Prepared for submission to JHEP

Lie Polynomials and a Penrose transform for scattering forms and amplitudes

Hadleigh Frost and Lionel Mason
The Mathematical Institute, University of Oxford,
AWB, ROQ, Oxford OX2 6GG, United Kingdom

Abstract: Lie polynomials underpin the structure of the so-called double copy relationship between gauge and gravity theories (and a network of other theories besides). Arkani-Hamed, Bai, He and Yan have recently shown that Lie polynomials arise naturally also in the geometry of $n-3$-forms on the space $\mathcal{K}_n$ of momentum invariants, Mandelstams, and they give a duality with certain associahedral $(n-3)$-planes. Lie polynomials also arise in the moduli space $\mathcal{M}_{0,n}$ of n points on a Riemann sphere up to Mobius transformations in the $n-3$-dimensional homology. We give a natural correspondendence between $\mathcal{K}_n$ and the bundle $T^*_D\mathcal{M}_{0,n}$ of 1-forms with logarithmic singularities on the divisor through which the relationships of some of these structures can be expressed. This gives a natural framework for CHY and ambitwistor-string formulae for scattering amplitudes of gauge and gravity theories. In particular we show that itgives a natural correspondence between CHY half-integrands and ABHY scattering forms. The double copy can then be expressed in this framework.

Contents

1 Introduction

The color-kinematics duality and the double copy [1, 2] have been a powerful influence on recent developments in scattering amplitudes. They stem from the KLT relations in string theory [3] between gravity and Yang-Mills tree-amplitudes and have been devloped as a tool for the study of multiloop gravity amplitudes and more recently for applications to perturbative classical gravity calculations in connection with gravitational waves [4]. This article develops the underpinning mathematical structures at tree level. We build on observations by Kapranov in an after dinner talk [5] concerning the relevance of Lie Polynomials, and recent work by Arkani-Hamed, Bai, He and Yan [6] that introduces differential forms in the space of kinematic invariants, $\mathcal{K}_n$.

At tree level, scattering amplitudes are rational functions of momentum invariants, or Mandelstam variables $\mathcal{K}_n$, and polynomial in invariants that incorporate the polarization data. The space of momentum invariants is $\mathcal{K}_n = \mathbb{R}^{n(n-3)/2}$ with coordinates s_{ij}, $i, j = 1, \dots, n$, $s_{ij} = s_{ji}$, $s_{ii} = 0$ and

{conservation}

$$\sum_{j=1}^{n} s_{ij} = 0. \tag{1.1}$$

These arise from the definition $s_{ij} = k_i \cdot k_j$ where k_i are n massless momenta in d-dimensions subject to momentum conservation $\sum_i k_i = 0$.

The key geometric structure in $\mathcal{K}_n$ as regards amplitudes are the factorization hyperplanes that depend on a subset $I \subset \{1, 2, \dots, n\}$ given by $s_I = 0$ where

$$s_I := \sum_{i,j \in I} s_{ij} = \left(\sum_i k_i \right)^2 . \tag{1.2}$$

We define $|I|$ to be the size of I and $\bar{I}$ to be its complement so that $s_{\bar{I}} = -s_I$ by (2.7). Locality states that the only singularities of tree amplitudes are simple poles on these hyperplanes.

2 The double copy and Lie polynomials

Colour structures for n-point amplitudes are degree n invariant polynomials of weight one in each of the n Lie algebra 'colours' of the external particles. These naturally arise in Feynman rules as trivalent Feynman diagrams whose vertices are the structure constants of some unspecified Lie algebra. If we fix the nth particle, and an invariant inner product on the Lie algebra, at tree-level, such a polynomial can be realized as the inner product of the nth colour with the Lie algebra element with a *Lie polynomial* formed by successive commutators of the $n-1$ other colours working through the diagram back from the nth particle. This section reviews material concerning such colour structures in the language of free Lie algebras and Lie polynomials together with their duality with words formed from permutations of the $n-1$ labels of the first $n-1$ external particles. A classic text on free Lie algebras is [7]. We then go on to formulate the BCJ and KLT relations in this framework.

2.1 An introduction to words, Lie polynomials and trees

The space of words $W(n-1)$ is the $(n-1)!$-dimensional linear span of words $a = a_1 a_2 \dots a_{n-1}$ where the letters $a_i \in \{1, \dots, n-1\}$ are all distinct, so that the a'a define permutations on $n-1$ letters. There is a natural inner product on $W(n-1)$ given on monomials a, b by $(a, b) = \delta_{ab}$, i.e., when every letter is the same.

$Lie(n-1)$ is the vector subspace of $W(n-1)$ generated by all Lie polynomials Γ of weight $n-1$ in the $n-1$ variables, $x_1, ..., x_{n-1}$ with weight one in each. A lie polynomial is formed from iterated commutators of the x_i, such as

$$\Gamma = [x_1, [..., [x_{n-1}, x_n]...]] + [x_n, [..., [x_{n-2}, x_{n-1}]...]] \in Lie(n-1),$$

where $[x_i, x_j] = x_i x_j - x_j x_i$ and so on. The commutator is as usual bilinear in its arguments, skew symmetric and satisfies the Jacobi relations

$$[x_i, x_j] = -[x_j, x_i]\,, \qquad [x_i, [x_j, x_k]] + \text{cyclic} = 0\,. \tag{2.1}$$

We will see that the dimension of $Lie(n-1)$ is $(n-2)!$. A possible basis[1], the *comb* basis, is given by the Lie monomials of the form

{comb}

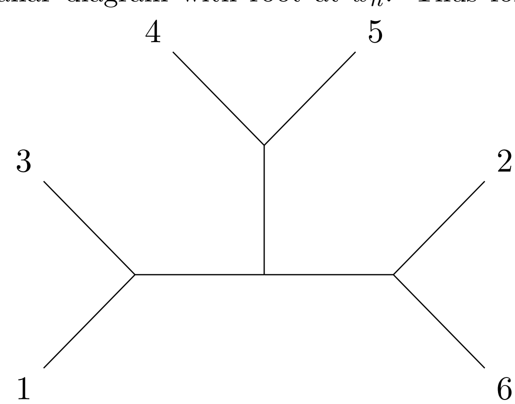
$$\Gamma_{1a} := [[\ldots[x_1, x_{a(2)}], ..., x_{a(n-2)}], x_{a(n-1)}], \tag{2.2}$$

for all $(n-2)!$ orderings a of $2, \ldots, n-1$. Every Lie monomial in $Lie(n-1)$ can be adjusted up to sign so that x_1 appears as the leftmost entry using the skew symmetry of the Lie bracket.

Assuming that the Lie algebra has an invariant inner product, we can take the inner product of Γ with an nth letter, x_n to give a colour invariant in n letters. We will denote such an invariant degree-n polynomial by c_Γ.

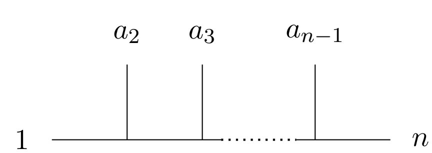
Such an invariant can be understood as arising from an oriented trivalent connected tree Γ with root x_n. By oriented, we mean each vertex is oriented, but two oriented diagrams are equivalent if they differ by an even number of orientation reversals at the vertices. The tree gives a Feynman diagram with structure constants for the Lie algebra at the trivalent vertices oriented by the orientation, which corresponds in turn to the ordering of the Lie brackets appearing in the monomial. The orientation can also be encoded by expressing the diagram as a planar diagram with root at x_n. Thus for example

$$\Gamma = [[[x_1, x_3], [x_4, x_5]], x_2] =$$



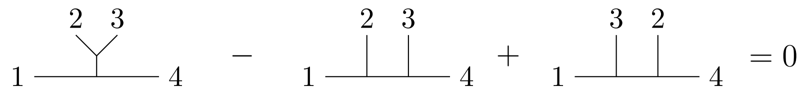
In the case of monomial (2.2), the trivalent diagram Γ_a in question is indeed a *comb* (or half-ladder) as its name is intended to express with x_1 and x_n at each end, and the remaining x_i being the intermediate teeth in the ordering a as in the following graph for the trivial permutation $a = 123\ldots n-1$

$$\Gamma_{1a} :=$$



[1]Such a basis is also known as a Hall basis [?] and is also that introduced by [8]

The Jacobi identity itself gives the vanishing of the sum of the three four-point graphs corresponding to an s, t and u-channel exchange graph.



We will typically denote three Lie polynomials or corresponding graphs that differ only on such a four point subgraph by Γ_s, Γ_t and Γ_u and we will consequently have

$$\Gamma_s + \Gamma_t + \Gamma_u = 0\,. \tag{2.3}$$

It is these relations reduce the dimension of $Lie(n-1)$ to $(n-2)!$ with basis given by the combs $c_{\Gamma_{1a}}$ where $1a$ is an $n-1$- word with first letter 1.

A tree Γ has *foliage* given by the a if it can be realized as a planar tree with external ordering given by the word an. This will be the case if the word a appears in this expansion of Γ in $W(n-1)$. We can define

Definition 2.1 *The duality between an oriented graph Γ and the word a is given by*

$$(\Gamma, a) = \begin{cases} \pm 1\,, & \text{if } \Gamma \text{ is planar with foliage ordered by } (an) \text{ with } \pm \text{ induced orientation,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

In terms of words $(\Gamma, a) = \pm 1$ if a appears with that coefficient in the expansion of its Lie monomial and 0 otherwise.

As an application of the notation, note that we can write

{gamma-a}

$$\Gamma = \sum_a (\Gamma, a) a\,. \tag{2.5}$$

This follows simply by expanding out the Lie polynomial based on Γ into commutators.

The space of Lie polynomials is $(n-2)!$-dimensional and there are many characterizations of $Lie(n-1)$ as a subspace of $W(n-1)$ [7]. These require the shuffle product

Definition 2.2 *The* shuffle product *of a pair of words b, c of length $|b|$ and $|c|$ is the linear combination of words $b \sqcup\!\sqcup c$ that is the sum over ordered permutations of the letters of b and c that preserve the original ordering of the letters in b and c.*

The following are characterisations of those words $w \in \subset W(n-1)$ that are in $Lie(n-1)$:

Lie-char

Proposition 2.1 *A word $w \in W[n-1]$ is the expansion of a Lie polynomial if either of the following is satisfied:*

1. $(w, a \sqcup\!\sqcup b) = 0$ *for all nontrivial shuffles $a \sqcup\!\sqcup b$.*

2. *The Kleiss-Kuijf relations [9]*

$$(w, a1b) = (-1)^{|a|}(w, 1\bar{a} \sqcup\!\sqcup\ b)\,. \tag{2.6}$$

Here $\bar{a}$ is the word a in reverse order and $|a|$ the length of a.

See [7] chapter 1 for the first (Ree's theorem from 1958) where there are a number of further characterizations. When $|a| = 1$ in the 2nd, these reduce to the $U(1)$-decoupling relation.

The last characterization in particular shows that words $1a$ span $Lie(n-1)^*$ where a is a permutation of $23\dots n-1$. That the dimension of $Lie(n-1)$ is $(n-2)!$ with dual basis given by combs Γ_{1a}, i.e. with 1 and n at each end follows from

Lemma 2.1 *For each Γ we can find an $n-2$-word a_Γ such that $(\Gamma, 1a_\Gamma) = \pm 1$. This word is unique for the comb basis (2.2) so that combs and words $1a$ form dual bases of $Lie(n-1)$ and $Lie(n-1)^*$ respectively.*

Proof: We can move the letter 1 to the left of each Lie bracket in which it appears using skew-symmetry of the bracket so that it ends up in the first position. After this process the letter 1 is on the left. For the comb basis there is no further freedom as the letter 1 is nested inside all the Lie brackets and so any use of skew symmetry will move 1 away from the first place and there are no further words in the expansion that have the letter 1 on the left. □

2.2 The geometry of $\mathcal{K}_n$

The space of Mandelstam variables $\mathcal{K}_n = \mathbb{R}^{n(n-3)/2}$ with coordinates s_{ij}, $i, j = 1, \dots, n$, $s_{ij} = s_{ji}$, $s_{ii} = 0$ and

{conservation}

$$\sum_{j=1}^{n} s_{ij} = 0. \tag{2.7}$$

The key geometric structure in $\mathcal{K}_n$ as regards amplitudes are the factorization hyperplanes that depend on a subset $I \subset \{1, 2, \dots, n\}$ given by $s_I = 0$ where

$$s_I := \sum_{i,j \in I} s_{ij} = \left(\sum_i k_i\right)^2 . \tag{2.8}$$

We define $|I|$ to be the size of I and $\bar{I}$ to be its complement so that $s_{\bar{I}} = -s_I$ by (2.7). Locality states that the only singularities of tree amplitudes are simple poles on these hyperplanes.

A key further requirement is that, poles along both $s_I = 0$ and $s_J = 0$ are only allowed if $I \subset J$ or $\bar{J}$. It follows that the allowed pole structure of a contribution to an n-point amplitude has poles along at most $n-3$ factorization hyperplanes $s_{I_p} = 0$ for $p = 1, \dots, n-3$. Such choices are in $1:1$ correspondence with trivalent tree graphs $\Gamma \in \mathcal{T}$ with n-leaves that we will take also, as before in the discussion of Lie polynomials, to be oriented about each vertex.

2.3 The double copy from biadjoint scalars to gauge and gravity theories

The double copy principle is that massless n-point tree amplitudes for a large web of important theories, including many gauge and gravity theories, can be expressed as a double copy in the form

$$\mathcal{M} = \sum_{\Gamma} \frac{N_\Gamma \tilde{N}_\Gamma}{d_\Gamma} \,. \tag{2.9}$$

Here the denominators

$$d_\Gamma = \prod_{r=1}^{n-3} s_{I_r} \tag{2.10}$$

are the propagator factors associated to the graph Γ thought of as a Feynman graph. Further, each trivalent diagram Γ has a pair of *numerator* factors N_Γ and $\tilde{N}_\Gamma$ that are functions of momenta, polarization data, flavour and colour which can therefore include lie polynomials c_Γ as invariants. Such factors are said to be *local* if they are polynomial, i.e., admit no spurious singularities.

The key additional feature required to be a BCJ numerator is that N_Γ and $\tilde{N}_\Gamma$ should represent homomorphisms from Lie polynomials to some vector space V of functions of the momenta and polarization and colour data

$$N : Lie(n-1) \to V \tag{2.11}$$

Thus for any three graphs Γ_s, Γ_t and Γ_u satisfying $[\Gamma_s] + [\Gamma_t] + [\Gamma_u] = 0$ as Lie polynomials, we must also have

{num-Lie}

$$N_{\Gamma_s} + N_{\Gamma_t} + N_{\Gamma_u} = 0 \,. \tag{2.12}$$

They are not uniquely determined: given a triple Γ_s, Γ_t and Γ_u we can perform the shift $\delta(N_{\Gamma_s}, N_{\Gamma_t}, N_{\Gamma_u}) = (s,t,u)A$ for any $A \ni V$. Such numerators can be determined from their values on a comb basis $N_{\Gamma_{1a}}$ by

{N-basis}

$$N_\Gamma = \sum_a (\Gamma, 1a) N_{\Gamma_{1a}} \,. \tag{2.13}$$

In the case of Yang mills, since the c_Γ are already Lie polynomials, the claim of BCJ [1] is that

$$\mathcal{A} = \sum_\Gamma \frac{N_\Gamma^{(\epsilon,k)} c_\Gamma}{d_\Gamma} \tag{2.14}$$

for some *kinematic numerators* $N_\Gamma^{(\epsilon,k)}$ depending on the polarization vectors ϵ_i and the momenta satisfying (2.12). The key nontrivial output of the double copy is that gravity amplitudes are obtained when $\tilde{N}_\Gamma = N_\Gamma^{(\epsilon,k)}$. The same numerators determine both the colour-ordered Yang-Mills amplitude with order a is then

{YM-N}

$$\mathcal{A}_a = \sum_\Gamma \frac{N_\Gamma(\Gamma, a)}{d_\Gamma} \tag{2.15}$$

and gravity amplitudes by

$$\mathcal{M} = \sum_{\Gamma} \frac{N_\Gamma N_\Gamma}{d_\Gamma}\,. \tag{2.16}$$

The most basic theory in this framework is the bi-adjoint scalar theory whose colour ordered amplitudes are given by

$$m(a,b) := \sum_{\Gamma} \frac{(\Gamma,a)(\Gamma,b)}{d_\Gamma}\,. \tag{2.17}$$

and we can introduce two underlying abstract amplitudes for these theories given by

$$m = \sum_{\Gamma} \frac{\Gamma\otimes\Gamma}{d_\Gamma} \in Lie(n-1)\otimes Lie(n-1)\,, \qquad m_a = \sum_{\Gamma} \frac{(\Gamma,a)\Gamma}{d_\Gamma} \in Lie(n-1). \tag{2.18}$$

Substituting (2.13) into (2.15) we obtain Yang-Mills amplitudes in terms of numerators and $m(a,b)$ by

{A-m-N}

$$\mathcal{A}_a = N(m_a) = \sum_{b} m(a,1b) N^{(\epsilon,k)}_{\Gamma_{1b}}\,, \tag{2.19}$$

with a similar form for gravity

$$\mathcal{M} = (N\otimes N)(m) = \sum_{a,b} m(1a,1b) N^{(\epsilon,k)}_{1a} \tilde{N}^{(\epsilon,k)}_{1b}\,. \tag{2.20}$$

We cannot simply invert (2.19) to obtain the N_{1b} as $m(1a,1b)$ is not invertible. Indeed, all theories that can be expressed in this double-copy format with one explicit Lie polynomial factor satisfy the fundamental BCJ relations [1]. There are many forms of the relations, one version being, for a word[2] $a \in W(n-2)$

$$\sum_{bc=a} s_{i,b_{|b|}} m((i \sqcup\!\sqcup b)c) = 0\,. \tag{2.21}$$

$$\sum_{bc=a} s_{b\,n-1} m(b(n-1)c) = 0\,, \qquad \text{where} \qquad s_{b,k} := \sum_{i=1}^{|b|} s_{b_i\,k}\,. \tag{2.22}$$

Thus (2.19) determines the N_{1a} only up to the addition of multiples of the BCJ relations. This freedom can be used to set all but $(n-3)!$ of the N_{1a} to zero, but this is at the expense of requiring numerators that are rational rather than polynomial in the momenta which will then have spurious poles.

[2] This is eq 3.8 of [?] together with the use of the $U(1)$-decoupling identity in the bracketed term.

2.4 Berends-Giele recursion and the momentum kernel

For Berends-Giele recursion [10] we extend the nth leg of the amplitude off-shell to allow diagrams to be glued together at the nth leg with a nontrivial propagator to give a Berends-Giele current. Let $\tilde{\mathcal{K}}_n$ be the $(n-1)(n-2)/2$-dimensional space of invariants of such momenta

$$\tilde{\mathcal{K}}_n = \{s_{ij} = s_{ji}, i < j, s_{nn} | \sum_i s_{ij} = 0\} . \tag{2.23}$$

We can then project $\tilde{\mathcal{K}}_n \to \tilde{\mathcal{K}}_m$ for $m < n$ by mapping k_i to itself for $i < m$ but taking $\sum_{i=m}^n k_i$ to k_m. Thus for $I \subset \{1, \ldots, m-1\}$, s_I pulls back to the corresponding s_I on $\tilde{\mathcal{K}}_n$, but if $m \in I$, then s_I pulls back to $s_{I'}$ where $I' = \{I, m+1, \ldots, n\}$. In particular $s_{m,m}$ pulls back to s_{I_m} where $I_m = \{m+1, \ldots, n\}$. We can write $s_n := s_{123\ldots n-1} \neq 0$. We must also consider different values of n, and embed the different $\tilde{\mathcal{K}}_n$ inside $\mathcal{K}_{n'}$ for $n' > n$ and so-on.

Definition 2.3 *The current with the $(|a|+1)$th leg off-shell, i.e., with $s_a \neq 0$ is:*

$$M_a := \frac{1}{s_a} \sum_\Gamma \frac{(\Gamma, a)\Gamma}{d_\Gamma} = \frac{1}{s_a} m_a .$$

Given two trivalent rooted diagrams Γ and Γ', then the diagram $\tilde{\Gamma}$ obtained by joining the roots of Γ sand Γ' together at a new trivalent vertex to give a new root is given in Lie polynomials simply by the commutator $\tilde{\Gamma} = [\Gamma, \Gamma']$. These summed together over all diagrams are what constitutes Berends-Giele recursion. This can be summarised in

$$M_a = \frac{1}{s_a} \sum_{bc=a} [M_b, M_c] . \tag{2.24}$$

This recursion will give the possibility of inductive proofs. If we deine also:

Definition 2.4 *The* momentum kernel $S(1a, 1b)$ *by*

$$S(1a, 1b) = \prod_{k=1}^{|a|} \left(s_{1a_k} + \sum_{l>k}^{n-3} \theta_{a_k, a_l} s_{a_k\, a_l}) \right) , \quad \theta_{a_k, a_l} = \begin{cases} 1, & \frac{a_k - a_l}{b_k - b_l} < 0, \\ 0, & o/wise . \end{cases}$$

Then we will be able to prove, following [11],

Proposition 2.2 *$S(1a, 1b)$ is an inverse for M_{1a}*

$$\Gamma_{1a} = \sum_{b \in W(n-2)} S(1a, 1b) M_{1b} .$$

However, we leave the proof until the next section so as to use the ABHY forms.

3 Scattering forms

ABHY [6] construct a homomorphism

$$Lie(n-1) \simeq \Omega_s^{n-3}\mathcal{K}_n \subset \Omega^{n-3}\mathcal{K}_n \tag{3.1}$$

{w-hom}

Here Ω_s^{n-3} is an $(n-2)!$-dimensional subbundle of the bundle of $(n-3)$-forms on $\mathcal{K}_N$. These can therefore be used as numerators and so provide scattering forms. We go on to use this further to refine the previous Lie polynomial version of Berends-Giele recursion in terms of forms.

3.1 ABHY Geometry of $\mathcal{K}_n$

ABHY introduce structures on cK_n as follows.

Definition 3.1 *On $\mathcal{K}_n$ we define the following $(n-3)$-forms and $(n-3)$-planes:*

- *Given an oriented trivalent graph Γ with propagators s_{I_p} define the $n-3$-form*

$$w_\Gamma = \mathrm{sgn}(\Gamma, I_p) \bigwedge_{p=1}^{n-3} ds_{I_p} \tag{3.2}$$

  Here the sign depends on the ordering of the propagators and hence I_p. ABHY [6] define it so that w_Γ provide the required homomorphism (3.1), i.e. (3.7) below. We give an alternative recursive definition in the next subsection.

- *For numerators N_Γ representing $Lie(n-1)$, i.e., satisfying stu-relations, define the scattering forms*

$$\Omega_N := \sum_\Gamma \frac{N_\Gamma w_\Gamma}{d_\Gamma}\,. \tag{3.3}$$

  As a particular example we define

$$\Omega_a = \sum_\Gamma \frac{(\Gamma, a) w_\Gamma}{d_\Gamma}\,. \tag{3.4}$$

- *Kinematic space is foliated by associahedral $n-3$ planes*

$$P_a = \{s_{a_i a_j} = const. | i < j < n, |i-j| \geq 2\}\,. \tag{3.5}$$

  An alternative definition is provided by the polyvector

$$P_a = \bigwedge_{i=2}^{n-2} \left(\frac{\partial}{\partial s_{a_{i-1}\, a_i}} - \frac{\partial}{\partial s_{a_i\, a_{i+1}}} \right)\,. \tag{3.6}$$

  (we ignore here the positivity aspects of the construction).

The main results from ABHY that we use here are

Proposition 3.1 *We have*

- *when* $[\Gamma_s+\Gamma_t+\Gamma_u]=0$ *as Lie polynomials we have*

{w-stu}

$$w_{\Gamma_s}+w_{\Gamma_t}+w_{\Gamma_u}=0 \tag{3.7}$$

- *The stu-relations on* N_Γ *imply projectivity, i.e.,*

$$\Upsilon \lrcorner \Omega_N=0, \qquad \Upsilon=\sum_{i<j} s_{ij}\frac{\partial}{\partial s_{ij}} \tag{3.8}$$

i.e., Υ *is the Euler homogeneity operator on* $\mathcal{K}_n$*.*

3.2 Berends-Giele recursion via forms

We can define w_Γ on $\mathcal{K}_n$ for any size of $\Gamma<n$ on these spaces by pullback from $\tilde{\mathcal{K}}_{|\Gamma|}$. In order to define a recursion we define the $n-2$-form

$$\tilde{w}_\Gamma=ds_\Gamma\wedge w_\Gamma \tag{3.9}$$

where s_Γ is the invariant for the foliage of Γ. We can then write the recursive definition

$$\tilde{w}_{[\Gamma,\Gamma']}=(-1)^{|\Gamma|}ds_{[\Gamma,\Gamma']}\wedge\tilde{w}_\Gamma\wedge\tilde{w}_{\Gamma'}\,. \tag{3.10}$$

The sign has been included to ensure that $w_{[\Gamma,\Gamma']}=-w_{[\Gamma',\Gamma]}$. **But I dont think that this expression works....**

On these spaces we can define scattering forms with one off-shell leg by

$$\tilde{\Omega}_a=\frac{ds_a}{s_a}\wedge\Omega_a=\sum_\Gamma(\Gamma,a)\frac{ds_a}{s_a}\wedge\frac{w_\Gamma}{d_\Gamma} \tag{3.11}$$

Such scattering forms on $\mathcal{K}_m$ can be pulled back to $\tilde{\mathcal{K}}_n$ for all $n>m$.

We then have the recursion relation

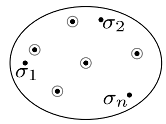
$$\tilde{\Omega}_a=\frac{ds_a}{s_a}\wedge\sum_{bc=a}\tilde{\Omega}_b\wedge\tilde{\Omega}_c\,. \tag{3.12}$$

The equation defining the momentum kernel then becomes

$$ds_{1a}\wedge w_{\Gamma_{1a}}=\sum_b S(1a,1b)\tilde{\Omega}_{1b}\,.$$

4 Trees and words in $\mathcal{M}_{0,n}$

In this section we consider the homology and cohomology of $\mathcal{M}^{\#}_{0,n}$ the space of n distinct points on the Riemann sphere $\mathbb{CP}^1$ up to Mobius transformations, together with its Deligne Mumford compactification $\mathcal{M}_{0,n}$.



We will see that Lie polynomials and their duals arise in the homology and cohomology of $\mathcal{M}_{0,n}$. We have

$$H_{n-3}(\mathcal{M}_{0,n} - D) = Lie(n-1), \tag{4.1}$$

and construct cycles corresponding to each Γ. Dually er have

$$\Gamma(\mathcal{M}_{0,n}, \Omega^{n-3}_D) = Lie(n-1)^*. \tag{4.2}$$

This dual space is generated by the familiar Parke-Taylor forms.

4.1 The cohomology of $\mathcal{M}_{0,n}$ and Parke-Taylors

The dimensions of the cohomology groups of $\mathcal{M}^{\#}_{0,n}$ are given by the Poincaré polynomial[3]

$$P(t) := \sum_i \dim(H^i(\mathcal{M}^{\#}_{0,n}))\, t^i = \prod_{k=2}^{n-2}(1+kt)\,. \tag{4.3}$$

The cohomology ring is generated by the $\mathrm{d}\log\sigma_{ij}$ in the standard gauge fixing subject to the quadratic relations

$$\frac{d\sigma_{ij}}{\sigma_{ij}} \wedge \frac{d\sigma_{jk}}{\sigma_{jk}} + \frac{d\sigma_{jk}}{\sigma_{jk}} \wedge \frac{d\sigma_{ki}}{\sigma_{ki}} + \frac{d\sigma_{ki}}{\sigma_{ki}} \wedge \frac{d\sigma_{ij}}{\sigma_{ij}} = 0\,. \tag{4.4}$$

{schouten}

This gives the dimension of $\Gamma(\Omega^1_D)$ as $\sum_{k=2}^{n-2} = n(n-3)/2$ as claimed earlier. Our focus will be on top degree holomorphic forms and their dual cycles.

The above shows that the cohomology $H^{n-3}(\mathcal{M}^{\#}_{0,n}) = \Gamma(\mathcal{M}_{0,n}, \Omega^{n-3}_D)$ and has dimension $(n-2)!$. A natural spanning set for $\Gamma(\Omega^{n-3}_D)$ is provided by the Parke-Taylor forms

$$PT(123\ldots n) = \frac{1}{\mathrm{Vol}SL(2)} \prod_{i=1}^{n} \frac{d\log\sigma_{i\,i+1}}{2\pi i}\,. \tag{4.5}$$

[3]Arnol'd [12] computes the Poincaré polynomial of the cohomology of the configuration space M_{n-1} of $n-1$ points in $\mathbb{C}$. This can be obtained inductively via the fibration $M_k \to M_{k-1}$ with fibre $\mathbb{C} - \{\sigma_1, \ldots, \sigma_{k-1}\}$ giving factors of $(1+(k-1)t)$. For $\mathcal{M}_{0,n}$ one needs to take the quotient by $\mathbb{C}^* \ltimes \mathbb{C}$ to get $\mathcal{M}_{0,n}$. Arnol'd's formula for the Poincaré polynomial of M_{n-1} must therefore be divided by $1+t$ to obtain the formula here.

In our gauge fixing

$$\frac{1}{\mathrm{Vol}SL(2)} = \frac{(2\pi i)^3 \sigma_{1\,n-1}\sigma_{n-1\,n}\sigma_{n1}}{d\sigma_1 d\sigma_{n-1} d\sigma_n} \tag{4.6}$$

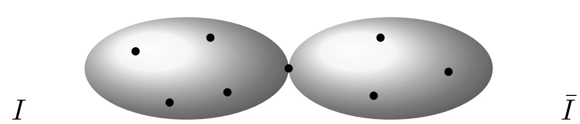
yielding now for a general choice of permutation a of $1,\dots,n-1$

$$PT_a := PT(an) = \frac{d^{n-3}\sigma}{\prod_{i=1}^{n-2}\sigma_{a(i)\,a(i+1)}}\,, \qquad d^{n-3}\sigma := \frac{1}{(2\pi i)^{n-3}}\prod_{i=2}^{n-2} d\sigma_i\,. \tag{4.7}$$

The Parke-Taylor forms can be acted on by permutations, but it is clear that cyclic permutations act trivially so we can always take n to be the last entry. Following Proposition 2.1, PT_a satisfy the shuffle relations identically $PT_{b\sqcup c} = 0$ for b, c nontrivial. Furthermore the Kleiss-Kuijf relations allows one to take 1 as the first entry yielding a basis for the $(n-2)!$-dimensional space $\Gamma(\mathbb{M}_{0,n}, \Omega_D^{n-3})$. It is naturally expressed in terms of permutations by permuting the remaining entries. Following [7], we will denote these as words a in the letters $\{1, 2, \dots, n-1\}$. The Kleiss Kuijf basis is made up of PT_a where the *word* a is a permutation of $\{1,\dots,n-1\}$ that fixes 1.

4.2 Homology of $\mathcal{M}_{0,n}$

We will adopt the traditional normalization of the σ_i under Mobius transformations so that $(\sigma_1, \sigma_{n-1}, \sigma_n) = (0, 1, \infty)$. This has a large open cell $\mathcal{M}_{0,n}^{\#}$ on which the n points are distinct, together with the boundary divisor D stratified by the codimension-1 loci D_I for subsets $I \subset \{1,\dots,n\}$, where the σ_i for $i \in I$ bubble off onto a new $\mathbb{CP}^1$ attached to the first by a node:



On intersections of strata we will have bubbles into more components connected at nodes. In order to see a stratum D_I directly, we need to introduce dihedral coordinates. These require the choice of an ordering for which I is a consecutive subset. In the case of the standard ordering we define the *chord coordinates*

$$u_{ij} = \frac{\sigma_{i\,j-1}\sigma_{i-1\,j}}{\sigma_{ij}\sigma_{i-1\,j-1}}\,, \qquad j > i+1\,, \qquad \sigma_{ij} = \sigma_i - \sigma_j\,. \tag{4.8}$$

When $I = \{1, 2, \dots, |I|\}$, with $|I|$ being the size of I, we take $u_{1\,i}$, $i = 3,\dots,n-1$ as our coordinates on $\mathcal{M}_{0,n}$ and in these coordinates, the D_I are given by $u_{1\,|I|+1} = 0$. To see this we need the 'non-crossing identity'

{non-cross}

$$u_{ij} = 1 - \prod_{(k,l)\in(i,j)^c} u_{kl}\,, \tag{4.9}$$

where for $k < l$, $(k,l) \in (i,j)^c$ means that the diagonal (k,l) of the polygon with vertices $\{1,\dots,n\}$ crosses the diagonal (i,j). This implies that on $u_{1\,|I|+1} = 0$, $u_{ij} = 1$ for $i \in I - 1$

and $j \in \bar{I} - \{|I|+1\}$ as then $u_{1\,|I|} \in (i,j)^c$ and this forces $\sigma_i \to 0$ for $1 < i \in I$ in our gauge fixing with the interpretation that the σ_i for $i \in I$ have bubbled off into their own Riemann sphere (fixing instead $(\sigma_1, \sigma_2, \sigma_n) = (0, 1, \infty)$ would instead have left $\sigma_i, i \in I$ finite but sent $\sigma_i, i \in \bar{I}$ to ∞. An often remarked feature of the boundary divisor is its recursive structure

$$D_I = \mathcal{M}_{0,|I|+1} \times \mathcal{M}_{0,|\bar{I}|+1}\,, \tag{4.10}$$

as the two components of the nodal sphere have $|I|+1$ and $|\bar{I}|+1$ marked points respectively with the additional points being the node that connects them.

It is clear that two divisor components D_I, D_J can only intersect if $I \subset J$ or $I \subset J^c$ the complement of J as otherwise the I cannot separate a bubbled-off component of J. Iterating we see that the maximal intersection of boundary components are points D_Γ that arise as the $n-3$-fold intersection of $n-3$ components D_{I_p} that are mutually compatible. Such compatible intersections are therefore naturally in $1:1$ correspondence with trivalent diagrams Γ as before in the $n-3$-fold intersection of compatible factorization hyperplanes in $\mathcal{K}_n$.

These dimension-0 strata D_Γ naturally give rise to cycles T_Γ $H_{n-3}(\mathcal{M}^{\#}_{0,n})$ given by $n-3$-dimensional tori. To be more explicit, Γ has $n-3$ internal propagators with the pth determining the partition of $\{1,\ldots,n\} = I_p \cup \bar{I}_p$ of leaves on either side of the propagator (which corresponds to a node on the curve separating it into the two corresponding components). To present T_Γ explicitly, choose a dihedral ordering so that Γ is planar. Relative to this we can introduce the chord coordinates u_{ij}. (A given tree, Γ, is compatible with 2^{n-3} distinct dihedral structures. Any choice of dihedral structure defines the same class in homology.) The propagators determine $n-3$ of these chords, u_{I_p}, $p = 1, \ldots, n-3$ whose vanishing gives D_Γ. Then the cycle is defined by

{cycle}

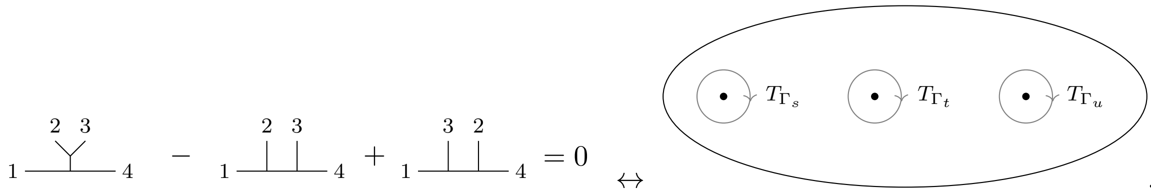
$$T_\Gamma = \{|u_{I_1}| = |u_{I_2}| = \ldots = |u_{I_{n-3}}| = \epsilon\} \subset \mathcal{M}_{0,n}\,, \tag{4.11}$$

for some small ϵ. We need to furthermore define an orientation for T_Γ. Each circle defined in (4.11) is oriented anticlockwise in its complex plane u_{I_p} around the origin. **we must furthermore choose an ordering of the circles by choosing the graph Γ to have a planar structure rooted at n so that we work round the propagators in order anti-clockwise from n.???**

These cycles are not linearly independent in homology. This is most easily seen at four points where $\mathcal{M}^{\#}_{0,n} = \mathbb{CP}^1 - \{0, 1, \infty\}$, with boundary points $D_s = 0$, $D_t = 1$, and $D_u = \infty$; the notation is intended to suggest four point graphs with propagators in the s, t and u channels (i.e., with $s = s_{12}, t = s_{14}$, and $u = s_{13}$). It is then easily seen that the three contours around these points add up to zero in homology

$$[T_s] + [T_t] + [T_u] = 0\,. \tag{4.12}$$

This gives the expected two-dimensions for $H_1(\mathcal{M}^{\#}_{0,4})$.



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This relation will arise similarly on any $\mathbb{CP}^1$ component of D: when four given trees are attached to the four legs of either an s, t or u exchange diagrams, we form trees Γ_s, Γ_t and Γ_u that sit inside such a $\mathbb{CP}^1 \subset D$. The same argument on this $\mathbb{CP}^1$ shows that

{stu-id}

$$[T_{\Gamma_s}] + [T_{\Gamma_t}] + [T_{\Gamma_u}] = 0 \,. \tag{4.13}$$

Thus

Lemma 4.1 *There is a canonical identification*

$$H_{n-3}(\mathcal{M}_{0,n}) = Lie(n-1) \tag{4.14}$$

given by identifying the cycle T_Γ with the monomial P_Γ corresponding to the planar tree Γ rooted at n.

Proof: The main task is to see that the skew symmetry is correctly encoded by the orientation and that the Jacobi identity by the *stu*-identity (4.13) but these are clear. □

Dually we have

T-PT-pairing

Proposition 4.1 *The natural integration pairing between $H_{n-3}(\mathcal{M}^{\#}_{0,n})$ and $\Gamma(\mathcal{M}_{0,n}, \Omega^{n-3}_D)$ is*

$$(T_\Gamma, PT_a) := \int_{T_\Gamma} PT_a = (\Gamma, a) \tag{4.15}$$

i.e., ± 1 if Γ is planar for the ordering a, otherwise zero.

Proof: The standard Parke-Taylor form can be written as

{comb-PT}

$$PT(12\dots n) = \prod_{i=3}^{n-1} \frac{1}{2\pi i} d\log \frac{u_{1i}}{1-u_{1i}} \,. \tag{4.16}$$

If $T_\Gamma = \{|u_{1i}| = \epsilon, i = 3, \dots, n-1\}$, i.e. Γ is the standard comb, the pairing then obviously gives 1. If we interchange 2 and 3 in Γ the u_{13} part of the contour will be $|\frac{1}{u_{13}}| = \epsilon$ but the Parke-Taylor has no pole at $u_{13} = \infty$ so the pairing will give zero. The *stu* identity will then give -1 for the planar graph for which $2, 3$ connect to a vertex, whose third propagator attaches to the standard comb with one fewer teeth (this is contour $|u_{13} - 1| = \epsilon$) that encircles the manifest pole with residue -1 at $u_{13} = 1$. **The general planar graph compatible with the ordering can be obtained by such *stu* moves acting on the propagator between $i-1$ and i by studying such moves on each u_{1i}-plane.** □

5 A Penrose transform for amplitudes

Our starting point is the observation that $\mathcal{K}_n$ also arises naturally from the geometry of $\mathcal{M}_{0,n}$ the Deligne-Mumford compactification of the moduli space of n points σ_i, $i = 1, \ldots, n$ on $\mathbb{CP}^1$ up to Möbius transformations.

Our 'twistor space' for the Penrose transform will be $\mathbb{T} = T^*_D\mathcal{M}_{0,n}$, the total space of the bundle of holomorphic 1-forms on $\mathcal{M}_{0,n}$ with $d\log$ singularities on D. The relationship with $\mathcal{K}_n$ is given by

$$\mathcal{K}_n = \Gamma(\mathcal{M}_{0,n}, T^*_D)\,. \tag{5.1}$$

This correspondence can be expressed by considering $d\log$ of the Koba-Nielsen factor

$$KN := \prod_{i<j} \sigma_{ij}^{s_{ij}}\,, \qquad \sigma_{ij} = \sigma_i - \sigma_j\,. \tag{5.2}$$

This gives the general section of $\tau \in \Gamma(T^*_D\mathcal{M}_{0,n})$ as

$$\tau = \sum_i E_i d\sigma_i := \sum_{i<j} s_{ij} d\log\sigma_{ij}\,, \qquad E_i = \sum_j \frac{s_{ij}}{\sigma_{ij}}\,. \tag{5.3}$$

Our normalizations $(\sigma_1, \sigma_{n-1}, \sigma_n) = (0, 1, \infty)$ gives the triviality of $d\log\sigma_{in}$ and $d\log\sigma_{1\,n-1}$ giving the correct dimensionality of the $d\log\sigma_{ij}$ basis. We remark that the vanishing of the E_i are the *scattering equations*.

To more clearly demonstrate the $d\log$ behaviour on D, given the choice of the standard ordering, we can also represent the Koba Nielsen factor as

{region-KN}

$$KN = \prod_{j>i+1} u_{ij}^{X_{ij}}\,, \qquad X_{ij} = \sum_{i\leq l<m<j} s_{lm}\,. \tag{5.4}$$

This gives the useful representation of the general section in terms of the $n(n-3)/2$ basis $d\log u_{ij}$

{tau-u}

$$\sum_i E_i d\sigma_i = \sum_{j>i+1} X_{ij} d\log u_{ij}. \tag{5.5}$$

This representation manifests the $d\log$ behaviour on the components of D manifested with this choice of ordering.

Finally, it will also be important to observe that the dual vector space $\mathcal{K}_n^*$ is isomorphic to $H_1(\mathcal{M}_{0,n})$. This follows from the fact that $\mathcal{K}_n = \Gamma(\mathcal{M}_{0,n}, T^*_D) \simeq H^1(\mathcal{M}_{0,n})$. More explicitly, we can regard $\mathcal{K}_n^*$ as the vector space $\mathbb{R}^{n(n-3)/2+1}$, generated by covectors e_{ij}, quotiented by the relations $\sum_j e_{ij} = 0$. In this presentation, we have an isomorphism that maps $e_{ij} \mapsto L_{ij}$, where L_{ij} is the class in H_1 of the loop around the divisor D_{ij}. This is an isomorphism because, in H_1, we have the relations $\sum_j L_{ij} = 0$.

5.1 The double fibration

The twistor correspondence arises from the following double fibration:

$$\begin{array}{ccc} & \mathcal{Y}_n = \mathcal{K}_n \times \mathcal{M}_{0,n},\ (s_{ij}, \sigma_j) & \\ & p \swarrow \qquad \searrow q & \\ (s_{ij}),\ \mathcal{K}_n & & \mathbb{T} = T^*_D\mathcal{M}_{0,n},\ (\tau_i, \sigma_i). \end{array} \tag{5.6}$$

where p forgets the second factor and and q is defined by the incidence relations

{incidence}

$$\tau_i = E_i(s_{kl}, \sigma_m) := \sum_j \frac{s_{ij}}{\sigma_{ij}}\,. \tag{5.7}$$

A point in $\mathcal{K}_n$ determines a section $\tau_i = E_i$ of $\mathbb{T} \to \mathcal{M}_{o,n}$. A special role is played by the zero-section $\mathbb{T}_0$ of $\mathbb{T}$ as it encodes the scattering equations: given generic s_{ij}, the section $\tau_i = E_i(\sigma)$ clearly intersects $\mathbb{T}_0$ at the $(n-3)!$ solutions to the scattering equations.

Conversely, a point $(\underline{\tau}, \underline{\sigma}) := (\tau_i, \sigma_i) \in \mathbb{T}$ determines a plane $\alpha_{(\underline{\tau},\underline{\sigma})}$ of co-dimension $n-3$ in $\mathcal{K}_n$ given by holding (τ_i, σ_i) fixed in (5.7). They are spanned by vector fields of the form

$$V_{ijk} = \sigma_{ij}\frac{\partial}{\partial s_{ij}} + \sigma_{jk}\frac{\partial}{\partial s_{jk}} + \sigma_{ki}\frac{\partial}{\partial s_{ki}}\,. \tag{5.8}$$

In order to see that they make sense with our gauge fixing and the relations between the s_{ij}, we first require $i, j, k < n$ and require s_{in} to be determined in terms of the remaining s_{ij} by $\sum_i s_{ij} = 0$. There is one remaining relation $\sum_i s_{in} = \sum_{i<j<n} s_{ij} = 0$ but it follows directly that V_{ijk} annihilates this relation and so restricts to this subspace. The vector fields V_{1ij} with $1 < i < j < n$ are a basis for the $(n-2)(n-3)/2$-planes that form the fibres of q.

Note that, dually, a point $(\underline{\tau}, \underline{\sigma})$ defines a dimension $n-3$ plane $\alpha^*_{(\underline{\tau},\underline{\sigma})}$ in $\mathcal{K}^*_n$, which we can regard as being defined by the equations $V_{1ij} \cdot W = 0$.

When $\underline{\sigma} \in D_I$, some boundary component, and $\underline{\tau} = 0$, these planes planes lie inside a factorisation hyperplane plane $s_I = 0$. We have

Lemma 5.1 *The factorization hyperplane $s_I = 0$ is ruled by the $\alpha_{(0,\underline{\sigma})}$ for $\underline{\sigma} \in D_I$.*

Proof: By an elementary calculation, we have the combination of scattering equations

$$E_I := \sum_{i\in I} \sigma_i E_i = \frac{s_I}{2} + \sum_{i\in I, j\in \bar{I}} \frac{s_{ij}\sigma_i}{\sigma_{ij}}\,. \tag{5.9}$$

Taking $I = \{1, 2, \ldots, |I|\}$, the divisor D_I is given by $\sigma_i = 0$ for $i \in I$, and $\sigma_j \neq 0$ for $j \in \bar{I}$, so for E_I to vanish, we must have $s_I = 0$. $\Box$

Note that E_I is often expressed in the more invariant cross-ratio form

$$E_I = \frac{s_I}{2} + \sum_{i\in I, j\in \bar{I}} s_{ij}\frac{\sigma_{i1}\sigma_{jn}}{\sigma_{ij}\sigma_{1n}}\,. \tag{5.10}$$

This can be extended to intersections of boundary components. We note that $D_{I_1} \cap D_{I_2} = \emptyset$ unless $I_1 \subset I_2$ or $I_1 \subset \bar{I}_2$ as only subsets of points on a Riemann sphere can bubble off. The zero-dimensional strata of the boundary are the intersection of $n-3$ such components. These boundary points D_Γ are in one to one correspondence with trivalent diagrams Γ that correspond to the bubbling of the Riemann sphere: each vertex corresponds to a $\mathbb{CP}^1$ component, and the propagators to the nodes connecting the spheres. (With only three points on each sphere no further bubbling is possible.) The total number of ternary trees is $(2n-5)!!$. According to the lemma above, each such point, $(\underline{\tau}, \underline{\sigma}) = (0, D_\Gamma)$, corresponds to the codimension-$n-3$ plane at the intersection of the $s_{I_p} = 0$ where $p = 1, \ldots, n-3$ enumerate the propagators in Γ and I_p is the subset of $\{1, \ldots, n\}$ corresponding to the momentum flowing through each propagator. More generally $(\underline{\tau}, D_\Gamma)$ is the translate of such a plane and we can characterise such planes as being those with normal $n-3$-form

$$w_\Gamma = \mathrm{sgn}(\Gamma) \bigwedge_{p=1}^{n-3} ds_{I_p} \tag{5.11}$$

where $\mathrm{sgn}(\Gamma)$ is a sign that will be specified later.

6 The symplectic form and the holomorphic volume form

The symplectic form on $T^*_D\mathcal{M}_{0,n}$ can be written explicitly as

$$\omega = \sum_{i \neq 1, n-1, n} d\tau_i \wedge d\sigma_i,$$

where the coordinates τ_i define a 1-form $\tau = \sum_i \tau_i d\sigma_i$.

Our most elementary starting point for a Penrose transform will be to consider the natural volume form ω^{n-3} on $T^*_D\mathcal{M}_{0,n}$ and to transform this to objects on $\mathcal{K}_n$. Pulling back ω^{n-3} to $\mathcal{Y}_n$, we can naturally decompose it into a sum over a basis of $\Gamma(\mathcal{M}_{0,n}, \Omega^{n-3}_D)$ with coefficients given by $n-3$-forms on $\mathcal{K}_n$. This gives a correspondence between $n-3$-forms on $\mathcal{K}_n$ and $(n-3)$-cycles in $\mathcal{M}_{0,n}$ and conversely between $(n-3)$-planes in $\mathcal{K}_n$ and $(n-3)$-forms on$\mathcal{M}_{0,n}$.

6.1 Forms on $\mathcal{K}_n$ from cycles in $\mathcal{M}_{0,n}$.

In the $n = 4$ case we find

$$q^*\omega = \frac{ds \wedge d\sigma}{\sigma} - \frac{dt \wedge d\sigma}{1-\sigma}.$$

In terms of Parke-Taylor factors, this form can be expressed in three different bases

$$\begin{aligned} q^*\omega &= -ds_{12} \wedge PT(2134) - ds_{23} \wedge PT(1324) \\ &= ds_{12} \wedge PT(1234) + ds_{13} \wedge PT(1324) \\ &= -ds_{23} \wedge PT(1234) + ds_{13} \wedge PT(2134). \end{aligned} \tag{6.1}$$

We have already introduced a natural class of integration cycles in order to push down to objects on $\mathcal{K}_n$. Define

$$w_\Gamma := \int_{T_\Gamma} q^*\omega^{n-3} \in \Omega^{n-3}(\mathcal{K}_n)\,. \tag{6.2}$$

{WG-def}

Then

Lemma 6.1 *We have*

$$w_\Gamma = \bigwedge_{p=1}^{n-3} ds_{I_p} \tag{6.3}$$

{WG-Kn}

where the s_{I_1} are ordered according to the definition of the orientation of T_Γ. If the diagrams Γ_s, Γ_t and Γ_u are as in (4.13) then

$$w_{\Gamma_s} + w_{\Gamma_t} + w_{\Gamma_u} = 0 \tag{6.4}$$

{WG-stu}

On $\mathcal{Y}_n$ we can write

$$q^*\omega^{n-3} = \sum_{a\in S_{n-2}} w_{\Gamma_{1a}} PT_{1a}\,. \tag{6.5}$$

{w-decomp}

Although we have used dual comb and KK bases here, this relation follows in any dual basis as w_Γ and PT_a furnish representations of $Lie(n-1)$ and $Lie(n-1)^*$ respectively so that(6.5) is an explicit of writing the Kronecker delta.

Proof: Although $w_{\Gamma_s} + w_{\Gamma_t} + w_{\Gamma_u} = 0$ follows directly from the definition (6.2) and (4.13), it also follows from the corresponding $s+t+u=0$ relations between the propagator factors on which they are distinct.

To prove (6.3), we use the representation (5.5) of $q^*\tau$ in a choice of dihedral coordinates u_{ij} for which Γ is planar. This gives

$$q^*\omega = d(q^*\tau) = \sum_{i+1<j} dX_{ij} \wedge \frac{du_{ij}}{u_{ij}} \tag{6.6}$$

It is then easily seen that integration of $q^*\omega^{n-3}$ over T_Γ picks out only those poles in u_{ij} that correspond to the defining equations $u_{I_p} = 0$ of D_{I_p} that correspond to propagators of Γ. The coefficients that are thereby picked out are the corresponding dX_{ij} that give the corresponding ds_{I_p} factors as desired **and signs?**.

To prove (6.5) we note that the fact that Γ_{1a} and $1a$ are dual bases implies

$$w_\Gamma = \sum_{a\in S_{N-2}} w_{\Gamma_{1a}}(\Gamma, 1a). \tag{6.7}$$

However, we have $\int_{T_\Gamma} PT_a = (\Gamma, a)$ from lemma 4.1 and so evaluating (6.5) on all T_Γ gives (6.2) and so (6.5) follows. $\square$

6.2 Associahedral $(n-3)$-planes in $\mathcal{K}_n$ and forms on $\mathcal{M}_{0,n}$.

An alternative way to study the correspondence is to restrict ω^{n-3} to different $(n-3)$-planes in $\mathcal{K}_n$. We first give a definition that appears natural within our construction. We then give the ABHY definition.

Definition 6.1 *Given a graph Γ we define the foliation by $(n-3)$-planes $P_\Gamma \subset \mathcal{K}_n$ in terms of the dihedral momentum invariants X_{ij} by*

{P-def}

$$P_\Gamma = \{X_J + \sum_{r, I_r^c \ni J} X_{I_r} = const. \,|J \neq I_r \in \Gamma, r = 1, \ldots, n-3\}\,. \tag{6.8}$$

With an abuse of notation we can denote the polyvector tangent to P_Γ also by P_Γ:

$$P_\Gamma = \bigwedge_{r=1}^{n-3} D_{I_r}\,, \qquad D_I := \frac{\partial}{\partial X_I} - \sum_{J \in I^c} \frac{\partial}{\partial X_J}\,. \tag{6.9}$$

n.b. In $\mathcal{K}_n^*$, the dual plane P_Γ^* is cut out by the following $n-3$ equations in the dual variables Y_I:

$$Y_{I_r} + \sum_{J \text{ crosses } I_r} Y_J = 0.$$

These equations resemble the non-crossing identity.

Lemma 6.2 *We have*

{PT-Gamma}

$$PT_\Gamma := P_\Gamma \lrcorner\, \omega^{n-3} = \bigwedge_{r=1}^{n-3} d\log\left(\frac{u_{I_r}}{1-u_{I_r}}\right)\,. \tag{6.10}$$

In particular, when Γ is a comb Γ_a, PT_{Γ_a} is the standard Parke-Taylor PT_a by (4.16).

Proof: This follows from the form (5.4) of the Koba-Nielsen factor in terms of the dihedral X_{ij} and cross-ratios u_{ij}, so that

$$D_I \log KN = \log u_I - \sum_{J \in I^c} \log u_J = \log \frac{u_I}{\prod_{J \in I^c} u_J} = \log \frac{u_I}{1-u_I}\,, \tag{6.11}$$

using the non-crossing identity (4.9) giving each factor of (6.10). Thus the pull back of ω^{n-3} to $\mathcal{Y}_n$ restricted to P_Γ gives the desired formula.$\Box$

Lemma 6.3 *If Γ and Γ' are related by a 'square move,' then $PT_\Gamma = -PT'_\Gamma$.*

Proof: Let Γ' be obtained from Γ by replacing some arc I with its square move flip I'. Then we have the following four non-crossing identities,

$$1 = u + AB, \quad 1 = u' + BC, \quad 1 = uu' + B, \quad 1 = AB + BC,$$

where

$$A = \prod_{J \text{ crossing } I \text{ but not } I'} u_J, \quad C = \prod_{J \text{ crossing } I' \text{ but not } I} u_J, \quad B = \prod_{J \text{ crossing } I \text{ and } I'} u_J.$$

It follows that

$$\frac{uu'}{1-u-u'+uu'} = \frac{1-B}{AB+BC-B} = 1,$$

and so

$$\log\left(\frac{u}{1-u}\right) + \log\left(\frac{u'}{1-u'}\right) = 0.$$

Notice that this lemma means that PT_Γ depends, up to a sign, only on the Dihedral structure. We can fix the sign as follows. It follows from (6.5) that for combs $P_a := P_{\Gamma_a}$ satisfy

$$(P_{1a}, w_{\Gamma_{1b}}) = (a, b), \tag{6.12}$$

and by the Jacobi relations amoung the $w_\Gamma' s$ it follows that

$$(P_{1a}, w_\Gamma) = (1a, \Gamma).$$

Our definition of associahedral $(n-3)$ planes is different from that of ABHY, but we have:

Proposition 6.1 *For the comb Γ_{1a}, $P_{\Gamma_{1a}}$ give the associahedral $n-3$-planes of ABHY.*

Proof: These too can also be determined by tangent $n-3$-polyvector $P \in \bigwedge^{n-3} \mathcal{K}_n$ in $\mathcal{K}_n$. In the standard case fixing a KK basis the definition of ABHY can be straightforwardly seen to correspond to the polyvector

$$P_a = \bigwedge_{i=2}^{n-2} \left(\frac{\partial}{\partial s_{a_{i-1}\, a_i}} - \frac{\partial}{\partial s_{a_i\, a_{i+1}}}\right). \tag{6.13}$$

It is then straightforward to check, using

$$\left(\frac{\partial}{\partial s_{i-1\, i}} - \frac{\partial}{\partial s_{i\, i+1}}\right) X_{jk} = \begin{cases} 1, & i = k-1, \\ -1\,, & i = j, \\ 0\,, & \text{otherwise.} \end{cases} \tag{6.14}$$

that P_a annihilates the relations (6.8) arising when Γ is the comb Γ_{1a}.□

6.3 Scattering forms and biadjoint scalars from CHY

The scattering forms of ABHY are defined by

$$\Omega_a = \sum_{\Gamma} \frac{w_\Gamma(a,\Gamma)}{s_\Gamma}\,, \tag{6.15}$$

where $s_\Gamma = \prod_{p=1}^{n-3} s_{I_p}$ and I_p are the propagators of Γ. The (a,Γ)-factor reduces the sum to one over planar diagrams for the ordering a.

The symplectic form discussed in the previous section gives rise to the ABHY scattering form on $\mathcal{K}_n$ from the Dolbeault formula

$$\Omega_a = \int_{p^{-1}(s_{ij})} q^* \left[\bar\delta^{n-3}(\underline{\tau}) \wedge \omega^{n-3} PT_a\right], \tag{6.16}$$

where

$$\bar\delta^{n-3}(\underline{\tau}) = \bigwedge_{i=1}^{n-3} \bar\delta(\tau_i)\,, \qquad \bar\delta(z) = \frac{1}{2\pi i}\bar\partial\frac{1}{z}\,. \tag{6.17}$$

This is the CHY formula in disguise using the formula we derived earlier for $q^*\omega^{n-3} = \sum$.

Lemma 6.4 $\Omega(b)$ *is a projective form, in the sense that it descends to the projective quotient of* $\mathcal{K}_n$*.*

Proof: This follows from the Dolbeault formula. On $\mathcal{Y}_n$, we have

$$\sum_{i<j} s_{ij}\frac{\partial}{\partial s_{ij}} = \sum_{i<j}\left(\frac{s_{ij}}{\sigma_{ij}}\frac{\partial}{\partial\tau_i} - \frac{s_{ij}}{\sigma_{ij}}\frac{\partial}{\partial\tau_j}\right) = \sum_i \tau_i\frac{\partial}{\partial\tau_i}.$$

Then, contracting,

$$\left(\sum_{i<j} s_{ij}\frac{\partial}{\partial s_{ij}}\right) \lrcorner\, \Omega_\alpha^{(n-3)} = \left(\sum_i \tau_i\frac{\partial}{\partial\tau_i}\right) \lrcorner\, \Omega_\alpha^{(n-3)} = 0,$$

on account of the delta functions in the integrand.

It might be interesting to note that the Dolbeault formula for $\Omega(\alpha)$ implicitly implements the Thom isomorphism. The form $\bar\delta(\tau)\wedge d\tau$ is a Dolbeault representative of the Poincaré dual of $\tau = 0$, the form

$$c = \bigwedge_{i=2}^{n-2} \bar\delta(\tau_i)\wedge d\tau_i$$

is a Dolbeault representative of the Thom class of $T^*\mathcal{M}_{0,n} \to \mathcal{M}_{0,n}$. With this understood, $\Omega(\alpha)$ is obtained from the map

$$H^{n-3}(\mathcal{M}_{0,n}) \to \wedge^{n-3}\mathcal{K}_n \;:\; [PT(\alpha)] \mapsto p_*q^*\left(c \cup [PT(\alpha)]\right).$$

$\Omega(\alpha)$ is the $n-3$-form on $\mathcal{K}_n$ obtained as the image of $PT(\alpha)$.

6.4 Open string integrals, and string KLT

Our starting points are:

- The open string integral $Z_a(b)$ is the pairing of $C(a)$ with $PT(b)$, where $C(a)$ is regarded as a cycle with values in a local system defined by $d + q^*\theta$, where $\theta = \tau_i d\sigma_i$. $Z_a(b)$ is a function on $\mathcal{K}_n$ with poles on the factorization hyperplanes D_I, for all I compatible with both a and b. In fact, $Z_a(b)$ also has poles at all points in the positive integer span of the e_I. For example, $Z_{1234}(1234)$ has simple poles when either s_{12} or s_{23} are nonnegative integers, and double poles at the points where both s_{12} and s_{23} are nonnegative integers.

- The string KLT kernel $K(a|b)$ is the pairing of $C(a)$ with $C(b)$, regarded as cycles with values in the local system. $K(a|b)$ is a function on $\mathcal{K}_n$ that is periodic under $s_I \mapsto s_I + 1$, for all I compatible with both a and b. In fact, $K(a|b)$ is the manifestly periodic sum $\sum_{n\geq 0} \exp(2\pi i n_I s_I)$, up to a normalization.

- The coefficient of $s_I^{m_I}$ in the Laurent expansion of $Z_a(b)$ around the origin is a multizeta value that we can write as a sum $\sum_{n\geq 0} 1/(n_{I_1}^{m_{I_1}} ...)$, up to a normalization.

- Integer shifts of the symplectic potential, $\theta \mapsto \theta + d\log\sigma_{ij}$ correspond to integer shifts $s_{ij} \mapsto s_{ij} + 1$.

We should like to be able to say:

- how the periodicity of $K(a|b)$ is related to $H_1(\mathcal{M}_{0,n})$

- how the poles of $Z_a(b)$ are inherited from the divisor structure

- how the formula $\int_{\mathcal{M}_{0,n}} PT(a) \wedge \bar{PT}(b) = Z_c(a)K(c|d)\bar{Z}_d(b)$ can be understood.

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