# Dynamic Programming: Placing Parentheses

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**Algorithmic Design and Techniques**
**Algorithms and Data Structures**

# Outline

How to place parentheses in an expression

$$1 + 2 - 3 \times 4 - 5$$

to maximize its value?

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- $((((1 + 2) - 3) \times 4) - 5) = -5$
- $((1 + 2) - ((3 \times 4) - 5)) = -4$

## Answer

$$((1 + 2) - (3 \times (4 - 5))) = 6$$

## Another example

What about

$$5 - 8 + 7 \times 4 - 8 + 9?$$

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## Soon

We'll design an efficient dynamic programming algorithm to find the answer.

# Outline

## Placing parentheses

Input: A sequence of digits $d_1, \ldots, d_n$ and a sequence of operations $op_1, \ldots, op_{n-1} \in \{+, -, \times\}$.

Output: An order of applying these operations that maximizes the value of the expression

$$d_1 \, op_1 \, d_2 \, op_2 \cdots op_{n-1} \, d_n \, .$$

# Intuition

- Assume that the last operation in an optimal parenthesizing of $5 - 8 + 7 \times 4 - 8 + 9$ is $\times$:

$$\left(5 - 8 + 7\right) \times \left(4 - 8 + 9\right).$$

# Intuition

- Assume that the last operation in an optimal parenthesizing of $5 - 8 + 7 \times 4 - 8 + 9$ is $\times$:

$$(5 - 8 + 7) \times (4 - 8 + 9).$$

- It would help to know optimal values for subexpressions $5 - 8 + 7$ and $4 - 8 + 9$.

## However

We need to keep track for both the minimal and the maximal values of subexpressions!

## Example: $(5 - 8 + 7) \times (4 - 8 + 9)$

$$\min(5 - 8 + 7) = (5 - (8 + 7)) = -10$$
$$\max(5 - 8 + 7) = ((5 - 8) + 7) = \phantom{-}4$$
$$\min(4 - 8 + 9) = (4 - (8 + 9)) = -13$$
$$\max(4 - 8 + 9) = ((4 - 8) + 9) = \phantom{-}5$$

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$$\max((5 - 8 + 7) \times (4 - 8 + 9)) = 130$$

# Subproblems

- Let $E_{i,j}$ be the subexpression

$$d_i \ op_i \cdots op_{j-1} \ d_j$$

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- Subproblems:

$$M(i,j) = \text{maximum value of } E_{i,j}$$
$$m(i,j) = \text{minimum value of } E_{i,j}$$

# Recurrence Relation

$$M(i,j) = \max_{i \le k \le j-1} \begin{cases} M(i,k) & op_k & M(k+1,j) \\ M(i,k) & op_k & m(k+1,j) \\ m(i,k) & op_k & M(k+1,j) \\ m(i,k) & op_k & m(k+1,j) \end{cases}$$

$$m(i,j) = \min_{i \le k \le j-1} \begin{cases} M(i,k) & op_k & M(k+1,j) \\ M(i,k) & op_k & m(k+1,j) \\ m(i,k) & op_k & M(k+1,j) \\ m(i,k) & op_k & m(k+1,j) \end{cases}$$

# Outline

## MinAndMax($i, j$)

```
min ← +∞
max ← −∞
for k from i to j − 1:
  a ← M(i, k)   op_k   M(k + 1, j)
  b ← M(i, k)   op_k   m(k + 1, j)
  c ← m(i, k)   op_k   M(k + 1, j)
  d ← m(i, k)   op_k   m(k + 1, j)
  min ← min(min, a, b, c, d)
  max ← max(max, a, b, c, d)
return (min, max)
```
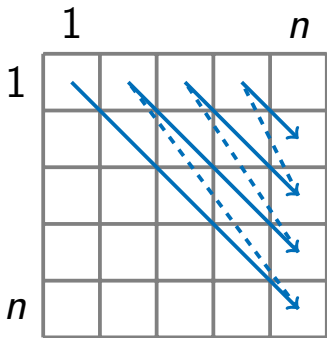
# Order of Subproblems

- When computing $M(i, j)$, the values of $M(i, k)$ and $M(k + 1, j)$ should be already computed.
- Solve all subproblems in order of increasing $(j - i)$.

# Possible Order

## Parentheses($d_1 \, op_1 \, d_2 \, op_2 \ldots d_n$)

```
for i from 1 to n:
    m(i, i) ← dᵢ,  M(i, i) ← dᵢ
for s from 1 to n − 1:
    for i from 1 to n − s:
        j ← i + s
        m(i, j), M(i, j) ← MinAndMax(i, j)
return M(1, n)
```

# Example: $5 - 8 + 7 \times 4 - 8 + 9$

| 5 | -3 | -10 | -55 | -63 | -94 |
|---|----|-----|-----|-----|------|
|   | 8  | 15  | 36  | -60 | -195 |
|   |    | 7   | 28  | -28 | -91  |
|   |    |     | 4   | -4  | -13  |
|   |    |     |     | 8   | 17   |
|   |    |     |     |     | 9    |

$m$

| 5 | -3 | 4  | 25 | 65 | 200 |
|---|----|----|----|----|-----|
|   | 8  | 15 | 60 | 52 | 75  |
|   |    | 7  | 28 | 20 | 35  |
|   |    |    | 4  | -4 | 5   |
|   |    |    |    | 8  | 17  |
|   |    |    |    |    | 9   |

$M$

# Outline

# Example: $5 - 8 + 7 \times 4 - 8 + 9$

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$m$

| | | | | | |
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| | | | | 8 | 17 |
| | | | | | 9 |

$M$