Greedy Algorithms: Grouping Children

Michael Levin

Higher School of Economics

Algorithmic Design and Techniques
Algorithms and Data Structures

Outline

1. The Problem
2. Naive Algorithm
3. Efficient Algorithm

Many children came to a celebration. Organize them into the minimum possible number of groups such that the age of any two children in the same group differ by at most one year.

Outline

1. The Problem
2. Naive Algorithm
3. Efficient Algorithm

MinGroups(C)

```
m ← len(C)
for each partition into groups
C = G_1 ∪ G_2 ∪ ··· ∪ G_k:
   good ← true
   for i from 1 to k:
      if max(G_i) − min(G_i) > 1:
         good ← false
   if good:
      m ← min(m, k)
return m
```

Running time

Lemma

The number of operations in `MinGroups`(C) is at least 2^n, where n is the number of children in C.

Proof

Proof

- Consider just partitions in two groups

Proof

- Consider just partitions in two groups
- $C = G_1 \cup G_2$

Proof

- Consider just partitions in two groups
- $C = G_1 \cup G_2$
- For each $G_1 \subset C$, $G_2 = C \setminus G_1$

Proof

- Consider just partitions in two groups
- $C = G_1 \cup G_2$
- For each $G_1 \subset C$, $G_2 = C \setminus G_1$
- Size of C is n

Proof

- Consider just partitions in two groups
- $C = G_1 \cup G_2$
- For each $G_1 \subset C$, $G_2 = C \setminus G_1$
- Size of C is n
- Each item can be included or excluded from G_1

Proof

- Consider just partitions in two groups
- $C = G_1 \cup G_2$
- For each $G_1 \subset C$, $G_2 = C \setminus G_1$
- Size of C is n
- Each item can be included or excluded from G_1
- There are 2^n different G_1

Proof

- Consider just partitions in two groups
- $C = G_1 \cup G_2$
- For each $G_1 \subset C$, $G_2 = C \setminus G_1$
- Size of C is n
- Each item can be included or excluded from G_1
- There are 2^n different G_1
- Thus, at least 2^n operations $\square$

Asymptotics

- Naive algorithm works in time $\Omega(2^n)$

Asymptotics

- Naive algorithm works in time $\Omega(2^n)$
- For $n = 50$ it is at least

$$2^{50} = 1125899906842624$$

operations!

Asymptotics

- Naive algorithm works in time $\Omega(2^n)$
- For $n = 50$ it is at least

$$2^{50} = 1125899906842624$$

  operations!
- We will improve this significantly

Outline

1. The Problem
2. Naive Algorithm
3. Efficient Algorithm

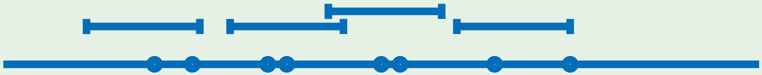
Covering points by segments

Input: A set of n points $x_1, \ldots, x_n \in \mathbb{R}$.

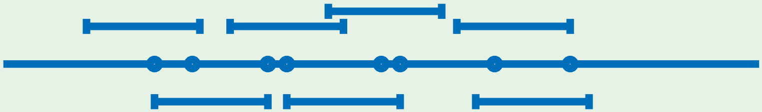
Output: The minimum number of segments of unit length needed to cover all the points.

Example

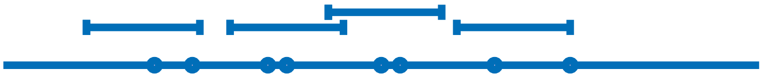
Example



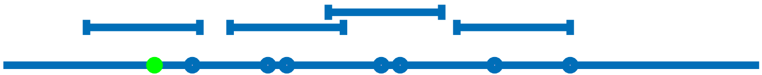
Example



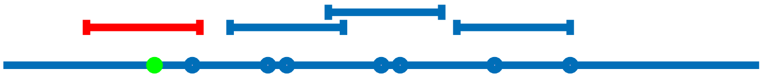
Safe move: cover the leftmost point with a unit segment which starts in this point.



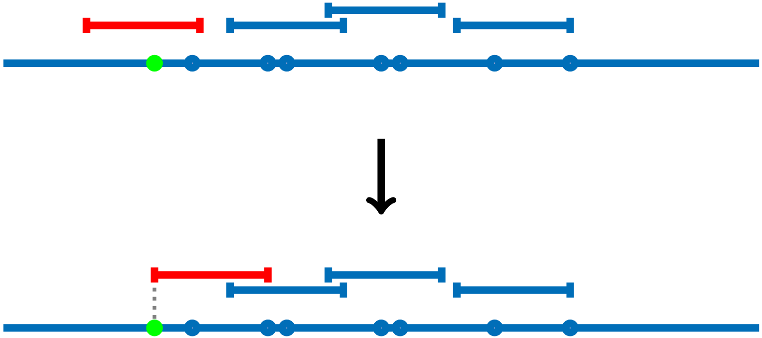
Safe move: cover the leftmost point with a unit segment which starts in this point.



Safe move: cover the leftmost point with a unit segment which starts in this point.



Safe move: cover the leftmost point with a unit segment which starts in this point.



Assume $x_1 \leq x_2 \leq \ldots \leq x_n$

```
PointsCoverSorted(x_1, ..., x_n)

R ← {}, i ← 1
while i ≤ n:
   [ℓ, r] ← [x_i, x_i + 1]
   R ← R ⋃ {[ℓ, r]}
   i ← i + 1
   while i ≤ n and x_i ≤ r:
      i ← i + 1
return R
```

Lemma

The running time of `PointsCoverSorted` is $O(n)$.

Lemma

The running time of `PointsCoverSorted` is $O(n)$.

Proof

- i changes from 1 to n

Lemma

The running time of `PointsCoverSorted` is $O(n)$.

Proof

- i changes from 1 to n
- For each i, at most 1 new segment

Lemma

The running time of `PointsCoverSorted` is $O(n)$.

Proof

- i changes from 1 to n
- For each i, at most 1 new segment
- Overall, running time is $O(n)$ $\square$

Total Running Time

- `PointsCoverSorted` works in $O(n)$ time

Total Running Time

- `PointsCoverSorted` works in $O(n)$ time
- Sort $\{x_1, x_2, \ldots, x_n\}$, then call `PointsCoverSorted`

Total Running Time

- `PointsCoverSorted` works in $O(n)$ time
- Sort $\{x_1, x_2, \ldots, x_n\}$, then call `PointsCoverSorted`
- Soon you'll learn to sort in $O(n \log n)$

Total Running Time

- `PointsCoverSorted` works in $O(n)$ time
- Sort $\{x_1, x_2, \ldots, x_n\}$, then call `PointsCoverSorted`
- Soon you'll learn to sort in $O(n \log n)$
- Sort + `PointsCoverSorted` is $O(n \log n)$

Asymptotics

- Straightforward solution is $\Omega(2^n)$

Asymptotics

- Straightforward solution is $\Omega(2^n)$
- Very long for $n = 50$

Asymptotics

- Straightforward solution is $\Omega(2^n)$
- Very long for $n = 50$
- Sort + greedy is $O(n \log n)$

Asymptotics

- Straightforward solution is $\Omega(2^n)$
- Very long for $n = 50$
- Sort + greedy is $O(n \log n)$
- Fast for $n = 10\,000\,000$

Asymptotics

- Straightforward solution is $\Omega(2^n)$
- Very long for $n = 50$
- Sort + greedy is $O(n \log n)$
- Fast for $n = 10\,000\,000$
- Huge improvement!

Conclusion

- Straightforward solution is exponential
- Important to reformulate the problem in mathematical terms
- Safe move is to cover leftmost point
- Sort in $O(n \log n)$ + greedy in $O(n)$