

Dynamic Programming: Knapsack

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Algorithmic Design and Techniques
Algorithms and Data Structures

Outline

- 1 Problem Overview
- 2 Knapsack with Repetitions
- 3 Knapsack without Repetitions
- 4 Final Remarks

TV commercial placement

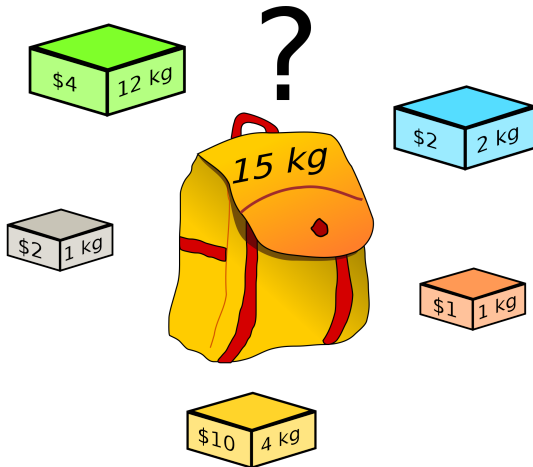
Select a set of TV commercials (each commercial has duration and cost) so that the total revenue is maximal while the total length does not exceed the length of the available time slot.

Optimizing data center performance

Purchase computers for a data center to achieve the maximal performance under limited budget.

Knapsack Problem

(knapsack is another word for backpack)



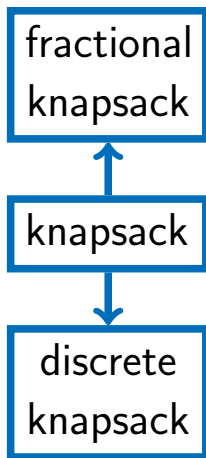
Goal

Maximize
value (\$)
while limiting
total
weight (kg)

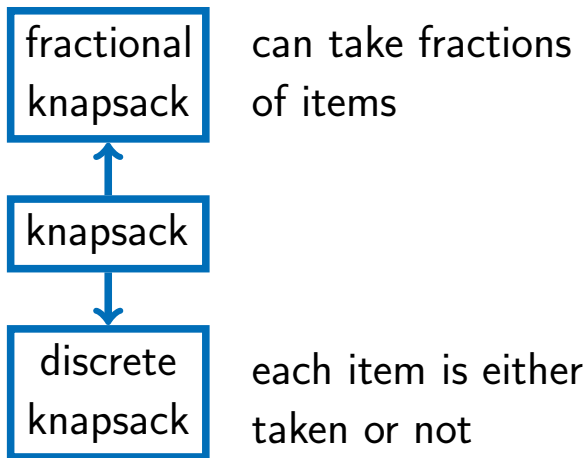
Problem Variations

knapsack

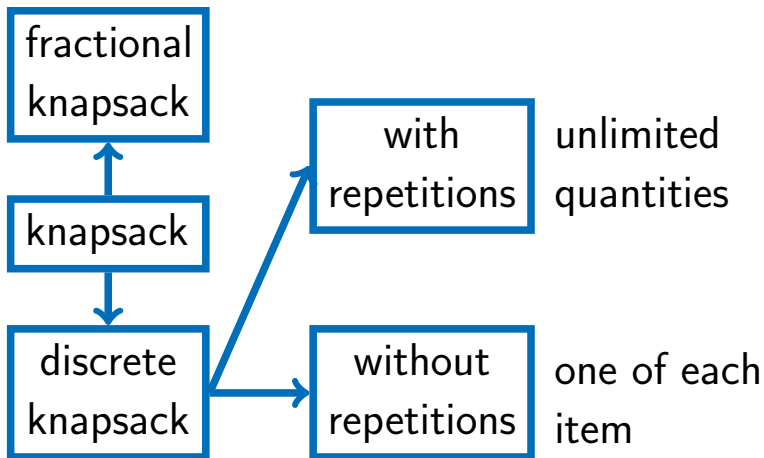
Problem Variations



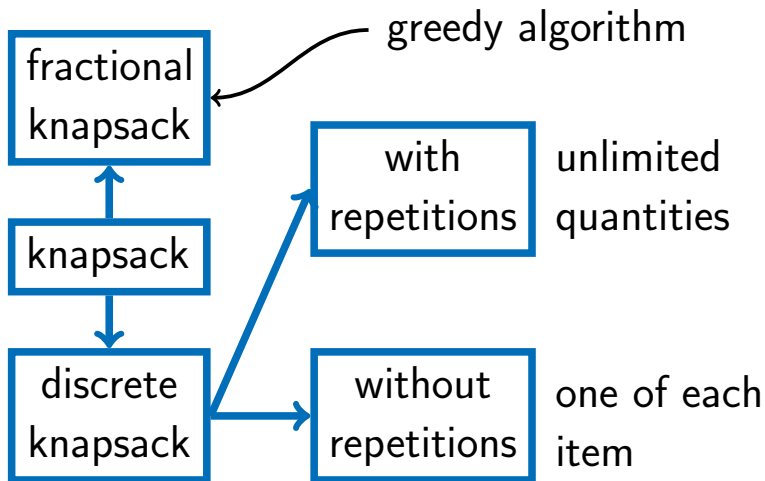
Problem Variations



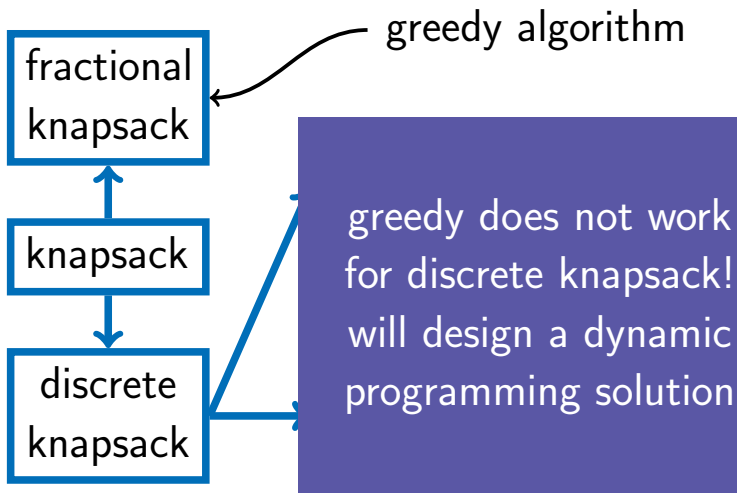
Problem Variations



Problem Variations



Problem Variations



Example

\$30

6

\$14

3

\$16

4

\$9

2

10

knapsack

Example

\$30



\$14



\$16



\$9



\$30

\$16

w/o repeats



total: \$46

Example

\$30



\$14



\$16



\$9



\$30

\$16

w/o repeats



total: \$46

\$30

\$9

\$9

w repeats



total: \$48

Example

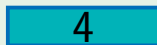
\$30



\$14



\$16



\$9



w/o repeats

\$30



\$16



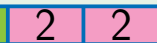
total: \$46

w repeats

\$30



\$9 \$9



total: \$48

fractional

\$30



\$14 \$4.5



total: \$48.5

Why does greedy fail for the discrete knapsack?

Example

\$30	\$14	\$16	\$9
6	3	4	2



Why does greedy fail for the discrete knapsack?

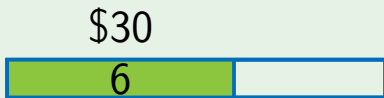
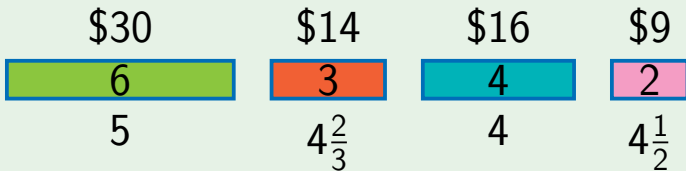
Example

\$30	\$14	\$16	\$9
6	3	4	2
5	$4\frac{2}{3}$	4	$4\frac{1}{2}$



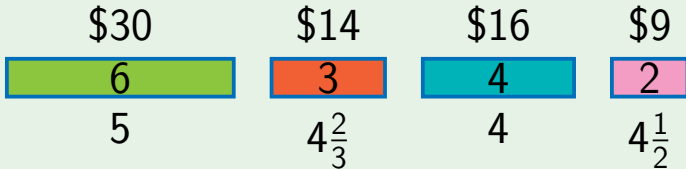
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Example



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Example

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taking an element of maximum value per unit of weight is not safe!

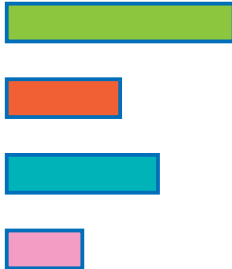
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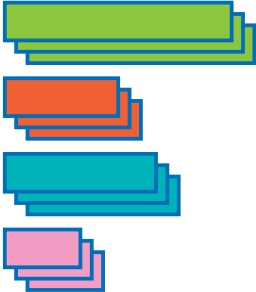
With repetitions:
unlimited quantities



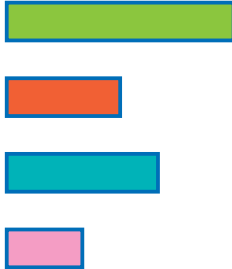
Without repetitions:
one of each
item



With repetitions:
unlimited quantities



Without repetitions:
one of each
item



Knapsack with repetitions problem

Input: Weights w_1, \dots, w_n and values v_1, \dots, v_n of n items; total weight W (v_i 's, w_i 's, and W are non-negative integers).

Output: The maximum value of items whose weight does not exceed W . Each item can be used any number of times.

Subproblems

- Consider an optimal solution and an item in it:



Subproblems

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- If we take this item out then we get an **optimal** solution for a knapsack of total weight $W - w_i$.

Subproblems

Let $value(w)$ be the maximum value of knapsack of weight w .

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$$value(w) = \max_{i: w_i \leq w} \{value(w - w_i) + v_i\}$$

Knapsack(W)

```
value(0)  $\leftarrow$  0
for  $w$  from 1 to  $W$ :
    value( $w$ )  $\leftarrow$  0
    for  $i$  from 1 to  $n$ :
        if  $w_i \leq w$ :
             $val \leftarrow value(w - w_i) + v_i$ 
            if  $val > value(w)$ :
                value( $w$ )  $\leftarrow$   $val$ 
return value( $W$ )
```

Example: $W = 10$

\$30	\$14	\$16	\$9
6	3	4	2

[illegible]

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0	1	2	3	4	5	6	7	8	9	10
0	0	9	0	0	0	0	0	0	0	0

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0	0	9	14	0	0	0	0	0	0	0

Example: $W = 10$

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6	3	4	2

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0	0	9	14	0	0	0	0	0	0	0

Example: $W = 10$

\$30	\$14	\$16	\$9
6	3	4	2

0	1	2	3	4	5	6	7	8	9	10
0	0	9	14	18	0	0	0	0	0	0

Example: $W = 10$

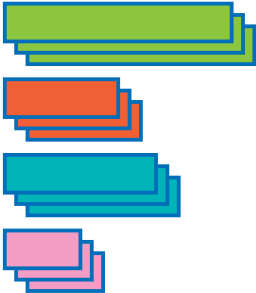
\$30	\$14	\$16	\$9
6	3	4	2

0	1	2	3	4	5	6	7	8	9	10
0	0	9	14	18	23	30	32	39	44	48

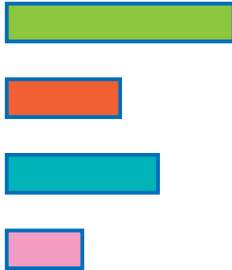
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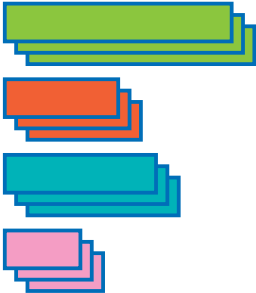
With repetitions:
unlimited quantities



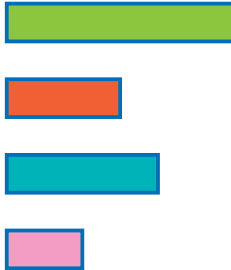
Without repetitions:
one of each
item



With repetitions:
unlimited quantities



Without repetitions:
one of each
item



Knapsack without repetitions problem

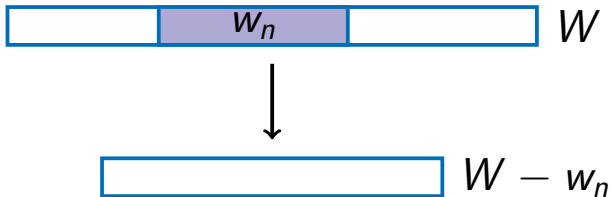
Input: Weights w_1, \dots, w_n and values v_1, \dots, v_n of n items; total weight W (v_i 's, w_i 's, and W are non-negative integers).

Output: The maximum value of items whose weight does not exceed W . Each item can be used at most once.

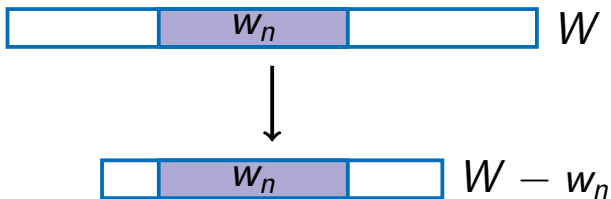
Same Subproblems?



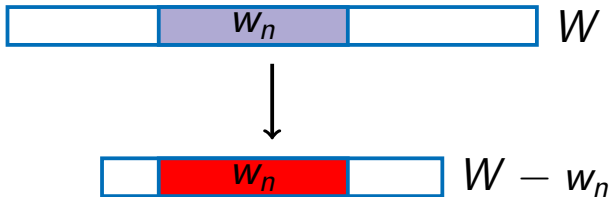
Same Subproblems?



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Same Subproblems?



Subproblems

- If the n -th item is taken into an optimal solution:



then what is left is an optimal solution for a knapsack of total weight $W - w_n$ using items $1, 2, \dots, n - 1$.

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then what is left is an optimal solution for a knapsack of total weight $W - w_n$ using items $1, 2, \dots, n - 1$.

- If the n -th item is not used, then the whole knapsack must be filled in optimally with items $1, 2, \dots, n - 1$.

Subproblems

For $0 \leq w \leq W$ and $0 \leq i \leq n$, $value(w, i)$ is the maximum value achievable using a knapsack of weight w and items $1, \dots, i$.

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For $0 \leq w \leq W$ and $0 \leq i \leq n$, $value(w, i)$ is the maximum value achievable using a knapsack of weight w and items $1, \dots, i$.

The i -th item is either used or not:
 $value(w, i)$ is equal to

$$\max\{value(w - w_i, i - 1) + v_i, value(w, i - 1)\}$$

Knapsack(W)

```
initialize all  $value(0, j) \leftarrow 0$   
initialize all  $value(w, 0) \leftarrow 0$   
for  $i$  from 1 to  $n$ :  
    for  $w$  from 1 to  $W$ :  
         $value(w, i) \leftarrow value(w, i - 1)$   
        if  $w_i \leq w$ :  
             $val \leftarrow value(w - w_i, i - 1) + v_i$   
            if  $value(w, i) < val$   
                 $value(w, i) \leftarrow val$   
return  $value(W, n)$ 
```

Example: reconstructing a solution

\$30	\$14	\$16	\$9
<input type="text" value="6"/>	<input type="text" value="3"/>	<input type="text" value="4"/>	<input type="text" value="2"/>

	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	30	30	30	30	30
2	0	0	0	14	14	14	30	30	30	44	44
3	0	0	0	14	16	16	30	30	30	44	46
4	0	0	9	14	16	23	30	30	39	44	46

Optimal solution:

1	2	3	4
<input type="text"/>	<input type="text"/>	<input type="text"/>	<input type="text"/>

Example: reconstructing a solution

\$30	\$14	\$16	\$9
6	3	4	2

	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	30	30	30	30	30
2	0	0	0	14	14	14	30	30	30	44	44
3	0	0	0	14	16	16	30	30	30	44	46
4	0	0	9	14	16	23	30	30	39	44	46

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1	2	3	4

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6	3	4	2

	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	30	30	30	30	30
2	0	0	0	14	14	14	30	30	30	44	44
3	0	0	0	14	16	16	30	30	30	44	46
4	0	0	9	14	16	23	30	30	39	44	46

Optimal solution:

1	2	3	4
			0

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6	3	4	2

	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	30	30	30	30	30
2	0	0	0	14	14	14	30	30	30	44	44
3	0	0	0	14	16	16	30	30	30	44	46
4	0	0	9	14	16	23	30	30	39	44	46

Optimal solution:

1	2	3	4
			0

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\$30	\$14	\$16	\$9
6	3	4	2

	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	30	30	30	30	30
2	0	0	0	14	14	14	30	30	30	44	44
3	0	0	0	14	16	16	30	30	30	44	46
4	0	0	9	14	16	23	30	30	39	44	46

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1	2	3	4
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0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	30	30	30	30	30
2	0	0	0	14	14	14	30	30	30	44	44
3	0	0	0	14	16	16	30	30	30	44	46
4	0	0	9	14	16	23	30	30	39	44	46

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1	0	0	0	0	0	0	30	30	30	30	30
2	0	0	0	14	14	14	30	30	30	44	44
3	0	0	0	14	16	16	30	30	30	44	46
4	0	0	9	14	16	23	30	30	39	44	46

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	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	30	30	30	30	30
2	0	0	0	14	14	14	30	30	30	44	44
3	0	0	0	14	16	16	30	30	30	44	46
4	0	0	9	14	16	23	30	30	39	44	46

Optimal solution:

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Example: reconstructing a solution

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6	3	4	2

	0	1	2	3	4	5	6	7	8	9	10
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	30	30	30	30	30
2	0	0	0	14	14	14	30	30	30	44	44
3	0	0	0	14	16	16	30	30	30	44	46
4	0	0	9	14	16	23	30	30	39	44	46

Optimal solution:

1	2	3	4
1	0	1	0

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Memoization

Knapsack(w)

```
if  $w$  is in hash table:
    return  $value(w)$ 
 $value(w) \leftarrow 0$ 
for  $i$  from 1 to  $n$ :
    if  $w_i \leq w$ :
         $val \leftarrow Knapsack(w - w_i) + v_i$ 
        if  $val > value(w)$ :
             $value(w) \leftarrow val$ 
insert  $value(w)$  into hash table with key
 $w$ 
return  $value(w)$ 
```

What Is Faster?

- If all subproblems must be solved then an iterative algorithm is usually faster since it has no recursion overhead.

What Is Faster?

- If all subproblems must be solved then an iterative algorithm is usually faster since it has no recursion overhead.
- There are cases however when one does not need to solve all subproblems: assume that W and all w_i 's are multiples of 100; then $value(w)$ is not needed if w is not divisible by 100.

Running Time

- The running time $O(nW)$ is not polynomial since the input size is proportional to $\log W$, but not W .

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- In other words, the running time is $O(n2^{\log W})$.

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- The running time $O(nW)$ is not polynomial since the input size is proportional to $\log W$, but not W .
- In other words, the running time is $O(n2^{\log W})$.
- E.g., for

$$W = 71\,345\,970\,345\,617\,824\,751$$

(twent digits only!) the algorithm needs roughly 10^{20} basic operations.

Running Time

- The running time $O(nW)$ is not

- later, we'll learn why solving this problem in polynomial time costs \$1M!

(twent digits only!) the algorithm needs roughly 10^{20} basic operations.