EE302 - Feedback Systems **Spring 2018**

Lecture 9

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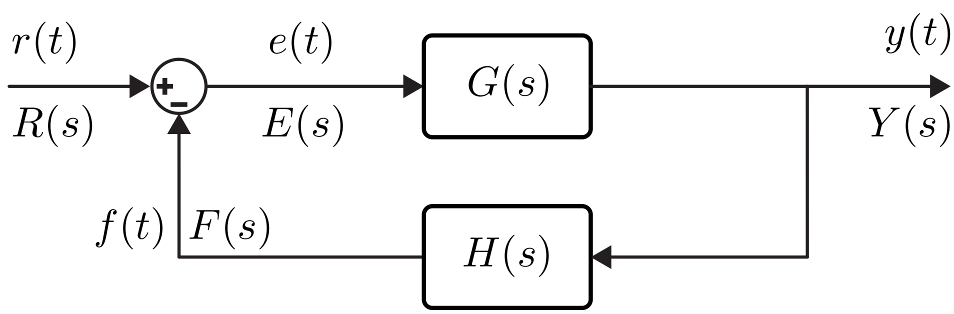
9.1 Steady-Sate Response Analysis

Fundamental concept that we need to perform stead-state response analysis of a control system is the final value theorem. Given a continuous time signal $x(t)$ and its Laplace transform $X(s)$, if $x(t)$ is convergent signal, final value theorem states that

$$\begin{aligned}\lim_{t\to\infty} x(t) &= \lim_{s\to 0}\left[sX(s)\right]\\ x_{ss} &= \lim_{s\to 0}\left[sX(s)\right]\end{aligned}$$

9.1.1 Tracking Performance

The most important steady-state performance condition for a control system is the tracking performance under steady-state conditions. Let's consider the following fundamental feedback topology.



In order to achieve a good tracking performance, obviously the error signal $e(t)$ need to be small. Accordingly, steady-state tracking performance is determined by the steady-state error of the closed-loop system, that we can compute using final value theorem as

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0}\left[se(s)\right]$$

Let's compute $E(s)/R(s)$, i.e. transfer function from the reference input to the error signal,

$$\begin{aligned}E(s) &= R(s) - E(s)G(s)H(s),\\ \frac{E(s)}{R(s)} &= \frac{1}{1+G(s)H(s)}\end{aligned}$$

Note that $G(s)H(s)$ is the transfer function from the error signal $E(s)$ to the signal which is fed to the negative terminal of the main difference operator, i.e. $F(s)$. This transfer function is called the feed-forward or open-loop pulse transfer function of the closed-loop control system. For this system,

$$\frac{F(s)}{E(s)} = G_{OL} = G(s)H(s)$$

Then $E(s)$ can be written as

$$E(s) = R(s)\frac{1}{1 + G_{OL}(s)}$$

It is obvious that first requirement on steady-state error performance is that closed-loop system have to be stable. Now let's analyze specific but fundamental input scenarios.

Unit-Step Input

We know that $r(t) = h(t)$ and $R(s) = \frac{1}{s}$ then we have

$$\begin{aligned} e_{ss} &= \lim_{s\to 0}\left[sR(s)\frac{1}{1+G_{OL}(s)}\right] \\ &= \lim_{s\to 0}\left[s\frac{1}{s}\frac{1}{1+G_{OL}(s)}\right] \\ e_{ss} &= \frac{1}{1+\lim_{s\to 0} G_{OL}(s)} \end{aligned}$$

If the DC gain of the system (also called static error constant) is constant, i.e. $\lim_{s\to 0} G_{OL}(s) = K_{DC}$ then the steady state error can be computed as

$$e_{ss} = \frac{1}{1+K_{DC}}$$

It is obvious that

$$\begin{aligned} e_{ss} &\neq 0 \quad \text{if} \quad |K_{DC}| < \infty \\ e_{ss} &\to 0 \quad \text{if} \quad K_{DC} \to \infty \end{aligned}$$

At this point, it could be helpful to introduce the concept of system *type*, to generalize the steady-state error analysis.

Definition: Let's write the open-loop transfer function of a closed-loop system in the following standard form

$$G_{OL}(s) = \frac{K}{s^N}\frac{b_0 s^m + \cdots + b_{m-1}s + 1}{a_0 s^n + \cdots + a_{n-1}s + 1}$$

The closed-loop system is called as **Type N** system, where N is the # if integrators in the open-loop transfer function (OLTF).

Based on these results, we can have the following conclusions regarding steady-state error for unit-step input

- If $G_{OL}(0) = K_P$, $|K_P| < \infty$, then

  $$e_{ss} = 1/(1 + K_P)$$

  These are **Type 0** (or **Type** $N \leq 0$) systems. We observe a bounded steady-state error and it is possible to reduce the error by increasing the static gain constant K_P.

- If $G_{OL}(0) = \infty$, then $e_{ss} = 0$. In other words, for **Type** $N > 0$ systems, the steady-state error is perfectly zero .

Now let's summarize the steady-state error conditions

- Type $N \leq 0$: $e_{ss} = \frac{1}{1+K_P}$
- Type $N > 0$: $e_{ss} = 0$

Unit-Ramp Input

We know that $r(t) = th(t)$ and $R(s) = \frac{1}{(s^2}$ then we have

$$\begin{aligned}
e_{ss} &= \lim_{s\to 0}\left[sR(s)\frac{1}{1+G_{OL}(s)}\right] \\
&= \lim_{s\to 0}\left[s\frac{1}{s^2}\frac{1}{1+G_{OL}(s)}\right] \\
&= \frac{1}{lim_{s\to 0}s^{-1}G_{OL}(s)} \\
e_{ss} &= \frac{1}{lim_{s\to 0}\frac{K}{s^{N-1}}\frac{b_0s^m+\cdots+b_{m-1}s+1}{a_0s^n+\cdots+a_{n-1}s+1}}
\end{aligned}$$

Based on this result we can have the following steady-state error conditions for the unit-ramp input based on the type condition of the system

- Type $N < 1$: $e_{ss} \to \infty$
- Type $N = 1$: $e_{ss} = \frac{1}{K_v}$
- Type $N > 1$: $e_{ss} = 0$

where K_v is called the velocity error constant.

Unit-Quadratic (Acceleration) Input

We know that $r(t) = \frac{1}{2}t^2h(t)$ and $R(s) = \frac{1}{s^3}$ then we have

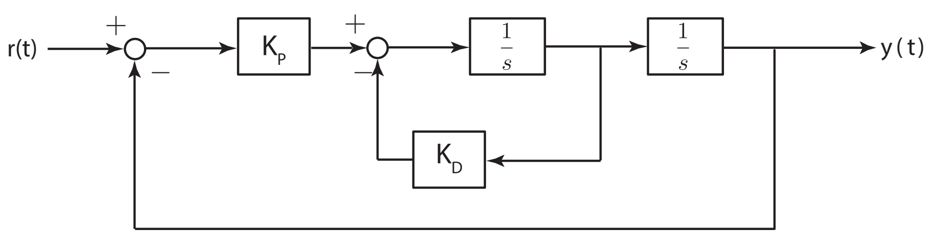
$$\begin{aligned}
e_{ss} &= \lim_{s\to 0}\left[sR(s)\frac{1}{1+G_{OL}(s)}\right] \\
&= \lim_{s\to 0}\left[s\frac{1}{s^3}\frac{1}{1+G_{OL}(s)}\right] \\
&= \frac{1}{lim_{s\to 0}s^{2}G_{OL}(s)} \\
e_{ss} &= \frac{1}{lim_{s\to 0}\frac{K}{s^{N-2}}\frac{b_0s^m+\cdots+b_{m-1}s+1}{a_0s^n+\cdots+a_{n-1}s+1}}
\end{aligned}$$

Based on this result we can have the following steady-state error conditions for the unit-ramp input based on the type condition of the system

- Type $N < 2$: $e_{ss} \to \infty$
- Type $N = 2$: $e_{ss} = \frac{1}{K_a}$
- Type $N > 2$: $e_{ss} = 0$

where K_a is called the acceleration (parabolic) error constant.

Example 1: Compute the $G_{OL}(s)$ for the following closed-loop system and define its **Type**. After that, compute the steady-state errors to unit-step, unit-ramp, a and unit-quadratic inputs.



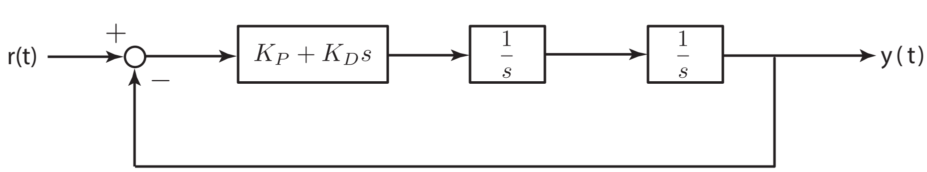
Solution:

$$G_{OL}(s) = \frac{K_P}{s(s+K_D)}$$
$$\text{Type 1} \quad , \; K_v = \frac{K_P}{K_D}$$

Then the steady-state errors are computed as

- Unit-step: $e_{ss} = 0$
- Unit-ramp: $e_{ss} = \frac{K_D}{K_P}$
- Unit-acceleration: $e_{ss} = \infty$

Example 2: Compute the $G_{OL}(s)$ for the following closed-loop system and define its **Type**. After that, compute the steady-state errors to unit-step, unit-ramp, a and unit-quadratic inputs.



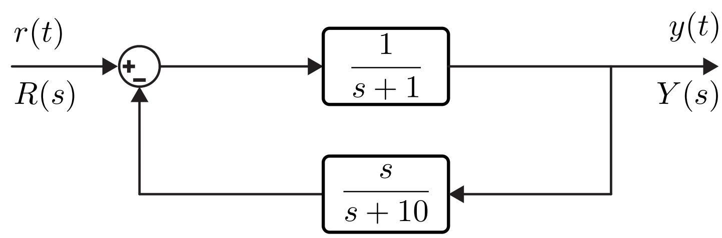
Solution:

$$G_{OL}(s) = \frac{K_P + K_D s}{s^2}$$
$$\text{Type 2} \quad , \; K_a = K_P$$

Then the steady-state errors are computed as

- Unit-step: $e_{ss} = 0$
- Unit-ramp: $e_{ss} = 0$
- Unit-acceleration: $e_{ss} = \frac{1}{K_a}$

Example 3: Compute the $G_{OL}(s)$ for the following closed-loop system and define its **Type**. After that, compute the steady-state errors to unit-step, unit-ramp, a and unit-quadratic inputs.



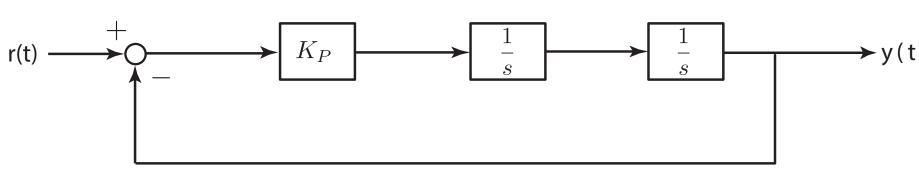
Solution:

$$G_{OL}(s) = \frac{s}{(s+1)(s+10)}$$
$$\text{Type } -\mathbf{1} \quad , \; K_P = 0$$

Then the steady-state errors are computed as

- Unit-step: $e_{ss} = 1$
- Unit-ramp: $e_{ss} = \infty$
- Unit-acceleration: $e_{ss} = \infty$

Example 4: Compute the steady-state error to unit-step input for the following system.



Bad Solution:

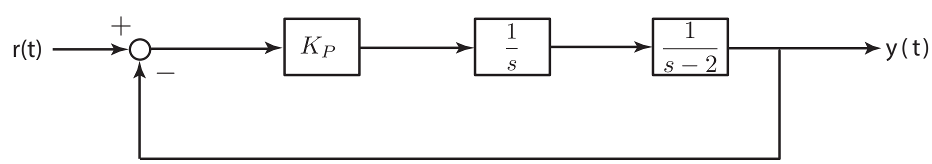
$$G_{OL}(s) = \frac{K_p}{s^2}$$
$$\text{Type } \mathbf{2}$$
$$e_{ss} = 0 \quad ??????$$

Good Solution: Let's compte $Y(s)$ and then $y(t)$,

$$Y(s) = \frac{\frac{K_p}{s^2}}{1+\frac{K_p}{s^2}} R(s) = \frac{K_p}{s(s^2+K_p)}$$
$$y(t) = 1 - \cos(Kt) \; t > 0$$

Error function takes the form $e(t) = \cos(Kt)$ which does not have a limit, i.e., there is no e_{ss}. If closed-loop transfer function has poles on imaginary axis then, we can not apply final value theorem.

Example 5: Compute the steady-state error to unit-step input for the following system when $K_P = 1$.



Good Solution :) Let's check if $y(t)$ is a convergent signal

$$Y(s) = \frac{\frac{1}{s(s-2)}}{1+\frac{1}{s(s-2)}} R(s) = \frac{1}{s(s^2-2s+1)}$$
$$y(t) = te^t - e^t + 1 \; t > 0$$

Error function takes the form $e(t) = e^t - te^t$, thus

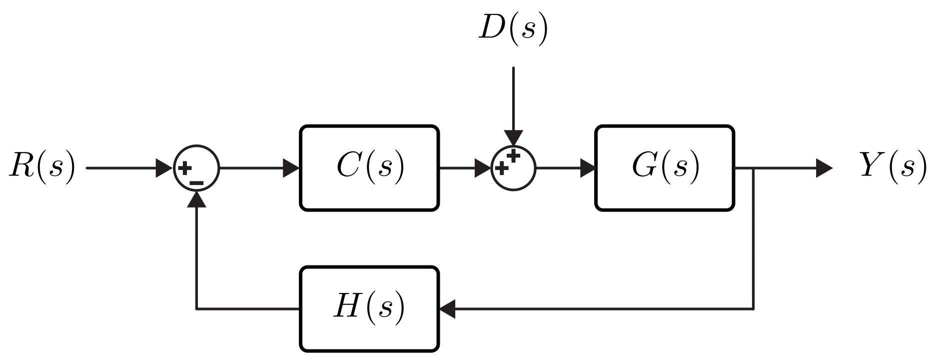
$$e_{ss} = |\lim_{t\to\infty} e(t)| = \infty$$

In conclusion, If closed-loop transfer function has poles on imaginary axis or open right half-plane then, we can not apply final value theorem.

9.1.2 Stead-State Response to Disturbances

When analyzing the steady-state response of a system in addition to the desired response to the reference input, it is also important to analyze the response to unwanted disturbances and noises.

Let's analyze the steady-state performance of the following topology which is perturbed by a disturbance input, $d(t)$.

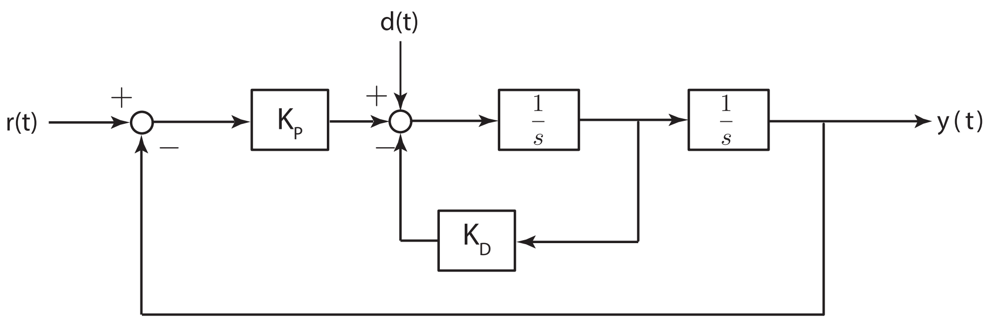


In order to analyze the response to the disturbance $d(t)$, we assume $r(t) = 0$ (which is just fine due to the linearity). Let's first find the pulse transfer function from $D(s)$ to $Y(s)$.

$$\begin{aligned} T_D(s) = \frac{Y(s)}{D(s)} &= \frac{G(s)}{1 + C(s)G(s)H(s)} \\ &= \frac{G(s)}{1 + G_{OL}(s)} \end{aligned}$$

Note that $Y(s)$ depends on both $G_{OL}(s)$ (OLTF) and $G(s)$ (Plant TF). If one wants to generalize the stead-state disturbance rejection performance, he/she needs to analyze the conditions for both $G_{OL}(s)$ and $G(s)$. Moreover, for a different topology and type of disturbance, we can have very different conditions. For this reason, in order to analyze steady-state disturbance/noise rejection performance, it is better to use fundamentals and apply final value theorem.

Example 6: The following closed-loop system is affected by a disturbance input $d(t)$. Compute the steady-state performance/response to a unit step disturbance input.



Solution: Lets compute $Y(s)/D(s)$

$$
\begin{aligned}
Y(s) &= (D(s) - Y(s)K_P)\,\frac{1}{s(s+K_D)} \\
Y(s)\left[1 + \frac{K_P}{s(s+K_D)}\right] &= D(s)\frac{1}{s(s+K_D)} \\
\frac{Y(s)}{D(s)} &= \frac{\frac{1}{s(s+K_D)}}{\frac{s^2+K_Ds+K_P}{s(s+K_D)}} = \frac{1}{s^2+K_Ds+K_P}
\end{aligned}
$$

Now let's compute y_{ss},

$$
\begin{aligned}
y_{ss} &= \lim_{s\to 0}[sY(s)] = \lim_{s\to 0}\left[sD(s)\frac{1}{s^2+K_Ds+K_P}\right] \\
&= \lim_{s\to 0}\left[s\frac{1}{s}\frac{1}{s^2+K_Ds+K_P}\right] \\
&= \frac{1}{K_P}
\end{aligned}
$$

We can see that even if same system has 0 steady-state error when the reference signal is step-like input, the error under unit-step disturbance is not zero, i.e., $y_{ss} = 1/K_P$. One can improve the disturbance rejection performance by increasing the K_P gain.