EE302 - Feedback Systems	**Spring 2019**

Lecture 15

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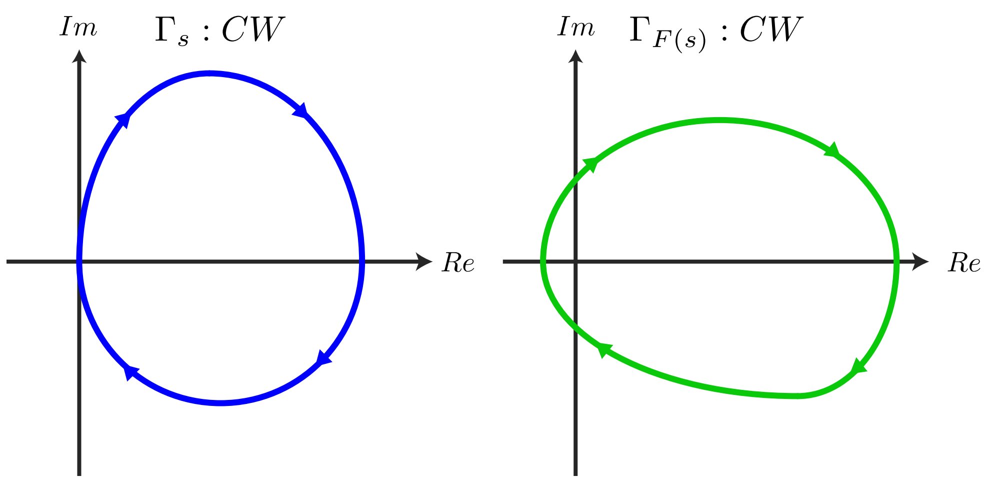
15.1 Nyquist Stability Criterion

Nyquist stability criterion is another method to investigate the stability (absolute and relative) of a dynamical system (feed-forward and feed-back). Its based on the frequency response characteristics of a system.

Definition: A contour Γ_s is a closed path with a direction in a complex plane.

Remark: A continuous function $F(s)$ maps a contour Γ_s in $s-$plane to another contour $\Gamma_{F(s)}$ in $F(s)$ plane. The figure below illustrates a clock-wise contour Γ_s and its map $\Gamma_{F(s)}$ which is also clock-wise in this example.



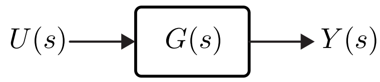
Theorem: Cauchy's Argument Principle

Given $F(s) = \frac{N(s)}{D(s)}$ and a CW contour Γ_s in $s-$ plane which does not cross any zeros (roots of $N(s)$) or poles (roots of $D(s)$) of $F(s)$, then the mapped contour $\Gamma_{F(s)}$ will encircle the origin

$$
\begin{aligned}
&N = Z - P \quad \text{times} \\
&N : \# \text{ CW encirclements of origin} \\
&Z : \# \text{ zeros } F(s) \\
&P : \# \text{ poles } F(s)
\end{aligned}
$$

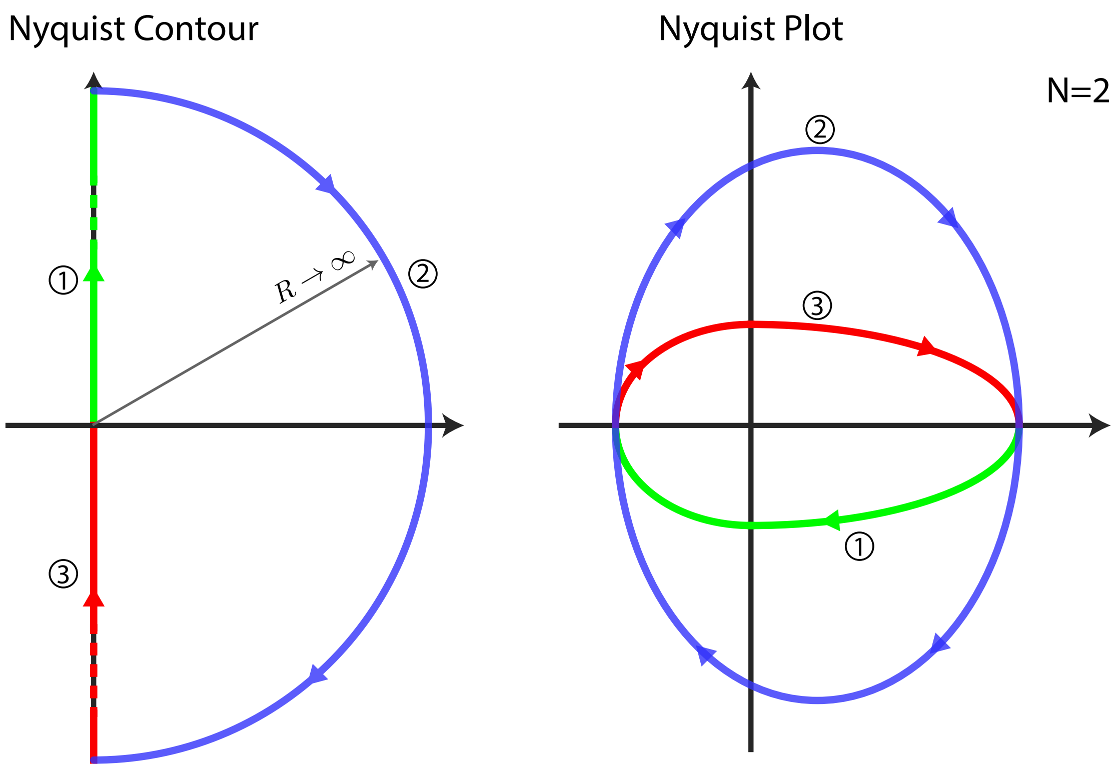
15.1.1 Nyquist Stability Criterion for Feedforward Systems

Let's assume that we want to analyze the stability of the following input-output system



and we know that $G(s) = \frac{N(s)}{D(s)}$ has no zeros or poles on the imaginary axis. Since we are intersted in finding the unstable poles, Nyquist contour/path, Γ_s, is defined in a way that it covers the whole open-right half plane. As illustrated in the Figure below, Nyquist contour is technically a half-circle for which the radius, $R \to \infty$. After that, one can draw the Nyquist plot, which is the mapped contour $\Gamma_{G(s)}$, and count the # of CW encirclements of the origin (N). Based on Cauchy's Argument Principle we know that

- $P = Z - N$, where
- N: # CW encirclements of origin by $\Gamma_{G(s)}$
- Z: # zeros of $G(s)$ with positive real parts
- P: # poles of $G(s)$ with positive real parts



In this example illustration $N = 2$, which implies that total number of unstable poles of $G(s)$ is given by $P = Z - 2$, if Z, i.e. number of zeros with positive real parts, is known than we can compute number of unstable poles P. Alternatively, since in a feedforward system only denominator part determines the stability, instead of $G(s)$, one can draw the Nyquist plot of $\bar{G}(s) = \frac{1}{D(s)}$.

Ex: Let's analyze the feedforward stability of $G(s) = \frac{s-1}{s+1}$ using Nyquist plot.

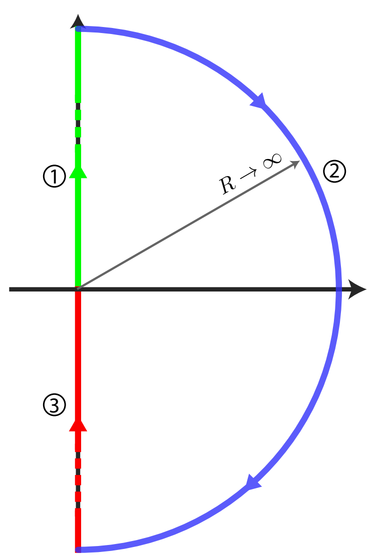
Solution: Based on the Nyquist contour we have three major paths

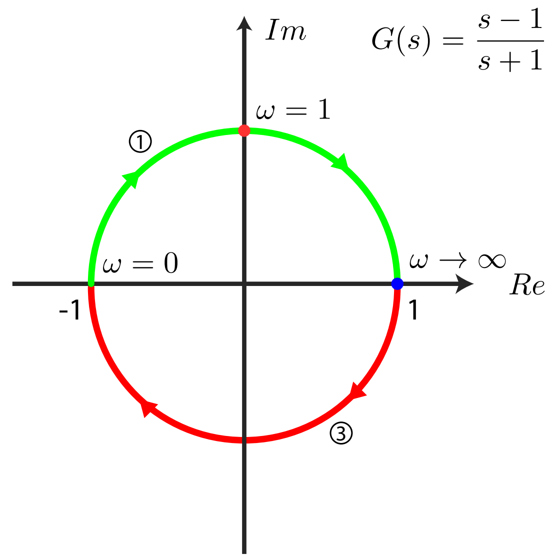
1. This is the polar plot that we covered in the previous lecture, where we plot $G(j\omega)$, where $\omega : 0 \to \infty$
2. This is the infinite radius circular path. In this case if we write s in polar form, we get $s = Re^{j\theta}$ where $R \to \infty$, and $\theta : \pi/2 \to -\pi/2$. Then we can derive that

$$\begin{aligned} G\left(Re^{j\theta}\right) &= \frac{Re^{j\theta}-1}{Re^{j\theta}+1} \approx \frac{Re^{j\theta}}{Re^{j\theta}} \\ \Rightarrow |G\left(Re^{j\theta}\right)| &\approx 1 \quad , \quad \angle[G\left(Re^{j\theta}\right)] \approx 0 \end{aligned}$$

   In other words, this whole path in the Nyquist plot is concentrated around $1+0j$ point in the complex plane.
3. This path is the mapping of the negative imaginary axis, i.e $G(-j\omega)$, where $\omega : \infty \to 0$. Obviously since $G(-j\omega)$ is complex conjugate of $G(j\omega)$ this path is symmetric to the polar plot with respect to the real axis. Note that direction of this path is reverse of the direction of the polar plot.

If we follow the procedure, we obtain the following Nyquist plot.





We can see from the derived Nyquist plot that $N = 1$, and we know that the system has 1 zero with positive real part $Z = 1$. The total number of unstable poles is equal to $P = Z - N = 1 - 1 = 0$, thus the system is obviously stable.

Ex: Let's analyze the feedforward stability of $G(s) = \frac{1}{s+1}$ using Nyquist plot.

Solution: Now let's analyze the Nyquist paths

1. This is the polar plot that we covered in the previous lecture, where we plot $G(j\omega)$, where $\omega : 0 \to \infty$. In this case, the behavior when $\omega \to \infty$ is important. Now let's assume that $\omega \to R$ and $R \gg 1$

$$G_1(jR) \approx \frac{1}{R^2} - \frac{1}{R}j$$
$$\angle[G_1(j\omega)] \approx -\pi/2$$

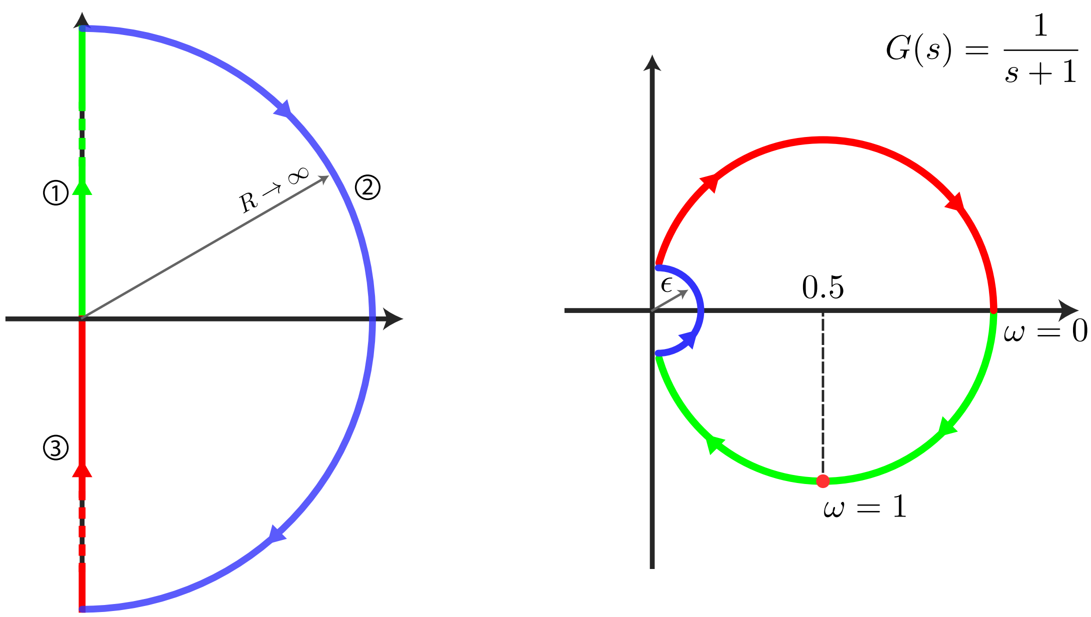
2. Now we should be careful with mapping the infinite radius circular path. Again let $s = Re^{j\theta}$ and $\theta : \pi/2 \to -\pi/2$. Then we can derive that

$$G\left(Re^{j\theta}\right) =\approx \frac{1}{Re^{j\theta}} = \frac{e^{j(-\theta)}}{R}$$
$$\Rightarrow |G\left(Re^{j\theta}\right)| \approx \epsilon \ll 1 \quad , \quad \angle[G\left(Re^{j\theta}\right)] \approx -\theta$$

   Note that when $\theta : \pi/2 \to -\pi/2$, the infinite-small contour around origin rotates in CCW direction.

3. Last path (mapping of negative imaginary axis) is again is the conjugate of polar plot with reverse direction.

If we follow the procedure, we obtain the following Nyquist plot.



We can see from the derived Nyquist plot that $N = 0$, and we know that the system has no zero with positive real part $Z = 0$. The total number of unstable poles is equal to $P = Z - N = 0 - 0 = 0$, thus the system is obviously stable.

Ex: Analyze the feedforward stability of $G(s) = \frac{1}{(s+1)^2}$ using Nyquist plot.

Solution: Firs let's analyze the Nyquist paths

1. This is the polar plot that we covered in the previous lecture, where we plot $G(j\omega)$, where $\omega : 0 \to \infty$. In this case, the behavior when $\omega \to \infty$ is important. Now let's assume that $\omega \to R$ and $R \gg 1$

$$G_1(jR) \approx -\frac{1}{R^2} - \frac{1}{R^3}j \quad , \quad \angle[G_1(j\omega)] \approx -\pi$$

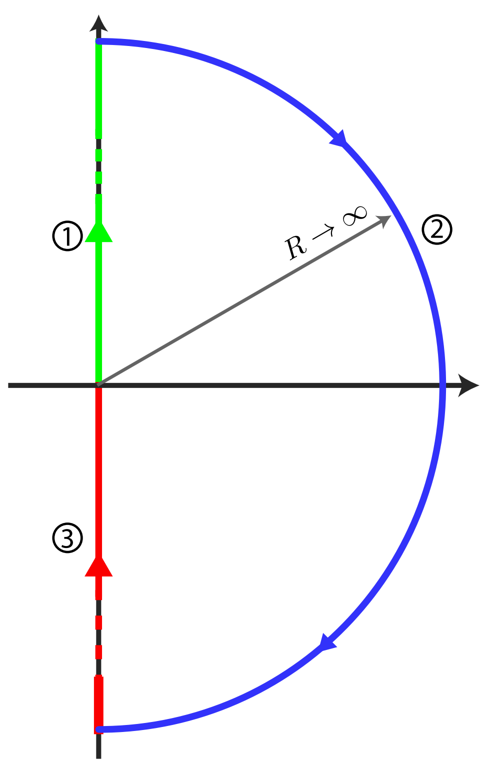
2. Again, we should be careful with mapping of the infinite radius circular path. $s = Re^{j\theta}$ and $\theta : \pi/2 \to -\pi/2$. Then we can derive that

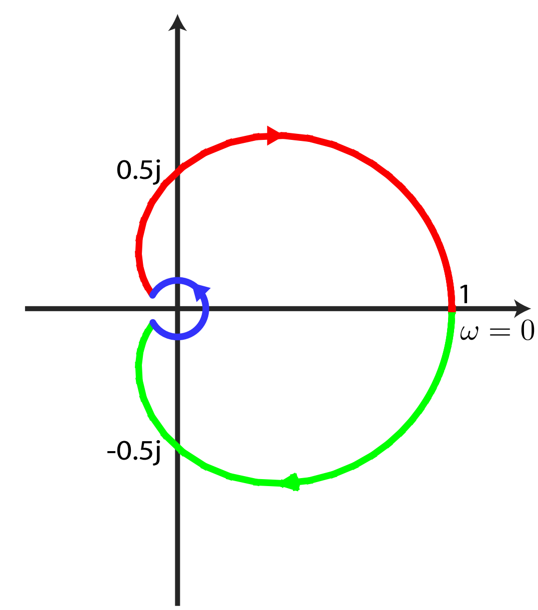
$$G\left(Re^{j\theta}\right) =\approx \frac{1}{R^2 e^{j2\theta}} = \frac{e^{j(-2\theta)}}{R^2}$$
$$\Rightarrow |G\left(Re^{j\theta}\right)| \approx \epsilon^2 \ll 1 \quad , \quad \angle[G\left(Re^{j\theta}\right)] \approx -2\theta$$

   Note that when $\theta : \pi/2 \to -\pi/2$, the infinite-small contour around origin rotates in CCW direction.

3. Last path (mapping of negative imaginary axis) is again the conjugate of polar plot with reverse direction.

If we follow the procedure, we obtain the following Nyquist plot.





We can see from the derived Nyquist plot that $N = 0$, and we know that the system has no zero with positive real part $Z = 0$. The total number of unstable poles is equal to $P = Z - N = 0 - 0 = 0$, thus the system is obviously stable.

Ex: Analyze the feedforward stability of $G(s) = \frac{1}{(s+1)^3}$ using Nyquist plot.

Solution: Firs let's analyze the Nyquist paths

1. This is the polar plot that we have covered in the previous lecture, where we plot $G(j\omega)$, where $\omega : 0 \to \infty$. In this case, the behavior when $\omega \to \infty$ is important. Now let's assume that $\omega \to R$ and $R \gg 1$

$$G(jR) \approx -\frac{3}{R^4} + \frac{1}{R^3}j \qquad \angle[G(j\omega)] \approx \pi/2$$

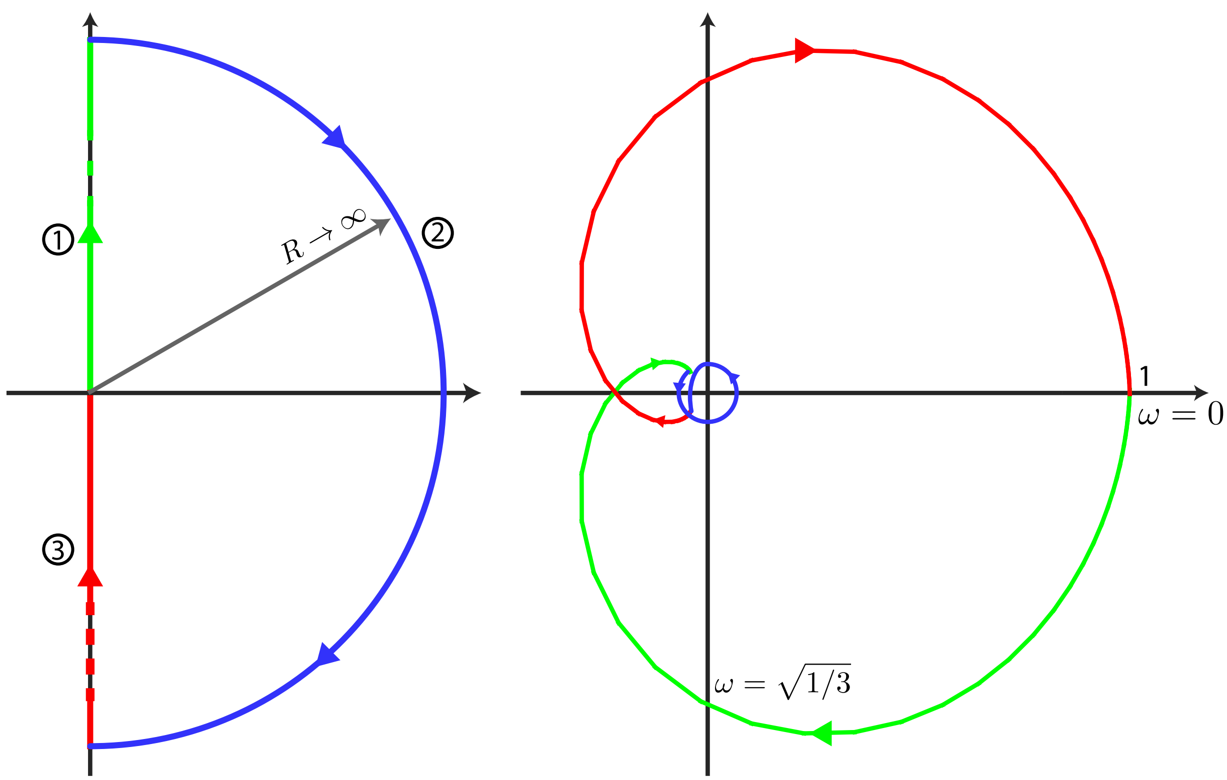
2. Again, we should be careful with mapping of the infinite radius circular path. $s = Re^{j\theta}$ and $\theta : \pi/2 \to -\pi/2$. Then we can derive that

$$G\left(Re^{j\theta}\right) =\approx \frac{1}{R^3 e^{j3\theta}} = \frac{e^{j(-3\theta)}}{R^3}$$
$$\Rightarrow |G\left(Re^{j\theta}\right)| \approx \epsilon^3 \ll 1 \quad , \quad \angle[G\left(Re^{j\theta}\right)] \approx -3\theta$$

Note that when $\theta : \pi/2 \to -\pi/2$, the infinite-small contour around origin rotates in CCW direction.

3. Last path (mapping of negative imaginary axis) is again the conjugate of polar plot with reverse direction.

If we follow the procedure, we obtain the following Nyquist plot.



If carefully analyze the encirclements around the origin, we can see that **net** encirclement around the origin is $N = 1 - 1$ (or $N = 2 - 2$). Based on this, we verify that the total number of unstable poles is equal to $P = Z - N = 0 - 0 = 0$, thus the system is stable.

Ex: Analyze the feedforward stability of $G(s) = \frac{1}{(s-1)(s+2)}$ using Nyquist plot.

Solution: In the polar plot we draw $G(j\omega)$ where $\omega : 0 \to \infty$

$$G(j\omega) = \frac{1}{(j\omega - 1)(j\omega + 2)} = \frac{-(j\omega + 1)(-j\omega + 2)}{(\omega^2 + 1)(\omega^2 + 4)} = \left[-(2 + \omega^2) - \omega j\right] \frac{1}{(\omega^2 + 1)(\omega^2 + 4)}$$
$$\forall \omega \in (0, \infty),\ Re\{G(j\omega)\} < 0 \ \&\ Im\{G(j\omega)\} < 0$$

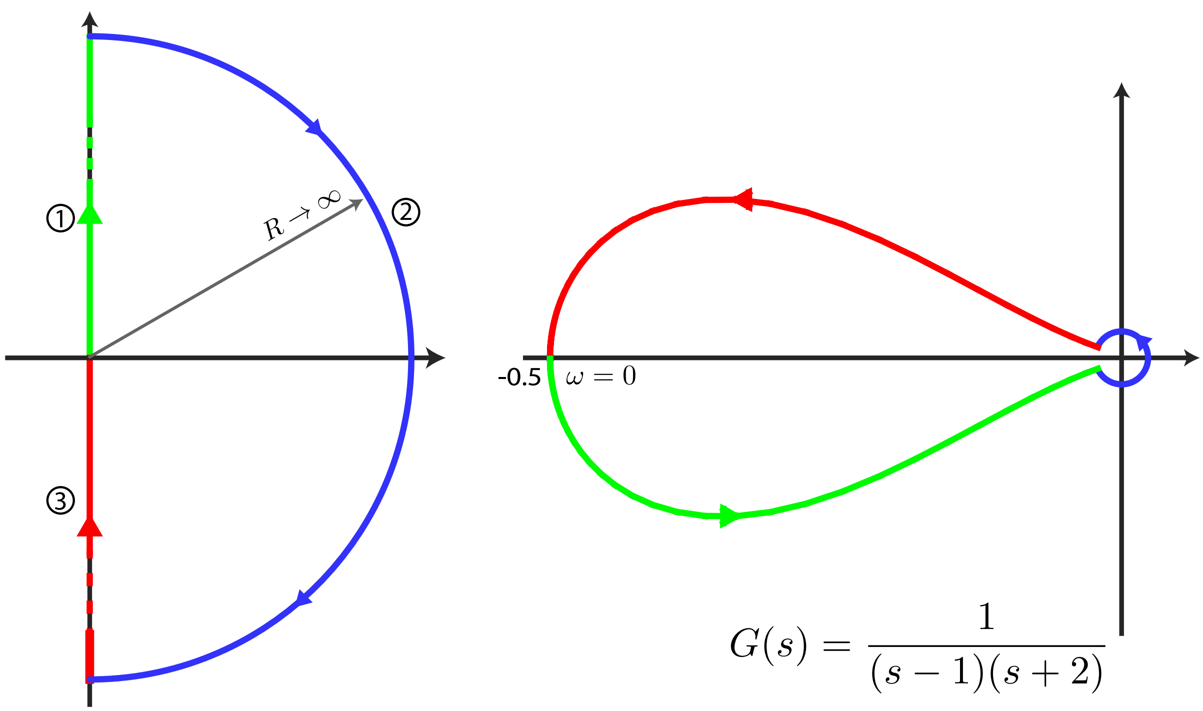
Let's derive where the polar plot approaches when $\omega \to 0$ and $\omega \to \infty$

$$\lim_{\omega \to 0} G(j\omega) = -0.5 + 0j$$
$$\lim_{\omega \to \infty} |G(j\omega)| = 0 \quad , \quad \lim_{\omega \to \infty} \angle[G(j\omega)] = -\pi$$

Now let's derive the Nyquist plot conditions for the infinite circle on the Nyquist contour. In this path $s = Re^{j\theta}$ where $R \to \infty$ (or $R \gg 1$) and $\theta : \pi/2 \to -\pi/2$ (in CW direction). Thus,

$$G(Re^{j\theta}) \approx \frac{1}{R^2 e^{j2\theta}} = \frac{1}{R^2} e^{j(-2\theta)}$$
$$|G(Re^{j\theta})| \approx \epsilon \ll 1$$
$$\angle[G(Re^{j\theta})] = -2\theta \quad \theta : \pi/2 \to -\pi/2$$

Note that associated Nyquist plot around origin turns approximately 2π radians in CCW direction. Last path is the mapping of negative imaginary axis which is simply conjugate of the polar plot with reverse direction.



If carefully analyze the encirclements around the origin, we can see that **net** CW encirclement around the origin is $N = -1$. Based on this, we verify that the total number of unstable poles is equal to $P = Z - N = 0 - (-1) = 1$, thus the system is indeed unstable.