The friendship problem on graphs

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Abstract

In this paper we provide a purely combinatorial proof of the Friendship Theorem, which has been first proven by P. Erdös et al. by using also algebraic methods. Moreover, we generalize this theorem in a natural way, assuming that every pair of nodes occupies $\ell \geq 2$ common neighbors. We prove that every graph, which satisfies this generalized ℓ-friendship condition, *is a regular graph.*

Keywords: Friendship Theorem, friendship graph, windmill graph, Kotzig's conjecture.

1 Introduction

A graph is called a *friendship graph* if every pair of its nodes has exactly one common neighbor. This condition is called the *friendship condition.* Furthermore, a graph is called a *windmill graph*, if it consists of $k \geq 1$ triangles, which have a unique common node, known as the "politician". Clearly, any windmill graph is a friendship graph. Erdös et al. [1] were the first who proved the Friendship Theorem on graphs:

Theorem 1 (Friendship Theorem) *Every friendship graph is a windmill graph.*

The proof of Erdös et al. used both combinatorial and algebraic methods [1]. Due to the importance of this theorem in various disciplines and applications except graph theory, such as in the field of block designs and coding theory [2], as well as in the set theory [3], several different approaches have been used to provide a simpler proof.

In 1971, Wilf provided a geometric proof of the Friendship Theorem by using projective planes [4], while in 1972, Longyear and Parsons gave a proof by counting neighbors, walks and cycles in regular graphs [3]. Both Longyear et al. and Wilf refer to an unpublished proof of G. Higman in lecture form at a conference on combinatorics in 1969; however, to the best of our knowledge, no known printed article of this proof exists. Hammersley avoided the use of eigenvalues and provided in 1983 a proof using numerical techniques [5]. He extended the Friendship Theorem to the so called "love problem", where self loops are allowed. In 2001, Aigner and Ziegler mentioned the Friendship Theorem in [6] as one of the greatest theorems of Erdös of all time. In the same year, West gave a proof similar to that in [3], counting common neighbors and cycles [7]. Finally, Huneke gave in 2002 two proofs, one being more combinatorial and one that combines combinatorics and linear algebra [8].

The friendship condition can be rewritten as follows: "For every pair of nodes, there is exactly one path of length two between them". In this direction, the friendship problem can be generalized as follows: *Find all graphs, in which every pair of nodes is connected with exactly ℓ paths of length k.* Such graphs are called *ℓ-regularly k-path connected* graphs, or simply $P_\ell(k)$-graphs [9]. The Friendship Theorem implies that the $P_1(2)$-graphs are exactly the windmill graphs. For the case of $P_1(k)$-graphs, where $k > 2$, Kotzig conjectured in 1974 that there exists no such graph (*Kotzig's conjecture*) [10] and he proved this conjecture for $3 \leq k \leq 8$ [11]. Kostochka proved in 1988 that the conjecture is true for $k \leq 20$ [12]. Furthermore, Xing and Hu proved the Kotzig's conjecture in 1994 for $k \geq 12$ [13] and Yang et al. in 2000 for the cases $k = 9, 10$ and 11 [14]. Thus, the Kotzig's conjecture is valid now as a theorem.

In Section 2 of this paper we propose a simple purely combinatorial proof of the Friendship Theorem. At first step, we prove that any graph G satisfying the friendship condition is a windmill graph, under the assumption that G has at least one node of degree at most two. At second step, we prove that G is a regular graph in the case that all its nodes have degree greater than two. Finally, we prove by contradiction that G has always a node of degree two, following a counting argument similar to [3].

In Section 3, we generalize the friendship condition in a natural way to the *ℓ-friendship condition*: "Every pair of nodes has exactly $\ell \geq 2$ common neighbors". The graphs that satisfy the ℓ-friendship condition are exactly the $P_\ell(2)$-graphs and they are called *ℓ-friendship graphs*. We prove that every ℓ-friendship graph is a regular graph, for every $\ell \geq 2$. This result implies that the ℓ-friendship graphs coincide with the class of *strongly regular graphs* $srg(n, k, \lambda, \mu)$ with $\lambda = \mu = \ell$, which correspond to symmetric balanced incomplete block designs [7]. This class of graphs has been extensively studied and several non-trivial examples of them are known in the literature [15, 16]. Finally, in Section 4 we summarize the results obtained in this paper.

2 A combinatorial proof of the Friendship Theorem

In this section we propose a purely combinatorial proof of the Friendship Theorem, i.e. that every friendship graph is a windmill graph. In the following, denote by C_4 a node-simple cycle on 4 nodes, by $N(v)$ the set of neighbors of v in G and $N[v] = N(v) \cup \{v\}$.

Proposition 1 *A friendship graph G contains no C_4 as a subgraph, as well as the distance between any two nodes in G is at most two.*

Proof. If G includes C_4 as a subgraph (not necessary induced), there are two nodes v and u with at least two common neighbors, as it is illustrated in Figure 1a. This is in contradiction to the friendship condition. On the other hand, if a pair (v, u) of G has distance at least three, then v and u have no common neighbor in G, which is also a contradiction. ■

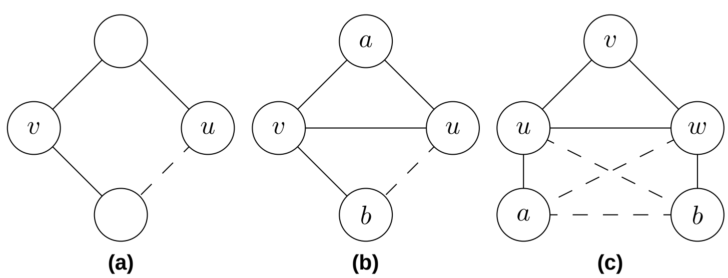

Figure 1: Three forbidden cases.

An arbitrary friendship graph has to be connected, since otherwise there are at least two nodes with no common neighbor, which is in contradiction to the friendship condition. Also, no node v of it may have $\deg(v) = 1$. Indeed, suppose otherwise that u is the unique neighbor of v. Then, v has no common neighbor with u, which is again a contradiction. It follows that $\deg(v) \geq 2$ for every node v of a friendship graph. Therefore, we may distinguish the nodes of a friendship graph by their degree, as Definition 1 states.

Definition 1 *In a friendship graph G, every node v with* $\deg(v) = 2$ *is called a* simple node*, otherwise it is called a* complex node.

Lemma 1 *For every node v of a friendship graph G, $N[v]$ induces a windmill graph.*

Proof. Consider two nodes v and $u \in N(v)$. Due to the assumption, they have a unique common neighbor a, as it is illustrated in Figure 1b. Consider now another node $b \in N(v) \setminus \{u, a\}$. If $b \in N(u)$, then G

includes a C_4 as a subgraph, which is a contradiction due to Proposition 1. Thus, $b \notin N(u)$. Since this holds for every node $b \in N(v) \setminus \{u, a\}$, it follows that every node $u \in N(v)$ produces with v exactly one triangle. Therefore, for every node v of G, $N[v]$ induces a windmill graph. ■

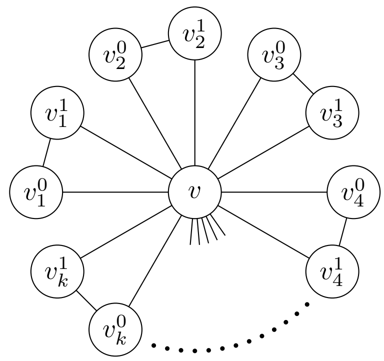

Figure 2: A non-trivial windmill graph.

Lemma 2 *If a friendship graph G has at least one simple node, then G is a windmill graph.*

Proof. Consider a simple node v of G with $N(v) = \{u, w\}$, as it is illustrated in Figure 1c. Due to Lemma 1, u and w are also neighbors. At first, since u and w have a unique common neighbor, all their neighbors are distinct, except v. In the case where G is constituted of only these three nodes, G is obviously a windmill graph. Otherwise, every other node of $V \setminus \{v, u, w\}$ is either neighbor of u or of w, since in the opposite case it would have no common neighbor with v, which is a contradiction. Finally, consider two nodes $a \in N(u) \setminus \{v, w\}$ and $b \in N(w) \setminus \{v, u\}$. Then, a and b are not neighbors, since otherwise u, w, b and a would induce a C_4, which is in contradiction to Proposition 1. It follows that the distance between a and b is three, which is also a contradiction. Thus, at least one node of $\{u, w\}$ is simple and the other one is neighbored to all other nodes in G. It follows that G is a windmill graph, due to Lemma 1. ■

Lemma 3 *If a friendship graph G has no simple node, then G is a regular graph.*

Proof. Suppose that all nodes of G are complex nodes, i.e. their degree is greater than two. Let v be such a node of G. Then, all the remaining nodes in $V \setminus \{v\}$ are partitioned into the sets $L = N(v)$ and $L' = V \setminus N[v]$.

Due to Lemma 1 and the assumption, $N[v]$ induces a non-trivial windmill graph, as it is illustrated in Figure 2. Suppose now that the windmill graph $N[v]$ has $k \geq 2$ triangles. Thus the graph induced by $N(v)$ is a perfect matching of size k with edges: $\{v_1^0, v_1^1\}, \{v_2^0, v_2^1\}, \ldots, \{v_k^0, v_k^1\}$. Now consider a node v_i^x of L , for some $i \in \{1, 2, \ldots, k\}$ and $x \in \{0, 1\}$. Denote by $N'(v_i^x) = N(v_i^x) \cap L'$ the set of nodes of the windmill graph $N[v_i^x]$ that belong to L', as it is illustrated in Figure 3. Due to the assumption it follows that $N'(v_i^x) \neq \emptyset$.

Due to the windmill structure of $N[v_i^x]$, $N'(v_i^x)$ constitutes a perfect matching of $k_i^x \geq 1$ pairs of nodes in L', denoted by $P_\ell(v_i^x)$, $\ell = 1, 2, \ldots, k_i^x$. Clearly, there is no edge connecting two nodes from two different pairs $P_a(v_i^x)$ and $P_b(v_i^x)$, since otherwise there exists a C_4, which is a contradiction due to Proposition 1. Similarly, an arbitrary node in $N'(v_i^x)$ does not have any other neighbor in L except v_i^x, since otherwise there exists again a C_4. Define now the i^{th} *block* $B_i := N'(v_i^0) \cup N'(v_i^1)$, as it is illustrated in Figure 3.

Since $k \geq 2$, there are at least two different blocks B_i and B_j in G. Consider now a node $q \in N'(v_j^0)$, as it is illustrated in Figure 4. Since the nodes q and v_i^0 have exactly one common neighbor, q has exactly one neighbor p in $N'(v_i^0)$. On the other hand, the only neighbor of p in $N'(v_j^0)$ is q, since otherwise p would have more than one common neighbor with v_j^0, which is a contradiction. Thus, the edges between $N'(v_i^0)$ and $N'(v_j^0)$ constitute a perfect matching. This holds similarly for the edges between $N'(v_i^x)$ and $N'(v_j^y)$ as well, where $x, y \in \{0, 1\}$ and, hence it holds $k_i^0 = k_i^1 =: k'$ for every $i \in \{1, 2, \ldots, k\}$.

Now, an arbitrary node $p \in N'(v_i^0)$ is a neighbor to *exactly* two nodes q and s of any block B_j, $j \neq i$, one in $N'(v_j^0)$ and one in $N'(v_j^1)$, as it is illustrated in Figure 4. Similarly, q and s are neighbors to exactly two

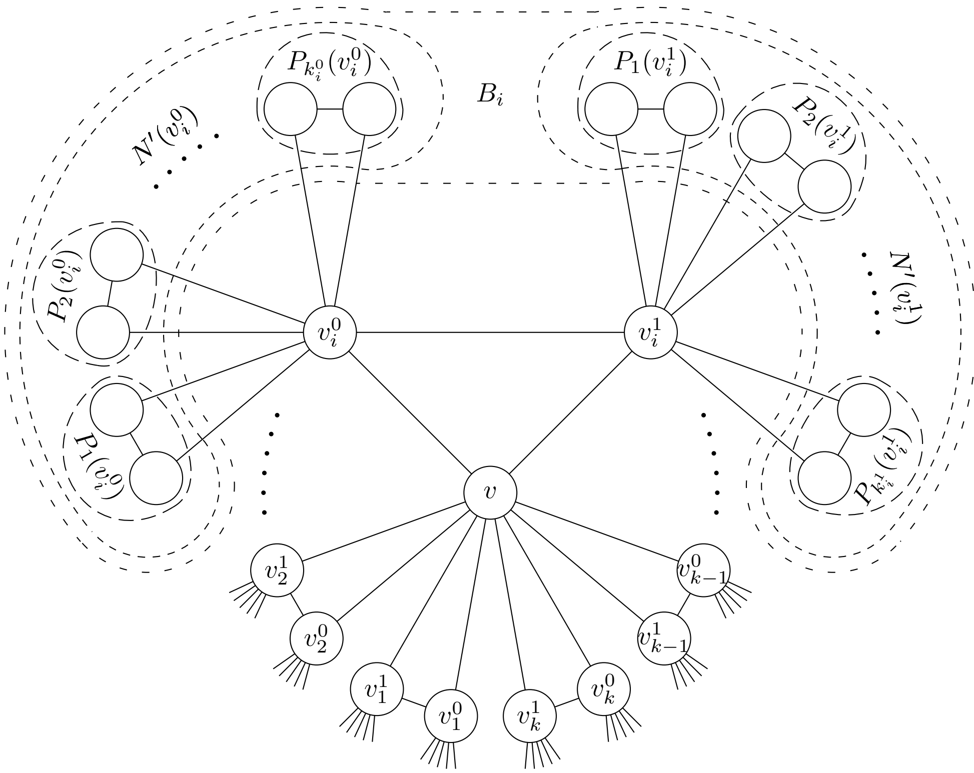

Figure 3: The $\mathrm{i^{th}}$ block $\mathrm{B_i}$.

nodes q' and s' of $N'(v_i^1)$ respectively. Consequently, since there are in total $2\,(k-1)$ sets $N'(v_j^0), N'(v_j^1)$ with $j \neq i$ and since the set $N'(v_i^1)$ has $2k'$ nodes, the assumption that p has exactly one common neighbor with every node of $N'(v_i^1)$ implies that $2\,(k-1) = 2k'$, i.e. $k' = k-1$. Thus, taking into account the two neighbors r and u_i^0 of p, it has exactly $2(k-1)+2 = 2k$ neighbors in G. Furthermore, any node v_i^x has $2k'+2 = 2k$ neighbors in G as well. Thus, since $\deg(v) = 2k$, it follows that G is a $2k$-regular graph. Finally, since the blocks B_i, $i \in \{1, 2, \ldots, k\}$ have $2k \cdot 2(k-1)$ nodes in total and since v has $2k$ neighbors, it follows that G has $n = 2k(2k-1)+1$ nodes. ∎

Lemma 4 *There is at least one simple node in any friendship graph G.*

Proof. The proof will be done by contradiction, following a counting argument similar to that used in [3]. Suppose that all nodes of G are complex, i.e. their degree is greater than two. Then, the proof of Lemma 3 implies that G is a $2k$-regular graph with $n = 2k(2k-1)+1$ nodes, for some $k \geq 2$. For an arbitrary natural number $\ell \geq 1$, let $T(\ell)$ be the set of all ordered ℓ-tuples $\langle v_1, v_2, \ldots, v_\ell \rangle$ of (not necessary distinct) nodes of G, such that v_i is neighbored with v_{i+1} for every $i \in \{1, 2, \ldots, \ell-1\}$. Since $n = 2k(2k-1)+1$, it holds that

$$|T(\ell)| = n \cdot (2k)^{\ell-1} \equiv 1 \bmod (2k-1) \tag{1}$$

for every $\ell \geq 1$. If the nodes v_ℓ and v_1 are neighbored, then the tuple $\langle v_1, v_2, \ldots, v_\ell \rangle$ constitutes a *closed ℓ-walk* in G. Let $C(\ell) \subseteq T(\ell)$ be the set of all closed ℓ-walks. Let furthermore $C^*(\ell) = \{\langle v_1, v_2, \ldots, v_{\ell-1}, v_\ell \rangle \in T(\ell) : v_\ell = v_1\}$ be the set of all closed $(\ell-1)$-walks in G.

Consider now the surjective mapping $f : C(\ell) \to T(\ell-1)$, such that $f(\langle v_1, v_2, \ldots, v_{\ell-1}, v_\ell \rangle) = \langle v_1, v_2, \ldots, v_{\ell-1} \rangle$. For every tuple $\langle v_1, v_2, \ldots, v_{\ell-1} \rangle$ of $T(\ell-1) \setminus C^*(\ell-1)$, i.e. with $v_{\ell-1} \neq v_1$, it holds that $\langle v_1, v_2, \ldots, v_{\ell-1} \rangle = f(\langle v_1, v_2, \ldots, v_{\ell-1}, y \rangle)$, where y is the unique common neighbor of $v_{\ell-1}$ and v_1 in G. On the other hand, for every tuple $\langle v_1, v_2, \ldots, v_{\ell-1} = v_1 \rangle$ of $C^*(\ell-1)$ it holds that $\langle v_1, v_2, \ldots, v_{\ell-1} = v_1 \rangle = f(\langle v_1, v_2, \ldots, v_{\ell-1} = v_1, z \rangle)$, where z is any of the $2k$ neighbors of v_1 in G. Since f

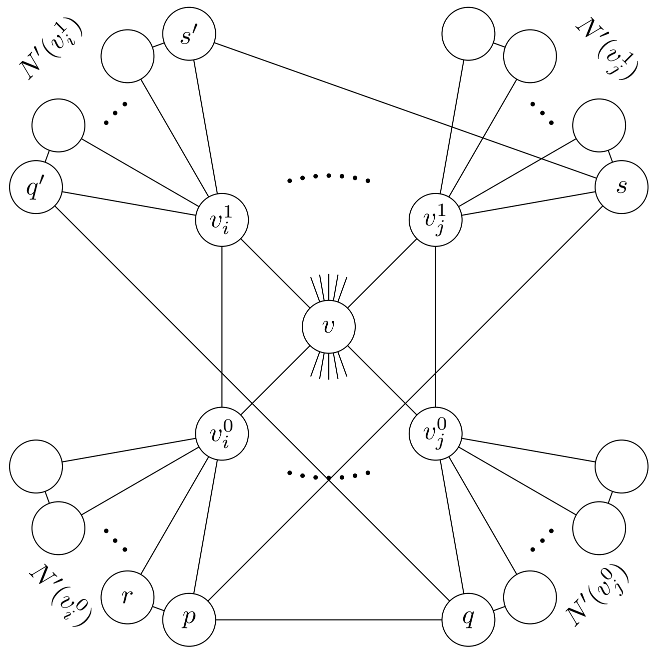

Figure 4: The regularity of the friendship graph G.

is surjective and due to (1), it follows that

$$\begin{aligned} |C(\ell)| &= 2k \cdot |C^*(\ell-1)| + |T(\ell-1) \setminus C^*(\ell-1)| \\ &\equiv |T(\ell-1)| \bmod (2k-1) \\ &\equiv 1 \bmod (2k-1) \end{aligned} \tag{2}$$

for every $\ell \geq 2$.

Now, for an arbitrary prime divisor p of $2k-1$, consider the bijective mapping (cyclic permutation) $\pi : C(p) \to C(p)$, with $\pi(\langle v_1, v_2, \ldots, v_p \rangle) = \langle v_2, \ldots, v_p, v_1 \rangle$. Since p is a prime number, all tuples $\pi^i(\langle v_1, v_2, \ldots, v_p \rangle)$, where $i \in \{1, 2, \ldots, p\}$ are distinct. The mapping π defines in a trivial way an equivalence relation: the tuples $\langle v_1, v_2, \ldots, v_p \rangle$ and $\langle w_1, w_2, \ldots, w_p \rangle$ are equivalent if there is a number $t \in \{1, 2, \ldots, p\}$, such that $\pi^t(\langle v_1, v_2, \ldots, v_p \rangle) = \langle w_1, w_2, \ldots, w_p \rangle$. This equivalence relation partitions $C(p)$ into equivalence classes of p elements each and thus, it holds that

$$C(p) \equiv 0 \bmod (p) \tag{3}$$

Since p is a prime divisor of $2k-1$, (3) is in contradiction to (2) for $\ell = p$. Thus, G is not a $2k$-regular graph and therefore it has at least one simple node. ■

The Friendship Theorem follows now directly from to Lemmas 1, 2, 3 and 4.

3 The generalized friendship problem

In this section we generalize the friendship condition, assuming that each pair of nodes occupies exactly $\ell \geq 2$ common neighbors. We prove that these graphs are d-regular, with $d \geq \ell + 1$.

Definition 2 *The condition: "Every pair of nodes has exactly ℓ common neighbors" is called the* ℓ-friendship condition*. The graphs that satisfy the ℓ-friendship condition are exactly the $P_\ell(2)$-graphs and they are called* ℓ-friendship graphs*.*

Lemma 5 *Every ℓ-friendship graph G is a regular graph, for $\ell \geq 2$.*

Proof. Consider a node $v \in V$ with $d = \deg(v)$. Similarly to Section 2, denote $L = N(v)$ and $L' = V \setminus N[v]$. Obviously, every node of the set L' has distance 2 from v. Consider now a node $a \in L$. It follows that a has exactly ℓ neighbors in L, since the pair $\{v, a\}$ has exactly ℓ common neighbors in G.

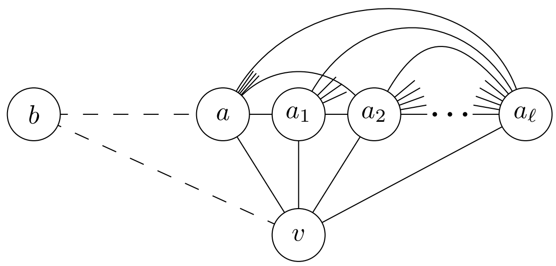

Figure 5: The case $\mathbf{L}' = \emptyset$.

Suppose at first that $L' = \emptyset$. Let $L \cap N(a) = \{a_1, a_2, \ldots, a_\ell\}$. For every $i \in \{1, 2, \ldots, \ell\}$, the pair $\{a, a_i\}$ has v as a common neighbor and $\ell - 1$ more common neighbors in L. It follows that $a_i \in N(a_j)$ for every $i \neq j \in \{1, 2, \ldots, \ell\}$, i.e. the tuple $\{v, a, a_1, \ldots, a_\ell\}$ constitutes an $(\ell+2)$-clique, as it is illustrated in Figure 5. Now, suppose that $L \setminus \{a, a_1, a_2, \ldots, a_\ell\} \neq \emptyset$ and consider a node $b \in L \setminus \{a, a_1, a_2, \ldots, a_\ell\}$. This node has no neighbor in the set $\{a, a_1, a_2, \ldots, a_\ell\}$, since otherwise at least one node of this set would have more than ℓ neighbors in L, which is a contradiction. Thus, the pair $\{a, b\}$ has v as the only common neighbor, which is also a contradiction, since $\ell \geq 2$. Therefore, if $L' = \emptyset$, then G is an $(\ell+2)$-clique and therefore an $(\ell+1)$-regular graph.

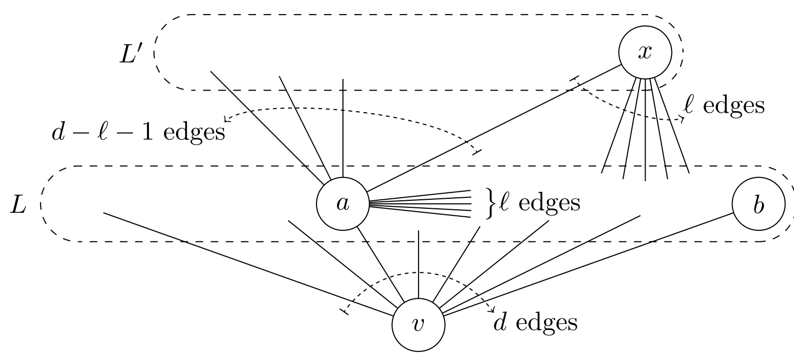

Figure 6: The case $\mathbf{L}' \neq \emptyset$.

Suppose now that $L' \neq \emptyset$. As it is illustrated in Figure 6, every node $x \in L'$ has exactly ℓ neighbors in L, since otherwise the pair $\{v, x\}$ would not have exactly ℓ common neighbors in G. If we fix the node $a \in L$, then there exist in G exactly $(d-1)\,\ell$ paths of length two with extreme nodes a and b, where $b \in L$, since there are $d - 1$ nodes $b \in L \setminus \{a\}$ and every such pair $\{a, b\}$ has exactly ℓ common neighbors in G. Among them, exactly $d - 1$ ones have v as the intermediate node. Furthermore, exactly $\ell(\ell - 1)$ ones have their intermediate node in L, since a has exactly ℓ neighbors in L and each of them has $\ell - 1$ other neighbors in L except a. Thus, each of the remaining

$$(d-1)\,\ell - (d-1) - \ell\,(\ell-1) = (d-\ell-1)\,(\ell-1)$$

paths has a node in L' as their intermediate node. Consider now a node $x \in L' \cap N(a)$. The edge between a and x is included in exactly $\ell - 1$ paths of length two with extreme nodes a and b, where $b \in L$, since x has exactly $\ell - 1$ other neighbors in L except a. Thus, every $a \in L$ is neighbored to exactly

$$\frac{(d-\ell-1)\,(\ell-1)}{(\ell-1)} = (d-\ell-1) \tag{4}$$

nodes in L'. It follows that

$$|L'| = \frac{d\,(d-\ell-1)}{\ell} \tag{5}$$

since L includes d nodes, each one of them has $d-\ell-1$ neighbors in L' and each node of L' is neighbored to ℓ nodes of L. Finally, since $|V| = |L| + |L'| + 1$ and $|L| = d$, it follows from (5) that

$$|V| = \frac{d\,(d-1)}{\ell} + 1 \tag{6}$$

Since (6) holds for the degree d of an arbitrary node $v \in V$, it results that every node v has equal degree d in G and therefore G is a d-regular graph. ■

Due to Lemma 5, the *ℓ-friendship graphs* coincide with the *strongly regular graphs* $srg(n, k, \lambda, \mu)$ with $\lambda = \mu = \ell$, which correspond to symmetric balanced incomplete block designs [7]. Several non-trivial examples of them are known in the literature, e.g. the line graph of K_6 with $n = 15, k = 8, \ell = 4$ [16], the cartesian product $K_4 \times K_4$ (or Shrikhande graph) with $n = 16, k = 6, \ell = 2$ and the halved 5-cube graph with $n = 16, k = 10, \ell = 6$, which is referred as Clebsch graph in [15].

4 Conclusion

In this paper we propose a purely combinatorial proof of the Friendship Theorem, originally proved by Erdös et al. Furthermore, we generalize the simple friendship condition in a natural way to the *ℓ-friendship condition*: "Every pair of nodes has exactly $\ell \geq 2$ common neighbors" and we prove that every graph which satisfies this condition is a regular graph. It remains open to characterize fully this class of graphs, which together with the recent proof of the Kotzig's conjecture, will complete the characterization of the graphs $P_\ell(2)$ and $P_1(k)$ that are the direct generalizations of the class $P_1(2)$ of the friendship graphs.

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