Lecture 14

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Asymptotic Stability of LTI Systems

Asymptotic Stability of CT-LTI Systems

Let's consider the state-representation of an CT LTI system

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Given CT-LTI system is called *asymptotically stable* if, with $u(t) = 0$ and $\forall x(0) \in \mathbb{R}^n$, we have

$$\lim_{t \to \infty} ||x(t)|| = 0$$

Example: Let's assume that system matrix A is diagonalizable and all eigenvalues are real. Find a necessary and sufficient condition such that the state-space representation is asymptotically stable.

Solution: When $u(t) = 0$, we have

$$\dot{x}(t) = Ax(t)$$
$$x(t) = e^{At}x_0$$

Since A is diagonalizable, we know that

$$A = P \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} P^{-1}$$

$$e^{At} = P \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} P^{-1}$$

Let $Px_0 = [e_1 \ e_2 \ \cdots \ e_n]^T, e_i \in \mathbb{R}$, then $x(t)$ can be expressed as

$$
x(t) = P \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_n \end{bmatrix}
$$

$$
x(t) = P \left(\begin{bmatrix} e_1 e^{\lambda_1 t} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ e_2 e^{\lambda_2 t} \\ \vdots \\ 0 \end{bmatrix} + \cdots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ e_n e^{\lambda_n t} \end{bmatrix} \right)
$$

Then it is easy to see that

$$
\forall x_0 \in \mathbb{R}^n, \lim_{t \to \infty} ||x(t)|| = 0 \iff \forall i \in \{1, \cdots, n\}, \lambda_i < 0
$$

Asymptotic Stability of DT-LTI Systems

Now, let's consider the state-representation of an DT LTI system

$$
x[k+1] = Gx[k] + Hu[k]
$$
$$
y[k] = Cx[k] + Du[k]
$$

Given DT-LTI system is called *asymptotically stable* if, with $u[k] = 0]$ and $\forall x(0) \in \mathbb{R}^n$, we have

$$
\lim_{k \to \infty} ||x[k]|| = 0
$$

Example: Let's assume that system matrix G is diagonalizable and all eigenvalues are real. Find a necessary and sufficient condition such that the state-space representation is asymptotically stable.

Solution: When $u[k] = 0$, we have

$$
x[k+1] = Gx[k]
$$
$$
x(t) = G^k x_0
$$

Since G is diagonalizable, we know that

$$
G = P \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} P^{-1}
$$

$$
G^k = P \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^l & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & & & \lambda_n^k \end{bmatrix} P^{-1}
$$

Let $Px_0 = [e_1 \; e_2 \; \cdots \; e_n]^T, e_i \in \mathbb{R}$, then $x[k]$ can be expressed as

$$x[k] = P\left(\begin{bmatrix} e_1\lambda_1^k \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ e_2\lambda_2^k \\ \vdots \\ 0 \end{bmatrix} + \cdots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ e_n\lambda_n^k \end{bmatrix}\right)$$

Then it is easy to see that

$$\forall x_0 \in \mathbb{R}^n, \lim_{k\to\infty} ||x[k]|| = 0 \iff \forall i \in \{1, \cdots, n\}, |\lambda_i| < 1$$

BIBO Stability of LTI Systems

A CT-system written in state-space form is stable if and only if the poles of $H(s) = C\left[sI - A\right]^{-1} B + D$ are located strictly in the open left half plane.

A DT-system written in state-space form is stable if and only if the poles of $H(z) = C\left[zI - G\right]^{-1} H + D$ are located strictly inside the unit circle.

We did not mention whether the system is SISO or MIMO.

- Do you think that BIBO notion is valid for MIMO systems?

- Do you think that checking "poles" of $H(s)$ or $H(z)$ is a valid method for checking stability?

- What are the poles of $H(s)$ and $H(z)$?

Example: Consider the following state-space form of a CT system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & -1 \end{bmatrix} x(t)$$

Is this system asymptotically stable?

Solution: Let's compute the eigenvalues of A

$$\det\left(\begin{bmatrix} \lambda & -1 \\ -1 & \lambda \end{bmatrix}\right) = \lambda^2 - 1$$
$$\lambda_{1,2} = \pm 1$$

Thus the system is NOT Asymptotically Stable.

Is this system BIBO stable?

Solution: Let's compute the $H(s)$

$$H(s) = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} s & -1 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s^2 - 1}$$

$$= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} s & 1 \\ 1 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{1}{s^2 - 1}$$

$$= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} \frac{1}{s^2 - 1}$$

$$= \frac{-(s-1)}{s^2 - 1}$$

$$= \frac{-1}{s + 1}$$

Indeed, the system is BIBO Stable.

In conclusion

- Asymptotic Stability $\rightarrow$ BIBO Stability

- BIBO Stability $\nrightarrow$ Asymptotic Stability