EE302 - Feedback Systems	**Spring 2019**

Lecture 13

Lecturer: Asst. Prof. M. Mert Ankarali

13.1 PID Control Analysis & Design with Root-Locus

Let's assume that we would like you to design a controller for the following second order plant transfer function

$$G(s) = \frac{1}{(s+1)(s+3)}$$

The requirements of the closed-loop system are

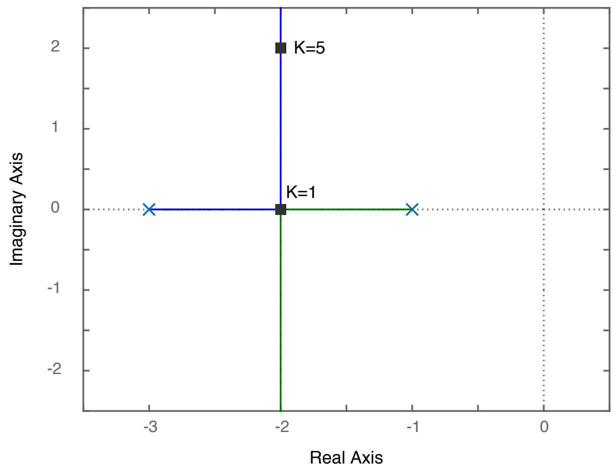
- Minimum possible settling time (%2)
- Minimum steady-state error
- Minimum possible over-shoot

13.1.1 Proportional (P) Controller

Let's first design a P controller, $C(s) = K$. Let's start with steady-state error performance. Plant is a type-one system and controller is a static gain thus unit step and unit ramp steady-state error can be computed as

$$\begin{aligned} \text{Step}: \quad & e_{ss} = \frac{1}{1 + K_P/3} \\ \text{Step}: \quad & e_{ss} = \infty \end{aligned}$$

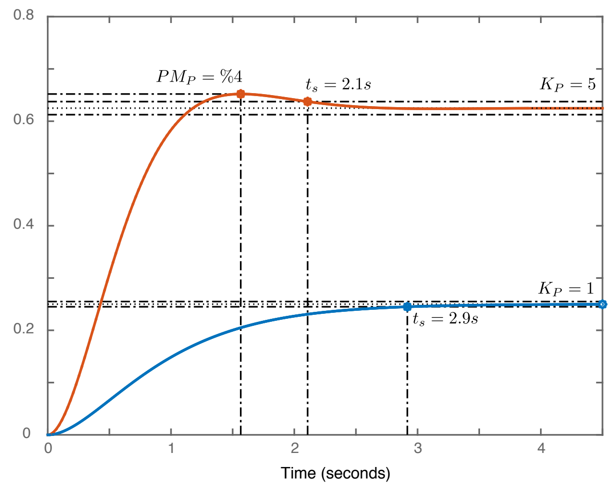
We can see that unit-ramp error is ∞ regardless of K_P, where as we can reduce the steady-state error by increasing proportional gain K_P. Now let's draw root locus and comment on settling time and over-shoot performance.



Based on root-locus plot we can see that best convergence rate achieved with $\sigma = -2$, for which the approximate settling time is $2s$. When $K = 1$, system becomes critically damped. When we further increase the gain, real part of the poles do not change, however oscillations starts and grows with K_P. Obviously, we should choose a gain $K_P > 1$, however after that point there is a trade-off between over-shoot and settling time performance. Let's choose two candidate locations, $p_{1,2}^{(1)} = -2$ and $p_{1,2}^{(1)} = -2 \pm 2j$. Table below details the gain values at these pole locations, unit-step steady-state errors, and estimated settling time and maximum over-shoot values.

$p_{1,2}$	K_P	e_{ss}	t_s	M_P
-2	1	0.75	2	0
-2 ± 2	5	0.375	2	0.04

We verify that both steady-state and transient performance increases with larger K_P. However, we can also see that settling time estimation for $K_P = 5$ has a larger error, which expected since the poles are close to each other thus violates the dominant pole assumption. If we consider the values in the table, it seems to be reasonable to choose $K_P = 5$ since it provides a much better steady-state performance, of course if existing of an small overs-shoot is not a major problem for the design. Now let's plot step-responses of the closed loop system for both cases.



We verify most of our theoretical findings with the only exception about the settling time for $K_P = 1$, which is larger then the estimation. In conclusion, $K_P = 5$ is a good choice for overall requirements.

13.1.2 Proportional Derivative (PD) Controller

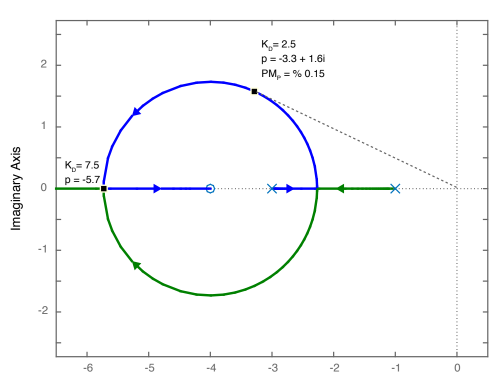
Now let's design a PD controller. First make some change of variables and re-write the PD controller in a different form.

$$C(s) = K_P + K_D s = K_D\left(s + \frac{K_P}{K_D}\right) = K_D(s + \alpha)$$

We can see that PD controller introduces an extra zero to the open loop transfer function. Let's compute the steady-state error performance.

- Unit step: $e_{ss} = \frac{1}{1+K_P/3} = \frac{1}{1+\alpha K_D/3}$,
- Unit ramp: $e_{ss} = \infty$

Technically in classical PD form K_D has no effect on steady-state performance. However if we adopt the second form with α and fix α first, then as we increase K_D steady-state error will decrease. Now let's fix $\alpha = 4 = \frac{K_P}{K_D}$, and draw the root-locus.

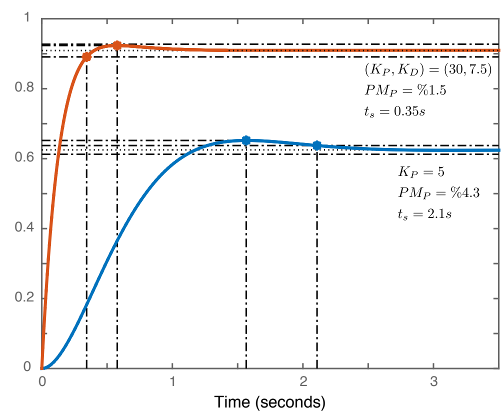


We think that the improvement of added zero is very clear. If we carefully look at the root-locus, we can see that at the pole location that gives the worst maximum over-shoot performance, maximum percent over-shoot is only %0.15, moreover overs-shoot completely disappears when $K_D \geq 7.5$. The best settling/convergence time performance occurs at the break-in point which has an approximate gain of $K_D = 7.5$,

Table below details the P and D gain values at this pole locations, closed-pole locations, unit-step steady-state error, and estimated settling time value.

K_P	K_D	p	e_{ss}	$t_s[s]$
30	7.5	-5.7	0.09	0.7

Now let's compare this PD controller (with $(K_P, K_D) = (30, 7.5)$) and the previously designed P controller (with $(K_P, K_D) = (5, 0)$) by simulating the step responses.



We can see that PD control policy outperforms the P controller in every category.

On the other hand, interestingly we expected no over-shoot with the PD controller, yet we still observe some over-shoot at the output. In addition to this, actual settling time is approximately half of our previous estimation. The discrepancies between the transient characteristics between simulation and estimated values occurs due the effects of extra zero in the closed-loop transfer function.

13.1.3 Integral (I) Controller

Now let's analyze how a pure integral controller affects the proposed system using root-locus analysis.

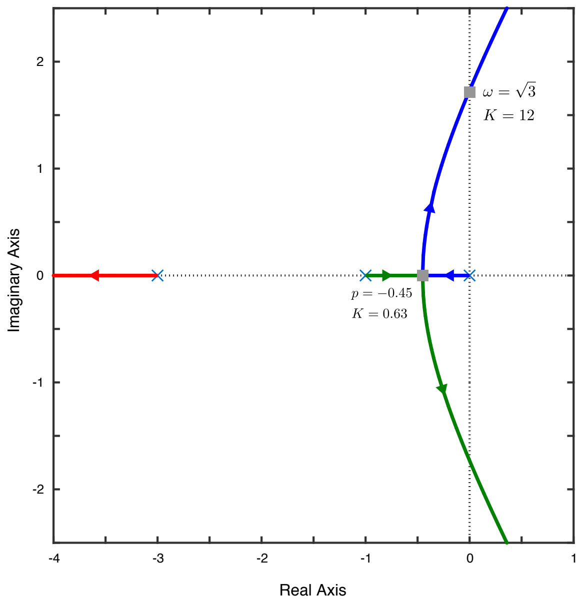
$$C(s) = \frac{K_I}{s} \quad , \quad G(s) = \frac{1}{(s+1)(s+3)}$$

$$G_{OL}(s) = \frac{1}{s(s+1)(s+3)} \quad , \quad K = K_I$$

We already know that steady-state performance is affected positively with an Integral controller

- Unit step: $e_{ss} = 0$
- Unit ramp: $e_{ss} = \frac{3}{K_I}$

Now let's analyze the affects on transient performance and stability using root-locus diagram



Break-away point

$$3\sigma_b^2 + 8\sigma_b + 3 = 0$$
$$\sigma_{b,1} = -0.45 \rightarrow \text{ OK} \quad , \quad K = 0.63$$
$$\sigma_{b,2} = -2.2 \rightarrow \text{ NO}$$

Imaginary axis crossing

$$D(j\omega) + K_I N(j\omega) = 0$$
$$(j\omega)^3 + 4(j\omega)^2 + 3(j\omega) + K_I = 0$$
$$(K - 4\omega^2) + (3\omega - \omega^3)j = 0$$
$$\Rightarrow \omega = \sqrt{3} \quad , \ K = 12$$

We can make the following observations

- Closed-loop system becomes unstable for $K_I > 6$. Note that with a P controller closed loop system was always stable for $K_P > 0$
- Best settling time that can be achieved with an I controller is approximately, $t_s = 9s$, which is quite bad compared to the simple P-controller.

In conclusion, integral action improves steady-state performance, but it degrades the stability and transient performance.

13.1.4 Proportional Integral (PI) Controller

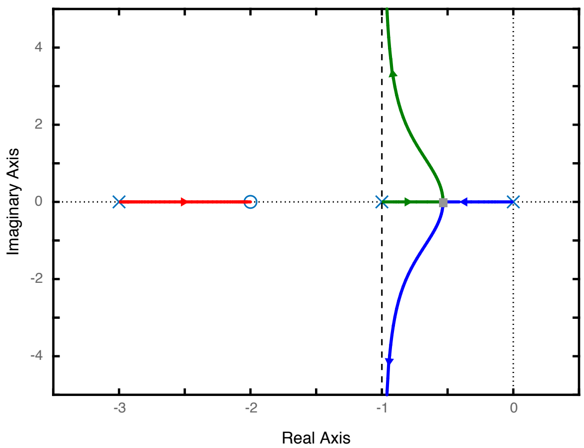
Now let's design a PI controller. First make some change of variables and re-write the PI controller in a different form.

$$C(s) = K_P + K_I \frac{1}{s} = K_P \left(\frac{s + K_I/K_P}{s} \right) = K_P \left(\frac{s+\alpha}{s} \right)$$

We can see that PI controller introduces an extra zero to the open loop transfer function and a pole at the origin. We know that the steady-state error performance characteristics. of a PI controller is same with an I controller.

Now let's fix $\alpha = 2 = \frac{K_I}{K_P}$, and draw the root-locus w.r.t K_P. Note that

$$G_{OL}(s) = \frac{s+2}{s(s+1)(s+3)}$$



Break-away point

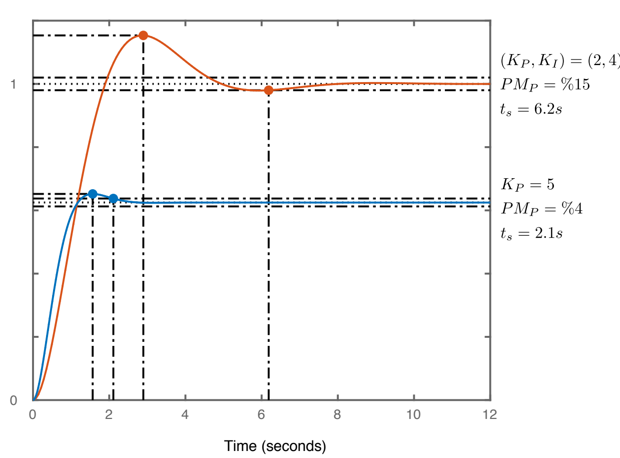
$$2\sigma_b^3 + 10\sigma_b^2 + 16\sigma_b + 6 = 0$$
$$\sigma_{b,1} = -0.53 \;\rightarrow\; \text{OK} \quad , \quad K = 0.42$$
$$\sigma_{b,2} = -2.2 \pm 0.8j \;\rightarrow\; \text{NO}$$

We can make the following observations

- Closed-loop system is stable for $K_P > 0$ for this choice of α.
- Best settling time that can be achieved with this PI controller is approximately, $t_s = 4s$, but this is achieved when $K \rightarrow \infty$.
- Best settling time value with this PI controller is the approximately double of the best settling time value of the P controller.
- System is approximately acts like a second-order over-damped system when $K_P \in (0, 0.42)$, where as acts like a a second-order under-damped system when $K_P \in (0.42, \infty)$.
- Oscillations and over-shoot increases as we increase K_P. This there is a trade-off between settling time and over-shoot performance.

In conclusion, PI controller has superior steady-state performance compared to P and PD controllers, however there exist substantial transient performance drop even compared to simple P controller.

Now let's choose $K_P = 2$, then we can find that $K_I = 4$. Now let's compare this PI controller $(K_P, K_I) = (2, 4)$ with our previously chosen P controller $(K_P, K_I) = (5, 0)$ by simulating the step-responses.



It is obvious that while PI controller has a superior steady-state performance, P controller provides much better transient characteristics.

Practice Question: Choose α from this set $\{4,\ -1.2,\ 0.8\}$, and draw the root-locus diagrams for each α. Comment on the results in terms of overall transient performance based on the root locus diagrams.

13.1.5 Proportional Integral Derivative (PID) Controller

If one is not satisfied with the transient characteristics of a P-controller but also would like to eliminate steady-state error fot the unit step input, he/she can design a PID controller.

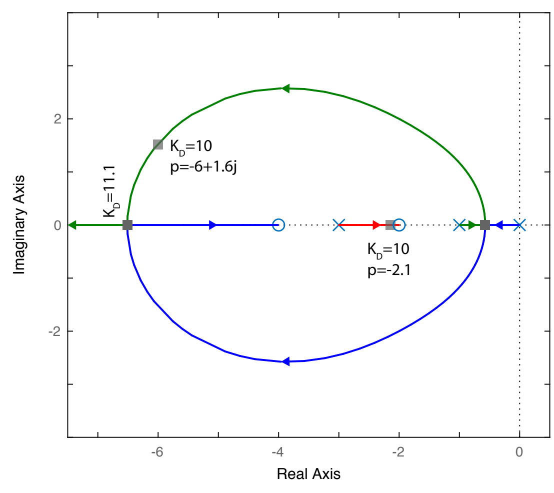
First make some change of variables and re-write the PID controller in a different form.

$$\begin{aligned} C(s) &= K_D s + K_P + K_I \frac{1}{s} = K_D \left(s + (K_P/K_D) + (K_I/K_D)\frac{1}{s} \right) \\ &= K_D \left(s^2 + (K_P/K_D)s + (K_I/K_D) \right) \frac{1}{s} \end{aligned}$$

We can see that a PID controller has a second order numerator dynamics, and it is possible to have two real zeros, or two complex conjugate zeros. In general, PID gains are chosen such that numerator has two zeros

$$C(s) = K_D \frac{(s+\alpha)(s+\beta)}{s}$$

We know that PID controller has same stead-state performance characteristics with PI and I controllers. Now let's choose $\alpha = 2$ and $\beta = 4$ and draw the root locus w.r.t. K_D.

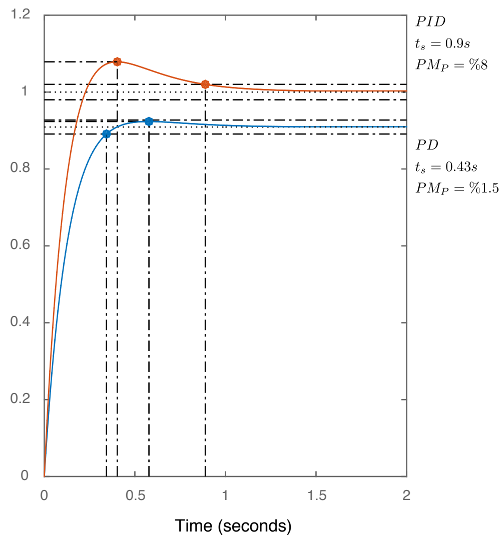


Unlike the root-locus plots of previous cases, it is harder to have an idea about what is good for transient performance. It seems that as $K_D \nearrow$, one pole on the real axis move towards imaginary axis and stops at $\sigma = -2$ when $K \to \infty$, where as other two poles deviates from the imaginary axis when $K_D \nearrow$. Let's choose $K_D = 10$ which marked on the root-locus plot. In this case poles of the closed-loop system takes the form $p_1 = -2.1$ and $p_{2,3} = -6 \pm 1.6j$. We may conclude that the system is approximately first order since the single pole is much closer to the real-axis. In this case, we expect settling time as $t \approx 1.9s$. Note that this settling time value is slightly better then the best settling time value that is satisfied with a P-controller. Moreover we expect that over-shoot would be negligible (less than %0.001).

When $K_D = 10$, we have

$$\begin{aligned} &s^2 + (K_P/K_D)s + (K_I/K_D) = (s+2)(s+4) = s^2 + 6s + 8 \\ &K_P = 60 \\ &K_I = 80 \end{aligned}$$

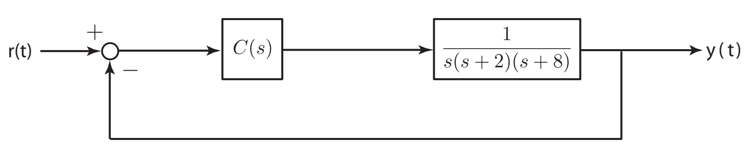
Now let's simulate this PID controller and previously designed PD controller and compare the transient and steady state performances.



We can clearly see the qualitative differences between the designed PD and PID controllers. For this specific controllers, we can see that PD has much better transient performance and PID is superior in steady-state performance. Note that PID completely eliminates the steady-state error and it is certainly possible to improve the steady-state performance of PD controller by increasing its gain.

On the other hand in PID controller, if we locate both zeros to the left of the pole at -3 location (**left as a practice**), we would obtain better root-locus picture (in terms of pole locations with better transient performance). However, the main trade-off will be the substantially increased gain values, which is already much larger then the designed P and PD controllers.

Example: Design a controller $C(s)$ for the following feedback-system such that dominant closed-loop poles has a damping ratio of $\zeta = 1/\sqrt{2}$ and damped natural frequency of $\omega_d = 2rad/s$.



Solution: Since there is no steady-state requirement and the goal is to locate the dominant poles to a specific location, first natural choice of a controller is a PD controller. We know that PD controller can be written in the following forms

$$C(s) = K_P + K_D s = K_D(s + \alpha)$$

and the open-loop transfer function takes the form

$$G_{OL}(s) = K_D \frac{s + \alpha}{s(s+2)(s+8)}$$

Desired pole location can be explicitly computed as

$$\tan\phi = \frac{\sqrt{1-\zeta^2}}{\zeta} = 1$$
$$p_{1,2} = -2 \pm 2j$$

Now let's try to compute α using the angle condition since we want the specified pole to be on the root-locus

$$\angle[G(s)]_{s=p} = (2k+1)\pi \ k \in \mathbb{Z}$$

,

$$\begin{aligned}\angle[G(p)] &= \angle[p+\alpha] - (\angle[p] + \angle[p+2] + \angle[p+8]) \\ &= \angle[(-2+\alpha) \pm 2j] - \left(\frac{3\pi}{4} + \frac{\pi}{2} + \arctan(1/3)\right) \\ &= \angle[(-2+\alpha) \pm 2j] + \frac{3\pi}{4} - \arctan(1/3)\end{aligned}$$

,

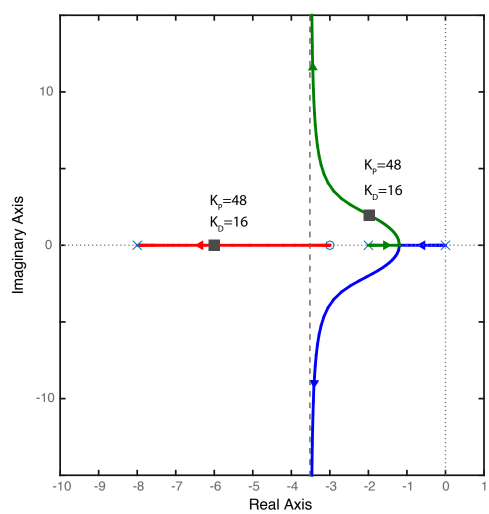
$$\angle[(-2+\alpha) \pm 2j] = \frac{\pi}{4} - \arctan(1/3)$$

,

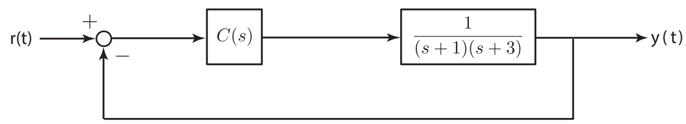
$$\Rightarrow \frac{2}{\alpha - 2} = \arctan\left(\frac{\pi}{4} + \arctan(1/3)\right) = 2$$

$$\Rightarrow \alpha = 3$$

Note that direct complex algebra could provide a simpler computational process. Now given that $\alpha = 3$, let's draw the root locus. We can see that the desire dominant pole locations are satisfied when $K_P = 48$ and $K_D = 16$.



Example: Design a controller for the following feedback-system such that dominant closed-loop poles has a damping ratio of $\zeta = 1/\sqrt{2}$ and damped natural frequency of $\omega_d = 2rad/s$, system has zero unit step steady-state error.



Solution: Since the requirement imposes zero steady-state error we have to implement an I action. Let's test a PI controller first.

$$C(s) = K_P + K_I \frac{1}{s} = K_I \frac{s+\alpha}{s}$$

and the open-loop transfer function takes the form

$$G_{OL}(s) = K_I \frac{s+\alpha}{s(s+1)(s+3)}$$

Now let's try to compute α using the angle condition since we want the specified pole to be on the root-locus

$$\angle[G(s)]_{s=p} = (2k+1)\pi \;\; k \in \mathbb{Z}$$

,

$$\begin{aligned}\angle[G(p)] &= \angle[p+\alpha] - (\angle[p] + \angle[p+1] + \angle[p+3]) \\ &= \angle[p+\alpha] - \left(\frac{3\pi}{4} + \pi - \arctan(2) + \arctan(2)\right) \\ &= \angle[p+\alpha] + \frac{\pi}{4}\end{aligned}$$

,

$$\angle[(-2+\alpha) \pm 2j] = \frac{3\pi}{4} \qquad \Rightarrow \alpha = 0 \text{ (????????)}$$

This implies that a PI controller can not satisfy the requirements. Now let's try to design a PID controller.

$$\begin{aligned}C(s) &= K_D s + K_P + K_I \frac{1}{s} == K_D \frac{s^2 + (K_P/K_D)s + (K_I/K_D)}{s} \\ &= K_D \frac{(s+\alpha)(s+\beta)}{s}\end{aligned}$$

Now we have two parameters to satisfy angle condition. Let's simplify the process and let $\alpha = \beta$, thus the PID controller has the following form

$$C(s) = K_D \frac{(s+\alpha)^2}{s}$$

Now let's try to compute α using the angle condition

$$\angle[G_{OL}(s)]_{s=p} = (2k+1)\pi \;\; k \in \mathbb{Z}$$

,

$$\begin{aligned}
\angle[G_{OL}(p)] &= 2\angle[p+\alpha] - (\angle[p] + \angle[p+1] + \angle[p+3]) \\
&= 2\angle[p+\alpha] - \left(\frac{3\pi}{4} + \pi - \arctan(2) + \arctan(2)\right) \\
&= 2\angle[p+\alpha] + \frac{\pi}{4}
\end{aligned}$$

,

$$\angle[(-2+\alpha) \pm 2j] = \frac{3\pi}{8} \qquad \Rightarrow \alpha = 2.8284$$

Now let's compute K_D using magnitude condition

$$K_D = \frac{1}{|G_{OL}(-2+2j)|} = 3.018$$

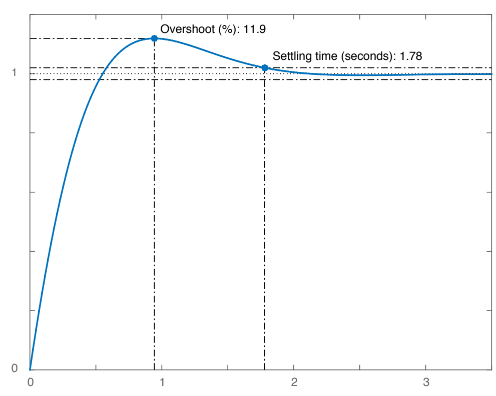
then we can compute K_P and K_I gains

$$\begin{aligned}
&K_D\left(s^2 + (K_P/K_D)s + (K_I/K_D)\right) = K_D(s+\alpha)^2 = K_D(s^2 + 5.657s + 8) \\
&K_I = 24.12 \\
&K_P = 17.07
\end{aligned}$$

Let's compute closed-loop transfer function and associated closed loop-poles

$$\begin{aligned}
T(s) &= \frac{3.018s^2 + 17.07s + 24.12}{s^3 + 7.018s^2 + 20.07s + 24.12} \\
p_{1,2} &= -2 \pm 2j \\
p_3 &= -3.0144
\end{aligned}$$

We can see that the poles that are closer to the origin are placed at desired locations, but we can also see that third pole is not far away enough to conclude the dominance. Now let's plot the step-response of the closed-loop system,



Based on the dominant pole locations, we would expect a settling time value of $t_s \approx 2s$, while the actual settling time is approximately $1.8s$. On the other hand the over-shoot expectation based on the pole location is $PM_P = \%4$, where as in the simulation we observe %12 maximum overs-hoot. Note that, we observed similar "errors" due to closed-loop zero dynamics even in the cases where system has only to complex-conjugate poles. For this reason, PID controller fairly satisfies the design requirements.