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$E(\phi\ U(\psi_1 \vee \psi_2))$ and $E(\phi\ U\,\psi_1) \vee E(\phi\ U\,\psi_2)$

In order to prove an equivalence we need to show that the forward and the converse implications are true.

Suppose we have for some atoms $\alpha,\ \beta$ the formula $E(\alpha\ U\ \beta)$. By definition, that is equivalent to $\pi \mid=^{\mathrm{M}}(\alpha\ U\ \beta)$ for some path π of M starting at s . Which is further equivalent to $\exists\pi$ path of M starting at s such that $\exists i \geq 0.(\mathrm{M}, \pi(i) \mid= \beta \wedge \forall j < i.\mathrm{M}, \pi(j) \mid= \alpha)$.

Applying this result to both sides we get paths π, π_1, π_2 ; qualified constants i, i_1, i_2 respectively for formulas $E(\phi\ U(\psi_1 \vee \psi_2))$, $E(\phi\ U\,\psi_1)$ and $E(\phi\ U\,\psi_2)$.

Either of π_1 and π_2 can be used as π along with its corresponding constant to make the converse implication true. On the same token π can be used as π_1 or π_2 along with its constant based on which one of ψ_1 and ψ_2 was true to make the forward implication true. Thus, we have logical equivalence between two sides of the formula.

QED

$A(\phi\ U(\psi_1 \vee \psi_2))$ and $A(\phi\ U\,\psi_1) \vee A(\phi\ U\,\psi_2)$

For these formulas, previous line of reasoning holds for the converse implication. However, it fails for the forward implication. Thus, these formulas are not equivalent.

As a basic overview of why that is the case, take ψ_1 and ψ_2 to be mutually exclusive. As in ψ_1 XOR ψ_2 always holds. Then, suppose $A(\phi\ U(\psi_1 \vee \psi_2))$ holds. Because ψ_1 and ψ_2 are mutually exclusive, neither of $A(\phi\ U\,\psi_1)$ and $A(\phi\ U\,\psi_2)$ can hold for all paths.

For the concrete counter-example, take

$$
\begin{aligned}
&\textit{atoms} = \{\psi_1, \psi_2\} \\
&\phi = \neg\psi_1 \wedge \neg\psi_2 \\
&\mathrm{M} = \{\{s_0, s_1, s_2\}, \{(s_0 \rightarrow s_1), (s_0 \rightarrow s_2), (s_1 \rightarrow s_2), (s_2 \rightarrow s_1)\}, L\} \\
&\text{where } L(s_0) = \phi, L(s_1) = \psi_1 \text{ and } L(s_2) = \psi_2
\end{aligned}
$$

Thus, we get a Kripke structure where ψ_1 and ψ_2 are mutually exclusive. Then, using

$$A(\alpha\ \mathrm{U}\ \beta) \equiv AF\beta \wedge \neg(E(\neg\beta\ \mathrm{U}\ (\neg\beta \wedge \neg\alpha)))$$

we transform formulas into their adequate forms such that

$$
\begin{aligned}
&A(\phi\ \mathrm{U}\ (\psi_1 \vee \psi_2)) \equiv AF(\neg\psi_1 \wedge \neg\psi_2) \wedge \neg E((\neg\psi_1 \wedge \neg\psi_2)\ \mathrm{U}\ \textit{false}) \\
&A(\phi\ \mathrm{U}\ \psi_1) \vee A(\phi\ \mathrm{U}\ \psi_1) \equiv \neg(\neg(AF\,\psi_1 \wedge \neg E(\neg\psi_1\ \mathrm{U}\ (\neg\psi_1 \wedge \neg\psi_2))) \vee \neg(AF\,\psi_2 \wedge \neg E(\neg\psi_2\ \mathrm{U}\ (\neg\psi_1 \wedge \neg\psi_2))))
\end{aligned}
$$

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Skipping to the conclusion for the sake of brevity, if we start from the state s_0 then we should get the desired result that these formulas are not equivalent. Since it would hold for the left hand side formula but not for the right hand side formula.

QED