Lecture:
Pixels and Filters

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

What we will learn today?

- Image sampling and quantization
- Image histograms
- Images as functions
- Linear systems (filters)
- Convolution and correlation

Some background reading:
Forsyth and Ponce, Computer Vision, Chapter 7

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Types of Images

Binary

Types of Images

Binary

Gray Scale

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Types of Images

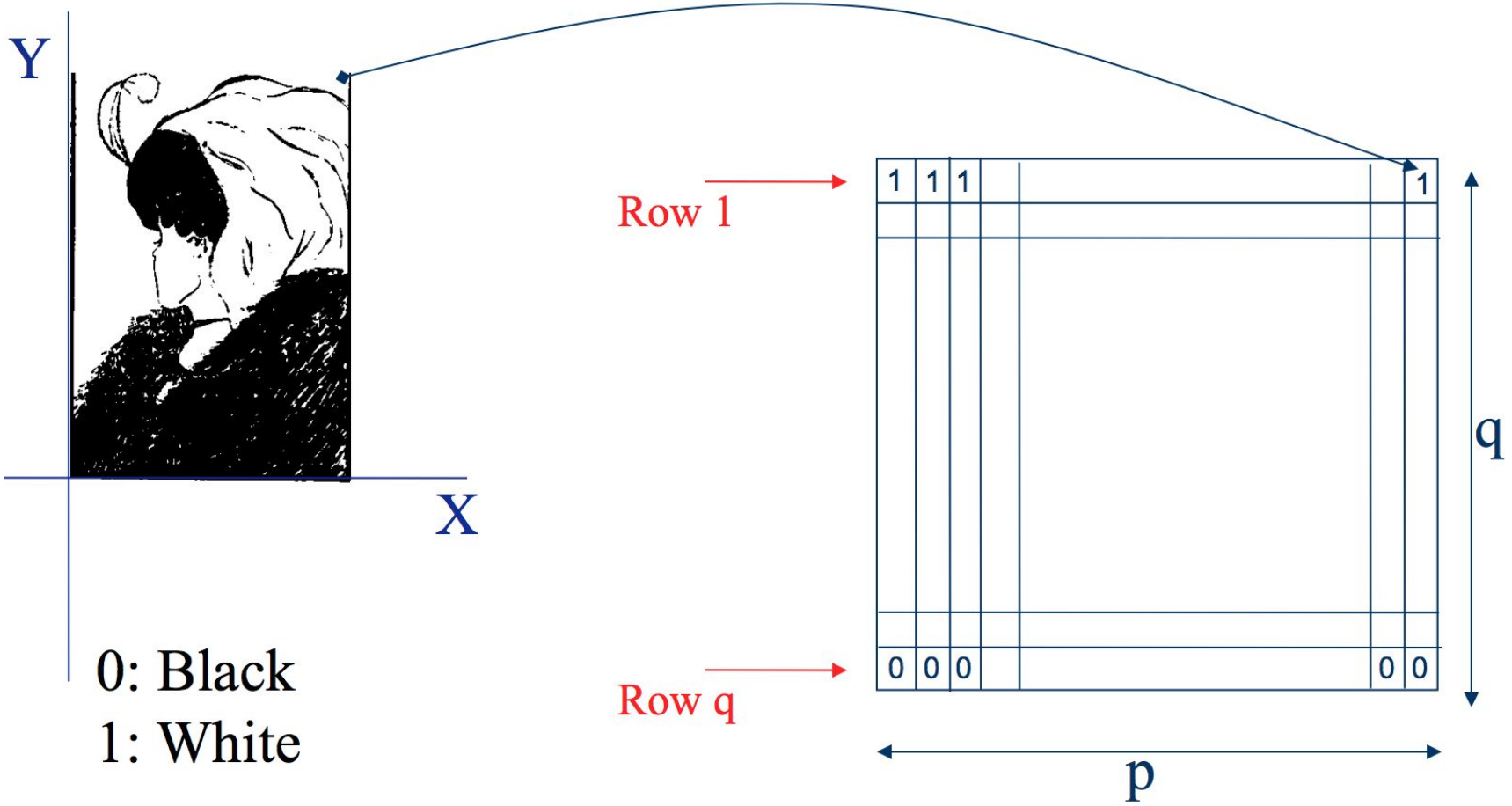
Binary

Gray Scale

Color

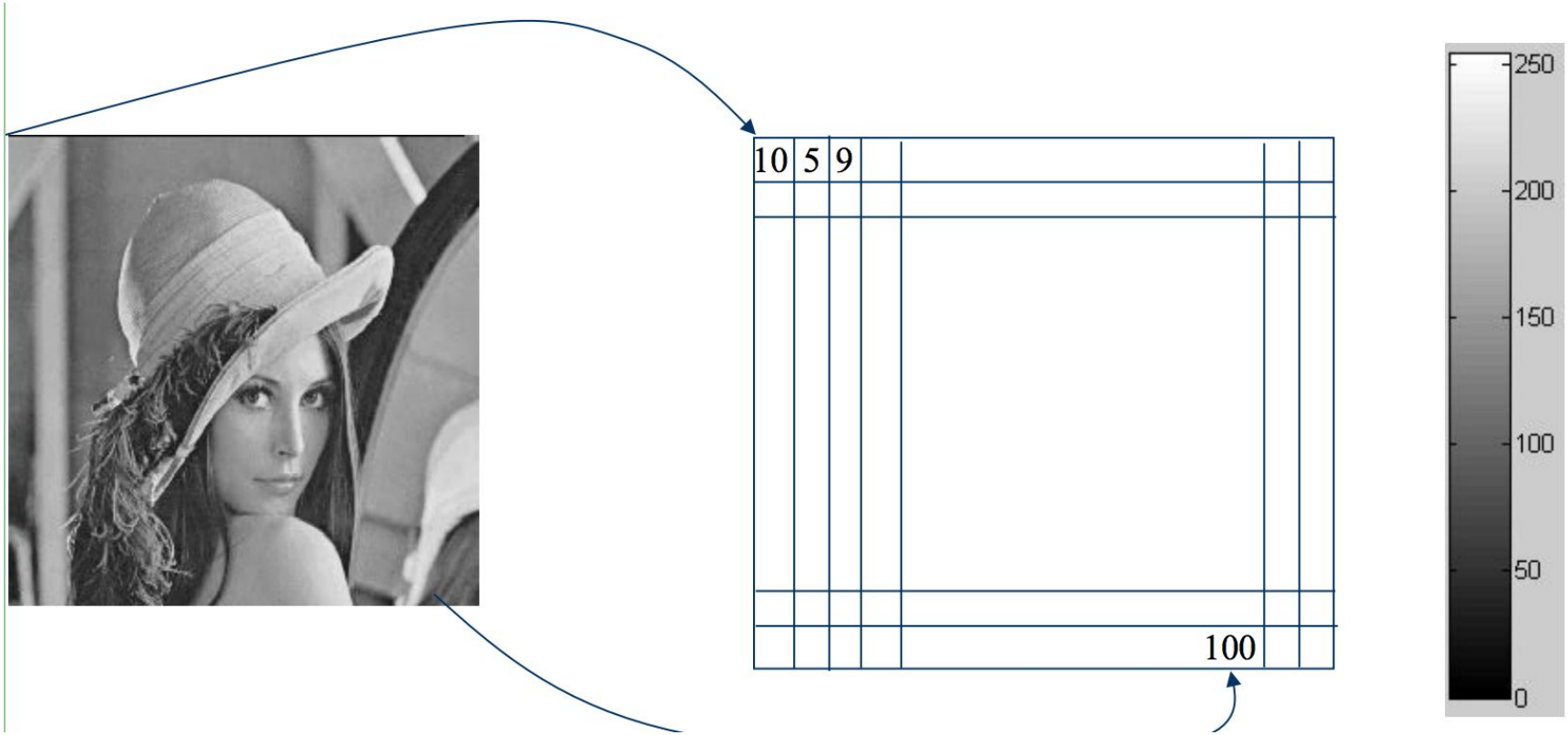
Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Binary image representation



Slide credit: Ulas Bagci

Grayscale image representation



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Color Image - one channel

Slide credit: Ulas Bagci

Color image representation

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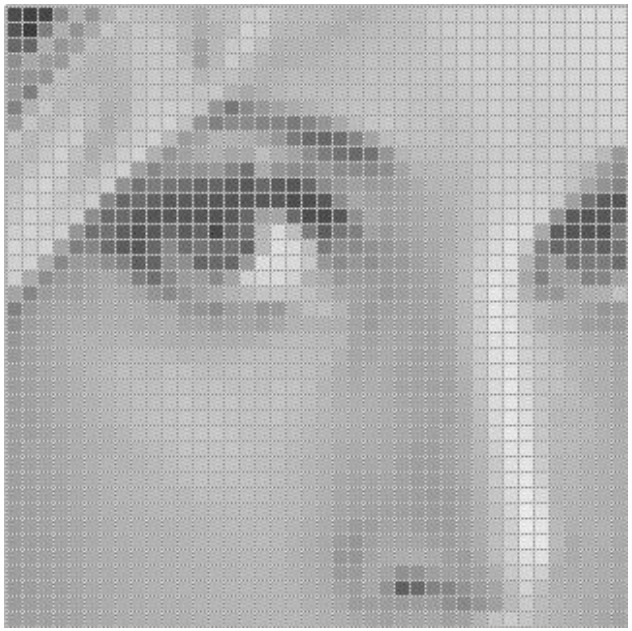
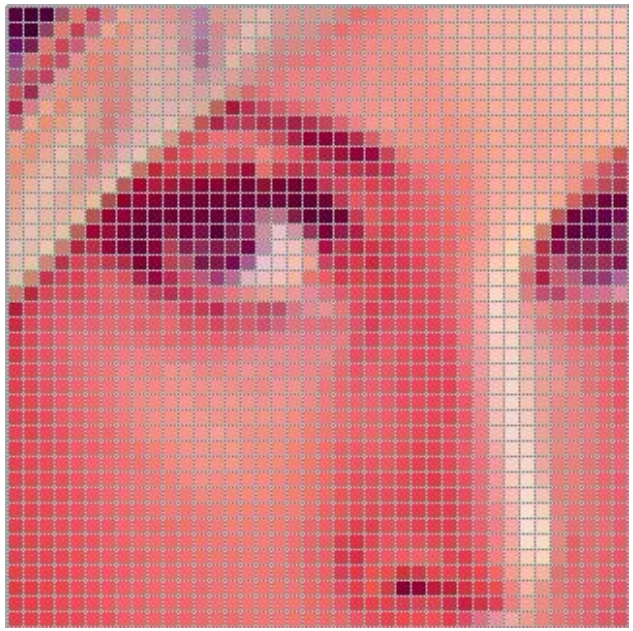
Images are sampled

What happens when we zoom into the images we capture?





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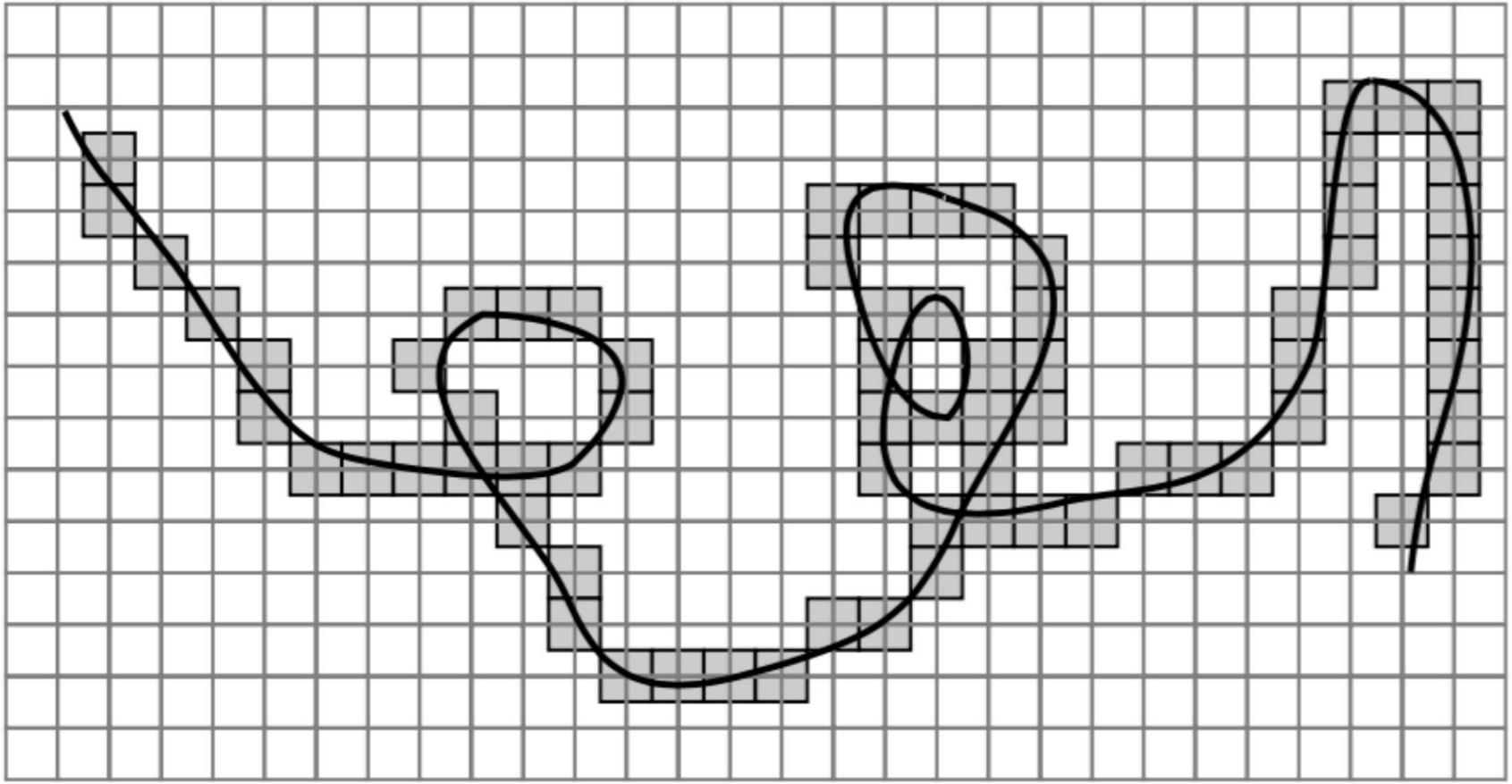
Errors due Sampling



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Resolution

is a **sampling** parameter, defined in dots per inch (DPI) or equivalent measures of spatial pixel density

- Resolution of the camera and the resolution of the screen can both affect the quality.

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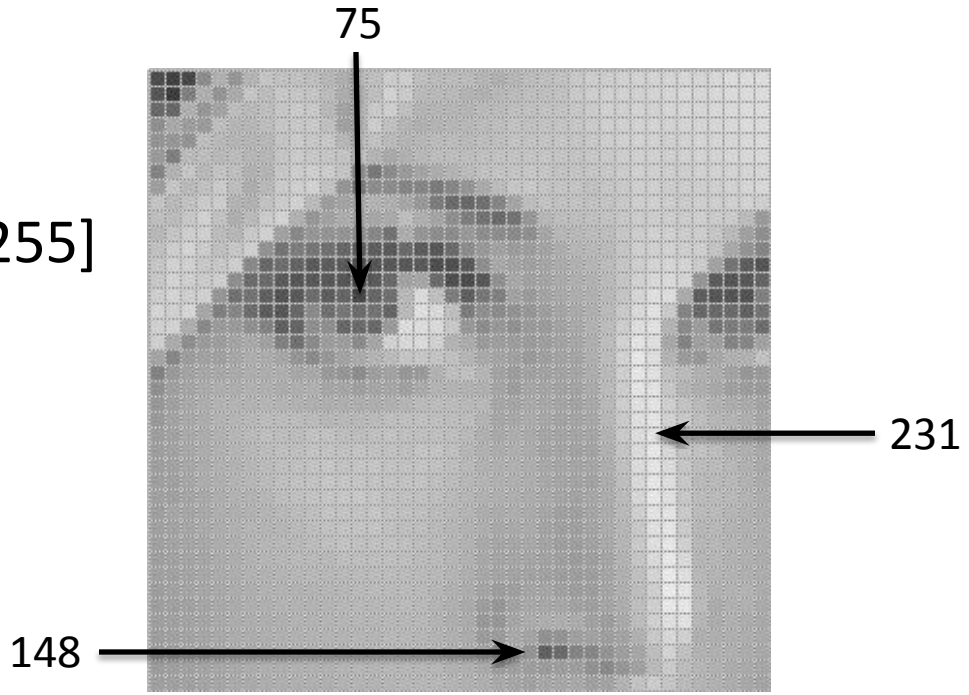
Images are Sampled and Quantized

- An image contains discrete number of pixels
  - A simple example
  - Pixel value:
    - "grayscale"

    (or "intensity"): [0,255]



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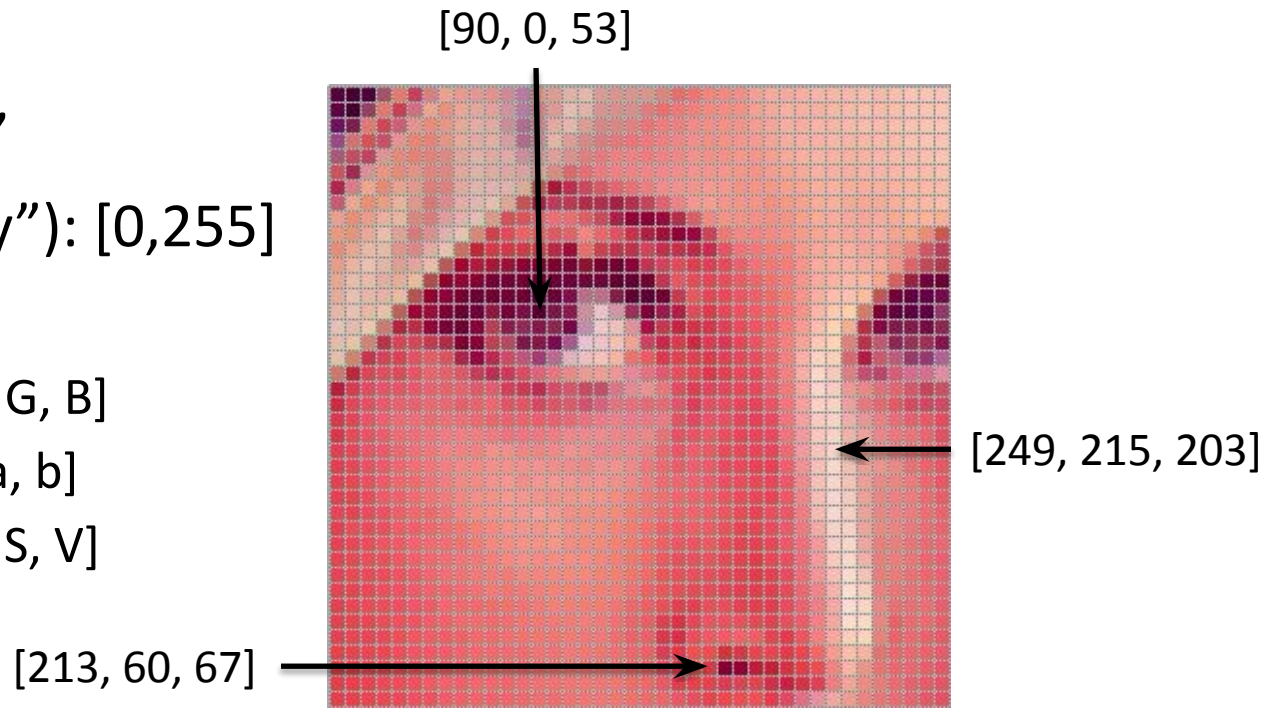
Images are Sampled and Quantized

- An image contains discrete number of pixels
  - A simple example
  - Pixel value:
    - "grayscale"

      (or "intensity"): [0,255]
    - "color"
      - RGB: [R, G, B]
      - Lab: [L, a, b]
      - HSV: [H, S, V]



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With this loss of information (from sampling and quantization),

Can we still use images for useful tasks?

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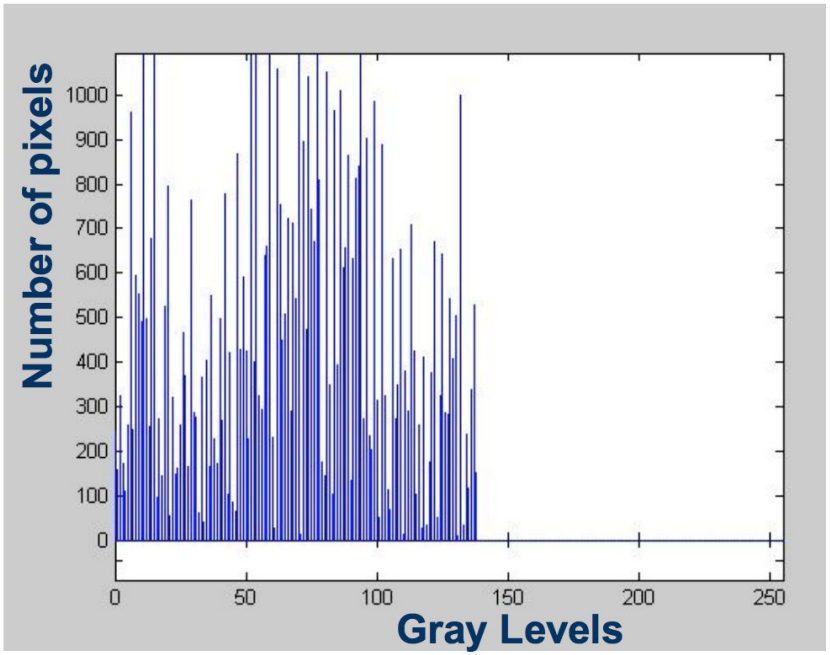
Histogram

- Histogram captures the distribution of "countable elements" in the image, e.g. how frequently each gray level occurs in the image

Histogram

- Grayscale image - intensity histogram: one of the simplest *image representations*



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Histogram

Example implementation of the most basic grayscale histogram:

```
def histogram(im):
    h = np.zeros(256)
    for row in range(im.shape[0]):
        for col in range(im.shape[1]):
            val = im[row, col]
            h[val] += 1
```

Note: Python for loops can be slow, a faster implementation: see *numpy.histogram*.

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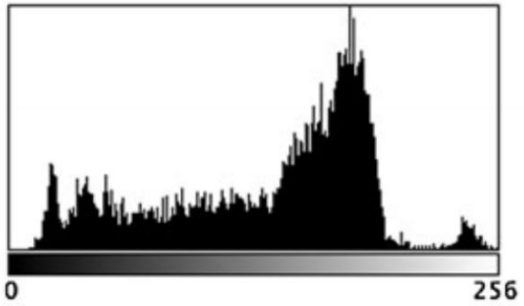
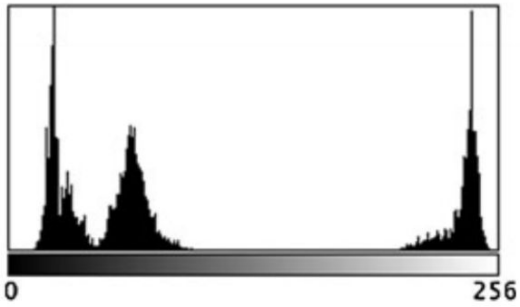
Histogram





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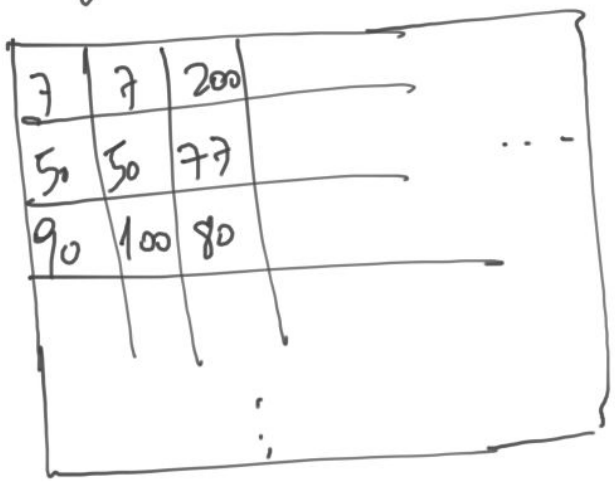
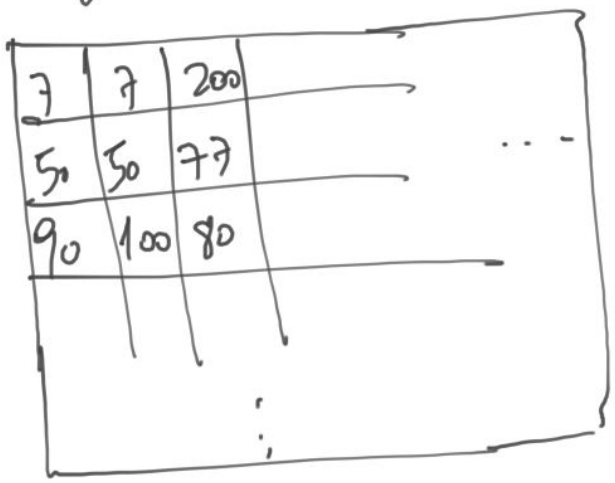
Histogram



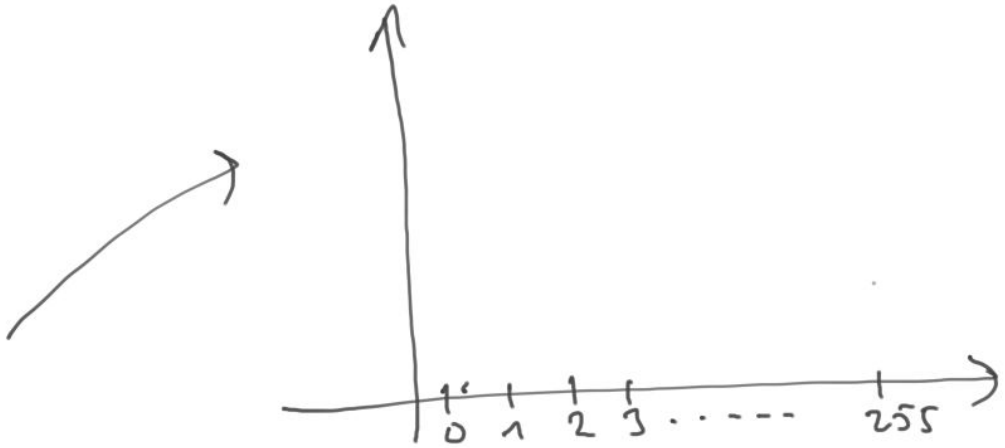
gray scale image

?

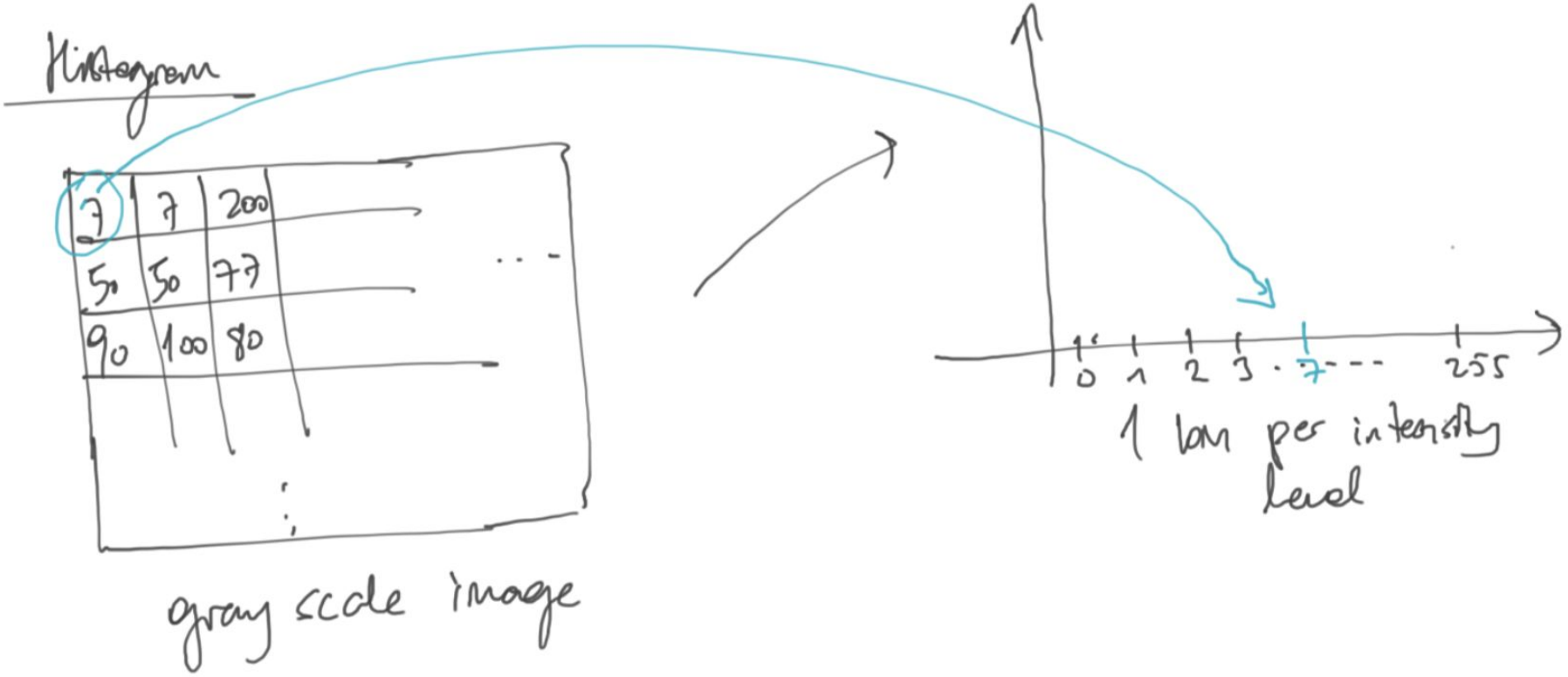
Histogram

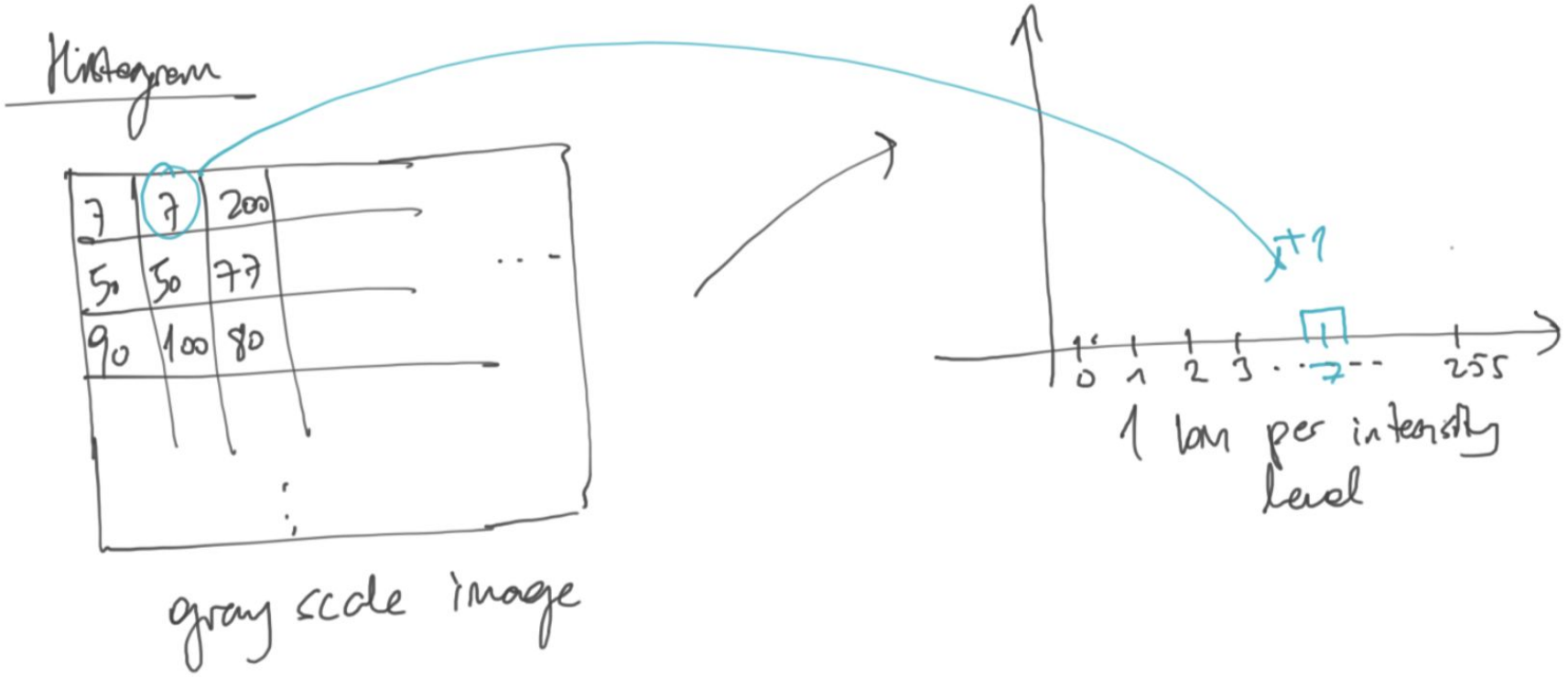


gray scale image

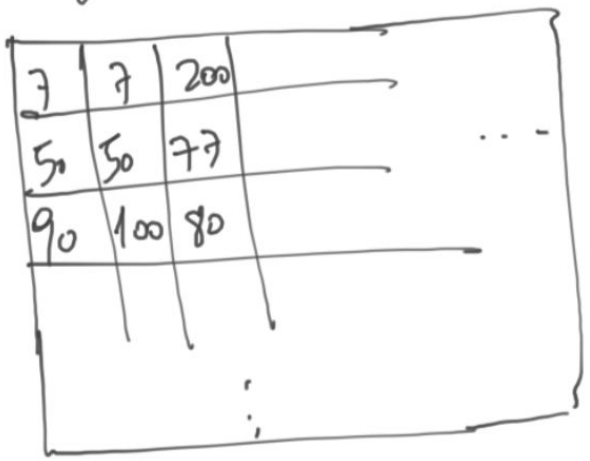


1 bin per intensity level

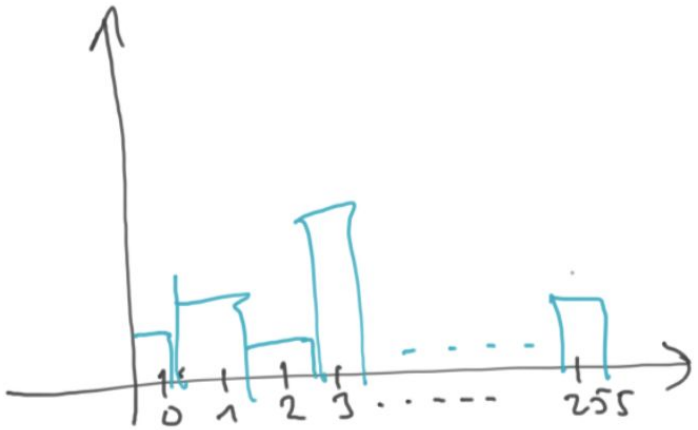

Histogram
7 7 200
50 50 77
90 100 80
gray scale image
0 1 2 3 7 255
1 bar per intensity level


Histogram
7
7
200
50
50
77
90
100
80
gray scale image
x+1
0
1
2
3
7
255
1 bin per intensity level

Histogram

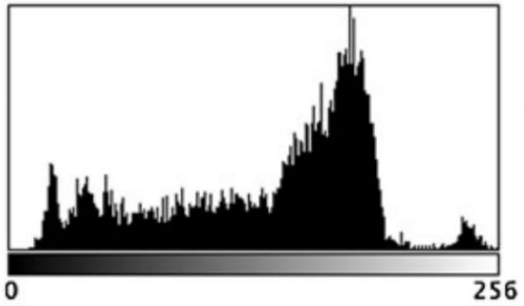


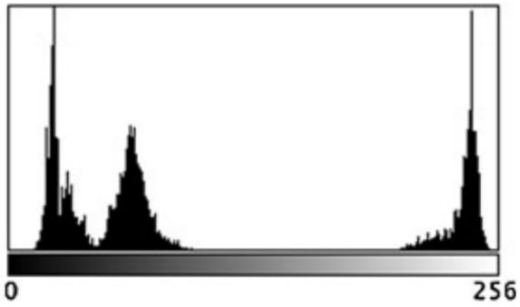
gray scale image



1 bar per intensity level

How to improve the histogram? (ie. What can go wrong with 1 bin per intensity level?)





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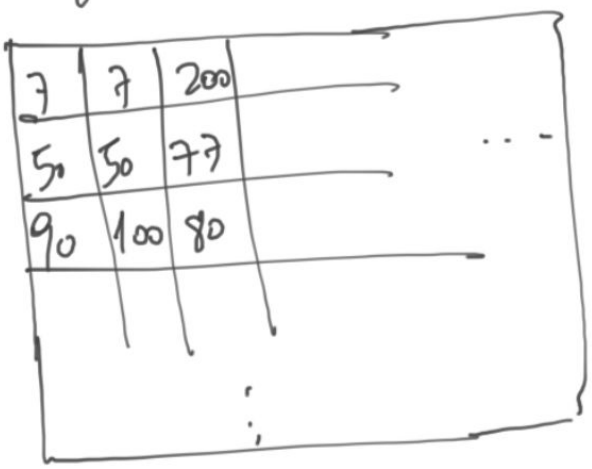
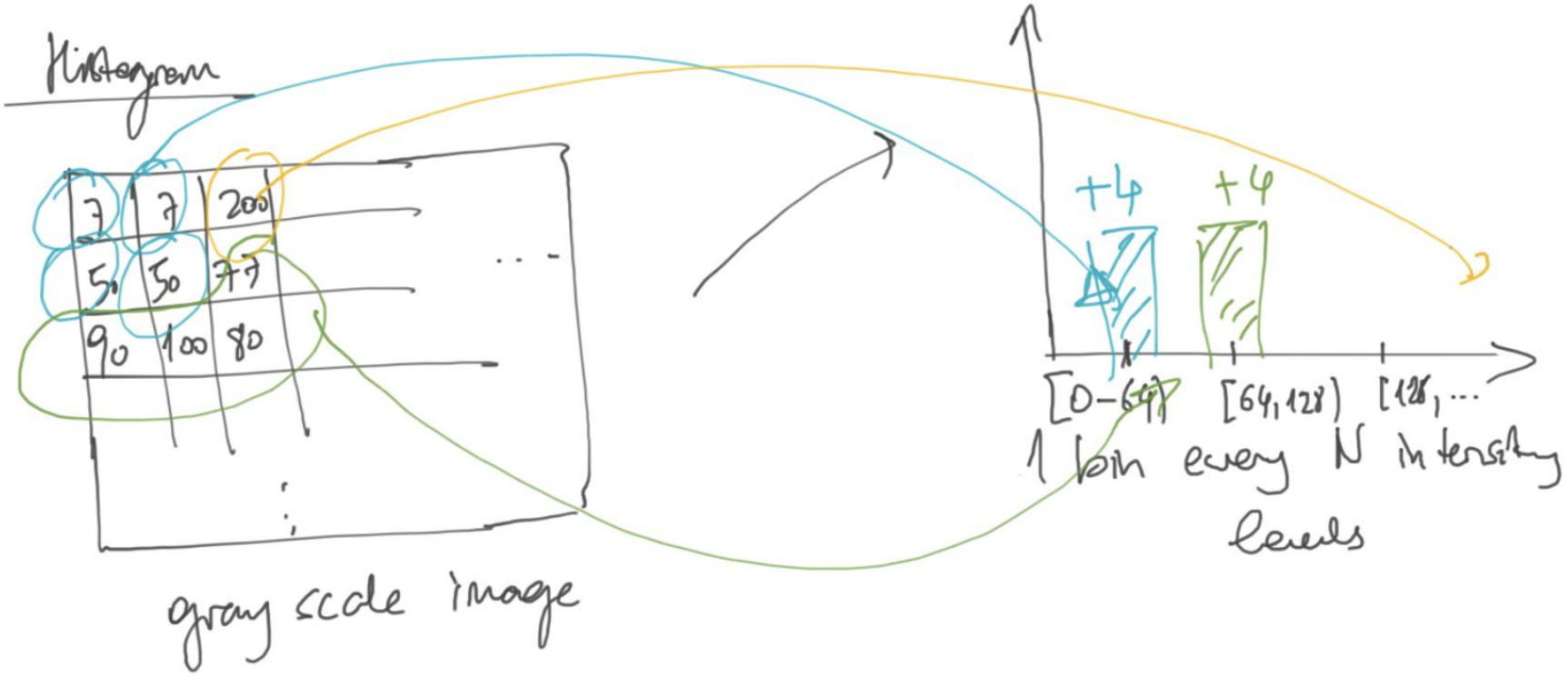
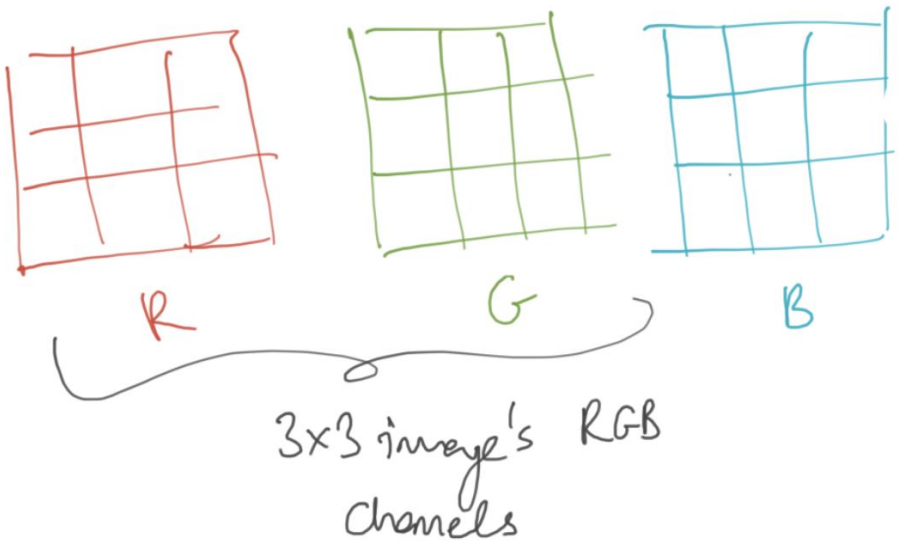
Histogram



gray scale image

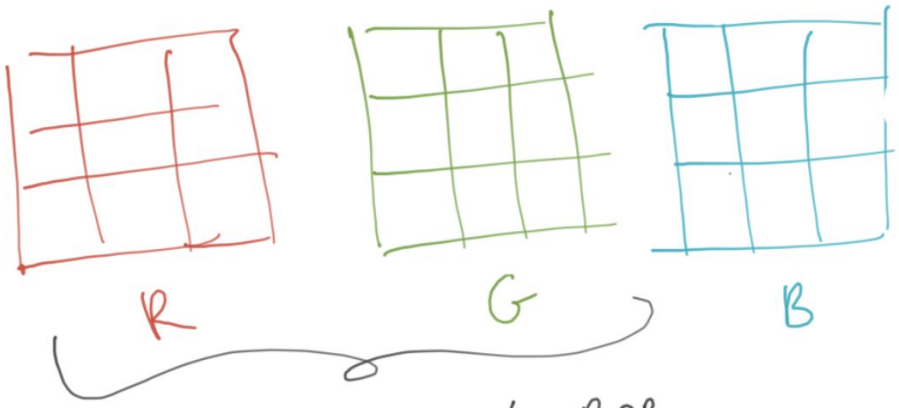

Histogram
7
7
200
50
50
77
90
100
80
+4
+4
[0-64)
[64,128)
[128,...
1 bin every N intensity levels
gray scale image

Color histogram

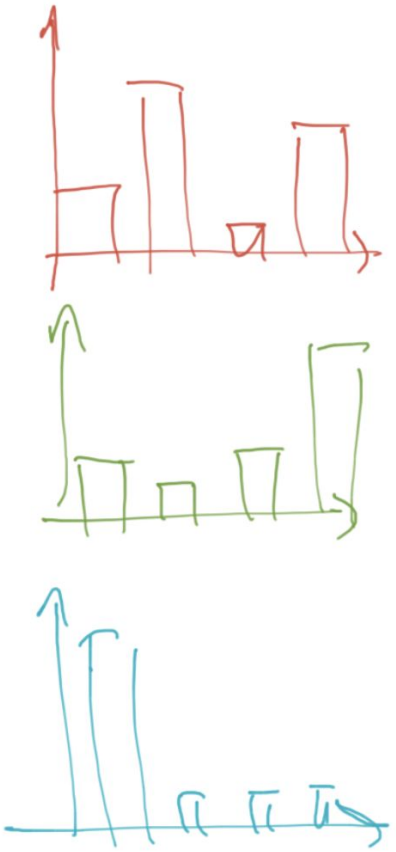


?

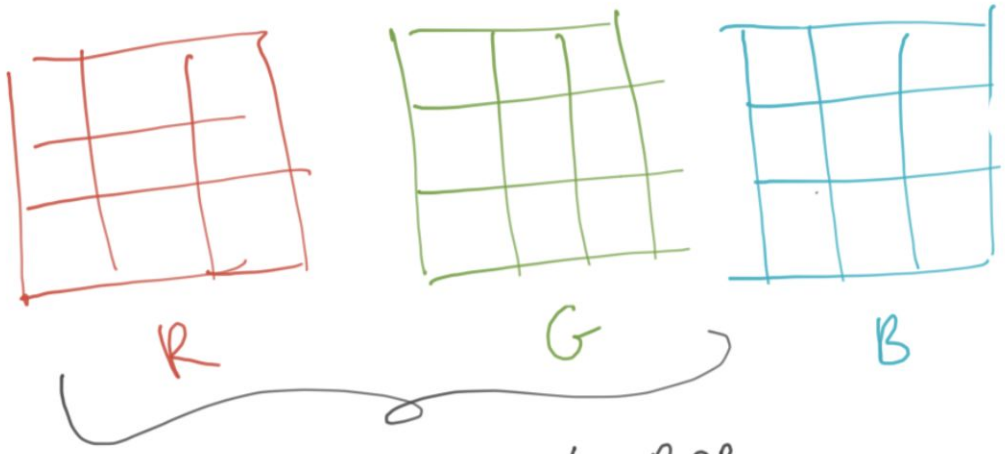
Color histogram



1 histogram per channel

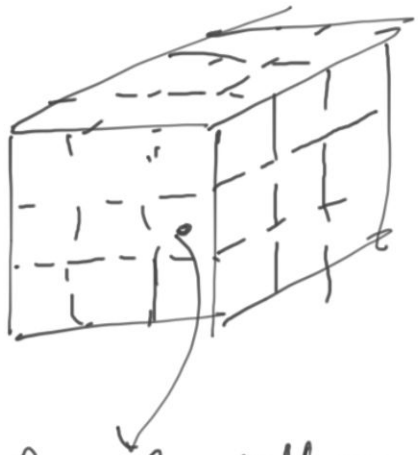


Color histogram



3x3 image's RGB channels

Joint histogram



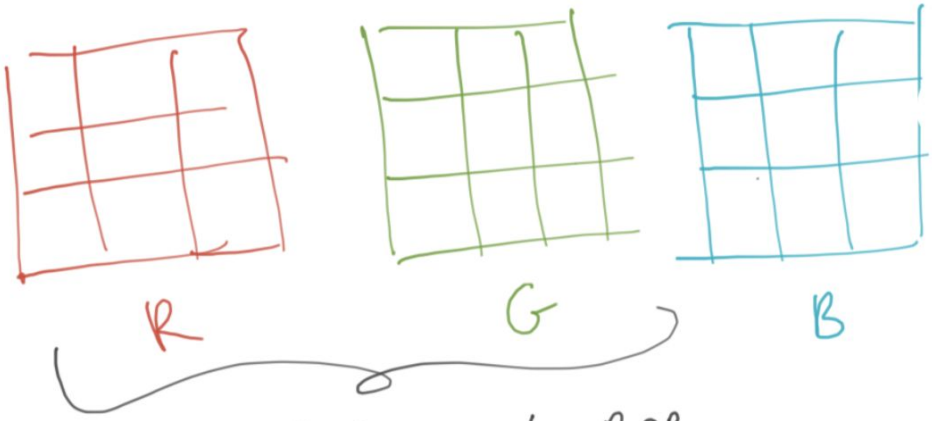
A single cell J:

Count of pixels in the range:

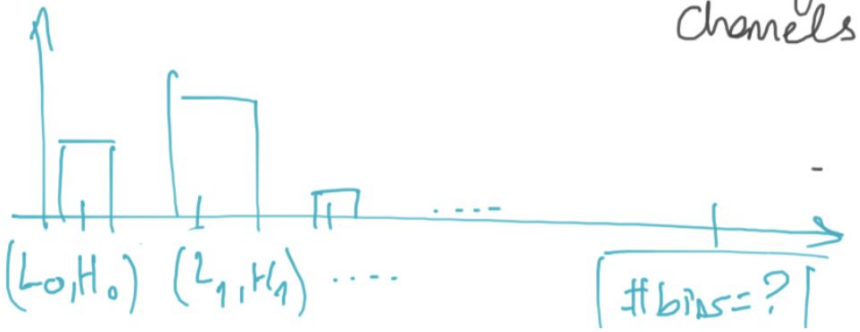
$$L_J^r \leq r < H_J^r, \; L_J^g \leq g < H_J^g$$

AND $L_J^b \leq b < H_J^b$

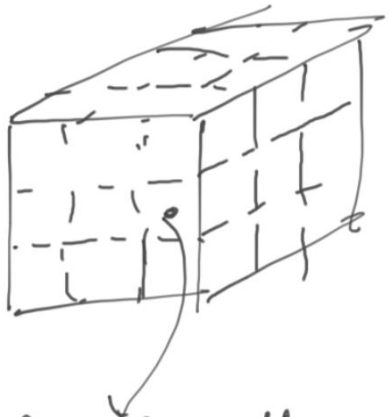
Color histogram



3×3 image's RGB channels



Joint histogram



A single cell J:

Count of pixels in the range:

$$\Leftarrow L_J^r \leq r < H_J^r,\ L_J^g \leq g < H_J^g$$

AND $L_J^b \leq b < H_J^b$

Color histogram

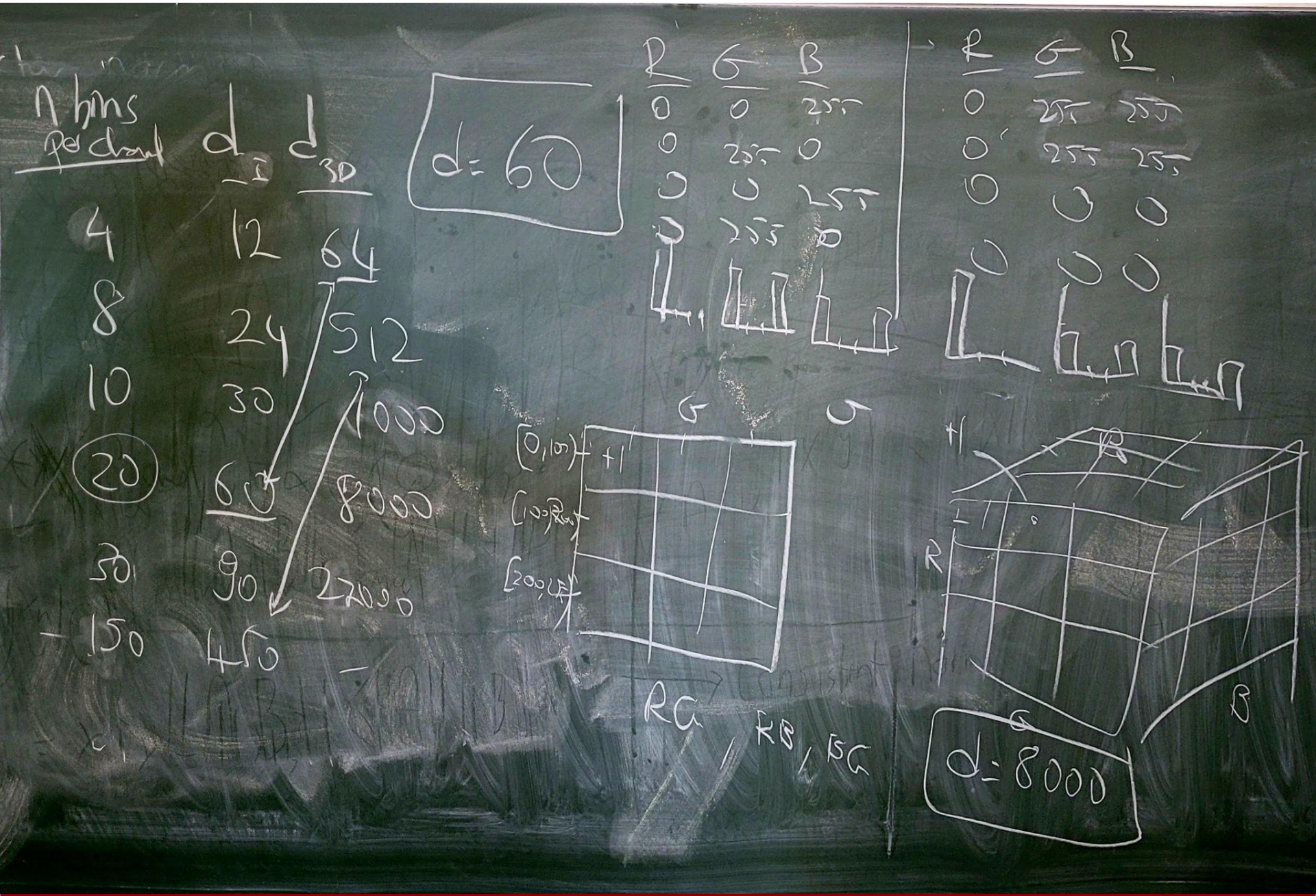
Joint histogram

THINK OF ADVANTAGES/DISADVANTAGES:

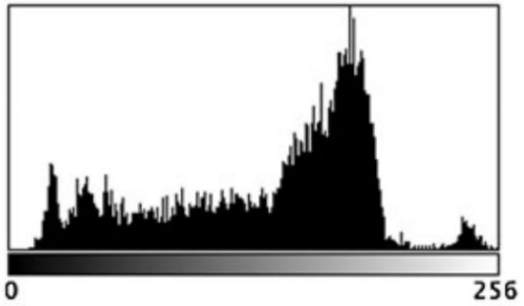
- Number of bins in grayscale/color histograms
- Grayscale vs color histogram
- Per-channel vs joint (n-dim) color histogram

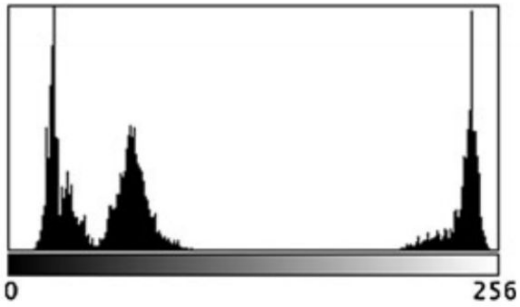
$\Leftarrow L_J^r \leq r < H_J^r, \; L_J^g \leq g < H_J^g$

AND $L_J^b \leq b < H_J^b$


n bins per channel
d_1
d_{3D}
4 12 64
8 24 512
10 30 1000
20 60 8000
50 150 27000
d = 60
R G B
0 0 255
0 255 0
255 0 0
R G B
0 255 255
255 255 0
0 0 0
G
R
B
d = 8000

Re-think of histogram's application





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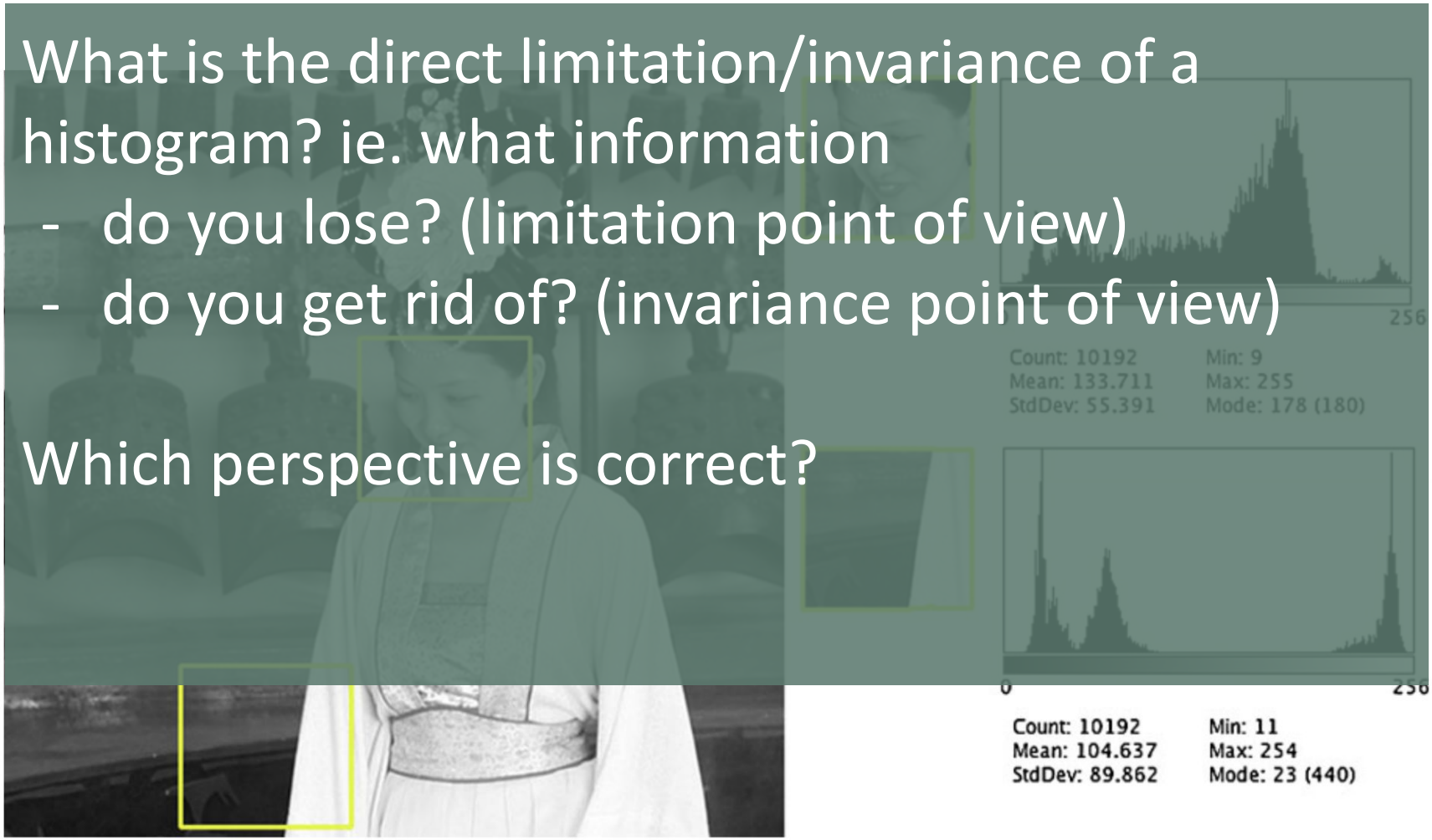
Re-think of histogram's application

What is the direct limitation/invariance of a histogram? ie. what information

- do you lose? (limitation point of view)
- do you get rid of? (invariance point of view)

Which perspective is correct?



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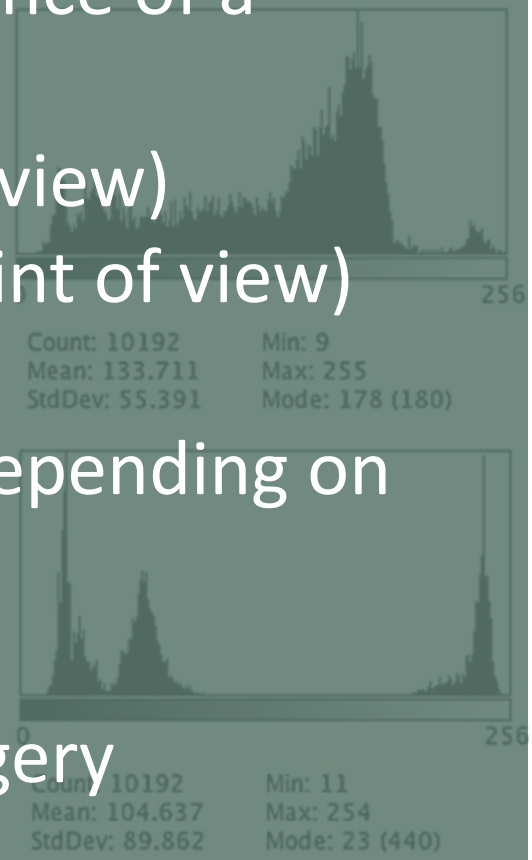
Re-think of histogram's application

What is the direct limitation/invariance of a histogram? ie. what information

- do you lose? (limitation point of view)
- do you get rid of? (invariance point of view)

Both points of views can be valid, depending on the task! Examples:

- detection of camera orientation
- detection of cars in satellite imagery



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Re-think of histogram's application

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What we will learn today?

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- Image histograms
- Images as functions
- Linear systems (filters)
- Convolution and correlation

Some background reading:
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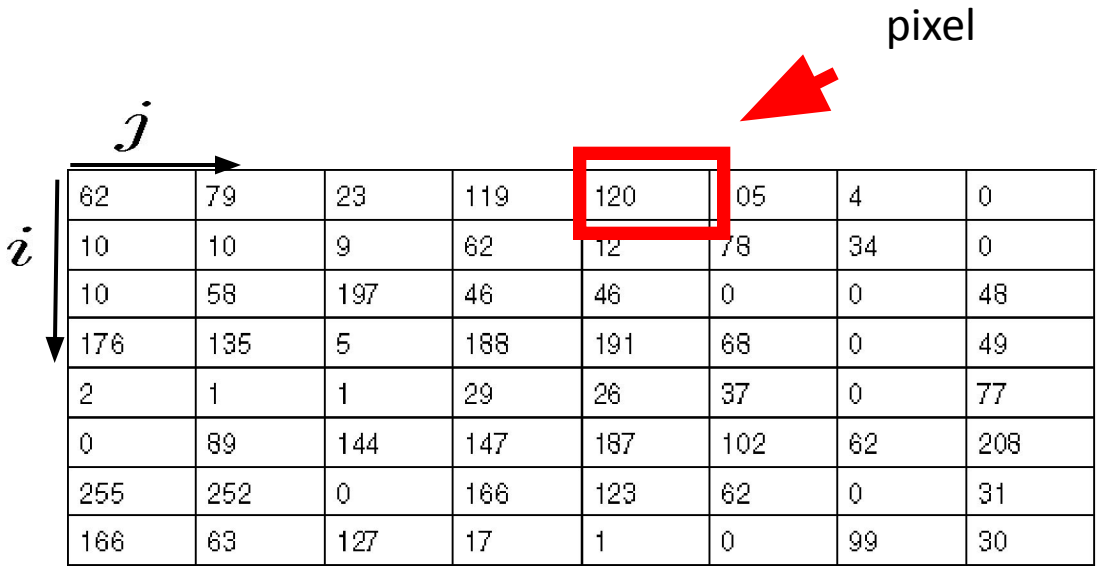
Images as discrete functions

- Images are usually **digital** (**discrete**):
  - **Sample** the 2D space on a regular grid

- Represented as a matrix of integer values



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Images as coordinates

Cartesian coordinates

$$f[n,m] = \begin{bmatrix} \ddots & \vdots & & \\ f[-1,-1] & f[0,-1] & f[1,-1] & \\ \dots \; f[-1,0] & \underline{f[0,0]} & f[1,0] & \dots \\ f[-1,1] & f[0,1] & f[1,1] & \\ & \vdots & & \ddots \end{bmatrix}$$

Notation for discrete functions

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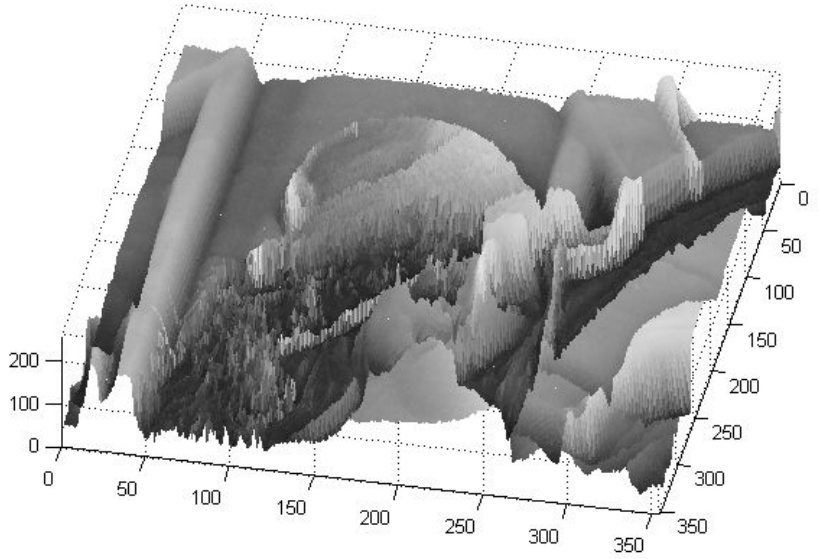
Images as functions

- **An Image** as a function *f* from R^2 to R^M:
  - $f(x, y)$ gives the **intensity** at position (x, y)
  - Defined over a rectangle, with a finite range:

$$f: [a,b] \times [c,d] \rightarrow [0,255]$$

Domain support

range



Adapted from slides hna

Images as functions

- **An Image** as a function f from R^2 to R^{M}:
  - $f(x, y)$ gives the **intensity** at position (x, y)
  - Defined over a rectangle, with a finite range:

$$f: [a,b] \times [c,d] \rightarrow [0,255]$$

Domain support

range

- A color image:

$$f(x,y) = \begin{bmatrix} r(x,y) \\ g(x,y) \\ b(x,y) \end{bmatrix}$$

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Histograms are a type of image function

- Histogram can be seen as another f: region → vector mapping.
- In recognition tasks for computer vision, *image representation* typically plays a critical role.
- Throughout the course, we'll progressively discuss **more powerful representation** extraction techniques.

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Histograms are a type of image function

- Histogram can be seen as another f: region → vector mapping.
- In recognition tasks for computer vision, *image representation* typically plays a critical role.
- Throughout the course, we'll progressively discuss **more powerful representation** extraction techniques.

More *powerful* is <u>not</u> necessarily more *detailed*. (Think of the trade-offs in histograms, as a good example.)

Utility of a representation is typically task specific and plays a central role in the success of computer vision approaches.

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

What we will learn today?

- Image sampling and quantization
- Image histograms
- Images as functions
- Linear systems (filters)
- Convolution and correlation

Some background reading:
Forsyth and Ponce, Computer Vision, Chapter 7

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Systems and Filters

Filtering:

- Forming a new image whose pixel values are transformed from original pixel values

Goals:

- Goal is to extract useful information from images, or transform images into another domain where we can modify/enhance image properties
  - Features (edges, corners, blobs…)
  - super-resolution; in-painting; de-noising

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

System and Filters

- We define a system as a unit that converts an input function f[n,m] into an output (or response) function g[n,m], where (n,m) are the independent variables.
  - In the case for images, (n,m) represents the **spatial position in the image.**

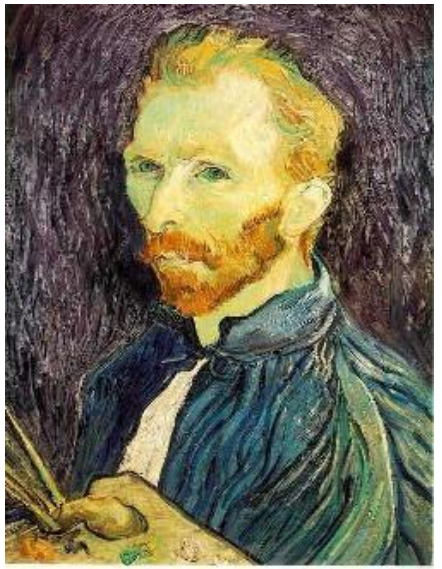
$$f[n,m] \rightarrow \boxed{\text{System } \mathcal{S}} \rightarrow g[n,m]$$

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

De-noising

Salt and pepper noise

Super-resolution



In-painting

Bertamio et al

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

2D discrete-space systems (**filters**)

A digital filter is a system that performs mathematical operations on a sampled, discrete-time signal. [Wikipedia]

- S is the **system operator**, defined as a mapping or assignment of a member of the set of possible outputs g[n,m] to each member of the set of possible inputs f[n,m].

$$f[n,m] \rightarrow \boxed{\text{System } \mathcal{S}} \rightarrow g[n,m]$$

$$g = \mathcal{S}[f], \quad g[n,m] = \mathcal{S}\{f[n,m]\}$$

$$f[n,m] \xrightarrow{\mathcal{S}} g[n,m]$$

Adapted from slides by Juan Carlos N

Filter example #1: Moving Average

2D discrete space moving average over a 3 × 3 window of neighborhood

$$g[n,m] = \frac{1}{9} \sum_{k=n-1}^{n+1} \sum_{l=m-1}^{m+1} f[k,l]$$

$$= \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} f[n-k, m-l]$$

h

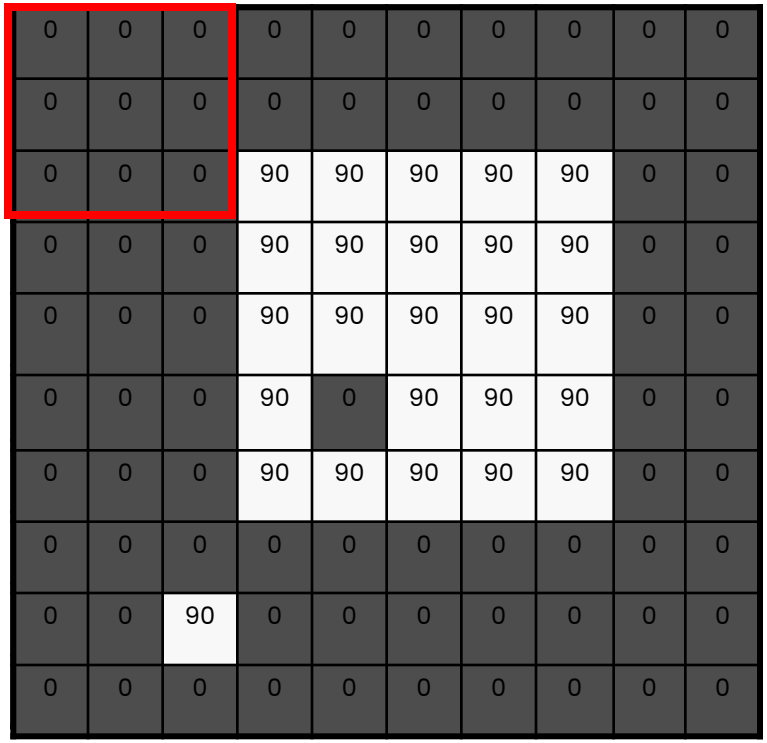
$\frac{1}{9}$

1	1	1
1	1	1
1	1	1

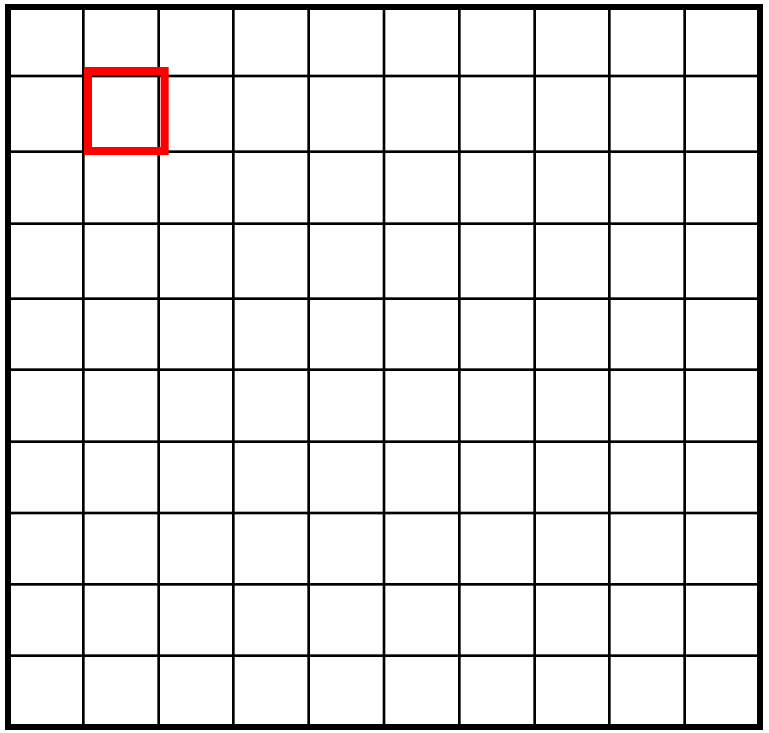
Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Filter example #1: Moving Average

$F[x, y]$



$G[x, y]$

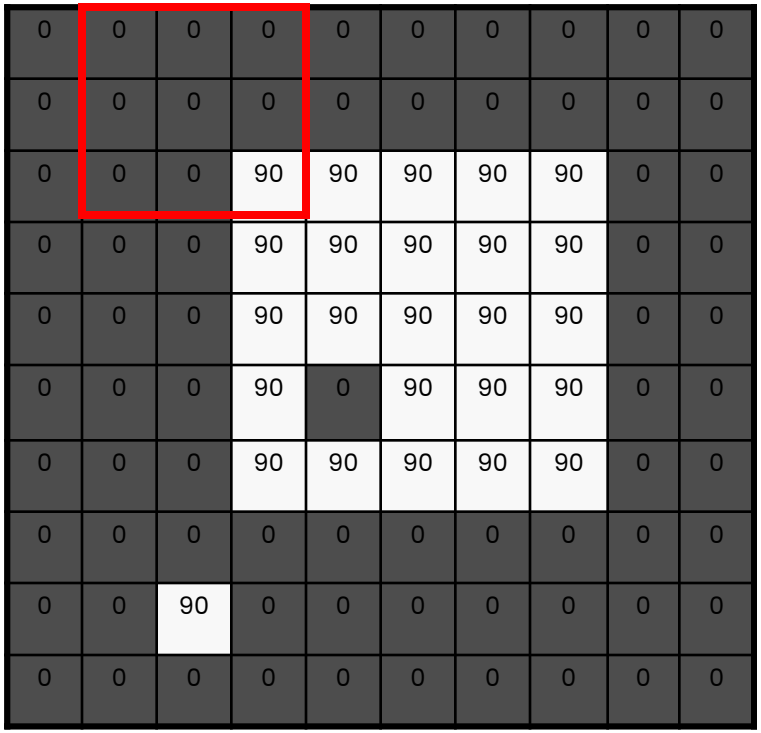


$$(f * h)[m, n] = \sum_{k,l} f[k, l]\, h[m - k, n - l]$$

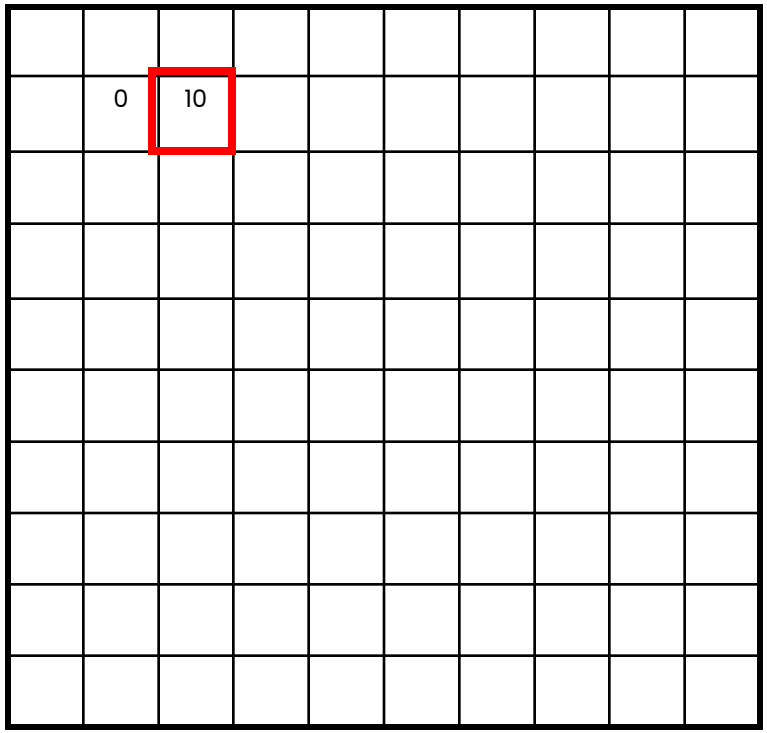
Courtesy of S. Seitz

Filter example #1: Moving Average

$F[x, y]$



$G[x, y]$



$$(f * h)[m, n] = \sum_{k,l} f[k, l]\, h[m - k, n - l]$$

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Filter example #1: Moving Average

$F[x, y]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$G[x, y]$

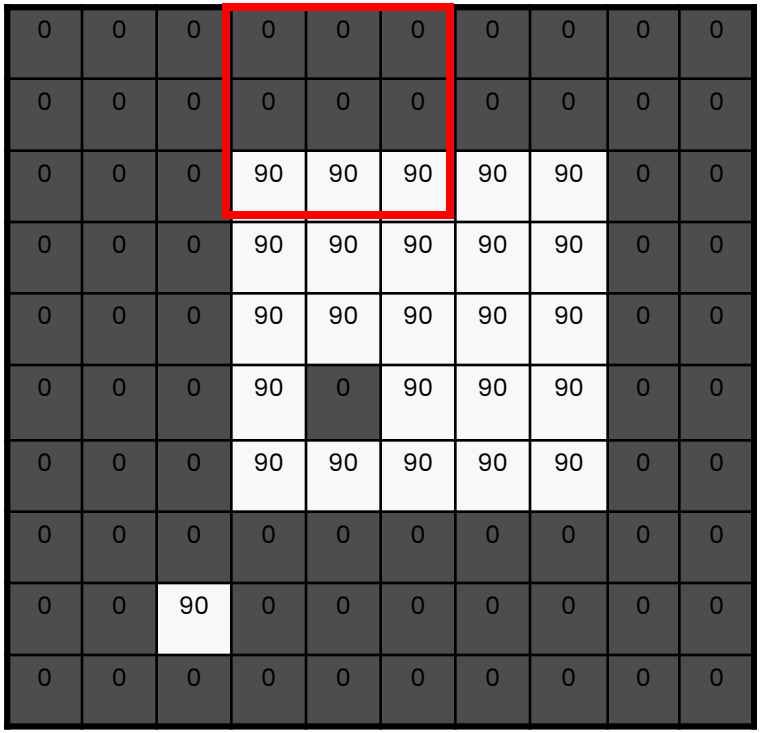
	0	10	20						

$$(f * h)[m, n] = \sum_{k,l} f[k, l]\, h[m - k, n - l]$$

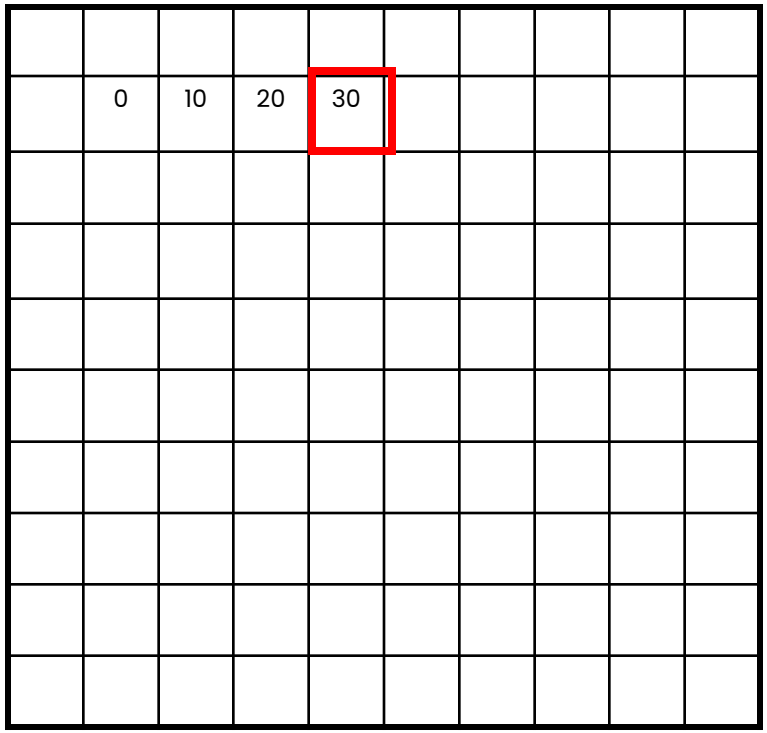
Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Filter example #1: Moving Average

$F[x, y]$



$G[x, y]$

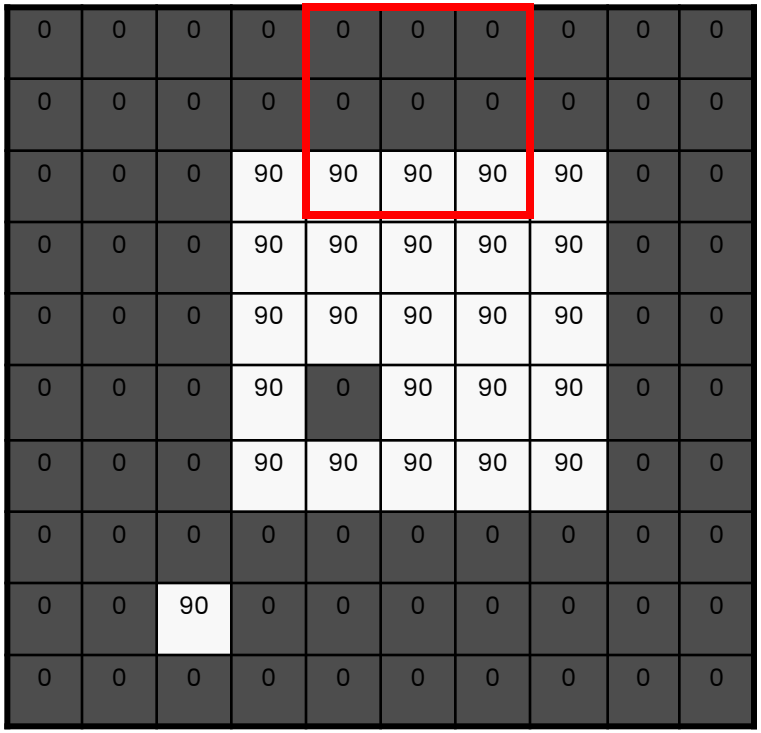


$$(f * h)[m, n] = \sum_{k,l} f[k, l]\, h[m - k, n - l]$$

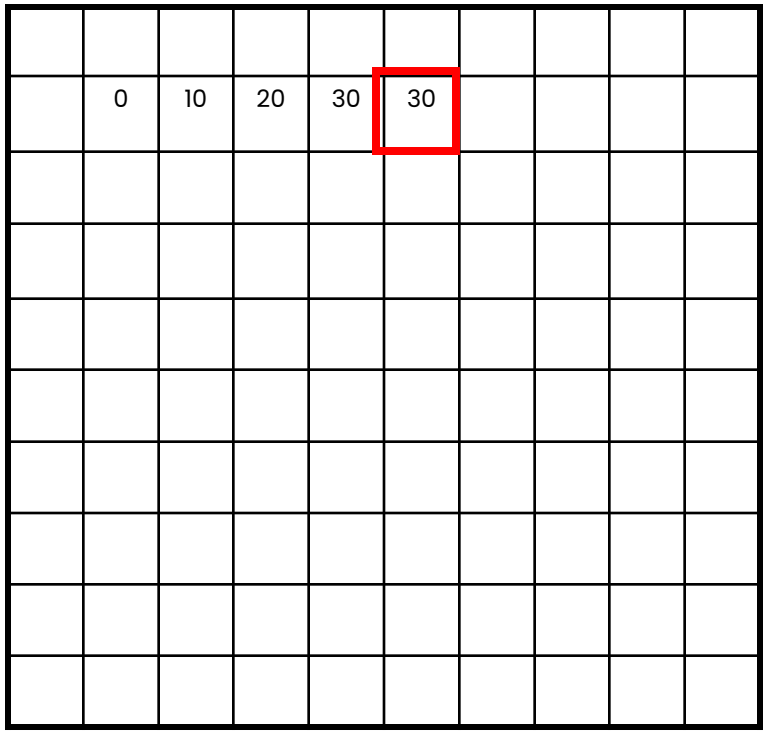
Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Filter example #1: Moving Average

$F[x, y]$



$G[x, y]$



$$(f * h)[m, n] = \sum_{k,l} f[k, l]\, h[m - k, n - l]$$

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Filter example #1: Moving Average

$F[x, y]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$G[x, y]$

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

$$(f * h)[m, n] = \sum_{k,l} f[k, l]\, h[m - k, n - l]$$

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Source: S. Seitz

Filter example #1: Moving Average

In summary:

- This filter "Replaces" each pixel with an average of its neighborhood.

- Achieve smoothing effect (remove sharp features)

$$h[\cdot,\cdot]$$

$$\frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Filter example #1: Moving Average

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Filter example #2: Thresholding

- Image segmentation based on a simple threshold:

$$g[n,m] = \begin{cases} 255, & f[n,m] > 100 \\ 0, & \text{otherwise.} \end{cases}$$

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Properties of linear systems

- Amplitude properties:
  - Additivity

$$S[f_i[n,m] + f_j[n,m]] = S[f_i[n,m]] + S[f_j[n,m]]$$

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Properties of linear systems

- Amplitude properties:
  - Additivity

$$S[f_i[n,m] + f_j[n,m]] = S[f_i[n,m]] + S[f_j[n,m]]$$

  - Homogeneity

$$S[\alpha f_i[n,m]] = \alpha S[f_i[n,m]]$$

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Properties of linear systems

- Amplitude properties:
  - Additivity

$$S[f_i[n,m] + f_j[n,m]] = S[f_i[n,m]] + S[f_j[n,m]]$$

  - Homogeneity

$$S[\alpha f_i[n,m]] = \alpha S[f_i[n,m]]]$$

  - Superposition

$$S[\alpha f_i[n,m] + \beta f_j[n,m]] = \alpha S[f_i[n,m]] + \beta S[f_j[n,m]]$$

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Properties of linear systems

- Amplitude properties:
  - Additivity

$$S[f_i[n,m] + f_j[n,m]] = S[f_i[n,m]] + S[f_j[n,m]]$$

  - Homogeneity

$$S[\alpha f_i[n,m]] = \alpha S[f_i[n,m]]]$$

  - Superposition

$$S[\alpha f_i[n,m] + \beta f_j[n,m]] = \alpha S[f_i[n,m]] + \beta S[f_j[n,m]]$$

  - Stability (bounded-input, bounded-output (BIBO) stability)

$$|f[n,m]| \leq k \implies |g[n,m]| \leq ck$$

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Properties of linear systems

- Amplitude properties:
  - Additivity [Required]

$$S[f_i[n,m] + f_j[n,m]] = S[f_i[n,m]] + S[f_j[n,m]]$$

  - Homogeneity [Required]

$$S[\alpha f_i[n,m]] = \alpha S[f_i[n,m]]]$$

  - Superposition (Generalizes additivity + homogeneity)

$$S[\alpha f_i[n,m] + \beta f_j[n,m]] = \alpha S[f_i[n,m]] + \beta S[f_j[n,m]]$$

  - Stability [extra]

$$|f[n,m]| \leq k \implies |g[n,m]| \leq ck$$

  - Invertibility [extra]

$$S^{-1}[S[f_i[n,m]]] = f[n,m]$$

Properties of linear systems

- Spatial properties
  - Causality [extra]

$$\text{for } n < n_0, m < m_0, \text{ if } f[n,m] = 0 \implies g[n,m] = 0$$

  - Shift invariance [extra]
    Idea = same input at a *later time* yields the same response

$$f[n - n_0, m - m_0] \xrightarrow{\mathcal{S}} g[n - n_0, m - m_0]$$

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Is the moving average system is shift invariant?

$$f[n,m] \quad \xrightarrow{S} \quad g[n,m] = \frac{1}{9} \sum_{k=-1}^{1} \sum_{l=-1}^{1} f[n-k, m-l]$$

$F[x,y]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$G[x,y]$

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Is the moving average system is shift invariant?

$$f[n,m] \quad \xrightarrow{\mathcal{S}} \quad g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$$

$$f[n-n_0, m-m_0]$$

$$\xrightarrow{\mathcal{S}} \quad \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[(n-n_0)-k, (m-m_0)-l]$$

$$= \quad g[n-n_0, m-m_0]$$

Yes!

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Is the moving average system is casual?

$$f[n,m] \xrightarrow{\mathcal{S}} g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$$

$F[x,y]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$G[x,y]$

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

$$\text{for } n < n_0, m < m_0, \text{ if } f[n,m] = 0 \implies g[n,m] = 0$$

Adapted from sli

Is the moving average system is casual?

$$f[n.m] \xrightarrow{S} g[n,m] = \frac{1}{9}\sum_{k=-1}^{1}\sum_{l=-1}^{1} f[n-k, m-l]$$

NO! It *looks* into future

$[x, y]$

0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	90	0	90	90	90	0	0
0	0	0	90	90	90	90	90	0	0
0	0	0	0	0	0	0	0	0	0
0	0	90	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$G[x, y]$

	0	10	20	30	30	30	20	10	
	0	20	40	60	60	60	40	20	
	0	30	60	90	90	90	60	30	
	0	30	50	80	80	90	60	30	
	0	30	50	80	80	90	60	30	
	0	20	30	50	50	60	40	20	
	10	20	30	30	30	30	20	10	
	10	10	10	0	0	0	0	0	

$$\text{for } n < n_0, m < m_0, \text{ if } f[n,m] = 0 \implies g[n,m] = 0$$

Adapted from sli

Linear Systems (filters)

$$f[n,m] \rightarrow \boxed{\text{System } \mathcal{S}} \rightarrow g[n,m]$$

- Linear filtering:
  - Form a new image whose pixels are a weighted sum of original pixel values
  - Use the same set of weights at each point

- **S** is a linear system (function) iff it *S satisfies*

$$S[\alpha f_i[n,m] + \beta f_j[h,m]] = \alpha S[f_i[n,m]] + \beta S[f_j[h,m]]$$

superposition property

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Linear Systems (filters)

$$f[n,m] \rightarrow \boxed{\text{System } \mathcal{S}} \rightarrow g[n,m]$$

- Is the moving average a linear system?

(Exercise)

- Is thresholding a linear system?
  - f1[n,m] + f2[n,m] > T
  - f1[n,m] < T
  - f2[n,m]<T

No!

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

A more interesting exercise

$y[n] = n \cdot x[n]$

- linear?
- shift/time-invariant?
- causal?
- stable?

An analysis can be found here:

https://www.youtube.com/watch?v=fO6RRXDacUw

What we will learn today?

- Images as functions
- Linear systems (filters)
- Convolution and correlation

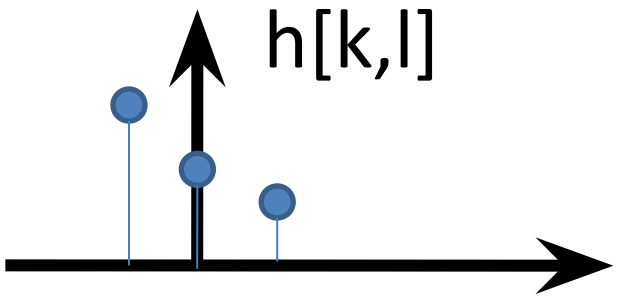
Some background reading:
Forsyth and Ponce, Computer Vision, Chapter 7

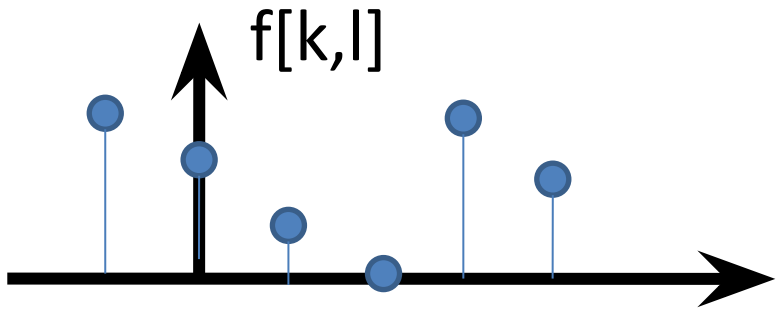
Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

1D Discrete convolution (symbol: $*$)

We are going to convolve a **function f** with a **filter h**.

$$g[n] = \sum_k f[k]h[n-k]$$





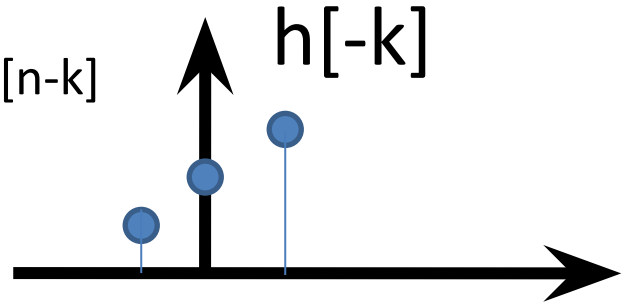
Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

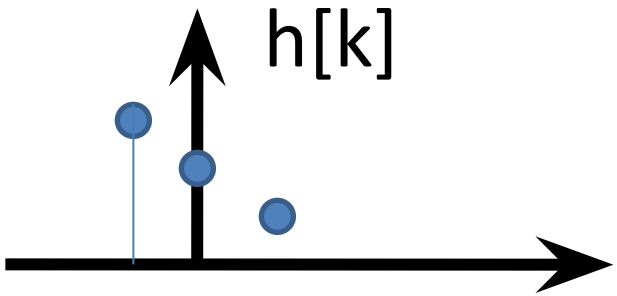
1D Discrete convolution (symbol $*$)

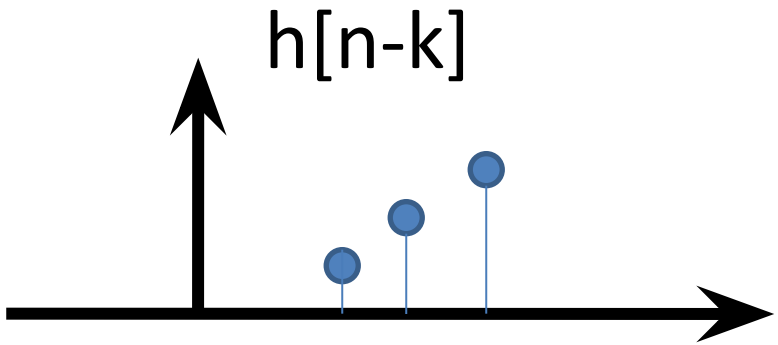
We are going to convolve a **function f** with a **filter h.**

$$g[n] = \sum_k f[k]h[n-k]$$

We first need to calculate the shifted filter h[n-k]







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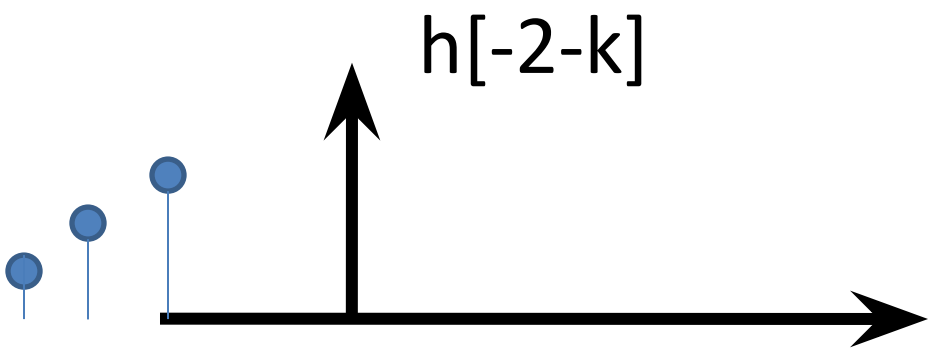
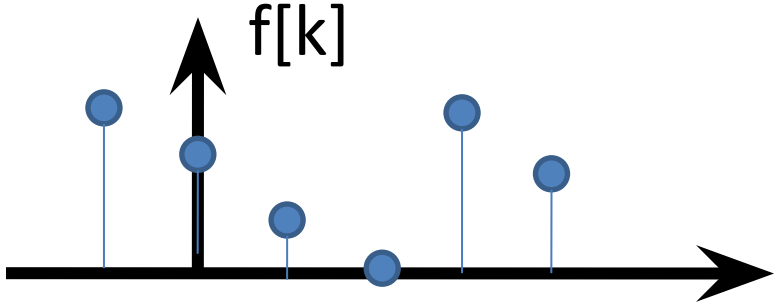
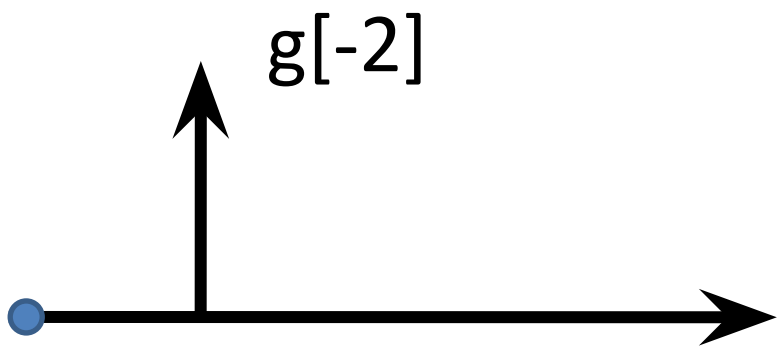
Discrete convolution (symbol: $*$)

We are going to convolve a function **f** with a filter **h.**

$$g[n] = \sum_k f[k]h[n-k]$$







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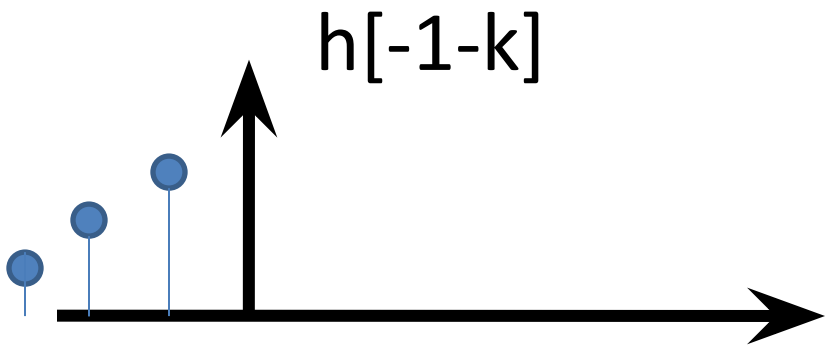
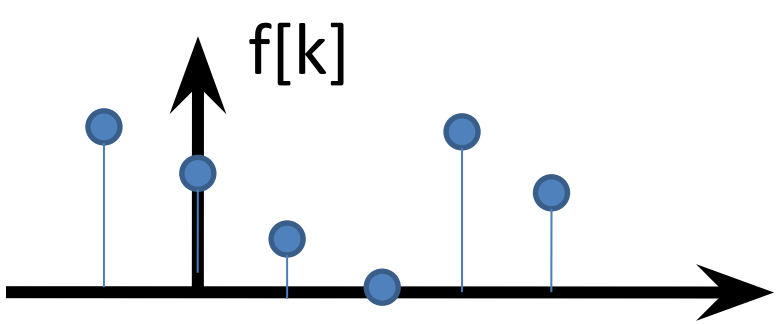
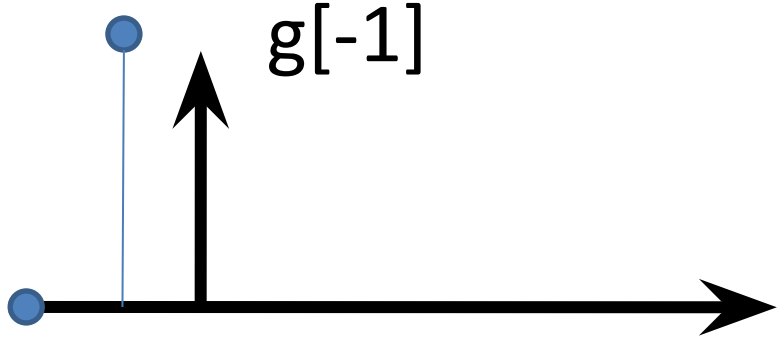
Discrete convolution (symbol: $*$)

We are going to convolve a function **f** with a filter **h.**

$$g[n] = \sum_k f[k]h[n-k]$$





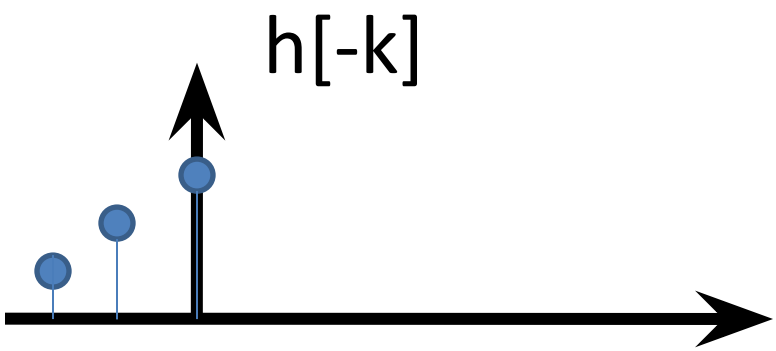


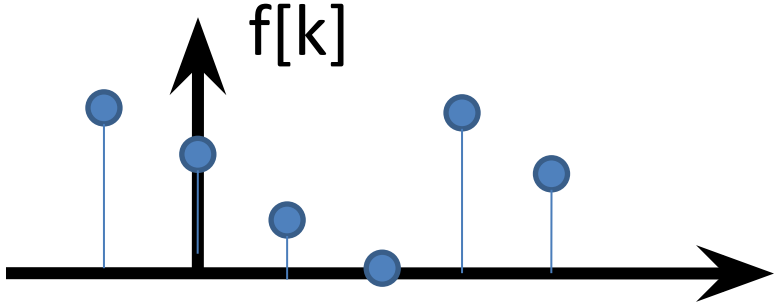
Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

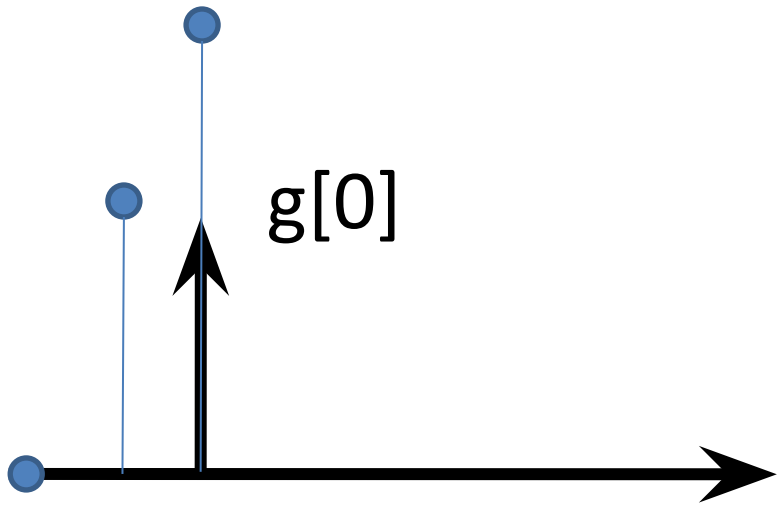
Discrete convolution (symbol: $*$)

We are going to convolve a function **f** with a filter **h.**

$$g[n] = \sum_k f[k]h[n-k]$$







Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

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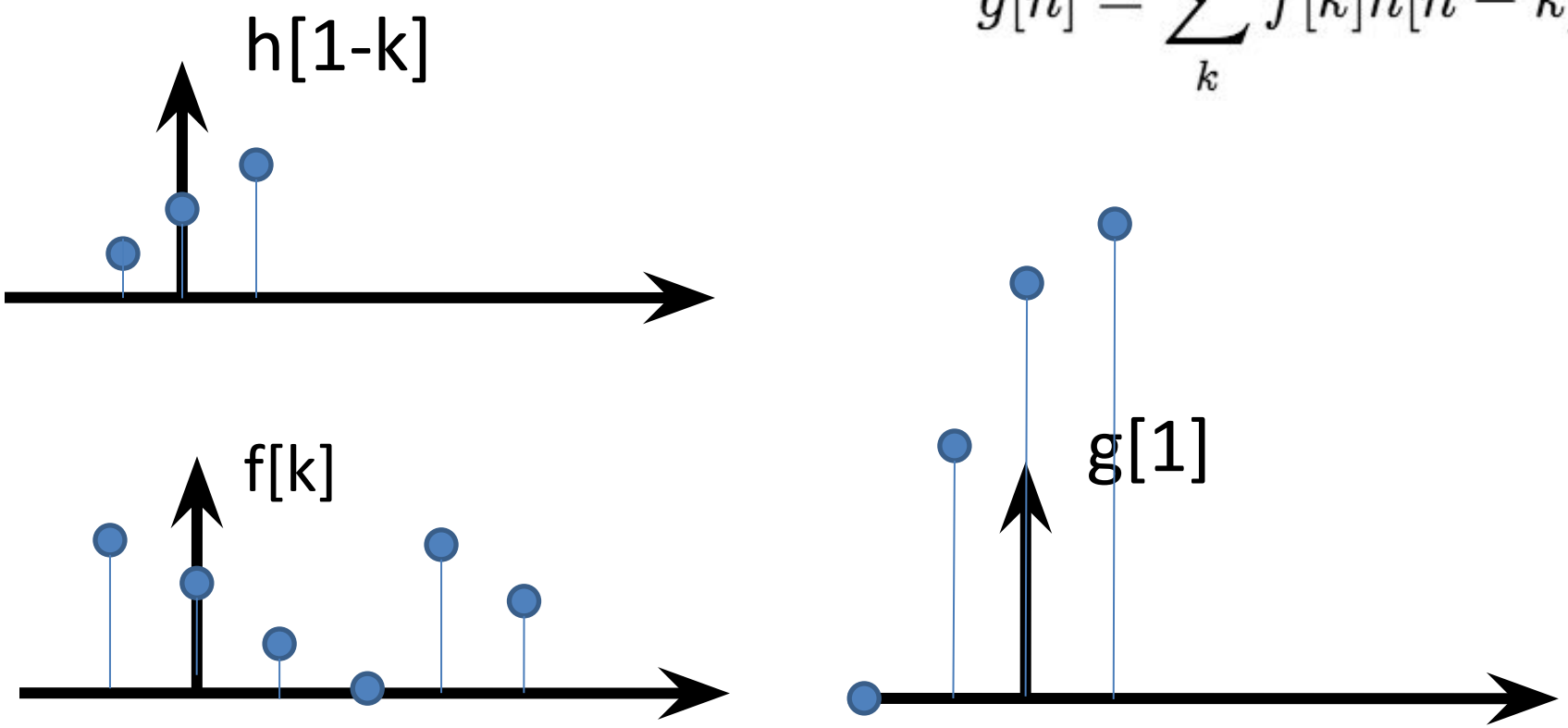

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

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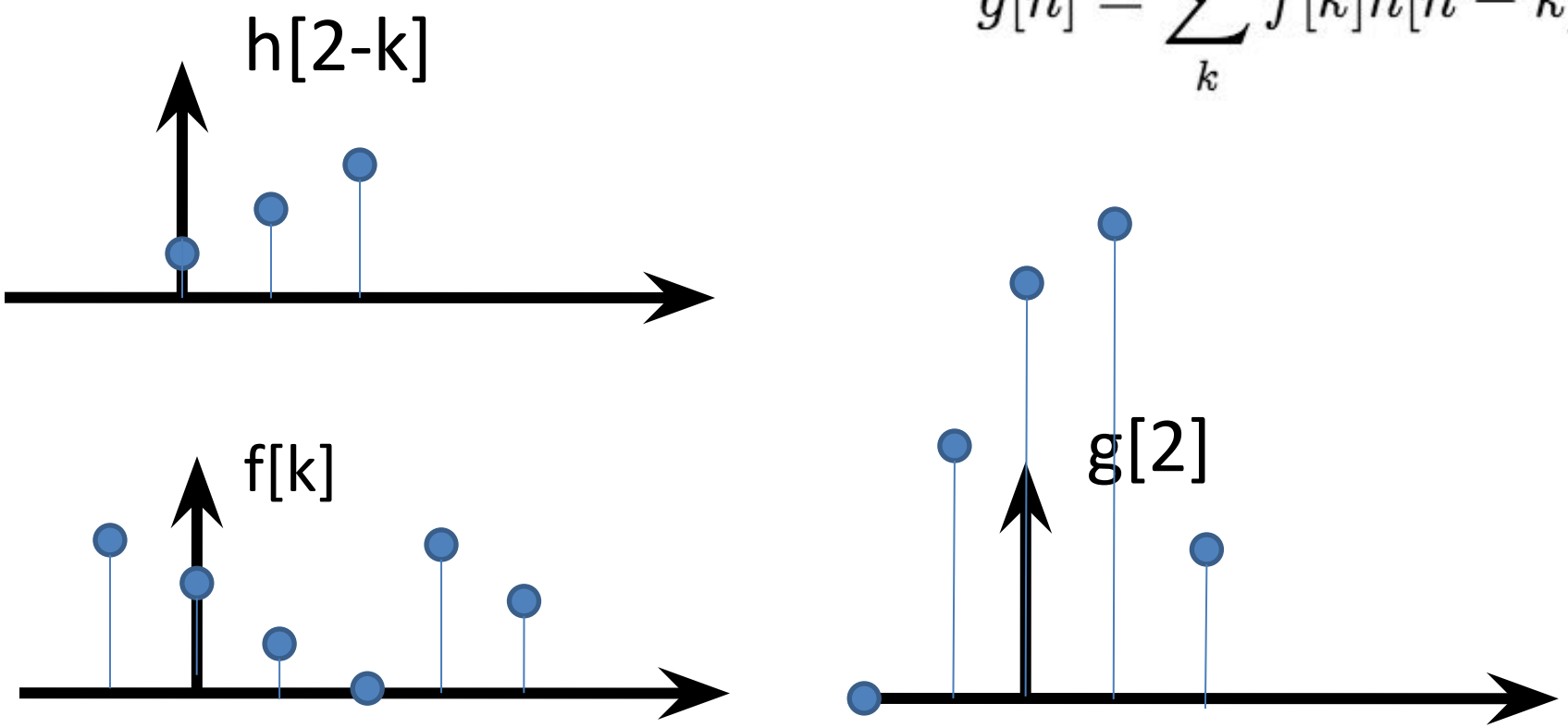

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Discrete convolution (symbol: $*$)

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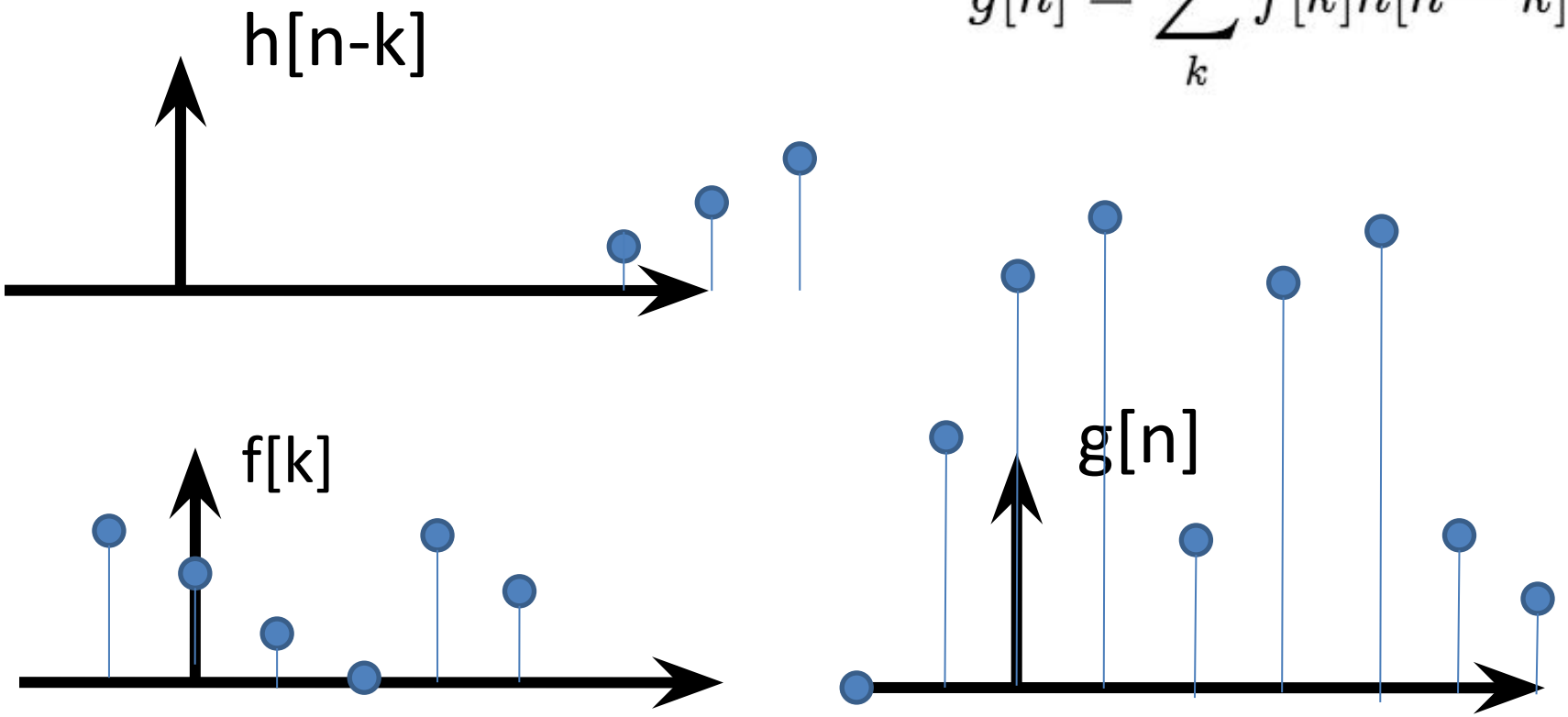

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Discrete convolution (symbol: $*$)

In summary, the steps for discrete convolution are:

- Fold h[k,l] about origin to form h[−k]
- Shift the folded results by n to form h[n − k]
- Multiply h[n − k] by f[k]
- Sum over all k
- Repeat for every n

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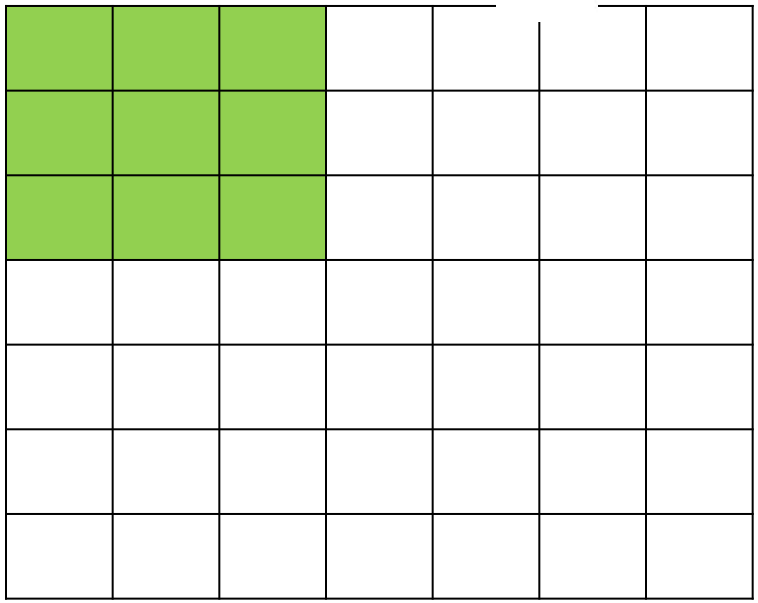
n

2D convolution

2D convolution is very similar to 1D.

- The main difference is that we now have to iterate over 2 axis instead of 1. (flip the kernel in 2 axis.)

$$f[n,m] * h[n,m] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} f[k,l]\, h[n-k, m-l]$$



Assume we have a filter(h[,]) that is 3x3. and an image (f[,]) that is 7x7.

n

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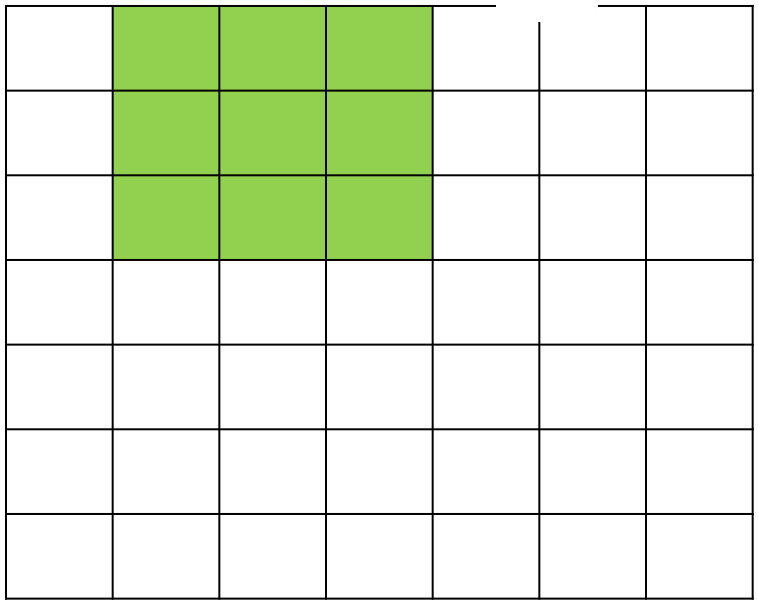

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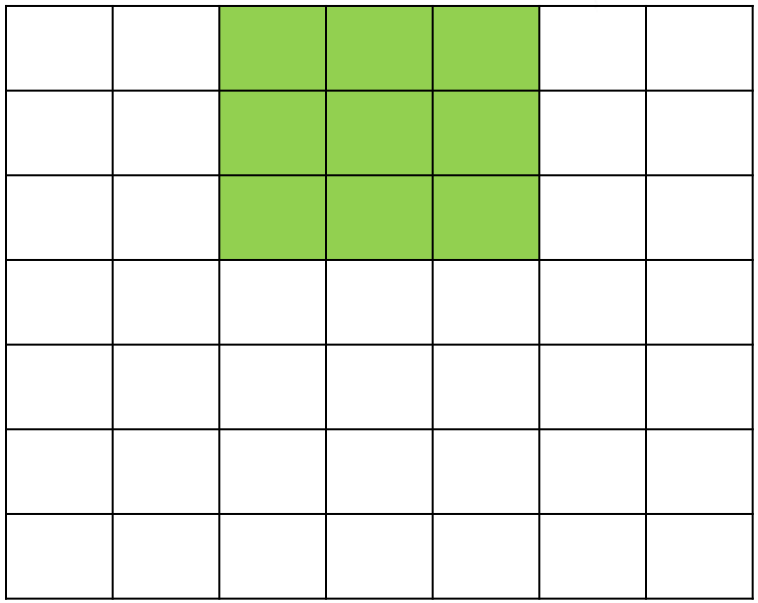

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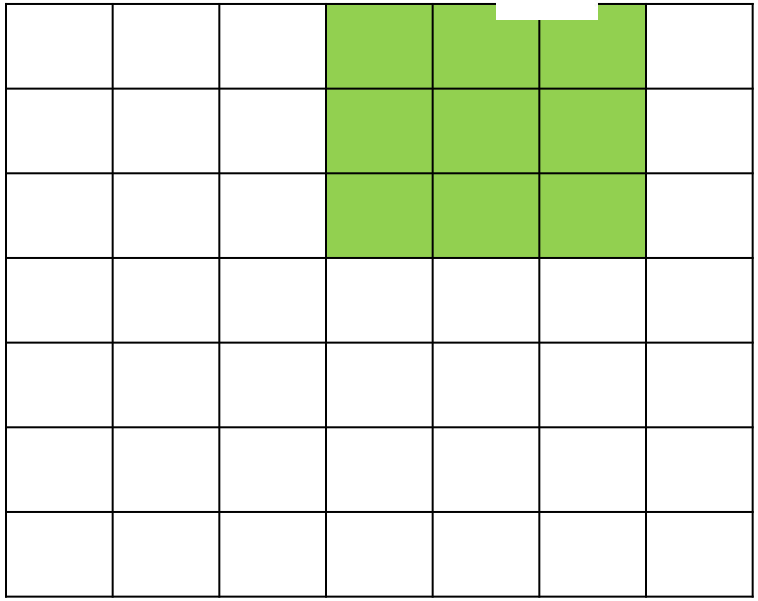

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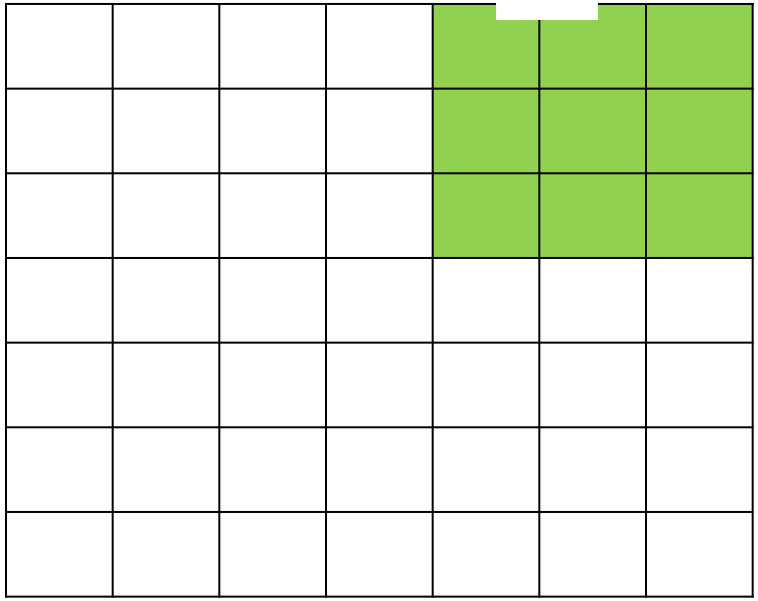

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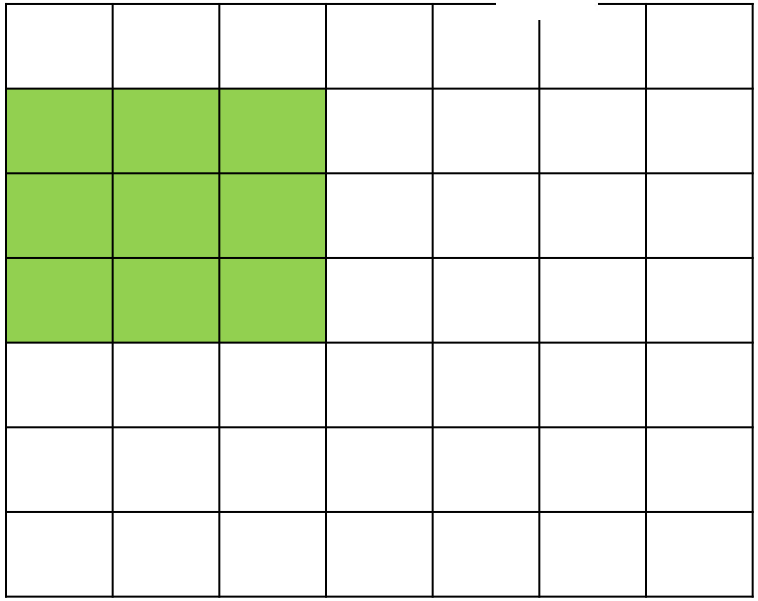
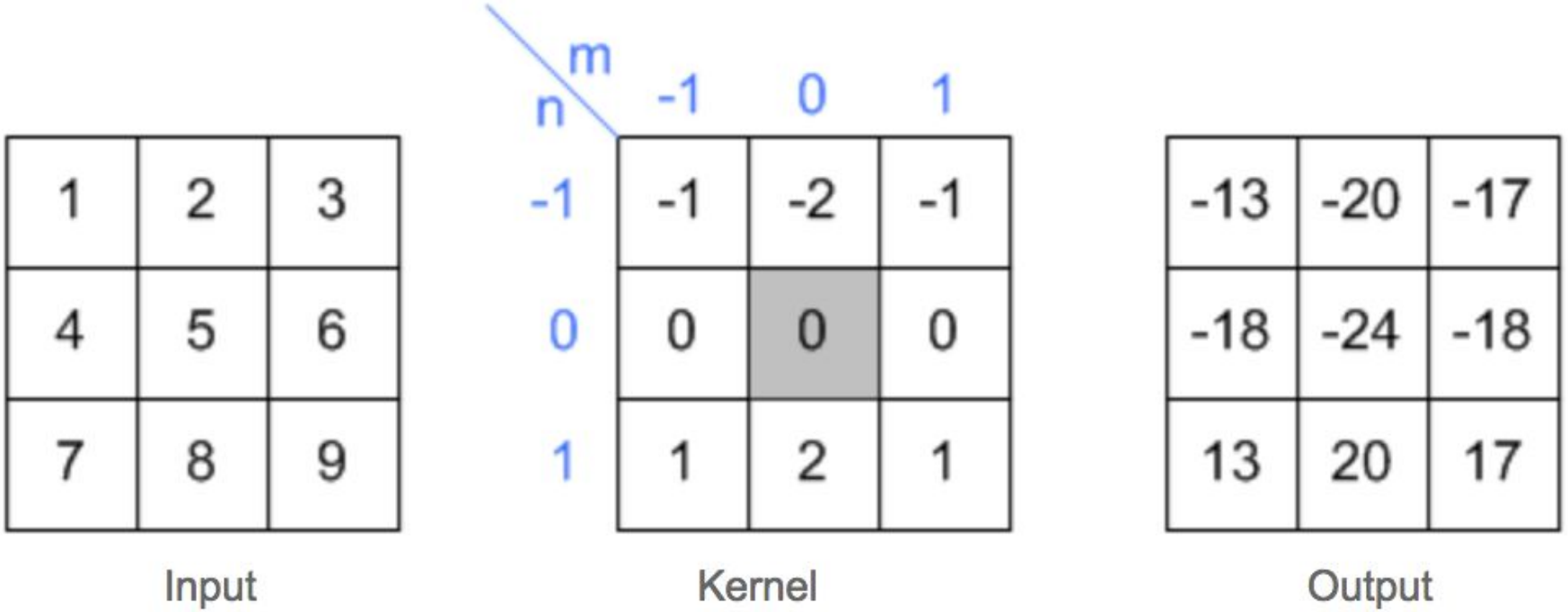


Assume we have a filter(h[,]) that is 3x3. and an image (f[,]) that is 7x7.

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

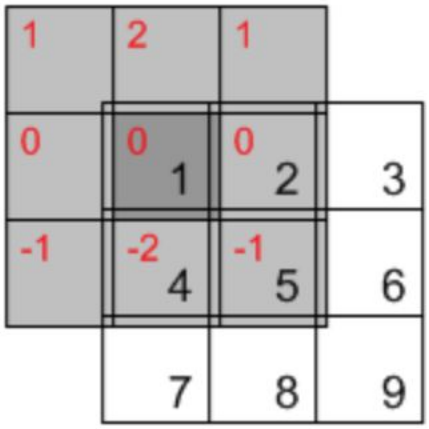
2D convolution example



Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Slide credit: Song Ho Ahn

2D convolution example



$$
\begin{aligned}
&= x[-1,-1]\cdot h[1,1] + x[0,-1]\cdot h[0,1] + x[1,-1]\cdot h[-1,1] \\
&\quad + x[-1,0]\cdot h[1,0] + x[0,0]\cdot h[0,0] + x[1,0]\cdot h[-1,0] \\
&\quad + x[-1,1]\cdot h[1,-1] + x[0,1]\cdot h[0,-1] + x[1,1]\cdot h[-1,-1] \\
&= 0\cdot 1 + 0\cdot 2 + 0\cdot 1 + 0\cdot 0 + 1\cdot 0 + 2\cdot 0 + 0\cdot(-1) + 4\cdot(-2) + 5\cdot(-1) = -13
\end{aligned}
$$

-13	-20	-17
-18	-24	-18
13	20	17

Output

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

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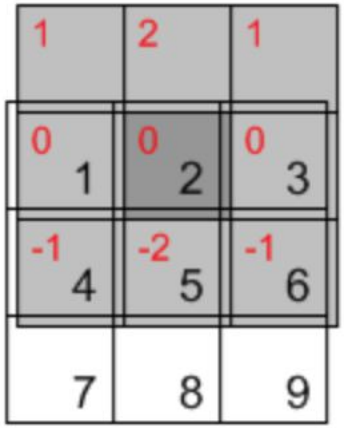
2D convolution example



$$
\begin{aligned}
&= x[0,-1]\cdot h[1,1] + x[1,-1]\cdot h[0,1] + x[2,-1]\cdot h[-1,1] \\
&\quad + x[0,0]\cdot h[1,0] + x[1,0]\cdot h[0,0] + x[2,0]\cdot h[-1,0] \\
&\quad + x[0,1]\cdot h[1,-1] + x[1,1]\cdot h[0,-1] + x[2,1]\cdot h[-1,-1] \\
&= 0\cdot 1 + 0\cdot 2 + 0\cdot 1 + 1\cdot 0 + 2\cdot 0 + 3\cdot 0 + 4\cdot(-1) + 5\cdot(-2) + 6\cdot(-1) = -20
\end{aligned}
$$

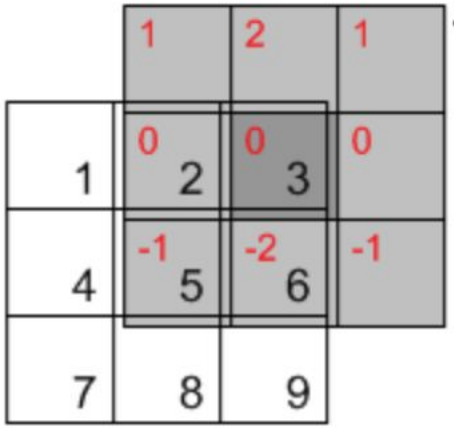
-13	-20	-17
-18	-24	-18
13	20	17

Output

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Slide credit: Song Ho Ahn

2D convolution example



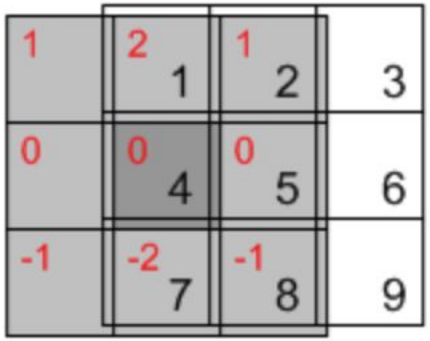
$$= x[1,-1]\cdot h[1,1] + x[2,-1]\cdot h[0,1] + x[3,-1]\cdot h[-1,1]$$
$$+ x[1,0]\cdot h[1,0] + x[2,0]\cdot h[0,0] + x[3,0]\cdot h[-1,0]$$
$$+ x[1,1]\cdot h[1,-1] + x[2,1]\cdot h[0,-1] + x[3,1]\cdot h[-1,-1]$$
$$= 0\cdot 1 + 0\cdot 2 + 0\cdot 1 + 2\cdot 0 + 3\cdot 0 + 0\cdot 0 + 5\cdot(-1) + 6\cdot(-2) + 0\cdot(-1) = -17$$

-13	-20	-17
-18	-24	-18
13	20	17

Output

Slide credit: Song Ho Ahn

2D convolution example



$$
\begin{aligned}
&= x[-1,0]\cdot h[1,1] + x[0,0]\cdot h[0,1] + x[1,0]\cdot h[-1,1] \\
&\quad + x[-1,1]\cdot h[1,0] + x[0,1]\cdot h[0,0] + x[1,1]\cdot h[-1,0] \\
&\quad + x[-1,2]\cdot h[1,-1] + x[0,2]\cdot h[0,-1] + x[1,2]\cdot h[-1,-1] \\
&= 0\cdot 1 + 1\cdot 2 + 2\cdot 1 + 0\cdot 0 + 4\cdot 0 + 5\cdot 0 + 0\cdot(-1) + 7\cdot(-2) + 8\cdot(-1) = -18
\end{aligned}
$$

-13	-20	-17
-18	-24	-18
13	20	17

Output

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

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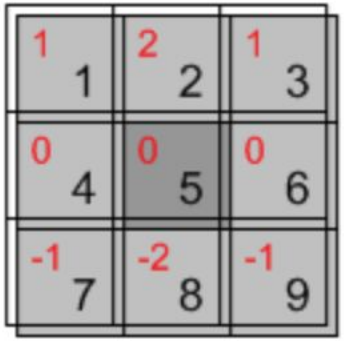
2D convolution example



$$
\begin{aligned}
&= x[0,0]\cdot h[1,1] + x[1,0]\cdot h[0,1] + x[2,0]\cdot h[-1,1] \\
&\quad + x[0,1]\cdot h[1,0] + x[1,1]\cdot h[0,0] + x[2,1]\cdot h[-1,0] \\
&\quad + x[0,2]\cdot h[1,-1] + x[1,2]\cdot h[0,-1] + x[2,2]\cdot h[-1,-1] \\
&= 1\cdot 1 + 2\cdot 2 + 3\cdot 1 + 4\cdot 0 + 5\cdot 0 + 6\cdot 0 + 7\cdot(-1) + 8\cdot(-2) + 9\cdot(-1) = -24
\end{aligned}
$$

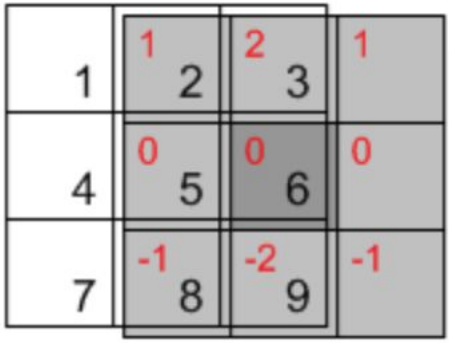
-13	-20	-17
-18	-24	-18
13	20	17

Output

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Slide credit: Song Ho Ahn

2D convolution example



$$
\begin{aligned}
&= x[1,0]\cdot h[1,1] + x[2,0]\cdot h[0,1] + x[3,0]\cdot h[-1,1] \\
&\quad + x[1,1]\cdot h[1,0] + x[2,1]\cdot h[0,0] + x[3,1]\cdot h[-1,0] \\
&\quad + x[1,2]\cdot h[1,-1] + x[2,2]\cdot h[0,-1] + x[3,2]\cdot h[-1,-1] \\
&= 2\cdot 1 + 3\cdot 2 + 0\cdot 1 + 5\cdot 0 + 6\cdot 0 + 0\cdot 0 + 8\cdot(-1) + 9\cdot(-2) + 0\cdot(-1) = -18
\end{aligned}
$$

-13	-20	-17
-18	-24	-18
13	20	17

Output

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Slide credit: Song Ho Ahn

Convolution in 2D - examples

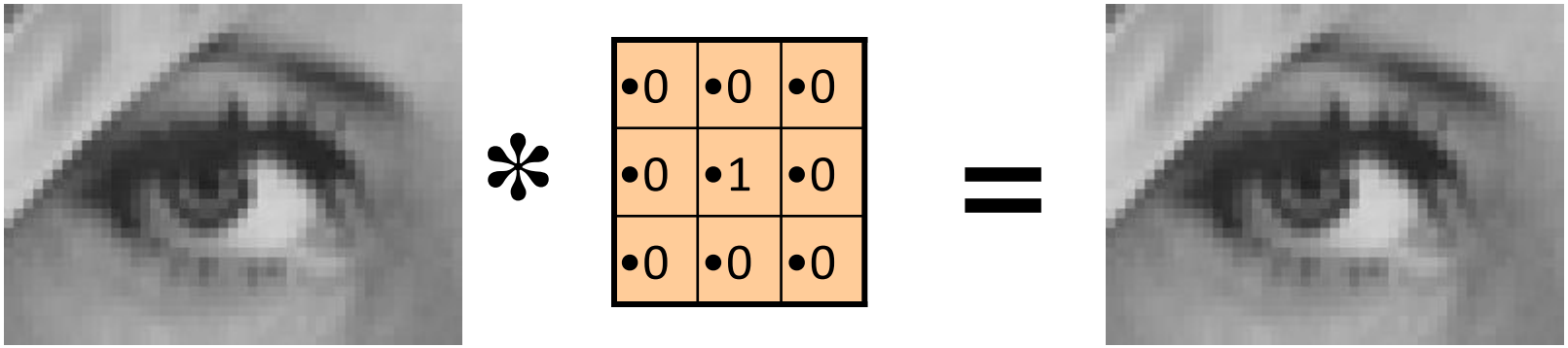
Original

$$*\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = ?$$

Courtesy of D Lowe

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Convolution in 2D - examples



Original

Filtered
(no change)

Courtesy of D Lowe

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Convolution in 2D - examples

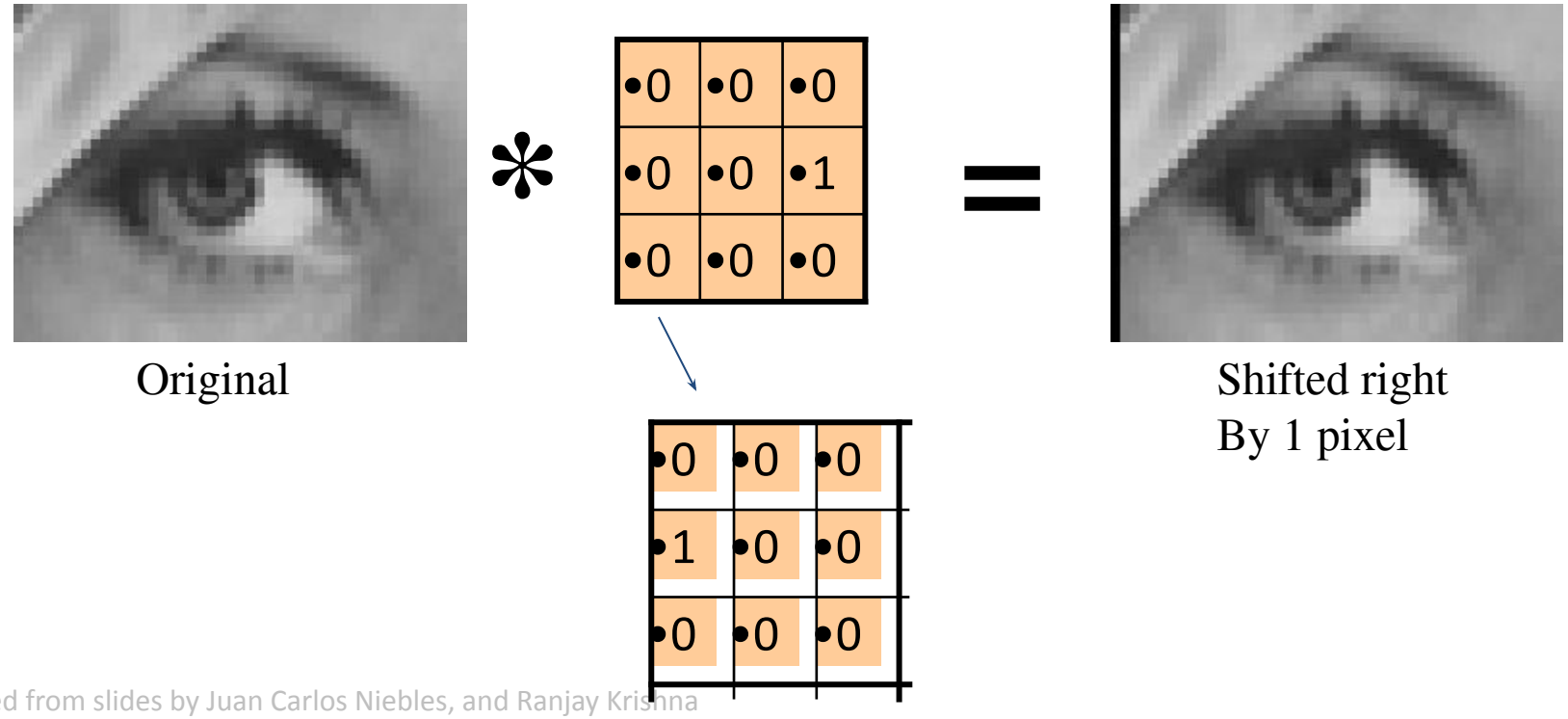
Original

$$*\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = ?$$

Courtesy of D Lowe

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Convolution in 2D - examples



Courtesy of D Lowe

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Convolution in 2D - examples

Original

$$* \frac{1}{9} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = ?$$

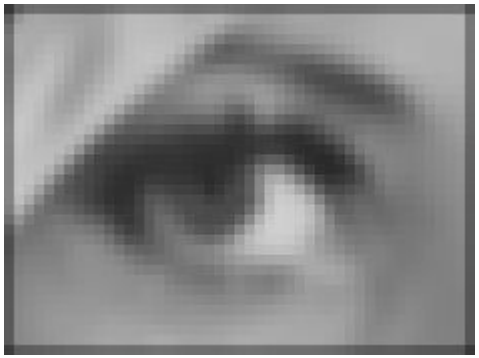
Courtesy of D Lowe

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Convolution in 2D - examples

Original

$$* \quad \frac{1}{9} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} =$$



Blur (with a box filter)

Courtesy of D Lowe

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Convolution in 2D - examples

Original

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = ?$$

(Note that filter sums to 1)

"details of the image"

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Courtesy of D Lowe

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

- What does blurring take away?

–

=

  - Let's add it back:

+ a

=

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Convolution in 2D – Sharpening filter

Original

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{1}{9}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} =$$

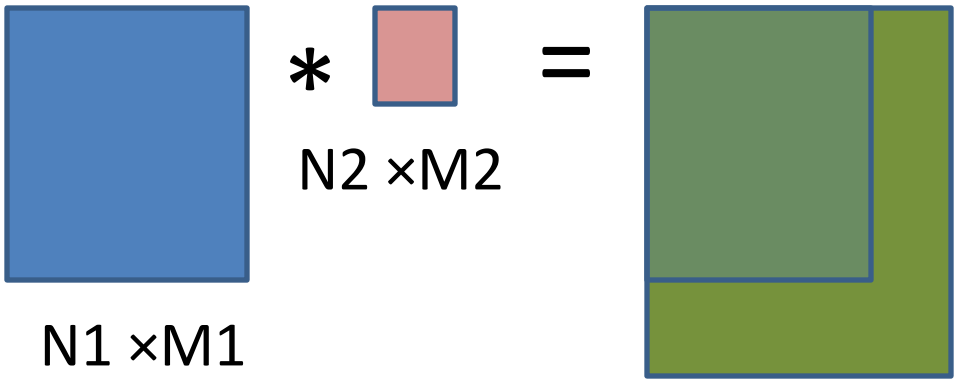
Sharpening filter: Accentuates differences with local average

Courtesy of D Lowe

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Image support and edge effect

- A computer will only convolve **finite support signals.**
  - That is: images that are zero for n,m outside some rectangular region
- numpy's convolution performs 2D convolution of finite-support signals.



(N1 - N2 + 1) × (M1 - M2 + 1)

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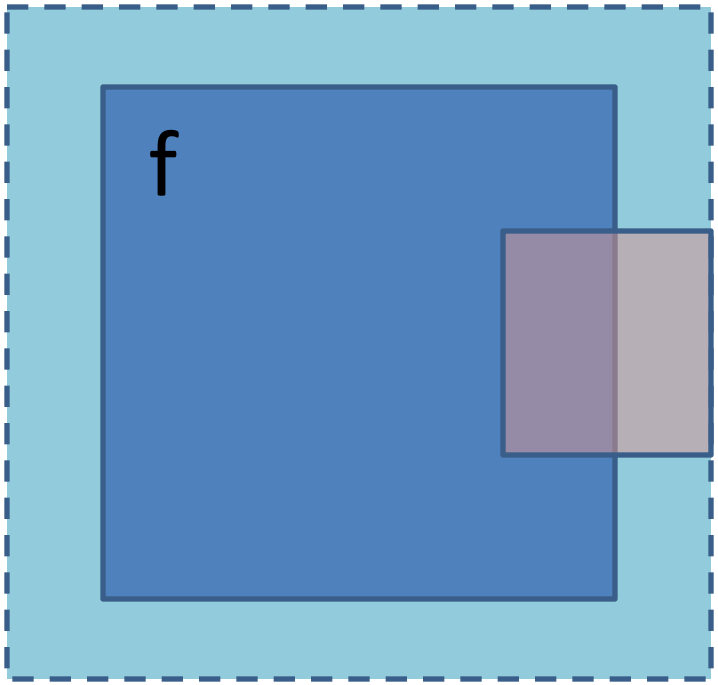
Image support and edge effect

- A computer will only convolve **finite support signals.**
- What happens at the edge?



h

- zero “padding”
- edge value replication
- flip extension
- more (beyond the scope of this class)

-> Matlab conv2 uses zero-padding

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

What we will learn today?

- Image sampling and quantization
- Image histograms
- Images as functions
- Linear systems (filters)
- Convolution and correlation

Some background reading:
Forsyth and Ponce, Computer Vision, Chapter 7

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

(Cross) correlation (symbol: $\star\star$)

Cross correlation of two 2D signals f[n,m] and g[n,m]

$$r_{fg}[k,l] \triangleq \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f[n,m]\, g^*[n-k, m-l]$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f[n+k, m+l]\, g^*[n,m], \quad k,l \in \mathbb{Z}$$

(k, l) is called the **lag**

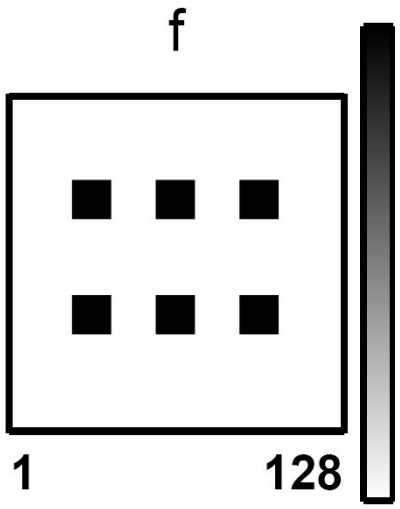
- Equivalent to a convolution without the flip

$$r_{fg}[n,m] = f[n,m] * g^*[-n,-m]$$

(g* is defined as the *complex conjugate* of g. In this class, g(n,m) are real numbers, hence g*=g.)

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

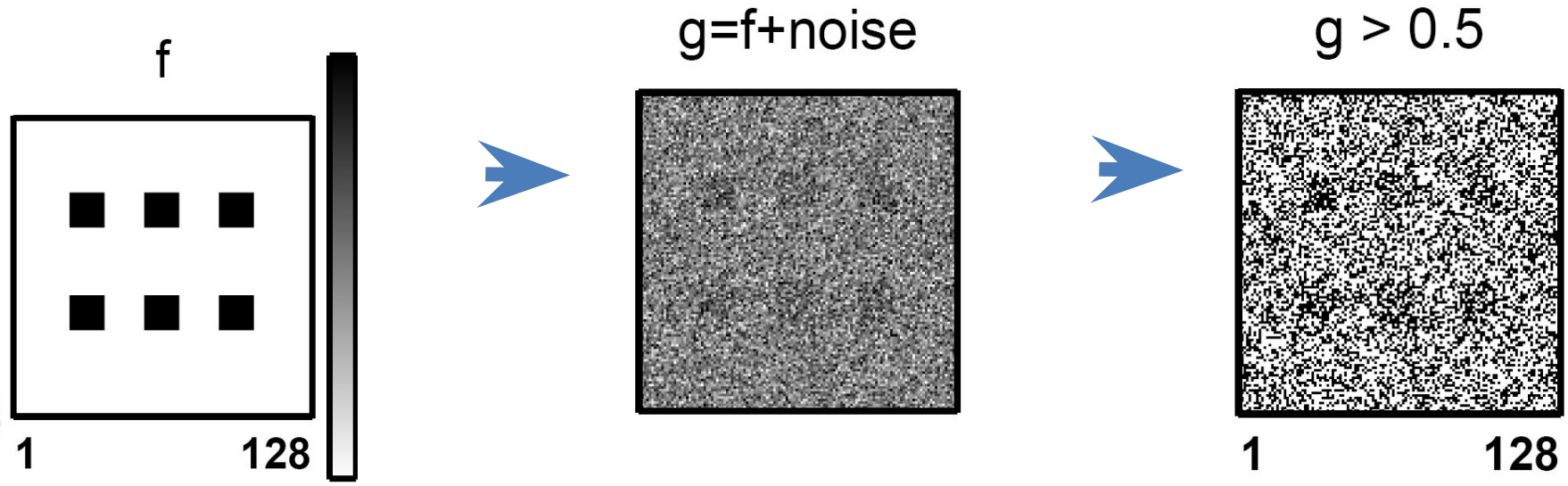
(Cross) correlation – example



Courtesy of J. Fessler

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

(Cross) correlation – example



Courtesy of J. Fessler

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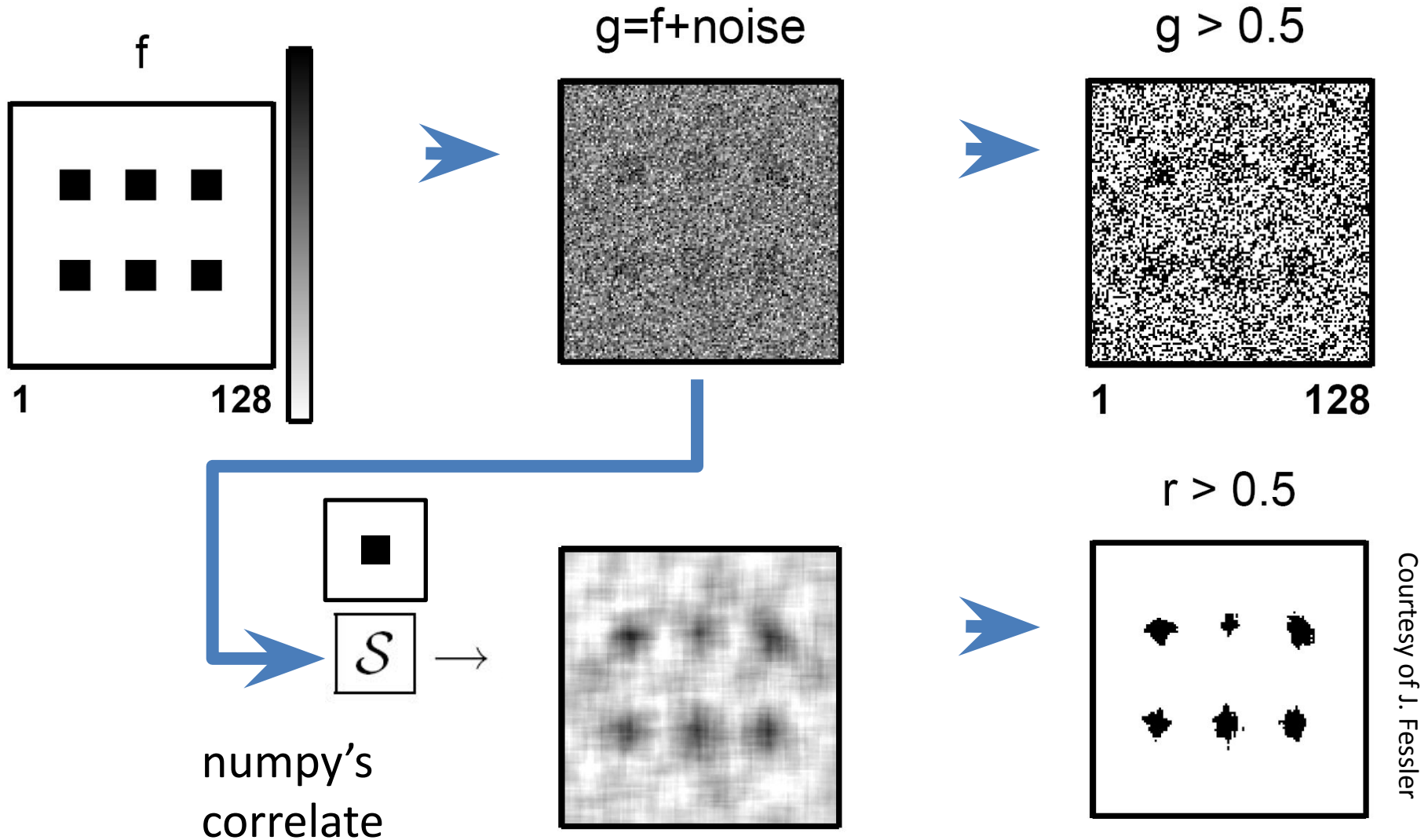
(Cross) correlation – example



Courtesy of J. Fessler

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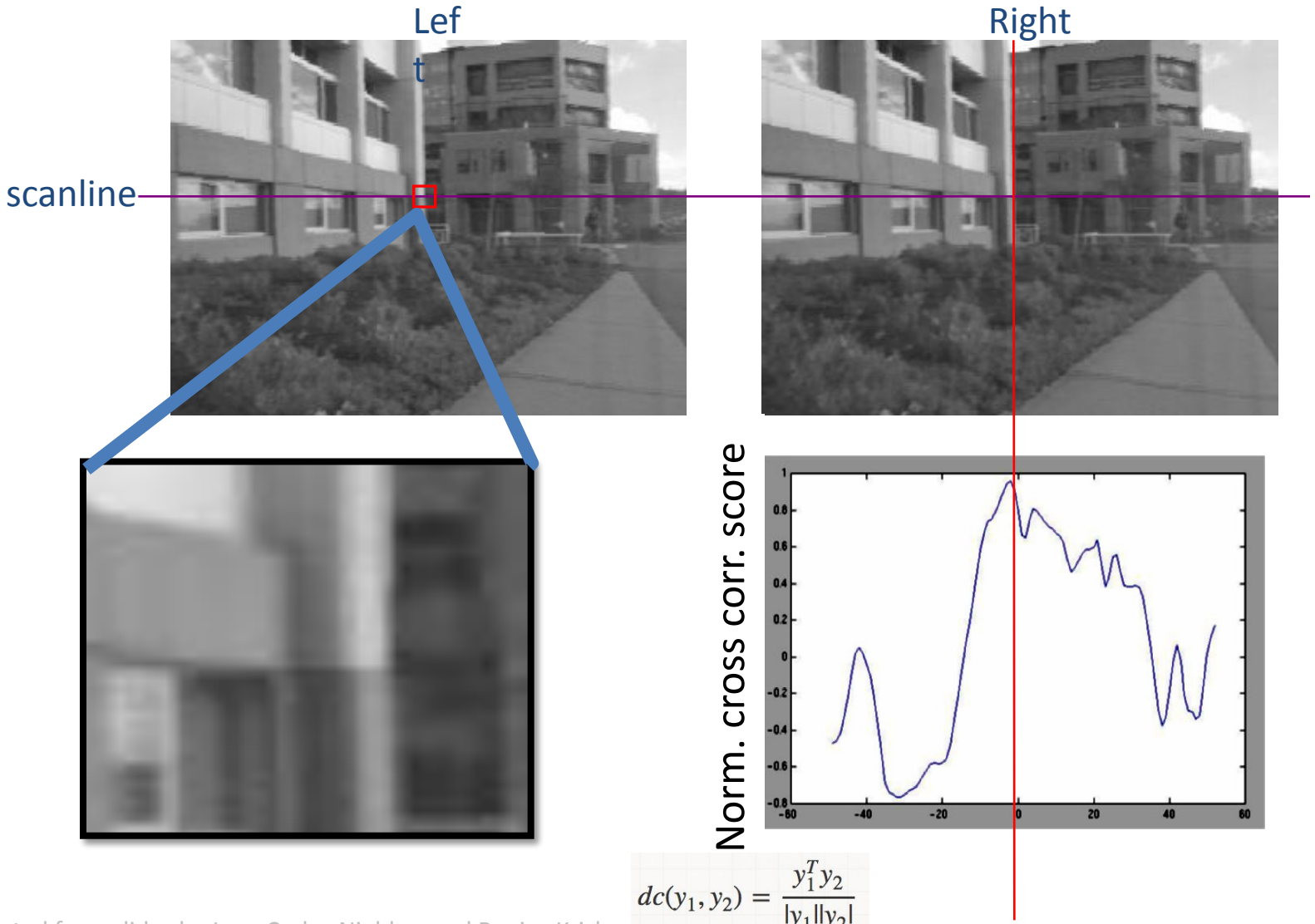
(Cross) correlation – example



$$dc(y_1, y_2) = \frac{y_1^T y_2}{|y_1||y_2|}$$

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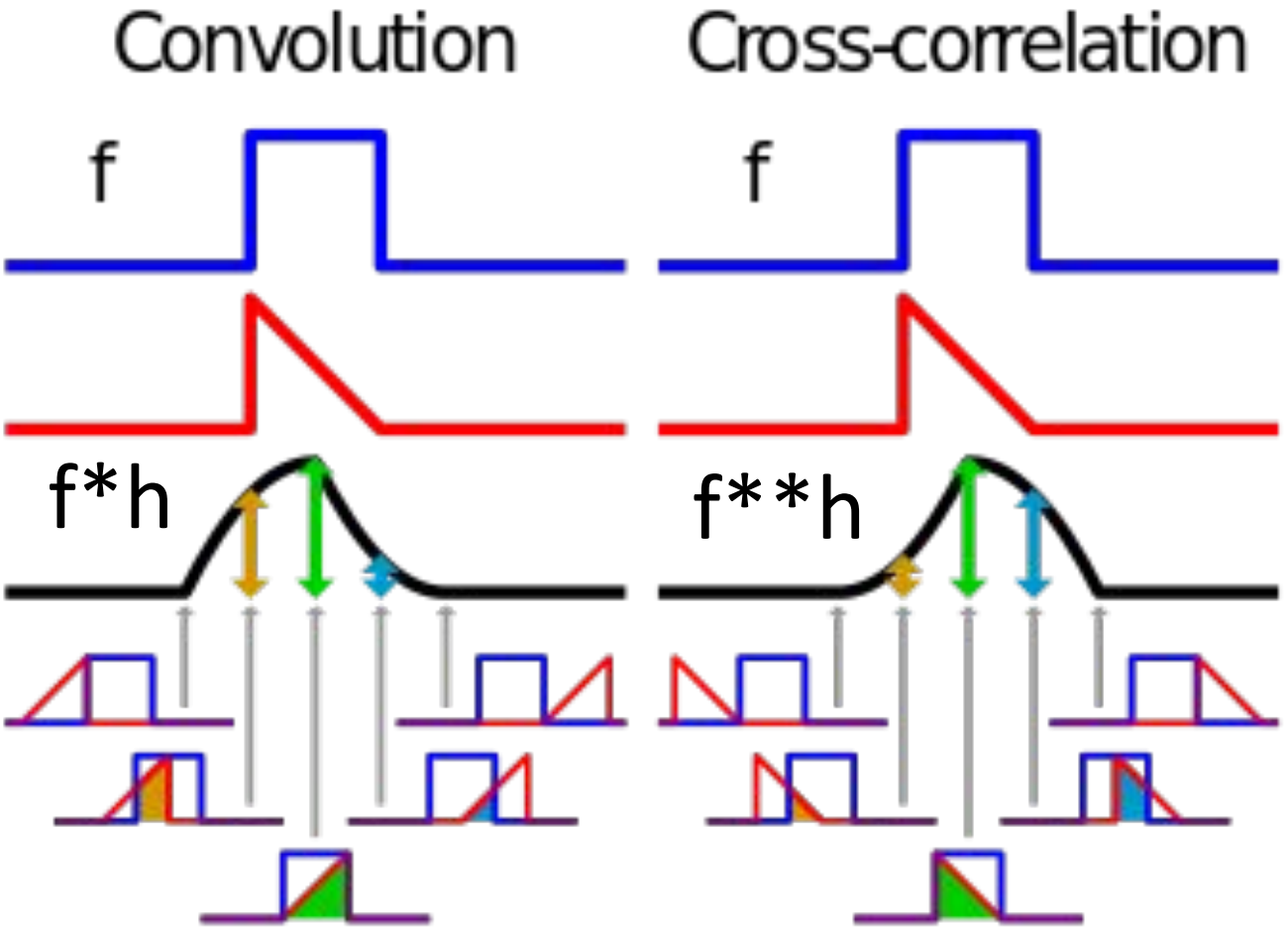
Convolution vs. (Cross) Correlation



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Cross Correlation Application: Vision system for TV remote control

- uses template matching

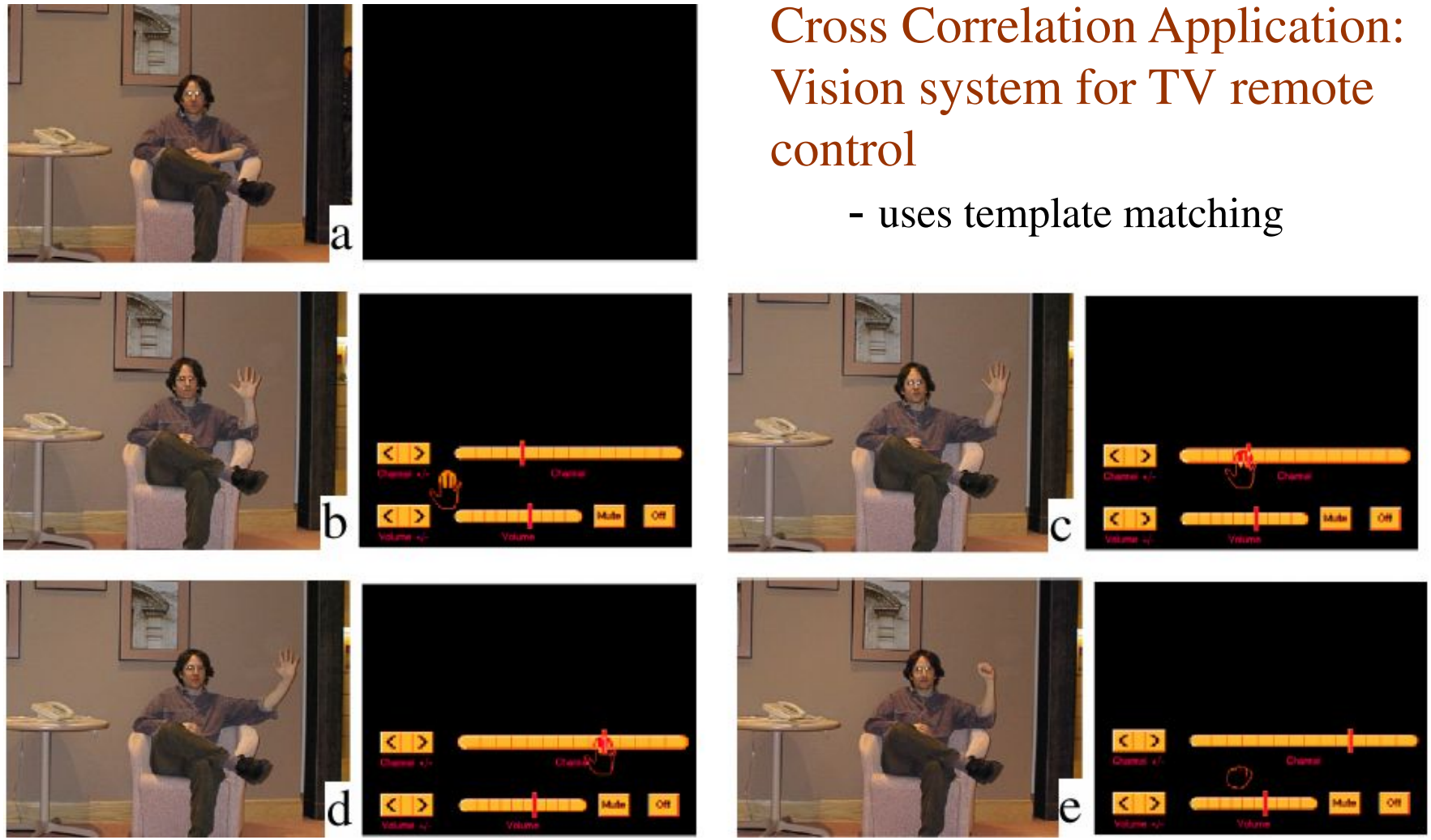

Figure from "Computer Vision for Interactive Computer Graphics," W.Freeman et al, IEEE Computer Graphics and Applications, 1998 copyright 1998, IEEE

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Convolution vs. (Cross) Correlation

- A **convolution** is an integral that expresses the amount of overlap of one function as it is shifted over another function.
  - convolution is a filtering operation

- **Correlation** compares the ***similarity*** *of* ***two*** *sets of* ***data.*** Correlation computes a measure of similarity of two input signals as they are shifted by one another. The correlation result reaches a maximum at the time when the two signals match best .
  - correlation is a measure of relatedness of two signals

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Two operations are very similar, why convolution?

Convolution is correlation with pre-flipping of h.
Seems more cumbersome, why bother?

Two operations are very similar, why convolution?

Convolution is correlation with pre-flipping of h.

Seems more cumbersome, why bother?

- Correlation (unlike conv) is:
  - NOT commutative

    (eg. [1 2 3] ** [0 1 0] ≠ [0 1 0] ** [1 2 3])
  - NOT associative: a**(b**c) ≠ (a**b)**c (in general)

Therefore, convolution is much more convenient for mathematical construction + close relation to Fourier transform.

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

What we have learned today?

- Image sampling and quantization
- Image histograms
- Images as functions
- Linear systems (filters)
- Convolution and correlation

Adapted from slides by Juan Carlos Niebles, and Ranjay Krishna

Optional reading

Sections 3.1 and 3.2 of Richard Szeliski's excellent book (https://szeliski.org/Book/).