Divide-and-Conquer: Polynomial Multiplication

Neil Rhodes

Department of Computer Science and Engineering University of California, San Diego

Algorithmic Design and Techniques Algorithms and Data Structures

Outline

- 1 Problem Overview
- 2 Naïve Algorithm
- 3 Naïve Divide and Conquer Algorithm
- 4 Faster Divide and Conquer

Uses of multiplying polynomials

- Error-correcting codes
- Large-integer multiplication
- Generating functions
- Convolution in signal processing

$$A(x) = 3x^2 + 2x + 5$$

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$$B(x) = 5x^2 + x + 2$$

$$A(x) = 3x^{2} + 2x + 5$$

$$B(x) = 5x^{2} + x + 2$$

$$A(x)B(x) = 15x^{4} + 13x^{3} + 33x^{2} + 9x + 10$$

Input: Two n-1 degree polynomials: $a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$ $b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1x + b_0$

Output:

Input: Two
$$n-1$$
 degree polynomials: $a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$
 $b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1x + b_0$

Output: The product polynomial:
$$c_{2n-2}x^{2n-2} + c_{2n-3}x^{2n-3} + \cdots + c_1x + c_0$$

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$$n-1$$
 degree polynomials: $a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$
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$$c_{2n-2}x^{2n-2} + c_{2n-3}x^{2n-3} + \cdots + c_1x + c_0$$
 where: $c_{2n-2} = a_{n-1}b_{n-1}$

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Output: The product polynomial: $c_{2n-2}x^{2n-2} + c_{2n-3}x^{2n-3} + \cdots + c_1x + c_0$ where:

$$c_{2n-2} = a_{n-1}b_{n-1}$$

 $c_{2n-3} = a_{n-1}b_{n-2} + a_{n-2}b_{n-1}$

Input: Two
$$n-1$$
 degree polynomials: $a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$
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 $c_{2n-3} = a_{n-1}b_{n-2} + a_{n-2}b_{n-1}$

$$\cdots$$

$$c_2 = a_2b_0 + a_1b_1 + a_0b_2$$

Input: Two
$$n-1$$
 degree polynomials: $a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$

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 $c_2 = a_2b_0 + a_1b_1 + a_0b_2$ $c_1 = a_1b_0 + a_0b_1$

 $c_0 = a_0 b_0$

Input: Two
$$n-1$$
 degree polynomials: $a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$ $b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \cdots + b_1x + b_0$

Output: The product polynomial: $c_{2n-2}x^{2n-2} + c_{2n-3}x^{2n-3} + \cdots + c_1x + c_0$ where: $c_{2n-2} = a_{n-1}b_{n-1}$ $c_{2n-3} = a_{n-1}b_{n-2} + a_{n-2}b_{n-1}$...

$$egin{aligned} c_{2n-2} &= a_{n-1}b_{n-1} \ c_{2n-3} &= a_{n-1}b_{n-2} + a_{n-2}b_{n-1} \ & \cdots \ c_2 &= a_2b_0 + a_1b_1 + a_0b_2 \ c_1 &= a_1b_0 + a_0b_1 \end{aligned}$$

Example

Input: n = 3, A = (3, 2, 5), B = (5, 1, 2)

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$$A(x)B(x) = 15x^{4} + 13x^{3} + 33x^{2} + 9x + 10$$

Example

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$$n = 3, A = (3, 2, 5), B = (5, 1, 2)$$

$$A(x) = 3x^{2} + 2x + 5$$

$$B(x) = 5x^{2} + x + 2$$

$$A(x)B(x) = 15x^{4} + 13x^{3} + 33x^{2} + 9x + 10$$

Output: C = (15, 13, 33, 9, 10)

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```
product \leftarrow Array[2n-1] for i from 0 to 2n-2: product[i] \leftarrow 0
```

```
product \leftarrow Array[2n-1]
```

for i from 0 to 2n-2: $product[i] \leftarrow 0$

for i from 0 to n-1:

for *i* from 0 to n-1:

 $product[i + j] \leftarrow product[i + j] + A[i] \times B[j]$

```
product \leftarrow Array[2n-1]
```

 $product[i] \leftarrow 0$

return product

for i from 0 to 2n-2:

for i from 0 to n-1:

for *i* from 0 to n-1:

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product[i] \leftarrow 0
for i from 0 to n-1:
product[i+j] \leftarrow product[i+j] + A[i] \times B[j]
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```

Runtime: $O(n^2)$

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Let
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 where $D_1(x) = a_{n-1}x^{\frac{n}{2}-1} + a_{n-2}x^{\frac{n}{2}-2} + ... + a_{\frac{n}{2}}$

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 where $D_1(x) = a_{n-1}x^{\frac{n}{2}-1} + a_{n-2}x^{\frac{n}{2}-2} + ... + a_{\frac{n}{2}}$

$$D_1(x) = a_{n-1}x^2 + a_{n-2}x^2 + ... + a_{\frac{n}{2}}$$

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$$E_0(x) = b_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + b_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + ... + b_0$$

$$AB = (D_1 x^{\frac{n}{2}} + D_0)(E_1 x^{\frac{n}{2}} + E_0)$$

$$= (D_1 E_1) x^n + (D_1 E_0 + D_0 E_1) x^{\frac{n}{2}} + D_0 E_0$$

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$$= (D_1 E_1) x^n + (D_1 E_0 + D_0 E_1) x^{\frac{n}{2}} + D_0 E_0$$

■ Calculate D_1E_1 , D_1E_0 , D_0E_1 , and D_0E_0

Let
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 where $D_1(x) = a_{n-1}x^{\frac{n}{2}-1} + a_{n-2}x^{\frac{n}{2}-2} + ... + a_{\frac{n}{2}}$ $D_0(x) = a_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + a_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + ... + a_0$

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$$AB = (D_1 x^{\frac{n}{2}} + D_0)(E_1 x^{\frac{n}{2}} + E_0)$$

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$$E_1(x) = b_{n-1}x^{\frac{n}{2}-1} + b_{n-2}x^{\frac{n}{2}-2} + \dots + b_{\frac{n}{2}}$$

$$E_0(x) = b_{\frac{n}{2}-1}x^{\frac{n}{2}-1} + b_{\frac{n}{2}-2}x^{\frac{n}{2}-2} + \dots + b_0$$

$$= (D_1E_1)x^n + (D_1E_0 + D_0E_1)x^{\frac{n}{2}} + D_0E_0$$
• Calculate D_1E_1, D_1E_0, D_0E_1 , and D_0E_0

Recurrence: $T(n) = 4T(\frac{n}{2}) + kn$.

Polynomial Mult: Divide & Conquer $A(x) = 4x^3 + 3x^2 + 2x + 1$

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

Polynomial Mult: Divide & Conquer $A(x) = \frac{4x^3 + 3x^2 + 2x + 1}{4x^3 + 3x^2 + 2x + 1}$

$$A(x) = \frac{4x^3 + 3x^2 + 2x + 1}{B(x) = x^3 + 2x^2 + 3x + 4}$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

Polynomial Mult: Divide & Conquer $A(x) = 4x^3 + 3x^2 + 2x + 1$

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$B(x) = 4x + 3x + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

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$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$B(x) = x^{3} + 2x^{2} + 3x + 4$$

$$D_{1}(x) = 4x + 3$$

$$D_{0}(x) = 2x + 1$$

 $E_1(x) = x + 2$

Polynomial Mult: Divide & Conquer $A(x) = 4x^3 + 3x^2 + 2x + 1$

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 $D_1(x) = 4x + 3$

$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

 $F_1(x) = x + 2$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$y = 4x + 3$$

 $y = x + 2$

$$=x+2$$

$$= 4x + 3$$
$$= x + 2$$

$$D_0(x) = 2x + 1$$

 $E_0(x) = 3x + 4$

$$D_0(x)=2x+1$$

 $E_0(x) = 3x + 4$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

 $D_1(x) = 4x + 3$ $D_0(x) = 2x + 1$

$$D_1(x) = 4x + 3$$

 $E_1(x) = x + 2$

 $D_1 E_1 = 4x^2 + 11x + 6$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$B(x) = x^3 + 2x^2 + 3x + D_1(x) = 4x + 3$$

 $D_1 E_1 = 4x^2 + 11x + 6$

$$=4x+3$$

$$D_0(x) = 2x + 1$$

 $E_0(x) = 3x + 4$

$$E_0(x) = 3$$

$$E_0(x) \equiv 3x + 4$$

 $D_1 E_0 = 12x^2 + 25x + 12$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

 $D_1 E_1 = 4x^2 + 11x + 6$

 $D_0E_1 = 2x^2 + 5x + 2$

 $E_1(x) = x + 2$

$$+2x^{2}+3x+$$

$$D_0(x) = 2x + 1$$

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 $D_1 E_0 = 12x^2 + 25x + 12$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

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 $D_0E_1 = 2x^2 + 5x + 2$

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$$E_1(x) = x + 2$$

 $D_1 E_1 = 4x^2 + 11x + 6$

$$E_0(x) = \frac{3x + 4}{D_1 E_0} = 12x^2 + \frac{3x + 4}{D_1 E_0}$$

$$D_1 E_0 = 12x^2 + 25x + 12$$

$$= 12x^2$$

 $D_0(x) = 2x + 1$

$$D_1 E_0 = 12x + 25x + 12$$
$$D_0 E_0 = 6x^2 + 11x + 4$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

 $D_0E_1 = 2x^2 + 5x + 2$

AB =

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 3$$
$$E_1(x) = x + 2$$

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 $D_1 E_1 = 4x^2 + 11x + 6$

$$D_1E_0=12x^2+25x+12$$

$$=1$$

 $D_0(x) = 2x + 1$ $E_0(x) = 3x + 4$

$$= 12x$$

$$D_1 E_0 = 12x^2 + 25x +$$

$$D_0 E_0 = 6x^2 + 11x + 4$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

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$$= x + 2$$

$$D_1 E_1 = 4x^2 + 11x + 6$$

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$$D_0E_1 = 2x^2 + 5x + 2$$

 $AB = (4x^2 + 11x + 6)x^4 + 6$

$$P_0 E_0 =$$

 $D_0(x) = 2x + 1$ $E_0(x) = 3x + 4$

$$D_0 E_0 = 6x^2 + 11x + 4$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

 $D_1 E_1 = 4x^2 + 11x + 6$

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 $E_1(x) = x + 2$

$$D_1(x) = 4x + 3$$

$$D_0 E_0 = 6x^2 + 11x + 4$$

$$_{0}E_{0} =$$

 $D_0(x) = 2x + 1$ $E_0(x) = 3x + 4$

$$D_1E_0=12x^2+25x+12$$

 $)x^{2} +$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

 $D_0E_1 = 2x^2 + 5x + 2$

 $AB = (4x^2 + 11x + 6)x^4 +$

 $(12x^2 + 25x + 12)$

 $E_1(x) = x + 2$

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 $D_1E_1 = 4x^2 + 11x + 6$

$$D_0 E_0 = 6x^2 + 11x + 4$$

 $D_0(x) = 2x + 1$ $E_0(x) = 3x + 4$

$$5x^2 + 1$$

 $)x^{2} +$

$$11x +$$

$$D_1 E_0 = 12x^2 + 25x + 12$$
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 $AB = (4x^2 + 11x + 6)x^4 +$

 $D_1(x) = 4x + 3$

 $E_1(x) = x + 2$

 $(12x^2 + 25x + 12 + 2x^2 + 5x + 2)x^2 +$

$$E_0(x)=3x+4$$

$$E_0($$

 $D_0(x) = 2x + 1$

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 $6x^2 + 11x + 4$

 $E_1(x) = x + 2$

$$B(x) = x^3 + 2x^2 + 3x +$$

$$D_1(x) = 4x + 3$$

 $(12x^2 + 25x + 12 + 2x^2 + 5x + 2)x^2 +$

$$E_0(x)=3x+4$$

$$D_1 E_0 = 12x^2 + 25x + 12$$

$$= 1$$

 $D_0(x) = 2x + 1$

$$D_0 E_0 = 6x^2 + 11x + 4$$



$$B(x) = x^3 + 2x^2 + 3x + 4$$

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 $D_1(x) = 4x + 3$ $D_0(x) = 2x + 1$
 $E_1(x) = x + 2$ $E_0(x) = 3x + 4$

$$E_1(x) = 4x + 3$$
 $E_0(x) = 2x + 1$
 $E_1(x) = x + 2$ $E_0(x) = 3x + 4$
 $E_1(x) = 4x^2 + 11x + 6$ $E_1(x) = 12x^2 + 11$

 $6x^2 + 11x + 4$

$$D_1E_1 = 4x^2 + 11x + 6$$
 $D_1E_0 = 12x^2 + 25x + 6$
 $D_0E_1 = 2x^2 + 5x + 2$ $D_0E_0 = 6x^2 + 11x + 6$

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 $D_1E_0 = 12x^2 + 25x + 12$
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$$D_1E_1 = 4x^2 + 11x + 6$$
 $D_1E_0 = 12x^2 + 25x +$
 $D_0E_1 = 2x^2 + 5x + 2$ $D_0E_0 = 6x^2 + 11x + 4$
 $AB = (4x^2 + 11x + 6)x^4 +$

$$D_0E_1 = 2x^2 + 5x + 2$$
 $D_0E_0 = 6x^2 + 11x + 4$
 $AB = (4x^2 + 11x + 6)x^4 + (12x^2 + 25x + 12 + 2x^2 + 5x + 2)x^2 + 6x^2 +$

 $=4x^{6}+11x^{5}+20x^{4}+30x^{3}+20x^{2}+11x+4$

Function Mult2 (A, B, n, a_l, b_l)

if n=1: $R[0]=A[a_l]*B[b_l]$; return R

if n = 1: $R[0] = A[a_I] * B[b_I]$; return R $R[0..n-2] = Mult2(A, B, \frac{n}{2}, a_I, b_I)$

if
$$n=1$$
: $R[0]=A[a_I]*B[b_I]$; return R

$$R[0..n-2] = Mult2(A, B, \frac{n}{2}, a_l, b_l)$$

 $R[n..2n-2] = Mult2(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l + \frac{n}{2})$

if
$$n = 1$$
:
$$R[0] = A[a_l] * B[b_l] ; return R$$

$$R[0..n-2] = Mult2(A, B, \frac{n}{2}, a_l, b_l)$$

 $R[n..2n-2] = Mult2(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l + \frac{n}{2})$

$$\frac{n}{2}$$
)
 $D_0E_1 = \text{Mult2}(A, B, \frac{n}{2}, a_l, b_l + \frac{n}{2})$

if
$$n = 1$$
:
$$R[0] = A[a_l] * B[b_l] ; return R$$

$$R[0..n-2] = Mult2(A, B, \frac{n}{2}, a_l, b_l)$$

 $R[n..2n-2] = Mult2(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l + \frac{n}{2})$

$\frac{n}{2}$)			,	, 2,	-
_	=	$Mult2(A, B, \frac{n}{2}, a_I,$	b_{l}	$+\frac{n}{2}$	
		$Mult2(A, B, \frac{n}{2}, a_I)$		_	

$$E_{0}^{\frac{n}{2}}$$
)
 $D_{0}E_{1} = \text{Mult2}(A, B, \frac{n}{2}, a_{l}, b_{l} + \frac{n}{2})$
 $D_{1}E_{0} = \text{Mult2}(A, B, \frac{n}{2}, a_{l} + \frac{n}{2}, b_{l})$
 $R[\frac{n}{2} \dots n + \frac{n}{2} - 2] + D_{1}E_{0} + D_{0}E_{1}$

if
$$n = 1$$
:
 $R[0] = A[a_I] * B[b_I]$; return R
 $R[0..n-2] = Mult2(A, B, \frac{n}{2}, a_I, b_I)$

$$R[n..2n-2] = \text{Mult2}(A, B, \frac{n}{2}, a_l, b_l)$$

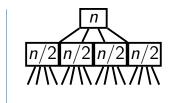
 $R[n..2n-2] = \text{Mult2}(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l + \frac{n}{2})$
 $D_0E_1 = \text{Mult2}(A, B, \frac{n}{2}, a_l, b_l + \frac{n}{2})$
 $D_1E_0 = \text{Mult2}(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l)$

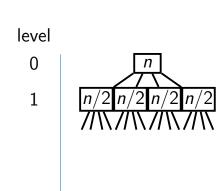
 $D_0 E_1 = \text{Mult2}(A, B, \frac{n}{2}, a_l, b_l + \frac{n}{2})$ $D_1 E_0 = \text{Mult2}(A, B, \frac{n}{2}, a_l + \frac{n}{2}, b_l)$ $R[\frac{n}{2}...n+\frac{n}{2}-2] += D_1E_0+D_0E_1$

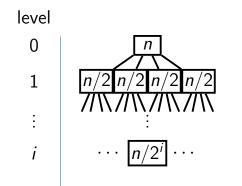
return R

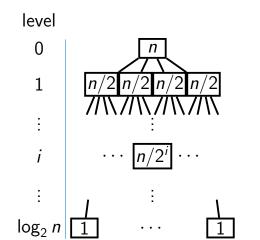


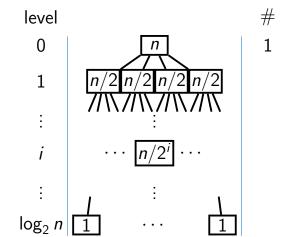
level

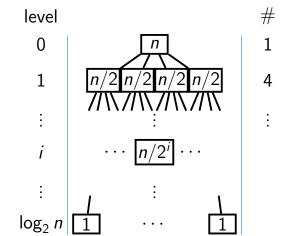


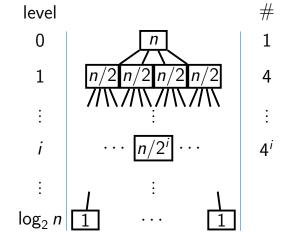


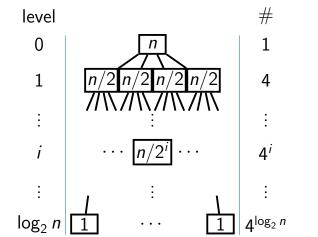


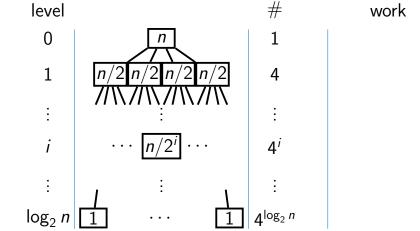


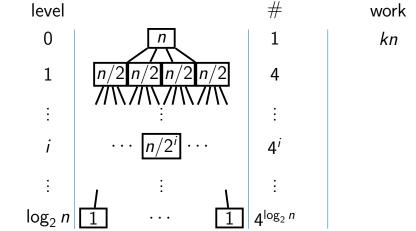


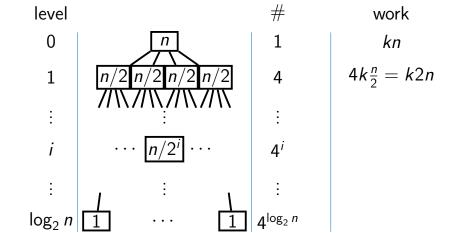


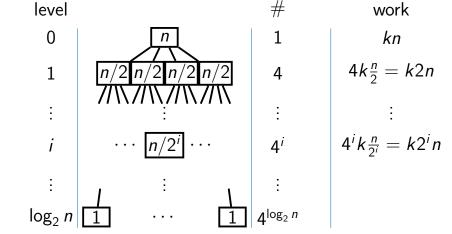












Outline

- Problem Overview
- 2 Naïve Algorithm
- 3 Naïve Divide and Conquer Algorithm
- 4 Faster Divide and Conquer

Karatsuba approach

$$A(x) = a_1x + a_0$$

$$A(x) = a_1x + a_0$$
$$B(x) = b_1x + b_0$$

$$A(x) = a_1x + a_0$$

$$B(x) = b_1 x + b_0$$

Needs 4 multiplications

$$A(x) = a_1x + a_0$$

$$=b_1 x +$$

$$B(x) = b_1 x + b_0$$

$$A(x) = a_1x + a_0$$

$$B(x) = b_1 x + b_0$$

$$B(x) = b_1 x + b_1 x^2$$

Rewrite as:

$$C(x) = a_1b_1x^2 + (a_1b_0 + a_0b_1)x + a_0b_0$$

Needs 4 multiplications

$$A(x) = a_1 x + a_0$$

$$B(x) = b_1 x + b_2$$

$$B(x) = b_1 x + b_0$$

 $C(x) = a_1 b_1 x^2 + b_0$

$$C(x) = a_1b_1x^2 + (a_1b_0 + a_0b_1)x + a_0b_0$$

Needs 4 multiplications

 $((a_1 + a_0)(b_1 + b_0) - a_1b_1 - a_0b_0)x +$

Rewrite as:

$$C(x) = a_1b_1x^2 +$$

 a_0b_0

$$A(x) = a_1 x + a_0$$

Rewrite as:

 $C(x) = a_1 b_1 x^2 +$

 a_0b_0

Needs 3 multiplications

$$B(x) = b_1 x + b_0$$

Needs 4 multiplications

 $((a_1 + a_0)(b_1 + b_0) - a_1b_1 - a_0b_0)x +$



$$A(x) = a_1 x + a_0$$

$$+ I$$

Needs 4 multiplications

Rewrite as:

 $C(x) = a_1 b_1 x^2 +$

 a_0b_0

Needs 3 multiplications

 $B(x) = b_1 x + b_0$

 $((a_1 + a_0)(b_1 + b_0) - a_1b_1 - a_0b_0)x +$

$$A(x) = a_1 x + a_0$$

$$+ I$$

Needs 4 multiplications

Rewrite as:

 $C(x) = a_1 b_1 x^2 +$

 a_0b_0

Needs 3 multiplications

 $B(x) = b_1 x + b_0$

 $((a_1 + a_0)(b_1 + b_0) - a_1b_1 - a_0b_0)x +$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$B(x) = \frac{4x^{3} + 3x^{2} + 2x + 3}{8(x)} = x^{3} + 2x^{2} + 3x + 4$$

$$D_{1}(x) = \frac{4x + 3}{8}$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

 $D_0(x) = 2x + 1$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = \frac{x}{2} + \frac{2}{3}$$

 $D_0(x) = 2x + 1$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

 $E_1(x) = x + 2$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

$$B(x) = x^3 + 2x^2 + 3x + 4$$
$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

 $E_1(x) = x + 2$

$$E_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$D_1 E_1 = 4x^2 + 11x + 6$$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$E_1(x) = 1x + 3$$

 $E_1(x) = x + 2$
 $D_1E_1 = 4x^2 + 11x + 6$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x)=4x+3$$

$$E_1(x) = x + 2$$

$$E_1(x) = x + 2$$

 $D_1E_1 = 4x^2 + 11x + 6$

$$D_1 E_1 = 4x^2 + 11x + 6$$

$$(D_1 + D_0)(E_1 + E_0) =$$

 $D_0(x) = 2x + 1$ $E_0(x) = 3x + 4$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

 $E_1(x) = x + 2$

$$E_1(x) = x + 2$$

 $D_1E_1 = 4x^2 + 11x + 6$

 $(D_1 + D_0)(E_1 + E_0) = (6x + 4)$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = \frac{x}{2} + \frac{2}{3}$$

$$E_1(x) = x + 2$$

 $D_1E_1 = 4x^2 + 11x + 6$

$$y = x + 2$$

 $y_1 = 4x^2 + 11x + 6$

$$11x + 6$$

 $(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$

$$E_0(x) = 3x + 4$$

 $D_0 F_0 = 6x^2 + 1$

 $D_0(x) = 2x + 1$

$$x^2 + 11x$$

$$D_0 E_0 = 6x^2 + 11x + 4$$

$$11x + \frac{1}{2}$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$E_1(x) = x + 2$$

 $D_1E_1 = 4x^2 + 11x + 6$

 $(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

 $= 24x^2 + 52x + 24$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$D_1 E_1 = 4x^2 + 11x + 6$$

 $AB = (4x^2 + 11x + 6)x^4 +$

$$E_1(x) = x + 2$$

 $D_1 E_1 = 4x^2 + 11x + 6$

$$G_1 = 4x^2 + 11x + 6$$

$$+11x+6$$

$$1x + 6$$

$$D_1E_1 = 4x^2 + 11x + 6$$
 $D_0E_0 = 6x^2 + 11x + 4$
 $(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$

$$= (6x + 4)(4x + 6)$$

$$= 24x^2 + 52x + 24$$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

 $AB = (4x^2 + 11x + 6)x^4 +$

$$D_1 E_1 = 4x^2 + 11x + 6$$

$$f_1 = 4x^2 + 11x + 6$$

- $f_2(E_1 + E_0) = (6x + 6)$

$$D_1E_1 = 4x^2 + 11x + 6$$
 $D_0E_0 = 6$
 $(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$

$$E_0) = (6x + 4x)^2$$

$$4)(4x+6)$$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

$$= (6x + 4)(4x + 6)$$
$$= 24x^2 + 52x + 24$$

 $D_0 E_0 = 6x^2 + 11x + 4$

 $)x^{2} +$

Karatsuba Example

$$A(x) = 4x^3 + 3x^2 + 2x + 1$$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

 $AB = (4x^2 + 11x + 6)x^4 +$

 $(24x^2 + 52x + 24)$

$$= x + 2$$

$$E_1(x) = x + 2$$

$$D_1 E_1 = 4x^2 + 11x + 6$$

$$11x + 6$$

$$(6x+4)$$

$$D_1E_1 = 4x^2 + 11x + 6$$
 $D_0E_0 = 6x^2 + 11x + 4$
 $(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$

$$= (6x + 4)(4x + 6)$$
$$= 24x^2 + 52x + 24$$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

 $)x^{2} +$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$f(x) = x + 2$$

 $f(x) = x + 2$

$$E_1(x) = x + 2$$

 $D_1 E_1 = 4x^2 + 11x + 6$

$$11x + 6$$

$$= (6x -$$

$$(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$$

= $24x^2 + 52x + 24$

$$= 24x^{2} + 52x + 24$$

$$AB = (4x^{2} + 11x + 6)x^{4} +$$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

 $D_0 E_0 = 6x^2 + 11x + 4$

$$4x^{2} +$$

 $(24x^2 + 52x + 24 - (4x^2 + 11x + 6))$

 $)x^{2} +$

$$B(x) = x^3 + 2x^2 + 3x + 4$$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$E_1(x) = x + 2$$

 $D_1 E_1 = 4x^2 + 11x + 6$

$$\overline{\xi}_1 = 4x^2 + 11x + 6$$

$$D_1E_1 = 4x^2 + 11x + 6$$
 $D_0E_0 = 6$
 $(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$

 $(24x^2 + 52x + 24 - (4x^2 + 11x + 6))$

$$+52x + 24$$

$$= 24x^{2} + 52x + 24$$

$$AB = (4x^{2} + 11x + 6)x^{4} + 4$$

 $-(6x^2+11x+4))x^2+$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

Karatsuba Example $A(x) = 4x^3 + 3x^2 + 2x + 1$ $B(x) = x^3 + 2x^2 + 3x + 4$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

 $AB = (4x^2 + 11x + 6)x^4 +$

 $6x^2 + 11x + 4$

$$E_1(x) = x + 2$$

 $E_1 = 4x^2 + 11x + 6$

$$D_1 E_1 = 4x^2 + 11x + 6$$

$$D_1E_1 = 4x^2 + 11x + 6$$
 $D_0E_0 = 6$
 $(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$

$$\begin{array}{l}
(1x+6) \\
E_0) = (6x+6)
\end{array}$$

$$6x + 6$$

 $(24x^2 + 52x + 24 - (4x^2 + 11x + 6))$

$$= (6x + 4)(4x + 6)$$
$$= 24x^2 + 52x + 24$$

 $-(6x^2+11x+4))x^2+$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

Karatsuba Example $A(x) = 4x^3 + 3x^2 + 2x + 1$ $B(x) = x^3 + 2x^2 + 3x + 4$

$$D_1(x) = 4x + 3$$

$$E_1(x) = x + 2$$

$$D_1 E_1 = 4x^2 + 11x + 6$$

$$D_1E_1 = 4x + 11x + 6$$
 $D_0E_0 = 6$
 $(D_1 + D_0)(E_1 + E_0) = (6x + 4)(4x + 6)$

$$(5x^2 + 6x^2 + 6)(4x + 6)$$

= $24x^2 + 52x + 24$

$$= 24x^2$$

$$= 24x^2$$

$$AB = (4x^2 + 11x + 6)x^4 + 6$$

$$(x^4 + x^4)$$

$$(4x^2 + 11x + 6)x + (24x^2 + 52x + 24 - (4x^2 + 11x + 6))$$

 $D_0(x) = 2x + 1$

 $E_0(x) = 3x + 4$

 $D_0 E_0 = 6x^2 + 11x + 4$

$$+0)$$

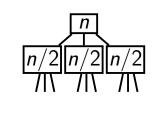
$$-(6x^2+11x+4))x^2+$$

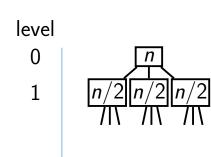
- $=4x^{6}+11x^{5}+20x^{4}+30x^{3}+20x^{2}+11x+4$

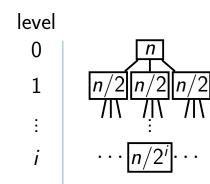
 $6x^2 + 11x + 4$

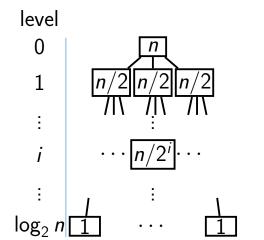


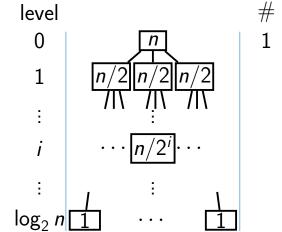
level

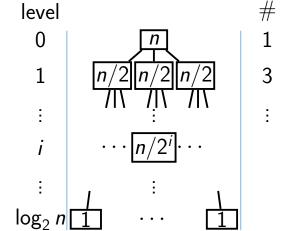


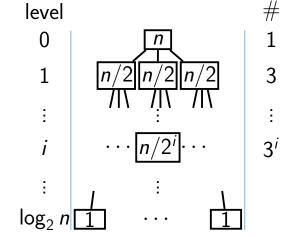


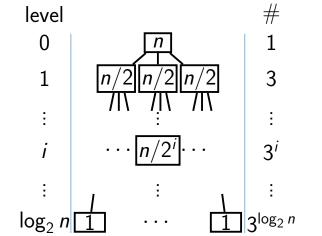


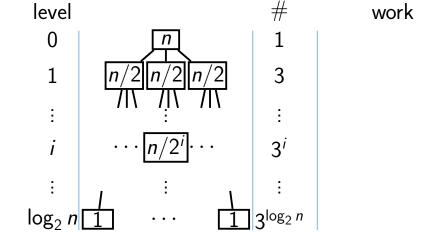


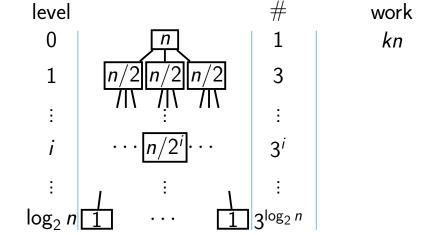


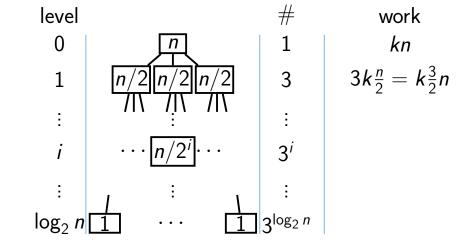


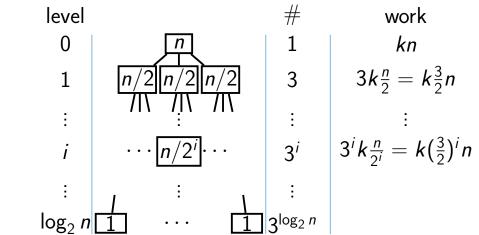












level

 $\log_2 n$

#

work

 $]3^{\log_2 n} k3^{\log_2 n} = kn^{\log_2 3}$

level # work

0 | n | 1 | kn

1 |
$$n/2$$
 | $n/2$ | $n/2$ | 3 | $3k\frac{n}{2} = k\frac{3}{2}n$
 \vdots | \vdots