Divide-and-Conquer: Quick Sort

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Algorithmic Design and Techniques
Algorithms and Data Structures

Outline

1. Overview
2. Algorithm
3. Random Pivot
4. Running Time Analysis
5. Equal Elements
6. Final Remarks

Quick Sort

- comparison based algorithm
- running time: $O(n \log n)$ (on average)
- efficient in practice

Example: quick sort

6	4	8	2	9	3	9	4	7	6	1

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partition with respect to $x = A[1]$

in particular, x is in its final position

1	4	2	3	4	6	6	9	7	8	9

≤ 6 > 6

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sort the two parts recursively

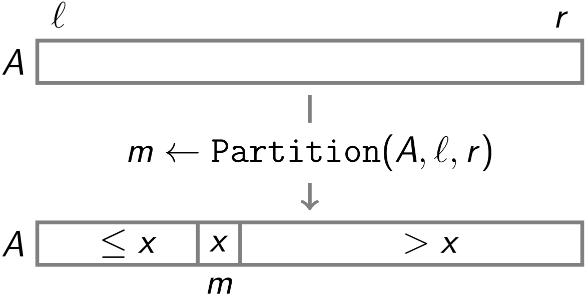
1	2	3	4	4	6	6	7	8	9	9

Outline

```
QuickSort(A, ℓ, r)

if ℓ ≥ r:
  return
m ← Partition(A, ℓ, r)
{A[m] is in the final position}
QuickSort(A, ℓ, m − 1)
QuickSort(A, m + 1, r)
```


ℓ
r
A



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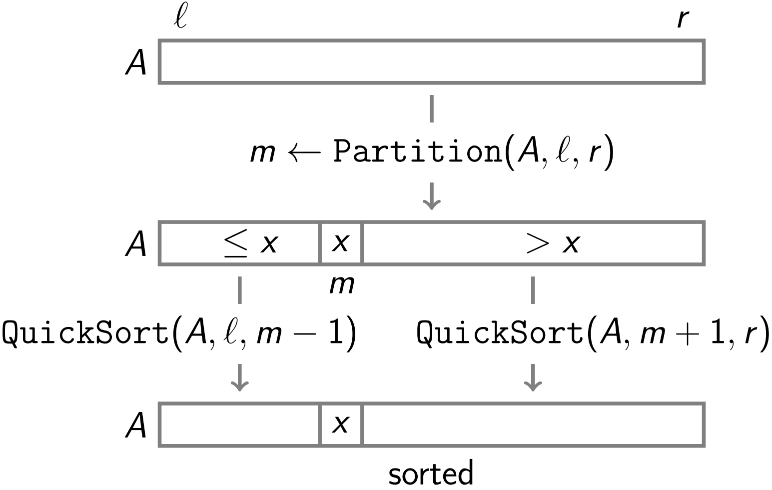
A
$\leq x$
x
$> x$
m



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$m \leftarrow \texttt{Partition}(A, \ell, r)$
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x
$> x$
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$\texttt{QuickSort}(A, \ell, m-1)$
$\texttt{QuickSort}(A, m+1, r)$
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x
sorted

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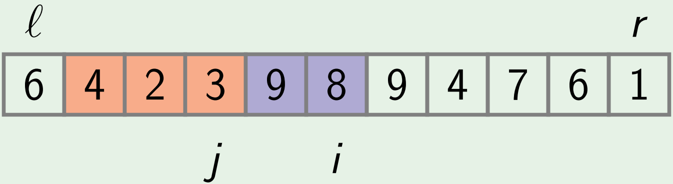

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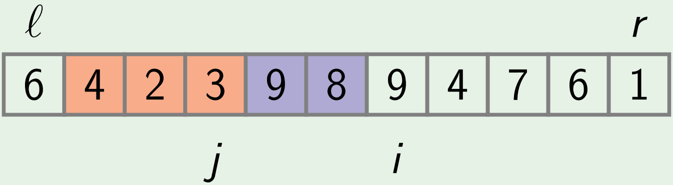

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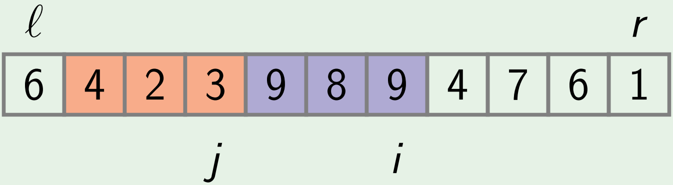

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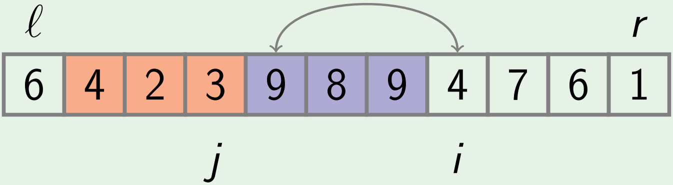

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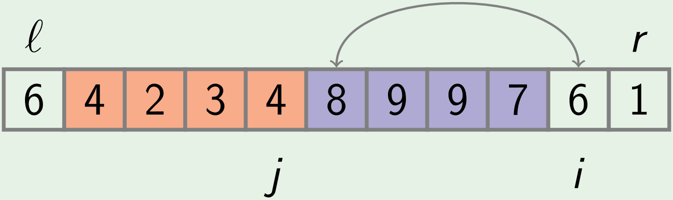

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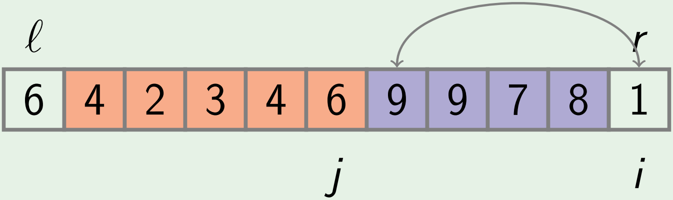

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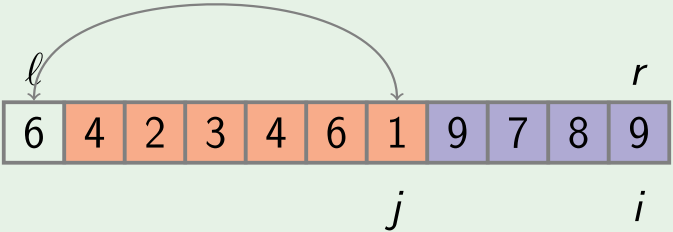

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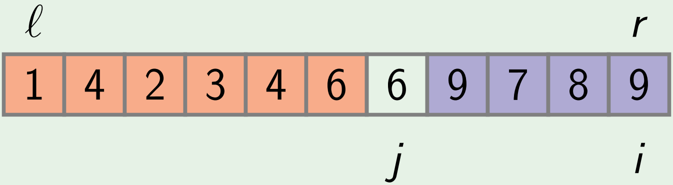

$\texttt{Partition}(A, \ell, r)$

```
x ← A[ℓ]        {pivot}
j ← ℓ
for i from ℓ+1 to r:
   if A[i] ≤ x:
      j ← j+1
      swap A[j] and A[i]
   {A[ℓ+1...j] ≤ x,  A[j+1...i] > x}
swap A[ℓ] and A[j]
return j
```

Outline

Unbalanced Partitions

- $T(n) = n + T(n-1)$:

  $T(n) = n+(n-1)+(n-2)+\cdots = \Theta(n^2)$

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- $T(n) = n + T(n-5) + T(4)$:

  $T(n) \geq n+(n-5)+(n-10)+\cdots = \Theta(n^2)$

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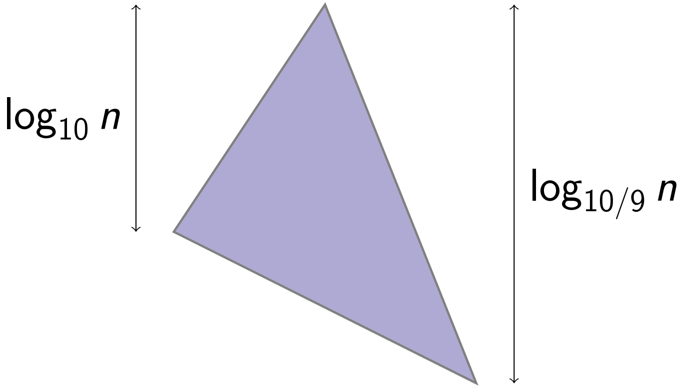
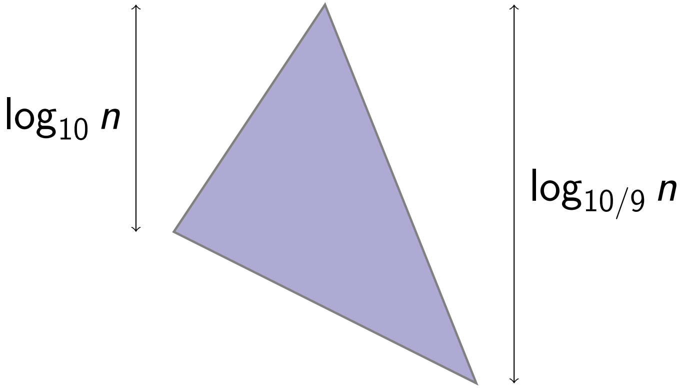


Balanced Partitions

$$T(n) = T(n/10) + T(9n/10) + O(n)$$



$$T(n) = O(n \log n)$$

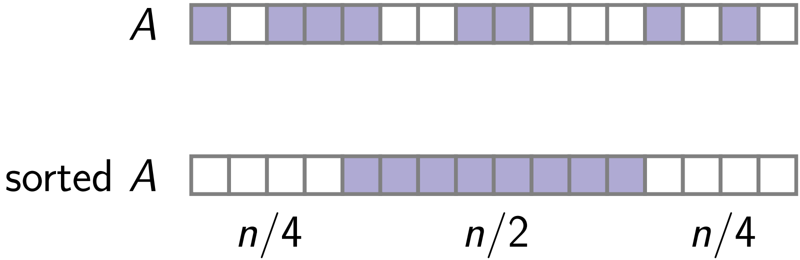
Random Pivot

`RandomizedQuickSort`(A, ℓ, r)

`if` $\ell \geq r$:
  `return`
$k \leftarrow$ `random number between` ℓ `and` r
`swap` $A[\ell]$ `and` $A[k]$
$m \leftarrow$ `Partition`(A, ℓ, r)
$\{A[m]$ `is in the final position`$\}$
`RandomizedQuickSort`$(A, \ell, m-1)$
`RandomizedQuickSort`$(A, m+1, r)$

Why Random?

half of the elements of A guarantees a balanced partition:



Theorem

Assume that all the elements of $A[1 \dots n]$ are pairwise different. Then the average running time of `RandomizedQuickSort`(A) is $O(n \log n)$ while the worst case running time is $O(n^2)$.

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Remark

Averaging is over random numbers used by the algorithm, but not over the inputs.

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Proof Ideas: Comparisons

- the running time is proportional to the number of comparisons made

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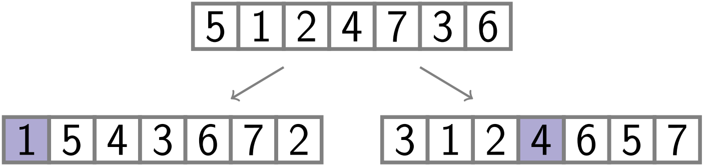
- balanced partition are better since they reduce the number of comparisons needed:



1 is min $\qquad 3, 1, 2 < 6, 5, 7$

Proof Ideas: Probability

A	5	1	8	9	2	4	7	3	6
A'	1	2	3	4	5	6	7	8	9

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$$\text{Prob}\,(3 \text{ and } 4 \text{ are compared}) =$$

Proof Ideas: Probability

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A'	1	2	3	4	5	6	7	8	9

$$\text{Prob}\left(1 \text{ and } 9 \text{ are compared}\right) = \frac{2}{9}$$

$$\text{Prob}\left(3 \text{ and } 4 \text{ are compared}\right) = 1$$

Proof

- let, for $i < j$,

$$\chi_{ij} = \begin{cases} 1 & A'[i] \text{ and } A'[j] \text{ are compared} \\ 0 & \text{otherwise} \end{cases}$$

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- for all $i < j$, $A'[i]$ and $A'[j]$ are either compared exactly once or not compared at all (as we compare with a pivot)
- this, in particular, implies that the worst case running time is $O(n^2)$

Proof (continued)

- crucial observation: $\chi_{ij} = 1$ iff the first selected pivot in $A'[i \ldots j]$ is $A'[i]$ or $A'[j]$

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- crucial observation: $\chi_{ij} = 1$ iff the first selected pivot in $A'[i \ldots j]$ is $A'[i]$ or $A'[j]$
- then $\mathrm{Prob}(\chi_{ij}) = \frac{2}{j-i+1}$ and $\mathrm{E}(\chi_{ij}) = \frac{2}{j-i+1}$

Proof (continued)

Then (the expected value of) the running time is

$$\begin{aligned}
\mathrm{E}\sum_{i=1}^{n}\sum_{j=i+1}^{n}\chi_{ij} &= \sum_{i=1}^{n}\sum_{j=i+1}^{n}\mathrm{E}(\chi_{ij}) \\
&= \sum_{i<j}\frac{2}{j-i+1} \\
&\le 2n\cdot\left(\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right) \\
&= \Theta(n\log n)
\end{aligned}$$

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- what if all the elements of the given array are equal to each other?

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- what if all the elements of the given array are equal to each other?
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- the array is always split into two parts of size 0 and $n-1$
- $T(n) = n + T(n-1) + T(0)$ and hence $T(n) = \Theta(n^2)$!

To handle equal elements, we replace the line

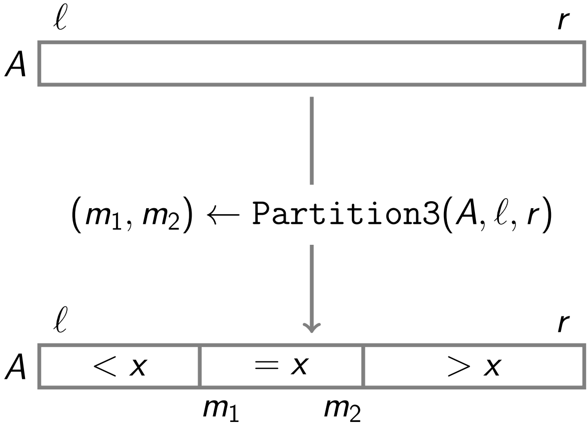
$$m \leftarrow \texttt{Partition}(A, \ell, r)$$

with the line

$$(m_1, m_2) \leftarrow \texttt{Partition3}(A, \ell, r)$$

such that

- for all $\ell \leq k \leq m_1 - 1$, $A[k] < x$
- for all $m_1 \leq k \leq m_2$, $A[k] = x$
- for all $m_2 + 1 \leq k \leq r$, $A[k] > x$



ℓ
r
A
(m1, m2) ← Partition3(A, ℓ, r)
ℓ
r
A
< x
= x
> x
m1
m2

RandomizedQuickSort(A, ℓ, r)

```
if ℓ ≥ r:
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(m1, m2) ← Partition3(A, ℓ, r)
{A[m1 ... m2] is in final position}
RandomizedQuickSort(A, ℓ, m1 − 1)
RandomizedQuickSort(A, m2 + 1, r)
```

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Tail Recursion Elimination

$\texttt{QuickSort}(A, \ell, r)$

$\texttt{while } \ell < r$:

  $m \leftarrow \texttt{Partition}(A, \ell, r)$

  $\texttt{QuickSort}(A, \ell, m - 1)$

  $\ell \leftarrow m + 1$

QuickSort(A, ℓ, r)

```
while ℓ < r:
    m ← Partition(A, ℓ, r)
    if (m − ℓ) < (r − m):
        QuickSort(A, ℓ, m − 1)
        ℓ ← m + 1
    else:
        QuickSort(A, m + 1, r)
        r ← m − 1
```

$\texttt{QuickSort}(A, \ell, r)$

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Worst-case space requirement: $O(\log n)$

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- if the recursion depth exceeds a certain threshold $c \log n$ the algorithm switches to heap sort
- the running time is $O(n \log n)$ in the worst case

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- Efficient in practice