EE302 - Feedback Systems **Spring 2019**

Lecture 3

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3.1 Modeling of Mechanical Systems and Their Electrical Analogy

We use Kirchhoff's Current and Voltage laws to derive the dynamical (and static) relationships in Electrical Circuits. Similarly, we utilize Newton's laws of motion to derive equations of motion in (rigid body) mechanical dynamical systems.

3.1.1 Mechanical vs. Electrical Analogy Between Dependent Variables

There exist two different analogies that we can construct between electrical and mechanical systems. Mathematically, there is no difference between the two approaches. In this lecture, we will learn one of these analogies.

In electrical circuits, the core variables are Voltage, V, and current, I, whereas in translational mechanical systems, core variables are translational velocity, ν, and force, f. Similarly, in rotational mechanical systems, the core variables are angular velocity, ωn, and torque τ.

Voltage, $V \iff$ **Velocity,** $\nu \iff$ **Angular Velocity,** ω

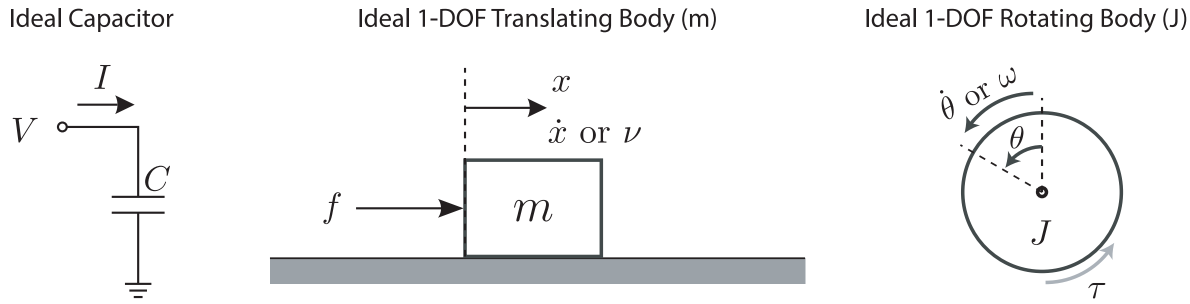
In electrical systems, voltage, also called electric potential difference, accounts for the difference in electric potential between two points. When we refer to the voltage of a node/point, we always measure it with respect to a reference point, e.g. *ground*. In mechanical systems, we measure the velocity either between two points in space, or (which is more general) with respect to an inertial reference frame, e.g. *ground* or *earth* in general. The analogy is similar with angular velocity. For this reason, we say that voltage, linear velocity, and angular velocity are the analog variables.

Current, $I \iff$ **force,** $f \iff$**Torque,** τ

In electrical systems, the current is the flow (or rate of change of) of electric charge and carried by electrons in motion. Roughly speaking, based on Newton's second law, the force acting upon a (rigid) body is equal to the rate of change of momentum related to this specific force component. Momentum can be considered as an analog of the electrical charge in this case. A similar analogy can also be constructed using Torque and Angular Momentum. For this reason, we say that Current, Force, and Torque are the analog variables.

3.1.2 Capacitor C, 1-DOF Translating Body with Mass m, and 1-DOF Rotating Body with Inertia J

If we follow the analogs between the variables, we can see that ideal capacitor for which one end is connected to the ground, 1-DOF translating body with a mass of m, and 1-DOF rotating body with an inertia of J analogs of each other. These are all ideal energy storage elements in their modeling domains and they are illustrated in the figure below.



The ODEs that govern the dynamics of these elements are provided below

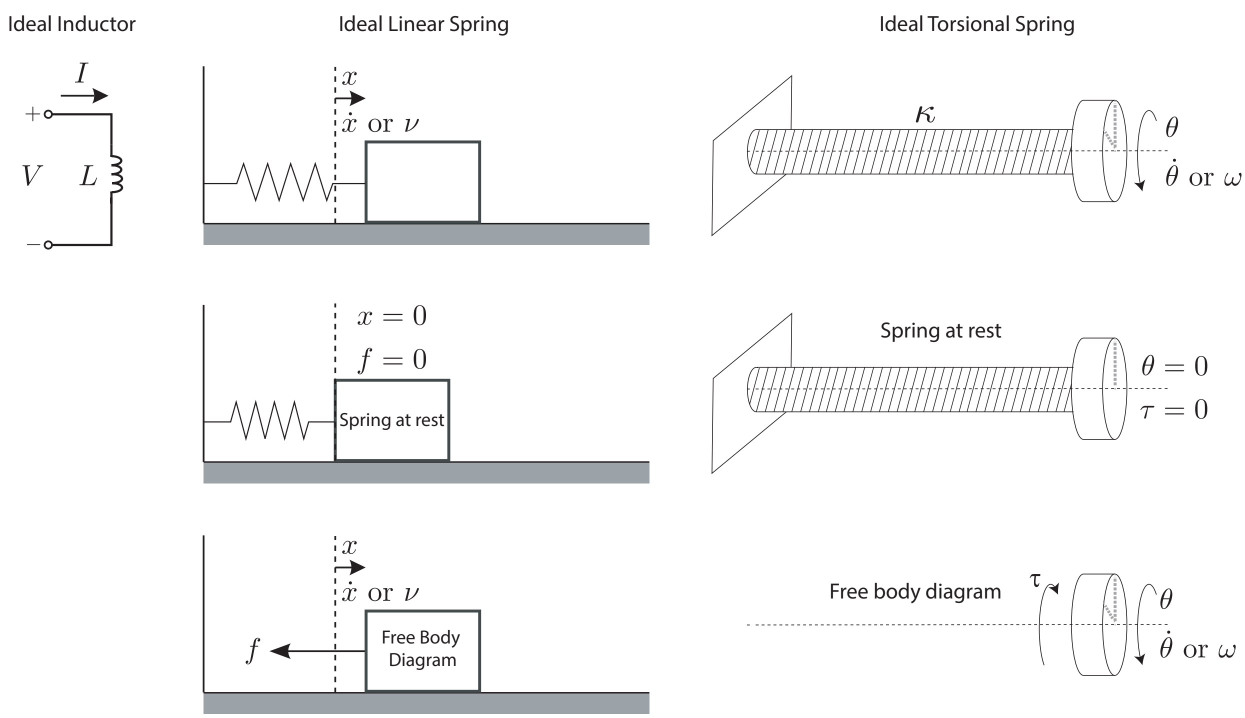
$$
\begin{aligned}
\text{Capacitor}: &\quad C\dot{V}(t) = I(t) \\
\text{Mass}: &\quad m\dot{\nu} = f(t) \text{ (or } m\ddot{x} = f(t)) \\
\text{Inertia}: &\quad J\dot{\omega} = \tau(t) \text{ (or } J\ddot{\theta} = \tau(t))
\end{aligned}
$$

Based on these equations we can reach the following (system) parameter analogy as

$$C \equiv m \equiv J$$

3.1.3 Inductor L, Translational Spring k, and Torsional Spring κ

If we follow the analogs between the variables we can see that Ideal inductor (L), linear translational spring (k), and linear torsional spring (κ) are analogs of each other. These are also ideal energy storage elements in their modeling domains and they are illustrated in the figure below.



The ODEs that govern the dynamics of these elements are provided below

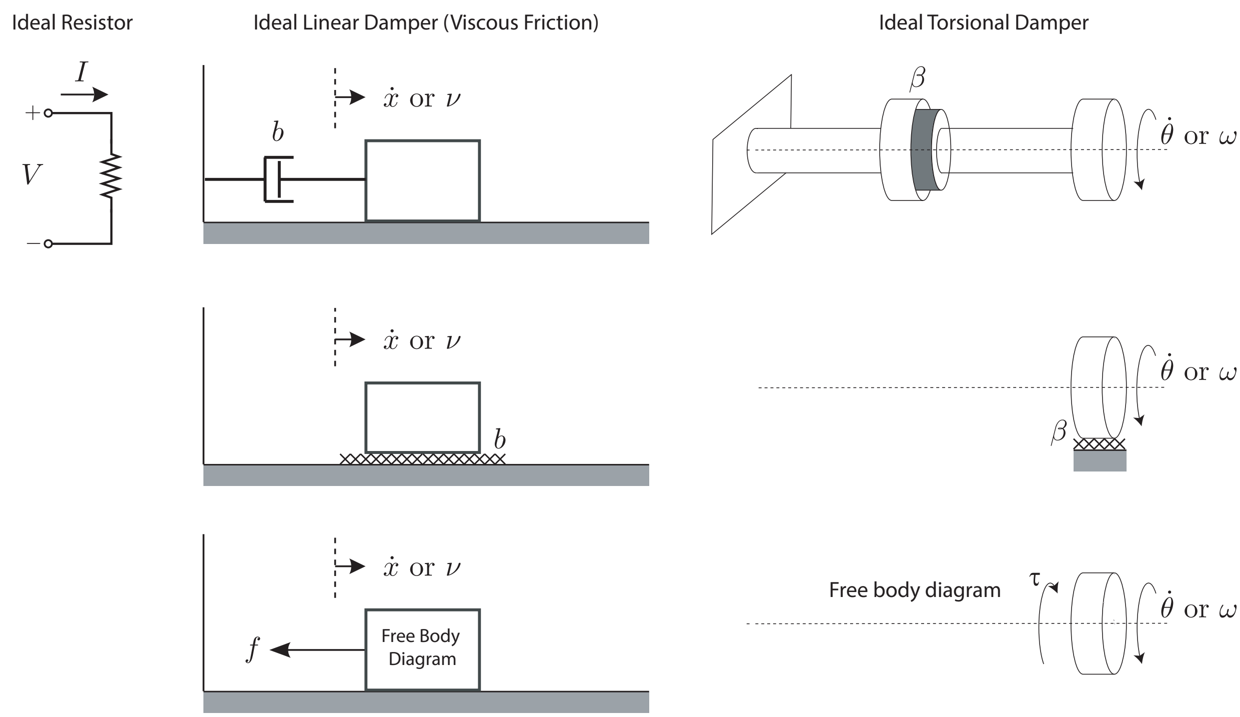
$$\begin{aligned}
\text{Induction}: &\quad L\dot{I}(t) = V(t) \\
\text{TranslationalSpring}: &\quad f(t) = kx(t) \;\rightarrow\; \frac{1}{k}\dot{f}(t) = \nu(t) \\
\text{TorsionalSpring}: &\quad \tau(t) = \kappa\theta(t) \;\rightarrow\; \frac{1}{\kappa}\dot{\tau}(t) = \omega(t)
\end{aligned}$$

Based on these equations we can reach the following (system) parameter analogy as

$$L \equiv \frac{1}{k} \equiv \frac{1}{\kappa}$$

3.1.4 Resistor R, Damper (Viscous Friction) b, and Torsional Damper β

If we follow the analogs between the variables we can see that Ideal resistor (R), linear translational damper (k), and linear torsional damper (κ) are analogs of each other. These elements are ideal fully passive dissipative elements. Thus, these are memoryless (static) components as opposed to the previous elements. These elements are illustrated in the figure below.



The algebraic equations that govern the statics of these elements are provided below

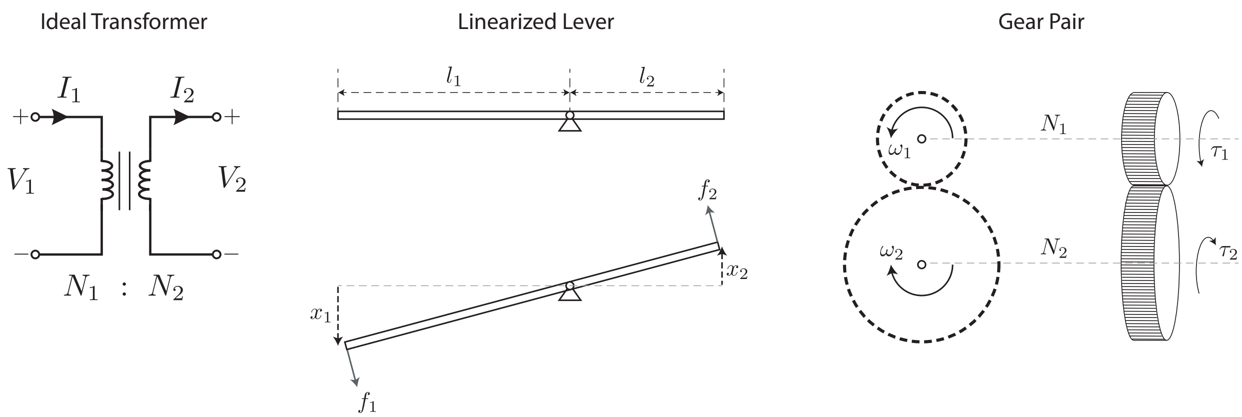
$$\begin{aligned}
\text{Resistor}: &\quad V(t) = RI(t) \\
\text{Translational Damper}: &\quad f(t) = b\dot{x}(t) \text{ or } \frac{1}{b}f(t) = \nu(t) \\
\text{Torsional Damper}: &\quad \tau(t) = \beta\dot{\theta}(t) \;\rightarrow\; \frac{1}{\beta}\tau(t) = \omega(t)
\end{aligned}$$

Based on these equations we can reach the following (system) parameter analogy as

$$R \equiv \frac{1}{b} \equiv \frac{1}{\beta}$$

3.1.5 Ideal Transformer, Linearized Lever, and Gear Pair

In both electrical and mechanical systems, we have transmission elements. In their ideal form, they conserve the energy after the transformation. In electrical systems, transformer is the component that achieves the transmission. In translational mechanical systems a linearized lever can achieve this under the assumption of small movements, where as for rotational systems a gear pair is one of the many solutions for mechanical transmission. These components are illustrated in the figure below.



The algebraic equations that govern the statics of these elements are provided below

$$\text{Electrical Transformer}: \quad \begin{aligned} \tfrac{V_1}{N_1} &= \tfrac{V_2}{N_2} \\ I_1 N_1 &= I_2 N_2 \end{aligned}$$

$$\text{Lever}: \quad \begin{aligned} \tfrac{\nu_1}{l_1} &= \tfrac{\nu_2}{l_2} \\ f_1 l_1 &= f_2 l_2 \end{aligned}$$

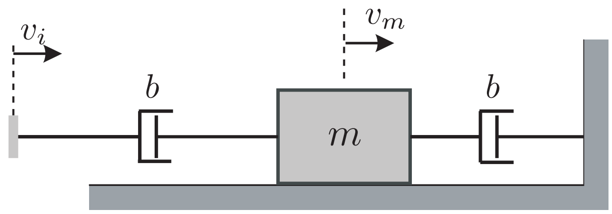
$$\text{Gear} - \text{Pair}: \quad \begin{aligned} \tfrac{\omega_1}{r_2} &= \tfrac{\omega_2}{r_1} \\ \tau_1 r_2 &= \tau_2 r_1 \end{aligned}$$

Based on these equations, it is to see that following (system) parameter analogy as

$$\frac{N_1}{N_2} \equiv \frac{l_1}{l_2} \equiv \frac{r_2}{r_1}$$

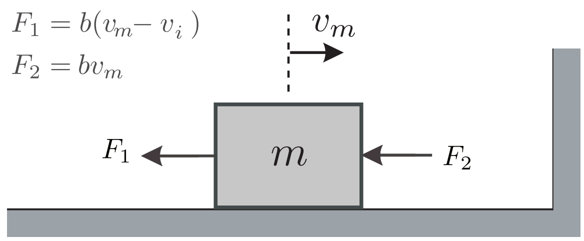
3.2 Examples

Ex 3.1. Let's consider the following translational mechanical system. The input of the system is $v_i(t)$ which is the velocity of the one side of the first damper. The output of the system is $v_m(t)$, i.e. the velocity of the mass.



1. Given that $u(t) = v_i(t)$ and $y(t) = v_m(t)$, find the transfer function $G(s) = \frac{Y(s)}{U(s)}$

   Let's first draw the FBD and then derive the equations of motion

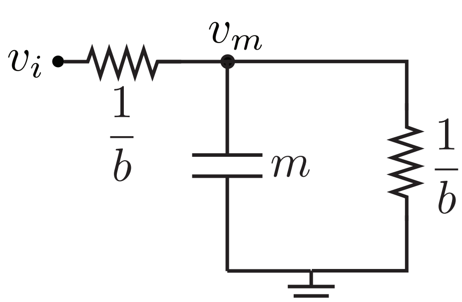


$$m\dot{v}_m = -F_1 - F_2 = bv_i - 2bv_m$$
$$m\dot{y} + 2by = bu$$
$$G(s) = \frac{Y(s)}{U(s)} = \frac{b}{ms + 2b} = \frac{b/m}{s + (2b)/m}$$

2. Now, let's construct an electrical circuit analog of the mechanical system.
   Let

$$V_i \equiv v_i \quad I_i \equiv f_i$$

   then we can build the circuit analog as in the illustration below.

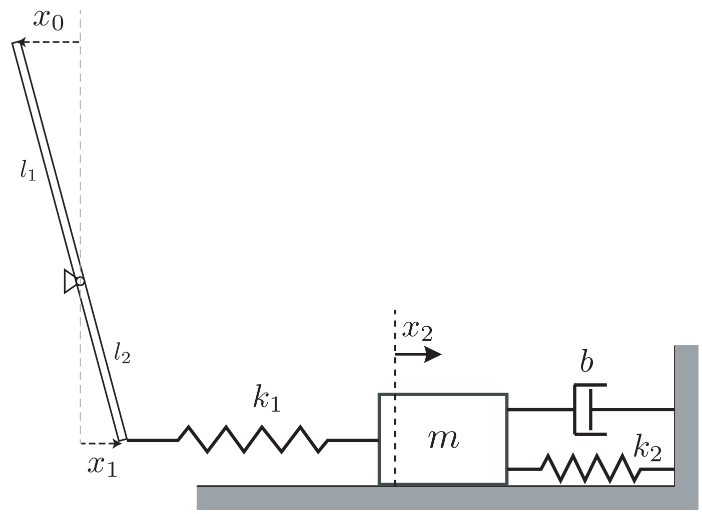


3. Finally, compute $G(s)$ by analyzing the electrical analog circuit.
   We simply apply KCL

$$\frac{v_i - v_m}{1/b} = I_m + \frac{v_m}{1/b} \quad \Rightarrow \quad I_m = m\dot{v}_m$$
$$bv_i = m\dot{v}_m + 2bv_m \quad \Rightarrow \quad bu = m\dot{y} + 2by$$
$$G(s) = \frac{Y(s)}{U(s)} = \frac{b/m}{s + (2b)/m}$$

Ex 3.2. Let's consider the following translational mechanical system. It is given that when the lever is in vertical position, $[x_0\ x_1\ x_2] = 0$ and springs are at their rest length positions.

1. Given that $u(t) = x_0(t)$ and $y(t) = x_2(t)$, find the ODE of the system dynamics.



$$m\ddot{x}_2 = -b\dot{x}_2 - k_2x_2 + k_1(x_1 - x_2)$$
$$x_1 = x_0\frac{l_2}{l_1}$$
$$m\ddot{x}_2 + b\dot{x}_2 + (k_1 + k_2)x_2 = k_1\frac{l_2}{l_1}x_0$$
$$\ddot{y} + \frac{b}{m}\dot{y} + \frac{k_1 + k_2}{m}y = \frac{k_1}{m}\frac{l_2}{l_1}u$$

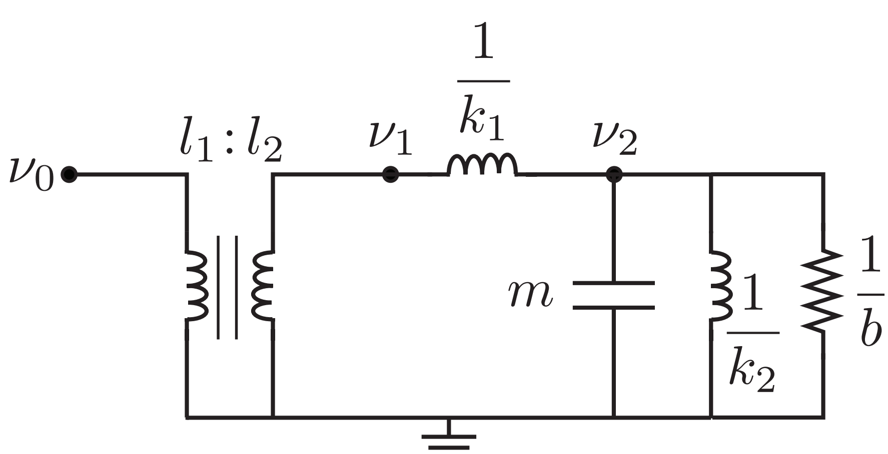
2. Find the transfer function for same input-output pair.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{\frac{k_1}{m}\frac{l_2}{l_1}}{s^2 + \frac{b}{m}s + \frac{k_1+k_2}{m}}$$

3. Now, let's construct an electrical circuit analog of the mechanical system.
   Let

$$V_i \equiv \dot{x}_i$$
$$I_i \equiv f_i$$

then we can build the circuit analog as in the illustration below.



Let's also compute $\frac{\mathcal{V}_2(s)}{\mathcal{V}_\prime(s)}$ using node voltage analysis in impedance domain.

$$\begin{aligned}
\frac{\mathcal{V}_2(s)-\mathcal{V}_1(s)}{\frac{s}{k_1}}+\frac{\mathcal{V}_2(s)}{\frac{1}{ms}}+\frac{\mathcal{V}_2(s)}{\frac{s}{k_2}}+\frac{\mathcal{V}_2(s)}{\frac{1}{b}}&=0\\
\mathcal{V}_2(s)\left[ms+b+\frac{k_1+k_2}{s}\right]&=V_1(s)\left[\frac{k_1}{s}\right]\\
\mathcal{V}_2(s)\left[ms^2+bs+(k_1+k_2)\right]&=V_1(s)\left[k_1\right]
\end{aligned}$$

Since ideal transformer has the following relation, $\mathcal{V}_1(s)=\frac{l_2}{l_1}\mathcal{V}_0(s)$, we have the following transfer function between $\mathcal{V}_0(s)$ and $\mathcal{V}_2(s)$

$$\frac{\mathcal{V}_2(s)}{\mathcal{V}_0(s)}=\frac{\frac{k_1}{m}\frac{l_2}{l_1}}{s^2+\frac{b}{m}s+\frac{k_1+k_2}{m}}$$

Obviously this transfer function is equal to $G(s)$ computed from directly mechanical system and considering positional variables.

4. Convert the derived ODE into a state-space form

   We will solve the problem using a different approach. First let's integrate the ODE twice

$$y=\int\left[-\frac{b}{m}y+\int\left\{-\frac{k_1+k_2}{m}y+\frac{k_1}{m}\frac{l_2}{l_1}u\right\}dt\right]dt$$

   Then let the stat variable definitions be

$$\begin{aligned}
x_1=y&=\int\left[-\frac{b}{m}y+\int\left\{-\frac{k_1+k_2}{m}y+\frac{k_1}{m}\frac{l_2}{l_1}u\right\}dt\right]dt\\
x_2&=\int\left\{-\frac{k_1+k_2}{m}y+\frac{k_1}{m}\frac{l_2}{l_1}u\right\}dt
\end{aligned}$$

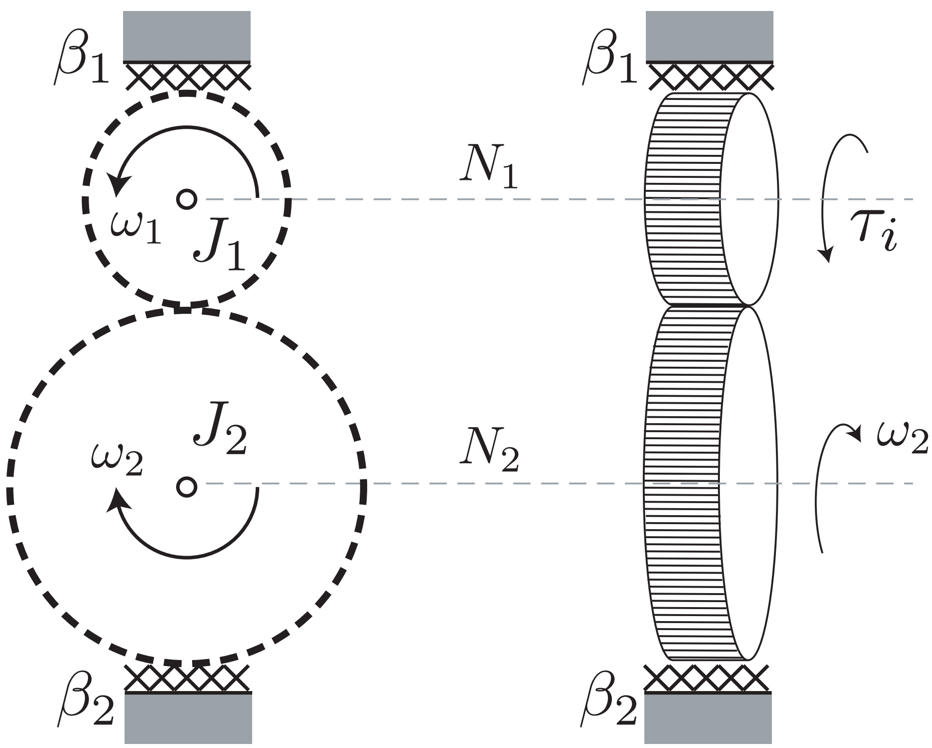
   Then state-equations take the form

$$\begin{aligned}
\dot{x}_1&=\left[-\frac{b}{m}y+\int\left\{-\frac{k_1+k_2}{m}y+\frac{k_1}{m}\frac{l_2}{l_1}u\right\}dt\right]\\
&=-\frac{b}{m}x_1+x_2\\
\dot{x}_2&=\left\{-\frac{k_1+k_2}{m}y+\frac{k_1}{m}\frac{l_2}{l_1}u\right\}\\
&=-\frac{k_1+k_2}{m}x_1+\frac{k_1}{m}\frac{l_2}{l_1}u
\end{aligned}$$

   If we gather the obtained state-equations in matrix form we obtain the following state-space representation

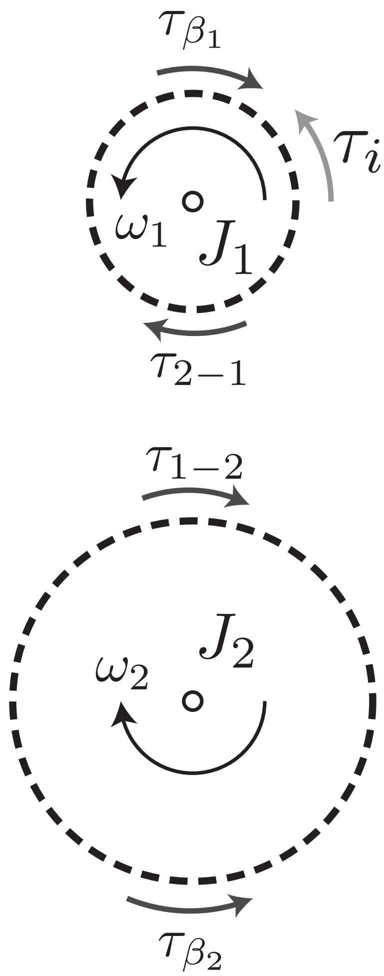
$$\begin{aligned}
\dot{x}&=\begin{bmatrix}-\frac{b}{m} & 1\\ -\frac{k_1+k_2}{m} & 0\end{bmatrix}x+\begin{bmatrix}0\\ \frac{k_1}{m}\frac{l_2}{l_1}\end{bmatrix}u\\
y&=\begin{bmatrix}1 & 0\end{bmatrix}x
\end{aligned}$$

Ex 3.3. Let's consider the following gear system. Unlike ideal gear pair case, now each gear has its own inertia, J_1 and J_2, as well as both gears are affected by viscous friction, β_1 and β_2 , due to mechanical contact with the environment.



1. Given that there is an external torque, τ_i, acting on the first gear is the input of the system, and the rotational speed of the second gear, ω_2, is the output of the system, find the ODE of the gear-box dynamics.

First let's draw the free-body diagrams of both gears separately and then write the equations of motion for each body.



$$
\begin{aligned}
J_1\dot{\omega}_1 &= \tau_i - \tau_{\beta_1} - \tau_{2-1}\\
&= \tau_i - \beta_1\omega_1 - \tau_{2-1}\\
J_2\dot{\omega}_2 &= -\tau_{\beta_2} + \tau_{1-2}\\
&= -\beta_2\omega_2 + \tau_{1-2}
\end{aligned}
$$

Based on the gear kinematics we know that

$$
\begin{aligned}
\omega_1 &= \frac{N_2}{N_1}\omega_2\\
\tau_{1-2} &= \frac{N_2}{N_1}\tau_{2-1}
\end{aligned}
$$

Thus we have the following derivations

$$
\begin{aligned}
\tau_{2-1} &= \tau_i - J_1\frac{N_2}{N_1}\dot{\omega}_2 - \beta_1\frac{N_2}{N_1}\omega_2\\
\tau_{1-2} &= \frac{N_2}{N_1}\tau_i - J_1\left(\frac{N_2}{N_1}\right)^2\dot{\omega}_2 - \beta_1\left(\frac{N_2}{N_1}\right)^2\omega_2
\end{aligned}
$$

Hence the ODE governing the dynamics can be formed as

$$
\left[J_2 + J_1\left(\frac{N_2}{N_1}\right)^2\right]\dot{y} + \left[\beta_2 + \beta_1\left(\frac{N_2}{N_1}\right)^2\right]y = \frac{N_2}{N_1}u
$$

It can be seen that the resultant ODE is a first order ODE. Ee can also consider the whole system as a single rotating body with an effective total inertia of $J_T = \left[J_2 + J_1\left(\frac{N_2}{N_1}\right)^2\right]$ and effective total viscous friction of $\beta_T = \left[\beta_2 + \beta_1\left(\frac{N_2}{N_1}\right)^2\right]$.

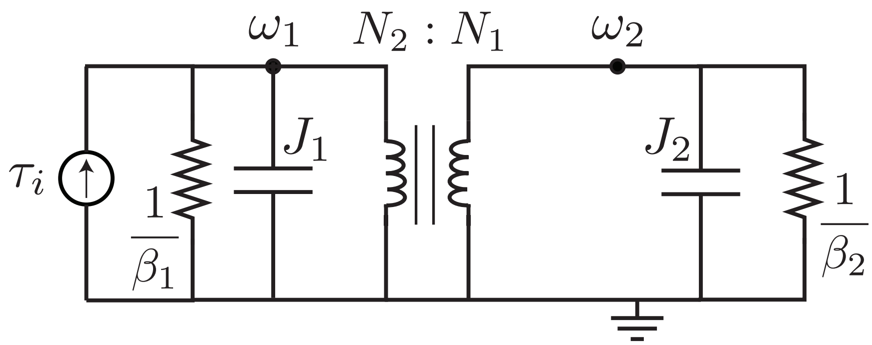
Take Home Problem: Now let's assume that output is $y(t) = \omega_1(t)$, and govern the ODE and re-compute the new effective inertia and viscous friction.

2. Compute the transfer function

$$
\begin{aligned}
G(s) = \frac{Y(s)}{U(s)} &= \frac{\frac{N_2}{N_1}}{\left[J_2 + J_1\left(\frac{N_2}{N_1}\right)^2\right]s + \left[\beta_2 + \beta_1\left(\frac{N_2}{N_1}\right)^2\right]}\\
&= \frac{\frac{N_2}{N_1}}{J_T s + \beta_T} = \frac{\frac{N_2}{N_1}\frac{1}{J_T}}{s + \frac{\beta_T}{J_T}}
\end{aligned}
$$

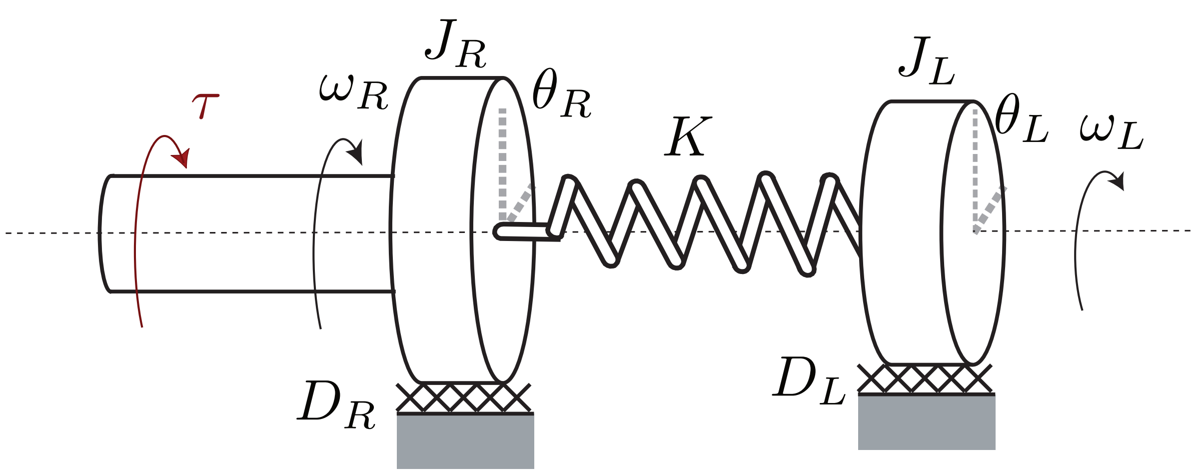
3. Now, build the electrical circuit analog of the gear-box system

The circuit diagram is given below



Take Home Problem: Solve the electrical circuit and compute the transfer function $G(s)$

Ex 3.4. The figure below illustrates the model of the target system. Input of the system is the external torque, $u(t) = \tau(t)$, acting on the R axis and the output of the system is the angular velocity of the Load, i.e. $y(t) = \omega_L(t)$



Let's use the following state-definition

$$x(t) = \begin{bmatrix} \omega_L \\ \omega_R \\ \theta_R - \theta_L \end{bmatrix}$$

and then find a state-space representation for the system under this assumption. Indeed, since output is simply equal to one of the states, output equation is trivial

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t) + [0]u(t)$$

Let's first write the differential equation on the motor axis

$$J_R \cdot \dot{\omega}_R = \tau - D_R\omega_R - \kappa(\theta_R - \theta_L)$$

If we replace the variables with state variables, we obtain.

$$J_R \cdot \dot{x}_2 = u - D_R x_2 - K x_3$$
$$\dot{x}_2 = \frac{1}{J_R}u - \frac{D_R}{J_R}x_2 - \frac{\kappa}{J_R}x_3$$

Now let's write the differential equation on the load axis and perform the same change of variables operation

$$J_L \cdot \dot{\omega}_L = \kappa(\theta_R - \theta_L) - D_L \omega_L$$
$$\dot{x}_1 = \frac{K}{J_L}x_3 - \frac{D_L}{J_L}x_1$$

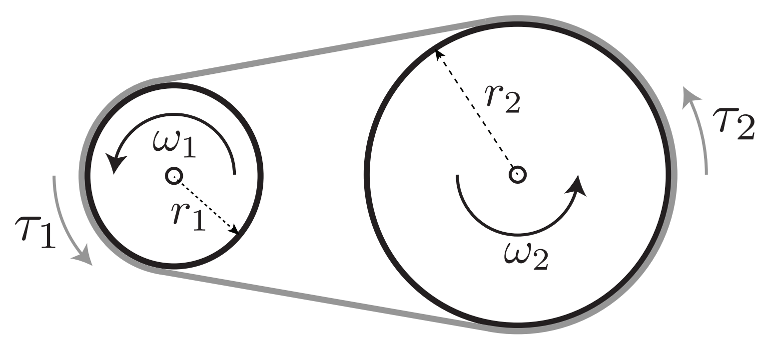
We need one more state-equation which can be drived as

$$\dot{x}_3 = \frac{d}{dt}(\theta_R - \theta_L) = \omega_R - \omega_L = x_2 - x_1$$

Based on our choice of state-definition, full state-space representation takes the form

$$\dot{x}(t) = \begin{bmatrix} \frac{-D_L}{J_L} & 0 & \frac{\kappa}{J_L} \\ 0 & -\frac{D_R}{J_R} & -\frac{\kappa}{J_R} \\ -1 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{J_R} \\ 0 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t) + [0]u(t)$$

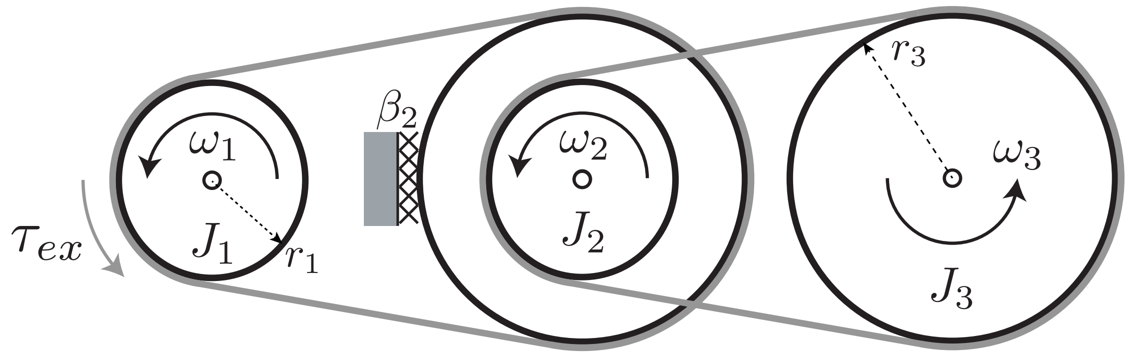
Ex 3.5. The mechanism illustration given below is an ideal belt-pulley mechanism. Fundamentally, it has the same kinematic relations with a gear pair system. The only difference is that, the direction of motion is preserved in a pulley system. r_1 and r_2 correspond to the radii of the first and second pulleys respectively.



$$\frac{\omega_1}{r_2} = \frac{\omega_2}{r_1}$$
$$\frac{\tau_1}{r_1} = \frac{\tau_2}{r_2}$$

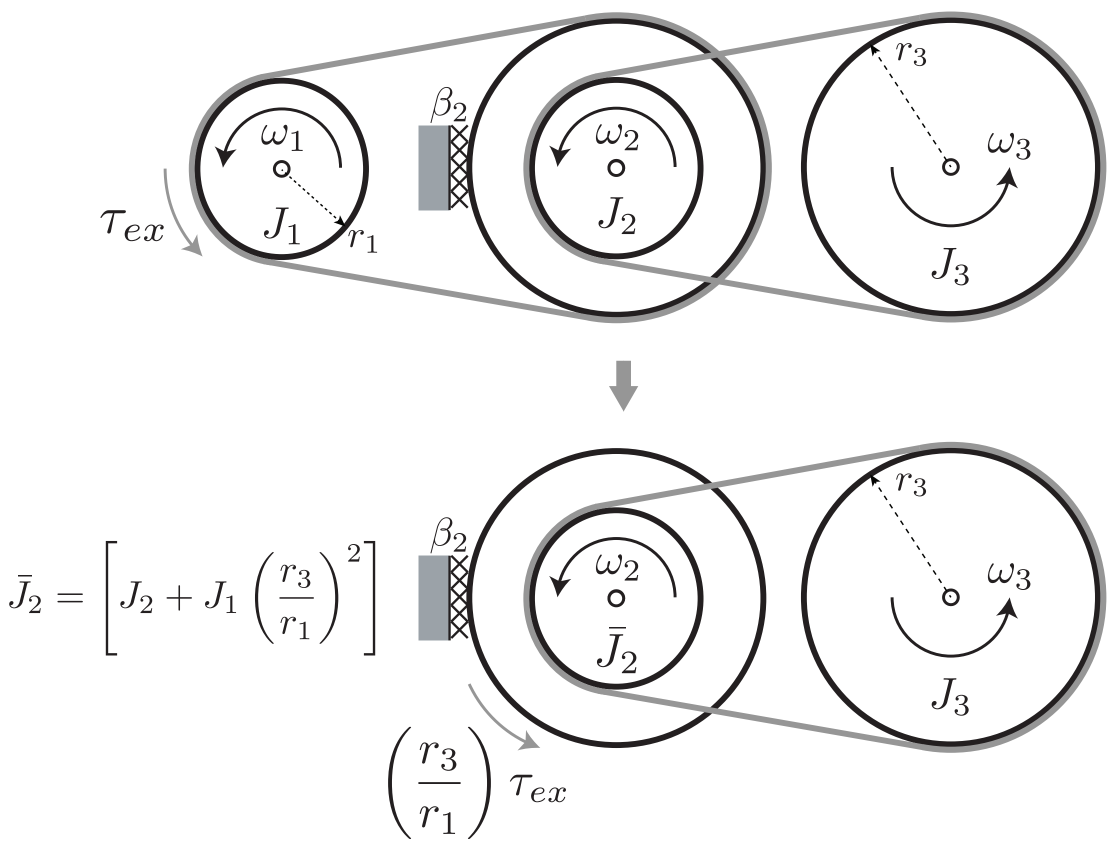
You will analyze the following belt-pulley system consisting of three pulleys and two belts. First pulley has a radius of r_1 and an inertia of J_1. Third pulley has a radius of r_3 and an inertia of J_3. Second pulley (in the middle) is connected to the first pulley through its outer disk, which has the same radius with the third pulley, i.e., $r_{2,o} = r_3$. Second pulley is also connected to the first pulley via its inner disk, which has the same radius with the first pulley, i.e., $r_{2,i} = r_1$. Second pulley has an inertia of J_2 (outer and inner disks move together). A linear rotational viscous damping with a damping constant β_2 also affects the motion of the second pulley.

Given that the external torque acting on the first pulley is the input, $u(t) = \tau_{ex}(t)$, and the angular velocity of the third pulley is the output, $y(t) = \omega_3(t)$, compute the transfer function of the system.



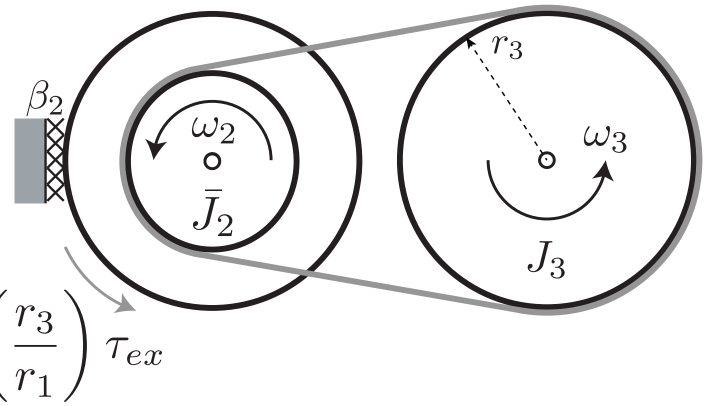
Let's solve this problem using the concept of reflected inertia, damping, and torque.

If we reflect the variables and parameters of first pullet to the second pulley we obtain

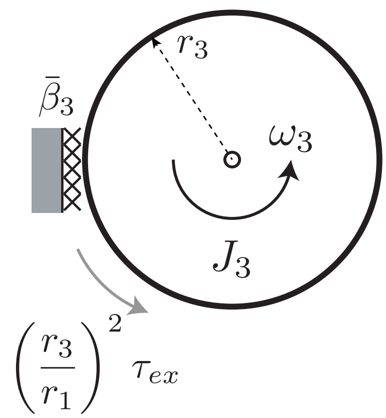


Now if we reflect the variables and parameters of the modified second pulley to the the third pulley we obtain

$$\bar{J}_2 = \left[J_2 + J_1 \left(\frac{r_3}{r_1} \right)^2 \right]$$



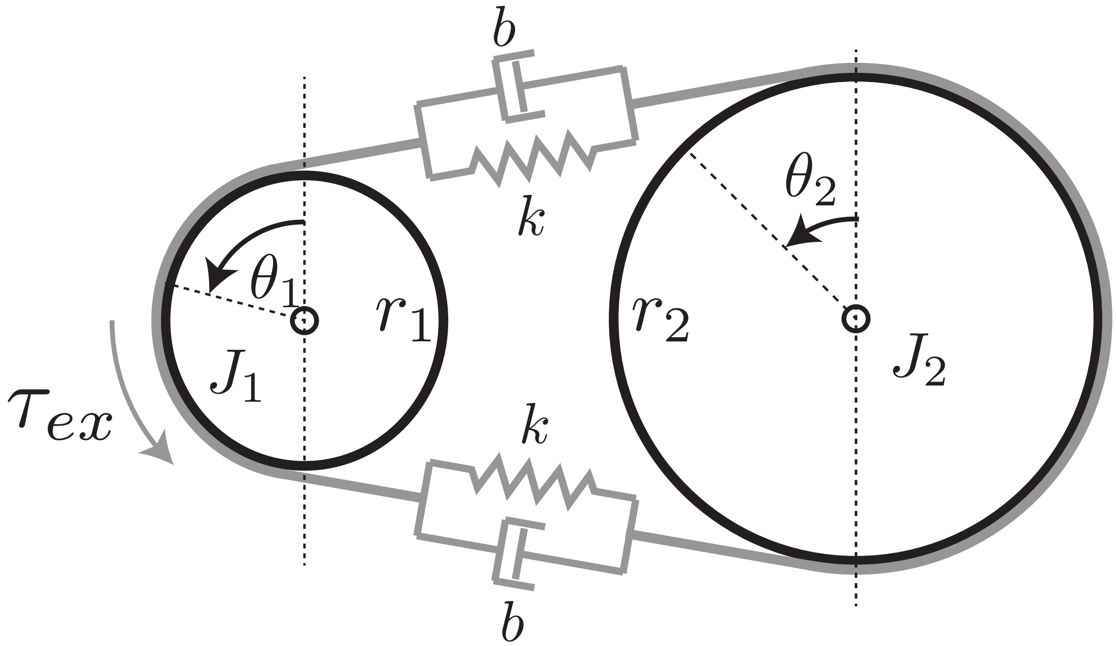
$$\bar{J}_3 = \left[J_3 + \bar{J}_2 \left(\frac{r_3}{r_1} \right)^2 \right]$$
$$= \left[J_3 + J_2 \left(\frac{r_3}{r_1} \right)^2 + J_1 \left(\frac{r_3}{r_1} \right)^4 \right]$$
$$\bar{\beta}_3 = \left[\beta_2 \left(\frac{r_3}{r_1} \right)^2 \right]$$



Hence, ode and transfer function of the system can be computed as

$$\bar{J}_3 \dot{\omega}_3 + \bar{\beta}_3 \omega_3 = \left(\frac{r_3}{r_1} \right)^2 \tau_{ex} \quad \rightarrow \quad \dot{y} + \frac{\bar{\beta}_3}{\bar{J}_3} = \frac{\left(\frac{r_3}{r_1} \right)^2}{\bar{J}_3} u$$
$$\frac{Y(s)}{U(s)} = \frac{\left(\frac{r_3}{r_1} \right)^2 \frac{1}{\bar{J}_3}}{s + \frac{\bar{\beta}_3}{\bar{J}_3}}$$

Ex 3.6. In some belt-pulley applications, ignoring the elasticity of the belt can be very crude and can lead to substantial modeling errors. In order to overcome this problem, a very common method is modeling the belt with a linear (translational) spring-damper as shown in the belt-pulley mechanism below. In this mechanism, first pulley has a radius of r_1 and inertia of J_1, where as the second pulley has a radius of r_2 and inertia of J_2. The spring-mass dampers (above and below) that model the elasticity of the belt have spring stiffnesses of k and damping constants of b.



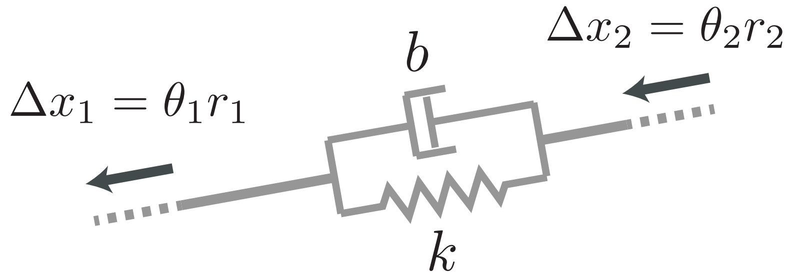
Given that the external torque acting on the first pulley is the input, $u(t) = \tau_{ex}(t)$, and the angular displacement of the second pulley is the output, $y(t) = \theta_2(t)$,

1. Find a state-space representation of the dynamics. (*Hint:* You can choose your state variables as $\mathbf{x} = \begin{bmatrix}\theta_1 \ \dot{\theta}_1 \ \theta_2 \ \dot{\theta}_2\end{bmatrix}^T$).
2. Compute the transfer function, $G(s) = Y(s)/U(s)$
3. Now, let the parameters of the system be equal to following numerical values

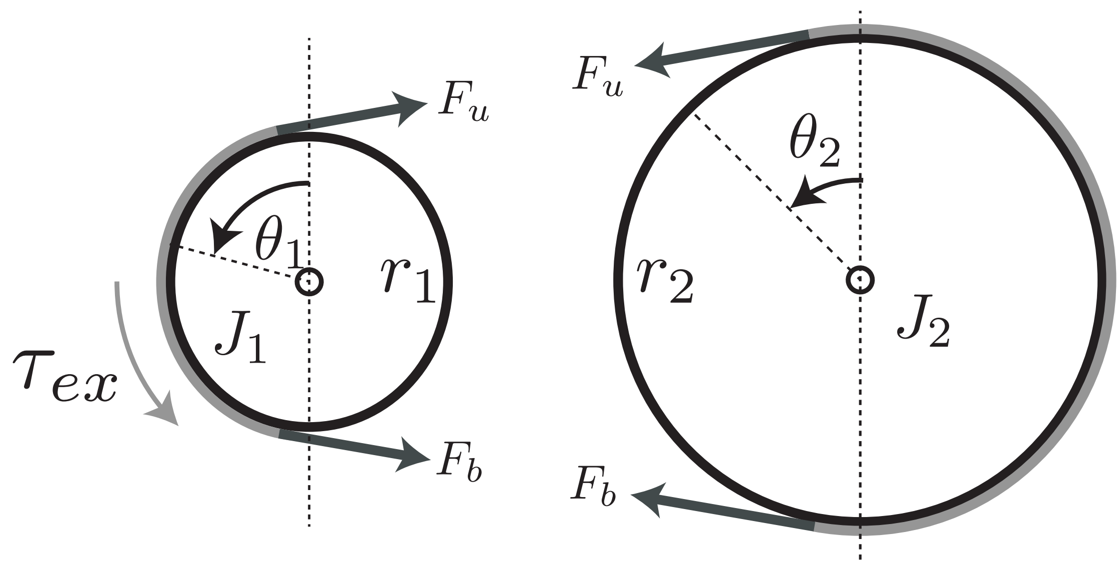
$$r_1 = 0.05\,m \ , \ r_2 = 0.1\,m \ , \ J_1 = 0.01\,kg \cdot m^2 \ , \ J_2 = 0.1\,kg \cdot m^2$$
$$k = 100\,N/m \ , \ b = 10\,N/(m \cdot s) \ .$$

Simplify both the state-space and transfer function representations using these numerical values. Finally convert the state-space form to the transfer function form and verify that converted transfer function is equal to the previously computed one. (*Hint:* You can use MATLAB's *ss2tf* command for conversion).

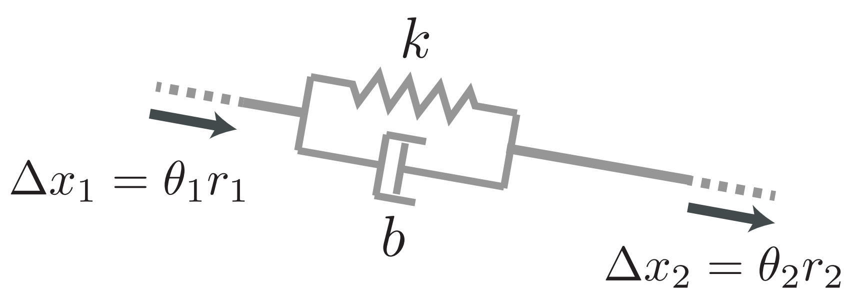
We assume that when $[\theta_1 \ \theta_2 \ \dot{\theta}_1 \ \dot{\theta}_2] = [0 \ 0 \ 0 \ 0]$ the mechanism is at rest condition. Then let's draw the free-body diagrams



b
$\Delta x_2 = \theta_2 r_2$
$\Delta x_1 = \theta_1 r_1$
k



F_u
F_u
θ_2
θ_1
r_1
r_2
J_2
J_1
τ_{ex}
F_b
F_b



k
$\Delta x_1 = \theta_1 r_1$
b
$\Delta x_2 = \theta_2 r_2$

Let's first write the spring force relations

$$
\begin{aligned}
F_u &= k\left(\Delta x_1 - \Delta x_2\right) + b\frac{d}{dt}\left(\Delta x_1 - \Delta x_2\right) \\
&= k\left(r_1\theta_1 - r_2\theta_2\right) + b\left(r_1\dot{\theta}_1 - r_2\dot{\theta}_2\right) \\
&= kr_1\theta_1 - kr_2\theta_2 + br_1\dot{\theta}_1 - br_2\dot{\theta}_2 \\
F_b &= k\left(-\Delta x_1 + \Delta x_2\right) + b\frac{d}{dt}\left(-\Delta x_1 + \Delta x_2\right) \\
&= k\left(-r_1\theta_1 + r_2\theta_2\right) + b\left(-r_1\dot{\theta}_1 + r_2\dot{\theta}_2\right) \\
&= -kr_1\theta_1 + kr_2\theta_2 - br_1\dot{\theta}_1 + br_2\dot{\theta}_2
\end{aligned}
$$

Now let's write the equations of motion of the individual bodies

$$
\begin{aligned}
J_1\ddot{\theta}_1 &= \tau_{ex} - F_u r_1 + F_b r_1 \\
&= \tau_{ex} + \left(-kr_1^2\theta_1 + kr_2r_1\theta_2 - br_1^2\dot{\theta}_1 + br_2r_1\dot{\theta}_2\right) + \left(-kr_1^2\theta_1 + kr_2r_1\theta_2 - br_1^2\dot{\theta}_1 + br_2r_1\dot{\theta}_2\right) \\
&= \tau_{ex} - 2kr_1^2\theta_1 + 2kr_2r_1\theta_2 - 2br_1^2\dot{\theta}_1 + 2br_2r_1\dot{\theta}_2 \\
J_2\ddot{\theta}_2 &= F_u r_2 - F_b r_2 \\
&= \left(kr_1r_2\theta_1 - kr_2^2\theta_2 + br_1r_2\dot{\theta}_1 - br_2^2\dot{\theta}_2\right) + \left(kr_1r_2\theta_1 - kr_2^2\theta_2 + br_1r_2\dot{\theta}_1 - br_2^2\dot{\theta}_2\right) \\
&= 2kr_1r_2\theta_1 - 2kr_2^2\theta_2 + 2br_1r_2\dot{\theta}_1 - 2br_2^2\dot{\theta}_2
\end{aligned}
$$

1. Let $\mathbf{x} = \left[\theta_1 \ \dot{\theta}_1 \ \theta_2 \ \dot{\theta}_2\right]^T$, then we can find a state-space representation

$$
\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{2kr_1^2}{J_1} & -\frac{2br_1^2}{J_1} & \frac{2kr_1r_2}{J_1} & \frac{2br_1r_2}{J_1} \\ 0 & 0 & 0 & 1 \\ \frac{2kr_1r_2}{J_2} & \frac{2br_1r_2}{J_2} & -\frac{2kr_2^2}{J_2} & -\frac{2br_2^2}{J_2} \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ \frac{1}{J_1} \\ 0 \\ 0 \end{bmatrix} u \\
y &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}\mathbf{x}
\end{aligned}
$$

2. Let's take the Laplace transform of the derived diffferential equations

$$
\begin{aligned}
\left[J_1 s^2 + 2br_1^2 s + 2kr_1^2\right]\Theta_1(s) &= U(s) + \left[2br_2r_1 s + 2kr_2r_1\right]Y(s) \\
\left[J_2 s^2 + 2br_2^2 s + 2kr_2^2\right]Y(s) &= \left[2kr_1r_2 + 2br_1r_2 s\right]\Theta_1(s)
\end{aligned}
$$

In order to simplify the expression let

$$
B_1 = 2br_1^2\ ,\ K_1 = 2kr_1^2\ ,\ B_2 = 2br_2^2\ ,\ K_2 = 2kr_2^2\ ,\ B_{12} = 2br_2r_1\ ,\ K_{12} = 2kr_1r_2
$$

Then we can have

$$
\begin{aligned}
&\left[J_1 s^2 + B_1 s + K_1\right]\Theta_1(s) = U(s) + \left[B_{12}s + K_{12}\right]Y(s)\\
&\left[J_2 s^2 + B_2 s + K_2\right]Y(s) = \left[B_{12}s + K_{12}\right]\Theta_1(s) \quad \rightarrow \quad \Theta_1(s) = \frac{\left[J_2 s^2 + B_2 s + K_2\right]}{\left[B_{12}s + K_{12}\right]}Y(s)\\
&Y(s)\left\{\left[J_1 s^2 + B_1 s + K_1\right]\frac{\left[J_2 s^2 + B_2 s + K_2\right]}{\left[B_{12}s + K_{12}\right]} - \left[B_{12}s + K_{12}\right]\right\} = U(s)\\
&Y(s)\left\{\left[J_1 s^2 + B_1 s + K_1\right]\frac{\left[J_2 s^2 + B_2 s + K_2\right]}{\left[B_{12}s + K_{12}\right]} - \left[B_{12}s + K_{12}\right]\right\} = U(s)
\end{aligned}
$$

Let $\frac{Y(s)}{U(s)} = \frac{N(s)}{D(s)}$, then

$$
\begin{aligned}
D(s) =& J_1 J_2 s^4 + (B_1 J_2 + B_2 J_1)s^3 + (-B_{12}^2 + B_1 B_2 + J_1 K_2 + J_2 K_1)s^2\\
&+ (B_1 K_2 + B_2 K_1 - 2B_{12}K_{12})s + (-K_{12}^2 + K_1 K_2)\\
D(s) =& J_1 J_2 s^4 + (B_1 J_2 + B_2 J_1)s^3 + (J_1 K_2 + J_2 K_1)s^2 + 0 + 0
\end{aligned}
$$

Finally the transfer function can be computed as

$$
\frac{Y(s)}{U(s)} = \frac{B_{12}s + K_{12}}{J_1 J_2 s^4 + (B_1 J_2 + B_2 J_1)s^3 + (J_1 K_2 + J_2 K_1)s^2}
$$

3. The state-space representation with the given coefficients take the form

$$
\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix} 0 & 100 & 0 & 0\\ -50 & -5 & 100 & 10\\ 0 & 0 & 0 & 1\\ 10 & 1 & -20 & -2 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0\\ 1\\ 0\\ 0 \end{bmatrix}u\\
y &= \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}\mathbf{x}
\end{aligned}
$$

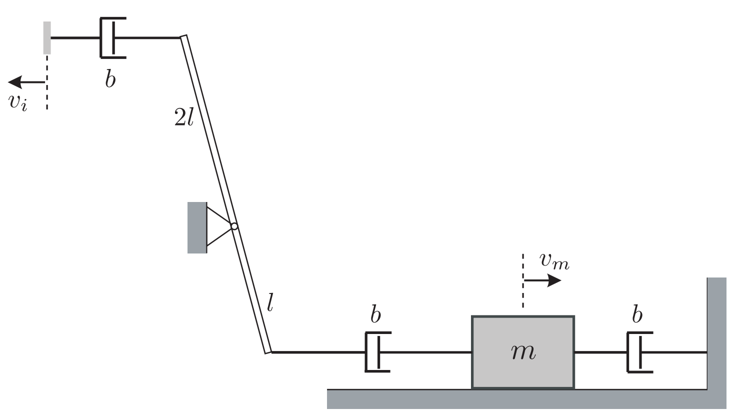
Transfer function with the given coefficients take the form

$$
G(s) = \frac{Y(s)}{U(s)} = \frac{100(s+10)}{s^4 + 7s^3 + 70s^2}
$$

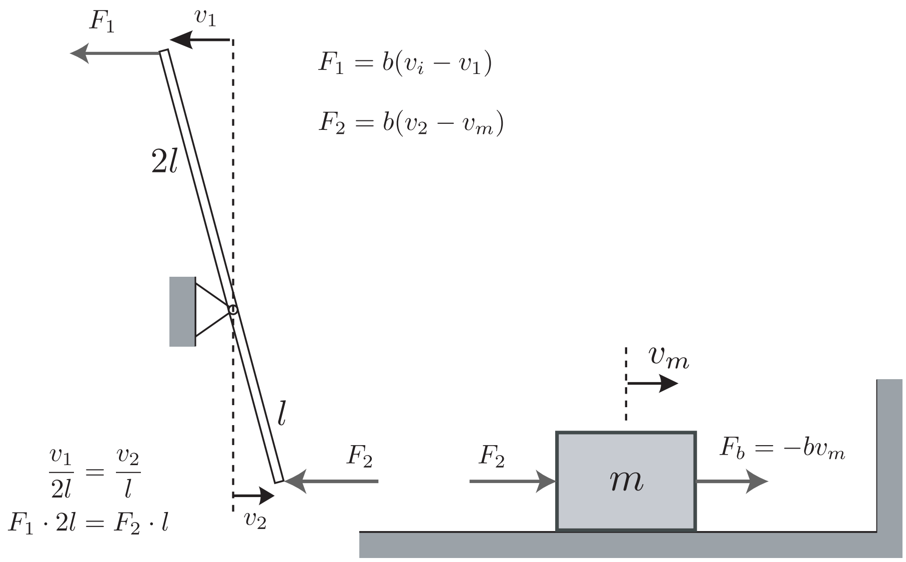
A sample MATLAB code piece which converts the state-space form to the transfer function form (in terms of numerator and denumerator coefficients) is provided below. It is clear that, the computed coefficients match the previous ones.

```
>> r1 = 0.05;
>> r2 = 0.1;
>> J1 = 0.01;
>> J2 = 0.1;
>> k = 100;
>> b = 10;
>>
>> A(1,:) = [0 1 0 0];
>> A(2,:) = [-2*k*r1^2/J1 -2*b*r1^2/J1 2*k*r1*r2/J1 2*b*r1*r2/J1];
>> A(3,:) = [0 0 0 1];
>> A(4,:) = [2*k*r1*r2/J2  2*b*r1*r2/J2 -2*k*r2^2/J2 -2*b*r2^2/J2];
>> A

A =

         0    1.0000         0         0
  -50.0000   -5.0000  100.0000   10.0000
         0         0         0    1.0000
   10.0000    1.0000  -20.0000   -2.0000

>> B = [0 ; 100 ; 0 ; 0];
>> C = [0 0  1 0];
>> D = 0;
>>
>> [num,denum] = ss2tf(A,B,C,D)

num =

   1.0e+03 *

         0         0         0    0.1000    1.0000


denum =

    1.0000    7.0000   70.0000    0.0000    0.0000

>> |
```

Ex 3.7. Mehcanical system with ideal lever



Input–output of the system is the are $u(t) = v_i(t)$, and $y(t) = v_m(t)$ respectively. If we draw the free body diagram of the mass and draw the kinematic realtions of the lever we obtain the following illsutration



FIrst let's concentrate on the lever side and try to eliminate intermediate variables

$$
\begin{aligned}
2F_1 &= F_2 \\
2(v_i - v_1) &= (v_2 - v_m) \\
v_2 + 2v_1 &= 2v_i + v_m \\
v_2 &= \frac{2}{5}v_i + \frac{1}{5}v_m
\end{aligned}
$$

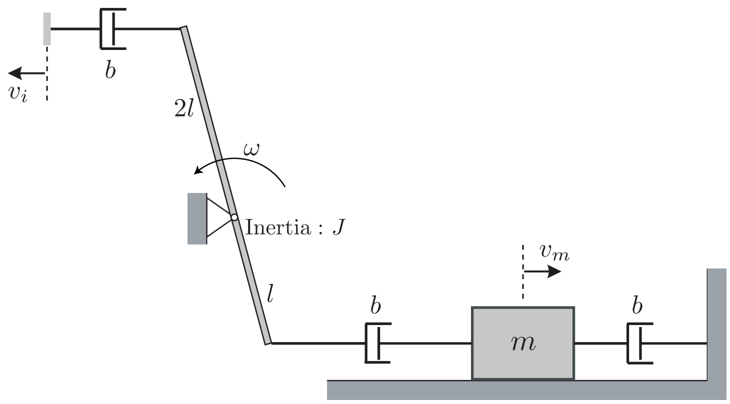
Now let's write equations of motion for the single body

$$
\begin{aligned}
m\dot{v}_M &= F_2 + F_b \\
m\dot{v}_M &= b(v_2 - v_m) - bv_m \\
\dot{v}_M &= \frac{2}{5}v_i + \frac{1}{5}v_m - v_m - v_m \\
\dot{v}_M + \frac{9}{5}v_m &= \frac{2}{5}v_i \\
\dot{y} + \frac{9}{5}y &= \frac{2}{5}u
\end{aligned}
$$

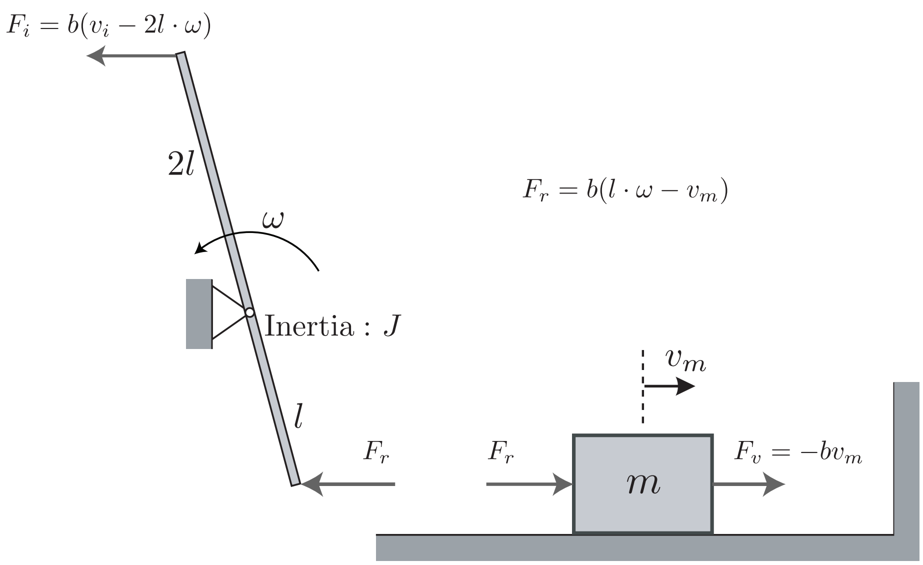
Transfer function simply takes the form

$$
\frac{Y(s)}{U(s)} = \frac{2/5}{s + 9/5}
$$

Ex 3.8. Mehcanical system with an non-ideal lever that has an inertia of $J = ml^2$



Input–output of the system is the are $u(t) = v_i(t)$, and $y(t) = v_m(t)$ respectively. If we draw the free body diagrams of the lever and the mass, and derive the force relations we obtain



If we write equations of motion for the lever and the mass, we obtain following differnetial equations

$$
\begin{aligned}
J\dot{\omega} &= F_i \times 2l - F_r \times l = (-5l^2 b)\omega + (2lb)v_i + (lb)v_m \\
\dot{\omega} + \frac{5b}{m}\omega &= \frac{b}{ml}v_m + \frac{2b}{ml}v_i \\
\dot{\omega} + 5\omega &= \frac{1}{l}v_m + \frac{2}{l}v_i
\end{aligned}
$$

,

$$
\begin{aligned}
m\dot{v}_m &= F_r + F_v = (-2b)v_m + (bl)\omega \\
\dot{v}_M + \frac{2b}{m}v_m &= \frac{bl}{m}\omega \\
\dot{v}_M + 2v_m &= l\,\omega
\end{aligned}
$$

If we take the Laplace transform of the derived differential equations

$$
\begin{aligned}
\Omega(s)\,[s+5] &= \frac{1}{l}Y(s) + \frac{2}{l}U(s) \\
Y(s)\,[s+2] &= l\,\Omega(s) \\
Y(s)\,[s+2]\,[s+5]\,\frac{1}{l} &= \frac{1}{l}Y(s) + \frac{2}{l}2U(s) \\
Y(s)(s+2)(s+5) &= Y(s) + 2U(s) \\
\frac{Y(s)}{U(s)} &= \frac{2}{s^2+7s+11}
\end{aligned}
$$

Now let's find a state-space representation for the given system. The ODE representation of the transfer function has the following form

$$\ddot{y} + 7\dot{y} + 11y = 2u$$

The system is a 2^{nd} order dynamical system. Let $\mathbf{x} = [y\ \dot{y}]^T$, then

$$
\begin{aligned}
\dot{x_1} &= \dot{y} = x_2 \\
\dot{x_2} &= \ddot{y} = -7\dot{y} - 11y + 2u = -7x_2 - 11x_1 + 2u \\
y &= x_1
\end{aligned}
$$

The state-representation with the chosedn state definition takes the form

$$
\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix} 0 & 1 \\ -11 & -7 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 0 \end{bmatrix}\mathbf{x} + [0]u
\end{aligned}
$$