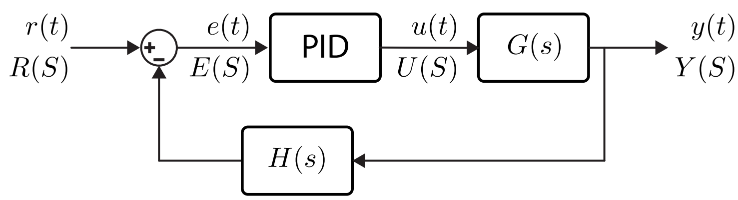
EE302 - Feedback Systems		Spring 2019
	Lecture 10	
Lecturer: Asst. Prof. M. Mert Ankarali		

10.1 PID Control Policy



A PID controller has the following forms in time and Laplace domain

$$u(t) = K_P e(t) + K_D \frac{d}{dt} e(t) + K_I \int_0^t e(t) dt$$
$$U(s) = \left[K_P + K_D s + K_I \frac{1}{s} \right] E(s)$$

In this lecture we will analyze the the effects of PID coefficients on the transient and steady-state performance on a 2^{nd} order plant.

10.1.1 Proportional (P) Controller

Let's assume that $H(s) = 1$ and $G(s)$ is a second order transfer function in general form

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Since $C(s) = K_P$, let's first analyze the steady-state error performance

$$G_{OL}(s) = \frac{K_P \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$
$$Type\ 0 \quad K_p = K_P$$

Thus steady-state error for unit-step and unit-ramp inputs can be find as

- Unit step: $e_{ss} = \frac{1}{1+K_P}$, i.e. $K_p \nearrow \Rightarrow \ e_{ss} \searrow$
  Unit ramp: $e_{ss} = \infty$

To sum up, higher K_P provides better steady-state performance. Now let's analyze transient performance.

$$\begin{aligned} T(s) &= \frac{Y(s)}{R(s)} = \frac{K_P\omega_n^2}{s^2 + 2\zeta\omega_n + (1+K_P)\omega_n^2} \\ \bar{\omega}_n &= \sqrt{1+K_P}\omega_n \\ \bar{\zeta} &= \zeta\frac{1}{\sqrt{1+K_P}} \end{aligned}$$

We can see that

$$\begin{aligned} K_p \nearrow\ &\Rightarrow\ \omega_n \nearrow \\ K_p \nearrow\ &\Rightarrow\ \zeta \searrow \end{aligned}$$

If the plant is an over-damped system (i.e. $\zeta > 1$), then increasing ω_n and decreasing ζ should has a positive net effect on the closed-loop performance.

On the other hand if the plant is an under-damped system, we can observe the following relations

$$\begin{aligned} \bar{\omega}_d &= \bar{\omega}_n\sqrt{1-\bar{\zeta}^2} = \omega_n\sqrt{1+K_P}\sqrt{1-\frac{\zeta^2}{1+K_P}} = \omega_n\sqrt{1+K_P-\zeta^2} \\ \bar{\zeta}\bar{\omega}_n &= \frac{\zeta}{\sqrt{1+K_P}}\omega_n\sqrt{1+K_P} = \zeta\omega_n \end{aligned}$$

We can see that

$$\begin{aligned} K_p \nearrow\ &\Rightarrow\ \omega_d \nearrow \\ K_p \nearrow\ &\Rightarrow\ \zeta\omega_n = \end{aligned}$$

In other words real part of the complex conjugate poles are unchanged, yet imaginary part deviates from the real axis. From previous lectures we kow that in this scenario

$$K_p \nearrow\ \Rightarrow\ M_p \nearrow$$

In conclusion, If the plant is an under-damped system (i.e. $\zeta < 1$), them increasing the P gains has an negative effect on the closed-loop performance (transient).

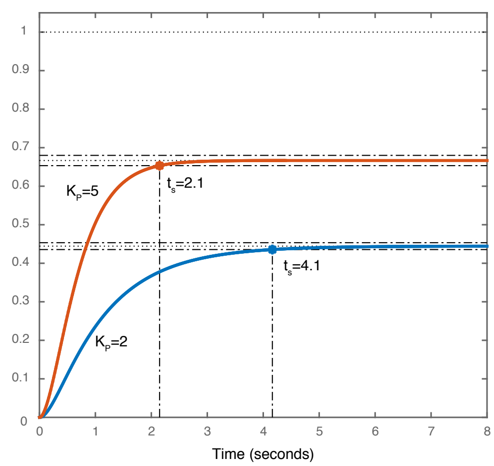
Example 1: Let $G(s) = \frac{1}{(s+0.5)(s+5)}$ (and over-damp plant), then compute the unit-step steady state error for $K_p = 2$ and $K_P = 5$.

$$\begin{aligned} e_2 &= \frac{1}{1+2/2.5} \approx 0.55\ , (\%55) \\ e_5 &= \frac{1}{1+5/2.5} \approx 0.33\ , (\%33) \end{aligned}$$

Obviously, stead-state performance is better with $K_p = 5$ compared to $K_p = 1$. Now compute the closed-loop poles for $K_p = 1$ and $K_P = 5$, and estimate associated settling times (%2).

$$\begin{aligned} T_2(s) &= \frac{2}{s^2+5.5s+4.5} \quad \rightarrow\ p_1^{(2)} = -1\ , p_2^{(2)} = -4.5 \quad t_s^{(2)} \approx 4s \\ T_5(s) &= \frac{5}{s^2+5.5s+7.5} \quad \rightarrow\ p_1^{(2)} = -2.5\ , p_2^{(2)} = -3 \quad t_s^{(2)} \approx 1.6s \end{aligned}$$

We can see that $K_P = 5$ provides a better transient performance compared to $K_P = 2$. Now let's draw step-responses and verify these observations



We verify that both steady-state and transient performance increases with larger K_P. However, we can also see that seetling time estimation for $K_P = 5$ has a larger error, which expected since the poles are close to each other thus violates the dominant pole assumption.

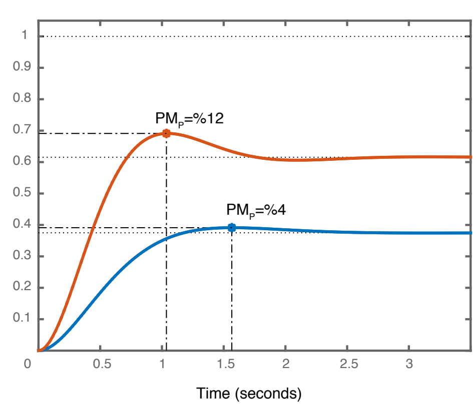
Example 2: Let $G(s) = \frac{1}{s^2+4s+5}$ (and under-damped plant), then compute the unit-step steady state error for $K_p = 3$ and $K_P = 8$.

$$e_3 = \frac{1}{1+3/5} \approx 0.625 \text{ , } (\%62.5)$$
$$e_8 = \frac{1}{1+8/5} \approx 0.385 \text{ , } (\%38.5)$$

Obviously, stead-state performance is better with $K_p = 8$ compared to $K_p = 3$. Now compute the closed-loop poles for $K_p = 3$ and $K_P = 8$, and estimate associated maximum overshoots

$$T_3(s) = \frac{3}{s^2+4s+8} \quad \to \; p_{1,2}^{(3)} = -2 \pm 2j \quad M_P = M_P = e^{-pi/\tan\phi_3} = e^{-pi} \approx 0.04 \; (\%4)$$
$$T_8(s) = \frac{5}{s^2+4s+13} \quad \to \; p_{1,1}^{(8)} = -2 \pm 3j \quad M_P = e^{-pi/\tan\phi_8} = e^{-pi2/3} \approx 0.12 \; (\%12)$$

We can see that $K_P = 3$ provides a better transient performance compared to $K_P = 8$, since both frequency oscillates and overs-shoot is increased with a higher P gain. Now let's draw step-responses and verify these observations.



We verify that stead-state performance is better with a larger P gain. However, transent performance is worse with a larger P-gain. This implies that if the plant is an under-damped plant, then P controller has some serious limitations.

10.1.2 Proportional Derivative (PD) Controller

In the classical form of PD controller, $C(s)$, takes the form

$$C(s) = K_P + K_D s$$

Let's first analyze the affect of K_D term on steady-state performance on the same case (2^{nd} order plant in standard form)

$$G_{OL}(s) = \frac{K_P \omega_n^2 + K_D \omega_n^2 s}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$Type\ 0 \quad K_p = K_P$$

Thus steady-state error for unit-step and unit-ramp inputs can be find as

- Unit step: $e_{ss} = \frac{1}{1+K_P}$, i.e. $K_p \nearrow \Rightarrow\ e_{ss} \searrow$
  Unit ramp: $e_{ss} = \infty$

Obviously, K_D has no effect on steady-state performance. Now let's analyze the closed-loop transfer function

$$\begin{aligned} T(s) = \frac{Y(s)}{R(s)} &= \frac{K_P \omega_n^2 + K_D \omega_n^2 s}{s^2 + 2\zeta\omega_n s + \omega_n^2 + K_D \omega_n^2 s + K_P \omega_n^2} \\ &= \frac{K_P \omega_n^2 + K_D \omega_n^2 s}{s^2 + (2\zeta\omega_n + K_D \omega_n^2)s + (1+K_P)\omega_n^2} \end{aligned}$$

,

$$\begin{aligned} \bar{\omega}_n &= \sqrt{1+K_P}\,\omega_n \\ \bar{\zeta} &= \frac{\zeta + K_D \omega_n/2}{\sqrt{1+K_P}} \end{aligned}$$

We can see that

$$\begin{aligned} K_P \nearrow\ &\Rightarrow\ \omega_n \nearrow\ \&\ \zeta \searrow \\ K_D \nearrow\ &\Rightarrow\ \omega_n =\ \&\ \zeta \nearrow \end{aligned}$$

In other words, for a second order system, we have full control on closed-loop pole locations with PD control policy. Since, a high K_P is required/preferred for steady-state performance (which causes the system to have overshoot and oscillatory behavior), K_D term can be used to suppress oscillations and overshoot. Note that in the closed-loop transfer function, numerator part has a zero due to $K_D \omega_n^2 s$ term, which implies that the closed-loop transfer function is not in standard 2^{nd} order form. One should note the fact that, the existing of closed-loop zero can affect the accuracy of our closed-loop transient performance metric calculations (most probably deviations will be minor).

Example 3: Let $G(s) = \frac{1}{s^2+4s+5}$ (and under-damp plant). Design a PD controller such that steady-state error to unit-step input is around %20 and the maximum percentage overshoot is less than %4.

Solution: We first design K_P based on steady-state requirement then choose K_D based on the over-shoot requirement.

$$e_{ss} = \frac{1}{1 + K_P/5} = 0.2$$
$$K_P = 20$$

Now let's compute the closed-loop transfer function with $K_P = 20$.

$$T(s) = \frac{20 + K_D s}{s^2 + (4 + K_D)s + 25}$$

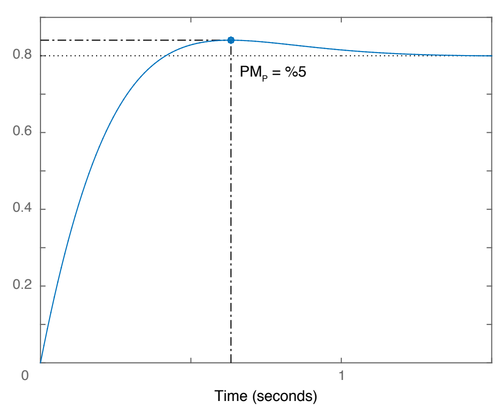
Let $K_D = 4$, than the closed-loop transfer function and associated poles are computed as

$$T(s) = \frac{20 + 4s}{s^2 + 8s + 25}$$
$$p_{1,2} = -4 \pm 3j$$

We can estimate the overshoot based on the pole locations

$$PM_P = \%100 e^{-\pi/\tan\phi} = e^{-\pi 4/3} = \%1.5 < \%4$$

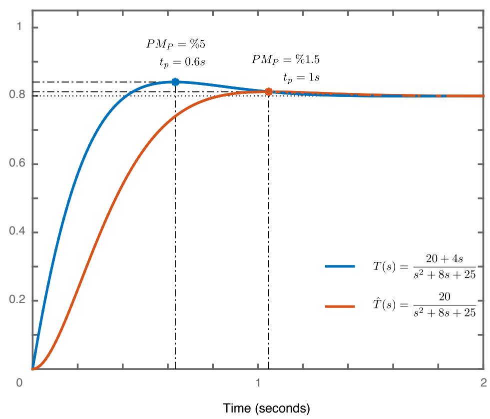
Let's plot the step-response of the resultant system and check if we can meet the specifications.



We can observe that the computed over-shoot is %5, and indeed higher than the requirement. Moreover, the gap between estimated and numerically computed over-shoot is around %3.5. Obviously, the $K_D s = 4s$ term in the numerator affects the output behavior and its affect should be maximum when $0 < t < t_s$. In order see how $K_D s = 4s$, let's compare step responses of following transfer functions

$$T(s) = \frac{20 + 4s}{s^2 + 8s + 25}$$
$$\hat{T}(s) = \frac{20}{s^2 + 8s + 25}$$

$T(s)$ and $\hat{T}(s)$ share the same poles and DC gain, but $\hat{T}(s)$ is in standard form, thus has no zeros.



We can see that there exist non-negligible differences between two transfer functions, which clearly shows that the numerator dynamics can substantially affect the response. In this context, we should think as the transient performance metrics and associated approximate estimation formulas as heuristics which guides the design of controllers.

10.1.3 Integral (I) Controller

Practical use of Integral controller (alone) is extremely rare, however we will analyze this case to better understand the effect of Integral action on more useful PI and PID topologies.

In the pure Integral controller, $C(s)$, takes the form

$$C(s) = K_I \frac{1}{s}$$

Let's first analyze the steady-state performance where the plant is a 2^{nd} system in standard

$$G_{OL}(s) = \frac{K_I \omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$
$$Type\ 1 \quad K_v = K_I$$

Thus steady-state error for unit-step and unit-ramp inputs can be find as

- Unit step: $e_{ss} = 0$
- Unit ramp: $e_{ss} = \frac{1}{K_I}$, i.e. $K_I \nearrow \Rightarrow \ e_{ss} \searrow$

Basic idea is very clear, Integral action increases the type of the system by introducing an extra pole at the origin (also increases the total system order). Thus for a Type 0 plant, it completely eliminates the steady-state error for step-like inputs, and provides a constant steady-state error for ramp-like inputs.

Let's compute the closed-loop transfer function

$$T(s) = \frac{K_I \omega_n^2}{s^3 + 2\zeta\omega_n s^2 + \omega_n^2 s + K_I \omega_n^2}$$

The closed loop system is now a third order system and thus harder to analyze (we need new analysis tools). Moroever, we have only one parameter K_I and the closed-loop system has three poles. One may easily guess that it is very hard to obtain a good performance with a pure Integral controller. Sometimes it may be even quite hard to obtain a stable behavior.

Example 4: Let's analyze the influence of an I controller on a first order plant in order to better understand the positive and negative effects of I action. Let $G(s) = \frac{1}{s+2}$.

The steady-state value of $y(t)$ (for unit-step input) for the uncontrolled plant is $y_{ss} = 1/2$, thus we can say that steady-state error under unit-step input is $e_{ss} = 0.5$, where as for unit ramp it is easy to show that $e_{ss} = \infty$. On the other hand, we can estimate the settling time (%2) for the uncontrolled plant as $t_s \approx 2$.

Now let's first analyze the setady-state performance under $C(s) = \frac{K_I}{s}$,

$$G_{OL}(s) = \frac{K_I}{s(s+2)}$$

$$\textit{Type } 1 \quad K_v = K_I/2$$

- Unit step: $e_{ss} = 0$
- Unit ramp: $e_{ss} = \frac{2}{K_I}$, i.e. $K_I \nearrow \Rightarrow \; e_{ss} \searrow$

Obviously, steady-state performance improvement is significant (structurally). Now, lets compute closed-lopp transfer function

$$T(s) = \frac{K_I}{s^2 + 2s + K_I}$$

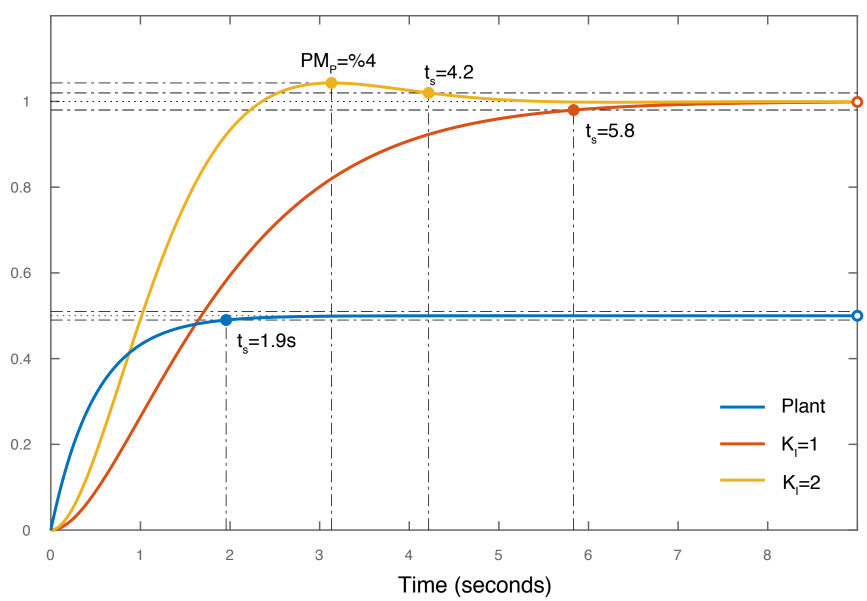
Now let's analyze the transient performance (settling time and maximum overshoot) for $K_I = 1$ and $K_I = 2$ and compare them w.r.t. uncontrolled plant

$$K_I = 1 \quad \rightarrow \quad \zeta = 1 \; , \; p_{1,2} = -1 \quad \rightarrow \quad M_P = 0 \; , \; t_s \approx 4s$$

$$K_I = 2 \quad \rightarrow \quad \zeta = 1/\sqrt{2} \; , \; p_{1,2} = -1 + \pm j \quad \rightarrow \quad M_P = 0.04 \; , \; t_s \approx 4s$$

We can clearly see that integral action has a negative effect on transient performance. $K_I = 1$ case has a worse settling time value than the original plant! Moreover we start to observe over-shoot at the output when we increase integral gain to $K_I = 2$ case.

In order to illustrate these analytic observations, we plotted the step responses of original plant, closed-loop system with $K_I = 1$ and closed-loop system with $K_I = 2$.



It is clear that I controller eliminates the steady-state error, but we can also observe that transient performance substantially degraded. In both cases settling time is worse than the original plant, and for case $K_I = 2$, overs-shoot is clear from the figure. One interesting result is that settling time performance for $K_I = 1$ (critically damped) is worse than $K_I = 2$ (under-damped) even though the approximate formula provides the same estimate. The reason is that for the critically damped case, the approximation under-estimates the settling time.

In this context, we can conclude that a little bit over-shoot could be good for the closed-loop system from the perspective of settling time.

10.1.4 Proportional Integral (PI) Controller

PI controller is commonly used in practical applications. In general, if a single P controller is satisfactory for transient requirements, but one seeks perfect steady-state performance for unit-step like inputs, PI is the first choice to test (compared to PID).

In the PI controller, $C(s)$, takes the form

$$C(s) = K_P + \frac{K_I}{s}$$

Let's first analyze the steady-state performance again with the plant is a 2^{nd} system in standard

$$G_{OL}(s) = \frac{K_P\omega_n^2 s + K_I\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$
$$Type\ 1 \quad K_v = K_I$$

Thus steady-state error for unit-step and unit-ramp inputs can be find as

- Unit step: $e_{ss} = 0$
- Unit ramp: $e_{ss} = \frac{1}{K_I}$, i.e. $K_I \nearrow \Rightarrow \ e_{ss} \searrow$

In other words PI and I controllers have same steady-state performance characteristics. Now, let's compte the closed-loop transfer function

$$T(s) = \frac{K_P\omega_n^2 s + K_I\omega_n^2}{s^3 + 2\zeta\omega_n s^2 + (1 + K_P)\omega_n^2 s + K_I\omega_n^2}$$

In this case, the closed-loop transfer function has three poles, and we have two parameters for tuning. Even though it provides a much better framework than an I controller, still we need different tools to tune the K_I and K_P gains.

Example 5: Let's compare PI and I controllers using the same plant in previous example, $G(s) = \frac{1}{s+2}$. We know that PI controller has the transfer function form $C(s) = \frac{K_I}{s}$ and steady-state error characteristics can be derived as

$$G_{OL}(s) = \frac{sK_P + K_I}{s(s+2)}$$
$$Type\ 1 \quad K_v = K_I/2$$
$$\text{Unit step} : e_{ss} = 0$$
$$\text{Unit ramp} : e_{ss} = e_{ss} = \frac{2}{K_I}$$

which are exactly same with the I controller. Now, lets compute closed-lopp transfer function

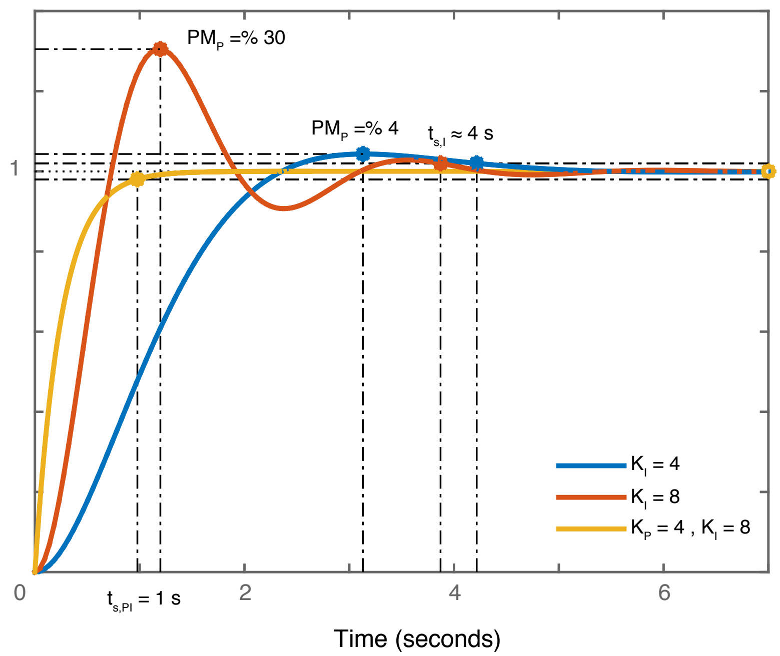
$$T(s) = \frac{K_D s + K_I}{s^2 + (2 + K_D)s + K_I}$$

Now let's choose $K_I = 8$ and $K_P = 4$, and estimate maximum over-shoot and settling time for the closed-loop plant.

$$p_1 = -2 \ , \ p_2 = -4 \quad \rightarrow \quad M_P = 0 \ , \ t_s \approx 2s$$

These PI gains can match the settling time value of the original plant without any over-shoot (since closed-loot TF is over-damped).

Let's illustrate this analytic observations, by plotting the step responses of the closed-loop system with only integral controller with $K_I = 2$, the closed-loop system with only integral controller with $K_I = 8$, and the closed-loop system with PI controller with $K_P = 4$, $K_I = 8$.



In this illustration, we can see that settling time for both I controllers is around $4s$, however settling time for the PI controller is $1s$ which is even better than our estimation. The gap between settling time estimates come form the affect of zero introduced by the PI controller. If one carefully, analyzes the closed-loop transfer function he/she can see that a pole-zero cancellation occurs (which may not be a good feature for practical reasons). Technically, the closed-loop transfer function is reduced to a first order system in this case (????).

We also see that when PI and I controllers has the same K_I gain, the over-shoot in I controller is %30 which is very high.

10.1.5 Proportional Integral Derivative (PID) Controller

PID controller technically combines the advantages of the PD and PI controllers, with the trade of increased parameter and implementation complexity. We know that PID controller has the following transfer function form

$$C(s) = K_P + K_D s + \frac{K_I}{s}$$

Let's first analyze the steady-state performance again where the plant is a 2^{nd} system in standard form

$$G_{OL}(s) = \frac{(K_D s^2 + K_P s + K_I)\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$
$$Type\ 1 \quad K_v = K_I$$

Thus steady-state error for unit-step and unit-ramp inputs can be find as

- Unit step: $e_{ss} = 0$
- Unit ramp: $e_{ss} = \frac{1}{K_I}$, i.e. $K_I \nearrow \Rightarrow\ e_{ss} \searrow$

In other words PID, PI, and I controllers share the same steady-state performance characteristics. Now, let's compute the closed-loop transfer function

$$T(s) = \frac{(K_D s^2 + K_P s + K_I)\omega_n^2}{s^3 + (2\zeta\omega_n + K_D\omega_n^2)s^2 + (1 + K_P)\omega_n^2 s + K_I}$$

We can see that the closed-loop transfer function has three poles, and we have three parameters to tune. If we have no limits on gains, we can place the closed-loop poles to any desired location. However, numerator has now two zeros, thus it is now harder to predict the affect of closed-loop zeros on the output behavior.

Example 6: Let $G(s) = \frac{1}{s^2+4s+5}$, Design a PID controller such that we observe zero unit-step steady state error, maximum percent overs-shoot is less than %5, and settling time is around $1s$.

Solution: We already know that PID controller for a Type 0 plant completely eliminates the unit-step steady state error. Now let's compute closed-loop transfer function

$$T(s) = \frac{K_D s^2 + K_P s + K_I}{s^3 + (4 + K_D)s^2 + (5 + K_P)s + K_I}$$

One way of choosing appropriate pole locations for a third-order closed-loop system is placing one of the poles (a real one) far away from the other poles such that closed-loop system shows a second order like behavior. We require that maximum over shoot of %5 and settling time is around $1s$. Let the dominant poles of the closed-loop system be

$$p_{1,2} = -4 \pm 3j$$

Settling time and maximum overs-shoot associated with this closed-loop poles can be estimated as

$$t_s \approx 1s$$
$$M_P \approx e^{-pi4/3} = 0.015$$

These estimates satisfy the requirements. Let $p_3 = 3(-4) = -12$, then desired characteristic equation takes the form

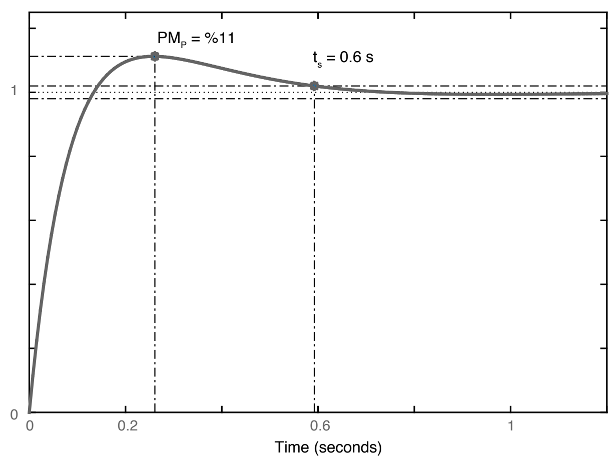
$$d^*(s) = (s + 12)(s^2 + 8s + 25) = s^2 + 20s + 121s + 300$$

Then, we can compute the PID gains as

$$
\begin{aligned}
K_P &= 116 \\
K_I &= 300 \\
K_D &= 16
\end{aligned}
$$

The first thing, we can observe is that the quantitative values of the gains seem to be much larger than the gains that we played before. In general, if we want to improve the performance of both steady-state and transient characteristics by implementing PID topology instead of P, PI, or PD the gain values would go up which may cause practical problems and potentially can be costly in terms of energetic performance.

Now let's plot the step response of the closed-loop system and try to verify these analytic observations



We can see that the settling time in numerical simulation is much better than our estimate $0.6s < 1s$, however numerical percentage over-shoot is higher than our estimate, $\%11 > \%1.5$, and does not meet our specifications. The core reason behind this is that it is basically harder to tune a PID controller compared to P, PD, and PI controllers due to increased parametric complexity and (may be more importantly) 2^{nd} numerator dynamics introduced with the PID control policy.