Temporal Core Decomposition

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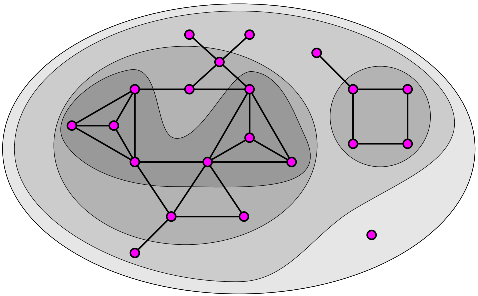
Background

Core decomposition

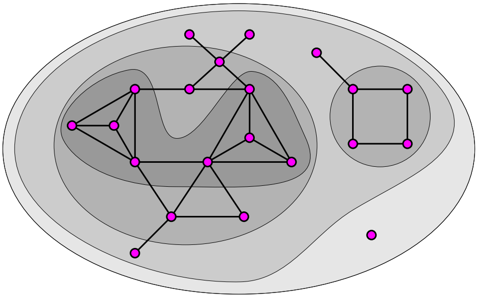
Definition

The **k-core** *(or core of order k) of a (non-temporal) graph $G = (V, E)$ is a maximal set of vertices $C_k \subseteq V$ such that $\forall u \in C_k : deg(C_k, u) \geq k$.*
The set of all k-cores $V = C_0 \supseteq C_1 \supseteq \cdots \supseteq C_{k^*}$ is the **core decomposition** *of G.*



- exact **linear-time** algorithm
- important tool to analyze and **visualize** networks
- **speed-ups** the extraction of dense subgraphs
- at the basis of **approximation algorithms** for, e.g., densest subgraph betweenness centrality

Core decomposition



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Temporal graphs

- A **temporal graph** is a representation of
  - **entities** (vertices)
  - their **relations** (links)
  - how these relations are **established/broken along time**

Temporal graphs

Definition

A **temporal graph** *is a triple* $G = (V, T, \tau)$, *where*

- V *is a set of vertices,*
- $T = [t_0, t_1, \ldots, t_{max}] \subseteq \mathbb{N}$ *is a discrete time domain,*
- $\tau : V \times V \times T \to \{0, 1\}$ *is a function defining for each* $u, v \in V$ *and each* $t \in T$ *whether edge* (u, v) *exists in* t.

t_0

t_1

$[t_0, t_1]$

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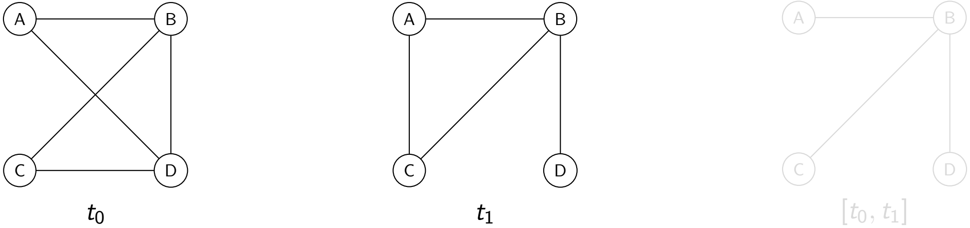

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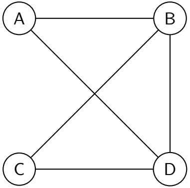
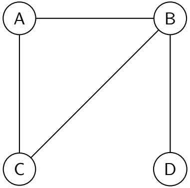
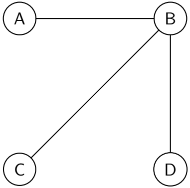


t_0



t_1



$[t_0, t_1]$

Span-core decomposition

Motivation

Extracting **dense structures** together with their **temporal span** is a key mining primitive

- anomaly detection in proximity networks
- quantify the transmission opportunities of respiratory infections
- identify events and buzzing stories
- understand the dynamics of collaboration in successful professional teams

Span-core decomposition

Definition

The $(\mathbf{k}, \boldsymbol{\Delta})$**-core** *of a temporal graph* $G = (V, T, \tau)$ *is a maximal and non-empty set of vertices* $\emptyset \neq C_{k,\Delta} \subseteq V$*, such that* $\forall u \in C_{k,\Delta} : deg_{\Delta}(C_{k,\Delta}, u) \geq k$*, where* $\Delta \sqsubseteq T$ *is a temporal interval and* $k \in \mathbb{N}^{+}$.

- $deg_{\Delta}(C_{k,\Delta}, u)$ represents the **degree of a vertex** u **in the subgraph induced by** $C_{k,\Delta}$ **within the temporal interval** Δ

Problem

Given a temporal graph G*, find the set of all* (k, Δ)*-cores of* G.

- the number of span-cores is $\mathcal{O}(|T|^2)$

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A naïve approach

Algorithm

- generate all temporal intervals $\Delta \sqsubseteq T$
- for each $\Delta \sqsubseteq T$, compute the subgraph $G_\Delta = (V, E_\Delta)$
- run a core-decomposition subroutine on each G_Δ

- $\mathcal{O}(|T|^2 \times |E|)$ time complexity

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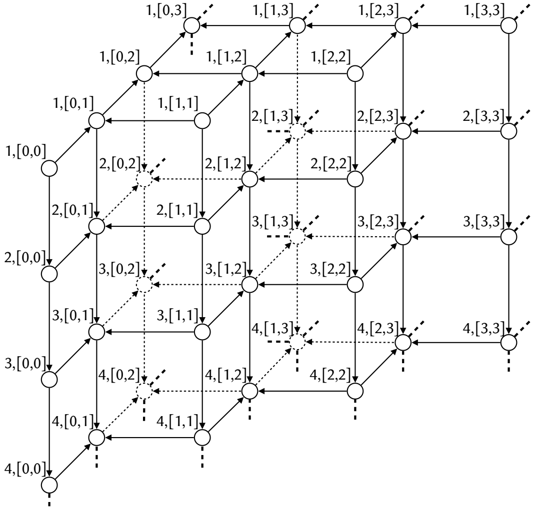
Span-core search space



Proposition

For any two span-cores $C_{k,\Delta}$, $C_{k',\Delta'}$ of a temporal graph G it holds that

$$k' \leq k \wedge \Delta' \sqsubseteq \Delta \;\Rightarrow\; C_{k,\Delta} \subseteq C_{k',\Delta'}.$$

Corollary

Given a temporal graph $G = (V, T, \tau)$, and a temporal interval $\Delta = [t_s, t_e] \sqsubseteq T$, let $\Delta_+ = [\min\{t_s + 1, t_e\}, t_e]$ and $\Delta_- = [t_s, \max\{t_e - 1, t_s\}]$. It holds that

$$C_{k,\Delta} \;\subseteq\; (C_{k,\Delta_+} \cap C_{k,\Delta_-}) \;=\; \bigcap_{\Delta' \sqsubseteq \Delta} C_{k,\Delta'}.$$

A more efficient algorithm

Algorithm

- generate temporal intervals $\Delta \sqsubseteq T$ of **increasing** size
- for each $\Delta \sqsubseteq T$ such that $|\Delta| > 1$, run a core-decomposition subroutine from $(C_{1,\Delta_+} \cap C_{1,\Delta_-})$
- if C_{1,Δ_+} or C_{1,Δ_-} does not exist, skip the core decomposition for Δ

- worst-case time complexity still $\mathcal{O}(|T|^2 \times |E|)$, but the algorithm is **much faster in practice** than the naïve one

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Maximal span-cores

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Definition

A span-core $C_{k,\Delta}$ of a temporal graph G is said **maximal** *if there does not exist any other span-core $C_{k',\Delta'}$ of G such that $k \leq k'$ and $\Delta \sqsubseteq \Delta'$.*

Problem

Given a temporal graph G, find the set of all maximal (k, Δ)-cores of G.

- the number of maximal span-cores is $\mathcal{O}(|T|^2)$
- experimentally, maximal span-cores are **at least one order of magnitude less** than the overall span-cores

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A filtering-based (naïve) approach

Algorithm

- equip the algorithm for span-core decomposition with a data structure $\mathcal{M}$ that
  - stores the span-core of the highest order for every temporal interval $\Delta \sqsubseteq T$
  - at the storage of a span-core $C_{k,\Delta}$, removes the span-cores dominated by $C_{k,\Delta}$
- return the span-cores retained by $\mathcal{M}$

- same running time as the algorithm for finding all the span-cores

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Properties of maximal span-cores

Lemma

Given a temporal graph $G = (V, T, \tau)$, let $\mathbf{C}_M$ be the set of all maximal span-cores of G, and $\mathbf{C}_{\mathbf{inner}} = \{C_{k^*}[G_\Delta] \mid \Delta \sqsubseteq T\}$ be the set of innermost cores of all graphs G_Δ. It holds that $\mathbf{C}_M \subseteq \mathbf{C}_{\mathbf{inner}}$.

- $\Delta = [t_s, t_e]$ yields a maximal span-core it suffices to start from a subgraph, which is composed of all the vertices whose temporal degree is larger than the maximum between the orders of the innermost cores of intervals $\Delta' = [t_s - 1, t_e]$ and $\Delta'' = [t_s, t_e + 1]$
- **Top-down** strategy: start from larger temporal intervals
- This also allows us to skip the computation of complete core decompositions of the whole "singleton-interval" graphs $\{G_{[t,t]}\}_{t \in T}$

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Properties of maximal span-cores

Lemma

Given a temporal graph $G = (V, T, \tau)$, and three temporal intervals $\Delta = [t_s, t_e] \sqsubseteq T$, $\Delta' = [t_s-1, t_e] \sqsubseteq T$, and $\Delta'' = [t_s, t_e+1] \sqsubseteq T$. The innermost core $C_{k^*}[G_\Delta]$ is a maximal span-core of G if and only if $k^* > \max\{k', k''\}$ where k' and k'' are the orders of the innermost cores of $G_{\Delta'}$ and $G_{\Delta''}$, respectively.

Lemma

Given G, Δ, Δ', Δ'', k', and k'' as in previous Lemma, let $\widetilde{V} = \{u \in V \mid deg_\Delta(V, u) > \max\{k', k''\}\}$, and let $C_{k^*}[G_\Delta[\widetilde{V}]]$ be the innermost core of $G_\Delta[\widetilde{V}]$. If $k^* > \max\{k', k''\}$, then $C_{k^*}[G_\Delta[\widetilde{V}]]$ is a maximal span-core; otherwise, no maximal span-core exists for Δ.

Efficient maximal-span-core finding

Algorithm

- consider intervals $\Delta = [t_s, t_e] \sqsubseteq T$, for increasing values of t_s and decreasing values of t_e
  - e.g., with $t_{max} = 10$, $\{[0,10],[0,9],\ldots,[0,0],[1,10],[1,9],\ldots,[1,1],[2,10],[2,9],\ldots\}$
  - this guarantees that once we consider Δ, no $\Delta' \sqsupseteq \Delta$ will be considered at later stage
- compute the **lower bound lb on the order** of a span-core in Δ to be recognized as maximal
- build the sets of vertices V_{lb} that have degree in Δ larger than lb
- extract the **innermost** core of the subgraph $(V_{lb}, E_\Delta[V_{lb}])$
- identify such a core as maximal if its order is actually larger than lb

- running time **much faster in practice** than the filtering-based algorithm

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Experiments

Datasets

dataset	$\|V\|$	$\|E\|$	$\|T\|$	window size (days)	domain
ProsperLoans	89k	3M	307	7	economic
Last.fm	992	4M	77	21	co-listening
WikiTalk	2M	10M	192	28	communication
DBLP	1M	11M	80	366	co-authorship
StackOverflow	2M	16M	51	56	question-and-answer
Wikipedia	343k	18M	101	56	co-editing
Amazon	2M	22M	115	28	co-rating
Epinions	120k	33M	25	21	co-rating

Evaluation

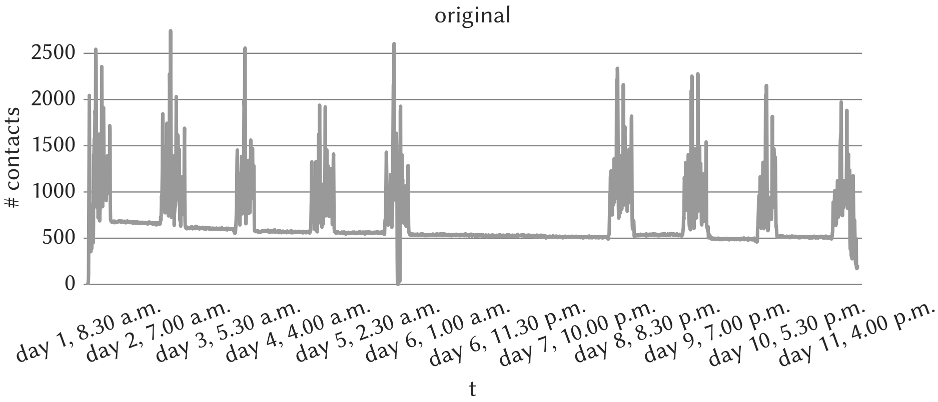
dataset	method	# output span-cores	time (s)	memory (GB)	# processed vertices
WikiTalk	Naïve-span-cores	19 693	322 302	36	25B
	Span-cores		1 084	36	555M
	Naïve-maximal-span-cores	632	1 194	36	555M
	Maximal-span-cores		126	35	2M
Wikipedia	Naïve-span-cores	125 191	17 155	4	1B
	Span-cores		522	4	35M
	Naïve-maximal-span-cores	2 147	537	4	35M
	Maximal-span-cores		201	4	320k
Amazon	Naïve-span-cores	29 318	10 415	18	2B
	Span-cores		409	18	247M
	Naïve-maximal-span-cores	303	580	18	247M
	Maximal-span-cores		123	18	688k

Applications

Datasets

- **face-to-face interaction networks** gathered by a proximity-sensing infrastructure in schools
  - PrimarySchool (242 individuals, 2 days)
  - HighSchool (327 individuals, 5 days)
  - HongKong (774 individuals, 11 days)
- window size of 5 minutes
- discarded span-cores of $|\Delta| = 1$

Anomaly detection



HongKong

Anomaly detection

1. find a set of anomalously long temporal intervals supporting maximal span-cores
   - find the set of temporal intervals $\mathcal{I} = \{\Delta \sqsubseteq T \mid C_{k,\Delta} \in \mathbf{C}_M \wedge |\Delta| > tr\}$ that are the span of a maximal span-core $C_{k,\Delta}$ with size longer than a certain threshold tr
   - we use $tr = 22$ (110 minutes)
2. identify anomalous vertices
   - for each timestamp $t \in T$, select as anomalous all those vertices that appear in the span-cores $\{C_{1,\Delta} \mid \Delta \in \mathcal{I} \wedge t \in \Delta\}$, i.e., the span-cores of $k = 1$ whose span is in $\mathcal{I}$ and contains t
3. filter out anomalous contacts
   - at each timestamp $t \in T$, filter out the contacts having at least an anomalous endpoint at time t.

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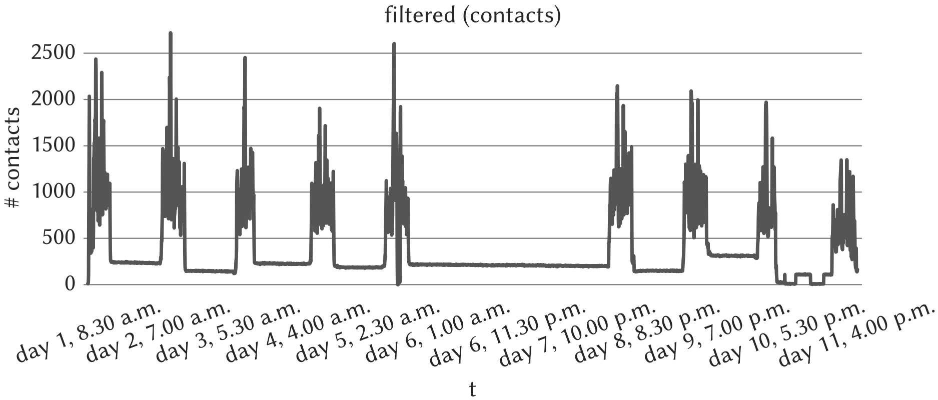

HongKong

- 0.91 precision, 0.64 recall

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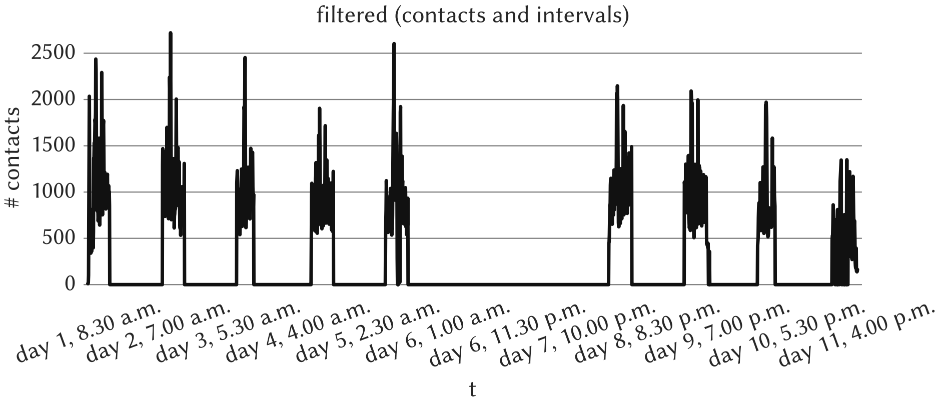

HongKong

- 0.93 precision, 0.99 recall

Conclusions

- introduced a notion of dense pattern in temporal networks that
  - takes into account the **sequentiality** of connections
  - is assigned with a clear **temporal collocation**
- developed efficient algorithms for computing all the span-cores, and only the maximal ones
- future work:
  - spreading processes analysis
  - temporal community search and temporal densest subgraph
  - network finger-printing

References

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Thanks!