EE302 - Feedback Systems **Spring 2019**

Lecture 18

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18.1 The Bode Plot

Previously, we showed how to illustrate the frequency response function of an LTI system, $G(j\omega)$, using the polar plot and the Nyquist plot. In the Bode Plot gain, $|G(j\omega)|$ and phase response, $\angle[G(j\omega)]$, of the system are illustrated separately as a function of frequency, ω. In both diagrams logarithmic scale is used for frequency axis. On the other hand, in magnitude axis we use a special logarithmic scale in dB units, where as for phase axis we use linear scale. Specifically Magnitude in dB scale, M_{dB} and phase response, ϕ of a transfer function, $G(j\omega)$ are computed as

$$\begin{aligned} M_{dB}(\omega) &= 20\log_{10}|G(j\omega)| \\ \phi(\omega) &= \angle[G(j\omega)] \end{aligned}$$

Now let's write $G(s)$ in pole-zero-gain form and analyze the magnitude (dB) and phase functions

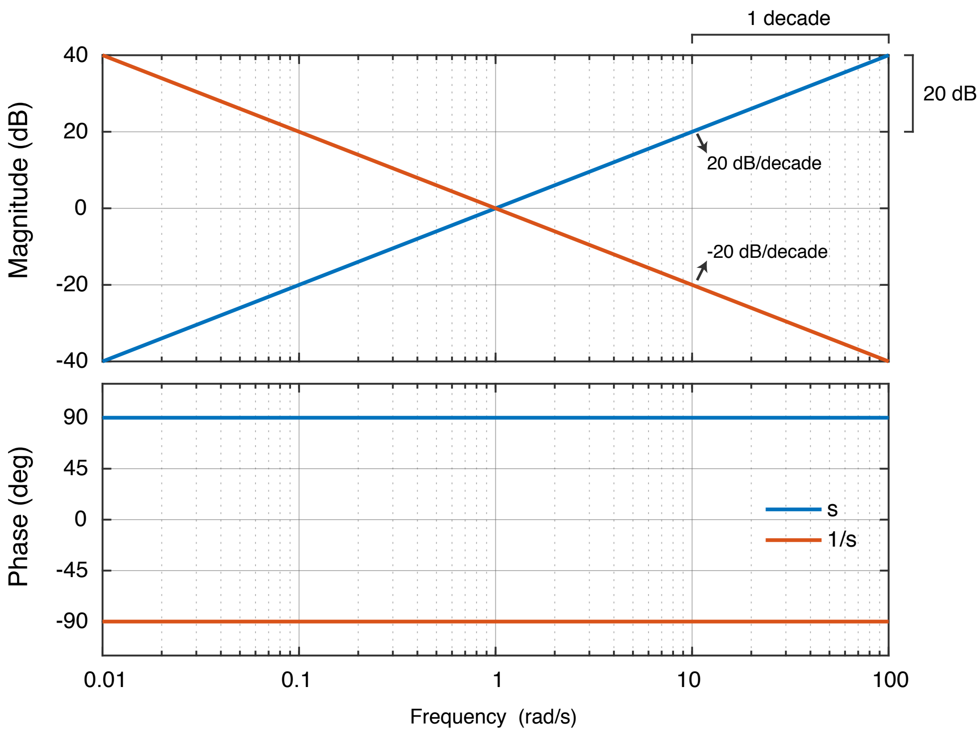
$$\begin{aligned} G(s) =& K\frac{(s-z_1)\cdots(s-z_N)}{(s-p_1)\cdots(s-p_N)} \\ &, \\ M_{dB}\{G(s)\} =& 20\log_{10}|G(j\omega)| \\ =& 20\log_{10}K + [20\log_{10}|(j\omega - z_1)| + \cdots + 20\log_{10}(j\omega - z_N)] \\ & - [20\log_{10}|(j\omega - p_1)| + \cdots + 20\log_{10}(j\omega - p_M)] \\ =& K_{dB} + [M_{dB}\{s-z_1\}\cdots M_{dB}\{s-z_N\}] - [M_{dB}\{s-p_1\}\cdots M_{dB}\{s-p_M\}] \\ &, \\ \phi\{G(s)\} =& \angle[G(j\omega)] \\ =& (\angle[j\omega - z_1)] + \cdots + \angle[j\omega - z_N]) - (\angle[j\omega - p_1] + \cdots + \angle[j\omega - p_M]) \\ =& (\phi\{s-z_1\} + \cdots + \phi\{s-z_N\}) - (\phi\{s-p_1\} + \cdots + \phi\{s-p_M\}) \end{aligned}$$

In conclusion, in order to obtain a bode diagram, we can first find phase and magnitude (dB) arguments associated with each pole/zero/gain separately for a given frequency. After that, final magnitude (dB) and phase arguments of $G(s)$ are found by simply adding (and subtracting) the individual components.

Bode Plots of s and $\frac{1}{s}$: Let's write the magnitude and phase functions

$$s \;\Rightarrow\; M_{dB}(\omega) = 20\log_{10}(\omega) \quad \& \quad \phi(\omega) = 90^0$$
$$\frac{1}{s} \;\Rightarrow\; M_{dB}(\omega) = -20\log_{10}(\omega) \quad \& \quad \phi(\omega) = -90^0$$

If we illustrate the responses in bode plot, we obtain the following Figure



Bode Plots of First-Order Forms

First let's analyze the phase and magnitude (dB) response of $G(s) = s + 1$

$$M_{dB}(\omega) = 20\log_{10}|G(j\omega)| = 20\log_{10}\left(\omega^2+1\right)^{1/2} = 10\log_{10}\left(\omega^2+1\right)$$
$$\phi(\omega) = \arctan\omega$$

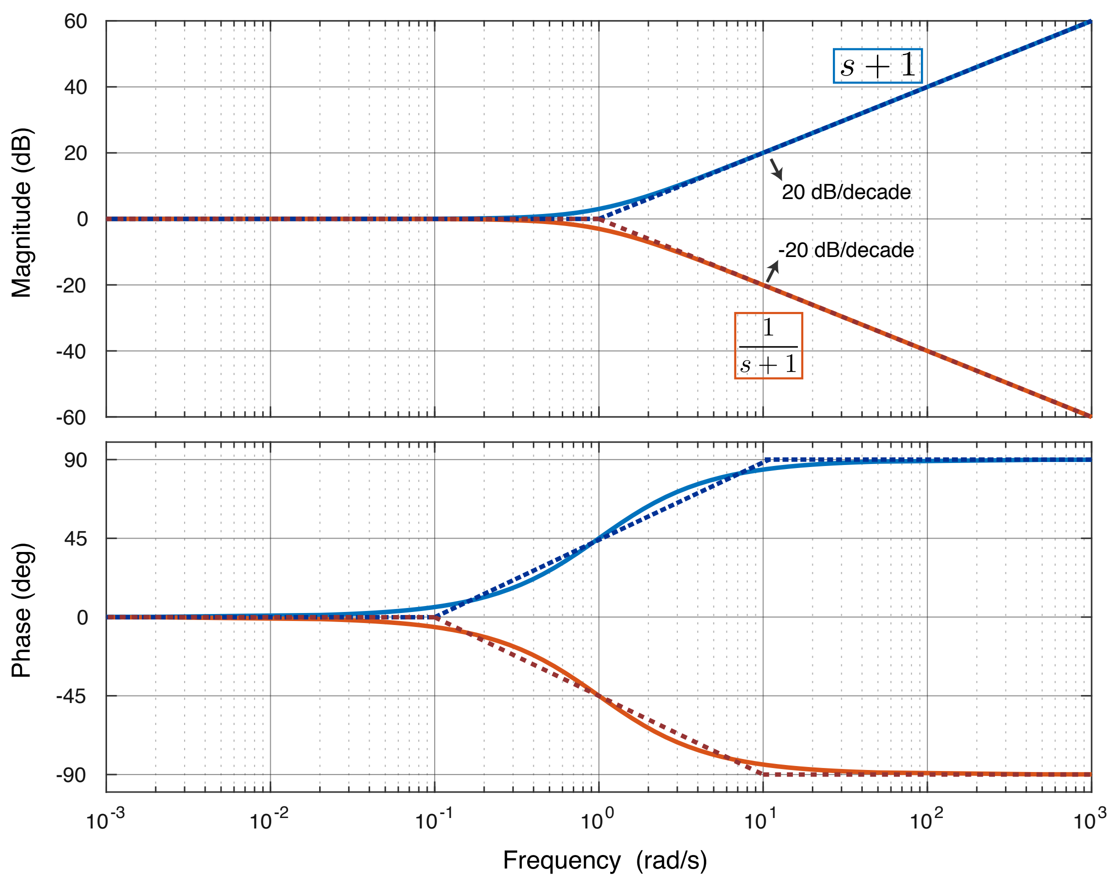
Now we will approximate the gain and phase curves using piece-wise continuous straight lines. Firs approximate the magnitude response

$$\text{Low} - \text{Frequency} \;\Rightarrow\; M_{dB}(\omega) \approx 0 \text{ dB}$$
$$\text{High} - \text{Frequency} \;\Rightarrow\; M_{dB}(\omega) \approx 20\log_{10}(\omega)$$

Note that high-frequency and low-frequency approximations intersect at $\omega = 1\ rad/s$ and $M_{dB} = 0$ dB point. Now let's approximate the phase response

$$\begin{aligned} \text{Low}-\text{Frequency} &\Rightarrow \phi \approx 0^o \\ \text{High}-\text{Frequency} &\Rightarrow \phi \approx 90^o \\ \text{Medium}-\text{Frequency} &\Rightarrow \phi \approx 45^0 + 45^0 \log_{10}(\omega) \end{aligned}$$

Note that, low-frequency and mid-frequency approximations intersect when $\omega = 0.1\ rad/s$, whereas high-frequency and mid-frequency approximations intersect when $\omega = 10\ rad/s$. The corner frequency of this "system" is $\omega_c = 1\ rad/s$. Note that it is very easy to obtain the bode approximations of $G(s) = \frac{1}{s+1}$, if we know the bode approximations of $G(s) = s+1$. We simple multiply both magnitude (dB) and phase responses of $s+1$ with (-1). The figure below illustrates the original bode plots (solid curves) of $G_1(s) = (s+1)$ and $G_2(s) = \frac{1}{s+1}$ as well as their approximations (dashed lines).



Now let's analyze the phase and magnitude (dB) response of

$$G(s) = Ts + 1 = \frac{s+a}{a} \text{ where } a = \frac{1}{T}$$

Magnitude and phase functions can be obtained as

$$\begin{aligned} M_{dB} &= 20\log_{10}|G(j\omega)| = 20\log_{10}\left(T^2\omega^2 + 1\right)^{1/2} = 10\log_{10}\left(T^2\omega^2 + 1\right) \\ \phi &= \arctan T\omega \end{aligned}$$

If we follow a similar approach, we can approximate the gain curves as

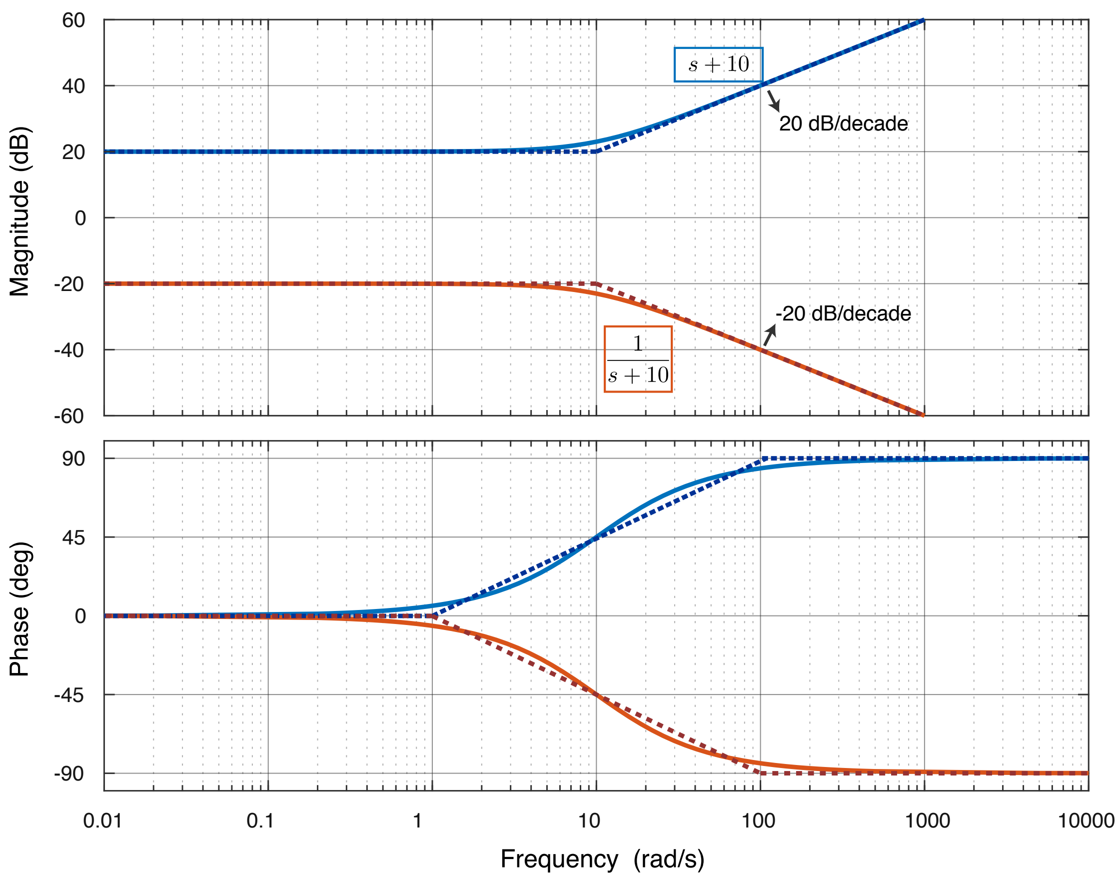
$$\mathrm{Low-Frequency}: \omega \leq \frac{1}{T} = a \;\Rightarrow\; M_{dB} \approx 0 \text{ dB}$$
$$\mathrm{High-Frequency}: \omega \geq \frac{1}{T} = a \;\Rightarrow\; M_{dB} \approx 20\log_{10}(T\omega)$$

Note that high-frequency and low-frequency approximations intersect at $\omega = a\ rad/s = \frac{1}{T}\ rad/s$ and $M_{dB} = 0$ dB point. Now let's approximate the phase response

$$\mathrm{Low-Frequency}: \omega \leq \frac{0.1}{T} = 0.1a \;\Rightarrow\; \phi \approx 0^o$$
$$\mathrm{High-Frequency}: \omega \geq \frac{10}{T} = 10a \;\Rightarrow\; \phi \approx 90^o$$
$$\mathrm{Medium-Frequency} \;\Rightarrow\; \phi \approx 45^0 + 45^0\log_{10}(\omega)$$

Note that low-frequency and mid-frequency approximations intersect when $\omega = 0.1a\ rad/s$, whereas high-frequency and mid-frequency approximations intersect when $\omega = 10a\ rad/s$. The corner frequency of this system is $\omega_c = a\ rad/s = 1/T\ rad/s$. Note that in order to obtain the bode plot of $Ts+1$, we simply shift the bode plot of $s+1$ (in ω axis).

Ex: The figure below illustrates the original bode plots (solid curves) of $G_1(s) = (s+10)$ and $G_2(s) = \frac{1}{s+10}$ as well as their approximations (dashed lines).



Bode Plots of Second-Order Forms

A unity gain standard second order system can be written in the form

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{1}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1}$$

Case 1 : $\zeta = 1$ First let's analyze the bode plots for the critically damped case,

$$G(s) = \frac{1}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1} = \frac{1}{(\frac{s}{\omega_n} + 1)^2}$$

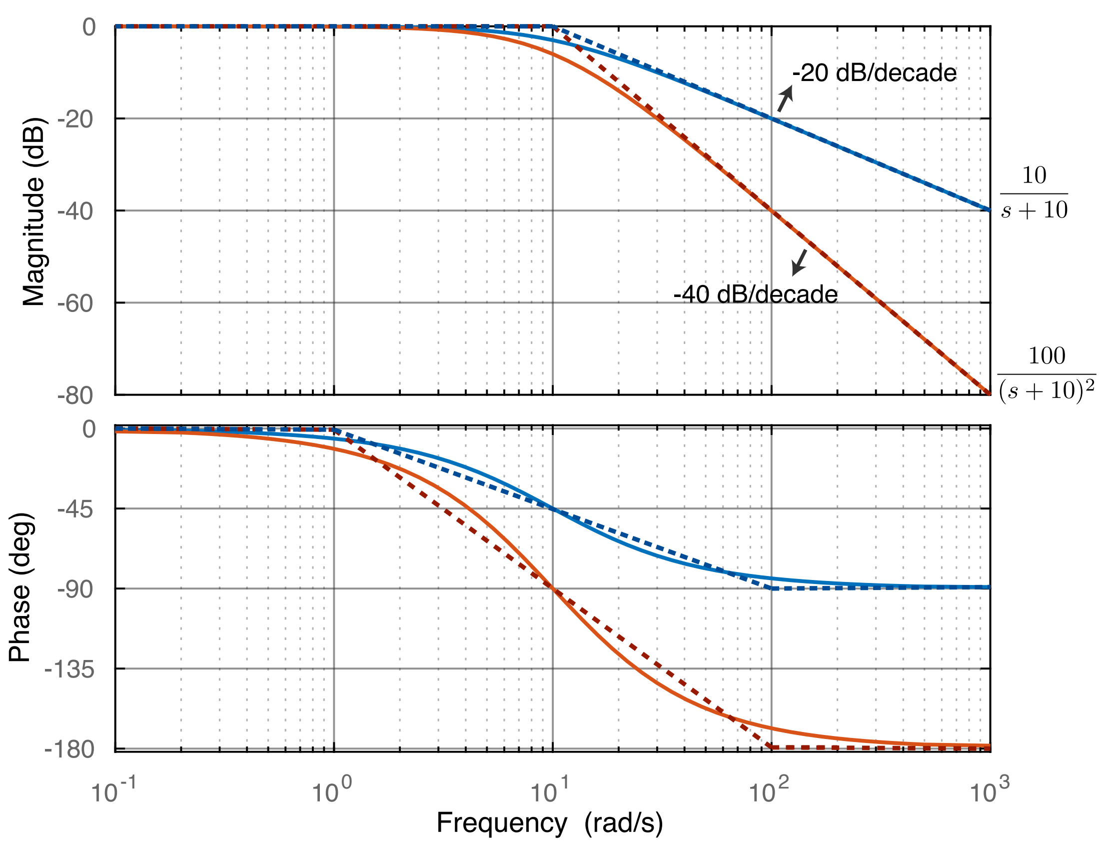
We can easily observe that Magnitude and phase functions can be obtained as

$$M_{dB}\{G(s)\} = 20\log_{10}|G(j\omega)| = 20\log_{10}\left|\frac{1}{\frac{s}{\omega_n} + 1}\right|^2$$

$$= 2M_{dB}\left\{\frac{1}{\frac{s}{\omega_n} + 1}\right\}$$

$$\phi\{G(s)\} = 2\phi\left\{\frac{1}{\frac{s}{\omega_n} + 1}\right\}$$

Ex: The figure below illustrates the original bode plots (solid curves) of $G_1(s) = \frac{10}{s+10}$ and $G_2(s) = \frac{100}{(s+10)^2}$ as well as their approximations (dashed lines).



Case 2: $\zeta > 1$ Over-damped case is simply the combination of two identical first order systems.

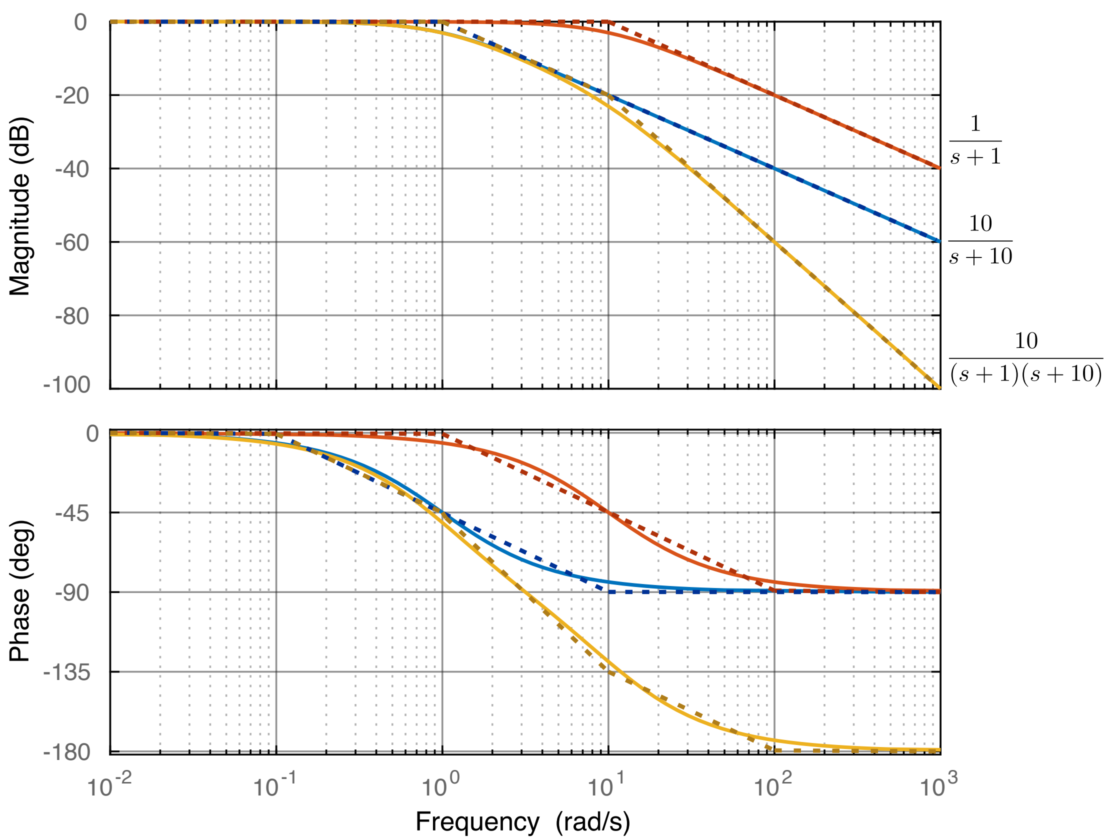
Ex: Let's analyze the the bode plots for the following system

$$G(s) = \frac{10}{(s+1)(s+10)} = \frac{1}{s+1}\frac{10}{s+10}$$

We can easily observe that

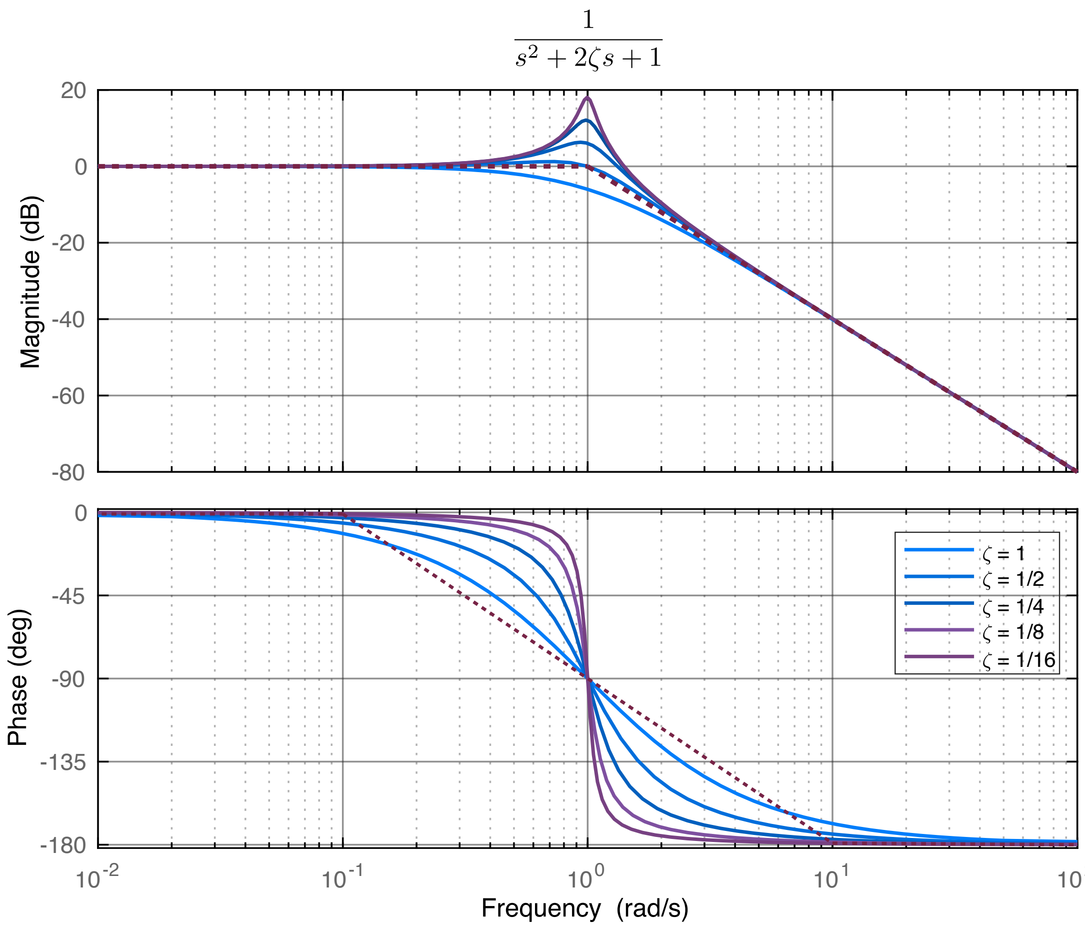
$$M_{dB}\{G(s)\} = M_{dB}\left\{\frac{1}{s+1}\right\} + M_{dB}\left\{\frac{10}{s+10}\right\}$$
$$\phi\{G(s)\} = \phi\left\{\frac{1}{s+1}\right\} + \phi\left\{\frac{10}{s+10}\right\}$$

The figure below illustrates the original bode plots (solid curves) of $G_1(s) = \frac{1}{s+1}$, $G_2(s) = \frac{10}{s+10}$ and $G_4(s) = \frac{10}{(s+1)(s+10)}$ as well as their approximations (dashed lines).



Case 3 : $\zeta < 1$ For under-damped systems, corner frequencies of the piece-wise linear approximations are unchanged. Thus, we use same approximation with the critically damped case. However, as damping ratio increases we may observe larger differences between the actual bode plot and the approximations.

Ex: The figure below illustrates the original bode plots (solid curves) of $G_1(s) = \frac{10}{s+10}$ and $G_2(s) = \frac{100}{(s+10)^2}$ as well as their approximations (dashed lines).



We can see that best phase matching between the actual bode plot and approximation is achieved when $\zeta = 1$, however surprisingly best magnitude matching between the actual bode plot and approximation is achieved when $\zeta = 2$. Indeed, best match between the actual and approximate bode plots in magnitude is achieved when $\zeta = 1/\sqrt{2}$.

18.2 Gain & Phase Margin from Bode Plots

We already know that for a feedback system, phase and gain margins can be computed based on the Frequency Response characteristics of the open-loop transfer function, $G_{OL}(j\omega)$ (under some assumption regarding the system properties).

Specifically, we can compute the phase crossover frequency, ω_{pc} and the gain margin, g_m (linear scale) and G_m (dB scale), as

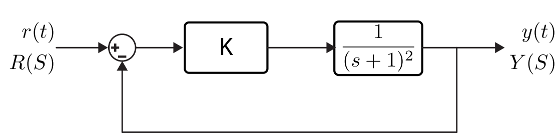
$$\angle[G_{OL}(j\omega_p)] = \pm - 180^0 \quad \Rightarrow \quad g_m = \frac{1}{|G_{OL}(j\omega_{pc})|} \quad \text{or} \quad G_m = -20\log_{10}|G_{OL}(j\omega_{pc})|$$

where as gain crossover frequency, ω_{gc}, and the phase margin can be computed as

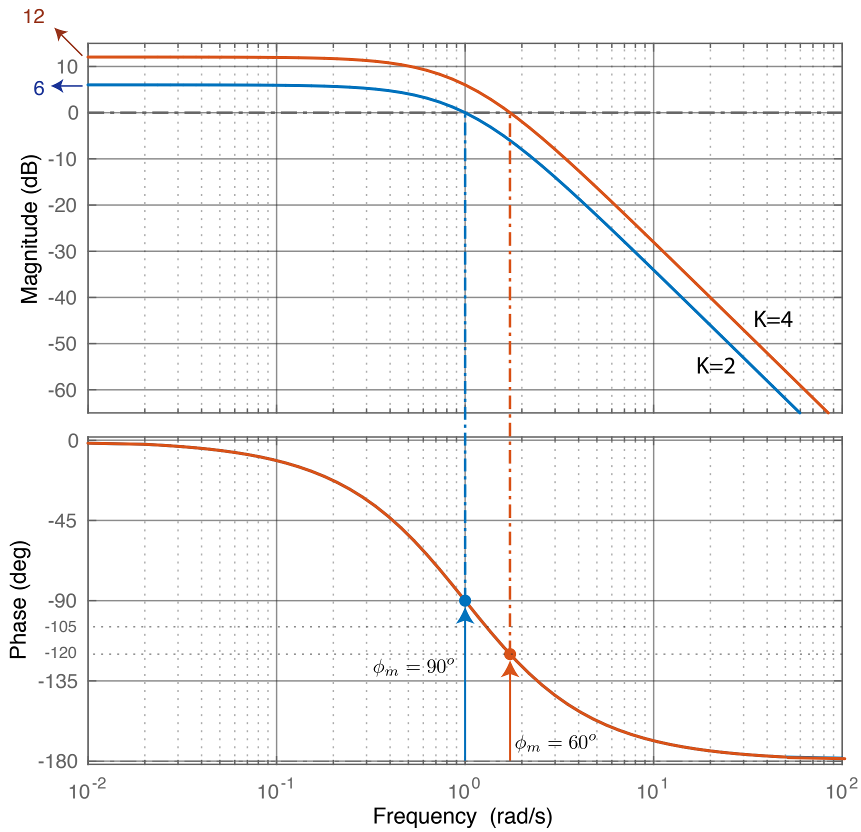
$$|G_{OL}(j\omega_{gc})| = 1 \quad \text{or} \quad M_{dB}\{G_{OL}(j\omega_{gc})\} = 0 \text{ dB} \quad \Rightarrow \quad \phi_m = \pi + \angle G_{OL}(j\omega_{gc})$$

Indeed it is generally easier to derive the phase and gain margin of a system from the bode plots compared to the Nyquist & polar plots.

Ex: Compute the phase margin for the following closed-loop system for $K = 2$ and $K = 4$, both from the approximate and actual bode plots.



Actual bode plots for both gains are illustrated below.

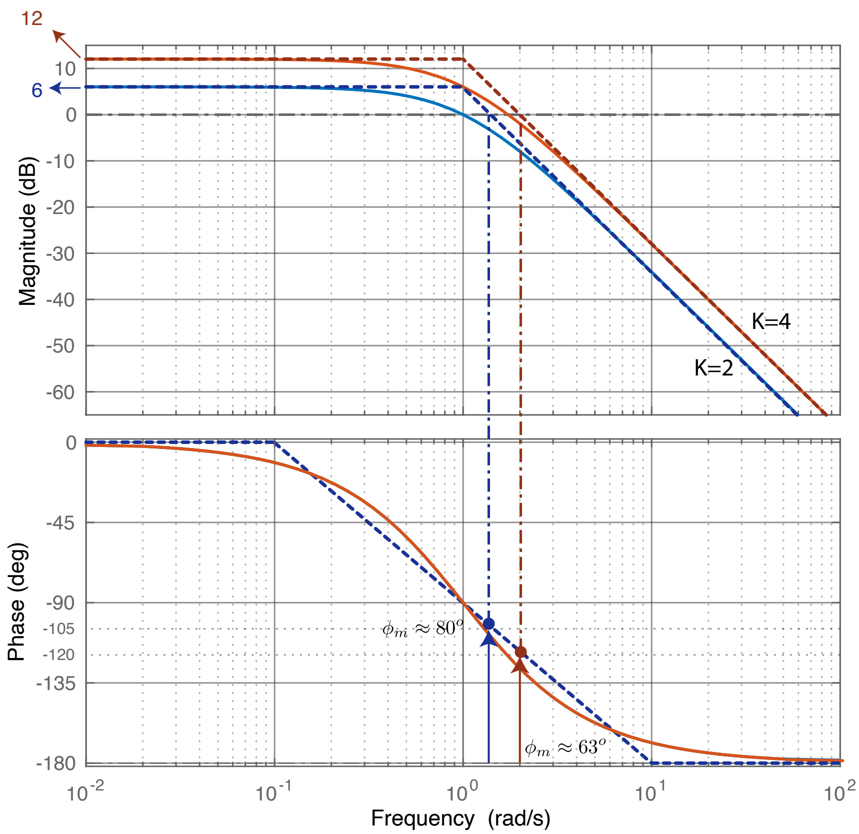


If we label the gain crossover frequencies and find the corresponding phase values, we can easily compute the phase margins as

$$K = 2 \Rightarrow \phi_m = 90^o \quad (\omega_{gc} = 1\ rad/s)$$
$$K = 4 \Rightarrow \phi_m = 60^o \quad (\omega_{gc} \approx 1.8\ rad/s)$$

These results verify the actual phase margin values that we previously computed using the Nyquist plot. Note that G_m is infinity for both cases.

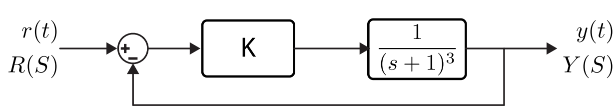
Now let's illustrate the approximate bode plots (dashed lines), which are illustrated in the Figure below on top of the actual ones



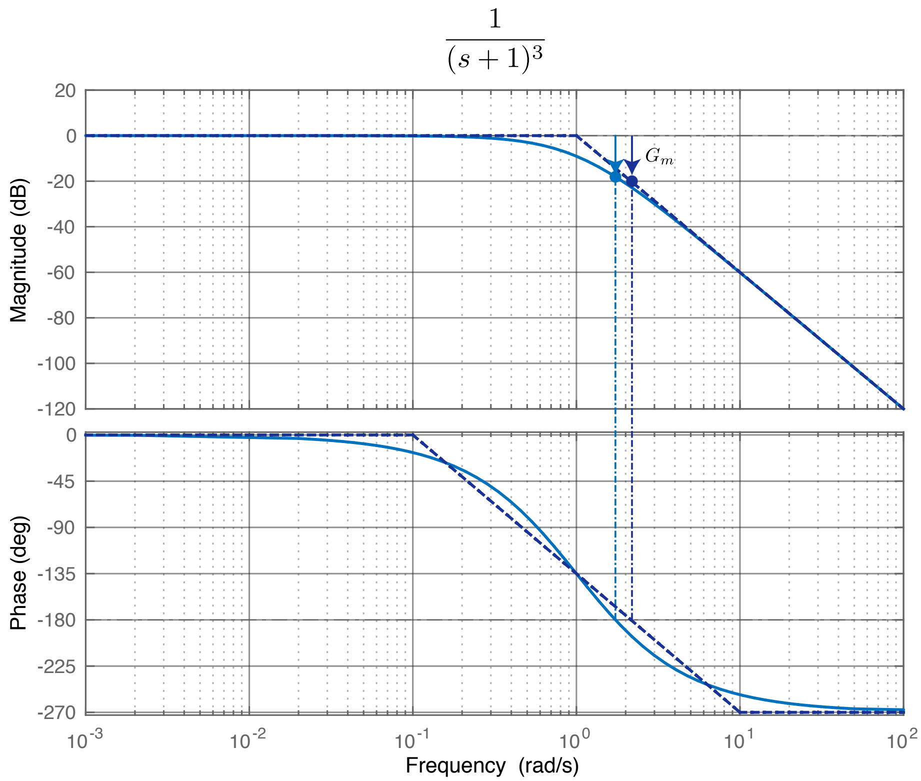
If we approximately compute the phase margins based on the approximate bode plots we obtain that

$$K = 2 \Rightarrow \phi_m \approx 80^o$$
$$K = 4 \Rightarrow \phi_m \approx 63^o$$

Ex: Compute the gain margin and phase margin for the following closed-loop system for $K = 1$ and $K = 8$ both from the actual and approximate bode plots.



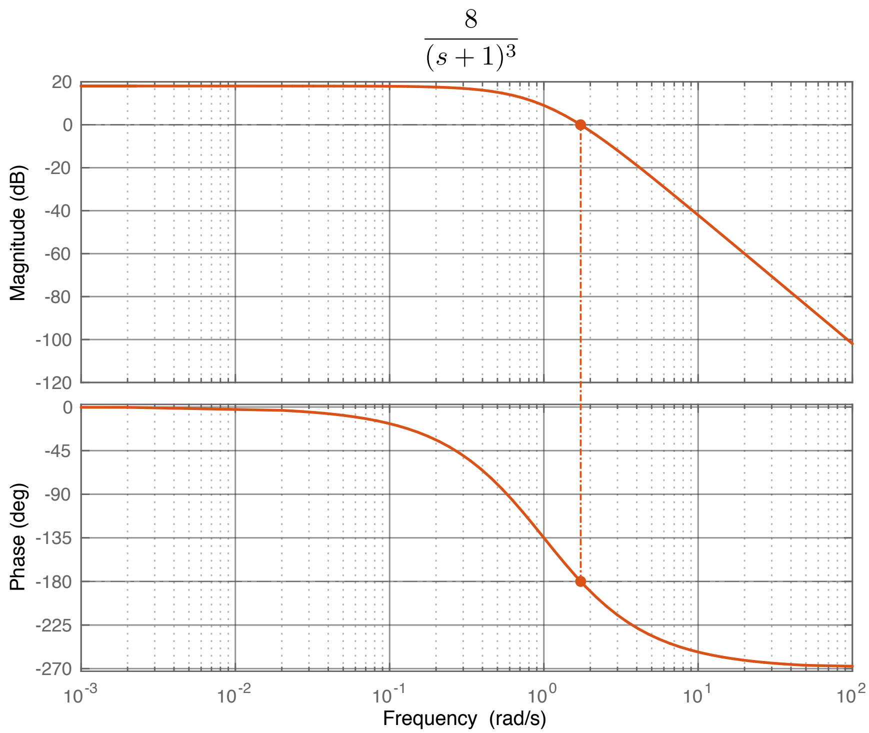
First let's analyze $K = 1$, the Figure below illustrates the actual and approximate bode plots of $G(s) = \frac{1}{(s+1)^3}$.



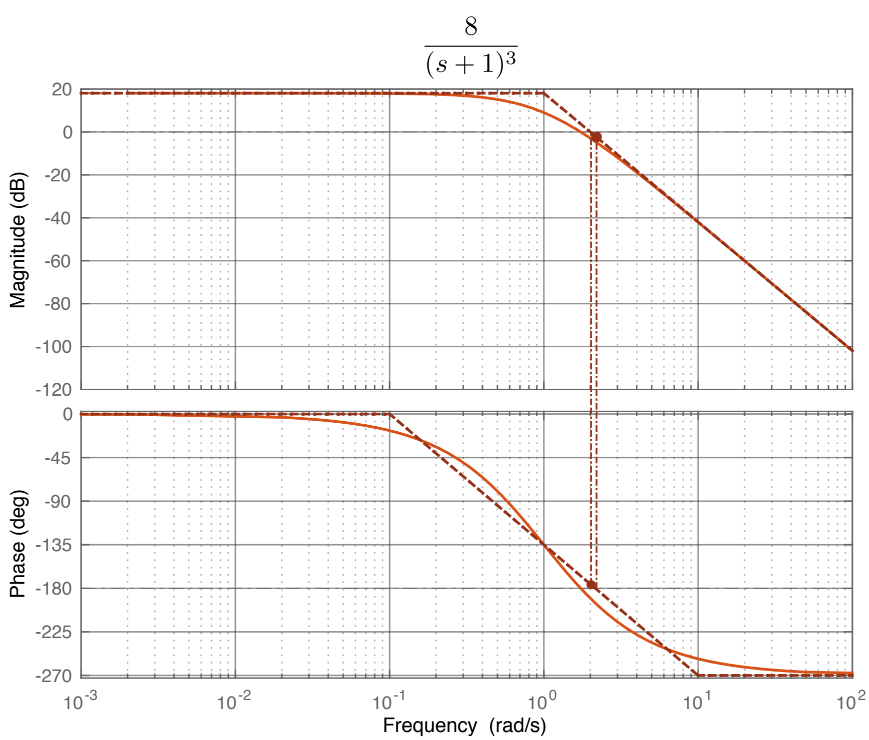
We can see that phase margin is 180^o for the given system, since $\omega_{gc} = 0 \;\to\; \phi(\omega_{gc}) = 0^o$. On the other hang we can derive the following gain margin estimates from the actual and approximate bode plots

$$\begin{aligned} \text{Actual}: &\quad G_m \approx 18 \text{ dB} \quad \to \quad g_m \approx 8 \\ \text{Approximate}: &\quad G_m \approx 20 \text{ dB} \quad \to \quad g_m \approx 10 \end{aligned}$$

Now let's analyze $K = 8$, the Figure below illustrates the actual Bode plots of $G(s) = \frac{8}{(s+1)^3}$.



We can see from the actual bode-plot that $G_m = 0$ dB and $\phi_m = 0^o$. However, if we draw the approximate bode plots

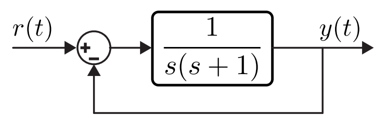


and estimate these margins we obtain

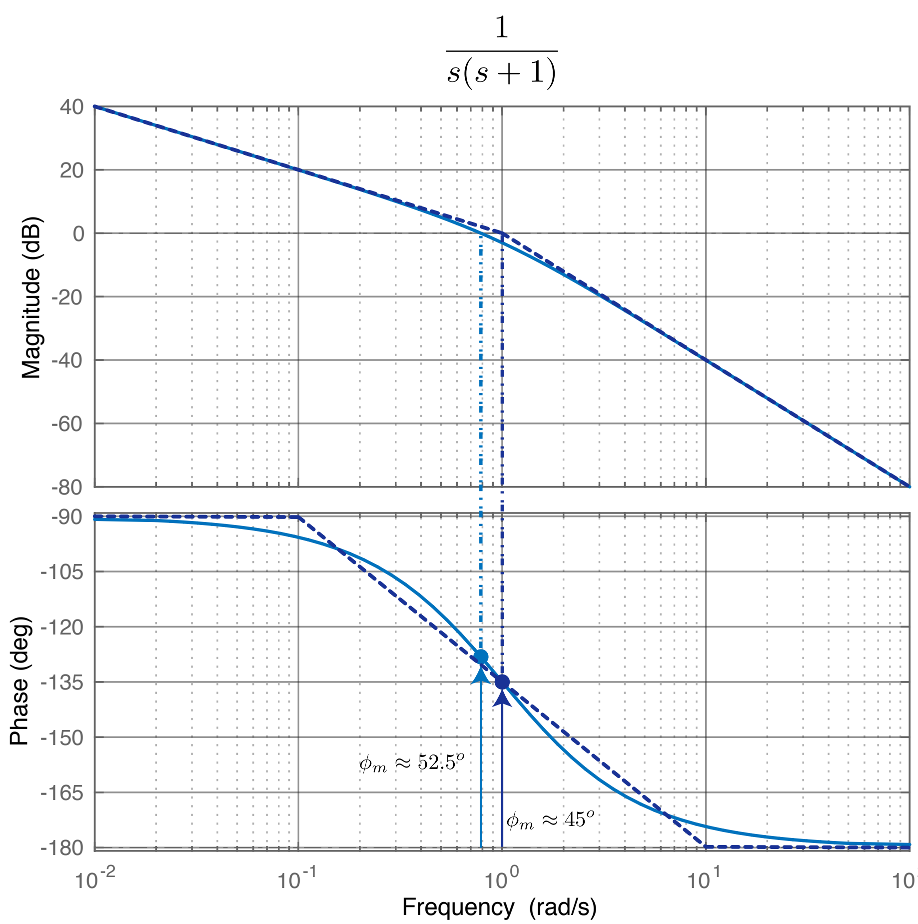
$$G_m \approx 2 \text{ dB} \quad \rightarrow \quad g_m \approx 1.2$$
$$\phi_m = 2^o$$

If we consider the bode plots that we draw based on asymptotic approximations and compute the phase and gain margins, we can assume that system is indeed stable. However, these margins are rather low. In this respect, we can conclude that if one analyzes the absolute stability of a system using approximately bode plots, in order to comment on stability, he/she should observe significant phase and gain margins.

Ex: Compute phase margin for the following closed-loop system using the actual and approximate bode plots



The Figure below illustrates the actual and approximate bode plots.



We can derive the following phase margin computations from the actual and approximate bode plots

$$\text{Actual}: \quad \phi_m \approx 52.5^0 \quad (\omega_{gc} \approx 0.8 \ rad/s)$$
$$\text{Approximate}: \quad \phi_m \approx 45^o \quad (\omega_{gc} = 1 \ rad/s)$$