Sparsification Techniques Preserving Temporal Connectivity

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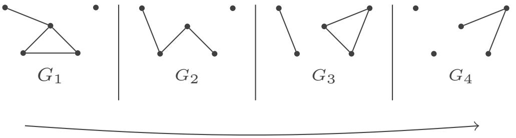
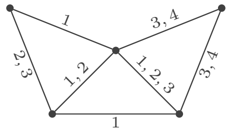
Algorithmic Aspects of Temporal Graphs II
(@ICALP'2019)

(arxiv.org/abs/1810.00104)

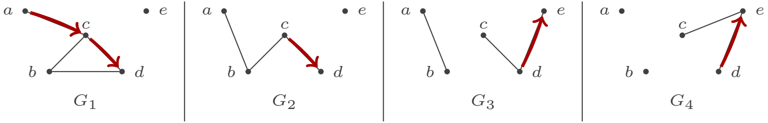
Temporal graphs (*a.k.a.* Time-varying graphs, Evolving graphs, Dynamic graphs)



the graph



Paths and connectivity:



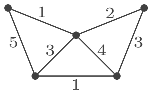
→ Temporal paths (*journey*): path labeled with non-decreasing times

→ *Strict vs. non-strict* journeys: allow (or not) consecutive hops in same time step?

→ *Temporal connectivity:* $\forall u, \forall v, \exists$ journey(u, v).

Simple temporal graph:

- Single presence time per edge
- Times are locally distinct (strict = non-strict)



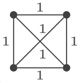
Temporal spanners

Original question

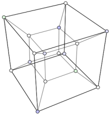
[Kempe, Kleinberg, Kumar, STOC'00]

"Given a temporally connected graph, is there always a subset of $O(n)$ edges that preserves temporal connectivity"? Followed by preliminary answers:

- If journeys are required to be strict $\rightarrow$ No, e.g.



- More generally? $\rightarrow$ No, e.g.



$O(n \log n)$ edges, but unsparsifiable.

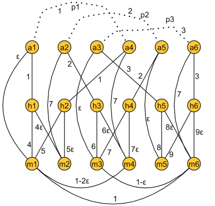
$\rightarrow$ *Relaxation:* How about a **sparse** (i.e. $o(n^2)$) subset of edges?

Theorem

[Axiotis, Fotakis, ICALP'16]

$\rightarrow$ **No!** $\exists$ non-sparsifiable graphs with $\Theta(n^2)$ many edges

(can be adapted to both strict and non-strict journeys)

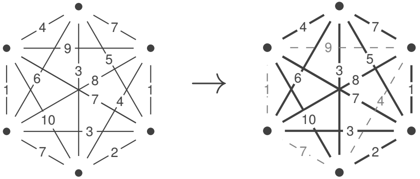


$\rightarrow$ More assumptions needed!

E.g. complete graphs?

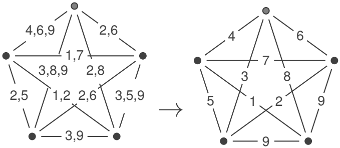
Temporal cliques

The model: Simple temporal cliques

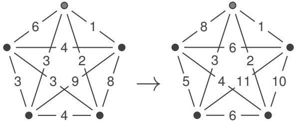


Applicability to more general cliques (by reduction):

- If the clique admits a simple sub-schedule ✓



- If non-strict journeys are allowed ✓



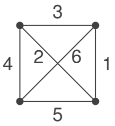
List of techniques:

- The K_4 technique	$\approx 5\binom{n}{2}/6$
- Pivotability	$2n-3$ (but not general)
- Delegation and dismountability	$2n-3$ (but not general)
- Fireworks (transitive delegations)	$\approx 3\binom{n}{2}/4$
- Bidirectional fireworks	$\approx \binom{n}{2}/2$
- Bidirectional fireworks + Dismountability + Partial delegation	$O(n \log n)$

The K_4 technique [Akrida, Gasieniek, Mertzios, Spirakis, ToCS'17]

The K_4 Lemma

Whatever the labeling, one can find an edge e such that $K_4 \setminus \{e\}$ is temporally connected.



Then...

Partition K_n into K_4's, remove an edge in each K_4 $\rightarrow$ removes $\lfloor n/4 \rfloor = \Theta(n)$ edges

Remark [C. Peters, Schoeters, ICALP'19]

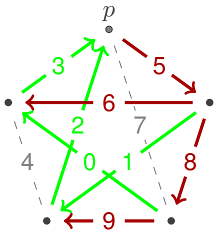
Can be improved by considering **edge-**disjoint K_4's

$\rightarrow$ $\binom{n}{2}/6 = \Theta(n^2)$ edges can be removed, resulting in spanner of size $5\binom{n}{2}/6$

(but still far from sparse...)

First attempt: Pivotability

A simple way to build linear spanners!



Motivation: Kosaraju's principle in directed graphs

$\rightarrow \exists v$ *s.t.* v can be reached by all others and v can reach all others $\Rightarrow$ strong connectivity

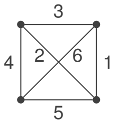
Temporal version:

$\rightarrow v$ reached by all others <u>before t</u> and reaching all others <u>after t</u> (for some t) $\Rightarrow$ temporal connectivity

Unfortunately: $\exists$ arbitrarily large non pivotable cliques

we need something else...

Delegation and dismountability



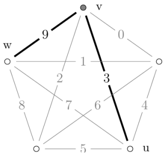
Delegation

If $uv = e^{-}(u)$, then v can reach all the vertices through u.

$\rightarrow$ We say that v can **delegate** its emissions to u

If $uv = e^{+}(u)$, then v can be reached by all the vertices through v.

$\rightarrow$ We say that v can **delegate** its receptions to u

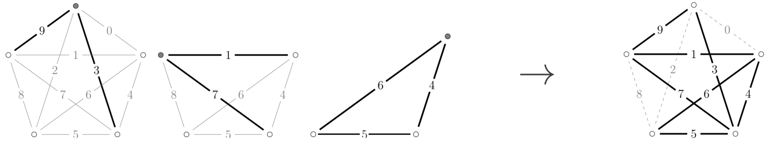


Dismountability

If v shares both the min edge of a neighbor u and the max edge of another neighbors w, then v is **dismountable**.

$\rightarrow$ a spanner for $K \setminus v$ + uv + vw is a spanner for K (self-reduction)

$\rightarrow$ Suggests recursion & a concept of *full-dismountability* (= recursively dismountable)



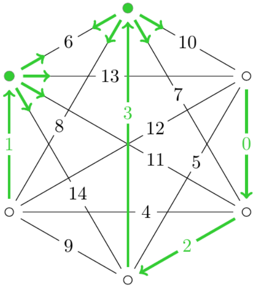
Relaxation

k-hop delegation, k-hop dismountability

Unfortunately (again!)

$\exists$ arbitrary large non (k-hop) dismountable cliques

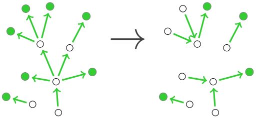
Transitive delegations (“fireworks”)



Principle:

- Min edges $\rightarrow$ “directed” forest
- Transitive delegations towards emitters (sinks)
- Spanner = min edges + all edges of emitters

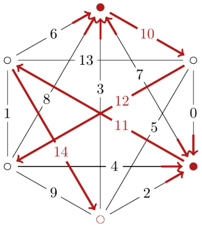
Wait a minute... possibly too many emitters!



$\rightarrow$ *Transformation of the forest*:

- At most $n/2$ emitters

Theorem: $\exists$ spanners of size $\frac{3}{4}\binom{n}{2} + O(n)$



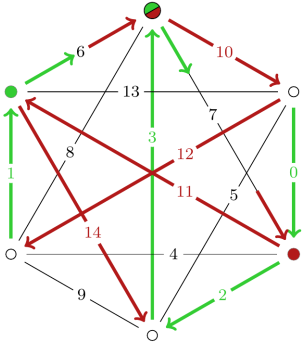
Note: also works for receptions (“backward fireworks”):

$\rightarrow$ Spanner = max edges + all edges of collectors

Combining both directions

Principle

- Every vertex can reach at least one emitter u through u's min edge
- Every vertex can reached by a collector v through v's max edge
- Every emitter can reach all collectors through direct edges



$\rightarrow$ Spanner = min edges + max edges

+ edges between emitters and collectors

Theorem:

At most $n/2$ emitters and $n/2$ collectors $\Rightarrow$ $\boxed{\exists \text{ Spanners of size } \binom{n}{2}/2 + O(n)}$

$\approx$ half of the edges

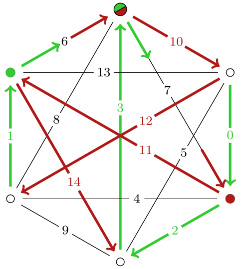
Recurse or iterate

(down to $O(n \log n)$)

Recurse or sparsify?

Two options:

- Case 1: emitters $\cup$ collectors $\subsetneq V$
- Case 2: emitters $\cup$ collectors $= V$

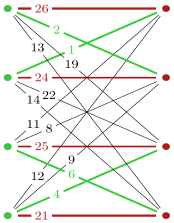


Case 1: One vertex v is neither emitter nor collector.

$\rightarrow \exists\ 2$-*hop dismountable* vertex
(select 4 edges selected, then recursion)

Case 2: emitters $\cup$ collectors $= V$

$\rightarrow$ All vertices are either emitters or collectors (not both)!

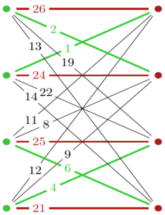


A lot of structure to work with:

- Complete bipartite graph $\mathcal{H}$ between emitters and collectors
- Min edges and max edges form two perfect matchings
- W.l.o.g. min edges (max edges) are *reciprocal* in $\mathcal{H}$

New objective:
$\rightarrow$ Sparsify $\mathcal{H}$ while preserving journeys from each emitter to all collectors

Sparsification of the bipartite graph



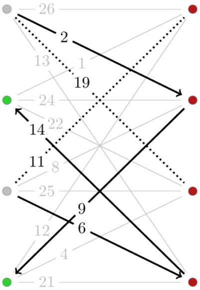
Technique: <u>Partial</u> delegations among emitters

- Find a 2-hop journey from one emitter to another, arriving through a “locally small” edge
- Pay extra edges to reach the missed collectors

Iterative procedure:

In each step i:

- Half of the emitters delegate to other half
- Some collectors are missed $\rightarrow$ pay extra edges (penalty)
- Penalty doubles in each iteration, but number of emitters halves



Cost:

$O(n)$ edges over $O(\log n)$ iterations $\rightarrow$ $\boldsymbol{O(n \log n)}$ edges.

Conclusion (entire algorithm):

- n_1 recursions due to 2-hop dismountability
- $n_2 = n - n_1$ vertices when meeting Case 2

$\rightarrow$ $\exists$ spanner of size $\Theta(n_1) + \Theta(n_2 \log n_2) = \boldsymbol{O(n \log n)}$. $\square$

Open questions

Open questions

Relaxing the complete graph assumption

- Can more general classes of dense graphs be sparsified?

  $\rightarrow$ Recall that $\exists$ unsparsifiable graphs of density $\Theta(n^2)$

  $\rightarrow$ Yes if $\Theta(1)$ edges are missing, can we do better?

Better spanners for temporal cliques

- Is $O(n \log n)$ optimal for cliques?

Experiments:

n	sparsest spanner (# edges)	
4	4 or 5	exhaustive search
5	6 or 7	exhaustive search
6	8 or 9	exhaustive search
7	10 or 11	millions random instances
...	...	
20	36 or 37	dozens random instances

$\rightarrow$ $2n - 4 \leq OPT \leq 2n - 3$?

Thank you!