EE502 - Linear Systems Theory II **Spring 2023**

Lecture 4

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4.1 Realization Theory

Definition: A transfer function (matrix) $G(s)$ is said to be realizable if there exists a (finite-dimensional) state-space realization of it.

Theorem: A Transfer-Function (matrix) $G(s)$ is realizable $\Leftrightarrow$ $G_{ij}(s)$ is a proper rational transfer function $\forall(i,j)$

Proof:

Part I: Show that $G(s)$ is realizable $\Rightarrow$ $G_{ij}(s)$ is proper $\forall(i,j)$

Assume $G(s)$ realizable $\exists(A,B,C,D)$ tuple such that $G(s) = C(sI-A)^{-1}B + D$

$$\lim_{s\to\infty} G(s) = \lim_{s\to\infty}\left[C\left(sI-A\right)^{-1}B\right] = \lim_{s\to\infty}\left[\frac{C\mathrm{Adj}\left(sI-A\right)B}{\mathrm{Det}\left(sI-A\right)} + D\right] = D\ ,\ \textit{where}$$

$|D_{ij}| < \infty$ $\forall(i,j) \Rightarrow G_{ij}(s)$ is proper $\forall(i,j)$

Part II: Show that $G_{ij}(s)$ is proper $\forall(i,j) \Rightarrow G(s)$ is realizable

The task is to find (A,B,C,D) tuple such that $G(s) = C(sI-A)^{-1}B + D$

4.1.1 Canonical State-Space Realizations of SISO Systems

For the sake of clarity, derivations are given for general 3^{rd} order LTI systems.

4.1.1.1 CT Reachable/Controllable Canonical Form

In this method of realization, we use the fact the system is LTI. Let's consider the transfer function of the system and let's perform some LTI operations.

$$\begin{aligned}
Y(s) &= \frac{b_3s^3+b_2s^2+b_1s+b_0}{s^3+a_2s^2+a_1s+a_0}U(s) \\
&= \left(b_3s^3+b_2s^2+b_1s+b_0\right)\frac{1}{s^3+a_2s^2+a_1s+a_0}U(s) \\
&= G_2(s)G_1(s)U(s) \text{ where} \\
G_1(s) &= \frac{H(s)}{U(s)} = \frac{1}{s^3+a_2s^2+a_1s+a_0} \\
G_2(s) &= \frac{Y(z)}{H(z)} = b_3s^3+b_2s^2+b_1s+b_0
\end{aligned}$$

As you can see, we introduced an intermediate variable $h(t)$ or with a Laplace transform of $H(s)$. The first transfer function has static input dynamics, operates on $u(t)$, and produces an output, i.e. $h(t)$. The second transfer function is a "non-causal" system and operates on $h(t)$ and produces output $y(t)$. If we write the ODEs of both systems, we obtain

$$\begin{aligned}\dddot{h} &= -a_2\ddot{h} - a_1\dot{h} - a_0 h + u \\ y &= b_3\,\dddot{h} + b_2\ddot{h} + b_1\dot{h} + b_0 h\end{aligned}$$

Now let the state-variables be $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} h \\ \dot{h} \\ \ddot{h} \end{bmatrix}$. Then, individual state equations take the form

$$\begin{aligned}\dot{x_1} &= x_2 \\ \dot{x_2} &= x_3 \\ \dot{x_3} &= -a_2 x_3 - a_1 x_2 - a_0 x_1 + u\end{aligned}$$

and the output equation takes the form

$$\begin{aligned}y &= b_3\left(-a_2 x_3 - a_1 x_2 - a_0 x_1 + u\right) + b_2 x_3 + b_1 x_2 + b_0 x_1 \\ &= (b_0 - b_3 a_0)x_1 + (b_1 - b_3 a_1)x_2 + (b_2 - b_3 a_2)x_3 + b_3 u\end{aligned}$$

If we re-write the equations in matrix form, we obtain the state-space representation as

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\ y &= \begin{bmatrix} (b_0 - b_3 a_0) & (b_1 - b_3 a_1) & (b_2 - b_3 a_2) \end{bmatrix} x + [b_3]\, u\end{aligned}$$

If we obtain a state-space model from this approach, the form will be in *controllable canonical form.*

For a general n^{th} order transfer function controllable canonical form has the following A , B , C , & D matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} (b_0 - b_n a_0) & (b_1 - b_n a_1) & \cdots & (b_{n-1} - b_n a_{n-1}) \end{bmatrix} \quad , \quad D = b_n$$

4.1.1.2 DT Reachable/Controllable Canonical Form

Similar to the CT case, we will use the fact the system is LTI. Let's consider the transfer function of the system and let's perform some LTI operations.

$$
\begin{aligned}
Y(z) &= \frac{b_3 + b_2 z^{-1} + b_1 z^{-2} + b_0 z^{-3}}{1 + a_2 z^{-1} + a_1 z^{-2} + a_0 z^{-3}} U(z) \\
&= \left(b_3 + b_2 z^{-1} + b_1 z^{-2} + b_0 z^{-3}\right) \frac{1}{1 + a_2 z^{-1} + a_1 z^{-2} + a_0 z^{-3}} X(z) \\
&= G_2(z) G_1(z) U(z) \text{ where} \\
G_1(z) &= \frac{H(z)}{U(z)} = \frac{1}{1 + a_2 z^{-1} + a_1 z^{-2} + a_0 z^{-3}} \\
G_2(z) &= \frac{Y(z)}{H(z)} = b_3 + b_2 z^{-1} + b_1 z^{-2} + b_0 z^{-3}
\end{aligned}
$$

As you can see, we introduced an intermediate variable $h[k]$ with a Z-transform of $H(z)$, the First transfer function, which is a system with static input dynamics, operates on $u[n]$ and produces an output. The second transfer function operates on $h[n]$ and produces output $y[n]$. If we write the difference equations of both systems, we obtain

$$
\begin{aligned}
h[k] &= -a_2 h[k-1] - a_1 h[k-2] - a_0 h[k-3] + u[k] \\
y[k] &= b_3 h[k] + b_2 h[k-1] + b_1 h[k-2] + b_0 h[k-3]
\end{aligned}
$$

As it can be seen the delay/shifting operations are only performed on the signal $h[k]$. Now let the state-variables be $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} h[k-3] \\ h[k-2] \\ h[k-1] \end{bmatrix}$. Then, individual state equations take the form

$$
\begin{aligned}
x_1[k+1] &= x_2[k] \\
x_2[k+1] &= x_3[k] \\
x_3[k+1] &= -a_2 x_3[k] - a_1 x_2[k] - a_0 x_1[k] + u
\end{aligned}
$$

and the output equation takes the form

$$
\begin{aligned}
y[k] &= b_3 \left(-a_2 x_3[k] - a_1 x_2[k] - a_0 x_1[k] + u[k]\right) + b_2 x_3[k] + b_1 x_2[k] + b_0 x_1[k] \\
&= (b_0 - b_3 a_0) x_1[k] + (b_1 - b_3 a_1) x_2[k] + (b_2 - b_3 a_2) x_3[k] + b_3 u[k]
\end{aligned}
$$

If we re-write the equations in matrix form, we obtain the state-space representation as

$$
\begin{aligned}
x[k+1] &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} x[k] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u[k] \\
y &= \begin{bmatrix} (b_0 - b_3 a_0) & (b_1 - b_3 a_1) & (b_2 - b_3 a_2) \end{bmatrix} x[k] + [b_3]\, u[k]
\end{aligned}
$$

It can be seen that reachability/controllability canonical forms for DT and CT systems are exactly the same. For a general n^{th} order transfer function controllable canonical form has the following A , B , C , & D matrices

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} (b_0 - b_n a_0) & (b_1 - b_n a_1) & \cdots & (b_{n-1} - b_n a_{n-1}) \end{bmatrix} \quad , \quad D = b_n$$

4.1.1.3 CT Observable Canonical Form

In this method will obtain a different minimal state-space realization, the form is called observable canonical form. The process is different and state-space structure will have a different topology. Let's start with a 3^{rd} transfer function and perform some grouping based on the s elements.

$$Y(s) = \frac{b_3 s^3 + b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0} U(s)$$
$$Y(s)\left(s^3 + a_2 s^2 + a_1 s + a_0\right) = \left(b_3 s^3 + b_2 s^2 + b_1 s + b_0\right) U(s)$$
$$s^3 Y(s) = b_3 s^3 U(s) + s^2\left(-a_2 Y(s) + b_2 U(s)\right) + s\left(-a_1 Y(s) + b_1 U(s)\right) + \left(-a_0 Y(s) + b_0 U(s)\right)$$

Let's multiply both sides with $\frac{1}{s^3}$ and perform further grouping

$$Y(s) = b_3 U(s) + \frac{1}{s}\left(-a_2 Y(s) + b_2 U(s)\right) + \frac{1}{s^2}\left(-a_1 Y(s) + b_1 U(s)\right) + \frac{1}{s^3}\left(-a_0 Y(s) + b_0 U(s)\right)$$
$$Y(s) = b_3 U(s) + \frac{1}{s}\left[\left(-a_2 Y(s) + b_2 U(s)\right) + \frac{1}{s}\left\{\left(-a_1 Y(s) + b_1 U(s)\right) + \frac{1}{s}\left(-a_0 Y(s) + b_0 U(s)\right)\right\}\right]$$

Let the Laplace domain representations of state variables $X(s) = \begin{bmatrix} X_1(s) \\ X_2(s) \\ X_3(s) \end{bmatrix}$ defined as

$$X_1(s) = \frac{1}{s}\left(-a_0 Y(s) + b_0 U(s)\right)$$
$$X_2(s) = \frac{1}{s}\left\{\left(-a_1 Y(s) + b_1 U(s)\right) + \frac{1}{s}\left(-a_0 Y(s) + b_0 U(s)\right)\right\}$$
$$X_3(s) = \frac{1}{s}\left[\left(-a_2 Y(s) + b_2 U(s)\right) + \frac{1}{s}\left\{\left(-a_1 Y(s) + b_1 U(s)\right) + \frac{1}{s}\left(-a_0 Y(s) + b_0 U(s)\right)\right\}\right]$$

In this context output equation in s and time domains simply takes the form

$$Y(s) = X_3(s) + b_3 U(S) \quad \rightarrow \quad y(t) = x_3(t) + b_3 u(t)$$

Dependently the state equations (in s and time domains) take the form

$$\begin{aligned} sX_1(s) &= -a_0 X_3(s) + (b_0 - a_0 b_3)U(s) & \rightarrow \quad \dot{x}_1 &= -a_0 x_3 + (b_0 - a_0 b_3)u \\ sX_2(s) &= X_1(s) - a_1 X_3(s) + (b_1 - a_1 b_3)U(s) & \rightarrow \quad \dot{x}_2 &= x_1 - a_1 x_3 + (b_1 - a_1 b_3)u \\ sX_3(s) &= X_2(s) - a_2 X_3(s) + (b_2 - a_2 b_3)U(s) & \rightarrow \quad \dot{x}_3 &= x_2 - a_2 x_3 + (b_2 - a_2 b_3)u \end{aligned}$$

If we re-write all equations in matrix form, we obtain the state-space representation as

$$\dot{x} = \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix} x + \begin{bmatrix} (b_0 - b_3 a_0) \\ (b_1 - b_3 a_1) \\ (b_2 - b_3 a_2) \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x + [b_3]\, u$$

If we obtain a state-space model from this method, the form will be in *observable canonical form*. Thus we can call this representation also as *observable canonical realization*. This form and representation is the dual of the previous representation.

For a general n^{th} order system controllable canonical form has the following A , B , C , & D matrices

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{n-2} \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} \quad , \quad B = \begin{bmatrix} (b_0 - b_n a_0) \\ (b_1 - b_n a_1) \\ \vdots \\ (b_{n-2} - b_n a_{n-2}) \\ (b_{n-1} - b_n a_{n-1}) \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \quad , \quad D = b_n$$

4.1.1.4 DT Observable Canonical Form

Let's start with the transfer function and perform some grouping based on the delay elements.

$$\begin{aligned} &Y(z)(1 + a_2 z^{-1} + a_1 z^{-2} + a_0 z^{-3}) = (b_3 + b_2 z^{-1} + b_1 z^{-2} + b_0 z^{-3})U(z) \\ &Y(z) = b_3 U(z) + z^{-1}\left(b_2 U(z) - a_2 Y(z)\right) + z^{-2}\left(b_1 U(z) - a_1 Y(z)\right) + z^{-3}\left(b_0 U(z) - a_0 Y(z)\right) \\ &Y(z) = b_3 U(z) + z^{-1}\left\{(b_2 U(z) - a_2 Y(z)) + z^{-1}\left[(b_1 U(z) - a_1 Y(z)) + z^{-1}\left(b_0 U(z) - a_0 Y(z)\right)\right]\right\} \end{aligned}$$

As you can see we have only z^{-1} terms in the representation there is a special topology embedded inside the expression. Let the Z-transform domain representations of state variables $X(z) = \begin{bmatrix} X_1(z) \\ X_2(z) \\ X_3(z) \end{bmatrix}$ defined as

$$\begin{aligned} X_1(z) &= \frac{1}{z}\left(-a_0 Y(z) + b_0 U(z)\right) \\ X_2(z) &= \frac{1}{z}\left\{(-a_1 Y(z) + b_1 U(z)) + \frac{1}{s}\left(-a_0 Y(z) + b_0 U(z)\right)\right\} \\ X_3(z) &= \frac{1}{z}\left[(-a_2 Y(z) + b_2 U(z)) + \frac{1}{z}\left\{(-a_1 Y(z) + b_1 U(z)) + \frac{1}{z}\left(-a_0 Y(z) + b_0 U(z)\right)\right\}\right] \end{aligned}$$

In this context output equation in z and time domains simply takes the form

$$Y(z) = X_3(z) + b_3 U(z) \quad \rightarrow \quad y[k] = x_3[k] + b_3 u[k]$$

Dependently the state equations (in z and time domains) take the form

$$\begin{aligned} zX_1(z) &= -a_0 X_3(z) + (b_0 - a_0 b_3)U(z) &\rightarrow\quad& x_1[k+1] = -a_0 x_3[k] + (b_0 - a_0 b_3)u[k] \\ zX_2(z) &= X_1(z) - a_1 X_3(z) + (b_1 - a_1 b_3)U(z) &\rightarrow\quad& x_2[k+1] = x_1[k] - a_1 x_3[k] + (b_1 - a_1 b_3)u[k] \\ zX_3(z) &= X_2(z) - a_2 X_3(z) + (b_2 - a_2 b_3)U(z) &\rightarrow\quad& x_3[k+1] = x_2[k] - a_2 x_3[k] + (b_2 - a_2 b_3)u[k] \end{aligned}$$

If we re-write all equations in matrix form, we obtain the state-space representation as

$$x[k+1] = \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix} x[k] + \begin{bmatrix} (b_0 - b_3 a_0) \\ (b_1 - b_3 a_1) \\ (b_2 - b_3 a_2) \end{bmatrix} u[k]$$
$$y[k] = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x[k] + [b_3]\, u[k]$$

If we obtain a state-space model from this method, the form will be in *observable canonical form.* Thus we can call this representation also as *observable canonical realization.* This form and representation is the dual of the controllable canonical form

For a general n^{th} order system observable canonical form has the following A , B , C , & D matrices

$$A = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & \cdots & 0 & -a_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -a_{n-2} \\ 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix} \quad , \quad B = \begin{bmatrix} (b_0 - b_n a_0) \\ (b_1 - b_n a_1) \\ \vdots \\ (b_{n-2} - b_n a_{n-2}) \\ (b_{n-1} - b_n a_{n-1}) \end{bmatrix}$$
$$C = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \quad , \quad D = b_n$$

4.1.1.5 CT Diagonal Canonical Form

If the transfer function of the CT-LTI system has distinct poles, we can expand it using partial fraction expansion

$$Y(s) = \left[b_3 + \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \frac{c_3}{s-p_3} \right] U(s)$$

Now let's concentrate on the candidate "state variables" and try to write state evaluation equations

$$\begin{aligned} X_1(s) &= \frac{1}{s-p_1} U(s) \quad \rightarrow \quad \dot{x}_1 = p_1 x_1 + u \\ X_2(s) &= \frac{1}{s-p_2} U(s) \quad \rightarrow \quad \dot{x}_2 = p_2 x_2 + u \\ X_3(s) &= \frac{1}{s-p_3} U(s) \quad \rightarrow \quad \dot{x}_3 = p_3 x_3 + u \end{aligned}$$

where as output equation can be derived as

$$y(t) = b_3 u(t) + c_1 x_1(t) + c_2 x_2(t) + c_3 x_3(t)$$

If we combine the state and output equations, we can obtain the state space form as

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \mathbf{x}(t) + b_3 u(t) \end{aligned}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \quad , \quad A = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad , \quad C = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \quad , \quad D = b_3$$

The form obtained with this approach is called diagonal canonical form. Obviously, this form is not applicable for "some" systems that has repeated roots.

For a general n^{th} order system with distinct roots diagonal canonical form has the following A , B , C , & D matrices

$$A = \begin{bmatrix} p_1 & 0 & \cdots & 0 & 0 \\ 0 & p_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & p_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & p_n \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} c_1 & c_2 & \cdots & c_{n-1} & c_n \end{bmatrix} \quad , \quad D = b_n$$

4.1.1.6 DT Diagonal Canonical Form

If the transfer function of the DT-LTI system has distinct poles, we can expand it using partial fraction expansion

$$Y(z) = \left[b_3 + \frac{c_1}{z - p_1} + \frac{c_2}{z - p_2} + \frac{c_3}{z - p_3} \right] U(s)$$

Now let's concentrate on the candidate "state variables" and try to write state evaluation equations

$$\begin{aligned} X_1(z) &= \frac{1}{z - p_1} U(z) \quad \rightarrow \quad x_1[k+1] = p_1 x_1[k] + u \\ X_2(z) &= \frac{1}{z - p_2} U(z) \quad \rightarrow \quad x_2[k+1] = p_2 x_2[k] + u \\ X_3(z) &= \frac{1}{z - p_3} U(z) \quad \rightarrow \quad x_3[k+1] = p_3 x_3[k] + u \end{aligned}$$

where as output equation can be derived as

$$y[k] = b_3 u[k] + c_1 x_1[k] + c_2 x_2[k] + c_3 x_3[k]$$

If we combine the state and output equations, we can obtain the state space form as

$$x[k+1] = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} u(t)$$
$$y[k] = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} x[k] + b_3 u[k]$$

where

$$A = \begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad , \quad C = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \quad , \quad D = b_3$$

The form obtained with this approach is called diagonal canonical form. Obviously, this form is not applicable for "some" systems that has repeated roots.

For a general n^{th} order system with distinct roots diagonal canonical form has the following A , B , C , & D matrices

$$A = \begin{bmatrix} p_1 & 0 & \cdots & 0 & 0 \\ 0 & p_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & p_{n-1} & 0 \\ 0 & 0 & \cdots & 0 & p_n \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} c_1 & c_2 & \cdots & c_{n-1} & c_n \end{bmatrix} \quad , \quad D = b_n$$

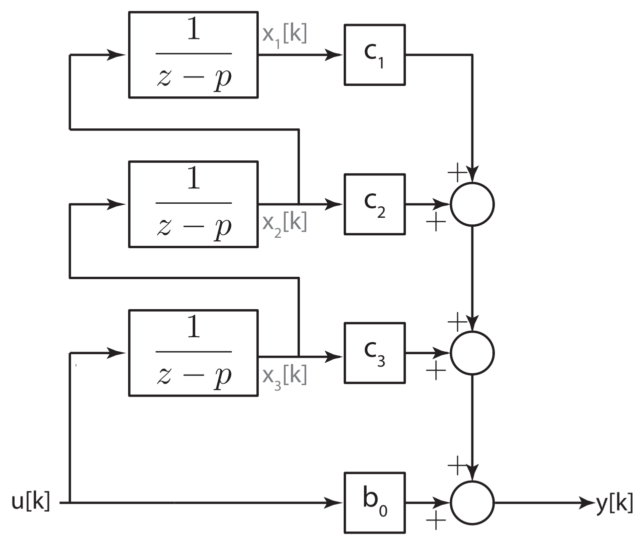
DT(CT) Jordan Canonical Form

Generalization of diagonal canonical farm is called Jordan canonical form which handles repeated roots.

In Jordan form the distinct roots has the same structure with Diagonal canonical form. Let's assume that the 3^{rd} order pulse transfer function has three repeated roots. In this case, we can expand it using partial fraction expansion

$$\begin{aligned} Y(z) &= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}} X(z) \\ &= \left(b_0 + \frac{c_1}{(z-p)^3} + \frac{c_2}{(z-p)^2} + \frac{c_3}{z-p} \right) X(z) \end{aligned}$$

It is possible to realize this expanded form in block diagram form as given in the Figure below.



Now let's concentrate on the candidate "state variables" and try to write state evaluation equations

$$\begin{aligned} X_1(z) &= \frac{1}{z-p} X_2(z) \quad \rightarrow \quad x_1[k+1] = p\ x_1[k] + x_2[k] \\ X_2(z) &= \frac{1}{z-p} X_3(z) \quad \rightarrow \quad x_2[k+1] = p\ x_2[k] + x_3[k] \\ X_3(z) &= \frac{1}{z-p} U(z) \quad \rightarrow \quad x_3[k+1] = p\ x_3[k] + u[k] \end{aligned}$$

where as output equation can be derived as

$$y[k] = b_0 u[k] + c_1 x_1[k] + c_2 x_2[k] + c_3 x_3[k]$$

If we combine the state and output equations, we can obtain the state space form as

$$\mathbf{x}[k+1] = \begin{bmatrix} p & 1 & 0 \\ 0 & p & 1 \\ 0 & 0 & p \end{bmatrix} \mathbf{x}[k] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u[k]$$
$$y[k] = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} + b_0 u[k]$$

where

$$\mathbf{x} = \begin{bmatrix} x_1[k] \\ x_2[k] \\ x_1[k] \end{bmatrix} \quad , \quad A = \begin{bmatrix} p & 1 & 0 \\ 0 & p & 1 \\ 0 & 0 & p \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad , \quad C = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix} \quad , \quad D = b_0$$

$A,\ B,\ \&\ C$ forms a Jordan block.

For a general n^{th} order system a Jordan block with m repeated roots inside a stat-space representation in Jordan canonical form looks like

$$A = \left[\begin{array}{c|ccccc|c} \ddots & & & & & & \\ \hline & \bar{p} & 1 & \cdots & 0 & 0 & \\ & 0 & \bar{p} & \cdots & 0 & 0 & \\ & & & \ddots & & & \\ & 0 & 0 & \cdots & \bar{p} & 1 & \\ & 0 & 0 & \cdots & 0 & \bar{p} & \\ \hline & & & & & & \ddots \end{array}\right] \quad , \quad B = \left[\begin{array}{c} \vdots \\ \hline 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ \hline \vdots \end{array}\right]$$
$$C = \left[\begin{array}{c|ccccc|c} \cdots & c_1 & c_2 & \cdots & c_{n-1} & c_n & \cdots \end{array}\right]$$

Ex 4.1 *Let* $G(s) = \frac{s^2+8s+10}{s^2+3s+2}$

find a controllable, observable, and diagonal canonical state-space representation of the given TF.

Solution:

If we follow the derivation of controllable canonical form for a second order system we obtain the following structure

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} (b_0 - b_2 a_0) & (b_1 - b_2 a_1) \end{bmatrix} x + [b_2]\, u$$

where

$$a_0 = 2\ ,\ a_1 = 3\ ,\ b_0 = 10\ ,\ b_1 = 8\ ,\&\ b_2 = 1$$

Thus, the state-space representation takes the form

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 8 & 5 \end{bmatrix} x + [1]u$$

Observable canonical form is the dual of the controllable canonical form thus for the given system, we know that

$$
\begin{aligned}
A_{OCF} &= A_{CCF}^T = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \\
B_{OCF} &= C_{CCF}^T = \begin{bmatrix} 8 \\ 5 \end{bmatrix} \\
C_{OCF} &= B_{CCF}^T = \begin{bmatrix} 0 & 1 \end{bmatrix} \\
D_{OCF} &= D_{CCF} = [1]
\end{aligned}
$$

In order to find the diagonal canonical form, we need to perform partial fraction expansion

$$
G(s) = \frac{s^2+8s+10}{s^2+3s+2} = 1 + \frac{3}{s+1} + \frac{2}{s+2}
$$

then SS matrices for the diagonal canonical form can be simply derived as

$$
\begin{aligned}
A_{DCF} &= \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \\
B_{DCF} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
C_{DCF} &= \begin{bmatrix} 3 & 2 \end{bmatrix} \\
D_{DCF} &= [1]
\end{aligned}
$$

Ex 4.2 *Consider the following general state-space representation*

$$
\begin{aligned}
\dot{x}(t) &= Ax(t) + Bu(t), \\
y(t) &= Cx(t) + Du(t)
\end{aligned}
$$

Now let's consider the following state-space representation

$$
\begin{aligned}
\dot{\bar{x}}(t) &= A^T\bar{x}(t) + C^Tu(t), \\
y(t) &= B^T\bar{x}(t) + Du(t)
\end{aligned}
$$

Show that these two state-space representations results in same transfer function form

Solution: For the second representation we have

$$
\begin{aligned}
\bar{G}(s) &= \bar{C}\left(sI - \bar{A}\right)^{-1}\bar{B} + D \\
&= B^T\left(sI - A^T\right)^{-1}C^T + D
\end{aligned}
$$

Since $\bar{G}(s)$ is a scalar quantity we can take its transpose

$$
\begin{aligned}
\bar{G}(s) = [\bar{G}(s)]^T &= [B^T\left(sI - A^T\right)^{-1}C^T + D]^T \\
&= (C^T)^T\left(\left(sI - A^T\right)^{-1}\right)^T(B^T)^T + D \\
&= C\left(\left(sI - A^T\right)^T\right)^{-1}B + D \\
&= C\left(sI - A\right)^{-1}B + D \\
\bar{G}(s) &= G(s)
\end{aligned}
$$

This result also shows that controllable and observable canonical representations are similar.

4.1.2 Canonical State-Space Realizations of SIMO Systems

For the sake of clarity, we will only consider double output systems, however generalization to higher dimensional outputs is straightforward.

4.1.2.1 CT Reachable/Controllable Canonical Form

Case I: Let transfer function matrix of a SIMO system be

$$G(s) = \begin{bmatrix} G_1(s) \\ G_2(s) \end{bmatrix} = \begin{bmatrix} d_1 + \frac{b_{1,2}s^2+b_{1,1}s+b_{1,0}}{s^3+a_2s^2+a_1s+a_0} \\ d_2 + \frac{b_{2,2}s^2+b_{2,1}s+b_{2,0}}{s^3+a_2s^2+a_1s+a_0} \end{bmatrix}$$

We can first obtain the D matrix as

$$D = \lim_{s\to\infty} G(s) = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

Now we concentrate the strictly proper parts of the transfer function matrix. Similar to SISO case we can organize the transfer function matrix equations as

$$\begin{aligned} Y(s) &= \begin{bmatrix} b_{1,2}s^2 + b_{1,1}s + b_{1,0} \\ b_{2,2}s^2 + b_{2,1}s + b_{2,0} \end{bmatrix} \frac{1}{s^3 + a_2s^2 + a_1s + a_0} U(s) = \begin{bmatrix} n_1(s) \\ n_2(s) \end{bmatrix} G_s(s)U(s) \text{ where} \\ Y(s) &= \begin{bmatrix} n_1(s) \\ n_2(s) \end{bmatrix} H(s) \\ G_s(s) &= \frac{H(s)}{U(s)} \end{aligned}$$

$G_s(s) = \frac{H(s)}{U(s)}$ defines a SISO system and we already know its state-update equation in controllable canonical form

$$\dot{x} = Ax + Bu \ , \text{ where } x = \begin{bmatrix} h \\ \dot{h} \\ \ddot{h} \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Not let's focus on $Y(s) = \begin{bmatrix} N_1(s) \\ N_2(s) \end{bmatrix} H(s)$ and take its inverse Laplace transform

$$y = \begin{bmatrix} b_{1,2}\ddot{h} + b_{1,1}\dot{h} + b_{1,0}h \\ b_{2,2}\ddot{h} + b_{2,1}\dot{h} + b_{2,0}h \end{bmatrix} = \begin{bmatrix} b_{1,0} & b_{1,1} & b_{1,2} \\ b_{2,0} & b_{2,1} & b_{2,2} \end{bmatrix} x \tag{4.1}$$

$$C = \begin{bmatrix} b_{1,0} & b_{1,1} & b_{1,2} \\ b_{2,0} & b_{2,1} & b_{2,2} \end{bmatrix} \tag{4.2}$$

It is quite straightforward to extend this operation to higher orders

Case II: Now consider a case where the transfer function matrix of a SIMO system has the following form

$$G(s) = \begin{bmatrix} G_1(s) \\ G_2(s) \end{bmatrix} = \begin{bmatrix} \frac{n_1(s)}{d_1(s)} \\ \frac{n_2(s)}{d_2(s)} \end{bmatrix} \ , \text{ where } d_1(s) \neq d_2(s)$$

Obviously we can not directly apply the standard approach for this problem. In this case, one first finds the *monic least common denominator* of all entries of the transfer function matrix (i.e. the least common multiple of the denominators of all the entries), $d_c(s)$ then we re-write the transfer function matrix as

$$G(s) = \begin{bmatrix} G_1(s) \\ G_2(s) \end{bmatrix} = \begin{bmatrix} \frac{n_1(s)}{d_1(s)} \\ \frac{n_2(s)}{d_2(s)} \end{bmatrix} = \begin{bmatrix} \frac{n_1(s)d_c(s)/d_1(s)}{d_c(s)} \\ \frac{n_2(s)d_c(s)/d_2(s)}{d_c(s)} \end{bmatrix}$$

and realize the transfer function using the controllable canonical form

4.1.3 Canonical State-Space Realizations of MISO Systems

For the sake of clarity, we will only consider double output systems, however generalization to higher dimensional outputs is straightforward.

4.1.3.1 DT Observable Canonical Form

Case I: Let transfer function matrix of a MISO system be

$$G(z) = \begin{bmatrix} G_1(z) & G_2(z) \end{bmatrix} = \begin{bmatrix} d_1 + \frac{b_{1,2}z^{-1}+b_{1,1}z^{-2}+b_{1,0}z^{-3}}{1+a_2z^{-1}+a_1z^{-2}+a_0z^{-3}} & d_2 + \frac{b_{2,2}z^{-1}+b_{2,1}z^{-2}+b_{2,0}z^{-3}}{1+a_2z^{-1}+a_1z^{-2}+a_0z^{-3}} \end{bmatrix}$$

We can first obtain the D matrix as

$$D = \lim_{z\to\infty} G(z) = \begin{bmatrix} d_1 & d_2 \end{bmatrix}$$

Now we concentrate the strictly proper parts of the transfer function matrix. Let's start with the frequency domain expression of the single output and perform some grouping based on the delay elements.

$$\begin{aligned} Y(z)(1 + a_2z^{-1} + a_1z^{-2} + a_0z^{-3}) &= (b_{1,2}z^{-1} + b_{1,1}z^{-2} + b_{1,0}z^{-3})U_1(z) + (b_{2,2}z^{-1} + b_{2,1}z^{-2} + b_{2,0}z^{-3})U_2(z) \\ Y(z) &= z^{-1}\left(b_{1,2}U_1(z) + b_{2,2}U_2(z) - a_2Y(z)\right) + z^{-2}\left(b_{1,1}U_1(z) + b_{2,1}U_2(z) - a_1Y(z)\right) \\ &\quad + z^{-3}\left(b_{1,0}U_1(z) + b_{2,0}U_2(z) - a_0Y(z)\right) \\ &= \frac{1}{z}\left\{(b_{1,2}U_1(z) + b_{2,2}U_2(z) - a_2Y(z)) + \frac{1}{z}\left[(b_{1,1}U_1(z) + b_{2,1}U_2(z) - a_1Y(z)) + \frac{1}{z}\left(b_{1,0}U_1(z) + b_{2,0}U_2(z) - a_0Y(z)\right)\right]\right\} \end{aligned}$$

As you can see we have only z^{-1} terms in the representation there is a special topology embedded inside the expression. Let the Z-transform domain representations of state variables $X(z) = \begin{bmatrix} X_1(z) \\ X_2(z) \\ X_3(z) \end{bmatrix}$ defined as

$$\begin{aligned} X_1(z) &= \frac{1}{z}\left(-a_0X_3(z) + b_{1,0}U_1(z) + b_{2,0}U_2(z)\right) \\ X_2(z) &= \frac{1}{z}\left\{(-a_1X_3(z) + b_{1,1}U_1(z) + b_{2,1}U_2(z)) + X_1(z)\right\} \\ X_3(z) &= \frac{1}{z}\left[(-a_2X_3(z) + b_{1,2}U_1(z) + b_{2,2}U_2(z)) + X_2(z)\right] \end{aligned}$$

In this context output equation in z and time domains simply takes the form

$$Y(z) = X_3(z) + D\begin{bmatrix} U_1(z) \\ U_2(z) \end{bmatrix} \quad \rightarrow \quad y[k] = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x[k] + D\begin{bmatrix} u_1[k] \\ u_2[k] \end{bmatrix}$$

State equations (in z and time domains) take the form

$$\begin{aligned}
zX_1(z) &= -a_0X_3(z) + b_{1,0}U_1(z) + b_{2,0}U_2(z) \;\rightarrow\; x_1[k+1] = -a_0x_3[k] + b_{1,0}u_1[k] + b_{2,0}u_2[k]\\
zX_2(z) &= X_1(z) - a_1X_3(z) + b_{1,1}U_1(z) + b_{2,1}U_2(z)U(z) \;\rightarrow\; x_2[k+1] = x_1[k] - a_1x_3[k] + b_{1,1}u_1[k] + b_{2,1}u_2[k]\\
zX_3(z) &= X_2(z) - a_2X_3(z) + b_{1,2}U_1(z) + b_{2,2}U_2(z)U(z) \;\rightarrow\; x_3[k+1] = x_2[k] - a_2x_3[k] + b_{1,2}u_1[k] + b_{2,2}u_2[k]
\end{aligned}$$

If we re-write all equations in matrix form, we obtain the state-space representation as

$$\begin{aligned}
x[k+1] &= \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix} x[k] + \begin{bmatrix} b_{1,0} & b_{2,0} \\ b_{1,1} & b_{2,1} \\ b_{1,2} & b_{2,2} \end{bmatrix} u[k]\\
y[k] &= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x[k] + \begin{bmatrix} d_1 & d_2 \end{bmatrix} u[k]
\end{aligned}$$

It is quite straightforward to extend this operation to higher orders. Note that for MISO systems with $d_1(s) \neq d_2(s)$ where $G_1(s) = \frac{n_1(s)}{d_1(s)}$ and $G_2(s) = \frac{n_2(s)}{d_2(s)}$ we can not directly apply same operations. However similar to the controllable canonical form in SIMO systems we can start by finding the *monic least common denominator* of all entries of the transfer function matrix (i.e. the least common multiple of the denominators of all the entries), $d_c(s)$ then we re-write the transfer function using the similar format in controllable canonical form. After that we can apply the MISO observable canonical form derivations to realize the state-space structure.

4.1.4 Canonical State-Space Realizations of MIMO Systems

4.1.4.1 Gilbert's Realization - MIMO Generalization of Diagonal Canonical Form

Suppose we have a transfer function matrix $G(s) \in \mathbb{T}^{q\times p}$ (where $\mathbb{T}$ is the vector space of all proper transfer functions). First find/compute the least common denominator, $d_c(s)$, of all the entries of $G(s)$. If $d_c(s)$ has no repeated poles, then we can find a *minimal* state-space realization in the form of generalized diagonal canonical form using Gilbert's method. Now, a partial fraction expansion to each of the elements of $G(s)$ and collect residue matrices, $R_i \in \mathbb{C}^{q\times p}$ for each distinct pole. Not we can re-write the transfer function matrix in the following form

$$G(s) = D + \sum_{i=1}^{N_p} R_i \frac{1}{(s-\rho_i)}$$

where $D = \lim_{s\to\infty} G(s)$, ρ_i's denotes the N_p distinct poles/roots, and $R_i \in \mathbb{C}^{q\times p}$ denotes the residue matrix for each ρ_i. Let $r_i = \text{Rank}(R_i)$, then r_i gives the number of poles required the mode associated with ρ_i. We know that (from Linear Algebra) we can write R_i in terms of multiplication of two full rank matrices

$$R_i = C_iB_i, \text{ where } C_i \in \mathbb{C}^{q\times r_i}\ ,\ B_i \in \mathbb{C}^{r_i\times p}$$

then each state-equation associated with mode ρ_i takes the form

$$\begin{aligned}
\dot{x}_i &= \begin{bmatrix} \rho_i & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & \rho_i \end{bmatrix} x_i + B_iu\ ,\ x_i \in \mathbb{R}^{r_i}\ ,\ u \in \mathbb{R}^q\\
y &= C_ix_i
\end{aligned}$$

whole state-space structure will be in diagonal canonical form

$$\dot{x} = \begin{bmatrix} A_1 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & A_{N_P} \end{bmatrix} x + \begin{bmatrix} B_1 \\ \vdots \\ B_{N_P} \end{bmatrix} u \ , \ x = \begin{bmatrix} x_i \\ \vdots \\ x_{N_P} \end{bmatrix}$$
$$y = \begin{bmatrix} C_1 & \cdots & C_{N_P} \end{bmatrix} x + D$$

Ex 4.3 *Let*

$$G(s) = \begin{bmatrix} G_1(s) \\ G_2(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{s^2+s} \\ \frac{1}{s} \end{bmatrix} = \begin{bmatrix} \frac{1}{s} - \frac{1}{s+1} \\ \frac{1}{s} \end{bmatrix}$$

Find reachable and diagonal canonical (both minimal) state-space representations of the given SIMO system

Solution: Let's start with reachable canonical realization. For the given system $d_c(s) = s(s+1)$ and hence we can re-write the transfer function matrix as

$$G(s) = \begin{bmatrix} \frac{1}{s^2+s} \\ \frac{s+1}{s^2+s} \end{bmatrix}$$

If we follow the reachable canonical forms derivation for SIMO systems we can obtain

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad , \quad C = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Now let's find the diagonal canonical form using Gilbert's method

$$G(s) = \begin{bmatrix} \frac{1}{s} - \frac{1}{s+1} \\ \frac{1}{s} \end{bmatrix} = \begin{bmatrix} \frac{-1}{s+1} \\ 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{s} \\ \frac{1}{s} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \frac{1}{s+1} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{s}$$
$$\rho_1 = -1 \ , \ R_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} = C_1 B_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} [1]$$
$$\rho_2 = 0 \ , \ R_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = C_2 B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1]$$

We can than find the state-space matrices

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad , \quad C = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

Ex 4.4 *Let*

$$G(s) = \begin{bmatrix} G_1(s) & G_2(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{s^2+s} & \frac{1}{s} \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{s} - \frac{1}{s+1}\right) & \frac{1}{s} \end{bmatrix}$$

Find observable and diagonal canonical (both minimal) state-space representations of the given MISO system

Solution: Let's start with observable canonical realization. For the given system $d_c(s) = s(s+1)$ and hence we can re-write the transfer function matrix as

$$G(s) = \begin{bmatrix} \frac{1}{s^2+s} & \frac{s+1}{s^2+s} \end{bmatrix}$$

If we follow the reachable canonical forms derivation for MISO systems we can obtain

$$A = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad , \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

Now let's find the diagonal canonical form using Gilbert's method

$$\begin{aligned} G(s) &= \begin{bmatrix} \frac{1}{s} - \frac{1}{s+1} & \frac{1}{s} \end{bmatrix} = \begin{bmatrix} \frac{-1}{s+1} & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{s} & \frac{1}{s} \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \end{bmatrix} \frac{1}{s+1} + \begin{bmatrix} 1 & 1 \end{bmatrix} \frac{1}{s} \\ \rho_1 &= -1 \ , \ R_1 = \begin{bmatrix} -1 & 0 \end{bmatrix} = C_1 B_1 = [1] \begin{bmatrix} -1 & 0 \end{bmatrix} \\ \rho_2 &= 0 \ , \ R_2 = \begin{bmatrix} 1 & 1 \end{bmatrix} = C_2 B_2 = [1] \begin{bmatrix} 1 & 1 \end{bmatrix} \end{aligned}$$

We can than find the state-space matrices

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \quad , \quad B = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \quad , \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Ex 4.5 *Let*

$$G(s) = \begin{bmatrix} G_{11}(s) & G_{12}(s) \\ G_{21}(s) & G_{22}(s) \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{s} - \frac{1}{s+1}\right) & \frac{1}{s} \\ \frac{1}{s} & 0 \end{bmatrix}$$

Find a diagonal canonical (minimal) state-space representations of the given MIMO system.

Solution: For the given system $d_c(s) = s(s+1)$. let's find the diagonal canonical form using Gilbert's method

$$\begin{aligned} G(s) &= \begin{bmatrix} \left(\frac{1}{s} - \frac{1}{s+1}\right) & \frac{1}{s} \\ \frac{1}{s} & 0 \end{bmatrix} = \begin{bmatrix} \frac{-1}{s+1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{s} & \frac{1}{s} \\ \frac{1}{s} & 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{s+1} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{s} \\ \rho_1 &= -1 \ , \ R_1 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} = C_1 B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \end{bmatrix} \ , \ \text{Rank}(R_1) = 1 \\ \rho_2 &= 0 \ , \ R_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = C_2 B_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \ , \ \text{Rank}(R_2) = 2 \end{aligned}$$

We can than find the state-space matrices

$$A = \left[\begin{array}{c|cc} -1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad , \quad B = \left[\begin{array}{cc} -1 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] \quad , \quad C = \left[\begin{array}{c|cc} 1 & 1 & 1 \\ 0 & 1 & 0 \end{array}\right]$$

4.2 Equivalent State-Space Representations

Definition: If two different quadruples (A, B, C, D) and $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ yield the same transfer function matrix, i.e. $C(zI - A)^{-1}B + D = \bar{C}(zI - \bar{A})^{-1}\bar{B} + \bar{D}$, then the systems

$$\{x[k+1] = Ax[k] + Bu[k]\ ,\ y[k] = Cx[k] + Du[k]\}\ \&\ \left\{\bar{x}[k+1] = \bar{A}\bar{x}[k] + \bar{B}u[k]\ ,\ y[k] = \bar{C}\bar{x}[k] + \bar{D}u[k]\right\}$$

are said to be *zero-state-equivalent* (ZSE).

Note that if two systems are ZSE, under zero-initial conditions, they give the same output response when excited by the same input (for all inputs).

Definition: Consider the n-dimensional LTI system

$$x[k+1] = Ax[k] + Bu[k]\ ,\ y[k] = Cx[k] + Du[k]\ (S_1)$$

The LTI system given as

$$\hat{x}[k+1] = \hat{A}\hat{x}[k] + \hat{B}u[k]\ ,\ y[k] = \hat{C}\hat{x}[k] + \hat{D}u[k]\ (S_2)$$

is said to be *algebraically equivalent* (AE) to S_1 if $D = \hat{D}$ and $\exists P \in \mathbb{C}^{n\times n}\quad \det(P) \neq 0$ that satisfies $\hat{A} = PAP^{-1}\ ,\ \hat{B} = PB\ ,\ \hat{C} = CP$.

Remark: S_2 corresponds to the state vector transformation $Px[k] = \hat{x}[k]$

$$\begin{aligned} P^{-1}\hat{x}[k+1] &= AP^{-1}\hat{x}[k] + Bu[k] \quad , \quad y[k] = CP^{-1}\hat{x}[k] + Du[k] \\ \hat{x}[k+1] &= PAP^{-1}\hat{x}[k] + PBu[k] \quad , \quad y[k] = CP^{-1}\hat{x}[k] + Du[k] \end{aligned}$$

where

$$\hat{A} = PAP^{-1}\ ,\ \hat{B} = PB\ ,\ \hat{C} = CP^{-1}\ ,\ \hat{D} = D$$

Since there exist infinitely many non-singular $n \times n$ matrices, for a given LTI system, there exist infinitely many different but AE state-space representations.

Ex 4.6 *Show that* $A \in \mathbb{R}^{n\times n}$ *and* $P^{-1}AP$*, where* $P \in \mathbb{R}^{n\times n}$ *and* $\det(P) \neq 0$*, have the same characteristic equation*

Solution:

$$\begin{aligned} \det\left(\lambda I - P^{-1}AP\right) &= \det\left(\lambda P^{-1}IP - P^{-1}AP\right) \\ &= \det\left(P^{-1}(\lambda I - A)P\right) \\ &= \det\left(P^{-1}\right)\det(\lambda I - A)\det(P) \\ &= \det\left(P^{-1}\right)\det(P)\det(\lambda I - A) \\ \det\left(\lambda I - P^{-1}AP\right) &= \det(\lambda I - A) \end{aligned}$$

Theorem: Algebraically equivalent systems has the same transfer function matrix

Consider the two different AE state-space representations

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k] & \qquad \hat{x}[k+1] &= \hat{A}\hat{x}[k] + \hat{B}u[k] \\ y[k] &= Cx[k] + Du[k] & \qquad y[k] &= \hat{C}x[k] + \hat{D}u[k] \end{aligned}$$

where they are related with the following similarity transformation

$$Px[k] = \hat{x}[k] \ , \ \hat{G} = PGP^{-1} \ , \ \hat{H} = PH \ , \ \hat{C} = CP^{-1} \ , \ \hat{D} = D$$

Let's compute the transfer function for the second representation

$$\begin{aligned}
\hat{T}(z) &= \left[\hat{C}\left(zI - \hat{G}\right)^{-1}\hat{H} + \hat{D}\right] \\
&= \left[CP^{-1}\left(zI - PGP^{-1}\right)^{-1}PH + D\right] \\
&= \left[CP^{-1}\left(P\left(zI - G\right)P^{-1}\right)^{-1}PH + D\right] \\
&= \left[CP^{-1}P\left(zI - G\right)^{-1}P^{-1}PH + D\right] \\
&= \left[C\left(zI - G\right)^{-1}H + D\right] \\
\hat{T}(z) &= T(z)
\end{aligned}$$