Competition and Regulation in a Wireless Operators Market: An Evolutionary Game Perspective

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***Abstract*—We consider a market where a set of wireless operators compete for a large common pool of users. The latter have a reservation utility of U_0 units or, equivalently, an alternative option to satisfy their communication needs. The operators must satisfy these minimum requirements in order to attract the users. We model the users decisions and interaction as an evolutionary game and the competition among the operators as a non cooperative price game which is proved to be a potential game. For each set of prices selected by the operators, the evolutionary game attains a different stationary point. We show that the outcome of both games depends on the reservation utility of the users and the amount of spectrum W the operators have at their disposal. We express the market welfare and the revenue of the operators as functions of these two parameters. Accordingly, we consider the scenario where a regulating agency is able to intervene and change the outcome of the market by tuning W and/or U_0. Different regulators may have different objectives and criteria according to which they intervene. We analyze the various possible regulation methods and discuss their requirements, implications and impact on the market.**

I. Introduction

Consider a city where 3 commercial operators (companies) and one municipal operator offer WiFi Internet access to the citizens (users). The companies charge for their services and offer better rates than the municipal WiFi service which however is given gratis. Users with high needs will select one of the companies. However, if they are charged with high prices, or served with low rates, a portion of them will eventually migrate to the municipal network. In other words, the municipal service constitutes an alternative choice for the users and therefore sets the minimum requirements which the commercial providers should satisfy. Apparently, the existence of the municipal network affects both the user decisions and the operators pricing policy. In different settings, the minimum requirement can be an inherent characteristic of the users as for example a lower bound on transmission rate for a particular application, an upper bound on the price they are willing to pay or certain combinations of both of these parameters. Again, the operators can attract the users only if they offer more appealing services and prices.

In this paper, we consider a general wireless communication services market where a set of *operators*, compete to sell their services to a common large pool of *users*. We assume that users have minimum requirements or alternative options to satisfy their needs which we model by introducing the reservation utility U_0, [1]. Users select an operator only if the offered service and the charged price ensure utility higher than U_0. We analyze the users strategy for selecting operator and the price competition among the operators under this constraint. We find that the market outcome depends on U_0 and on the amount of spectrum each operator has at his disposal W. Accordingly, we consider the existence of a regulating agency who is interested in affecting the market and enforcing a more desirable outcome, by tuning either W or U_0. For example, consider the municipal WiFi provider who is actually able to set U_0 and bias the competition among the commercial providers. This is of crucial importance since in many cases the competition of operators may yield inefficient allocation of the network resources, [1] or even reduced revenue for them, [2]. We introduce a rigorous framework that allows us to analyze the various methods through which the regulator can intervene and affect the market outcome according to his objective.

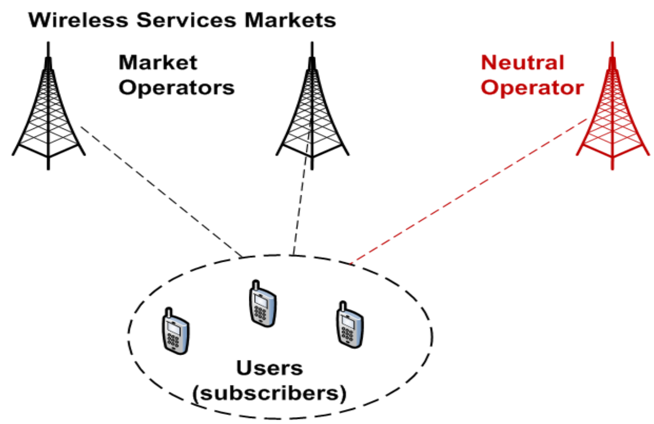

Fig. 1. The market consists of a set of operators competing over a common pool of users. Each user selects one of the market operators or opts to abstain from the market and be associated with the neutral operator. The latter models the alternative out-of-the-market users option or their minimum service-price requirements.

Our model captures many different settings such as a WiFi market in a city, a mobile/cell-phone market in a country or even a secondary spectrum market where primary users lease their spectrum to secondary users. In order to make our study more realistic, we adopt a macroscopic perspective and analyze the interaction of the operators and users in a large time scale, for large population of users, and under limited information. The operators are not aware of the users specific

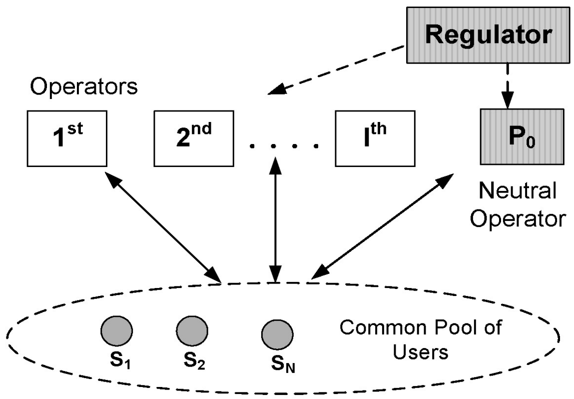

Fig. 2. The oligopoly market consists of I operators and N users (S). Each user is associated with one operator at each specific time slot. Every operator $i = 1, 2, \ldots, I$ can serve more than one users at a certain time slot. The users that fail to satisfy their minimum requirements, $U_i \le U_0$, $\forall i \in \mathcal{I}$, abstain from the market and select the neutral operator P_0.

needs and the latter cannot predict in advance the exact level of service they will receive. Each operator has a total resource at his disposal (e.g. the aggregate service rate) which is on average equally allocated to his subscribers, [1], [4]. This is due to the various network management and load balancing techniques that the operators employ, or because of the specific protocol that is used, [5]. Each user selects the operator that will provide the optimal combination of service quality and price. Apparently, the decision of each user affects the utility of the other users. We model this interdependency as an evolutionary game, [3] the stationary point of which represents the users distribution among the operators and depends on the charged prices. This gives rise to a non cooperative price competition game among the operators who strive to maximize their profits.

Central to our analysis is the concept or the *neutral operator* P_0 which provides to the users a constant and given utility of U_0 units. The P_0 can be a special kind of operator, like the municipal WiFi provider in the example above, or it can simply model the user choice to abstain from the market. This way, we can directly calculate how many users are served by the market and how many abstain from it and select P_0. Moreover, P_0 allows us to introduce the role of a regulating agency who can intervene and bias the market outcome through the service U_0. We show that P_0 can be used to increase the revenue of the operators or the efficiency of the market. In some cases, both of these metrics can be simultaneously improved at a cost which is incurred by the regulator. Alternatively, the outcome of the market can be regulated by changing the amount of spectrum each operator has at his disposal. Different regulating methods give different results and entail different cost for the regulator.

A. Related Work and Contribution

The competition of sellers for attracting buyers has been studied extensively in the context of network economics, [6], [7], both for the Internet and more recently for wireless systems. In many cases, the competition results in undesirable outcome. For example, in [1] the authors consider an oligopoly communication market and show that it yields inefficient resource allocation for the users. From a different perspective it is explained in [2], that selfish pricing strategies may also decrease the revenue of the sellers-providers. In these cases, the strategy of each node (buyer) affects the performance of the other nodes by increasing the delay of the services they receive, [1] (*effective cost*) or, equivalently, decreasing the resource the provider allocates to them, [4] (*delivered price*). This equal-resource sharing assumption represents many different access schemes and protocols (TDMA, CSMA/CA, etc), [5].

More recently, the competition of operators in wireless services markets has been studied in [4], [13], [14], [15], [21]. The users can be charged either with a usage-based pricing scheme, [13], or on a per-subscription basis, [14], [21]. We adopt the latter approach since it is more representative of the current wireless communication systems. We assume that users may migrate (churn) from one operator to the other, [21], and we use evolutionary game theory (EVGT) to model this process, [9]. This allows us to capture many realistic aspects and to analyze the interaction of very large population of users under limited information. The motivation for using EVGT in such systems is very nicely discussed in [15]. Due to the existence of the *neutral operator*, the user strategy is updated through a hybrid scheme based on imitation and direct selection of P_0. We define a new revision protocol to capture this aspect and we derive the respective system dynamic equations.

Although the regulation has been discussed in context of networks, [6], it remains largely unexplored. Some recent work [8], [11] study how a regulator or an *intervention device* may affect a non-cooperative game among a set of players (e.g. operators). However, these works do not consider hierarchical systems, with large populations and limited information. Our contribution can be summarized as follows: **(i)** we model the wireless service market using an evolutionary game where the users employ a new hybrid revision protocol, based both on imitation and direct selection of a specific choice, namely the P_0. We derive the differential equations that describe the evolution of this system and find the stationary points, **(ii)** we define the price competition game for I operators and the particular case that users have minimum requirements, or equivalently, alternative choices/offers, **(iii)** we prove that this is a Potential game and we analytically find the Nash equilibria, **(iv)** we introduce the concept of the neutral operator who represents the system/state regulator or the minimum users requirement, and **(v)** we discuss different regulation methods and analyze their efficacy, implications and the resources that are required for their implementation.

The rest of this paper is organized as follows. In Section II we introduce the system model and in Section III we analyze the dynamics of the users interaction and find the stationary point of the market. In Section IV we define and solve the price competition game among the operators and in Section V we discuss the relation between the revenue of the operators and the efficiency of the market and their dependency on the system parameters. Accordingly, we analyze various regulation methods for different regulation objectives and give related numerical examples. We conclude in Section VI.

II. System Model

We consider a wireless service market (hereafter referred to as a *market*) with a very large set of users $\mathcal{N} = (1, 2, \ldots, N)$ and a set of operators $\mathcal{I} = (1, 2, \ldots, I)$, which is depicted in Figure 2. We assume a time slotted operation. Each user cannot be served by more than one operator simultaneously. However, users can switch in each slot t between operators or even they can opt to refrain and not purchase services from anyone of the I operators. The net utility perceived by each user who is served by operator i in time slot t is:

$$U_i(W_i, n_i(t), \lambda_i) = V_i(W_i, n_i(t)) - \lambda_i \qquad (1)$$

where $n_i(t)$ are the users served by this specific operator in slot t, W_i the total spectrum at his disposal, and λ_i the charged price. In order to describe the market operation we introduce the users vector $\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_I(t), x_0(t))$, where the i^{th} component $x_i(t) = n_i(t)/N$ represents the portion of users that have selected operator $i \in \mathcal{I}$. Additionally, with $x_0(t) = n_0(t)/N$ we denote the portion of users that have selected neither of the I operators. We assume that the number of users N is very large, $N >> 1$ and therefore the variable $x_i(t) = n_i(t)/N$ is considered continuous. In other words, we assume that there exist a continuum of users partitioned among the different operators.

1) Valuation function: The function $V_i(\cdot)$ represents the value of the offered service for each user associated with operator $i \in \mathcal{I}$. Users are considered homogeneous: all the users served by a certain operator are charged the same price and perceive the same utility. We consider the following particular valuation function:

$$V_i(W_i, x_i(t)) = \log \frac{W_i}{N x_i(t)}, \; x_i(t) > 0 \qquad (2)$$

Since N is given, we use $x_i(t)$ instead of $n_i(t)$. This function has the following desirable properties: **(i)** the valuation for each user decreases with the total number of served users by the specific operator due to congestion, **(ii)** increases with the amount of available spectrum W_i, and **(iii)** it is a concave function and therefore captures the saturation of the user satisfaction as the allocated resource increases, i.e. it satisfies the principle of diminishing marginal returns, [6].

A basic assumption in our model is that users served by the same operator are allocated an equal amount of resource. We want to stress that this assumption captures many different settings in wireline, [1], or wireless networks, [4], [5], [14], [15], [18]. Some examples where the equal resource sharing assumption holds are the following:

- **FDMA - TDMA:** If the operator uses a multiple access scheme like Frequency Division Multiple Access (FDMA) or Time Division Multiple Access (TDMA), then the equal resource sharing assumption holds by default, [5]. The users served by a certain operator receive an equal share of his total available spectrum or an equal time share of the operator's channel.
- **CSMA/CA:** A similar result holds for the Carrier Sense Multiple Access scheme with Collision Avoidance, [16], that is used in IEEE 802.11 protocols. Users trying to access the channel receive an equal share of it and achieve - on average - the same transmission rate. Additionally, as it was shown in [17], even if the radio transmitters are controlled by selfish users, they can achieve this fair resource sharing.
- **Random access of multiple channels:** Even in more complicated access schemes as in the case, for example, where many different users iteratively select the least congested channel among a set of available channels, it is proved that each user receives asymptotically an equal share of the channel bandwidth, [18].

Additionally, the macroscopic perspective and the large time scale that we consider in this problem , ensure that spatiotemporal variations in the quality of the offered services will be smoothed out due to load balancing and other similar network management techniques that the operators employ. Therefore, users of each operator are treated in equal terms.

2) Neutral Operator: Variable $x_0(t)$ represents the portion of users that do not select anyone of the I operators. Namely, a user in each time slot t is willing to pay operator $i \in \mathcal{I}$ only if the offered utility $U_i(W_i, x_i(t), \lambda_i)$ is greater than a threshold $U_0 \geq 0$. If all operators fail to satisfy this minimum requirement then the user abstains from the market and is associated with the *Neutral Operator* P_0, Figure 2. In other words, P_0 represents the choice of selecting neither of the I operators and receiving utility of U_0 units. Technically, as it will be shown in the sequel, the inclusion of P_0 affects both the user decision process for selecting operator and the competition among the operators.

From a modeling perspective, the neutral operator may be used to represent different realistic aspects of the wireless service market. First, P_0 can be an actual operator owned by the state, as the public/municipal WiFi provider we considered in the introductory example. In this case, through the gratis U_0 service, the state intervenes and regulates the market as we will explain in Section V. Additionally, U_0 can be indirectly imposed by the state (the regulator) through certain rules such as the minimum amount of spectrum/rate per user. Finally, it can represent the users reluctancy to pay very high prices for poor QoS, similarly to the individual rationality constraint in mechanism design. We take these realistic aspects into account and moreover, by using $x_0(t)$, we find precisely how many users are not satisfied by the market of the I operators.

Unlike the valuation $V_i(\cdot)$ of the service offered by each operator $i \in \mathcal{I}$, U_0 is considered constant. When U_0 represents users minimum requirements or respective restrictions imposed by regulatory rules, this assumption follows directly and actually is imperative. In case U_0 models the service offered by the neutral operator (e.g. the municipal WiFi network), the constant value of U_0 means that it is independent of the number of users and hence non-congestible. We follow this assumption for the following two reasons:**(i)** U_0 is a free of charge service which in general is low and hence can be ensured for a large number of users. **(ii)** The state agency (i.e. the regulator) who provides U_0, is able to increase his resource in order to ensure a constant value for U_0. As we will explain in next sections, this latter aspect captures the cost of regulation, i.e. the cost of serving users through the neutral operator. Finally, notice that our model can be easily extended for the case that U_0 is a congestible service.

3) Revenue: Each operator $i \in \mathcal{I}$ determines the price $\lambda_i \in R^+$ that he will charge to his clients. The decisions of the operators are realized in a different time scale than the decisions of the users. Namely, each operator i determines his price in the beginning of each time epoch $\mathcal{T}$ which consists of T slots, while users update their operator association decision in each slot. Let us define the price vector $\lambda = (\lambda_i : i = 1, 2, \ldots, I)$ and the vector of the $I-1$ prices of operators other than i as $\lambda_{-i} = (\lambda_j : j \in \mathcal{I} \setminus i)$. We assume that T is large enough so that for each price vector λ set at the beginning of an epoch, the market of the users reaches a stationary point - if such a point is attainable - during this epoch. The objective of each operator $i \in \mathcal{I}$ is to maximize his revenue during each epoch $\mathcal{T}$:

$$R_i(x_i(t), \lambda_i) = \lambda_i x_i(t) N \tag{3}$$

In these markets there are no service level agreements (SLAs) or any other type of QoS guarantees and hence the operators are willing to admit and serve as many users as it is required to achieve their goal.

III. User Strategy and Market Dynamics

A. Evolutionary Game $\mathcal{G}_{\mathcal{U}}$ among Users

In order to select the optimal operator that maximizes eq. (1), each user must be aware of all system parameters, i.e. the spectrum W_i, the number of served users n_i and the charged price λ_i for each $i \in \mathcal{I}$. However, in realistic settings this information will not be available in advance. Given these restrictions and the large number of users, we model their interaction and the operator selection process by defining an evolutionary game, $\mathcal{G}_{\mathcal{U}}$, as follows:

- Players: the set of the N users, $\mathcal{N} = (1, 2, \ldots, N)$.
- Strategies: each user selects a certain operator $i \in \mathcal{I}$ or the neutral operator P_0.
- Population State: the users distribution over the I operators and the neutral operator, $\mathbf{x}(t) = (x_1(t), x_2(t), \ldots, x_I(t), x_0(t))$.
- Payoff: the user's net utility $U_i(W_i, x_i(t), \lambda_i)$ when he selects operator $i \in \mathcal{I}$, or U_0 when he selects P_0.

To facilitate our analysis we make the following assumptions:

- **Assumption 1**: The number of users N is very large, $N >> 1$ and therefore the variable $x_i(t) = n_i(t)/N$ is considered continuous.
- **Assumption 2**: The initial distribution of users over the I operators is non zero: $x_i(0) > 0$, $\forall i \in \mathcal{I}$. It directly follows that $x_0(0) < 1$.

In the sequel we explain how each user selects his strategy under this limited information and what is the outcome of this game.

B. User Strategy Update

A basic component of every evolutionary game is the *revision protocol*, [3]. It captures the dynamics of the interaction among the users and describes in detail the process according to which a player iteratively updates his strategy. There exist many different options for the revision protocol, depending on the modeling assumptions of the specific problem. These assumptions are mainly related to how sophisticated, informed and rational are the players. On the one extreme, fully rational and informed players update their choices according to a best response strategy like in the typical (non-evolutionary) strategic games. This means that players make a *direct selection* of the best available strategy. On the other extreme, players follow an imitation strategy. In this case a player (A) selects randomly another player (B) and if the utility of the latter is higher, (A) imitates his strategy with a probability that is proportional to the anticipated utility improvement. This modeling option is suitable for imperfectly informed players, or players with bounded rationality who update their strategy based on a better (instead of best) response strategy. Between these two extremes, there are many different options. For example, a player may update his strategy with a *hybrid* protocol based partially on imitation and on direct selection, [3].

In this work, we assume that each user updates his strategy by a special type of hybrid revision protocol which is a combination of imitation of other users associated with operators from the set $\mathcal{I}$ (market operators) and direct selection of the neutral operator P_0. The imitation component captures the lack of information users have at their disposal about the market. On the other hand, each user is aware of the exact value of U_0 and hence this choice is always available through direct selection. Notice that the considered revision protocol is not a typical hybrid protocol since the direct selection is related only to the selection of P_0 and not to the other operators.

In detail, the proposed revision protocol can be described by the following actions that each user may take in each slot t:

1) A user associated with an operator $i \in \mathcal{I}$, selects randomly another user who is associated with an operator $j \in \mathcal{I}$, $j \neq i$, and if $U_j > U_i$ imitates his strategy with a probability that is proportional to the difference $(U_j - U_i)$.
2) A user associated with the neutral operator P_0, selects randomly another user associated with operator $j \in \mathcal{I}$ and if $U_j > U_0$, imitates his strategy with a probability that is proportional to the difference $(U_j - U_0)$.
3) A user associated with operator $i \in \mathcal{I}$ selects the neutral operator P_0 with probability that is proportional to the

difference $(U_0 - U_i)$.

Options 1 and 2, stem from the replicator dynamics introduced by Taylor and Jonker in [10] and are based on imitation of users with better strategies. On the other hand, option 3 is based on direct selection of better strategies, known also as pairwise dynamics, introduced by Smith in [23].

After defining the revision protocol, we can calculate the rate at which users switch from one strategy (operator) to another strategy (operator). In particular, the switch rate of users migrating from operator i to operator $j \in \mathcal{I} \setminus i$ in time slot t, is:

$$\rho_{ij}(t) = x_j(t)[U_j(t) - U_i(t)]_+ \tag{4}$$

where $x_j(t)$ is the portion of users already associated with operator j. For simplicity, we express the user utilities as a function with a single argument, the time t. Additionally, the users switch rate from operator i to neutral operator P_0, is:

$$\rho_{i0}(t) = \gamma[U_0 - U_i(t)]_+ \tag{5}$$

Notice the difference between imitation and direct selection [3]. Instead of multiplying the utilities difference with the population $x_0(t)$, we use a constant multiplier $\gamma \in R$. This is due to the model assumption that switching to the neutral operator is not accomplished through imitation and hence does not depend on the portion of users already been associated with P_0. The probabilistic aspect captures the bounded rationality, the inertia of the users and other similar realistic aspects of these markets. Finally, the switch rate of users leaving P_0 and returning to the market (option 2) is:

$$\rho_{0i}(t) = x_i(t)[U_i(t) - U_0]_+ \tag{6}$$

Variables ρ_{ij}, ρ_{i0} and ρ_{0i} represent the rates at which users migrate from one operator to another, including the neutral operator P_0. It is interesting to notice that if these rates are normalized properly, they can be interpreted as the probabilities with which users update their operator selection strategy. This approach is discussed in [3]. In the sequel we use these rates to derive the ordinary differential equations (ODE) that describe the evolution of the population of users.

C. Market Stationary Points

The new type of hybrid revision protocol introduced above, results in user market dynamics that cannot be expressed with the known differential equations of replicator dynamics or other similar scheme, [3]. In Section A of the Appendix we prove that the mean dynamics of the system are:

$$\begin{aligned} \frac{dx_i(t)}{dt} &= x_i(t)[U_i(t) - U_{avg}(t) - x_0(t)(U_i(t) - U_0) \quad (7)\\ &- \gamma(U_0 - U_i(t))_+ + x_0(t)(U_i(t) - U_0)_+], \forall i \in \mathcal{I} \end{aligned}$$

where $U_{avg}(t) = \sum_{i\in\mathcal{I}} x_i(t)U_i(t)$ is the average utility of the market in each slot t. The user population associated with P_0 is:

$$\frac{dx_0(t)}{dt} = x_0 \sum_{i\in\mathcal{I}^+} x_i(U_0 - U_i) + \gamma \sum_{j\in\mathcal{I}^-} x_j(U_0 - U_j) \tag{8}$$

where $\mathcal{I}^+$ is the subset of operators offering utility $U_i(t) > U_0$, and $\mathcal{I}^-$ is the subset of operators offering utility $U_i(t) < U_0$, at slot t.

The important thing is that despite its different evolution, as we prove in Section B, this system has the same stationary points as the systems that are described by the classical replicator dynamic equations:

$$\dot{x}_i(t) = 0 \Rightarrow x_i(t)[U_i(t) - U_{avg}(t)] = 0, \forall i \in \mathcal{I} \tag{9}$$

and

$$\dot{x}_0(t) = 0 \Rightarrow x_0(t)[U_0 - U_{avg}(t)] = 0 \tag{10}$$

The user state vector $\mathbf{x}^*$ and the respective user utility U_i^*, $i \in \mathcal{I}$, that satisfy these stationary conditions can be summarized in the following 3 cases:

- **Case A**: x_i^*, $x_0^* > 0$ and $U_i^* = U_0$, $i \in \mathcal{I}$.
- **Case B**: x_i^*, $x_j^* > 0$, $x_0^* = 0$ and $U_i^* = U_j^*$, with U_i^*, $U_j^* > U_0, \forall\, i,\, j \in \mathcal{I}$.
- **Case C**: x_i^*, $x_j^* > 0$, $x_0^* = 0$ and $U_i^* = U_j^* = U_0$, $\forall\, i,\, j \in \mathcal{I}$.

Case A corresponds to the scenario where all operators offer to their clients net utility which is equal to the value of the service offered by the neutral operator. On the other hand, in case B the market operators offer higher utility than the neutral operator and hence all users are served by the market. Finally, in case C, the I operators offer marginal services, i.e. equal to U_0, but they have attracted all the users.

It is interesting to compare the above results with the Wardrop model and the Wardrop equilibrium, [20]. The market stationary points for **Case A** and **Case C** satisfy the *Wardrop first principle* and yield an equilibrium where the available strategy options (”operators” in our problem) result in equal utility for the players (”users”). However, this does not hold for **Case B** where operators other than P_0 offer higher utility. This emerges due to the fact that the alternative option (or reservation utility) is non-congestible, i.e. independent of x_0. The evolutionary game allows us to provide a richer model than the typical Wardrop model and more importantly to capture the users interaction and dynamics.

Before calculating the stationary point $\mathbf{x}^*$ for each case, and in order to facilitate our analysis, we define the scalar parameter $\alpha_i = W_i/(Ne^{U_0})$ for each operator $i \in \mathcal{I}$ and the respective vector $\alpha = (\alpha_i : i = 1, 2, \ldots, I)$. As it will be explained in the sequel, these parameters determine the operators and users interaction and will help us to explain the role of the regulator. We can find the stationary points for **Case A** by using equation $U_i(W_i, x_i^*, \lambda_i) = U_0$ and imposing the constraint $x_0^* > 0$. Apparently, the state vector $\mathbf{x}^*$ depends on the price vector λ. Therefore, we define the set of all possible **Case A** stationary points, X_A, as follows (see Section B for details):

$$X_A = \left\{ x_i^* = \alpha_i e^{-\lambda_i}, \forall i \in \mathcal{I}, x_0^* = 1 - \sum_{i=1}^{I} \alpha_i e^{-\lambda_i} : \lambda \in \Lambda_A \right\}$$

TABLE I
WIRELESS SERVICE MARKET STATIONARY POINTS.

	X_A	X_B	X_C
x_i^*	$\alpha_i e^{-\lambda_i}$	$\frac{\alpha_i}{e^{\lambda_i}\sum_{j=1}^{I}\alpha_j e^{-\lambda_j}}$	$\alpha_i e^{-\lambda_i}$
x_0^*	$1-\sum_{i=1}^{I}\alpha_i e^{-\lambda_i}$	0	0
Cond.	$\lambda \in \Lambda_A$	$\lambda \in \Lambda_B$	$\lambda \in \Lambda_C$

where Λ_A is the set of prices for which a stationary point in X_A is attainable, i.e. for which it holds $x_0^* > 0$:

$$\Lambda_A = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_I) : \sum_{i=1}^{I} \alpha_i e^{-\lambda_i} < 1 \right\}$$

Recall that due to the very large number of users, we consider x_i a continuous variable.

Similarly, for **Case B**, we calculate the stationary points by using the set of equations $U_i(W_i, x_i^*, \lambda_i) = U_j(W_j, x_j^*, \lambda_j)$, $\forall\, i, j \in \mathcal{I}$:

$$X_B = \left\{ x_i^* = \frac{\alpha_i}{e^{\lambda_i}\sum_{j=1}^{I}\alpha_j e^{-\lambda_j}}, \forall i \in \mathcal{I}, x_0^* = 0 :\ \lambda \in \Lambda_B \right\}$$

where Λ_B is the set of prices for which a stationary point in X_B is feasible, i.e. $U_i^* > U_0$:

$$\Lambda_B = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_I) : \sum_{i=1}^{I} \alpha_i e^{-\lambda_i} > 1 \right\}$$

Finally, the stationary points for the **Case C** solution must satisfy the constraint $\sum_{i=1}^{I} \alpha_i e^{-\lambda_i} = 1$ which yields:

$$X_C = \left\{ x_i^* = \alpha_i e^{-\lambda_i}, \forall\, i \in \mathcal{I}, x_0^* = 0 :\ \lambda \in \Lambda_C \right\}$$

with

$$\Lambda_C = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_I) : \sum_{i=1}^{I} \alpha_i e^{-\lambda_i} = 1 \right\}$$

Notice that the stationary point sets X_A, X_B and X_C and the respective price sets, Λ_A, Λ_B, and Λ_C depend on the vector α. These results are summarized in Table I. For each operators price profile λ, the evolutionary game admits a unique stationary point $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_I^*, x_0^*)$ which belongs in the respective set X_A, X_B, or X_C. The utility of the users is equal to U_0 for the **Case A** and **Case C**, while for **Case B** it depends on λ.

1) Stability of Stationary Points: Now that we found the stationary points of the hybrid revision protocol, it is important to characterize their stability. We prove in the sequel that these points are Evolutionary Stable Strategies (ESS) and hence they are locally asymptotically stable, i.e. stable within a limited region. ESS and replicator dynamics are the two concepts used for studying evolutionary games. Unlike the replicator dynamics, ESS is a static concept which requires that the strategy of players in the equilibrium is stable when it is invaded by a small population of players playing a different strategy, [22]. When the players population is homogeneous, as we assumed in our model, an ESS is stable in the replicator dynamic, but not every stable steady state is an ESS. Additionally, every ESS is Nash, and hence ESS is a refinement of the Nash equilibrium.

Let us first give a simple definition of the ESS, tailored to our system model. Assume that the users market has reached the stationary state described by vector $\mathbf{x}^*$. Suppose now that a small portion $\epsilon > 0$ of the users population deviates from their decision in the stationary state (i.e. selects another operator) and selects another operator $j \in \mathcal{I}$ or the neutral operator. This yields a new distribution of users which we denote by $\mathbf{x}_\epsilon = (x_1^\epsilon, x_2^\epsilon, \ldots, x_I^\epsilon, x_0^\epsilon)$. We say that $\mathbf{x}^*$ is an ESS if (i) users that deviate from $\mathbf{x}^*$ receive lower utility in the new system state $\mathbf{x}_\epsilon$ or, (ii) the utility of the deviating users in $\mathbf{x}_\epsilon$ is the same as in the previous state $\mathbf{x}^*$, but the utility of the legitimate users (those insisting in their initial decisions) is higher in $\mathbf{x}_\epsilon$ than in $\mathbf{x}^*$. In both cases, the deviating users worsen their obtained utility. The stationary points derived above satisfy these conditions and hence they are ESS.

In detail, assume that the system has a stationary point $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_I^*, x_0^*) \in X_B$, with $x_0^* = 0$. Suppose that a user who is associated with operator $i \in \mathcal{I}$ deviates and selects another operator $j \in \mathcal{I}$. In this case, the population of users in operator i decreases, $x_i^\epsilon < x_i^*$ and the population of users in operator j increases, $x_j^\epsilon > x_j^*$. Initially, these two operators offered identical utility, $U_i(W_i, x_i^*, \lambda_i) = U_j(W_j, x_j^*, \lambda_j)$ but after the decision of the deviating user it becomes $U_i(W_i, x_i^\epsilon, \lambda_i) > U_j(W_j, x_j^\epsilon, \lambda_j)$. Clearly, the deviating user obtains less utility and hence there is no incentive to deviate. Similarly, if a user deviates and selects the neutral operator, he will receive reduced utility since when $\mathbf{x}^* \in X_B$, it is $U_i^* > U_0$, $\forall i \in \mathcal{I}$.

Assume now that the system attains a stationary point $\mathbf{x}^* \in X_A$. Similarly to the previous analysis, it is straightforward that a user who deviates from $\mathbf{x}^*$ and moves from an operator $i \in \mathcal{I}$ to another operator $j \in \mathcal{I}$ will decrease his utility. If the user migrates to the neutral operator, his utility will not be reduced because U_0 is constant (non-congestible). However, in this case, the users that will insist in their initial choice of operator i will now receive higher utility due to the move of the deviator. Due to the ESS definition and specifically according to Smith's second condition, [24], this is not a preferable choice for the deviator and hence $\mathbf{x}^* \in X_B$ is an ESS.

Finally, when $\mathbf{x}^* \in X_C$, user deviation from a market operator $i \in \mathcal{I}$ to another market operator $j \in \mathcal{I}$ or to P_0 is not beneficial for the deviator, either because it decreases his utility or because it increases the utility of other users. In conclusion, the stationary points of the proposed revision protocol are ESS equilibriums and hence locally stable.

IV. PRICE COMPETITION AMONG OPERATORS

In the previous section we analyzed the stationary points of users interaction and showed that they depend on the prices selected by the operators. Each operator anticipates the users strategy and chooses accordingly for each epoch $\mathcal{T}$ the price that maximizes his revenue. This gives rise to a non-cooperative price competition game $\mathcal{G}_\mathcal{P}$ among the operators that is played in the beginning of each time epoch $\mathcal{T}$. We assume that operators are aware of the parameters of the users market and also know the values of parameters α_i, $i \in \mathcal{I}$ and U_0. Specifically, we model the operators competition as a static simultaneous move normal form game of complete information, following the Bertrand competition model [6]. We are interested not only in finding the Nash equilibriums (NE) of this game but also to understand if and how the game converges to them.

We prove that $\mathcal{G_P}$ is a potential game and hence if it is played in many rounds and operators choose their prices based on the previous prices of the other operators, the game converges to a NE. In other words, we analyze the dynamics induced by the repeated play of the same game assuming that operators follow simple myopic rules. We show that the equilibrium of the competition game depends on vector α and the value of U_0. For certain combinations of these parameters, the game admits a unique equilibrium while for other combinations, it reaches one of the infinitely many equilibriums depending on the initial prices.

A. Price Competition Game $\mathcal{G_P}$

Before analyzing this game, it is important to emphasize that the revenue function depends on the price vector λ. In particular, using equation (3), we can calculate the revenue of operator i when $\lambda \in \Lambda_A$, when $\lambda \in \Lambda_B$, and when $\lambda \in \Lambda_C$, denoted as $R_i^A(\cdot)$, $R_i^B(\cdot)$ and $R_i^C(\cdot)$ respectively:

$$R_i^A(\lambda_i) = R_i^C(\lambda_i) = \alpha_i \lambda_i N e^{-\lambda_i} \tag{11}$$

and

$$R_i^B(\lambda_i, \lambda_{-i}) = \frac{\alpha_i \lambda_i N}{e^{\lambda_i} \sum_{i=1}^{I} \alpha_i e^{-\lambda_i}} \tag{12}$$

$R_i^A(\cdot)$ and $R_i^C(\cdot)$ depend only on the price selected by operator i, while $R_i^B(\cdot)$ depends on the entire price vector λ. However, in all cases, the price set (Λ_A, Λ_B or Λ_C) to which the price vector $\lambda = (\lambda_i, \lambda_{-i})$ belongs, is determined jointly by all the I operators.

Let us now define the non-cooperative **Pricing Game** among the I operators, $\mathcal{G_P} = (\mathcal{I}, \{\lambda_i\}, \{R_i\})$:

- The set of *Players* is the set of the I operators $\mathcal{I} = (1, 2, \ldots, I)$.
- The *strategy space* of each player i is its price $\lambda_i \in [0, \lambda_{max}]$, $\lambda_{max} \in \mathcal{R}^+$, and the strategy profile is the price vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_I)$ of the operators.
- The payoff function of each player is his revenue $R_i : (\lambda_i, \lambda_{-i}) \to \mathcal{R}$, where $R_i = R_i^A$ or R_i^B or R_i^C.

The particular characteristic of this game is that each operator has 2 different payoff functions depending on the price profile. Despite this characteristic, the payoff function is continuous and quasi-concave as we prove in the Appendix, Section C. In the sequel, we analyze the best response of each operator which constitutes a reaction curve to the prices set by the other operators. The equilibrium of the game $\mathcal{G_P}$ is the intersection of the reaction curves of the operators.

B. Best Response Strategy of Operators

The best response of each operator i, λ_i^*, to the prices selected by the other $I-1$ operators, λ_{-i}, depends on the users market stationary point. Notice that for certain λ_{-i}, operator i may be able to select a price such that $(\lambda_i, \lambda_{-i})$ belongs to any price set (Λ_A, Λ_B or Λ_C) while for some λ_{-i} the operator choice will be restricted in two or even a single price set.

Best Response when $\lambda \in \Lambda_A$**:** If the $I-1$ operators $j \in \mathcal{I} \backslash i$ select such prices, λ_{-i}, that the market stationary point is $\mathbf{x}^* \in X_A$, then operator i finds the price λ_i^* that maximizes his revenue $R_i^A(\cdot)$ by solving the following constrained optimization problem ($\mathbf{P}_i^A$):

$$\max_{\lambda_i \geq 0} \alpha_i \lambda_i N e^{-\lambda_i} \tag{13}$$

s.t.

$$\sum_{j=1}^{I} \alpha_j e^{-\lambda_j} < 1 \tag{14}$$

The objective function of this problem is quasi-concave, [25]. However, the constraint defines an open set and hence uniqueness of optimal solution is not ensured. To overcome this obstacle we substitute constraint eq. (14) with the closed set:

$$\lambda_i \geq \log \frac{\alpha_i}{1 - \sum_{j \neq i} \alpha_j e^{-\lambda_j}} + \epsilon \tag{15}$$

where $\epsilon > 0$ is an arbitrary small constant number. This inequality stems from eq. (14) by solving for λ_i and adding ϵ. It does not affect the problem definition and formulation nor the obtained results since, as we will prove in the sequel, operators do not select a price in the lower bound of the constraint set. After this transformation the problem has a unique optimal solution which is equal to the solution of the respective unconstrained problem, $\lambda_i^* = 1$, if $(1, \lambda_{-i}) \in \Lambda_A$.

Best Response when $\lambda \in \Lambda_B$**:** Similarly, when λ_{-i} is such that operator i can select a price λ_i^* with $(\lambda_i^*, \lambda_{-i}) \in \Lambda_B$, then his revenue is given by the function $R_i^B(\cdot)$ and is maximized by the solution of problem ($\mathbf{P}_i^B$):

$$\max_{\lambda_i \geq 0} \frac{\lambda_i \alpha_i N}{e^{\lambda_i} \sum_{j \in \mathcal{I}} \alpha_j e^{-\lambda_j}} \tag{16}$$

s.t.

$$\sum_{j=1}^{I} \alpha_j e^{-\lambda_j} > 1 \tag{17}$$

This is also a concave problem which would have a unique solution if the constraint set was closed and compact. Again, we substitute the constraint with the (almost) equivalent inequality:

$$\lambda_i \leq \log \frac{\alpha_i}{1 - \sum_{j \neq i} \alpha_j e^{-\lambda_j}} - \epsilon \tag{18}$$

Now, the problem has a unique solution which coincides with the solution of the respective unconstrained problem, denoted μ_i^*, if $(\mu_i^*, \lambda_{-i}) \in \Lambda_B$ as we explain in detail in Section D.

Best Response when $\lambda \in \Lambda_C$**:** In this special case, the price of each operator i is directly determined by the prices that the other operators have selected. Namely, given the vector λ_{-i}, each operator i has only one feasible solution (otherwise λ does not belong to Λ_C):

$$\lambda_i^* = \log \frac{\alpha_i}{1 - \sum_{j\neq i} \alpha_j e^{-\lambda_j}} \tag{19}$$

Whether each operator i will agree and adopt this price or not, depends on the respective accrued revenue $R_i^C(\lambda_i^*, \lambda_{-i})$.

We can summarize the best response price strategy of each operator $i \in \mathcal{I}$, by defining his revenue function as follows:

$$R_i(\lambda_i, \lambda_{-i}, \alpha) = \begin{cases} \frac{\alpha_i \lambda_i N}{\sum_{j=1}^{I} \alpha_j e^{\lambda_i - \lambda_j}} & \text{if } \lambda_i < l_0, \\ \alpha_i \lambda_i N e^{-\lambda_i} & \text{if } \lambda_i \geq l_0. \end{cases} \tag{20}$$

where $l_0 = \log(\alpha_i/(1 - \sum_{j\neq i} \alpha_j e^{-\lambda_j}))$. Clearly, the optimal price λ_i^* depends both on the prices of the other operators λ_{-i} and on parameters α_i, $i = 1, 2, \ldots, I$:

$$\lambda_i^* = arg \max_{\lambda_i} R_i(\lambda_i, \lambda_{-i}, \alpha) \tag{21}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_I)$. Clearly, each operator needs to know the vector α and to be able to observe the other operators prices in order to calculate his best response.

For each possible price vector λ_{-i} of the $\mathcal{I} \setminus i$ operators, operator i will solve all the above optimization problems and find the solution that yields the highest revenue. In Section D we prove that this results in the following best response strategy:

$$\lambda_i^*(\lambda_{-i}, \alpha) = \begin{cases} 1 & \text{if } (1, \lambda_{-i}) \in \Lambda_A, \\ \mu_i^* & \text{if } (\mu_i^*, \lambda_{-i}) \in \Lambda_B, \\ l_0 & \text{otherwise.} \end{cases} \tag{22}$$

These options are mutually exclusive. Moreover, if $\sum_{j\neq i} \alpha_j / e^{\lambda_j} \geq 1$, the only feasible response is $\lambda_i^* = \mu_i^*$. The dependence of λ_i^* on parameters $\alpha_i = W_i/(N e^{U_0})$, $i = 1, 2, \ldots, I$, has interesting implications and brings into the fore the role of the regulator. Finally, observe that the transformation of the constraint set of problems $\mathbf{P}_i^A$ and $\mathbf{P}_i^B$ did not affect the best response strategy of operator i since he only selects the solution of the respective unconstrained problems.

C. Equilibrium Analysis of $\mathcal{G}_{\mathcal{P}}$

The price competition game $\mathcal{G}_{\mathcal{P}}$ is a finite ordinal potential game and therefore not only has pure Nash equilibria but also the players can reach them under any best response strategy. That is, if we consider that $\mathcal{G}_{\mathcal{P}}$ is played repeatedly by the operators who update their strategy with a myopic best response method, we can show that the convergence to the equilibriums is ensured under any finite improvement path (FIP), [12]. The potential function is:

$$\mathcal{P}(\lambda) = \begin{cases} \sum_{j=1}^{I} [\log \lambda_j - \lambda_j], & \text{if} \sum_{j=1}^{I} \alpha_j e^{-\lambda_j} \leq 1, \\ \sum_{j=1}^{I} [\log \lambda_j - \lambda_j] - \log\left(\sum_{j=1}^{I} \alpha_j e^{-\lambda_j}\right), & \text{else.} \end{cases}$$

The detailed proof is given in Section E. In order to find the NE we solve the system of equations (22), $i = 1, 2, \ldots, I$ and specifically we use the iterated dominance method (Section F).

TABLE II
EQUILIBRIUMS OF I OPERATORS COMPETITION FOR DIFFERENT VALUES OF α.

Prices/Rev.	$\alpha \in A_1$	$\alpha \in A_2$	$\alpha \in A_3$
λ_i^*	1	$\lambda_i \neq \lambda_j$ or $\lambda_i = \lambda_j = \log I\alpha$	$\frac{I}{I-1}$
R_i^*	$\frac{\alpha N}{e}$	$R_i \neq R_j$ or $R_i = R_j = \frac{N}{I} \log I\alpha$	$\frac{N}{I-1}$
$\mathbf{x}^*$	X_A	X_C	X_B

The outcome of the game $\mathcal{G}_{\mathcal{U}}$ affects the strategy of operators and therefore the outcome of the game $\mathcal{G}_{\mathcal{P}}$. A price vector $(\lambda_i^*, \lambda_{-i}^*)$ is an equilibrium of the game $\mathcal{G}_{\mathcal{P}}$, parameterized by the vector $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_I)$, if it satisfies:

$$R_i(\lambda_i^*, \lambda_{-i}^*, \alpha) \geq R_i(\lambda_i, \lambda_{-i}^*, \alpha), \forall\, i \in \mathcal{I},\ \forall \lambda_i \geq 0,\, \forall\, \mathbf{x}^*$$

In order to simplify our study and focus on the results and implications of our analysis, we assume that all operators have the same amount of available spectrum $W_i = W$ and therefore it is also $\alpha_i = \alpha$, $\forall i \in \mathcal{I}$.

The equilibrium of the price competition game and subsequently the market stationary point $\mathbf{x}^*$, depend on the value of α. These results are summarized in Table II and stem from the following Theorem:

Theorem IV.1. *The non-cooperative game $\mathcal{G}_{\mathcal{P}}$ where operators select their strategy in order to maximize their revenue, converges to one of the following pure Nash equilibria:*

- *If $\alpha \in A_1 = (0, e/I)$, there is a unique Nash Equilibrium $\lambda^* \in \Lambda_A$, with $\lambda^* = (\lambda_i^* = 1 : i = 1, 2, \ldots, I)$ and the respective unique market stationary point is $\mathbf{x}^* \in X_A$.*
- *If $\alpha \in A_3 = (e^{\frac{I}{I-1}}/I, \infty)$, there is a unique Nash Equilibrium $\lambda^* \in \Lambda_B$, with $\lambda^* = (\lambda_i^* = \frac{I}{I-1} : i = 1, 2, \ldots, I)$, which induces a unique respective market stationary point $\mathbf{x}^* \in X_B$.*
- *If $\alpha \in A_2 = [e/I, e^{\frac{I}{I-1}}/I]$, there exist infinitely many equilibria, $\lambda^* \in \Lambda_C$, and each one of them yields a respective market stationary point $\mathbf{x}^* \in X_C$.*

Proof: In Section E of the Appendix we provide the detailed proof according to which $\mathcal{G}_{\mathcal{P}}$ is a potential game and in Section F we use iterated strict dominance to find the Nash equilibrium λ^* which depends on parameter α. ■

In conclusion, $\mathcal{G}_{\mathcal{P}}$ is a non-cooperative game of complete information that attains certain pure Nash equilibriums (NE) which depend on parameters α_i, $i = 1, 2, \ldots, I$. It is proved to be a potential game and hence the equilibriums can be reached if $\mathcal{G}_{\mathcal{P}}$ is played repeatedly and operators update their strategy by simple best response or other similar utility improvement methods. If α_i parameters are equal, i.e. $\alpha_i = \alpha$, $\forall i \in \mathcal{I}$, then the NE is unique for $\alpha \in A_1$ or $\alpha \in A_3$. For the case $\alpha \in A_2$, the reached equilibrium depends on the initial price vector.

V. Market Outcome and Regulation

The outcome of the users and operators interaction can be characterized by the following two fundamental criteria: the efficiency of the users market and the total revenue the operators accrue. We show that both of them depend on parameter α and we further explore the impact of W and U_0 on them. Accordingly, we analyze the problem from a mechanism design perspective and explain how a regulator, as the municipal WiFi provider in the introductory example, can bias the market operation (outcome) by adjusting the value of α. We consider different regulation methods and discuss their implications.

A. Market Outcome and Regulation Criteria

1) Market Efficiency: A market is efficient if the users enjoy high utilities in the stationary point. However, in certain scenarios, the services provided by the P_0 may impose an additional cost to the system (e.g. the cost of the municipal WiFi provider is borne by the citizens) and hence it would be preferable to have all the users served by the I operators. Therefore, we use the following two metrics to characterize the efficiency of the market: **(i)** the aggregate utility (U_{agg}) of users in the stationary point $\mathbf{x}^*$, and **(ii)** the cost $J_0 = x_0 N U_0$ incurred by the neutral operator P_0 for serving the portion x_0 of the users. Both of these metrics depend on parameter α and hence on system parameters W and U_0.

In detail:

- When $\alpha \in A_1 = (0, e/I)$, it is $\mathbf{x}^* \in X_A$, which means that a portion of users $x_0^* >$ selects P_0. The latter incurs cost of $J_0 = x_0^* N U_0$ units. All users receive utility of U_0 units and hence the aggregate utility is $U_{agg} = NU_0$.
- On the other hand, when $\alpha \in A_2 = [e/I, e^{I/(I-1)}/I]$, it is $\mathbf{x}^* \in X_C$. In this case, all users are served by the I operators with marginal utility, i,e. $U_i^* = U_0$ for $i = 1, 2, \ldots, I$. There is no cost for P_0, i.e. $J_0 = 0$. Again, it is $U_{agg} = NU_0$ but unlike the previous case, there is no cost for P_0.
- Finally, if $\alpha \in A_3 = (e^{I/(I-1)}/I, \infty)$ it is $\mathbf{x}^* \in X_B$. All users are served by the I operators, i.e. $x_0^* = 0$ and $J_0 = 0$, and receive high utilities $U_i^* > U_0$, $i = 1, 2, \ldots, I$. The welfare is higher in this case, i.e. $U_{agg} > NU_0$.

In summary, the aggregate utility of the users changes with α as follows:

$$U_{agg} = \begin{cases} NU_0, & \text{if } \alpha \in A_1 \cup A_2, \\ N(log(\frac{WI}{N}) - \frac{I}{I-1}), & \text{if } \alpha \in A_3. \end{cases} \tag{23}$$

It can be easily verified that U_{agg} is a continuous function.

We have expressed U_{agg} in terms of W and U_0 in order to investigate the impact of the system parameters in the market. When $\alpha \in A_1 \cup A_2$, U_{agg} increases with U_0 and is independent of the spectrum W. On the contrary, when $\alpha \in A_3$, U_{agg} increases with W and is independent of U_0. Notice that when the value of α changes from A_1 to interval A_2, U_{agg} remains the same but the other metric of efficiency, the cost of neutral operator J_0, is improved:

$$J_0 = \begin{cases} \frac{\alpha I N U_0}{e}, & \text{if } \alpha \in A_1, \\ 0, & \text{if } \alpha \in A_2 \cup A_3. \end{cases} \tag{24}$$

2) Revenue of Operators: When α lies in the interval A_1, the optimal prices are $\lambda_i^* = 1$, $\forall i \in \mathcal{I}$ and all the operators accrue the same revenue $R_i^* = \alpha N e^{-1} = W e^{-(U_0+1)}$, which is proportional to α, increases with the available spectrum W, decreases with U_0 and is independent of the number N of users. In Figure 4 we depict the revenue of each operator for different values of α, in a duopoly market. Notice that the revenue increases linearly with $\alpha \in (0, e/2)$.

When $\alpha \in A_2$, the competition of the operators may attain different equilibria, $\lambda^* \in \Lambda_C$, depending on the initial prices and on the sequence the operators update their prices. In Figure 5 we present the revenue of two operators (duopoly) at the equilibrium, for various initial prices and for $\alpha = e \in A_2$. Here we assume that the 1^{st} operator is able to set his price $\lambda_1(0)$ before the 2^{nd} operator. Also, in Figure 4 we illustrate the dependence of the revenue of the operators on the value of α when it lies in A_2, given that $\lambda_1(0) = 1.1$. For certain prices, e.g. when $\lambda_1(0) = \log 2\alpha$, both operators accrue the same revenue at the equilibrium, $R_1^* = R_2^* = \frac{N \log 2\alpha}{2}$.

If $\alpha \in A_3 = (e^{I/(I-1)}, \infty)$ all operators set their prices to $\lambda_i^* = I/(I-1)$ and get $R_i^* = N/(I-1)$ units, as shown in Table II. Figure 6 depicts the competition of two operators and the convergence to the respective Nash equilibria for $\alpha = e^3 \in A_3$. We assume that both operators have selected prices $\lambda_1(0) = \lambda_2(0) = \log 2\alpha \approx 3.7$. However, this price vector does not constitute a NE and hence an operator (e.g. the 1^{st}) can temporarily increase his revenue by decreasing his price to $\lambda_1 = 3$. Accordingly, the other operator (2^{nd}) will react by reducing his price to $\lambda_2 = 2.5$. Gradually, the competition of the operators will converge to the NE where both of them will set $\lambda_1^* = \lambda_2^* = 2/(2-1) = 2$ and will have revenue $R_1^* = R_2^* = 1$. Interestingly, the revenue of both operators in the equilibrium is lower than their initial revenue when they did not compete. Finally, notice that, unlike the aggregate utility U_{agg}, the revenue of the operators depends only on $\alpha = W/(Ne^{U_0})$ and not the specific values of W and U_0.

Before we proceed, let us summarize the above results:

- If $\alpha \in A_1 = (0, e/I)$, it is $R_i^* = \alpha N e^{-1} = W e^{-(U_0+1)}$, $i = 1, 2, \ldots, I$. Operators receive equal revenue which is **(i)** proportional to W, **(ii)** inversely proportional to U_0 and **(iii)** independent of the number of users N.
- If $\alpha \in A_2 = [e/I, \frac{e^{I/(I-1)}}{I})$, R_i^* depends on the initial prices operators select. In the particular case that a single operator i sets first his price λ_i so as to be $\lambda_i(0) = \log I\alpha$, then all operators obtain finally equal revenue $R_i^* = \frac{N \log I\alpha}{I}$.
- If $\alpha \in A_3 = [\frac{e^{I/(I-1)}}{I}, \infty)$, it is $R_i^* = \frac{N}{I-1}$. Operators receive equal revenue which is **(i)** proportional to N, **(ii)** independent of U_0 and W.

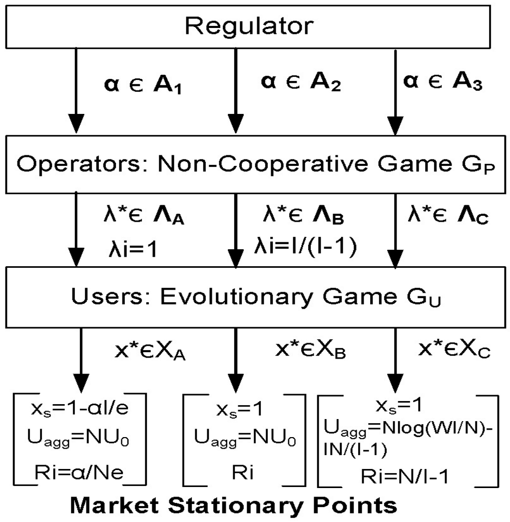

Fig. 3. The regulator selects parameter α, the operators compete and select the respective optimal prices λ_i^*, and then, the users are divided among the operators.

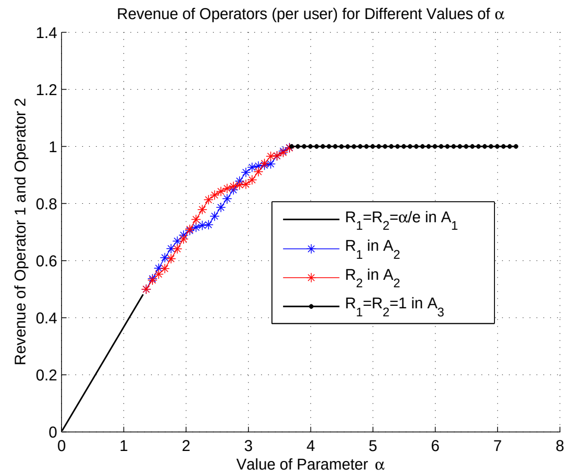

Fig. 4. The outcome of the operator competition ($\mathcal{G}_\mathcal{P}$ equilibrium) for different values of parameter α, i.e. in different intervals.

B. Regulation of the Wireless Service Market

Since both the market efficiency and the operator revenue depend on α and system parameters W and U_0, a regulating agency can act as a *mechanism designer* and steer the outcome of the market in a more desirable equilibrium according to his objective. This can be achieved by determining directly or indirectly (e.g. through pricing) the amount of spectrum W each operator has at his disposal, or by intervening in the market and setting the value U_0 as the example with the municipal WiFi Internet provider. This process is depicted in Figure 3.

1) Regulating to Increase Market Efficiency: First, we highlight the impact of parameters W and U_0 on the efficiency metrics. This is of crucial importance because tuning W or U_0 has different implications for the regulator and the market. For example, as it is explained below, the regulator can achieve the same level of market efficiency either by selling more spectrum to operators, e.g. by decreasing the spectrum price, or by allocating more spectrum to the neutral operator:

- *Assume that U_0 is fixed*. As the allocated spectrum W to each operator increases, aggregate utility U_{agg} remains constant until parameter α increases up to $\alpha \geq \frac{e^{I/(I-1)}}{I}$. When $\alpha \in A_3$, U_{agg} is log-proportional to W. Also, the cost J_0 increases with W, as long as $\alpha \in A_1$, and becomes zero for larger values of α.
- *Assume that W is fixed*. U_{agg} increases with U_0 as long as $\alpha \in A_1 \cup A_2$. For larger values of α, U_{agg} does not depend directly on U_0. Additionally, the cost J_0 increases with U_0 as long as $\alpha \in A_1$ while for larger values of α it becomes zero.

Let us now give a specific scenario for regulation. Assume that initially $\alpha \in A_1 = (0, e/I)$. Hence, a portion of users is not served by anyone of the I operators, $x_0^* > 1$ and all the users receive utility equal to U_0. The regulator can improve the market efficiency, i.e. increase U_{agg} and decrease J_0, by increasing the value of α. This can be achieved either by increasing W or decreasing U_0. Let us assume that the regulator selects the first method. For example, he can change the price of W and allow the operators to acquire more spectrum. If W is increased until $\alpha = e/I$, then the market stationary point $\mathbf{x}^*$ switches to X_B. In this case, all users are served by the market, $x_0^* = 0$, but they still receive only marginal utility, $U_{agg} = NU_0$. If the regulator provides even more spectrum W to operators so as $\alpha > e^{I/(I-1)}/I$, then $x_0^* = 0$ and moreover the users perceive higher utility because U_{agg} increases proportional to $\log W$, eq. (23). Obviously, the improvement in market efficiency comes at the cost (*opportunity cost*) of the additional spectrum the regulator must provide to operators.

On the other hand, the regulator may prefer to directly intervene in the market through P_0 and tune U_0. If U_0 decreases, the value of x_0^* decreases and users return to the market (to the I operators). The portion of users x_0^* becomes zero when $\alpha = e/I$. This way, the cost of the regulator J_0 decreases (since P_0 serves less users) but at the same time the aggregate utility, $U_{agg} = NU_0$, is also reduced. Namely, U_{agg} decreases linearly with U_0 until $\alpha = e^{I/(I-1)}/I$ and remains constant for larger values of U_0, eq. (23). Again, the decision of the regulator depends on his cost and on the efficiency he wants to achieve. In conclusion, depending on they system parameters (N, W, I) the efficiency of the market may be improved either by increasing the resources of operators (sell more spectrum) or by rendering highly competitive the services provided by the neutral operator P_0.

2) Regulating for Revenue: As illustrated in Table II, the revenue of the operators increases proportionally to α for $\alpha \in A_1$, and proportionally to $\log \alpha$ for $\alpha \in A_2$, while it remains constant when $\alpha \in A_3$. Notice that the revenue, unlike the market efficiency, depends on the value of α and

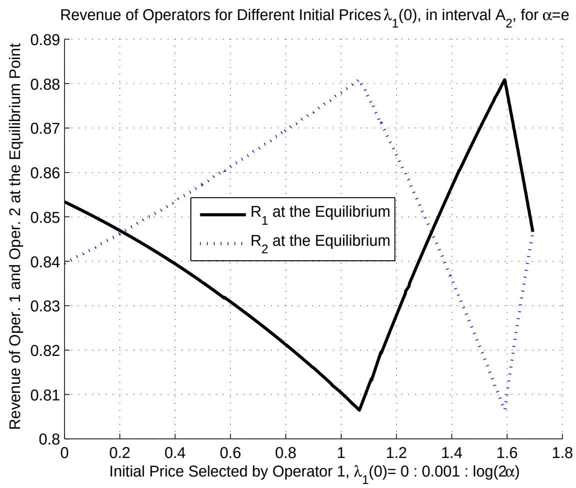

Fig. 5. The outcome of the competition of two operators, with $\alpha = e \in A_2$ and $N = 1000$. Operator 1 is assumed to set his price $\lambda_1(0)$ first. R_1^* and R_2^* depend on $\lambda_1(0)$.

not on the specific combination of W and U_0. These results are presented in Figure 7 for a market with $I = 3$ operators and $N = 1000$ users. In the upper plot, it is $U_0 = 0.1$ and the regulator increases the value of α by increasing W. The aggregate utility is constant and equal to $U_{agg} = NU_0 = 100$ for $\alpha < e^{3/(3-1)}/3 \approx 1.5$ while it increases proportionally to $\log W$ for $\alpha > 1.5$. Obviously, increasing the spectrum of operators improves both their revenue and the efficiency of the market.

In the lower plot, the spectrum at the disposal of each operator is constant, $W = 5000$, and the regulator increases the value of α by decreasing U_0. In this case, the total revenue increases but at the expense of market efficiency. When $\alpha \in A_1 \cup A_2 = (0, e^{1.5}/3]$, the aggregate utility U_{agg} is reduced as U_0 decreases but for $\alpha > e^{1.5}/3$ it remains constant. Notice that for very small values of α, U_{agg} is large. However, this desirable result comes at a cost for the regulator. Namely, in this case only a small portion of users are served by the market, while the rest of them select P_0. Therefore, the incurred cost J_0 for the regulator is high.

Another interesting point in Figure 7 is the following. In the upper subplot, for $U_0 = 0.1$, the total operators revenue is $R_{tot} = 1500$ units and the aggregate utility is $U_{agg} = 100$, achieved by increasing the spectrum W until $W = 1657.8$ units, which yields $\alpha = 1.5$. In the lower plot, the same total revenue is reached for $W = 5000$, and decreasing U_0 until $U_0 = 1.204$ units. In this case, the aggregate utility is $U_{agg} = 1204$ units. If, for example, the regulator is interesting only in maximizing the revenue of operators, then he would prefer the first method since it requires less spectrum and lower value for U_0.

VI. Conclusions

In this paper, we studied the operators price competition in a wireless services market where users have a certain reservation utility U_0. We modeled the users interaction as an evolutionary

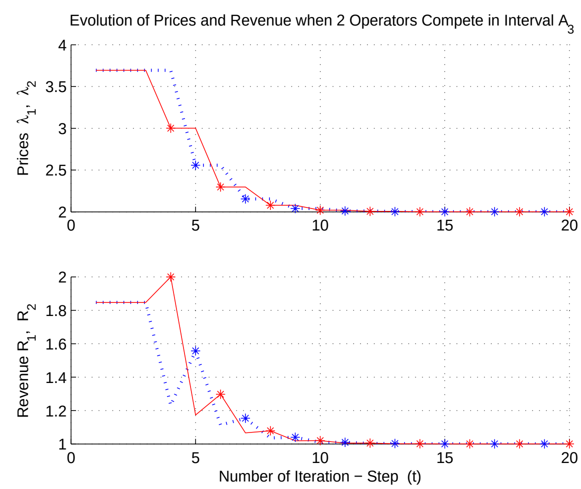

Fig. 6. Evolution of operator competition for $\alpha = e^3 \in A_3$. The game is played repeatedly and operators updated myopically their price based on the previous strategy of the other operators.

game and the competition of the operators as a non cooperative game of complete information. We proved that the latter is a potential game and hence has pure Nash equilibriums. The two games are realized in different time scale but they are interrelated. Additionally, both of them depend on the reservation utility U_0 and the amount of spectrum W each operator has at his disposal. Accordingly, we considered a regulating agency and discussed how he can intervene and change the outcome of the market by tuning either U_0 or W. Various regulation methods yield different market outcomes and induce a different cost for the regulator.

References

[1] D. Acemoglou, and A. Ozdaglar, “Price Competition in Communication Networks”, *IEEE Infocom*, 2006.

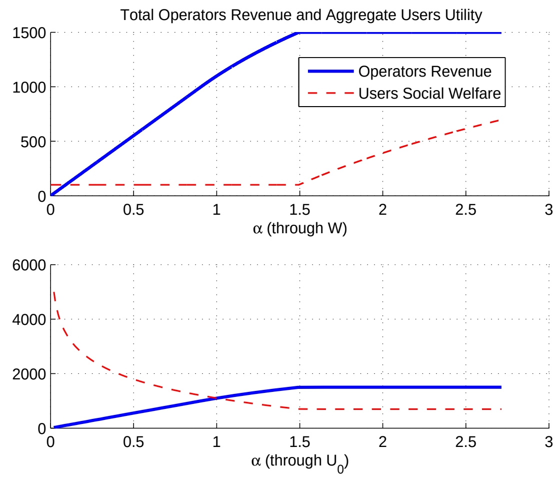

Fig. 7. Total revenue of operators and aggregate utility of the users for different values of parameter α. In the upper plot, the value of α changes through W while in the lower plot it changes through the tuning of U_0.

[2] L. He, and J. C. Walrand, "Pricing and revenue sharing strategies for Internet service providers", *IEEE JSAC,* 2006.

[3] W. H. Sandholm, "Pairwise Comparison Dynamics and Evolutionary Foundations for Nash Equilibrium", *Games,* vol. 1, pp. 3-17, 2010.

[4] T. Nguyen, H. Zhou, R. Berry, M. Honig and R. Vohra, "The Impact of Additional Unlicensed Spectrum on Wireless Services Competition", *IEEE Dyspan,* 2011.

[5] L. Gao, X. Wang, and Y. Xu, "Multiradio Channel Allocation in Multihop Wireless Networks", *IEEE Transactions on Mobile Computing,* 2009.

[6] C. Courcoubetis, and R. R. Weber, "Pricing Communication Networks: Economics, Technology and Modelling", *Willey,* 2003.

[7] S. Shakkottai, and R. Srikant, "Economics of Network Pricing With Multiple ISPs", *IEEE/ACM Transactions on Networking,* vol. 14, no. 6, 2006.

[8] K. Leung, and J. Huang, "Regulating Wireless Access Pricing", *IEEE ICC,* 2011.

[9] D. Niyato, and E. Hossain, "Modeling User Churning Behavior in Wireless Networks Using Evolutionary Game Theory", *IEEE WCNC,* 2008.

[10] P. D. Taylor, and L. Jonker, "Evolutionarily Stable Strategies and Game Dynamics", *Mathematical Biosciences,* vol. 40, pp. 145-156, 1978.

[11] J. Park, and M. van der Schaar, "The Theory of Intervention Games for Resource Sharing in Wireless Communications", *IEEE JSAC,* vol. 30, pp. 165-175, 2011.

[12] D. Monderer, and L. S. Shapley, "Potential Games", *Games and Economic Behavior,* vol. 14, 1996.

[13] G. Vojislav, J. Huang, and B. Rimoldi, "Competition of Wireless Providers for Atomic Users: Equilibrium and Social Optimality", *Allerton Conference,* 2009.

[14] D. Niyato, E. Hossain, and Z. Han, "Dynamics of Multiple-Seller and Multiple-Buyer Spectrum Trading in Cognitive Radio Networks: A Game-Theoretic Modeling Approach", *IEEE Transactions on Mobile Computing,* vol. 8, no. 8, 2009.

[15] D. Niyato, and E. Hossain, "Dynamics of Network Selection inH eterogeneous Wireless Networks: An Evolutionary Game Approach", *IEEE Transactions on Vehicular Technology,* 2009.

[16] G. Bianchi, "Performance Analysis of the IEEE 802.11 Distributed Coordination Function", *IEEE Journal on Selected Areas in Communication,* vol. 18, no. 3, 2000.

[17] M. Cagalj, S. Ganeriwal, and I. Aad, "On Selfish Behavior in CSMA/CA Networks", *IEEE Infocom ,* 2005.

[18] X. Chen, and J. Huang, "Evolutionarily Stable Spectrum Access in a Many-Users Regime", *IEEE Globecom,* 2011.

[19] O. Korcak, G. Iosifidis, T. Alpcan, and I. Koutsopoulos, "Competition and Regulation in Wireless Services Markets", *Technical Report, Arxiv, http://arxiv.org/abs/1112.2437*, 2012.

[20] J. G. Wardrop, "Some theoretical aspects of road traffic research", *Proceedings of the Institution of Civil Engineers,* 1952.

[21] P. Maille, and B. Tuffin, "Price war in heterogeneous wireless networks", *Computer Networks,* vol. 54, pp. 2281-2292, 2010.

[22] D. Fudenberg, and D. K. Levine, "Learning in Games", *MIT Press,* 1998.

[23] M. J. Smith, "The Stability of a Dynamic Model of Traffic Assignment An Application of a Method of Lyapunov", *Transactions of Science,* vol. 18, pp. 245-252, 1984.

[24] M. J. Smith, "Evolution and the Theory of Games", *Cambridge University Press,* 1982.

[25] S. Boyd, and L. Vandenberghe, "Convex Optimization", *Cambridge University Press,* 2004.

APPENDIX

A. Derivation of Evolutionary Dynamics

Here, we derive the new differential equations that describe the evolution of the market of the users under the new introduced revision protocol. Recall that, the latter is described by the following equations:

$$\rho_{ij}(t) = x_j(t)[U_j(t) - U_i(t)]_+, \forall i, j \in \mathcal{I} \tag{25}$$

$$\rho_{i0}(t) = \gamma[U_0 - U_i(t)]_+, \forall i \in \mathcal{I} \tag{26}$$

$$\rho_{0i}(t) = x_i(t)[U_i(t) - U_0]_+, \forall i \in \mathcal{I} \tag{27}$$

where $\rho_{ij}(t)$ is the rate at which users associated with operator i switch to operator j in time slot t, $\rho_{i0}(t)$ is the switch rate from operator i to neutral operator P_0 and $\rho_{0i}(t)$ the rate at which users return from P_0 to an operator $i \in \mathcal{I}$ in the market. The constant value $\gamma \in R^+$ represents the frequency of the direct selection.

For imitation-based revision protocols, the dynamics of the system can be described with the well-known replicator dynamics [3]. The hybrid revision protocol defined in equations (25), (26) and (27) is in part imitation-based ($\rho_{ij}(t)$ and $\rho_{0i}(t)$) and in part a probabilistic direct selection of the neutral operator ($\rho_{i0}(t)$). Therefore, the respective evolutionary dynamics of the system cannot be described by the replicator dynamic equations which correspond to the pure imitation mechanism. We have to stress that the hybrid protocol that we introduce, differs from the hybrid protocol in [3] in that users select directly only the neutral operator and not the other I operators.

The portion of users x_i who are associated with operator i changes from time t to the time $t+\delta t$, according to the following equation:

$$\begin{aligned} x_i(t+\delta t) &= x_i(t) - x_i(t)\delta t \sum_{j\neq 0} x_j(t)(U_j(t)-U_i(t))_+ \\ &- x_i(t)\delta t\gamma(U_0(t)-U_i(t))_+ \\ &+ \sum_{j=0}^{I} \delta t x_j(t) x_i(t)(U_i(t)-U_j(t))_+ \end{aligned} \tag{28}$$

for $\delta t \to 0$ we obtain the derivative:

$$\begin{aligned} \frac{dx_i(t)}{dt} &= x_i(t)[\sum_{j\neq 0} x_j(t)U_i(t) - \sum_{j\neq 0} x_j(t)U_j(t) \\ &- \gamma(U_0-U_i)_+ + x_0(t)(U_i-U_0)_+] \end{aligned}$$

or, if we omit the time index and rewrite the equation:

$$\begin{aligned} \frac{dx_i(t)}{dt} &= x_i[U_i - U_{avg} - x_0(U_i-U_0) - \gamma(U_0-U_i)_+ \\ &+ x_0(U_i-U_0)_+] \end{aligned}$$

which can be analyzed in:

$$\frac{dx_i(t)}{dt} = x_i(U_i - U_{avg}),\ \forall i \in \mathcal{I}^+ \tag{29}$$

$$\frac{dx_j(t)}{dt} = x_j[U_j - U_{avg} - (\gamma - x_0)(U_0 - U_j)],\ \forall j \in \mathcal{I}^- \tag{30}$$

where $\mathcal{I}^+$ is the set of operators offering utility $U_i(t) \geq U_0$, and $\mathcal{I}^-$ is the set of operators offering utility $U_j(t) < U_0$.

The dynamics of the population x_0 can be derived in a similar way:

$$\begin{aligned} x_0(t &+ \delta t) = x_0(t) - x_0(t)\delta t \sum_{i\neq 0} x_i(t)(U_i-U_0)_+ \\ &+ \sum_{i\neq 0} x_i(t)\delta t\gamma(U_0-U_i)_+ \end{aligned} \tag{31}$$

which can be written as:

$$\frac{dx_0(t)}{dt} = (x_0 \sum_{i\in I^+} x_i(U_0-U_i) + \gamma \sum_{j\in\mathcal{I}^-} x_j(U_0-U_j)) \tag{32}$$

Equations (29), (30) and (32) describe the evolutionary dynamics of game $\mathcal{G}_{\mathcal{U}}$.

B. Analysis of Stationary Points

Despite the different dynamics, the system reaches the same stationary points as if users where employing the typical imitation revision protocol. In detail, the market state vector at a fixed point, $\mathbf{x}^* = (x_i^*,\, x_j^*,\, x_0^*\text{: } \forall\, i \in \mathcal{I}^+,\, \forall\, j \in \mathcal{I}^-)$, can be found by the following set of equations:

$$\frac{dx_i(t)}{dt} = \frac{dx_j(t)}{dt} = \frac{dx_0(t)}{dt} = 0\ \forall i \in \mathcal{I}^+,\, j \in \mathcal{I}^- \tag{33}$$

Lemma A.1. *The stationary points of the evolutionary dynamics defined in equations (29), (30) and (32) are identical to the stationary points of the ordinary replicator dynamics [3] given by:*

$$\dot{x}_i(t) = 0 \Rightarrow x_i(t)[U_i(t) - U_{avg}(t)] = 0,\ \forall\, i \in \mathcal{I} \tag{34}$$

and

$$\dot{x}_0(t) = 0 \Rightarrow x_0(t)[U_0 - U_{avg}(t)] = 0 \tag{35}$$

Proof: First we prove that, in any stationary point, $x_j^*,\, j \in \mathcal{I}^-$ should be equal to zero. We prove this claim by contradiction. Assume that $x_j^* > 0$. Since $U_{avg} \geq U_0 > U_j$, this implies that there should be at least one operator i with $U_i > U_{avg}$ and $x_i^* > 0$. Therefore $(U_i - U_{avg})$ cannot be equal to zero $\forall i \in \mathcal{I}^+$, and $\dot{x}_i$ will be nonzero for at least one operator. Therefore (33) cannot be satisfied, if $x_j^* \neq 0$.

When $x_j = 0$, the evolutionary dynamics given by eq. (29), (30) and (32) reduce to ordinary replicator dynamics:

$$\begin{aligned} \dot{x}_i(t) &= x_i(t)[U_i(t) - U_{avg}(t)]\ \forall i \in \mathcal{I} \\ \dot{x}_0(t) &= x_0(t)[U_0 - U_{avg}(t)] \end{aligned} \tag{36}$$

Stationary points are identical to the stationary points of the typical replicator dynamics, [3]. ■

Due to this lemma, the stationary points for the users population associated with each operator $i \in \mathcal{I}$ should satisfy one of the following conditions: (i) $x_i^* = 0$, or (ii) $x_i^* > 0$ and $U_i^* = U_{avg}$. Similarly, for the neutral operator P_0, eq. (35), it must hold: (i) $x_0^* = 0$ and $U_0 < U_{avg}$, (ii) $x_0^* > 0$ and $U_0 = U_{avg}$ or (iii) $x_0^* = 0$ and $U_0 = U_{avg}$. The case $x_i^* = 0$ implies zero revenue for the i^{th} operator and hence case (i) does not constitute a valid choice. Therefore, there exist in total 3 possible combinations (cases) that will satisfy the stationarity properties given by eq. (34) and (35):

- **Case A**: $x_i^*,\, x_0^* > 0$ and $U_i^* = U_0,\, i \in \mathcal{I}$.
- **Case B**: $x_i^*,\, x_j^* > 0$, $x_0^* = 0$ and $U_i^* = U_j^*$, with $U_i^*,\, U_j^* > U_0, \forall\, i,\, j \in \mathcal{I}$.
- **Case C**: $x_i^*,\, x_j^* > 0$, $x_0^* = 0$ and $U_i^* = U_j^* = U_0$, $\forall\, i,\, j \in \mathcal{I}$.

We find now the exact value of the market state vector at the equilibrium (stationary point) $\mathbf{x}^*$ for each case. First, we define for every operator $i \in \mathcal{I}$ the scalar parameter $\alpha_i = W_i/(Ne^{U_0})$ and the respective vector $\alpha = (\alpha_i : i \in \mathcal{I})$.

We can find the stationary points for **Case A** by using the equation $U_i(W_i, x_i^*, \lambda_i) = U_0$ and imposing the constraint $x_0^* > 0$:

$$\begin{aligned} U_i(W_i, x_i^*, \lambda_i) &= \log\frac{W_i}{Nx_i^*} - \lambda_i = U_0 \\ &\Rightarrow\ x_i^* = \frac{W_i}{Ne^{\lambda_i + U_0}} = \alpha_i e^{-\lambda_i} \end{aligned} \tag{37}$$

and

$$x_0^* > 0 \Rightarrow 1 - \sum_{i=1}^{I} \alpha_i e^{-\lambda_i} > 0 \Rightarrow \sum_{i=1}^{I} \alpha_i e^{-\lambda_i} < 1 \tag{38}$$

Apparently, the state vector $\mathbf{x}^*$ depends on the operators' price vector λ. Therefore, we define the set of all possible **Case A** stationary points, X_A, as follows:

$$X_A = \left\{ x_i^* = \alpha_i e^{-\lambda_i}, \forall i \in \mathcal{I}, x_0^* = 1 - \sum_{i=1}^{I} \alpha_i e^{-\lambda_i} \,:\, \lambda \in \Lambda_A \right\}$$

where Λ_A is the set of prices for which a stationary point in X_A is reachable, i.e. $x_0^* > 0$:

$$\Lambda_A = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_I) \,:\, \sum_{i=1}^{I} \alpha_i e^{-\lambda_i} < 1 \right\}$$

Similarly, for **Case B**, we calculate the stationary points by using the set of equations $U_i(W_i, x_i^*, \lambda_i) = U_j(W_j, x_j^*, \lambda_j)$, $\forall\, i, j \in \mathcal{I}$, which yields:

$$\log \frac{W_1}{N x_1^*} - \lambda_1 = \log \frac{W_2}{N x_2^*} - \lambda_2 = \ldots = \log \frac{W_i}{N x_i^*} - \lambda_i \quad (39)$$

or, equivalently:

$$x_j^* = x_i^* \frac{e^{\lambda_i} \alpha_j}{e^{\lambda_j} \alpha_i} \quad \forall i, j \in \mathcal{I} \quad (40)$$

Moreover since $x_0^* = 0$ for **Case B**, the following holds:

$$\sum_{i \in \mathcal{I}} x_i^* = 1 \quad (41)$$

Using (40) and (41),

$$x_i^* = \frac{\alpha_i}{e^{\lambda_i} \sum_{j \in \mathcal{I}} \alpha_j e^{-\lambda_j}} \quad (42)$$

Additionally, $U_i > U_0$ implies that:

$$\log \frac{W_i}{N x_i^*} - \lambda_i > U_0 \Rightarrow x_i^* < \alpha_i e^{-\lambda_i} \quad (43)$$

Using (41) and (43),

$$\sum_{i=1}^{I} x_i^* < \sum_{i=1}^{I} \alpha_i e^{-\lambda_i} \Rightarrow \sum_{i=1}^{I} \alpha_i e^{-\lambda_i} > 1 \quad (44)$$

Therefore, according to (42) and (44), we define the set of all possible **Case B** stationary points, X_B, as follows:

$$X_B = \left\{ x_i^* = \frac{\alpha_i}{e^{\lambda_i} \sum_{j=1}^{I} \alpha_j e^{-\lambda_j}}, \forall i \in \mathcal{I}, x_0^* = 0 \,:\, \lambda \in \Lambda_B \right\}$$

where Λ_B is the set of prices for which a stationary point in X_B is feasible, $U_i^* > U_0$:

$$\Lambda_B = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_I) \,:\, \sum_{i=1}^{I} \alpha_i e^{-\lambda_i} > 1 \right\}$$

Finally, the stationary points for the **Case C** solution must satisfy the following:

$$U_i = U_0, \quad x_0^* = 0 \quad (45)$$

which yields:

$$x_i^* = \alpha_i e^{-\lambda_i}, \quad \sum_{i=1}^{I} \alpha_i e^{-\lambda_i} = 1 \quad (46)$$

Therefore, we define the set of all possible **Case C** stationary points, X_C, as follows:

$$X_C = \left\{ x_i^* = \alpha_i e^{-\lambda_i}, \forall\, i \in \mathcal{I}, x_0^* = 0 \,:\, \lambda \in \Lambda_C \right\}$$

with

$$\Lambda_C = \left\{ (\lambda_1, \lambda_2, \ldots, \lambda_I) \,:\, \sum_{i=1}^{I} \alpha_i e^{-\lambda_i} = 1 \right\}$$

First, we show that the revenue of each operator $i \in \mathcal{I}$ is a continuous and a quasi-concave function. Secondly, we analyze best response pricing in game $\mathcal{G}_{\mathcal{P}}$. Then, we derive the Nash equilibriums (NEs) of the game using iterated strict dominance. Finally, we prove convergence to these equilibriums by showing that $\mathcal{G}_{\mathcal{P}}$ is a potential game.

C. Properties of the Revenue Function

The revenue function of each operator i is given by the following equation:

$$R_i(\lambda_i, \lambda_{-i}) = \begin{cases} \dfrac{\alpha_i \lambda_i N}{e^{\lambda_i} \sum_{j=1}^{I} \alpha_j e^{-\lambda_j}} & \text{if } \lambda_i < l_0, \\ \alpha_i \lambda_i N e^{-\lambda_i} & \text{if } \lambda_i \geq l_0. \end{cases} \quad (47)$$

where $l_0 = \log(\alpha_i / (1 - \sum_{j \neq i} \alpha_j e^{-\lambda_j}))$.

Each component (for each case) is a positive function which is also *log-concave*. This means that it is a quasiconcave function and hence uniqueness of optimal solution is ensured for a proper constraint set. Namely, it is:

$$f_A(\lambda_i) = \log \alpha_i \lambda_i N e^{-\lambda_i} = \log \alpha_i \lambda_i N - \lambda_i \quad (48)$$

and

$$f_A(\lambda_i)^{(1)} = \frac{1}{\lambda_i} - 1 \Rightarrow f_A(\lambda_i)^{(2)} = \frac{-1}{\lambda_i^2} < 0 \quad (49)$$

Hence, $f_A(\cdot)$ which is the log-function of $R_i^A(\cdot)$, is concave which means that the later is log-concave and since it is $R_i^A(\lambda_i) > 0$, it is also quasi-concave. Similarly, for the other component of the revenue function:

$$f_B(\lambda_i, \lambda_{-i}) = \log \frac{\alpha_i \lambda_i N}{\alpha_i + \beta e^{\lambda_i}} = \log \alpha_i \lambda_i N - \log \alpha_i + \beta e^{\lambda_i} \quad (50)$$

where $\beta = \sum_{j \neq i} \alpha_j e^{-\lambda_i}$. The second derivative is:

$$f_B(\lambda_i, \lambda_{-i})^{(2)} = \frac{-1}{\lambda_i^2} - \frac{\alpha_i \beta e^{\lambda_i}}{(\alpha_i + \beta e^{\lambda_i})^2} < 0 \quad (51)$$

Hence, $R_i^B(\cdot)$ is also quasiconcave. Finally, it is easy to see that the function is continuous:

$$R_i^A(l_0, \lambda_{-i}) = R_i^B(l_0, \lambda_{-i}) = N(1 - \beta) \log \frac{\alpha_i}{1 - \beta} \quad (52)$$

D. Best Response Pricing in $\mathcal{G}_{\mathcal{P}}$

Each operator i finds his best response price λ_i^* for each price profile of the other $I-1$ operators by solving the following optimization problems. For the case the price vector belongs to the set Λ_A, $\lambda \in \Lambda_A$, ($\mathbf{P}_i^A$):

$$\max_{\lambda_i \geq 0} \alpha_i \lambda_i N e^{-\lambda_i} \tag{53}$$

s.t.

$$\sum_{j=1}^{I} \alpha_j e^{-\lambda_j} < 1 \tag{54}$$

In order to ensure the uniqueness of the problem solution, we transform the constraint set to a closed and compact set as follows:

$$\lambda_i \geq \log \frac{\alpha_i}{1-\sum_{j\neq i} \alpha_j e^{-\lambda_j}} + \epsilon \tag{55}$$

where $\epsilon > 0$ is an arbitrary small positive constant number. As we will show immediately this transformation of the constraint set does not affect the solution of the game. The problem now is quasi-concave with a closed and compact constraint set and hence it has a unique optimal solution, [25] which we denote λ_i^A and it is:

$$\lambda_i^A = 1, \texttt{ or } \lambda_i^A = \log \frac{\alpha_i}{1-\sum_{j\neq i} \alpha_j e^{-\lambda_j}} + \epsilon \tag{56}$$

The value $\lambda_i^A = 1$ is the optimal solution of the respective unconstrained problem, which yields optimal revenue $R_i^A = \alpha N/e$, and it is feasible if $\lambda = (1, \lambda_{-i}) \in \Lambda_A$. Otherwise, since $R_i^A(\cdot)$ is a decreasing function of λ_i, operator i can only select the minimum price λ_i^A such that $(\lambda_i^A, \lambda_{-i}) \in \Lambda_A$.

Similarly, when the price vector belongs to the set Λ_B, i.e. $\lambda \in \Lambda_B$, the revenue maximization problem for each operator $i \in \mathcal{I}$ ($\mathbf{P}_i^B$) is:

$$\max_{\lambda_i \geq 0} \frac{\lambda_i \alpha_i N}{e^{\lambda_i} \sum_{j\in\mathcal{I}} \alpha_j e^{-\lambda_j}} \tag{57}$$

s.t.

$$\sum_{j\in\mathcal{I}} \alpha_j e^{-\lambda_j} > 1 \tag{58}$$

Similarly to the previous analysis, we transform the constraint set to a closed and compact set by using the following inequality:

$$\lambda_i \leq \log \frac{\alpha_i}{1-\sum_{j\neq i} \alpha_j e^{-\lambda_j}} - \epsilon \tag{59}$$

This is also a concave problem which has unique solution and can be either the optimal solution of the respective unconstrained problem, λ_i^* if $(\lambda_i^*, \lambda_{-i}) \in \Lambda_B$, or the maximum price for which the price vector belongs to Λ_B ($R_i^B(\cdot)$ increases with λ_i:

$$\lambda_i^B = \mu_i^*, \texttt{ or } \lambda_i^B = \log \frac{\alpha_i}{1-\sum_{j\neq i} \alpha_j e^{-\lambda_j}} - \epsilon \tag{60}$$

Finally, for the special case that $\lambda \in \Lambda_C$, the price of each operator i is directly determined by the prices that the other operators have selected. Namely:

$$\lambda_i^C = \log \frac{\alpha_i}{1-\sum_{j\neq i} \alpha_j e^{-\lambda_j}} \tag{61}$$

Whether each operator i will agree and adopt this price or not, depends on the respective accrued revenue, $R_i^C(\lambda_i^C, \lambda_{-i})$.

In the sequel, we examine and analyze jointly the solutions of the above optimization problems and derive the exact best response of the i^{th} operator for each vector λ_{-i} of the $I-1$ prices.

Lemma A.2. *For each operator $i \in \mathcal{I}$, if $(1, \lambda_{-i}) \notin \Lambda_A$, then there is no best response price λ_i^*, such that $(\lambda_i^*, \lambda_{-i}) \in \Lambda_A$. That is, operator i will not select Λ_A.*

Proof: Given that the price vector $\lambda \in \mathbf{\Lambda_A}$, best response price is:

$$\lambda_i^A = \begin{cases} 1 & \text{if } (1, \lambda_{-i}) \in \Lambda_A, \\ l_0 + \epsilon & \text{if } (1, \lambda_{-i}) \notin \Lambda_A. \end{cases} \tag{62}$$

where $l_0 = \lambda_i^C$ is the price operator i selects when $\lambda \in \mathbf{\Lambda_C}$.

If $(1, \lambda_{-i}) \notin \Lambda_A$, then $l_0 + \epsilon > 1$. Otherwise price vector $(l_0 + \epsilon, \lambda_{-i})$ will not belong to Λ_A. Therefore, $R_i^A(\cdot)$ is a decreasing function at the point $\lambda_i = \lambda_0 + \epsilon$ due to quasi-concavity property. Therefore, if $(1, \lambda_{-i}) \notin \Lambda_A$, then $R_i^C(l_0) = R_i^A(l_0) > R_i^A(l_0 + \epsilon)$ which means that λ_i^C always gives better response than λ_i^A. ∎

Lemma A.3. *Let us denote with μ_i^* the optimal solution of the unconstraint problem P_i^B. For each operator $i \in \mathcal{I}$, if $(\mu_i^*, \lambda_{-i}) \notin \Lambda_B$, then there is no best response price λ_i^*, such that $(\lambda_i^*, \lambda_{-i}) \in \Lambda_B$.*

Proof: Given that the price vector $\lambda \in \mathbf{\Lambda_B}$, best response price is:

$$\lambda_i^B = \begin{cases} \mu_i^* & \text{if } (\mu_i^*, \lambda_{-i}) \in \Lambda_B, \\ l_0 - \epsilon & \text{if } (\mu_i^*, \lambda_{-i}) \notin \Lambda_B. \end{cases} \tag{63}$$

and recall that $\lambda_i^C = l_0$. If $(\mu_i^*, \lambda_{-i}) \notin \Lambda_B$, then $l_0 - \epsilon < \mu_i^*$. Otherwise, the price vector $(l_0 - \epsilon, \lambda_{-i})$ cannot be in Λ_B. Therefore, $R_i^B(\cdot)$ is an increasing function at the point $\lambda_i = \lambda_0 - \epsilon$ due to quasi-concavity property. Therefore, if $(\mu_i^*, \lambda_{-i}) \notin \Lambda_B$, then $R_i^C(l_0) = R_i^B(l_0) > R_i^B(l_0 - \epsilon)$ which means that λ_i^C always gives better response than λ_i^B. ∎

In other words, the previous two Lemmas state that the only eligible best response for each operator $i \in \mathcal{I}$ in the price sets Λ_A and Λ_B are prices $\lambda_i^* = 1$ and and $\lambda_i^* = \mu_i^*$ respectively.

Theorem A.4. *The best response price of an operator i is:*

$$\lambda_i^* = \begin{cases} 1, & \textit{if } (1, \lambda_{-i}) \in \Lambda_A, \\ \mu_i^*, & \textit{if } (\mu_i^*, \lambda_{-i}) \in \Lambda_B, \\ \lambda_i^C = l_0, & \textit{otherwise}. \end{cases} \tag{64}$$

Proof: First we prove that $(1, \lambda_{-i}) \in \Lambda_A$ and $(\mu_i^*, \lambda_{-i}) \in \Lambda_B$ cannot be true at the same time. Since μ_i^* is the optimal solution of unconstraint R_i^B:

$$\frac{dR_i^B(\lambda_i)}{d\lambda_i} = 0 \Rightarrow e^{\mu_i^*}(\mu_i^* - 1) = \frac{\alpha_i}{\sum_{j\neq i}\alpha_j e^{-\lambda_j}} \tag{65}$$

It is obvious that equation (65) can only hold when $\mu_i^* > 1$. Note that if $(\mu_i^*, \lambda_{-i}) \in \Lambda_B$, the vector $\lambda = (l, \lambda_{-i}) \in \Lambda_B$ for any price $l < \mu_i^*$. Hence, it should also hold that $\lambda = (1, \lambda_{-i}) \in \Lambda_B$. With a similar reasoning, when $(1, \lambda_{-i}) \in \Lambda_A$, $(l, \lambda_{-i}) \in \Lambda_A$ holds for any price $l > 1$ and therefore $(\mu_i^*, \lambda_{-i}) \in \Lambda_A$. Also, if $(1, \lambda_{-i}) \in \Lambda_A$, λ_i^C cannot be a best response, because $R_i^A(1) > R_i^A(\lambda_i^C) = R_i^C(\lambda_i^C)$. Similarly, if $(\mu_i^*, \lambda_{-i}) \in \Lambda_B$, λ_i^C is not a best response.

Finally, from Lemma A.2 and Lemma A.3, we can say that λ_i^C dominates all other prices if $(1, \lambda_{-i}) \notin \Lambda_A$ and $(\mu_i^*, \lambda_{-i}) \notin \Lambda_B$ which concludes the proof. ∎

E. Existence and Convergence Analysis of Nash Equilibriums

In the previous section, we derived the best response strategy for each player of the game $\mathcal{G}_\mathcal{P}$. The next important steps are **(i)** to explore the existence of Nash Equilibriums (NE) for $\mathcal{G}_\mathcal{P}$, and **(ii)** to study if the convergence to them is guaranteed. In [12], it is proven that if the game can be modeled as a potential game, not only the existence of pure NEs are ensured, but also convergence to them is guaranteed under any finite improvement path. In other words, a potential game always converges to pure NE when the players adjust their strategies based on accumulated observations as game unfolds. In this section, we provide the necessary definitions for ordinal potential games, and we prove that game $\mathcal{G}_\mathcal{P}$ belongs in this class of games.

Definition A.5. *A game $(\mathcal{I}, \lambda, \{R_i\})$ is an **ordinal potential game**, if there is a potential function $\mathcal{P} : [0, \lambda_{max}] \to \mathbb{R}$ such that the following condition holds:*

$$\begin{aligned}&\text{sgn}(\mathcal{P}(\lambda_i, \lambda_{-i}) - \mathcal{P}(\lambda_i', \lambda_{-i})) = \\ &\text{sgn}(R_i(\lambda_i, \lambda_{-i}) - R_i(\lambda_i', \lambda_{-i}))\forall i \in \mathcal{I},\ \lambda_i, \lambda_i' \in [0, \lambda_{max}]\end{aligned} \tag{66}$$

where sgn*(·) is the sign function.*

Lemma A.6. *The game $\mathcal{G}_\mathcal{P}$ is an ordinal potential game.*

Proof: We define the potential function as:

$$\mathcal{P}(\lambda) = \begin{cases} \sum_{j=1}^I (\log\lambda_j - \lambda_j), \text{ if} \sum_{j=1}^I \alpha_j e^{-\lambda_j} \le 1, \\ \sum_{j=1}^I (\log\lambda_j - \lambda_j) - \log(\sum_{j=1}^I \alpha_j e^{-\lambda_j}), \text{ else.}\end{cases} \tag{67}$$

Therefore,

$$\mathcal{P} \quad (\lambda_i, \lambda_{-i}) - \mathcal{P}(\lambda_i', \lambda_{-i}) =$$

$$= \begin{cases} \log\frac{\lambda_i e^{\lambda_i'}}{\lambda_i' e^{\lambda_i}}, \text{ if } \lambda_i, \lambda_i' \ge l_0 \\ \log\dfrac{\lambda_i e^{\lambda_i'}(\alpha_i e^{-\lambda_i'} + \sum_{j\neq i}\alpha_j e^{-\lambda_j})}{\lambda_i' e^{\lambda_i}(\alpha_i e^{-\lambda_i} + \sum_{j\neq i}\alpha_j e^{-\lambda_j})}, \text{ if } \lambda_i, \lambda_i' < l_0 \\ \log\dfrac{\lambda_i e^{\lambda_i'}}{\lambda_i' e^{\lambda_i}(\alpha_i e^{-\lambda_i} + \sum_{j\neq i}\alpha_j e^{-\lambda_j})}\lambda_i < l_0, \text{ if } \lambda_i' \ge l_0 \\ \log\dfrac{\lambda_i e^{\lambda_i'}(\alpha_i e^{-\lambda_i'} + \sum_{j\neq i}\alpha_j e^{-\lambda_j})}{e^{\lambda_i}\lambda_i'} \text{ if } \lambda_i \ge l_0,\ \lambda_i' < l_0 \end{cases}$$

where $l_0 = \log(\alpha_i/(1 - \sum_{j\neq i}(\alpha_j e^{-\lambda_j})))$. Moreover, using (47),

$$\log R_i = \begin{cases} \log\frac{\lambda_i}{e^{\lambda_i}} + \log\alpha_i N & \text{if } \lambda_i \ge l_0, \\ \log\dfrac{\lambda_i}{e^{\lambda_i}(\frac{\alpha_i}{e^{\lambda_i}} + \sum_{j\neq i}\frac{\alpha_j}{e^{\lambda_j}})} + \log\alpha_i N & \text{if } \lambda_i < l_0. \end{cases}$$

Now, it is straightforward to show that $\mathcal{P}(\lambda_i, \lambda_{-i}) - \mathcal{P}(\lambda_i', \lambda_{-i}) = \log R_i(\lambda_i, \lambda_{-i}) - \log R_i(\lambda_i', \lambda_{-i})$ for any operator $i \in \mathcal{I}$ and for any $\lambda_i, \lambda_i' \in [0, \lambda_{max}]$. Since $\log R_i(\lambda_i, \lambda_{-i}) - \log R_i(\lambda_i', \lambda_{-i})$ has always same sign as $R_i(\lambda_i, \lambda_{-i}) - R_i(\lambda_i', \lambda_{-i})$, condition given in (66) is satisfied, and game $\mathcal{G}_\mathcal{P}$ is an ordinal potential game. ∎

F. Detailed Analysis of Nash Equilibriums

In the previous section, we proved the existence of pure NE and convergence to them. In this section, we extend our analysis further in order to find these NEs. For the sake of simplicity, we consider the case where all the operators have same amount of available spectrum $W_i = W$ and hence $\alpha_i = \alpha$, $\forall i \in \mathcal{I}$.

Before starting our analysis, we rewrite constraint of set Λ_A given in eq. (39) as follows:

$$\alpha \le \frac{1}{\sum_{j\in\mathcal{I}} e^{-\lambda_j}} = \frac{H(\{e^{\lambda_j}|j \in \mathcal{I}\})}{I} \tag{68}$$

where $H(\cdot)$ is the harmonic mean function of the variables $(e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_I}) = (\{e^{\lambda_j}|j \in \mathcal{I}\})$:

$$H(\{e^{\lambda_j}|j \in \mathcal{I}\}) = \frac{I}{e^{-\lambda_1} + e^{-\lambda_2} + \ldots + e^{-\lambda_I}} \tag{69}$$

Therefore, if $\lambda \in \Lambda_A$, it is:

$$H(\{e^{\lambda_j}|j \in \mathcal{I}\}) \ge \alpha I \tag{70}$$

Similarly, according to (45), if $\lambda \in \Lambda_B$ then:

$$H(\{e^{\lambda_j}|j \in \mathcal{I}\}) \le \alpha I \tag{71}$$

and finally, if $\lambda \in \Lambda_C$:

$$H(\{e^{\lambda_j}|j \in \mathcal{I}\}) = \alpha I \tag{72}$$

Next, we define a new variable, h as the natural logarithm of the harmonic mean:

$$h = \log H(\{e^{\lambda_j}|j \in \mathcal{I}\}) \tag{73}$$

Note that, since e^h is the harmonic mean of $\{e^{\lambda_j}|j \in \mathcal{I}\}$, we can say that one of the following should hold:

1) Every operator $i \in \mathcal{I}$ adopts the same price $\lambda_i = h$.
2) If one operator $j \in \mathcal{I}$ selects a price $\lambda_j < h$, then there must be at least one other operator $k \in \mathcal{I}$ who will adopt a price $\lambda_k > h$.

Additionally, we define the variable h_{-i} which is similar to h except that price of the i^{th} operator is excluded. That is:

$$h_{-i} = \log(H(\{e^{\lambda_j}|j \in \mathcal{I} \setminus i\})) \tag{74}$$

It is obvious that if $\lambda_i > h$, then $h_{-i} < h$, if $\lambda_i < h$, then $h_{-i} > h$, and if $\lambda_i = h$, then $h_{-i} = h$.

Lemma A.7. *If $\alpha \in A_1 = (0, e/I)$, there is a unique NE $\lambda^* \in \Lambda_A$, with $\lambda^* = (\lambda_i^* = 1 : i \in \mathcal{I})$*

Proof: First, we prove that the NE cannot be in Λ_B or Λ_C ($\lambda^* \notin \Lambda_B \cup \Lambda_C$) if $\alpha \in A_1 = (0, e/I)$. Notice that, when the price vector is not in Λ_A, $h \leq \log(\alpha I) < 1$ for given α values. Therefore there exists at least one operator with price less than one. Since R_i^B is an increasing function between $\lambda_i \in (0, 1)$, operators with $\lambda_i < 1$ would gain more revenue by unilaterally increasing their prices. Therefore λ^* can only be in Λ_A. According to Theorem A.4, given that the price vector is in Λ_A, optimal price for any operator i can only be $\lambda_i^A = 1$ if $(1, \lambda_{-i}) \in \Lambda_A$. Since $\lambda^* = (\lambda_i^* = 1 : i \in \mathcal{I}) \in \Lambda_A$ when $\alpha \in A_1$, it is a feasible and unique solution. ■

Lemma A.8. *μ_i^* is always between $\frac{I}{I-1}$ and h_{-i}*

Proof: We can rewrite equation (65) as follows:

$$e^{\mu_i^*}(\mu_i^* - 1) = \frac{e^{h_{-i}}}{I - 1} \tag{75}$$

where h_{-i} is defined in equation (74). Now, if $h_{-i} < \frac{I}{I-1}$, or equivalently if $h_{-i} - 1 < \frac{1}{I-1}$, then λ_i^* should be greater than h_{-i} in order to satisfy (75). Moreover, if $\lambda_i^* > h_{-i}$, then $\lambda_i^* - 1$ should be less than $\frac{1}{I-1}$ in order to satisfy (75). Therefore, $h_{-i} < \lambda_i^* < \frac{I}{I-1}$. Similarly, if $h_{-i} \geq \frac{I}{I-1}$, then $\frac{I}{I-1} \leq \lambda_i^* \leq h_{-i}$, which proves the lemma. ■

Lemma A.9. *If $\alpha \in A_3 = (e^{I/(I-1)}, \infty)$, there is a unique NE $\lambda^* \in \Lambda_B$, with $\lambda^* = (\lambda_i^* = I/(I-1) : i \in \mathcal{I})$*

Proof: First we prove that there is no NE in Λ_A if $\alpha \geq e/I$ (i.e. if $\alpha \in A_2 \cup A_3$). According to Theorem A.4, optimal price for any operator i can only be $\lambda_i^A = 1$ if $(1, \lambda_{-i}) \in \Lambda_A$. Otherwise λ_i^C dominates λ_i^A. Since $\lambda^* = (\lambda_i^* = 1 : i \in \mathcal{I}) \notin \Lambda_A$ when $\alpha \in A_2 \cup A_3$, there is no NE in Λ_A.

Secondly, we prove that there is no NE in Λ_C if $\alpha \in A_3 = (e^{I/(I-1)}, \infty)$. Recall that, when the price vector is in Λ_C, $h = \log(\alpha I) > I/(I-1)$, which means that there exists at least one operator with price $\lambda_i^C > I/(I-1)$ and $\lambda_i^C \geq h$. Remember that if $\lambda_i \geq h$, then $h_{-i} \leq h$, so $\lambda_i \geq h_{-i}$. Therefore, for an operator i, λ_i^C is greater than both h_{-i} and $I/(I-1)$. According to Theorem A.4 and Lemma A.8, when $(\mu_i^*, \lambda_{-i}) \in \Lambda_B$, best response price of operator i is μ_i^* which is between h_{-i} and $I/(I-1)$. This means that for at least one operator, λ_i^C is greater than μ_i^*, which implies that $(\mu_i^*, \lambda_{-i}) \in \Lambda_B$. This operator can increase his revenue by reducing his price to μ_i^*. Therefore, there is no NE in λ_C for the given α values, and we proved that the NE can only be in Λ_B.

Finally, we prove that the only NE is $\lambda^* = (\lambda_i^* = I/(I-1) : i \in \mathcal{I})$, if $\alpha \in A_3$. According to Lemma A.8, μ_i^* is between h_{-i} and $I/(I-1)$ for all operators. h can be greater than or less than $I/(I-1)$. If $h \geq I/(I-1)$, unless all of the operators set their prices to $I/(I-1)$, there exists at least one operator i with price λ_i greater than both h_{-i} and $I/(I-1)$. Hence, λ_i is also greater than μ_i^* and this operator can increase his revenue by reducing his price to μ_i^*. Similarly, if $h < I/(I-1)$, there exists at least one operator with price λ_i less than both h_{-i} and $I/(I-1)$. This operator can increase his revenue by increasing his price. If all the operators set their prices to $\lambda_i = I/(I-1)$, the price vector is in Λ_B and none of the operators can increase his revenue by unilaterally changing his price. Therefore, the only NE is $\lambda^* = (\lambda_i^* = I/(I-1) : i \in \mathcal{I})$.

Hence the lemma is proved. ■

Lemma A.10. *If $\alpha \in A_2 = [e/I, e^{I/(I-1)}]$, a NE can only be in Λ_C.*

Proof: In the proof of the Lemma A.9, we showed that there is no NE in Λ_A, if $\alpha \in A_2$. We can also prove that there is no NE in Λ_B for the given range of α values. If the price vector is in Λ_B and $\alpha < \frac{e^{I/(I-1)}}{I}$, then $h < I/(I-1)$. Therefore, there is at least one operator with price $\lambda_i \leq h_{-i}$ and $\lambda_i < I/(I-1)$, who can increase his revenue by increasing his price. So we conclude that, if $\alpha \in A_2 = [e/I, e^{I/(I-1)}]$, there is no NE in Λ_A or Λ_B. ■

Lemma A.11. *If $\alpha \in A_2 = [e/I, e^{I/(I-1)}]$, $\lambda^* \in \Lambda_C$ with $\lambda^* = (\lambda_i^* = \log(I\alpha) : i \in \mathcal{I})$ is a NE.*

Proof: When all the operators set the same price $\lambda_i^* = \log(I\alpha)$ and $\alpha \in A_2 = [e/I, e^{I/(I-1)}]$, $\log(I\alpha)$ is between 1 and μ_i^* for all operators (this can be verified through equation (65)). Therefore, for any operator i, $(1, \lambda_{-i}) \notin \Lambda_A$ and $(\mu_i^*, \lambda_{-i}) \notin \Lambda_B$. Then, according to Theorem A.4, best response price is λ_i^C which is equal to $\log(I\alpha)$. Hence, no operator can gain more revenue by unilaterally changing his price, and $\lambda^* = (\lambda_i^* = \log(I\alpha) : i \in \mathcal{I})$ is a NE. ■

Finally we analyze the NE for boundary values of A_2, i.e. for $\alpha = e/I$ and $\alpha = e^{I/(I-1)}/I$. In the previous lemma, it is proven that $\lambda^* = (\lambda_i^* = \log(I\alpha) : i \in \mathcal{I})$ is a NE if $\alpha \in A_2$. We can also prove that it is the only NE for these boundary values. In Lemma A.10, it is proven that any NE is in Λ_C for these α values. So, when $\alpha = e/I$, $h = \log(I\alpha) = 1$, which means that unless all of the users set their prices to one, there exists some operators with $\lambda_i < 1$. These operators would gain more revenue by setting their prices to one. Hence, the only NE is $\lambda_i^* = \log(I\alpha) = 1$. Similarly, when $\alpha = e^{I/(I-1)}/I$, $h = \log(I\alpha) = I/(I-1)$. This means that unless all of the users set their prices to $I/(I-1)$, there exists some operators with λ_i greater than both h_{-i} and $M/(M-1)$. These operators would gain more revenue by reducing their prices. Hence, the only NE is $\lambda_i^* = \log(I\alpha) = I/(I-1)$.

We also show that when $\alpha \in (e/I, e^{I/(I-1)})$, there can be infinitely many NEs, all in Λ_C, via numerical simulations. For different initial price settings, the game converges to different NE.

Theorem A.12. *The game $\mathcal{G}_{\mathcal{P}}$ attains a pure NE which depends on the value of parameter α as follows:*

- *If $\alpha \in A_1 = (0, e/I)$, there is a unique NE $\lambda^* \in \Lambda_A$, with $\lambda^* = (\lambda_i^* = 1 : i \in \mathcal{I})$ and respective market equilibrium $\mathbf{x}^* \in X_A$.*
- *If $\alpha \in A_3 = (e^{\frac{I}{I-1}}/I, \infty)$, there is a unique NE $\lambda^* \in \Lambda_B$, with $\lambda^* = (\lambda_i^* = \frac{I}{I-1} : i \in \mathcal{I})$, which induces a*

respective market equilibrium $\mathbf{x}^* \in X_B$.

- *If* $\alpha \in A_2 = [e/I, e^{\frac{I}{I-1}}/I]$, *there exist infinitely many NEs,* $\lambda^* \in \Lambda_C$, *and each one of them yields a respective market stationary point* $\mathbf{x}^* \in X_C$.

Proof: Lemma A.6 proves that the game $\mathcal{G}_{\mathcal{P}}$ is a finite ordinal potential game. Therefore, it always attains a pure NE. Lemma A.7 proves the case for $\alpha \in A_1 = (0, e/I)$. Lemma A.9 proves the case for $\alpha \in A_3 = (e^{\frac{I}{I-1}}/I, \infty)$. The case for $\alpha \in A_2 = [e/I, e^{\frac{I}{I-1}}/I]$ is proven in Lemma A.10 and Lemma A.11. ■