EE502 - Linear Systems Theory II **Spring 2023**

Lecture 3

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3.1 State-Space Representation to Frequency Domain

In this lecture we will cover the conversion from state-space representations to frequency domain representations ($s-$domain for CT systems and $z-$domain for DT systems) and analyze the connections between two representations.

3.1.1 CT State-Space to $s-$domain

Note that a SS representation of an n^{th} order CTI-LTI system has the from below.

$$\begin{aligned}
\text{Let } & x(t) \in \mathbb{R}^n \ , \ y(t) \in \mathbb{R}^q \ , \ u(t) \in \mathbb{R}^p, \\
& \dot{x}(t) = Ax(t) + Bu(t), \\
& y(t) = Cx(t) + Du(t), \\
\text{where } & A \in \mathbb{R}^{n \times n} \ , \ B \in \mathbb{R}^{n \times p} \ , \ C \in \mathbb{R}^{q \times n} \ , \ D \in \mathbb{R}^{q}
\end{aligned}$$

In order to convert state-space to frequency domain, we start with taking the Laplace transform of the both sides of the state-equation

$$\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t) \\
sX(s) - x_0 &= AX(s) + BU(s) \\
sX(s) - AX(s) &= x_0 + BU(s) \\
(sI - A)\, X(s) &= x_0 + BU(s) \\
X(s) &= (sI - A)^{-1} x_0 + (sI - A)^{-1} BU(s)
\end{aligned}$$

Now let's concentrate on the output equation

$$\begin{aligned}
y(t) &= Cx(t) + Du(t) \\
Y(s) &= C\,(sI - A)^{-1} x_0 + \left[C\,(sI - A)^{-1} B + D\right] U(s)
\end{aligned}$$

where $C\,(sI - A)^{-1} x_0$ corresponds to the initial-condition response and when $u(t) = 0$ we have

$$\begin{aligned}
Y(s) &= \left[C\,(sI - A)^{-1} B + D\right] U(s) \\
G(s) &= C\,(sI - A)^{-1} B + D
\end{aligned}$$

where $G(s)$ is called the **transfer function matrix** which has the following form for a general $p-$input–$q-$output MIMO system

$$G(s) = \begin{bmatrix} G_{11}(s) & \cdots & G_{1p}(s) \\ \vdots & & \vdots \\ G_{q1}(s) & \cdots & G_{qp}(s) \end{bmatrix}$$

Definiton: $G_{ij}(s) = \frac{n_{ij}(s)}{d_{ij}(s)}$ is classified as follows

- $G_{ij}(s)$ is *proper* $\Leftrightarrow \deg(n_{ij}(s)) \leq \deg(d_{ij}(s)) \Leftrightarrow G_{ij}(\infty) = \lim_{s\to\infty} G_{ij}(s) = C$ where $|C| < \infty$
- $G_{ij}(s)$ is *strictly proper* $\Leftrightarrow \deg(n_{ij}(s)) < \deg(d_{ij}(s)) \Leftrightarrow G_{ij}(\infty) = \lim_{s\to\infty} G_{ij}(s) = 0$
- $G_{ij}(s)$ is *bi-proper* $\Leftrightarrow \deg(n_{ij}(s)) = \deg(d_{ij}(s)) \Leftrightarrow G_{ij}(\infty) = C$ where $|C| < \infty$ & $C \neq 0$
- $G_{ij}(s)$ is *improper* $\Leftrightarrow \deg(n_{ij}(s)) > \deg(d_{ij}(s)) \Leftrightarrow |G_{ij}(\infty)| \to \infty$

Remark: $G_{ij}(s)$ is strictly proper $\forall(i,j)$ iff $D = \mathbf{0}$

$$\begin{aligned} G(s) &= C\,(sI-A)^{-1}\,B = \frac{C\mathrm{Adj}\,(sI-A)\,B}{\mathrm{Det}\,(sI-A)} \text{ , where} \\ &\det\left(\mathrm{Det}\,(sI-A)\right) = n \\ &\mathrm{Adj}\,(sI-A) = \left[\mathrm{Cofactor}\,(sI-A)\right]^T \end{aligned}$$

Let Cofactor $(sI-A) = Co$ then $Co_{ij} = (-1)^{i+j}\mathrm{Det}\,(M_{ij})$, where $\mathrm{Det}\,(M_{ij})$ is called the minor of $(sI-A)_{ij}$ and is the the determinant of the submatrix formed by deleting the ith row and jth column. Note that $\deg(Co_{ij}) \leq (n-1)\ \forall(i,j)$ which implies that $G_{ij}(s)$ is strictly proper.

Definition:

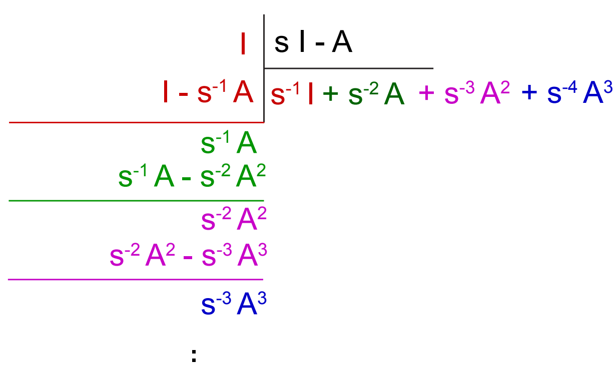
- A scalar $\lambda \in \mathbb{C}$ is called a pole of $G_{ij}(s)$ if $|G_{ij}(\lambda)| \to \infty$
- A scalar $\gamma \in \mathbb{C}$ is called a zero of $G_{ij}(s)$ if $|G_{ij}(\gamma)| = 0$

Definition: Two polynomials are said to be coprime of they have no common root.

Remark:

- $\lambda \in \mathbb{C}$ is a pole of $G_{ij}(s) = \frac{n_{ij}(s)}{d_{ij}(s)}$ if $d_{ij}(s)$ and $n_{ij}(s)$ are coprime and $d_{ij}(\lambda) = 0$
- $\lambda \in \mathbb{C}$ is a zero of $G_{ij}(s) = \frac{n_{ij}(s)}{d_{ij}(s)}$ if $d_{ij}(s)$ and $n_{ij}(s)$ are coprime and $n_{ij}(\lambda) = 0$

It seems that if we can find $\Phi(t) = \mathcal{L}\{(sI-A)^{-1}\}$, it would be helpful to find both the initial condition response and forced response in time-domain. Let's first expand $(sI-A)^{-1}$ by long "division"



If we follow the path we can find that

$$(sI - A)^{-1} = s^{-1}I + s^{-2}A + s^{-3}A^2 + s^{-4}A^3 + s^{-5}A^4 + + \cdots$$

$$\Psi(t) = \mathcal{L}^{-1}\left[(sI - A)^{-1}\right] = I + At + A^2\frac{t^2}{2} + A^3\frac{t^3}{3!} + A^4\frac{t^3}{4!} + \cdots = e^{At}$$

We can then find the initial condition only response and impulse response matrix of the system using $\Psi(t) = e^{At}$

- Initial Condition Only Response : $x(t) = e^{At}x_0$
- Impulse Response Matrix : $G(t) = Ce^{At}B + D\delta(t)$

In that respect general solution can be written as

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

$$y(t) = Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)$$

3.1.2 DT State-Space to $z-$domain

Note that a SS representation of an n^{th} order DTI-LTI system has the from below.

$$\begin{aligned}
\text{Let } x[k] \in \mathbb{R}^n\ ,\ y[k] \in \mathbb{R}^q\ ,\ u[k] \in \mathbb{R}^p, \\
x[k+1] = Ax[k] + Bu[k], \\
y[k] = Cx[k] + Du[k], \\
\text{where } A \in \mathbb{R}^{n\times n}\ ,\ B \in \mathbb{R}^{n\times p}\ ,\ C \in \mathbb{R}^{q\times n}\ ,\ D \in \mathbb{R}^q
\end{aligned}$$

In order to convert state-space to frequency domain, we start with taking the $Z-$transform of the both sides of the state-equation, where Z-transform of a unilateral (causal) discrete time signal $w[k]$ is given by

$$W(z) = \mathcal{Z}\{w[k]\} = \sum_{k=0}^{\infty} w[k]z^{-k}$$

$$\begin{aligned}
x[k+1] &= Ax[k] + Bu[k] \\
zX(z) - zx[0] &= AX(z) + BU(z) \\
zX(z) - AX(z) &= zx[0] + BU(z) \\
(zI - A)\,X(Z) &= zx[0] + BU(z) \\
X(z) &= z\,(zI - A)^{-1}\,x[0] + (zI - A)^{-1}\,BU(z)
\end{aligned}$$

I recommend to those of you not familiar with Z-transform operation on difference equations to read *shifting theorem* in EE402 Lecture Notes (Lecture # 2), indeed going over the whole Lecture would be very helpful.

Now let's concentrate on the output equation

$$\begin{aligned}
y[k] &= Cx[k] + Du[k] \\
Y(z) &= zC\,(zI - A)^{-1}\,x[0] + \left[C\,(zI - A)^{-1}\,B + D\right]U(z)
\end{aligned}$$

where $zC\,(zI-A)^{-1}\,x[0]$ corresponds to the initial-condition response and when $u[k]=0$ we have

$$
\begin{aligned}
Y(z) &= \left[C\,(zI-A)^{-1}\,B+D\right]U(z)\\
G(z) &= C\,(zI-A)^{-1}\,B+D
\end{aligned}
$$

similar to the CT case $G(z)$ is called the **transfer function matrix**. Note that resultant frequency domain solution in DT systems is very similar to the solution in CT systems (except the initial condition response). Without a big surprise state-space to transfer function related definitions of CT systems generally holds also for DT systems, Such as *properness, poles, zeros* etc.

Similar to the CT case, if we expand $z\,(zI-A)^{-1}$ by long "division" we may find expressions for time-domain solutions.

$$
\begin{array}{r|l}
zI & zI-A \\ \cline{2-2}
zI-A & I + z^{-1}A + z^{-2}A^2 + z^{-3}A^3 + \ldots \\ \cline{1-1}
A & \\
A - z^{-1}A^2 & \\ \cline{1-1}
z^{-1}A^2 & \\
z^{-1}A^2 - z^{-2}A^3 & \\ \cline{1-1}
z^{-2}A^3 & \\
\vdots &
\end{array}
$$

$$
\begin{aligned}
z\,(zI-A)^{-1} &= I + z^{-1}A + z^{-2}A^2 + z^{-3}A^3 + \cdots\\
\Psi[k] = \mathcal{Z}^{-1}\left[z\,(zI-A)^{-1}\right] &= I\,\delta[k] + A\,\delta[k-1] + A^2\,\delta[k-2] + A^3\,\delta[k-3] + \cdots\\
&= A^k
\end{aligned}
$$

Similarly initial condition only response simple becomes

$$
\begin{aligned}
x[k] &= A^k x[0]\\
y[k] &= CA^k x[0]
\end{aligned}
$$

whereas we can derive the impulse response as

$$
\begin{aligned}
G[k] &= \mathcal{Z}^{-1}\{C\,(zI-A)^{-1}\,B+D\}\\
&= C\mathcal{Z}^{-1}\{z^{-1}\left[z\,(zI-A)^{-1}\right]\}B + D\delta[k]\\
&= CA^{k-1}Bh[k-1] + D\delta[k]
\end{aligned}
$$

where $h[n]$ is the unit-step function.