Lecture 12

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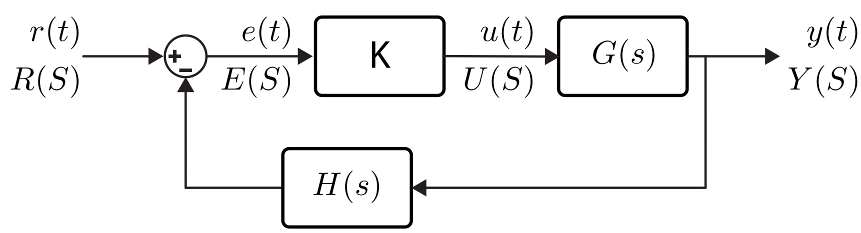
12.1 Root Locus

In control theory, root locus analysis is a graphical analysis method for investigating the change of closed-loop poles/roots of a system with respect to the changes of a system parameter, commonly a gain parameter $K > 0$.

In order to better understand the root locus and derive fundamental rules, we start with the following basic feedback topology where the controller is a P-controller with a gain K.



The closed loop transfer function of this basic control system is

$$\frac{Y(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)H(s)} = \frac{KG(s)}{1 + KG_{OL}(s)}$$

where the poles of the closed loop system are the roots of the characteristic equation

$$1 + KG_{OL}(s) = 0$$
$$1 + K\frac{n(s)}{d(s)} = 0$$

The goal is deriving the qualitative and quantitive behavior of closed-loop pole "paths" for **positive** gain K that solves the equation $1 + KG_{OL}(s) = 0$ (or $1 + K\frac{n(s)}{d(s)} = 0$).

12.1.1 Angle and Magnitude Conditions

Let's analyze the characteristic equation

$$KG_{OL}(s) = -1 \quad ,\text{or} \quad K\frac{n(s)}{d(s)} = -1 \quad ,\text{or} \quad K\frac{(s-z_1)\cdots(s-z_M)}{(s-p_1)\cdots(s-p_N)} = -1$$

Let's derive the **magnitude condition** given that $K > 0$,

$$K|G_{OL}(s)| = 1 \quad ,\text{or} \quad K\left|\frac{n(s)}{d(s)}\right| = 1 \quad ,\text{or} \quad K\frac{|s-z_1|\cdots|s-z_M|}{|s-p_1|\cdots|s-p_N|} = 1$$

Now let's derive the **angle condition** given that $K > 0$,

$$\begin{aligned} \angle[G_{OL}(s)] = \pi(2k+1) \quad &,\text{or} \quad \angle[n(s)] - \angle[d(s)] = \pi(2k+1) \\ &,\text{or} \quad \angle[s-z_1]\cdots\angle[s-z_M] - (\angle[s-p_1]\cdots\angle[s-p_N]) = \pi(2k+1), \quad k \in \mathbb{Z} \end{aligned}$$

For a given K, s values that satisfy both magnitude and angle conditions are located on the root loci. These constitutes the most fundamental knowledge regarding the root locus analysis.

How we can check whether a candidate s^* is in the root -locus or not. If we analyze the angle condition, we can see that it is independent from the parameter K. However, If we focus on the magnitude condition, we can see that

$$K = \frac{1}{|G_{OL}(s^*)|} = \left|\frac{d(s^*)}{d(s^*)}\right| = \frac{|s^*-p_1|\cdots|s^*-p_N|}{|s^*-z_1|\cdots|s^*-z_M|}$$

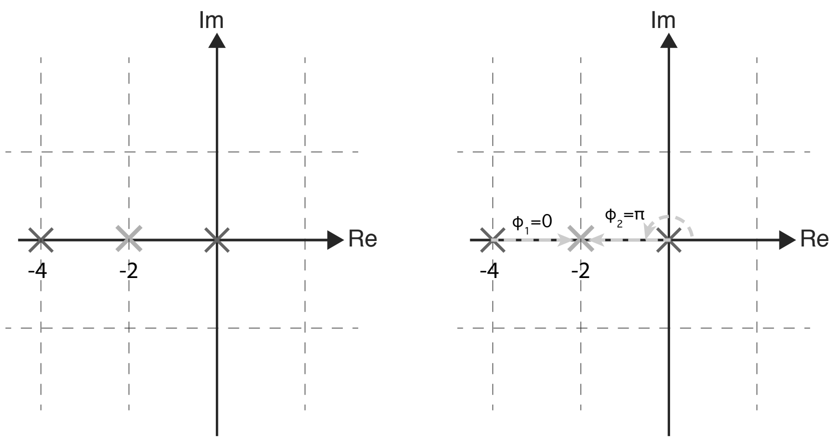
which implies that for every s^* candidate (that is not a pole or zero), we can indeed compute a gain K value.

In conclusion, only angle condition is used for testing whether a point is in the root-locus or not. On the other hand, will use the magnitude condition to compute the value of gain K, if we find that a candidate s^* is in the root locus based on the angle condition.

Ex: It is given that $G_{OL}(s) = \frac{1}{s(s+4)}$. Determine if the following pole candidates are on the root-locus or not

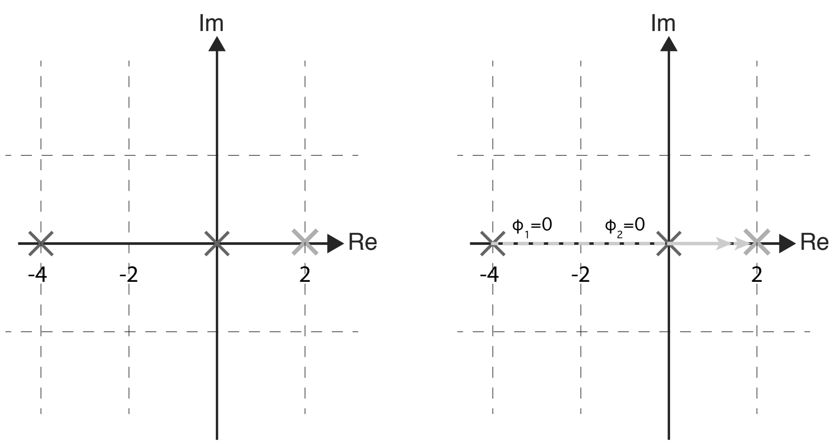
$$p_1^* = -2 \ , \ p_2^* = 2 \ \ p_3^* = -2 + 2j$$

Solution: We only test the angle condition. Solutions are illustrated on the $s-$planes provided below



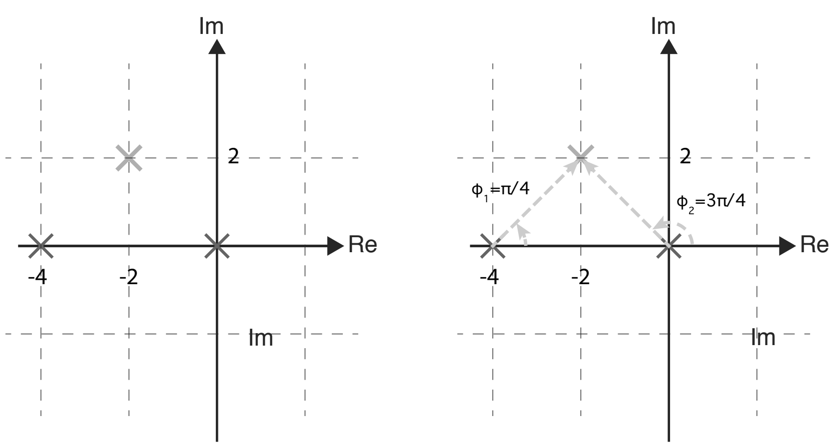
√

$\angle G_{OL}(p^*) = -\pi$



×

$\angle G_{OL}(p^*) = 0$



√

$\angle G_{OL}(p^*) = -\pi$

12.1.2 Rules and procedure for constructing root loci

1. Compute the zeros poles of the open-loop transfer function and write the characteristic eqiation of closed-loop system.

$$
\begin{aligned}
1 + KG_{OL}(s) &= 0 \\
1 + K\frac{n(s)}{d(s)} &= 0 \\
1 + K\frac{(zs - z_1)\cdots(zs - z_M)}{(s - p_1)\cdots(s - p_N)} &= 0
\end{aligned}
$$

2. Root loci has N separate branches. Since,

$$
\begin{aligned}
[(s - p_1)\cdots(s - p_N)] + K\left[(zs - z_1)\cdots(zs - z_M)\right] &= 0 \\
d(s) + Kn(s) &= 0
\end{aligned}
$$

   has N number of roots for all K.

3. Root loci starts from poles of $G_{OL}(s)$ and
   (a) M branches terminates at the zeros of $G_{OL}(s)$
   (b) $N - M$ branches terminates at ∞ (implicit zeros of $G_{OL}(s)$)

   It is relatively easy to understand this

$$
\begin{aligned}
d(s) + Kn(s) &= 0 \\
K \to 0 \;\Rightarrow\; [d(s) + Kn(s) = 0 \;\to\; [d(s) &= 0] \\
K \to \infty \;\Rightarrow\; [d(s) + Kn(s) = 0 \;\to\; [n(s) &= 0]
\end{aligned}
$$

4. Root loci on the real axis determined by open-loop zeros and poles. $s = \sigma \in \mathbb{R}$ then, based on the angle condition we have

$$
\mathrm{Sign}[G_{OL}(\sigma)] = -1
$$

   Let's first analyze the effect of complex conjugate pole/zero (and double pole/zero on real axis) pairs on the equation above. Let $\sigma^* \in \mathbb{R}$ is the candidate location and complex conjugate poles has the following form $p_{1,2} = \sigma \pm j\omega$

$$
\mathrm{Sign}[(\sigma^* + \sigma - j\omega)(\sigma^* + \sigma + j\omega)] = \mathrm{Sign}[(\sigma^* + \sigma)^2 + \omega^2] = 1
$$

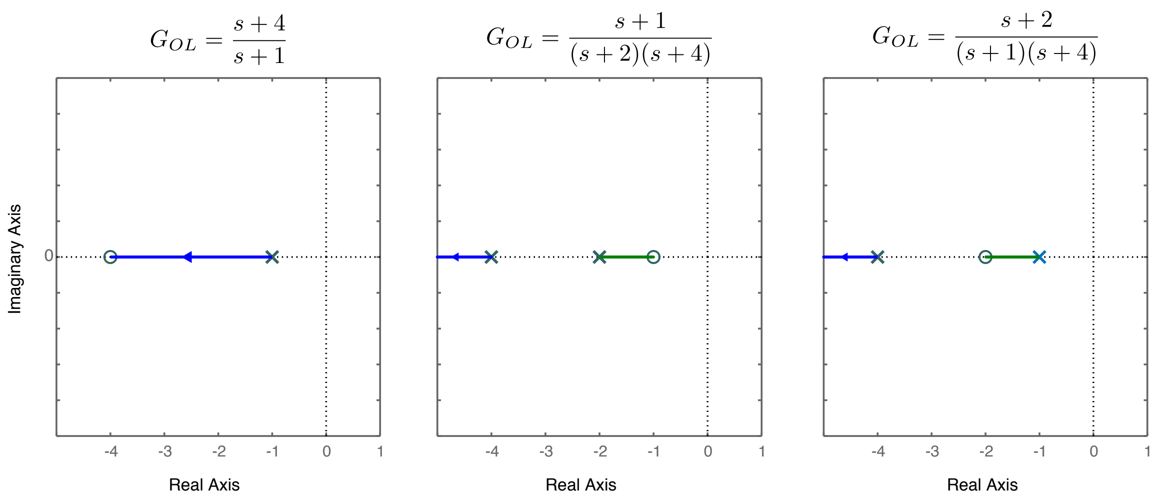
   We can see that complex conjugate zero/pole pairs have not effect on angle condition for the roots on the real axis. Then for the remaining ones we can derive the following condition

$$
\mathrm{Sign}[G_{OL}(\sigma)] = \prod_{i=1}^{\bar{M}} \mathrm{Sign}[\sigma - z_i] \prod_{j=1}^{\bar{N}} \mathrm{Sign}[\sigma - p_j] = -1
$$

   which means that for ODD number of poles + zeros $\mathrm{Sign}[\sigma - p_i]$ and $\mathrm{Sign}[\sigma - z_i]$ must be negative for satisfying this condition for that particular σ to be on the root-locus. We can summarize the rule as

   If the test point σ on real axis has ODD numbers of poles and zeros in its right, then this point is located on the root-locus.

Ex: The figure below illustrates the root locus plots of three different transfer functions.



5. Asymptotes

- $N - M$ branches goes to infinity. Thus, there exist $N - M$ many asymptotes
- For large s we can have the following approximation

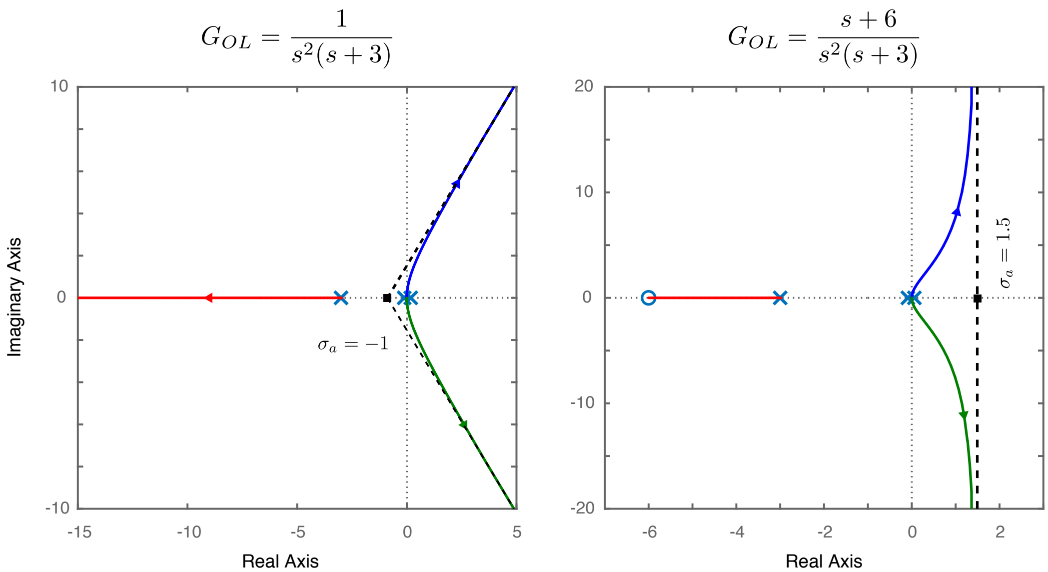
$$K\frac{(s-z_1)\cdots(s-z_M)}{(s-p_1)\cdots(s-p_N)} \approx \frac{K}{s^{N-M}}$$
$$\angle\left[\frac{K}{s^{N-M}}\right] = -(N-M)\angle[s] = \pi(2k+1),\ k \in \mathbb{Z}$$
$$\phi_a = \frac{\pm\pi(2k+1)}{N-M},\ k \in \{1, \cdots, N-M\}$$

- Real axis intercept σ_c can be computed as

$$\sigma_c = \frac{\sum p_i - \sum z_i}{N-M}$$

This can be derived via a different approximation (see textbook)

Ex: The figure below illustrates the root locus plots of two different transfer functions.



6. Breakaway and break-in points on real axis. When s is real $s = \sigma,\ \sigma \in \mathbb{R}$, we have

$$1 + KG_{OL}(\sigma) = 0$$

Note that break-in and breakaway points corresponds to double roots. Thus, if σ_b is a break-away or break-in point we have

$$1 + KG_{OL}(\sigma_b) = 0$$
$$K\left[\frac{d}{d\sigma}G_{OL}(\sigma)\right]_{\sigma_b} = 0$$

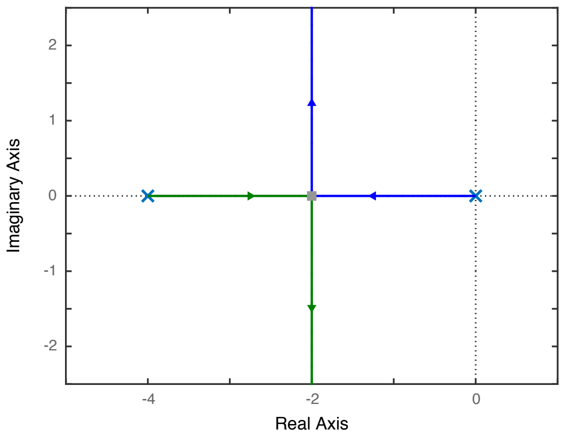
Thus, we conclude that break-in or break-away points satisfy the following conditions

$$\left[\frac{dG_{OL}(\sigma)}{d\sigma}\right]_{\sigma=\sigma_b} = 0 \quad , K(\sigma_B) = \frac{-1}{G_{OL}(\sigma_b)} \quad , K(\sigma_b) > 0$$

We can derive two corollary conditions for computing σ_b as

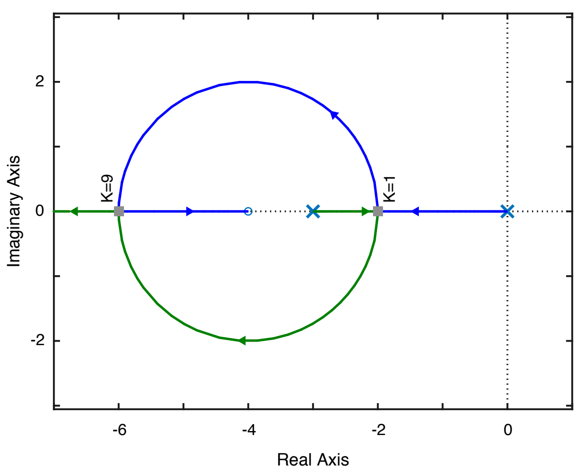
$$\left[\frac{d}{d\sigma}\frac{1}{G_{OL}(\sigma)}\right]_{\sigma=\sigma_b} = 0 \quad or, \quad \left[\left(\frac{d}{d\sigma}N(\sigma)\right)D(\sigma) - \left(\frac{d}{d\sigma}D(\sigma)\right)N(\sigma)\right]_{\sigma_b} = 0$$

Ex: Draw the root locus diagram for $G_{OL}(s) = \frac{1}{s(s+4)}$, compute the real axis intercept σ_c and break away point (with the associated gain value).



$$\sigma_c = -2$$
$$\sigma_{ba} = -2 \quad , \ K(\sigma_{ba}) = 4$$

Ex: Draw the root locus diagram for $G_{OL}(s) = \frac{s+4}{s(s+3)}$, compute the break-away and break-in points (with the associated gain values).



$$\sigma_b^2 + 8\sigma_b + 12 = 0$$
$$\sigma_{b-a} = -2 \quad , \ K(\sigma_{b-a}) = 1$$
$$\sigma_{b-in} = -6 \quad , \ K(\sigma_{b-in}) = 9$$

7. Finding the imaginary axis crossings.

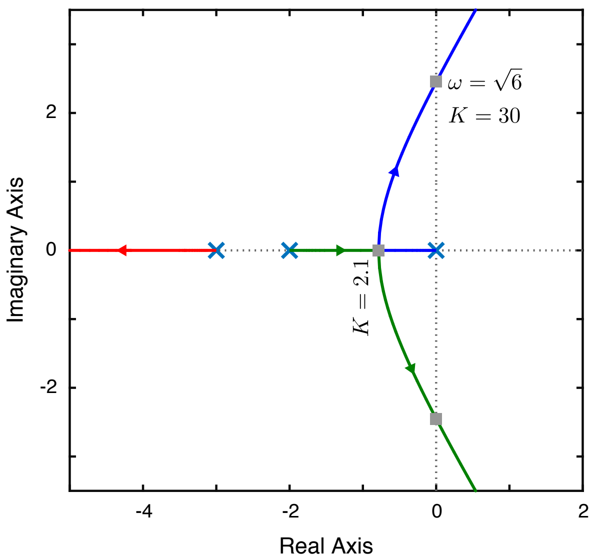
   These crossings are particularly important, since at these crossings (generally) stability changes. At these points the poles become purely imaginary $p_{1,2} = \pm j\omega$. If we insert this into characteristic equation we get

$$1 + KG_{OL}(j\omega) = 0$$
$$D(j\omega) + KN(j\omega) = 0$$
$$,$$
$$\text{Re}\{D(j\omega) + KN(j\omega)\} = 0$$
$$\text{Im}\{D(j\omega) + KN(j\omega)\} = 0$$

   Note that depending on the order of the system, solving the above equation can be most computationally very heavy.

Second way of finding the imaginary axis crossing is to apply the Routh-Hurwitz criteria. Note that at these crossings the system becomes unstable. Using this fact, we can first construct a Routh table for the closed-loop characteristic equation and then derive the K values where a change of stability occurs. After that, we can use the computed critical K values to derive the pole locations on the imaginary axis..

Ex: Draw the root locus diagram for $G_{OL}(s) = \frac{1}{s(s+2)(s+3)}$, compute the break-away point and imaginary axis crossings (with the associated gain values).



Break-away point

$$3\sigma_b^2 + 10\sigma_b + 6 = 0$$
$$\sigma_{b,1} = -0.8 \quad , \; K(\sigma_{b,1}) = 2.1 > 0 \rightarrow \text{OK}$$
$$\sigma_{b,2} = -2.5 \quad , \; K(\sigma_{b,2}) = -0.6 < 0 \rightarrow \text{NO}$$

Imaginary axis crossing

$$D(j\omega) + K(j\omega) = 0$$
$$(j\omega)^3 + 5(j\omega)^2 + 6(j\omega) + K = 0$$
$$(K - 5\omega^2) + (6\omega - \omega^3)j = 0$$
$$\Rightarrow \omega = \sqrt{6} \quad , \; K = 30$$

Now let's find the imaginary axis crossings using the Routh table. The characteristic equation for this system is $s^3 + 5s^2 + 6s + K$, then the Routh table takes the form

s^3	1	6	
s^2	5	K	
s^1	$\frac{30-K}{5}$	0	
s^0	K	0	

We know that in order for the system to stable $K \in (0, 30)$, since we only consider positive K values, when $K = 30$ system stability changes (from stable to unstable). Let $K = 30$ and re-form the Routh table.

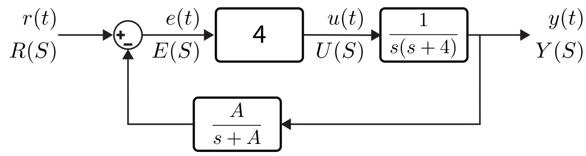
s^3	1	6	
s^2	5	30	$\rightarrow A(s) = 5s^2 + 30$
s^1	10	0	$\leftarrow A'(s) = 10s$
s^0	30	0	

Based on the Routh table, we conclude that when $K = 30$, system becomes unstable, the unstable poles are located on the imaginary axis, and their locations can be find using the Auxiliary polynomial as

$$A(s) = 0 \rightarrow p_{1,2} = \pm\sqrt{6}j$$

12.1.3 Root-locus with respect to different parameters

Let's consider the following feedback control system. We wonder the location of closed-loop poles with respect to the parameter A which does not directly fit to the classical form of root-locus.



Let's first compute the closed-loop TF and analyze the characteristic equation.

$$\begin{aligned}\frac{Y(s)}{R(s)} &= \frac{KG(s)}{1+KG(s)H(s)} \\ &= \frac{s+A}{s(s+4)(s+A)+4A} \\ &= \frac{s+A}{s^3+(A+4)s^2+4As+4A}\end{aligned}$$

Now let's organize the characteristic equation

$$(s^3+4s^2)+A(s^2+4s+4)=0$$

If we divide the characteristic equation by (s^3+4s^2) we obtain

$$\begin{aligned}1+A\frac{s^2+4s+4}{s^3+4s^2} &= 0 \\ 1+A\bar{G}_{OL}(s) &= 0\end{aligned}$$

Now if we consider $\bar{G}_{OL}(s)$ as the open-loop transfer function and draw the root-locus, then we would derive the dependence of the roots to the parameter A.

Root-locus of the system w.r.t parameter $A>0$ is given below.