x + a_0 b_0$$

Needs 4 multiplications

Rewrite as:

$$C(x) = a_1 b_1 x^2 + ((a_1 + a_0)(b_1 + b_0) - a_1 b_1 - a_0 b_0)x + a_0 b_0$$

Karatsuba approach

$$A(x) = a_1 x + a_0$$

$$B(x) = b_1 x + b_0$$

$$C(x) = a_1 b_1 x^2 + (a_1 b_0 + a_0 b_1)x + a_0 b_0$$

Needs 4 multiplications

Rewrite as:

$$C(x) = a_1 b_1 x^2 + ((a_1 + a_0)(b_1 + b_0) - a_1 b_1 - a_0 b_0)x + a_0 b_0$$

Needs 3 multiplications

Karatsuba approach

$$A(x) = a_1 x + a_0$$
$$B(x) = b_1 x + b_0$$
$$C(x) = a_1 b_1 x^2 + (a_1 b_0 + a_0 b_1)x + a_0 b_0$$

Needs 4 multiplications

Rewrite as:

$$\begin{aligned} C(x) = & a_1 b_1 x^2 + \\ & ((a_1 + a_0)(b_1 + b_0) - a_1 b_1 - a_0 b_0)x + \\ & a_0 b_0 \end{aligned}$$

Needs 3 multiplications

Karatsuba approach

$$A(x) = a_1 x + a_0$$

$$B(x) = b_1 x + b_0$$

$$C(x) = a_1 b_1 x^2 + (a_1 b_0 + a_0 b_1)x + a_0 b_0$$

Needs 4 multiplications

Rewrite as:

$$C(x) = a_1 b_1 x^2 + ((a_1 + a_0)(b_1 + b_0) - a_1 b_1 - a_0 b_0)x + a_0 b_0$$

Needs 3 multiplications

Karatsuba Example

$A(x) = 4x^3 + 3x^2 + 2x + 1$

$B(x) = x^3 + 2x^2 + 3x + 4$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$
$$B(x) = x^3 + 2x^2 + 3x + 4$$
$$D_1(x) = 4x + 3$$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

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$$D_1 E_1 = 4x^2 + 11x + 6$$

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$$E_1(x) = x + 2 \qquad E_0(x) = 3x + 4$$

$$D_1 E_1 = 4x^2 + 11x + 6 \qquad D_0 E_0 = 6x^2 + 11x + 4$$

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$$E_1(x) = x + 2 \qquad E_0(x) = 3x + 4$$

$$D_1E_1 = 4x^2 + 11x + 6 \qquad D_0E_0 = 6x^2 + 11x + 4$$

$$(D_1 + D_0)(E_1 + E_0) =$$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3 \qquad D_0(x) = 2x + 1$$

$$E_1(x) = x + 2 \qquad E_0(x) = 3x + 4$$

$$D_1 E_1 = 4x^2 + 11x + 6 \qquad D_0 E_0 = 6x^2 + 11x + 4$$

$$(D_1 + D_0)(E_1 + E_0) = (6x + 4)$$

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$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3 \qquad D_0(x) = 2x + 1$$

$$E_1(x) = x + 2 \qquad E_0(x) = 3x + 4$$

$$D_1 E_1 = 4x^2 + 11x + 6 \qquad D_0 E_0 = 6x^2 + 11x + 4$$

$$(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$$

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$$A(x) = 4x^3 + 3x^2 + 2x + 1$$
$$B(x) = x^3 + 2x^2 + 3x + 4$$
$$D_1(x) = 4x + 3 \qquad D_0(x) = 2x + 1$$
$$E_1(x) = x + 2 \qquad E_0(x) = 3x + 4$$
$$D_1 E_1 = 4x^2 + 11x + 6 \qquad D_0 E_0 = 6x^2 + 11x + 4$$
$$\begin{aligned}(D_1 + D_0)(E_1 + E_0) &= (6x + 4)(4x + 6) \\ &= 24x^2 + 52x + 24\end{aligned}$$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

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$$D_1 E_1 = 4x^2 + 11x + 6 \qquad D_0 E_0 = 6x^2 + 11x + 4$$

$$\begin{aligned}(D_1 + D_0)(E_1 + E_0) &= (6x + 4)(4x + 6) \\ &= 24x^2 + 52x + 24\end{aligned}$$

$$AB = (4x^2 + 11x + 6)x^4 +$$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

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$$D_1E_1 = 4x^2 + 11x + 6 \qquad D_0E_0 = 6x^2 + 11x + 4$$

$$\begin{aligned}(D_1 + D_0)(E_1 + E_0) &= (6x + 4)(4x + 6) \\ &= 24x^2 + 52x + 24\end{aligned}$$

$$\begin{aligned}AB = {} & (4x^2 + 11x + 6)x^4 + \\ & (\qquad\qquad)x^2 +\end{aligned}$$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

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$$\begin{aligned}AB = {} & (4x^2 + 11x + 6)x^4 + \\ & (24x^2 + 52x + 24 \\ & \qquad)x^2 +\end{aligned}$$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

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$$E_1(x) = x + 2 \qquad E_0(x) = 3x + 4$$

$$D_1E_1 = 4x^2 + 11x + 6 \qquad D_0E_0 = 6x^2 + 11x + 4$$

$$\begin{aligned}(D_1 + D_0)(E_1 + E_0) &= (6x + 4)(4x + 6)\\ &= 24x^2 + 52x + 24\end{aligned}$$

$$\begin{aligned}AB = {}&(4x^2 + 11x + 6)x^4 +\\ &(24x^2 + 52x + 24 - (4x^2 + 11x + 6)\\ &\qquad)x^2 +\end{aligned}$$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3 \qquad D_0(x) = 2x + 1$$

$$E_1(x) = x + 2 \qquad E_0(x) = 3x + 4$$

$$D_1 E_1 = 4x^2 + 11x + 6 \qquad D_0 E_0 = \color{red}{6x^2 + 11x + 4}$$

$$\begin{aligned}(D_1 + D_0)(E_1 + E_0) &= (6x + 4)(4x + 6)\\ &= 24x^2 + 52x + 24\end{aligned}$$

$$\begin{aligned}AB = {} & (4x^2 + 11x + 6)x^4 + \\ & (24x^2 + 52x + 24 - (4x^2 + 11x + 6) \\ & \qquad - (\color{red}{6x^2 + 11x + 4}))x^2 +\end{aligned}$$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$
$$B(x) = x^3 + 2x^2 + 3x + 4$$
$$D_1(x) = 4x + 3 \qquad D_0(x) = 2x + 1$$
$$E_1(x) = x + 2 \qquad E_0(x) = 3x + 4$$
$$D_1 E_1 = 4x^2 + 11x + 6 \qquad D_0 E_0 = 6x^2 + 11x + 4$$
$$\begin{aligned}(D_1 + D_0)(E_1 + E_0) &= (6x + 4)(4x + 6) \\ &= 24x^2 + 52x + 24\end{aligned}$$
$$\begin{aligned}AB = {} & (4x^2 + 11x + 6)x^4 + \\ & (24x^2 + 52x + 24 - (4x^2 + 11x + 6) \\ & \qquad - (6x^2 + 11x + 4))x^2 + \\ & 6x^2 + 11x + 4\end{aligned}$$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$
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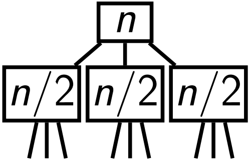
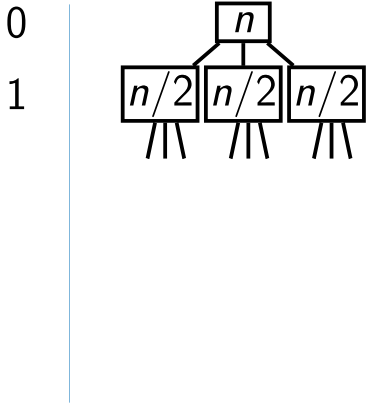
$$\begin{aligned}(D_1 + D_0)(E_1 + E_0) &= (6x + 4)(4x + 6)\\ &= 24x^2 + 52x + 24\end{aligned}$$
$$\begin{aligned}AB = {} & (4x^2 + 11x + 6)x^4 + \\ & (24x^2 + 52x + 24 - (4x^2 + 11x + 6) \\ & \qquad - (6x^2 + 11x + 4))x^2 + \\ & 6x^2 + 11x + 4 \\ = {} & 4x^6 + 11x^5 + 20x^4 + 30x^3 + 20x^2 + 11x + 4\end{aligned}$$

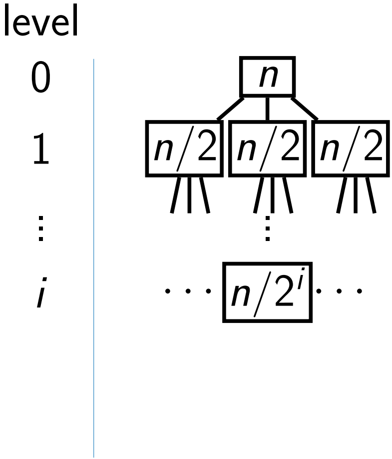
$\boxed{n}$

level



level

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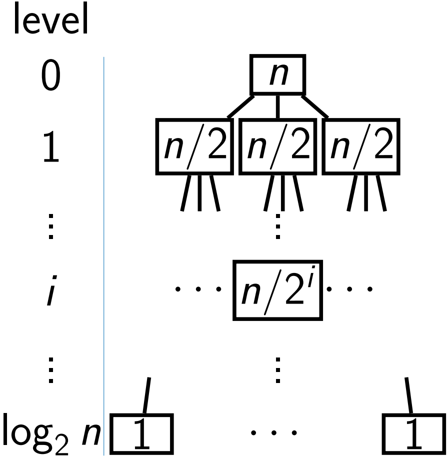
n/2
n/2
⋮
n/2^i


level
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$n/2$
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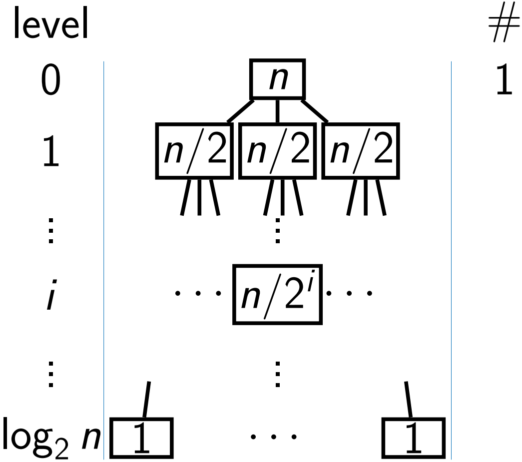
$\cdots$ $n/2^i$ $\cdots$
⋮
1
$\cdots$
1


level
#
0
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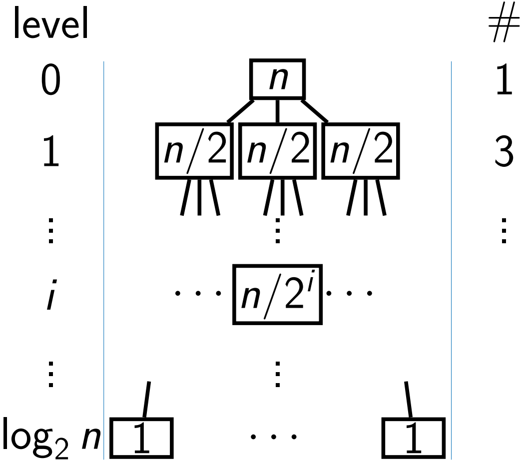
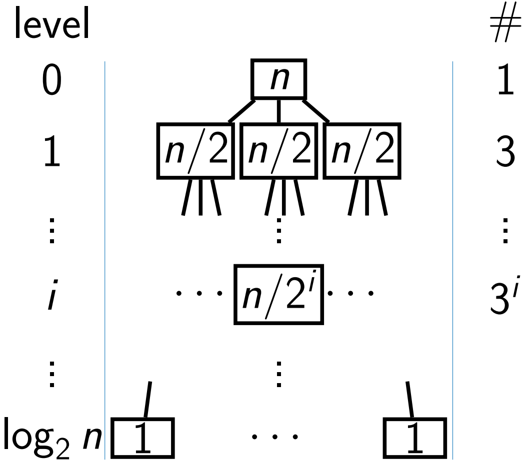
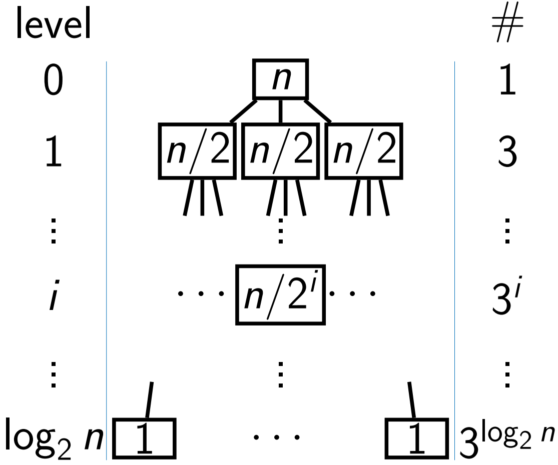
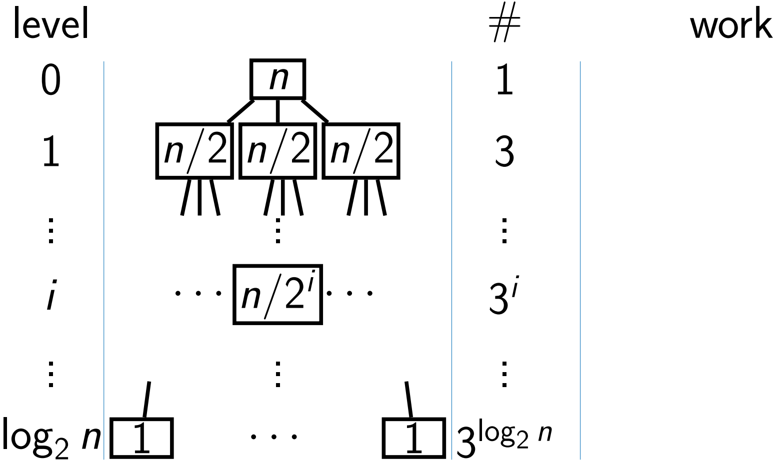
1
n
n/2
n/2
n/2
i
n/2^i
log_2 n
1
1

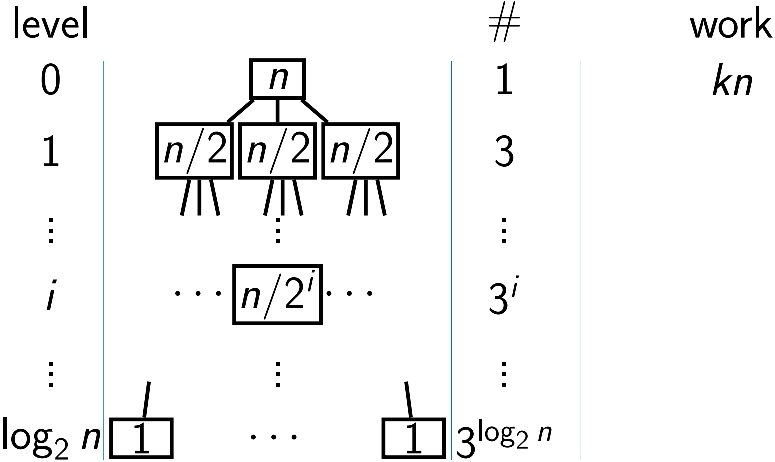


level
#
0
n
1
1
$n/2$
$n/2$
$n/2$
3
i
$\cdots$ $n/2^i$ $\cdots$
$\log_2 n$
1
$\cdots$
1

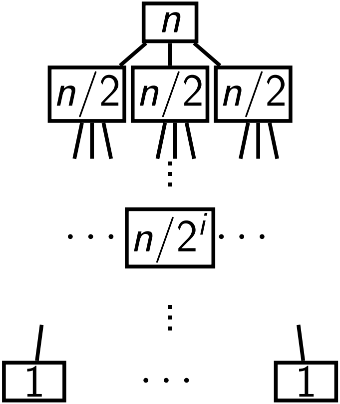

level
#
0
n
1
1
n/2
n/2
n/2
3
i
n/2^i
3^i
log_2 n
1
1


level
#
0
n
1
1
$n/2$
$n/2$
$n/2$
3
i
$n/2^i$
3^i
$\log_2 n$
1
1
$3^{\log_2 n}$

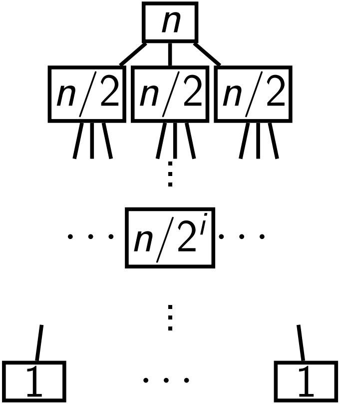

level
#
work
0
n
1
1
$n/2$ $n/2$ $n/2$
3
i
$\cdots$ $n/2^i$ $\cdots$
3^i
$\log_2 n$
1 $\cdots$ 1
$3^{\log_2 n}$



level
#
work
0
n
1
kn
1
n/2
n/2
n/2
3
i
n/2^i
3^i
log_2 n
1
1
3^{log_2 n}

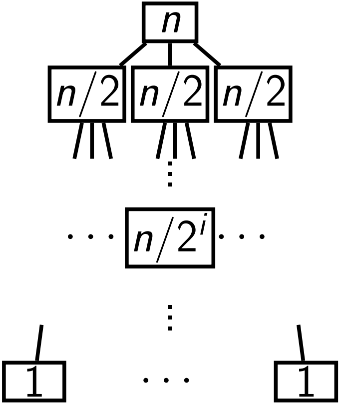


level	#	work
0	1	kn
1	3	$3k\frac{n}{2} = k\frac{3}{2}n$
$\vdots$	$\vdots$	
i	3^i	
$\vdots$	$\vdots$	
$\log_2 n$	$3^{\log_2 n}$	



level	#	work
0	1	kn
1	3	$3k\frac{n}{2} = k\frac{3}{2}n$
$\vdots$	$\vdots$	$\vdots$
i	3^i	$3^i k \frac{n}{2^i} = k\left(\frac{3}{2}\right)^i n$
$\vdots$	$\vdots$	
$\log_2 n$	$3^{\log_2 n}$	

level		#	work
0	$\boxed{n}$	1	kn
1	$\boxed{n/2}\ \boxed{n/2}\ \boxed{n/2}$	3	$3k\frac{n}{2} = k\frac{3}{2}n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
i	$\cdots \boxed{n/2^i} \cdots$	3^i	$3^i k \frac{n}{2^i} = k(\frac{3}{2})^i n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\log_2 n$	$\boxed{1}\ \cdots\ \boxed{1}$	$3^{\log_2 n}$	$k3^{\log_2 n} = kn^{\log_2 3}$

level		#	work
0		1	kn
1		3	$3k\frac{n}{2} = k\frac{3}{2}n$
$\vdots$		$\vdots$	$\vdots$
i		3^i	$3^i k \frac{n}{2^i} = k(\frac{3}{2})^i n$
$\vdots$		$\vdots$	$\vdots$
$\log_2 n$		$3^{\log_2 n}$	$k3^{\log_2 n} = kn^{\log_2 3}$

Total: $\sum_{i=0}^{\log_2 n} 3^i k \frac{n}{2^i} = \Theta(n^{\log_2 3})$

$= \Theta(n^{1.58})$