Arithmetic From Logic

Digital Systems 2022/23

Dr James Stovold

Boolean Logic

- We have seen how to represent a logical circuit using Boolean logic operations (AND/OR/NOT etc.)
- A reasonable question might be: can we also implement arithmetic operations?

Binary Arithmetic

- Let's consider base 10 addition:

$$1 + 1 = 2$$

- Convert to base 2:

$$01 + 01 = 10_2$$

Binary Arithmetic

Truth Table

- Two binary variables, A and B:

A	B	add(A,B)
0	0	0
0	1	1
1	0	1
1	1	0

+ 1 carry bit

Binary Arithmetic

Truth Table

- Or three binary variables?

A	B	C	O1	O2
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

Binary Arithmetic

Truth Table

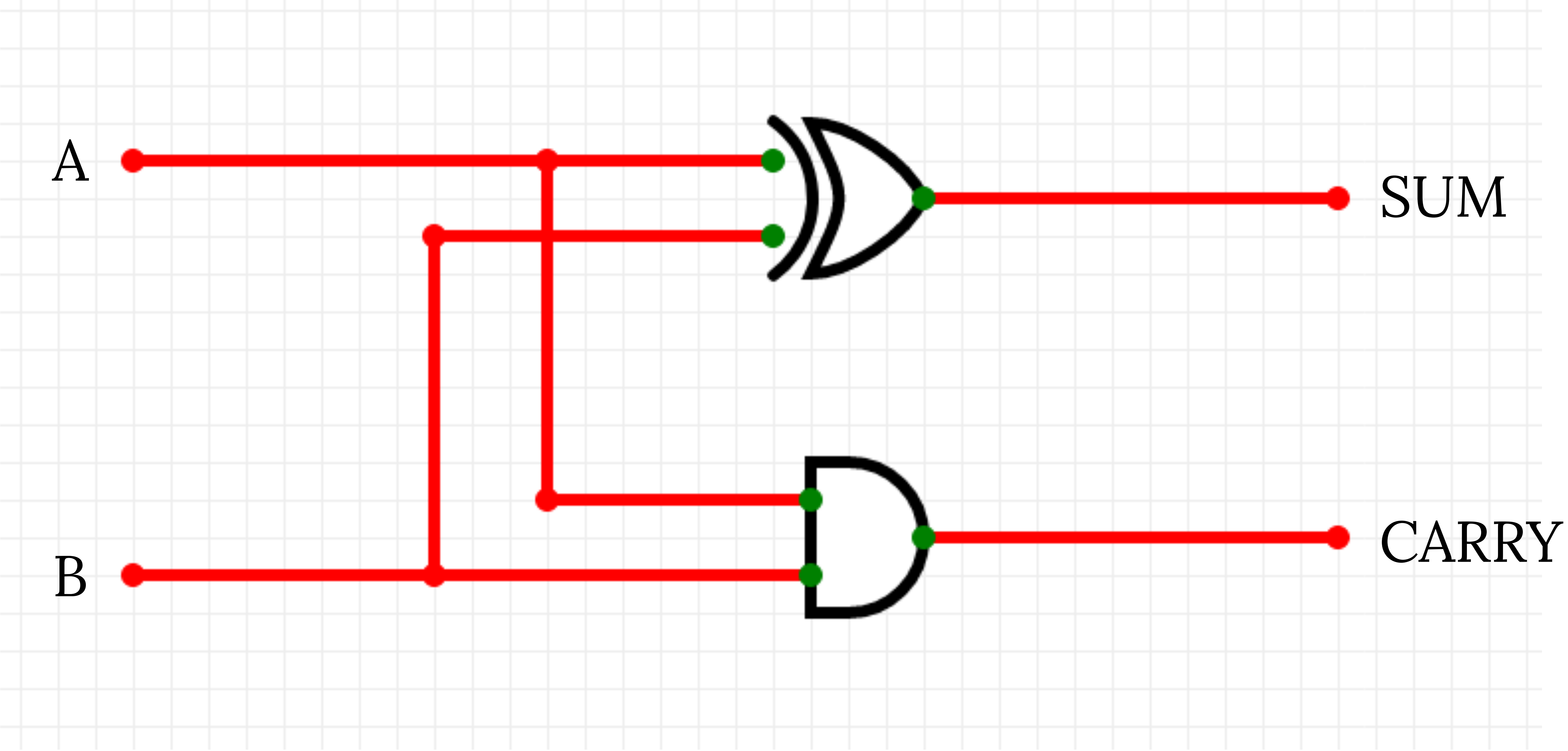
- Two binary variables, A and B:

A	B	SUM
0	0	0
0	1	1
1	0	1
1	1	0

A	B	CARRY
0	0	0
0	1	0
1	0	0
1	1	1

Half adder

Half Adder



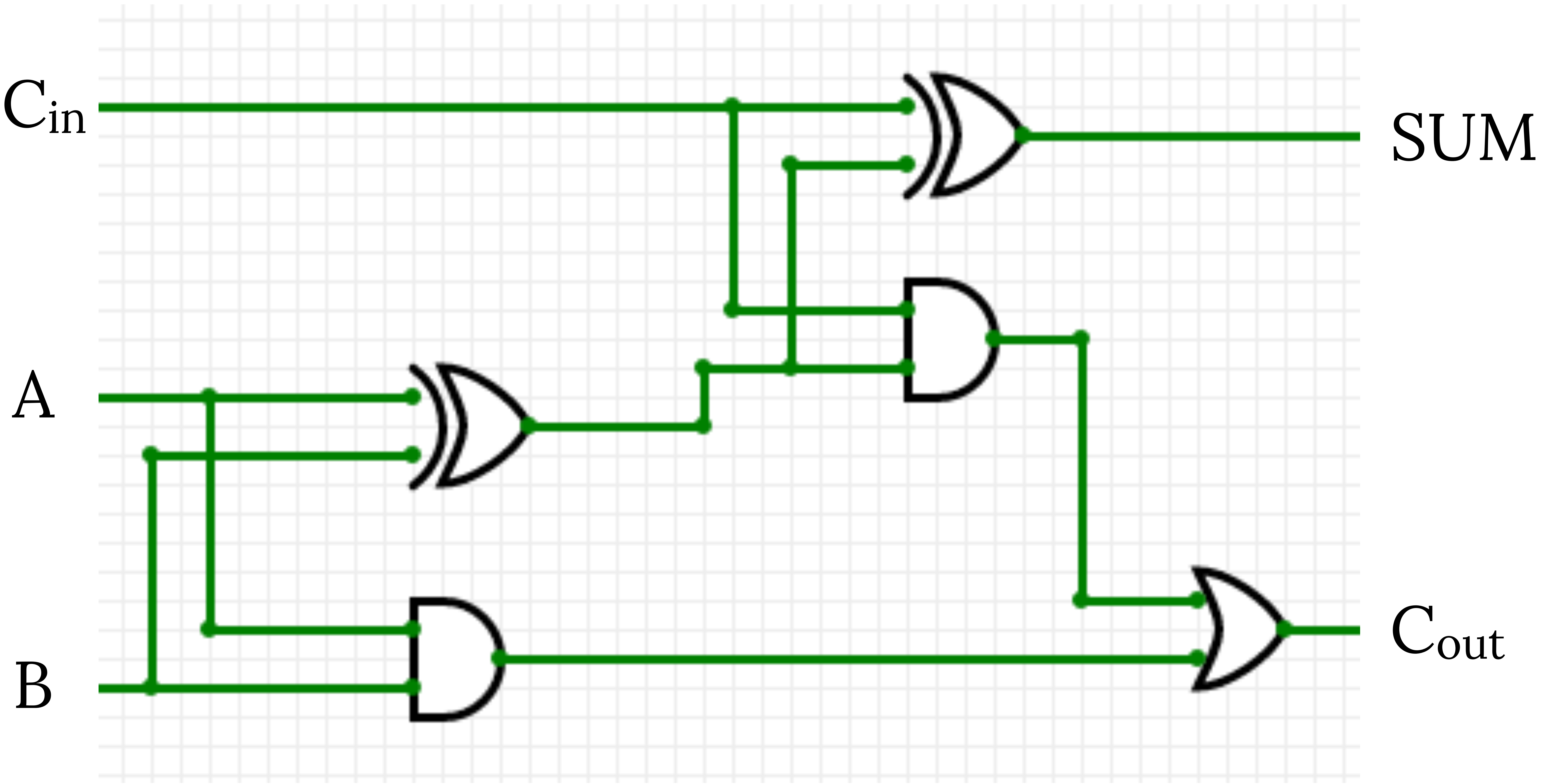
Testing the Circuit

https://circuitverse.org/

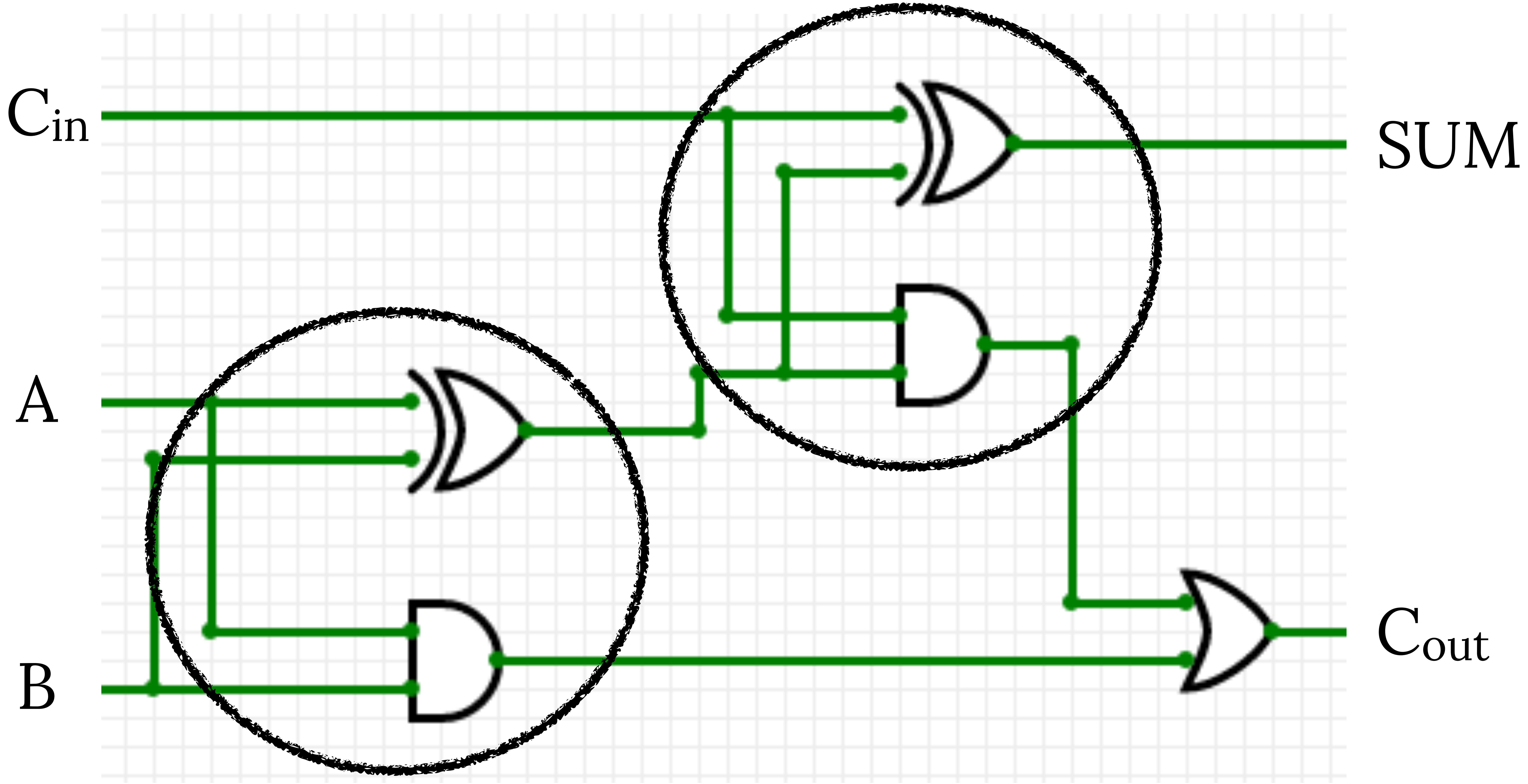
Half Adder

- A half adder only adds two bits
- When adding multi-digit numbers consisting of multiple bits, we need to consider the CARRY bits.
  - Need to be able to add 3 bits: two input bits and a possible CARRY bit.
  - If all three are 1, we should get a max output of 3 (i.e. sum(1,1,1) = 11_2)
- We can use two half adders to make a **full adder.**

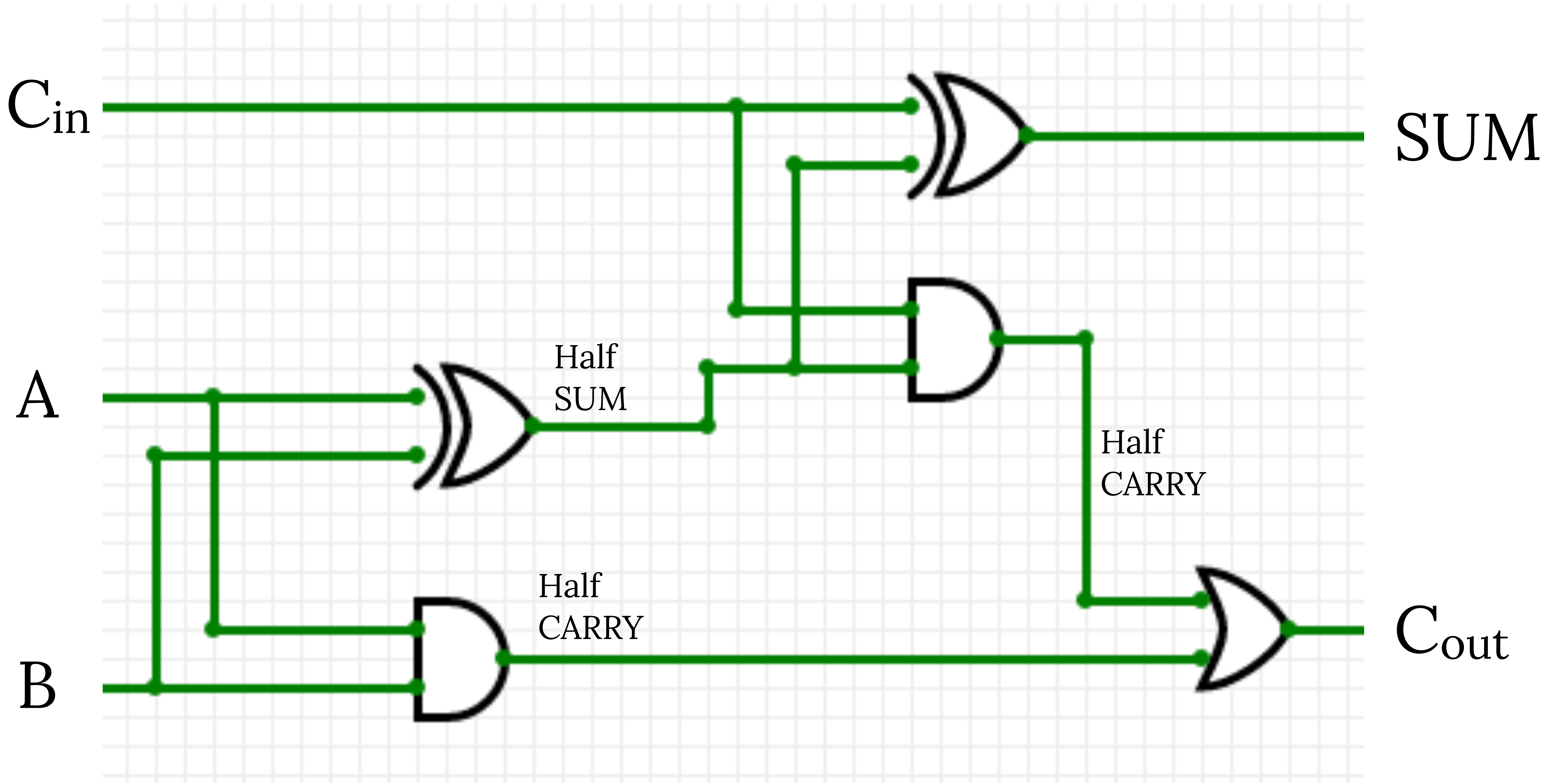
The Full Adder



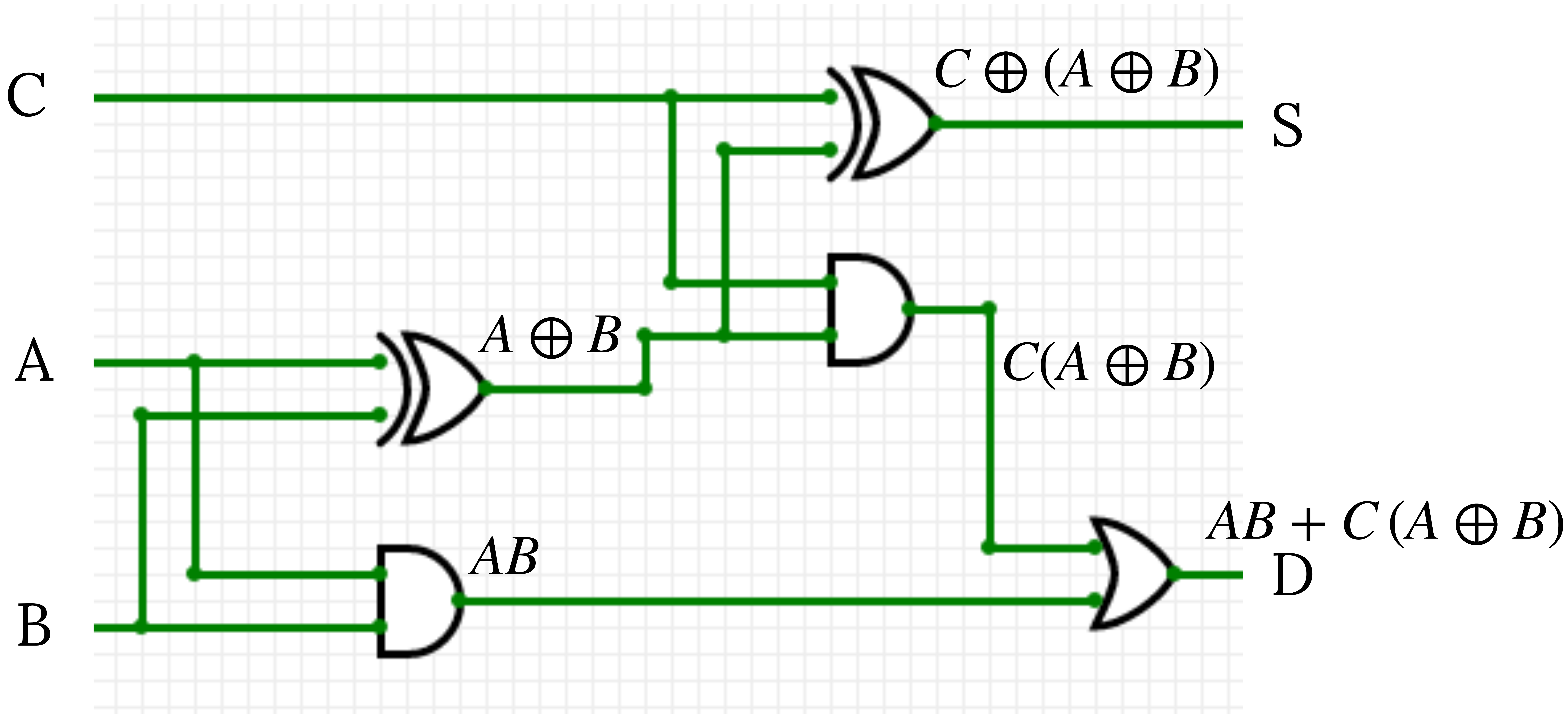
The Full Adder



The Full Adder



Why does this work?



$$S = C \oplus (A \oplus B) \qquad D = AB + C(A \oplus B)$$

Why does this work?

Proof by Perfect Induction

C	A	B	S	D	$A \oplus B$	$C \oplus (A \oplus B)$	AB	$C(A \oplus B)$	$AB + C(A \oplus B)$
0	0	0							
0	0	1							
0	1	0							
0	1	1							
1	0	0							
1	0	1							
1	1	0							
1	1	1							

$$S = C \oplus (A \oplus B) \qquad D = AB + C(A \oplus B)$$

Why does this work?

Proof by Perfect Induction

C	A	B	S	D	$A \oplus B$	$C \oplus (A \oplus B)$	AB	$C(A \oplus B)$	$AB + C(A \oplus B)$
0	0	0			0				
0	0	1			1				
0	1	0			1				
0	1	1			0				
1	0	0			0				
1	0	1			1				
1	1	0			1				
1	1	1			0				

$$S = C \oplus (A \oplus B)$$

$$D = AB + C(A \oplus B)$$

Why does this work?

Proof by Perfect Induction

C	A	B	S	D	$A \oplus B$	$C \oplus (A \oplus B)$	AB	$C(A \oplus B)$	$AB + C(A \oplus B)$
0	0	0			0		0		
0	0	1			1		0		
0	1	0			1		0		
0	1	1			0		1		
1	0	0			0		0		
1	0	1			1		0		
1	1	0			1		0		
1	1	1			0		1		

$$S = C \oplus (A \oplus B)$$

$$D = AB + C(A \oplus B)$$

Why does this work?

Proof by Perfect Induction

C	A	B	S	D	$A \oplus B$	$C \oplus (A \oplus B)$	AB	$C(A \oplus B)$	$AB + C(A \oplus B)$
0	0	0			0	0	0		
0	0	1			1	1	0		
0	1	0			1	1	0		
0	1	1			0	0	1		
1	0	0			0	1	0		
1	0	1			1	0	0		
1	1	0			1	0	0		
1	1	1			0	1	1		

$$S = C \oplus (A \oplus B) \qquad D = AB + C(A \oplus B)$$

Why does this work?

Proof by Perfect Induction

C	A	B	S	D	$A \oplus B$	$C \oplus (A \oplus B)$	AB	$C(A \oplus B)$	$AB + C(A \oplus B)$
0	0	0			0	0	0	0	
0	0	1			1	1	0	0	
0	1	0			1	1	0	0	
0	1	1			0	0	1	0	
1	0	0			0	1	0	0	
1	0	1			1	0	0	1	
1	1	0			1	0	0	1	
1	1	1			0	1	1	0	

$$S = C \oplus (A \oplus B)$$

$$D = AB + C(A \oplus B)$$

Why does this work?

Proof by Perfect Induction

C	A	B	S	D	$A \oplus B$	$C \oplus (A \oplus B)$	AB	$C(A \oplus B)$	$AB + C(A \oplus B)$
0	0	0			0	0	0	0	0
0	0	1			1	1	0	0	0
0	1	0			1	1	0	0	0
0	1	1			0	0	1	0	1
1	0	0			0	1	0	0	0
1	0	1			1	0	0	1	1
1	1	0			1	0	0	1	1
1	1	1			0	1	1	0	1

$$S = C \oplus (A \oplus B)$$

$$D = AB + C(A \oplus B)$$

Why does this work?

Proof by Perfect Induction

C	A	B	S	D	$A \oplus B$	$C \oplus (A \oplus B)$	AB	$C(A \oplus B)$	$AB + C(A \oplus B)$
0	0	0	0		0	0	0	0	0
0	0	1	1		1	1	0	0	0
0	1	0	1		1	1	0	0	0
0	1	1	0		0	0	1	0	1
1	0	0	1		0	1	0	0	0
1	0	1	0		1	0	0	1	1
1	1	0	0		1	0	0	1	1
1	1	1	1		0	1	1	0	1

$$S = C \oplus (A \oplus B)$$

$$D = AB + C(A \oplus B)$$

Why does this work?

Proof by Perfect Induction

C	A	B	S	D	$A \oplus B$	$C \oplus (A \oplus B)$	AB	$C(A \oplus B)$	$AB + C(A \oplus B)$
0	0	0	0	0	0	0	0	0	0
0	0	1	1	0	1	1	0	0	0
0	1	0	1	0	1	1	0	0	0
0	1	1	0	1	0	0	1	0	1
1	0	0	1	0	0	1	0	0	0
1	0	1	0	1	1	0	0	1	1
1	1	0	0	1	1	0	0	1	1
1	1	1	1	1	0	1	1	0	1

$$S = C \oplus (A \oplus B)$$

$$D = AB + C(A \oplus B)$$

Why does this work?

Proof by Perfect Induction

C	A	B	S	D	$A \oplus B$	$C \oplus (A \oplus B)$	AB	$C(A \oplus B)$	$AB + C(A \oplus B)$
0	0	0	0	0	0	0	0	0	0
0	0	1	1	0	1	1	0	0	0
0	1	0	1	0	1	1	0	0	0
0	1	1	0	1	0	0	1	0	1
1	0	0	1	0	0	1	0	0	0
1	0	1	0	1	1	0	0	1	1
1	1	0	0	1	1	0	0	1	1
1	1	1	1	1	0	1	1	0	1

$$S = C \oplus (A \oplus B) \qquad D = AB + C(A \oplus B)$$

Why does this work?

Proof by Perfect Induction

C	A	B	D	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

Testing the Circuit

https://circuitverse.org/

Chaining Full Adders

- There are some issues with chaining multiple full adders together.
- We'll cover this when we build our ALU.