6. BÖLÜM

LAPLACE DÖNÜŞÜMLERİ

Tanım: $f(x)$, $0 \leq x < \infty$ için tanımlı olsun. ve s keyfi bir reel değişkeni göstersin. $f(x)$'in $L\{f(x)\}$ veya $F(s)$ ile gösterilen Laplace dönüşümü

$$L\{f(x)\} = F(s) = \int_0^\infty e^{-sx} \cdot f(x)\,dx$$

ile verilir. Burada

$$\int_0^\infty e^{-sx} f(x)\,dx = \lim_{R\to\infty} \int_0^R e^{-sx} f(x)\,dx$$

şeklinde tanımlı limit varsa $f(x)$'in Laplace dönüşümü vardır. Aksi halde Laplace dönüşümü yoktur.

Laplace Dönüşümünün Özellikleri

(1) $L\{f(x)\} = F(s)$ ve $L\{g(x)\} = G(s)$ ise o zaman herhangi iki c_1 ve c_2 sabiti için

$$L\{c_1 f(x) + c_2 g(x)\} = c_1 L\{f(x)\} + c_2 L\{g(x)\}$$ dir.

(2) $L\{f(x)\} = F(s)$ ise herhangi bir a sabiti için

$$L\{e^{ax} f(x)\} = F(s-a)$$

(3) $L\{f(x)\} = F(s)$ ise $n \in \mathbb{Z}^+$ için

$$L\{x^n f(x)\} = (-1)^n \frac{d^n}{ds^n}\left[F(s)\right]$$

(4) $L\{f(x)\} = F(s)$ ise $L\{\frac{1}{x} f(x)\} = \int_s^\infty F(t)\,dt$

(5) $L\{f(x)\} = F(s)$ ise $L\{\int_0^x f(t)\,dt\} = \frac{1}{s} F(s)$ dir.

ÖRNEK : $f(x)=1$ fonksiyonunun Laplace dönüşümünü bulunuz.

Çözüm : $L\{1\} = F(s)$ olmak üzere $L\{1\} = ?$

$$F(s) = L\{1\} = \int_0^\infty e^{-sx} \cdot 1 \cdot dx = \lim_{R\to\infty} \int_0^R e^{-sx} \cdot 1 \cdot dx \quad \text{dir.}$$

$\int_0^R e^{-sx}\,dx$ integralinde $-sx=u$ olsun. $-s\,dx = du \Rightarrow dx = -\frac{du}{s}$

$$\Rightarrow \int_0^R e^{-sx}dx = -\frac{1}{s}\int_0^R e^u du = -\frac{1}{s} e^u \Big|_{x=0}^{x=R} = -\frac{1}{s} e^{-sx}\Big|_0^R$$

$$= -\frac{1}{s}e^{-sR} - \left(-\frac{1}{s}e^{-s.0}\right) = \frac{1}{s} - \frac{1}{s}e^{-sR} = \frac{1}{s} - \frac{1}{s e^{sR}}$$

$$\Rightarrow \lim_{R\to\infty} \int_0^R e^{-sx}dx = \lim_{R\to\infty}\left(\frac{1}{s} - \frac{1}{s e^{sR}}\right) = \frac{1}{s} \quad \text{olur.}$$

O halde $L\{1\} = \frac{1}{s}$ dir. $(s>0)$

ÖRNEK : $L\{e^{ax}\} = ?$

Çözüm : $$L\{e^{ax}\} = \int_0^\infty e^{-sx} \cdot e^{ax}dx = \lim_{R\to\infty}\int_0^R e^{(a-s)x}dx$$

$$\Rightarrow (a-s)x = u \Rightarrow (a-s)dx = du \Rightarrow dx = \frac{du}{a-s}$$

$$\lim_{R\to\infty}\int_0^R e^{(a-s)x}dx = \lim_{R\to\infty}\int_0^R \frac{e^u du}{a-s} = \lim_{R\to\infty}\left(\frac{e^{(a-s)x}}{a-s}\right)_{x=0}^{x=R}$$

$$= \lim_{R\to\infty}\left(\frac{e^{(a-s)R}}{a-s} - \frac{e^{(a-s).0}}{a-s}\right) = \lim_{R\to\infty}\left[\frac{e^{(a-s)R}-1}{a-s}\right]$$

$$= \frac{1}{s-a} \quad (s>a \text{ için})$$

Bazı Laplace Dönüşümleri

	$f(x)$	$L\{f(x)\} = F(s)$
1	1	$\frac{1}{s}$
2	x	$\frac{1}{s^2}$
3	x^n	$\frac{n!}{s^{n+1}}$
4	$\sqrt{x}$	$\frac{1}{2}\sqrt{\pi}\, s^{-3/2}$
5	e^{ax}	$\frac{1}{s-a}$
6	$\sin ax$	$\frac{a}{s^2+a^2}$
7	$\cos ax$	$\frac{s}{s^2+a^2}$
8	$e^{bx} \cdot \sin ax$	$\frac{a}{(s-b)^2+a^2}$
9	$e^{bx} \cdot \cos ax$	$\frac{s-b}{(s-b)^2+a^2}$
10	$x \cdot \sin ax$	$\frac{2as}{(s^2+a^2)^2}$
11	$x \cdot \cos ax$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
12	$x^n \cdot e^{ax}$	$\frac{n!}{(s-a)^{n+1}}$
13	$\sin^2 ax$	$\frac{2a^2}{s(s^2+4a^2)}$
14	$\sin ax - ax\cos ax$	$\frac{2a^3}{(s^2+a^2)^2}$

ÖRNEK : $L\{3+2x^2\} = ?$

Çözüm : $L\{3+2x^2\} = L\{3.1\} + L\{2x^2\}$

$= 3L\{1\} + 2L\{x^2\} = 3.\left(\frac{1}{s}\right) + 2.\left(\frac{2}{s^3}\right) = \frac{3}{s} + \frac{4}{s^3}$

ÖRNEK : $L\{5\sin 3x - 17e^{-2x}\} = ?$

Çözüm : $L\{5\sin 3x - 17e^{-2x}\} = 5L\{\sin 3x\} - 17L\{e^{-2x}\}$

$= 5.\left(\frac{3}{s^2+3^2}\right) - 17\left(\frac{1}{s-(-2)}\right) = \frac{15}{s^2+9} - \frac{17}{s+2}$

ÖRNEK : $L\{2\sin x + 3\cos 2x\} = ?$

Çözüm : $L\{2\sin x + 3\cos 2x\} = 2L\{\sin x\} + 3L\{\cos 2x\}$

$= 2.\frac{1}{s^2+1^2} + 3\frac{s}{s^2+2^2} = \frac{2}{s^2+1} + \frac{3s}{s^2+4}$

ÖRNEK : $L\{xe^{4x}\} = ?$

Çözüm : (I) : 12. formülde $n=1$, $a=4$ alınırsa

$$L\{xe^{4x}\} = \frac{1}{(s-4)^2}$$

(II) : 2. özellik kullanılırsa $L\{e^{ax}f(x)\} = F(s-a)$ idi.

$$F(s) = L\{f(x)\} = L\{x\} = \frac{1}{s^2}$$

ve

$$L\{e^{4x}.x\} = F(s-4) = \frac{1}{(s-4)^2}$$

bulunur.

ÖRNEK : $L\{e^{-2x} \cdot \sin 5x\} = ?$

Çözüm (I) : Tabloda 8. formülde $b=-2$ ve $a=5$ için

$$L\{e^{-2x}\sin 5x\} = \frac{5}{[s-(-2)]^2+5^2} = \frac{5}{(s+2)^2+25}$$

(II) : $L\{\sin 5x\} = \dfrac{5}{s^2+25}$ ve

$$L\{e^{-2x}\sin 5x\} = F(s-(-2)) = F(s+2) = \frac{5}{(s+2)^2+25}$$

ÖRNEK : $L\{x \cos\sqrt{7}\,x\} = ?$

Çözüm : (I) : Tabloda 11. formülde $a=\sqrt{7}$ alınırsa

$$L\{x\cos\sqrt{7}\,x\} = \frac{s^2-(\sqrt{7})^2}{[s^2+(\sqrt{7})^2]^2} = \frac{s^2-7}{(s^2+7)^2}$$

(II) :

$$L\{\cos\sqrt{7}\,x\} = \frac{s}{s^2+(\sqrt{7})^2} = \frac{s}{s^2+7}$$

ve

$$L\{x\cos\sqrt{7}\,x\} = -\frac{d}{ds}\left(\frac{s}{s^2+7}\right) = \frac{s^2-7}{(s^2+7)^2}$$

ÖRNEK : $L\{e^{-x} \cdot x\cos 2x\} = ?$

Çözüm : $L\{x\cos 2x\} = \dfrac{s^2-4}{(s^2+4)^2}$ dır.

$$L\{e^{-x}x\cos 2x\} = \frac{(s+1)^2-4}{[(s+1)^2+4]^2}$$

ÖRNEK : $L\left\{ \frac{\sin 3x}{x} \right\} = ?$

Çözüm : $f(x) = \sin 3x$ alınırsa

$$F(s) = \frac{3}{s^2+9} \quad \text{veya} \quad F(t) = \frac{3}{t^2+9} \quad \text{bulunur.}$$

4. özellik kullanılırsa

$$L\left\{ \frac{\sin 3x}{x} \right\} = \int_s^\infty \frac{3}{t^2+9}\,dt = \lim_{R\to\infty} \int_s^R \frac{3}{t^2+9}\,dt$$

$$= \lim_{R\to\infty} \arctan \frac{t}{3} \Big|_s^R = \lim_{R\to\infty} \left(\arctan \frac{R}{3} - \arctan \frac{s}{3} \right)$$

$$= \frac{\pi}{2} - \arctan \frac{s}{3}$$

TERS LAPLACE DÖNÜŞÜMLERİ

$F(s)$ nin $L^{-1}\{F(s)\}$ ile gösterilen ters Laplace dönüşümü $L\{f(x)\} = F(s)$ özelliğine sahip bir $f(x)$ fonksiyonudur. Eğer $F(s)$ belirli biçimlerden birine sahip değilse bazen cebirsel işlemlerle böyle bir biçime dönüştürülebilir.

Paydalar genellikle iki metotla bilinen biçimlere dönüştürülür. Bunlar kareye tamamlama ve Basit kesirler metodudur.

Kareye tamamlama metodunda, paydadaki polinom karelerin toplamı şeklinde yazılmaya çalışılır.

Basit kesirler metodunda $\frac{a(s)}{b(s)}$ biçimindeki bir fonksiyon diğer kesirlerin toplamı haline çevrilir. Eğer $b(s)$ ifadesi $(s-a)^m$ şeklindeyse

$$\frac{A_1}{s-a} + \frac{A_2}{(s-a)^2} + \dots + \frac{A_n}{(s-a)^n}$$

şeklinde kesirler toplamı atanır.

ÖRNEK : $L^{-1}\left\{\frac{1}{s}\right\} = ?$

Çözüm : $L\{1\} = \frac{1}{s}$ olduğundan $L^{-1}\left\{\frac{1}{s}\right\} = 1$ dir.

ÖRNEK : $L^{-1}\left\{\frac{1}{s-8}\right\} = ?$

Çözüm : $L\{e^{8x}\} = \frac{1}{s-8}$ olduğundan $L^{-1}\left\{\frac{1}{s-8}\right\} = e^{8x}$ dir.

ÖRNEK : $L^{-1}\left\{\frac{s}{s^2+6}\right\} = ?$

Çözüm : $L\{\cos\sqrt{6}\,x\} = \frac{s}{s^2+(\sqrt{6})^2} = \frac{s}{s^2+6}$ old.

$$L^{-1}\left\{\frac{s}{s^2+6}\right\} = \cos\sqrt{6}\,x \text{ dir.}$$

ÖRNEK : $L^{-1}\left\{\frac{5s}{(s^2+1)^2}\right\} = ?$

Çözüm :

$$L^{-1}\left\{\frac{5s}{(s^2+1)^2}\right\} = L^{-1}\left\{\frac{\frac{5}{2}\cdot 2s}{(s^2+1)^2}\right\}$$

$$= \frac{5}{2}L^{-1}\left\{\frac{2s}{(s^2+1)^2}\right\} = \frac{5}{2}x\sin x$$

ÖRNEK : $L^{-1}\left\{\frac{s+1}{s^2+9}\right\} = ?$

Çözüm

$$L^{-1}\left\{\frac{s+1}{s^2+9}\right\} = L^{-1}\left\{\frac{s}{s^2+9}\right\} + L^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$= \cos 3x + L^{-1}\left\{\frac{1}{3}\,\frac{3}{s^2+9}\right\}$$

$$= \cos 3x + \frac{1}{3}\sin 3x$$

ÖRNEK : $L^{-1}\left\{\dfrac{s}{(s-2)^2+9}\right\} = ?$

Çözüm : $L^{-1}\left\{\dfrac{s}{(s-2)^2+9}\right\} = L^{-1}\left\{\dfrac{(s-2)+2}{(s-2)^2+9}\right\}$

$$= L^{-1}\left\{\frac{s-2}{(s-2)^2+9}\right\} + L^{-1}\left\{\frac{2}{(s-2)^2+9}\right\}$$

$$= e^{2x}\cos 3x + L^{-1}\left\{\frac{2}{3}\cdot\frac{3}{(s-2)^2+9}\right\}$$

$$= e^{2x}\cos 3x + \frac{2}{3}e^{2x}\sin 3x$$

ÖRNEK : $L^{-1}\left\{\dfrac{1}{s^2-2s+9}\right\} = ?$

Çözüm : $s^2-2s+9 = (s^2-2s+1)+8$

$$= (s-1)^2+8 = (s-1)^2+(\sqrt{8})^2$$

$$L^{-1}\left\{\frac{1}{s^2-2s+9}\right\} = L^{-1}\left\{\frac{1}{(s-1)^2+(\sqrt{8})^2}\right\}$$

$$= L^{-1}\left\{\frac{1}{\sqrt{8}}\,\frac{\sqrt{8}}{(s-1)^2+(\sqrt{8})^2}\right\}$$

$$= \frac{1}{\sqrt{8}}\,L^{-1}\left\{\frac{\sqrt{8}}{(s-1)^2+(\sqrt{8})^2}\right\}$$

$$= \frac{1}{\sqrt{8}}\cdot e^{x}\sin\sqrt{8}\,x$$

ÖRNEK : $L^{-1}\left\{\dfrac{s+2}{s^2-3s+4}\right\}=?$

ÇÖZÜM : $s^2-3s+4=\left(s^2-3s+\frac{9}{4}\right)+\underbrace{\frac{7}{4}}_{4-\frac{9}{4}=\frac{7}{4}}=\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2$

$$L^{-1}\left\{\frac{s+2}{s^2-3s+4}\right\}=L^{-1}\left\{\frac{s+2}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}\right\}$$

$$=L^{-1}\left\{\frac{s-\frac{3}{2}+\frac{7}{2}}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}\right\}=L^{-1}\left\{\frac{s-\frac{3}{2}}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}\right\}+L^{-1}\left\{\frac{\frac{7}{2}}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}\right\}$$

$$=L^{-1}\left\{\frac{s-\frac{3}{2}}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}\right\}+\sqrt{7}\,.\,L^{-1}\left\{\frac{\frac{\sqrt{7}}{2}}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{7}}{2}\right)^2}\right\}$$

$$=e^{\frac{3}{2}x}\cos\frac{\sqrt{7}}{2}x+\sqrt{7}\,e^{\frac{3}{2}x}\sin\frac{\sqrt{7}}{2}x$$

ÖRNEK ! $L^{-1}\left\{\dfrac{s+3}{(s-2)(s+1)}\right\}=?$

ÇÖZÜM ! $\dfrac{s+3}{(s-2)(s+1)}\equiv\underset{(s+1)}{\dfrac{A}{s-2}}+\underset{(s-2)}{\dfrac{B}{s+1}}$

$$s+3\equiv A(s+1)+B(s-2)=(A+B)s+A-2B\Rightarrow A=\frac{5}{3},\quad B=-\frac{2}{3}$$

$$L^{-1}\left\{\frac{s+3}{(s-2)(s+1)}\right\}=L^{-1}\left\{\frac{\frac{5}{3}}{s-2}+\frac{-\frac{2}{3}}{s+1}\right\}$$

$$=\frac{5}{3}L^{-1}\left\{\frac{1}{s-2}\right\}-\frac{2}{3}L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$=\frac{5}{3}e^{2x}-\frac{2}{3}e^{-x}$$

ÖRNEK : $L^{-1}\left\{\dfrac{1}{(s+1)(s^2+1)}\right\}=?$

ÇÖZÜM : $$\frac{1}{(s+1)(s^2+1)} \equiv \underset{(s^2+1)}{\frac{A}{s+1}} + \underset{(s+1)}{\frac{Bs+C}{s^2+1}}$$

$$1 \equiv A(s^2+1)+(Bs+C)(s+1) = \underbrace{(A+B)}_{=0}s^2+\underbrace{(B+C)}_{=0}s+\underbrace{(A+C)}_{=1}$$

$$A=\frac{1}{2}, \quad B=-\frac{1}{2}, \quad C=\frac{1}{2}$$

$$L^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\} = L^{-1}\left\{\frac{\frac{1}{2}}{s+1}\right\}+L^{-1}\left\{\frac{-\frac{1}{2}s+\frac{1}{2}}{s^2+1}\right\}$$

$$=\frac{1}{2}L^{-1}\left\{\frac{1}{s+1}\right\}-\frac{1}{2}L^{-1}\left\{\frac{s}{s^2+1}\right\}+\frac{1}{2}L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$=\frac{1}{2}e^{-x}-\frac{1}{2}\cos x+\frac{1}{2}\sin x$$

ÖRNEK : $L^{-1}\left\{\dfrac{1}{s(s^2+4)}\right\}=?$

ÇÖZÜM : $$\frac{1}{s(s^2+4)} \equiv \underset{(s^2+4)}{\frac{A}{s}} + \underset{(s)}{\frac{Bs+C}{s^2+4}}$$

$$1 \equiv A(s^2+4)+(Bs+C)\cdot s$$

$$1 \equiv (A+B)s^2+Cs+4A \Rightarrow A=\frac{1}{4}, \quad B=-\frac{1}{4}, \quad C=0$$

$$\frac{1}{s(s^2+4)} = \frac{\frac{1}{4}}{s}+\frac{\left(-\frac{1}{4}\right)s}{s^2+4}$$

$$L^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \frac{1}{4}L^{-1}\left\{\frac{1}{s}\right\}-\frac{1}{4}L^{-1}\left\{\frac{s}{s^2+4}\right\}$$

$$=\frac{1}{4}\cdot 1-\frac{1}{4}\cos 2x$$

ÖRNEK : $L^{-1}\left\{ \dfrac{8}{s^3(s^2-s-2)} \right\} = ?$

Çözüm : $s^2 - s - 2 = (s-2)(s+1)$ old.

$$\frac{8}{s^3(s^2-s-2)} = \frac{8}{s^3(s-2)(s+1)} \equiv \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s-2} + \frac{E}{s+1}$$

$s^2(s-2)(s+1)$ $\quad s(s-2)(s+1)$ $\quad (s-2)(s+1)$ $\quad s^3(s+1)$ $\quad s^3(s-2)$

$$8 \equiv A s^2(s-2)(s+1) + B s(s-2)(s+1) + C(s-2)(s+1) + D s^3(s+1) + E(s-2)s^3$$

Sırasıyla $s=-1$, $s=2$ ve $s=0$ alınırsa

$E = \frac{8}{3}$, $D = \frac{1}{3}$ ve $C = -4$ elde edilir.

Daha sonra $s=1$ ve $s=-2$ alınırsa ($s=-1, 2, 0$ hariç)

$A + B = -1$ ve $2A - B = -8$

$\Rightarrow$ $A = 3$ ve $B = 2$ bulunur.

$$\frac{8}{s^3(s^2-s-2)} = -\frac{3}{s} + \frac{2}{s^2} - \frac{4}{s^3} + \frac{\frac{1}{3}}{s-2} + \frac{\frac{8}{3}}{s+1}$$

$$\Rightarrow L^{-1}\left\{ \frac{8}{s^3(s^2-s-2)} \right\} = -3L^{-1}\left\{\frac{1}{s}\right\} + 2L^{-1}\left\{\frac{1}{s^2}\right\} - 2L^{-1}\left\{\frac{2}{s^3}\right\} + \frac{1}{3}L^{-1}\left\{\frac{1}{s-2}\right\} + \frac{8}{3}L^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= -3 + 2x - 2x^2 + \frac{1}{3}e^{2x} + \frac{8}{3}e^{-x}$$