

# On the Intersection of Tolerance and Cocomparability Graphs

George B. Mertzios\*

Shmuel Zaks†

## Abstract

Tolerance graphs have been extensively studied since their introduction, due to their interesting structure and their numerous applications, as they generalize both interval and permutation graphs in a natural way. It has been conjectured by Golumbic, Monma, and Trotter in 1984 that the intersection of tolerance and cocomparability graphs coincides with bounded tolerance graphs. Since cocomparability graphs can be efficiently recognized, a positive answer to this conjecture in the general case would enable us to efficiently distinguish between tolerance and bounded tolerance graphs, although it is NP-complete to recognize each of these classes of graphs separately. The conjecture has been proved under some –rather strong– *structural* assumptions on the input graph; in particular, it has been proved for complements of trees, and later extended to complements of bipartite graphs, and these are the only known results so far. Furthermore, it is known that the intersection of tolerance and cocomparability graphs is contained in the class of trapezoid graphs. Our main result in this article is that the above conjecture is true for every graph  $G$  that admits a tolerance representation with exactly one unbounded vertex; note here that this assumption concerns only the given tolerance *representation*  $R$  of  $G$ , rather than any structural property of  $G$ . Moreover, our results imply as a corollary that the conjecture of Golumbic, Monma, and Trotter is true for every graph  $G = (V, E)$  that has no three independent vertices  $a, b, c \in V$  such that  $N(a) \subset N(b) \subset N(c)$ ; this is satisfied in particular when  $G$  is the complement of a triangle-free graph (which also implies the above-mentioned correctness for complements of bipartite graphs). Our proofs are constructive, in the sense that, given a tolerance representation  $R$  of a graph  $G$ , we transform  $R$  into a bounded tolerance representation  $R^*$  of  $G$ . Furthermore, we conjecture that any *minimal* tolerance graph  $G$  that is not a bounded tolerance graph, has a tolerance representation with exactly one unbounded vertex. Our results imply the non-trivial result that, in order to prove the conjecture of Golumbic, Monma, and Trotter, it suffices to prove our conjecture.

**Keywords:** Tolerance graphs, cocomparability graphs, 3-dimensional intersection model, trapezoid graphs, parallelogram graphs.

## 1 Introduction

A simple undirected graph  $G = (V, E)$  on  $n$  vertices is called a *tolerance* graph if there exists a collection  $I = \{I_u \mid u \in V\}$  of closed intervals on the real line and a set  $t = \{t_u \mid u \in V\}$  of positive numbers, such that for any two vertices  $u, v \in V$ ,  $uv \in E$  if and only if  $|I_u \cap I_v| \geq \min\{t_u, t_v\}$ . The pair  $\langle I, t \rangle$  is called a *tolerance representation* of  $G$ . A vertex  $u$  of  $G$  is called a *bounded vertex* (in a certain tolerance representation  $\langle I, t \rangle$  of  $G$ ) if  $t_u \leq |I_u|$ ; otherwise,  $u$  is called an *unbounded vertex* of  $G$ . If  $G$  has a tolerance representation  $\langle I, t \rangle$  where all vertices are bounded, then  $G$  is called a *bounded tolerance* graph and  $\langle I, t \rangle$  a *bounded tolerance representation* of  $G$ .

Tolerance graphs find numerous applications in constrained-based temporal reasoning, data transmission through networks to efficiently scheduling aircraft and crews, as well as contributing to genetic analysis and studies of the brain [12, 13]. This class of graphs has been introduced in 1982 [10] in order to generalize some of the well known applications of interval graphs. The main motivation was in the context of resource allocation and scheduling problems, in which resources, such as rooms and vehicles, can tolerate sharing among users [13]. Since then, tolerance graphs

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\*School of Engineering and Computing Sciences, Durham University, United Kingdom. Email: [george.mertzios@durham.ac.uk](mailto:george.mertzios@durham.ac.uk)

†Department of Computer Science, Technion, Haifa, Israel. Email: [zaks@cs.technion.ac.il](mailto:zaks@cs.technion.ac.il)

have attracted many research efforts [2, 4, 8, 11–14, 16, 18–20], as they generalize in a natural way both interval graphs (when all tolerances are equal) and permutation graphs [10] (when  $t_i = |I_i|$  for every  $i = 1, 2, \dots, n$ ); see [13] for a detailed survey.

Given an undirected graph  $G = (V, E)$  and a vertex subset  $M \subseteq V$ ,  $M$  is called a *module* in  $G$ , if for every  $u, v \in M$  and every  $x \in V \setminus M$ ,  $x$  is either adjacent in  $G$  to both  $u$  and  $v$  or to none of them. Note that  $\emptyset$ ,  $V$ , and all singletons  $\{v\}$ , where  $v \in V$ , are trivial modules in  $G$ . A *comparability* graph is a graph which can be transitively oriented. A *cocomparability* graph is a graph whose complement is a comparability graph. A *trapezoid* (resp. *parallelogram* and *permutation*) graph is the intersection graph of trapezoids (resp. parallelograms and line segments) between two parallel lines  $L_1$  and  $L_2$  [9]. Such a representation with trapezoids (resp. parallelograms and line segments) is called a *trapezoid* (resp. *parallelogram* and *permutation*) *representation* of this graph. A graph is bounded tolerance if and only if it is a parallelogram graph [2]. Permutation graphs are a strict subset of parallelogram graphs [3]. Furthermore, parallelogram graphs are a strict subset of trapezoid graphs [23], and both are subsets of cocomparability graphs [9, 13]. On the other hand, not every tolerance graph is a cocomparability graph [9, 13].

Cocomparability graphs have received considerable attention in the literature, mainly due to their interesting structure that leads to efficient algorithms for several NP-hard problems, see e.g. [5, 6, 13, 17]. Furthermore, the intersection of the class of cocomparability graphs with other graph classes has interesting properties and coincides with other widely known graph classes. For instance, their intersection with chordal graphs is the class of interval graphs [9], while their intersection with comparability graphs is the class of permutation graphs [9, 22]. These structural characterizations find also direct algorithmic implications to the recognition problem of interval and permutation graphs, respectively, since the class of cocomparability graphs can be recognized efficiently [9, 24]. In this context, the following conjecture has been made in 1984 [11]:

**Conjecture 1 ([11])** *The intersection of cocomparability graphs with tolerance graphs is exactly the class of bounded tolerance graphs.*

Note that the inclusion in one direction is immediate: every bounded tolerance graph is a cocomparability graph [9, 13], as well as a tolerance graph by definition. Conjecture 1 has been proved for complements of trees [1], and later extended to complements of bipartite graphs [21], and these are the only known results so far. Furthermore, it has been proved that the intersection of tolerance and cocomparability graphs is contained in the class of trapezoid graphs [8]. Since cocomparability graphs can be efficiently recognized [24], a positive answer to Conjecture 1 would enable us to efficiently distinguish between tolerance and bounded tolerance graphs, although it is NP-complete to recognize each of these classes of graphs separately [19]. Only little is known so far about the separation of tolerance and bounded tolerance graphs; a recent work can be found in [7]. An intersection model for general tolerance graphs has been recently presented in [18], given by 3-dimensional parallelepipeds. This *parallelepiped representation* of tolerance graphs generalizes the parallelogram representation of bounded tolerance graphs; the main idea is to exploit the third dimension to capture the information given by unbounded tolerances. Furthermore, this model proved to be a powerful tool for designing efficient algorithms for general tolerance graphs [18].

**Our contribution.** Our main result in this article is that Conjecture 1 is true for every graph  $G$ , for which there exists a tolerance representation with exactly one unbounded vertex. Furthermore, we state a new conjecture regarding the *minimal* separating examples between tolerance and bounded tolerance graphs (cf. Conjecture 2 below). That is, unlike Conjecture 1, this conjecture does not concern any other class of graphs, such as cocomparability or trapezoid graphs. In order to state Conjecture 2, we first define a graph  $G$  to be a *minimally unbounded tolerance* graph, if  $G$  is tolerance but not bounded tolerance, while  $G$  becomes a bounded tolerance graph if we remove any vertex of  $G$ .

**Conjecture 2** *Any minimally unbounded tolerance graph has a tolerance representation with exactly one unbounded vertex.*

Our results imply the non-trivial result that, in order to prove Conjecture 1, it suffices to prove Conjecture 2. To the best of our knowledge, Conjecture 2 is true for all known examples of minimally unbounded tolerance graphs in the literature (see e.g. [13]).

All our results are based (a) on the 3-dimensional parallelepiped representation of tolerance graphs [18] and (b) on the fact that every graph  $G$  that is both a tolerance and a cocomparability graph, has necessarily a trapezoid representation  $R_T$  [8]. Specifically, in order to prove our results, we define three conditions on the unbounded vertices of  $G$  (in the parallelepiped representation  $R$  of  $G$ ). Condition 1 states that  $R$  has exactly one unbounded vertex. Condition 2 states that, for every unbounded vertex  $u$  of  $G$  (in  $R$ ), there exists no unbounded vertex  $v$  whose neighborhood is strictly included in the neighborhood of  $u$ . Note that both Conditions 1 and 2 concern only the parallelepiped representation  $R$ ; furthermore, Condition 2 is weaker than Condition 1. Then, Condition 3 (which has a more complicated statement, cf. Section 3.2) concerns also the position of the unbounded vertices in the trapezoid representation  $R_T$  of  $G$ , and it is weaker than both Conditions 1 and 2.

Consider a graph  $G$  that is both tolerance and cocomparability, and thus  $G$  is also a trapezoid graph [8], i.e.  $G$  has both a parallelepiped representation  $R$  and a trapezoid representation  $R_T$ . Assuming that  $G$  satisfies Condition 3, we construct a parallelogram representation of  $G$ , thus proving that  $G$  is a bounded tolerance graph. Therefore, since Condition 3 is weaker than both Conditions 1 and 2, the same result immediately follows by assuming that the graph  $G$  satisfies Conditions 1 or Condition 2. In particular, this immediately implies our main result of this paper, i.e. that Conjecture 1 is true for every graph  $G$  that admits a tolerance representation with exactly one unbounded vertex (i.e. when Condition 1 is satisfied). Moreover, our results imply easily (cf. Corollary 2) that Conjecture 1 is true for every graph  $G = (V, E)$  that has no three independent vertices  $a, b, c \in V$  such that the neighborhood of  $a$  is strictly included in the neighborhood of  $b$ , which in turn is strictly included in the neighborhood of  $c$ . This is a consequence of the fact that, if a graph  $G$  has no such triple of vertices  $\{a, b, c\}$ , then Condition 2 is satisfied. Thus, in particular, Conjecture 1 is true for all complements of triangle-free graphs (which also implies the above-mentioned correctness for complements of trees [1] and complements of bipartite graphs [21]).

The main idea of the proofs is to iteratively “eliminate” the unbounded vertices of the parallelepiped representation  $R$ . That is, assuming that the input representation  $R$  has  $k \geq 1$  unbounded vertices, we choose an unbounded vertex  $u$  in  $R$  and construct a parallelepiped representation  $R^*$  of  $G$  with  $k - 1$  unbounded vertices; specifically,  $R^*$  has the same unbounded vertices as  $R$  except for  $u$  (which becomes bounded in  $R^*$ ). As a milestone in the above construction of the representation  $R^*$ , we construct an induced subgraph  $G_0$  of  $G$  that includes  $u$ , with the property that the vertex set of  $G_0 \setminus \{u\}$  is a module in  $G \setminus \{u\}$ . The presented techniques are new and provide geometrical insight for the graphs that are both tolerance and cocomparability.

**Organization of the paper.** We first review in Section 2 some properties of tolerance and trapezoid graphs. Then we define the notion of a *projection representation* of a tolerance graph  $G$ , which is an alternative way to think about a parallelepiped representation of  $G$ . Furthermore, we introduce the *right* and *left border properties* of a vertex in a projection representation, which are crucial for our analysis. In Section 3 we prove our main results. Specifically, we first consider in Section 3.1 the case where the graph  $G$  has at least one unbounded vertex  $u$  with the right or with the left border property in its projection representation, and then we consider in Section 3.2 the case that  $G$  has no such unbounded vertex. Next we discuss in Section 3.3 how these results reduce Conjecture 1 to Conjecture 2. Finally, we discuss the presented results and further research in Section 4.

## 2 Definitions and basic properties

**Notation.** We consider in this article simple undirected graphs with no loops or multiple edges. In a graph  $G = (V, E)$ , the edge between vertices  $u$  and  $v$  is denoted by  $uv$ , and in this case  $u$  and  $v$  are called *adjacent* in  $G$ . Given a vertex subset  $S \subseteq V$ ,  $G[S]$  denotes the induced subgraph of  $G$  on

the vertices in  $S$ . Whenever it is clear from the context, we may not distinguish between a vertex set  $S$  and the induced subgraph  $G[S]$  of  $G$ . In particular, if  $M$  is a module in  $G$ , we may also say that the induced subgraph  $G[M]$  is a module in  $G$ . Furthermore, we denote for simplicity the induced subgraph  $G[V \setminus S]$  by  $G \setminus S$ . Denote by  $N(u) = \{v \in V \mid uv \in E\}$  the set of neighbors of a vertex  $u$  in  $G$ , and  $N[u] = N(u) \cup \{u\}$ . For a subset  $U$  of vertices of  $G$ , denote  $N(U) = \bigcup_{u \in U} N(u) \setminus U$ . For any  $k$  vertices  $u_1, u_2, \dots, u_k$  of  $G$ , denote for simplicity  $N[u_1, u_2, \dots, u_k] = N[u_1] \cup N[u_2] \cup \dots \cup N[u_k]$ , i.e.  $N[u_1, u_2, \dots, u_k] = N(\{u_1, u_2, \dots, u_k\}) \cup \{u_1, u_2, \dots, u_k\}$ . For any two sets  $A$  and  $B$ , we will write  $A \subseteq B$  if  $A$  is included in  $B$ , and  $A \subset B$  if  $A$  is strictly included in  $B$ .

Consider a trapezoid graph  $G = (V, E)$  and a trapezoid representation  $R_T$  of  $G$ , where for any vertex  $u \in V$  the trapezoid corresponding to  $u$  in  $R_T$  is denoted by  $T_u$ . Since trapezoid graphs are also cocomparability graphs [9], we can define the partial order  $(V, \ll_{R_T})$ , such that  $u \ll_{R_T} v$ , or equivalently  $T_u \ll_{R_T} T_v$ , if and only if  $T_u$  lies completely to the left of  $T_v$  in  $R_T$  (and thus also  $uv \notin E$ ). Note that there are several trapezoid representations of a particular trapezoid graph  $G$ . Given one such representation  $R_T$ , we can obtain another one  $R'_T$  by *vertical axis flipping* of  $R_T$ , i.e.  $R'_T$  is the mirror image of  $R_T$  along an imaginary line perpendicular to  $L_1$  and  $L_2$ .

Let us now briefly review the parallelepiped representation model of tolerance graphs [18]. Consider a tolerance graph  $G = (V, E)$  and let  $V_B$  and  $V_U$  denote the set of bounded and unbounded vertices of  $G$  (for a certain tolerance representation), respectively. Consider now two parallel lines  $L_1$  and  $L_2$  in the plane. For every vertex  $u \in V$ , consider a parallelogram  $\bar{P}_u$  with two of its lines on  $L_1$  and  $L_2$ , respectively, and  $\phi_u$  be the (common) slope of the other two lines of  $\bar{P}_u$  with  $L_1$  and  $L_2$ . For every unbounded vertex  $u \in V_U$ , the parallelogram  $\bar{P}_u$  is trivial, i.e. a line. In the model of [18], every bounded vertex  $u \in V_B$  corresponds to the parallelepiped  $P_u = \{(x, y, z) \mid (x, y) \in \bar{P}_u, 0 \leq z \leq \phi_u\}$  in the 3-dimensional space, while every unbounded vertex  $u \in V_U$  corresponds to the line  $P_u = \{(x, y, z) \mid (x, y) \in \bar{P}_u, z = \phi_u\}$ . The resulting set  $\{P_u \mid u \in V\}$  of parallelepipeds in the 3-dimensional space constitutes the *parallelepiped representation* of  $G$ . In this model, two vertices  $u, v$  are adjacent if and only if  $P_u \cap P_v \neq \emptyset$ . That is,  $R$  is an intersection model for  $G$ . For more details we refer to [18].

An example of a tolerance graph  $G$  is given in Figure 1(a) (in this example,  $G$  is the induced path  $P_4 = (z, u, v, w)$  with four vertices). Furthermore, a parallelepiped representation  $R$  is illustrated in Figure 1(b). In particular, vertex  $w$  is unbounded in the parallelepiped representation  $R$ , while the vertices  $z, u, v$  are bounded in  $R$ . In the following, let  $V_B$  and  $V_U$  denote the sets of bounded and unbounded vertices of a tolerance graph  $G$  (for a certain parallelepiped representation), respectively.

**Definition 1 ([18])** *An unbounded vertex  $v \in V_U$  of a tolerance graph  $G$  is called inevitable (in a certain parallelepiped representation  $R$ ), if making  $v$  a bounded vertex in  $R$ , i.e. if replacing  $P_v$  with  $\{(x, y, z) \mid (x, y) \in \bar{P}_v, 0 \leq z \leq \phi_v\}$ , creates a new edge in  $G$ .*

**Definition 2 ([18])** *A parallelepiped representation  $R$  of a tolerance graph  $G$  is called canonical if every unbounded vertex in  $R$  is inevitable.*

For example, the parallelepiped representation of Figure 1(b) is canonical, since  $w$  is the only unbounded vertex and it is inevitable. A canonical representation of a tolerance graph  $G$  always exists, and can be computed in  $O(n \log n)$  time, given a parallelepiped representation of  $G$ , where  $n$  is the number of vertices of  $G$  [18].

Given a parallelepiped representation  $R$  of the tolerance graph  $G$ , we define now an alternative representation, as follows. Let  $\bar{P}_u$  be the projection of  $P_u$  to the plane  $z = 0$  for every  $u \in V$ . Then, for two bounded vertices  $u$  and  $v$ ,  $uv \in E$  if and only if  $\bar{P}_u \cap \bar{P}_v \neq \emptyset$ . Furthermore, for a bounded vertex  $v$  and an unbounded vertex  $u$ ,  $uv \in E$  if and only if  $\bar{P}_u \cap \bar{P}_v \neq \emptyset$  and  $\phi_v > \phi_u$ . Moreover, two unbounded vertices  $u$  and  $v$  of  $G$  are never adjacent (even in the case where  $\bar{P}_u$  intersects  $\bar{P}_v$ ). In the following, we will call such a representation a *projection representation* of a tolerance graph. Note that  $\bar{P}_u$  is a parallelogram (resp. a line segment) if  $u$  is bounded (resp. unbounded). The projection representation that corresponds to the parallelepiped representation of Figure 1(b) is presented in Figure 1(c). In the sequel, we will say that a vertex  $u$  is *adjacent* to a vertex  $v$

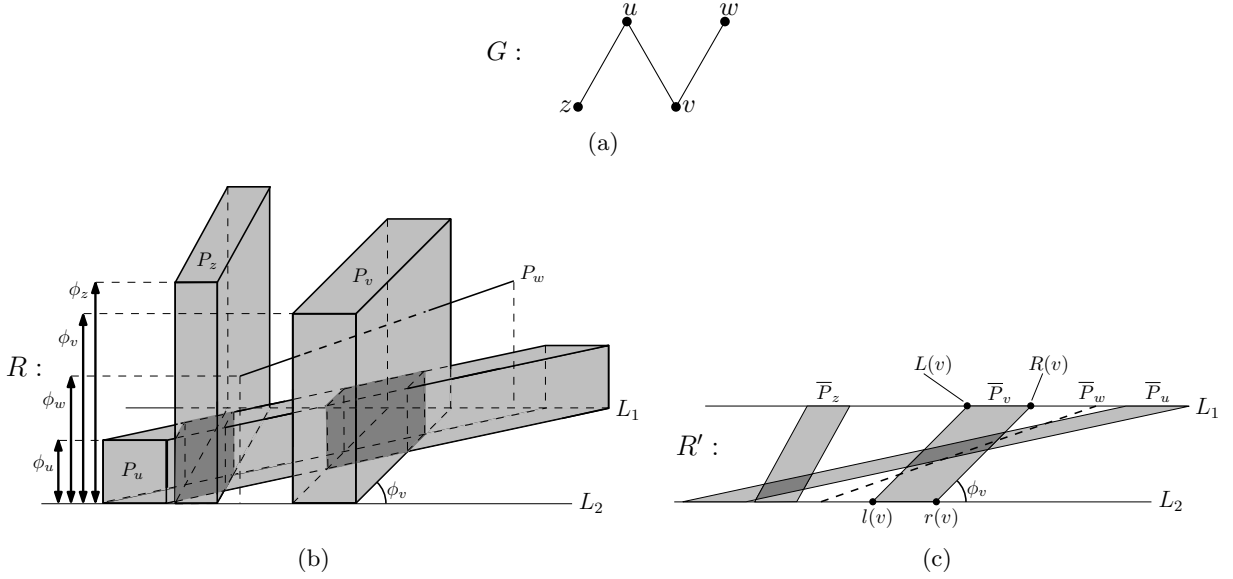


Figure 1: (a) A tolerance graph  $G$  (the induced path  $P_4 = (z, u, v, w)$  with four vertices), (b) a parallelepiped representation  $R$  of  $G$ , and (c) the corresponding projection representation  $R'$  of  $G$ .

in a projection representation  $R$ , if  $u$  is adjacent to  $v$  in the tolerance graph  $G_R$  induced by  $R$ . Furthermore, given a tolerance graph  $G$ , we will call a projection representation  $R$  of  $G$  a *canonical representation* of  $G$ , if  $R$  is the projection representation that is implied by a canonical parallelepiped representation of  $G$ . In the example of Figure 1, the projection representation  $R'$  is canonical, since the parallelepiped representation  $R$  is canonical as well.

Let  $R$  be a projection representation of a tolerance graph  $G = (V, E)$ . For every parallelogram  $\bar{P}_u$  in  $R$ , where  $u \in V$ , we define by  $l(u)$  and  $r(u)$  (resp.  $L(u)$  and  $R(u)$ ) the lower (resp. upper) left and right endpoint of  $\bar{P}_u$ , respectively (cf. the parallelogram  $\bar{P}_v$  in Figure 1(c)). Note that  $l(u) = r(u)$  and  $L(u) = R(u)$  for every unbounded vertex  $u$ . Furthermore, we denote by  $\phi_u$  the (common) slope of the lines of  $\bar{P}_u$  in  $R$  that do not lie on  $L_1$  or on  $L_2$  (cf. the parallelepiped  $P_v$  in Figure 1(b) and the parallelogram  $\bar{P}_v$  in Figure 1(c)). We assume throughout the paper w.l.o.g. that all endpoints and all slopes of the parallelograms in a projection representation are distinct [13, 15, 18]. For simplicity of the presentation, we will denote in the following  $\bar{P}_u$  just by  $P_u$  in any projection representation. Throughout the paper, given a projection representation  $R$ , we will often need to transform  $R$  to another projection representation  $R'$  by moving endpoints of some parallelograms of  $R$ . After such a transformation, we say that the endpoint  $a$  on  $L \in \{L_1, L_2\}$  lies in  $R'$  *immediately before* (resp. *immediately after*) the endpoint  $b$  on  $L$ , if there is no other endpoint between  $a$  and  $b$  in  $R'$ , and additionally if  $a = b - \varepsilon$  (resp.  $a = b + \varepsilon$ ) on  $L$ , where  $\varepsilon > 0$  is a sufficiently small positive number. Similarly, given a set  $A$  of points on  $L \in \{L_1, L_2\}$ , we say that  $A$  lies in  $R'$  *immediately before* (resp. *immediately after*) the endpoint  $b$  on  $L$ , if for every  $a \in A$  there is no endpoint  $c \notin A \cup \{b\}$  between  $a$  and  $b$  in  $R'$ , and additionally if  $a \in (b - \varepsilon, b)$  (resp.  $a \in (b, b + \varepsilon)$ ) on  $L$ , where  $\varepsilon > 0$  is a sufficiently small positive number. The exact value of  $\varepsilon > 0$  will be chosen each time appropriately, such that certain conditions hold.

Similarly to a trapezoid representation, we can define the relation  $\ll_R$  also for a projection representation  $R$ . Namely,  $P_u \ll_R P_v$  if and only if  $P_u$  lies completely to the left of  $P_v$  in  $R$ . Otherwise, if neither  $P_u \ll_R P_v$  nor  $P_v \ll_R P_u$ , we will say that  $P_u$  *intersects*  $P_v$  in  $R$ , i.e.  $P_u \cap P_v \neq \emptyset$  in  $R$ . Furthermore, we define the total order  $<_R$  on the lines  $L_1$  and  $L_2$  in  $R$  as follows. For two points  $a$  and  $b$  on  $L_1$  (resp. on  $L_2$ ), if  $a$  lies to the left of  $b$  on  $L_1$  (resp. on  $L_2$ ), then we will write  $a <_R b$ . Note that, for two vertices  $u$  and  $v$  of a tolerance graph  $G = (V, E)$ ,  $P_u$  may intersect  $P_v$  in a projection representation  $R$  of  $G$ , although  $u$  is not adjacent to  $v$  in  $G$ , i.e.  $uv \notin E$ . Thus, a projection representation  $R$  of a tolerance graph  $G$  is *not* necessarily an intersection model for  $G$ .

Let  $R$  be a projection representation of a tolerance graph  $G = (V, E)$  and  $S \subseteq V$  be a set

of vertices of  $G$ . We denote by  $R \setminus S$  the representation that we obtain by removing the parallelograms  $\{P_u \mid u \in S\}$  from  $R$ . Then,  $R \setminus S$  is a projection representation of the induced subgraph  $G \setminus S = G[V \setminus S]$  of  $G$ . Furthermore, similarly to the trapezoid representations, there are several projection representations of a particular tolerance graph  $G$ . In the next two definitions, we correspond to every projection representation of a tolerance graph  $G$  another projection representation of the same graph  $G$  with special properties.

**Definition 3** Let  $R$  be a projection representation. The reverse representation  $\widehat{R}$  of  $R$  is obtained as the rotation of  $R$  by the angle  $\pi$ .

As an example, given the projection representation  $R'$  presented in Figure 1(c), its reverse representation  $\widehat{R}'$  is illustrated in Figure 2(a). It is easy to see that if  $R$  is a projection representation of a tolerance graph  $G$ , then for any two vertices  $u$  and  $v$  of  $G$ ,  $P_u \ll_R P_v$  if and only if  $P_v \ll_{\widehat{R}} P_u$ , and that  $P_u \cap P_v \neq \emptyset$  in  $\widehat{R}$  if and only if  $P_u \cap P_v \neq \emptyset$  in  $R$ . Furthermore, the slope  $\phi_u$  in  $\widehat{R}$  equals the slope  $\phi_u$  in  $R$ , for every vertex  $u$  of  $G$ . Therefore, reverse representation  $\widehat{R}$  of  $R$  is also a projection representation of the same graph  $G$ .

**Definition 4** Let  $L_1$  and  $L_2$  be two parallel lines and  $\ell$  be a line segment with endpoints  $a_\ell$  and  $b_\ell$  on  $L_1$  and on  $L_2$ , respectively, and  $\varepsilon > 0$  be arbitrary. A projection representation  $R_\ell$  between  $L_1$  and  $L_2$  is  $\varepsilon$ -squeezed with respect to  $\ell$ , if all endpoints of  $R_\ell$  on  $L_1$  and on  $L_2$  lie in the intervals  $[a_\ell - \frac{\varepsilon}{2}, a_\ell + \frac{\varepsilon}{2}]$  and  $[b_\ell - \frac{\varepsilon}{2}, b_\ell + \frac{\varepsilon}{2}]$ , respectively.

As an example, given the projection representation  $R'$  presented in Figure 1(c), the  $\varepsilon$ -squeezed representation  $R'_\ell$  of  $R'$  with respect to a line  $\ell$  is illustrated in Figure 2(b). It can be easily seen that, given a projection representation  $R$  of a tolerance graph  $G$ , a line segment  $\ell$  with endpoints on  $L_1$  and on  $L_2$ , and any  $\varepsilon > 0$ , there clearly exists an  $\varepsilon$ -squeezed projection representation  $R_\ell$  of  $G$  with respect to  $\ell$ ; however, we will apply this squeezing operation in a rather delicate way (cf. the proof of Theorem 2) to only some of the parallelograms in a given representation, in order to get some desired properties.

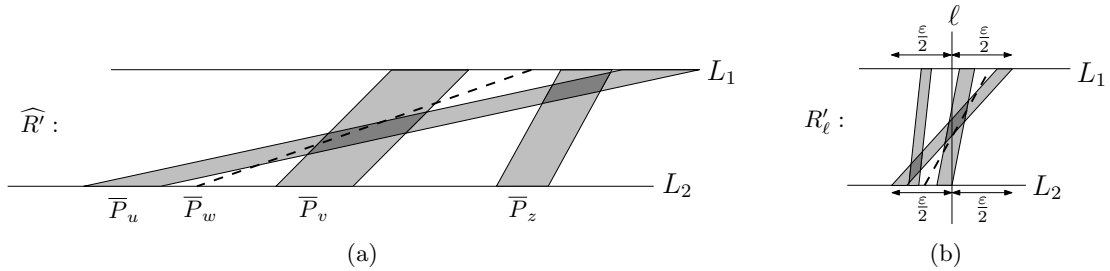


Figure 2: (a) The reverse representation  $\widehat{R}'$  of the projection representation  $R'$  of Figure 1(c), and (b) the  $\varepsilon$ -squeezed representation  $R'_\ell$  of  $R'$  with respect to the line  $\ell$ .

**Lemma 1** Let  $G$  be a tolerance graph and  $u$  be an unbounded vertex of  $G$  in a projection representation  $R$  of  $G$ . Then,  $r(u) <_R r(v)$ ,  $L(v) <_R L(u)$ , and  $v$  is a bounded vertex in  $R$ , for every  $v \in N(u)$ .

**Proof.** Let  $v \in N(u)$ . Then, since  $u$  is unbounded, and since no two unbounded vertices are adjacent,  $v$  is a bounded vertex in  $R$  and  $\phi_v > \phi_u$ . Moreover,  $P_u$  intersects  $P_v$  in the projection representation  $R$ . Suppose that  $r(u) = l(u) >_R r(v)$  (resp.  $L(v) >_R L(u) = R(u)$ ). Then, since  $P_u$  intersects  $P_v$  in  $R$ , it follows that  $L(u) = R(u) <_R R(v)$  (resp.  $l(v) <_R r(u) = l(u)$ ), and thus  $\phi_v < \phi_u$ , which is a contradiction. Therefore,  $r(u) <_R r(v)$  and  $L(v) <_R L(u)$ . ■

**Lemma 2** Let  $G$  be a tolerance graph and  $u$  be an unbounded vertex of  $G$  in a projection representation  $R$  of  $G$ . Then,  $l(v) <_R l(u)$  and  $R(u) <_R R(v)$  for every vertex  $v \neq u$ , such that  $P_v$  intersects  $P_u$  in  $R$  and  $\phi_v < \phi_u$ .

**Proof.** Suppose first that  $l(u) <_R l(v)$ . Then, since by assumption  $P_v$  intersects  $P_u$  in  $R$ , it follows that  $L(v) <_R L(u)$ , and thus  $\phi_v > \phi_u$  in  $R$ , which is a contradiction. Thus,  $l(v) <_R l(u)$ . Similarly, if  $R(v) <_R R(u)$ , then  $r(u) <_R r(v)$ , since  $P_v$  intersects  $P_u$  in  $R$ , and thus  $\phi_v > \phi_u$  in  $R$ , which is again a contradiction. Thus,  $R(u) <_R R(v)$ . ■

In Figure 2(a) an example for Lemma 1 (resp. Lemma 2) is illustrated, where  $w$  is the unbounded vertex and  $v \in N(w)$  (resp.  $u$  is a vertex, such that  $P_u$  intersects  $P_w$  in  $R$  and  $\phi_u < \phi_w$ ).

**Lemma 3** *Let  $G = (V, E)$  be a tolerance graph,  $R$  be a projection representation of  $G$ , and  $u, v$  be two vertices of  $G$ . If  $uv \notin E$ ,  $P_u$  intersects  $P_v$  in  $R$ , and  $\phi_v < \phi_u$  in  $R$ , then  $N(u) \subseteq N(v)$ .*

**Proof.** Suppose first that  $u$  is a bounded vertex in  $R$ . Then, in both cases where  $v$  is bounded or unbounded,  $u$  is adjacent to  $v$  in  $R$ , since  $P_v \cap P_u \neq \emptyset$  and  $\phi_v < \phi_u$ . This is a contradiction, since  $vu \notin E$ , and thus  $u$  is an unbounded vertex of  $R$ . If  $v$  is a bounded vertex, then  $l(v) <_R l(u)$  and  $R(u) <_R R(v)$  by Lemma 2. Suppose that  $v$  is unbounded. If  $l(u) <_R l(v)$ , then  $L(v) <_R L(u)$ , since  $P_u$  intersects  $P_v$  in  $R$ , and thus  $\phi_v > \phi_u$ , which is a contradiction to the assumption. Therefore  $l(v) <_R l(u)$ , and thus also  $R(u) = L(u) <_R L(v) = R(v)$ , since  $P_u$  intersects  $P_v$  in  $R$ . Summarizing,  $l(v) <_R r(u) = l(u)$  and  $R(u) = L(u) <_R R(v)$  in both cases where  $v$  is bounded and unbounded. Consider now a vertex  $w \in N(u)$ . Then,  $w$  is a bounded vertex in  $R$ ,  $r(w) >_R r(u)$ , and  $L(w) <_R L(u)$  by Lemma 1. Furthermore,  $\phi_w > \phi_u > \phi_v$ . Therefore,  $r(w) >_R l(v)$  and  $L(w) <_R R(v)$ , and thus  $P_w$  intersects  $P_v$  in  $R$ . Thus, since also  $\phi_w > \phi_v$ , it follows that  $w \in N(v)$ . Therefore,  $N(u) \subseteq N(v)$ . ■

In [12, 18] the *hovering set* of an unbounded vertex in a tolerance graph has been defined. According to these definitions, the hovering set depends on a particular representation of the tolerance graph. In the following, we extend this definition to the notion of *covering* vertices of an arbitrary graph  $G$ , which is independent of any representation of  $G$ .

**Definition 5** *Let  $G = (V, E)$  be an arbitrary graph and  $u \in V$  be a vertex of  $G$ . Then,*

- *the set  $\mathcal{C}(u) = \{v \in V \setminus N[u] \mid N(u) \subseteq N(v)\}$  is the covering set of  $u$ , and every vertex  $v \in \mathcal{C}(u)$  is a covering vertex of  $u$ ,*
- *$V_0(u)$  is the set of connected components of  $G \setminus N[u]$  that have at least one covering vertex  $v \in \mathcal{C}(u)$  of  $u$ .*

Now, similarly to [12], we state the following auxiliary lemma.

**Lemma 4** *Let  $G = (V, E)$  be a tolerance graph and  $R$  be a canonical representation of  $G$ . Then, for every unbounded vertex  $u$  of  $G$  in  $R$ , there exists a covering vertex  $u^*$  of  $u$  in  $G$ , such that  $u^*$  is bounded in  $R$ ,  $P_{u^*}$  intersects  $P_u$  in  $R$ , and  $\phi_{u^*} < \phi_u$ . Thus, in particular  $V_0(u) \neq \emptyset$ .*

**Proof.** Let  $u$  be an arbitrary unbounded vertex of  $G$  in  $R$ . Since  $R$  is a canonical representation of  $G$ , if we make  $u$  a bounded vertex in  $R$ , then we introduce at last one new adjacency  $uu^*$  in  $G$  by Definitions 1 and 2. That is, there exists at least one vertex  $u^*$ , such that  $P_{u^*}$  intersects  $P_u$  in  $R$ ,  $\phi_{u^*} < \phi_u$ , and  $uu^* \notin E$ . Then, Lemma 3 implies that  $N(u) \subseteq N(u^*)$ , i.e.  $u^*$  is a covering vertex of  $u$ .

Suppose now that every covering vertex  $v$  of  $u$ , such that  $P_v$  intersects  $P_u$  in  $R$  and  $\phi_v < \phi_u$ , is unbounded, and let  $u^*$  be the vertex with the smallest slope  $\phi_{u^*}$  among them in  $R$ . Then, since  $P_{u^*}$  intersects  $P_u$  in  $R$  and  $\phi_{u^*} < \phi_u$ , it follows that  $l(u^*) = r(u^*) <_R l(u) = r(u)$  and  $L(u^*) = R(u^*) >_R L(u) = R(u)$ . Furthermore, since  $u^*$  is assumed to be unbounded, there exists similarly to the previous paragraph at least one vertex  $u^{**}$ , such that  $P_{u^{**}}$  intersects  $P_{u^*}$  in  $R$  and  $\phi_{u^{**}} < \phi_{u^*}$ , and thus  $N(u^*) \subseteq N(u^{**})$  by Lemma 3. Thus  $N(u) \subseteq N(u^{**})$ , since also  $N(u) \subseteq N(u^*)$ . Furthermore,  $l(u^{**}) <_R l(u^*)$  and  $R(u^*) <_R R(u^{**})$  by Lemma 2. That is,  $l(u^{**}) <_R l(u^*) <_R l(u)$  and  $R(u) <_R R(u^*) <_R R(u^{**})$ , and thus  $P_{u^{**}}$  intersects  $P_u$  in  $R$ . Moreover  $uu^{**} \notin E$ , since  $u$  is unbounded and  $\phi_{u^{**}} < \phi_{u^*} < \phi_u$ .

Summarizing,  $u^{**}$  is a covering vertex of  $u$ ,  $P_{u^{**}}$  intersects  $P_u$  in  $R$  and  $\phi_{u^{**}} < \phi_u$ . This is a contradiction, since  $\phi_{u^{**}} < \phi_{u^*}$ , and since  $u^*$  has by assumption the smallest slope  $\phi_{u^*}$  among the covering vertices  $v$  of  $u$ , such that  $P_v$  intersects  $P_u$  in  $R$  and  $\phi_v < \phi_u$ . Therefore, there exists for every unbounded vertex  $u$  at least one covering vertex  $u^*$  of  $u$ , such that  $P_{u^*}$  intersects  $P_u$  in  $R$ ,  $\phi_{u^*} < \phi_u$ , and  $u^*$  is bounded in  $R$ . Furthermore, note that  $u^* \in V_0(u)$ , and thus  $V_0(u) \neq \emptyset$ . This completes the proof of the lemma. ■

In the following, for simplicity of the presentation, we may not distinguish between the connected components of  $V_0(u)$  and the vertex set of these components. Note here that  $V_0(u) \neq \emptyset$  for every unbounded vertex  $u$  in a canonical representation  $R$ , as we proved in Lemma 4. In the next definition we introduce the notion of the right (resp. left) border property of a vertex  $u$  in a projection representation  $R$  of a tolerance graph  $G$ . This notion is of particular importance for the remainder of the paper.

**Definition 6** *Let  $G = (V, E)$  be a tolerance graph,  $u$  be an arbitrary vertex of  $G$ , and  $R$  be a projection representation of  $G$ . Then,  $u$  has the right (resp. left) border property in  $R$ , if there exists no pair of vertices  $w \in N(u)$  and  $x \in V_0(u)$ , such that  $P_w \ll_R P_x$  (resp.  $P_x \ll_R P_w$ ).*

Observe that, if a vertex  $u$  has the left border property in a projection representation  $R$  of a tolerance graph  $G$ , then  $u$  has the right border property in the reverse representation  $\hat{R}$  of  $R$ . We denote in the following by TOLERANCE the class of tolerance graphs, and we use the corresponding notations for the classes of bounded tolerance, cocomparability, and trapezoid graphs.

Let  $G \in \text{TOLERANCE} \cap \text{COCOMPARABILITY}$ . Then  $G$  is also a trapezoid graph [8]. Thus, since  $\text{TRAPEZOID} \subseteq \text{COCOMPARABILITY}$ , it follows that  $\text{TOLERANCE} \cap \text{COCOMPARABILITY} = \text{TOLERANCE} \cap \text{TRAPEZOID}$ . Furthermore, clearly  $\text{BOUNDED TOLERANCE} \subseteq (\text{TOLERANCE} \cap \text{TRAPEZOID})$ , since  $\text{BOUNDED TOLERANCE} \subseteq \text{TOLERANCE}$  and  $\text{BOUNDED TOLERANCE} \subseteq \text{TRAPEZOID}$ . In what follows, we consider a graph  $G \in (\text{TOLERANCE} \cap \text{TRAPEZOID}) \setminus \text{BOUNDED TOLERANCE}$ , assuming that one exists, and our aim is to get to a contradiction; namely, to prove that  $(\text{TOLERANCE} \cap \text{TRAPEZOID}) = \text{BOUNDED TOLERANCE}$ .

Now we state two lemmas that are of crucial importance for the proof of Theorems 1 and 2, (in Sections 3.1 and 3.2, respectively).

**Lemma 5** *Let  $G \in (\text{TOLERANCE} \cap \text{TRAPEZOID}) \setminus \text{BOUNDED TOLERANCE}$  with the smallest number of vertices and  $u$  be a vertex of  $G$ . Then, either  $V_0(u) = \emptyset$  or  $V_0(u)$  is connected.*

**Proof.** For the sake of contradiction, suppose that  $V_0(u)$  has at least two connected components, for some vertex  $u$  of  $G$ . Let  $v_1$  and  $v_2$  be two covering vertices of  $u$  that belong to two different connected components of  $V_0(u)$ . Since  $G$  has the smallest number of vertices in the class  $(\text{TOLERANCE} \cap \text{TRAPEZOID}) \setminus \text{BOUNDED TOLERANCE}$ ,  $G \setminus \{u\}$  is a bounded tolerance graph. Let  $R$  be any parallelogram representation of  $G \setminus \{u\}$ , and  $R'$  be the representation of  $G \setminus N[u]$  obtained by  $R$  if we remove all parallelograms that correspond to vertices of  $N(u)$ . Since  $v_1$  and  $v_2$  belong to different connected components of  $G \setminus N[u]$ , there is at least one line segment  $\ell$  between the connected components of  $v_1$  and  $v_2$  in  $G \setminus N[u]$ , which does not intersect any parallelogram of  $R'$ . Since  $N_G(u) \subseteq N_G(v_1)$  and  $N_G(u) \subseteq N_G(v_2)$ , and since  $\ell$  lies between  $P_{v_1}$  and  $P_{v_2}$  in  $R'$ , it follows that exactly the parallelograms of the vertices of  $N(u)$  intersect  $\ell$  in  $R$ . Thus, we can add the trivial parallelogram  $P_u = \ell$  to  $R$ , obtaining thus a parallelogram representation of  $G$ . Thus,  $G$  is a parallelogram graph, i.e. a bounded tolerance graph, which is a contradiction to the assumption. Therefore, either  $V_0(u) = \emptyset$  or  $V_0(u)$  is connected, for any vertex  $u$  of  $G$ . This completes the proof of the lemma. ■

The next lemma follows now easily by Lemmas 4 and 5.

**Lemma 6** *Let  $G \in (\text{TOLERANCE} \cap \text{TRAPEZOID}) \setminus \text{BOUNDED TOLERANCE}$  with the smallest number of vertices and  $v_1, v_2$  be distinct unbounded vertices of  $G$  in a canonical projection representation  $R$  of  $G$ . Then  $N(v_1) \neq N(v_2)$ .*



**Proof.** Suppose otherwise that  $N(v_1) = N(v_2)$  for two unbounded vertices  $v_1$  and  $v_2$  in  $R$ , i.e.  $v_2$  is a covering vertex of  $v_1$  and  $v_1$  is a covering vertex of  $v_2$ . Furthermore,  $v_1$  is an isolated vertex in  $G \setminus N[v_2]$ . Recall now by Lemma 4 that there exists at least one covering vertex  $v_2^*$  of  $v_2$  in  $R$ , such that  $v_2^*$  is bounded in  $R$ . Then, since  $v_1$  is unbounded and  $v_2^*$  is bounded in  $R$ , it follows that the covering vertices  $v_1$  and  $v_2^*$  of  $v_2$  do not lie in the same connected component of  $G \setminus N[v_2]$ . That is,  $V_0(v_2)$  is not connected, which is a contradiction by Lemma 5. Thus,  $N(v_1) \neq N(v_2)$ . ■

### 3 Main results

In this section we present our main results. Consider a graph  $G$  that is both a tolerance and a trapezoid graph, where  $R$  is a projection representation of  $G$ . Then, we choose a certain unbounded vertex  $u$  in  $R$  and we “eliminate”  $u$  in  $R$  in the following sense: assuming that  $R$  has  $k \geq 1$  unbounded vertices, we construct a projection representation  $R^*$  of  $G$  with  $k-1$  unbounded vertices, where all bounded vertices remain bounded and  $u$  is transformed to a bounded vertex. In Section 3.1 we deal with the case where the unbounded vertex  $u$  has the right or the left border property in  $R$ , while in Section 3.2 we deal with the case where  $u$  has neither the left nor the right border property in  $R$ . Finally we combine these two results in Section 3.3, in order to eliminate all  $k$  unbounded vertices in  $R$ , regardless of whether or not they have the right or left border property.

#### 3.1 The case where $u$ has the right or the left border property

In this section we consider an arbitrary unbounded vertex  $u$  of  $G$  in the projection representation  $R$ , and we assume that  $u$  has the right or the left border property in  $R$ . Then, as we prove in the next theorem, there is another projection representation  $R^*$  of  $G$ , in which  $u$  has been replaced by a bounded vertex.

**Theorem 1** *Let  $G = (V, E) \in (\text{TOLERANCE} \cap \text{TRAPEZOID}) \setminus \text{BOUNDED TOLERANCE}$  with the smallest number of vertices. Let  $R$  be a projection representation of  $G$  with  $k$  unbounded vertices and  $u$  be an unbounded vertex in  $R$ . If  $u$  has the right or the left border property in  $R$ , then there exists a projection representation  $R^*$  of  $G$  with  $k-1$  unbounded vertices.*

**Proof.** If  $R$  is not a canonical representation of  $G$ , then there exists a projection representation  $R^*$  of  $G$  with  $k-1$  unbounded vertices by Definition 2. Suppose in the sequel that  $R$  is a canonical representation of  $G$ . Then, for the unbounded vertex  $u$  of  $G$  in  $R$ , there exists at least one bounded covering vertex  $u^*$  of  $u$  by Lemma 4. Therefore  $V_0(u) \neq \emptyset$ , and thus  $V_0(u)$  is connected by Lemma 5. The proof is done constructively. Namely, we will construct the projection representations  $R'$ ,  $R''$ , and  $R'''$ , by applying to  $R$  sequentially the Transformations 1, 2, and 3, respectively. Finally,  $R'''$  is a projection representation of the same graph  $G$  with  $k-1$  unbounded vertices, where  $u$  is represented as a bounded vertex in  $R'''$ .

For simplicity reasons, we add in  $G$  an isolated bounded vertex  $t$ . This vertex  $t$  corresponds to a parallelogram  $P_t$ , such that  $P_v \ll_R P_t$  for every vertex  $v$  of  $G$ . Recall that  $V_B$  and  $V_U$  denote the sets of bounded and unbounded vertices of  $G$  in  $R$ , respectively (note that  $t \in V_B$ ). First, we define for every  $w \in N(u)$  the value  $L_0(w) = \min_R \{L(x) \mid x \in V_B \setminus N(u), P_w \ll_R P_x\}$ . Note that the value  $L_0(w)$  is well defined for every  $w \in N(u)$ , since in particular  $t \in V_B \setminus N(u)$  and  $P_w \ll_R P_t$ . Moreover, for every  $w \in N(u)$ ,  $w$  is a bounded vertex and  $\phi_w > \phi_u$ . For every vertex  $x \in V_B \setminus N(u)$ , such that  $P_w \ll_R P_x$  for some  $w \in N(u)$ , it follows that  $x \notin V_0(u)$  by Definition 6, since  $u$  has the right border property in  $R$  by assumption. Thus, for every  $w \in N(u)$ ,  $L_0(w) = \min_R \{L(x) \mid x \in (V_B \setminus N(u)) \setminus V_0(u), P_w \ll_R P_x\}$ . Define now the value  $\ell_0 = \max_R \{L(x) \mid x \in V_0(u)\}$  and the subset  $N_1 = \{w \in N(u) \mid r(w) <_R \ell_0\}$  of neighbors of  $u$ .

An example of a projection representation  $R$  of a tolerance graph  $G$  with seven vertices is illustrated in Figure 3(a). In this figure, the parallelogram  $P_u$  of the unbounded vertex  $u$  is illustrated by a bold and dotted line. The transparent parallelograms  $P_{w_1}$  and  $P_{w_2}$  correspond to the neighbors  $N(u) = \{w_1, w_2\}$  of  $u$  in  $G$ , the light colored parallelograms  $P_{u^*}$  and  $P_x$  correspond to the vertices

of  $V_0(u) = \{u^*, x\}$ , and the dark colored parallelograms  $P_y$  and  $P_t$  correspond to the vertices of  $(V \setminus N[u]) \setminus V_0(u) = \{y, t\}$ . In this example,  $L_0(w_1) = L(t)$ ,  $L_0(w_2) = L(y)$ , and  $\ell_0 = l(x)$ , while  $N_1 = \{w_1, w_2\}$ .

We construct now the projection representation  $R'$  from  $R$  as follows.

**Transformation 1** *For every  $w \in N_1$ , move the right line of  $P_w$  parallel to the right, until either  $r(w)$  comes immediately after  $\ell_0$  on  $L_2$ , or  $R(w)$  comes immediately before  $L_0(w)$  on  $L_1$ . Denote the resulting projection representation by  $R'$ .*

Note that the left lines of all parallelograms do not move during Transformation 1. Thus, in particular, the value of  $\ell_0$  is the same in  $R$  and in  $R'$ , i.e.  $\ell_0 = \max_{R'} \{l(x) \mid x \in V_0(u)\}$ . As we will prove in Lemma 8, the representation  $R'$  is a projection representation of the same graph  $G$ , and thus the parallelograms of two bounded vertices intersect in  $R$  if and only if they intersect also in  $R'$ . Therefore, for every  $w \in N(u)$  the value  $L_0(w)$  remains the same in  $R$  and in  $R'$ , i.e.  $L_0(w) = \min_{R'} \{L(x) \mid x \in (V_B \setminus N(u)) \setminus V_0(u), P_w \ll_{R'} P_x\}$  for every  $w \in N(u)$ . Define now the subset  $N_2 = \{w \in N(u) \mid \ell_0 <_{R'} r(w)\}$  of neighbors of  $u$ . If  $N_2 \neq \emptyset$ , we define the value  $r_0 = \min_{R'} \{r(w) \mid w \in N_2\}$ . Then,  $r_0 >_{R'} r(u)$  by Lemma 1, since  $N_2 \subseteq N(u)$ . Since the lower right endpoint  $r(w)$  of all parallelograms  $P_w$  in  $R'$  is greater than or equal to the corresponding value  $r(w)$  in  $R$ , it follows that  $N(u) \setminus N_1 = \{w \in N(u) \mid \ell_0 <_R r(w)\} \subseteq \{w \in N(u) \mid \ell_0 <_{R'} r(w)\} = N_2$ . Thus,  $N(u) \setminus N_2 \subseteq N_1$  and  $N_2 \cup (N_1 \setminus N_2) = N(u)$ .

Define now the value  $L_0 = \min_{R'} \{L(x) \mid x \in (V_B \setminus N(u)) \setminus V_0(u), P_u \ll_{R'} P_x\}$ ; again,  $L_0$  is well defined, since in particular  $t \in (V_B \setminus N(u)) \setminus V_0(u)$  and  $P_u \ll_{R'} P_t$ . The following property of the projection representation  $R'$  can be obtained easily by Transformation 1.

**Lemma 7** *For all vertices  $w \in N_1 \setminus N_2$ , for which  $R(w) <_{R'} L_0$ , the values  $R(w)$  lie immediately before  $L_0$  in  $R'$ .*

**Proof.** Let  $w \in N_1 \setminus N_2$ . By definition of the sets  $N_1$  and  $N_2$ , it follows that  $r(w) <_R \ell_0$  and  $r(w) <_{R'} \ell_0$  in both  $R$  and  $R'$ . Thus,  $R(w)$  comes immediately before  $L_0(w)$  in  $R'$  during Transformation 1. Consider now a vertex  $x \in (V_B \setminus N(u)) \setminus V_0(u)$ , such that  $P_w \ll_R P_x$ , i.e.  $r(w) <_R l(x)$  and  $R(w) <_R L(x)$ . Then  $r(u) <_R l(x)$ , since  $r(u) <_R r(w)$  by Lemma 1. Suppose that  $L(x) <_R R(u)$ . Then,  $P_x$  intersects  $P_u$  in  $R$  and  $\phi_x > \phi_u$ . Thus, since  $x$  is assumed to be bounded, it follows that  $x \in N(u)$ , which is a contradiction. Therefore  $R(u) <_R L(x)$ , and thus  $P_u \ll_R P_x$ , since also  $r(u) <_R l(x)$ . Furthermore, also  $P_u \ll_{R'} P_x$ , since  $P_u$  and  $P_x$  remain the same in both  $R$  and  $R'$ . That is,  $P_u \ll_{R'} P_x$  for every  $x \in (V_B \setminus N(u)) \setminus V_0(u)$ , such that  $P_w \ll_R P_x$ . Therefore, it follows by the definitions of  $L_0$  and of  $L_0(w)$  that  $L_0 \leq L_0(w)$ . Thus, since  $R(w)$  comes immediately before  $L_0(w)$  in  $R'$  during Transformation 1, it follows that either  $R(w)$  comes immediately before  $L_0$  in  $R'$  during Transformation 1 (in the case where  $L_0 = L_0(w)$ ) or  $R(w) >_{R'} L_0$  (in the case where  $L_0 < L_0(w)$ ). This completes the proof of the lemma. ■

For the example of Figure 3, the projection representation  $R'$  is illustrated in Figure 3(b). In this figure,  $L_0 = L(y)$  and  $r_0 = r(w_1)$ , while  $N_2 = \{w_1\}$  and  $N_1 \setminus N_2 = \{w_2\}$ .

If  $N_2 = \emptyset$ , then we set  $R'' = R'$ ; otherwise, if  $N_2 \neq \emptyset$ , we construct the projection representation  $R''$  from  $R'$  as follows.

**Transformation 2** *For every  $v \in V_0(u) \cap V_B$ , such that  $r(v) >_{R'} r_0$ , move the right line of  $P_v$  in  $R'$  parallel to the left, such that  $r(v)$  comes immediately before  $r_0$  in  $L_2$ . Denote the resulting projection representation by  $R''$ .*

Since by Transformation 2 only some endpoints of vertices  $v \in V_0(u) \cap V_B$  are moved, it follows that the value  $L_0$  does not change in  $R''$ , i.e.  $L_0 = \min_{R''} \{L(x) \mid x \in (V_B \setminus N(u)) \setminus V_0(u), P_u \ll_{R''} P_x\}$ . The next property of the projection representation  $R''$  follows by Lemma 7.

**Corollary 1** *For all vertices  $w \in N_1 \setminus N_2$ , for which  $R(w) <_{R''} L_0$ , the values  $R(w)$  lie immediately before  $L_0$  in  $R''$ .*

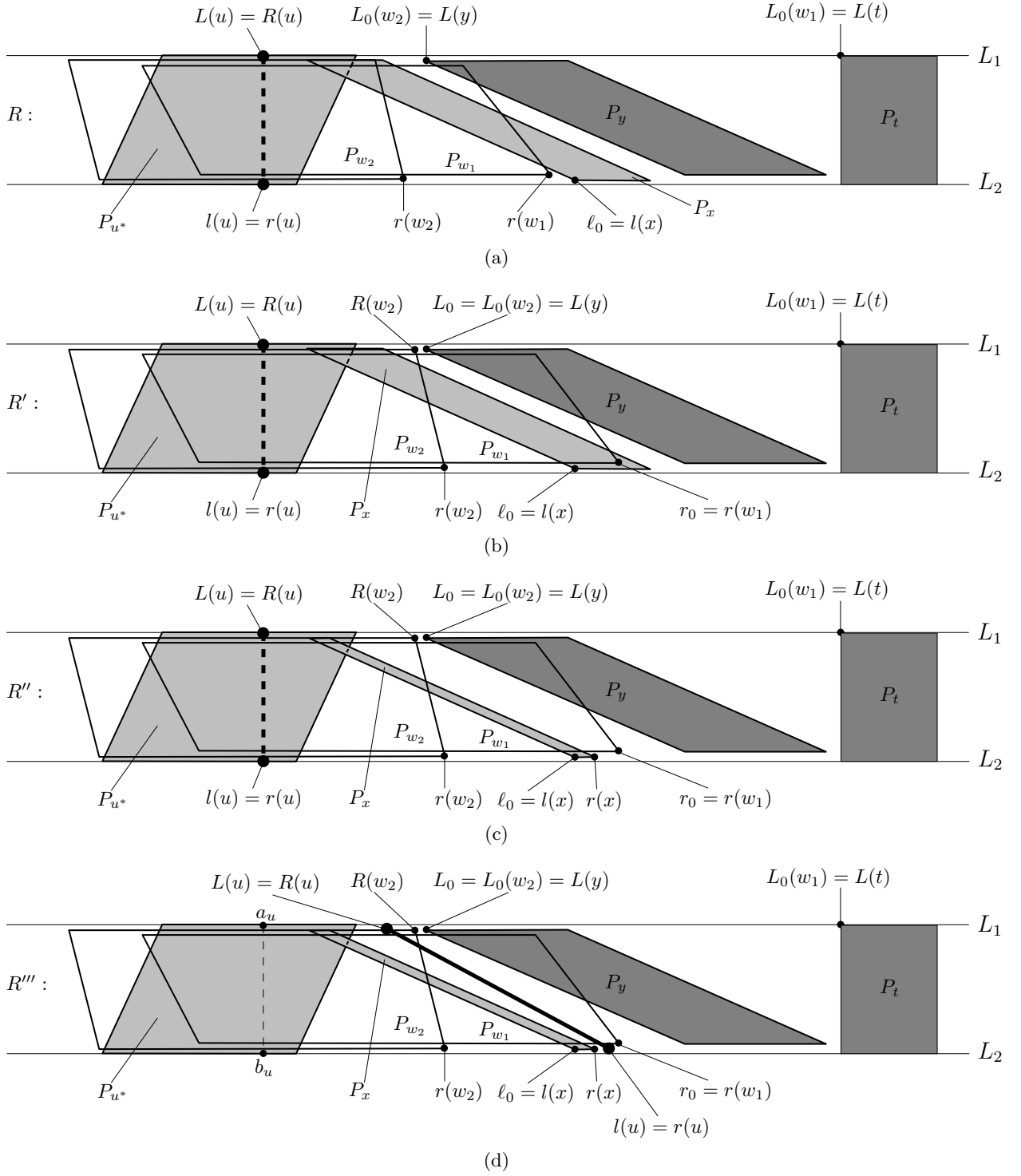


Figure 3: (a) The projection representation  $R$  of a tolerance graph  $G$  with seven vertices, and the projection representations (b)  $R'$  after Transformation 1, (c)  $R''$  after Transformation 2, and (d)  $R'''$  after Transformation 3.

**Proof.** Let  $x_0$  be the vertex of  $(V_B \setminus N(u)) \setminus V_0(u)$ , such that  $L_0 = L(x_0)$ . Recall by Lemma 7 that for all vertices  $w \in N_1 \setminus N_2$ , for which  $R(w) <_{R'} L_0$ , the values  $R(w)$  lie immediately before  $L_0$  in  $R'$ . Furthermore, note that the parallelograms of all neighbors  $w \in N(u)$  of  $u$  do not move by Transformation 2. Therefore, since also the value  $L_0$  is the same in both  $R'$  and  $R''$ , it suffices to prove that there do not exist vertices  $v \in V_0(u) \cap V_B$  and  $w \in N_1 \setminus N_2$ , such that  $R(w) <_{R''} R(v) <_{R''} L_0$  in  $R''$ . Suppose otherwise that  $R(w) <_{R''} R(v) <_{R''} L_0 = L(x_0)$  for two vertices  $v \in V_0(u) \cap V_B$  and  $w \in N_1 \setminus N_2$ . Thus, since only the right lines of some parallelograms  $P_v$ ,

where  $v \in V_0(u) \cap V_B$ , are moved to the left by Transformation 2, it follows that  $R(w) <_{R'} L_0 = L(x_0) <_{R'} R(v)$  in  $R'$ . Therefore, in particular  $P_v$  intersects  $P_{x_0}$  in  $R'$ , and thus  $v \in N(x_0)$ , since both  $v$  and  $x_0$  are bounded. Thus  $x_0 \in V_0(u)$ , since also  $v \in V_0(u)$ . This is a contradiction, since  $x_0 \in (V_B \setminus N(u)) \setminus V_0(u)$ . This completes the proof of the corollary. ■

The projection representation  $R''$  for the example of Figure 3 is illustrated in Figure 3(c). We construct now the projection representation  $R'''$  from  $R''$  as follows.

**Transformation 3** *Move the line  $P_u$  in  $R''$ , such that its upper endpoint  $L(u) = R(u)$  comes immediately before  $\min_{R''}\{L_0, R(w) \mid w \in N_1 \setminus N_2\}$  and its lower endpoint  $l(u) = r(u)$  comes immediately after  $\max_{R''}\{r(v) \mid v \in V_0(u) \cap V_B\}$ . Finally, make  $u$  a bounded vertex. Denote the resulting projection representation by  $R'''$ .*

The resulting projection representation  $R'''$  has  $k - 1$  unbounded vertices, since  $u$  is represented in  $R'''$  as a bounded vertex. The projection representation  $R'''$  for the example of Figure 3 is illustrated in Figure 3(d). In this figure, the new position of the trivial parallelogram (i.e. line)  $P_u$  that corresponds to the (bounded) vertex  $u$  is drawn in bold. Furthermore, for better visibility, the position of  $P_u$  in the previous projection representations  $R$ ,  $R'$ , and  $R''$  is pointed by a non-bold dashed line; in this figure,  $a_u$  and  $b_u$  denote the endpoints of this old position of  $P_u$  on  $L_1$  and on  $L_2$ , respectively.

In the following three lemmas, we prove sequentially that  $R'$ ,  $R''$ , and  $R'''$  are all projection representations of the same tolerance graph  $G$ , and thus  $R^* = R'''$  is a projection representation of  $G$  with  $k - 1$  unbounded vertices.

**Lemma 8**  *$R'$  is a projection representation of  $G$ .*

**Proof.** Denote by  $x_0$  the vertex of  $V_0(u)$ , such that  $\ell_0 = l(x_0)$ . Recall by Lemma 4 that there exists a covering vertex  $u^*$  of  $u$  in  $G$ , such that  $u^*$  is bounded in  $R$ . Since we move the right line of some parallelograms to the right, i.e. we increase some parallelograms, all adjacencies of  $R$  are kept in  $R'$ . Suppose that  $R'$  has the new adjacency  $wv$  that is not an adjacency in  $R$ , for some  $w \in N_1$ . Therefore, since we perform parallel movements of lines, i.e. since every slope  $\phi_z$  in  $R'$  equals the value of  $\phi_z$  in  $R$  for every vertex  $z$  of  $G$ , it follows that  $P_w \ll_R P_v$  and  $P_w$  intersects  $P_v$  in  $R'$ . Thus  $v \notin V_0(u)$ , since  $u$  has the right border property in  $R$  by assumption. Furthermore  $r(w) <_R \ell_0 = l(x_0)$ , since  $w \in N_1$ . However, since  $x_0 \in V_0(u)$ , and since  $u$  has the right border property in  $R$ , it follows that  $P_w$  intersects  $P_{x_0}$  in  $R$ , and thus  $L(x_0) <_R R(w)$ .

Moreover,  $r(u) <_R r(w) <_R l(x_0)$  and  $L(w) <_R L(u)$  by Lemma 1. Suppose that  $L(x_0) <_R L(u) = R(u)$ . Then,  $P_u$  intersects  $P_{x_0}$  in  $R$  and  $\phi_{x_0} > \phi_u$ . Thus,  $x_0$  is unbounded, since otherwise  $x_0 \in N(u)$ , which is a contradiction. Furthermore,  $N(x_0) \subseteq N(u)$  by Lemma 3, and thus  $x_0$  is an isolated vertex of  $G \setminus N[u]$ . Therefore, since  $x_0$  is unbounded and  $u^*$  is bounded in  $R$ , it follows that  $x_0$  and  $u^*$  do not lie in the same connected component of  $G \setminus N[u]$ . That is,  $V_0(u)$  is not connected, which is a contradiction. Thus,  $L(u) = R(u) <_R L(x_0)$ , i.e.  $R(u) <_R L(x_0) <_R R(w) <_R L(v)$  and  $r(u) <_R r(w) <_R l(v)$ , which implies that  $P_u \ll_R P_v$ , and thus  $v \notin N(u)$ .

Consider now the projection representation  $R'$  constructed by Transformation 1. Let first  $r(w) <_{R'} l(v)$ . Then, since  $P_w$  intersects  $P_v$  in  $R'$ , it follows that  $L(v) <_{R'} R(w)$ , and thus  $\phi_v > \phi_w$ . If  $v$  is an unbounded vertex, then  $w$  is not adjacent to  $v$  in  $R'$ , which is a contradiction to the assumption. Thus,  $v$  is a bounded vertex. Recall that  $P_w \ll_R P_v$  and that  $v \notin V_0(u)$  and  $v \notin N(u)$ , i.e.  $v \in (V_B \setminus N(u)) \setminus V_0(u)$ , and thus  $L_0(w) \leq_R L(v)$  in  $R$  by definition of  $L_0(w)$ . Furthermore, since the left lines of the parallelograms in  $R$  do not move during Transformation 1, it remains also  $L_0(w) \leq_{R'} L(v)$  in  $R'$ . Therefore, since  $R(w) <_{R'} L_0(w)$  by definition of Transformation 1, it follows that  $R(w) <_{R'} L(v)$ , which is a contradiction, since  $L(v) <_{R'} R(w)$ , as we proved above in this paragraph.

Let now  $l(v) <_{R'} r(w)$ . Suppose that  $l(x_0) <_{R'} l(v)$ . Then, since  $r(w)$  comes in  $R'$  at most immediately after  $\ell_0 = l(x_0)$  on  $L_2$ , it follows that also  $r(w) <_{R'} l(v)$ , which is a contradiction. Therefore,  $l(v) <_{R'} l(x_0)$ , and thus since the left lines of the parallelograms in  $R$  do not move

during Transformation 1, it follows that also  $l(v) <_R l(x_0)$ . Furthermore, since  $L(x_0) <_R R(w)$  and  $P_w \ll_R P_v$ , it follows that  $L(x_0) <_R R(w) <_R L(v)$ , and thus  $P_{x_0}$  intersects  $P_v$  in  $R$  and  $\phi_{x_0} > \phi_v$ . Now, if  $x_0$  is bounded, then  $x_0v \in E$ . Thus,  $v \in V_0(u)$ , since  $x_0 \in V_0(u)$  and  $v \notin N(u)$ , which is a contradiction. Therefore,  $x_0$  is unbounded, and thus  $x_0v \notin E$ . Then, since  $P_{x_0}$  intersects  $P_v$  in  $R$  and  $\phi_{x_0} > \phi_v$ , it follows that  $N(x_0) \subseteq N(v)$  by Lemma 3. Recall now that there exists a bounded covering vertex  $u^*$  of  $u$  in  $G$ , and thus  $u^*, x_0 \in V_0(u)$ . Furthermore  $u^* \neq x_0$ , since  $u^*$  is bounded and  $x_0$  is unbounded. Therefore, since  $V_0(u)$  is connected,  $x_0$  is adjacent to at least one other vertex  $y \in V_0(u)$ , and thus  $y \in N(v)$ , since  $N(x_0) \subseteq N(v)$ . It follows now that  $v \in V_0(u)$ , since  $y \in V_0(u)$  and  $v \notin N(u)$ , which is again a contradiction.

Therefore,  $R'$  has no new adjacency  $wv$  that is not an adjacency in  $R$ , for any  $w \in N_1$ , i.e.  $R'$  is a projection representation of  $G$ . This completes the proof of the lemma. ■

**Lemma 9**  $R''$  is a projection representation of  $G$ .

**Proof.** Denote by  $w_0$  the vertex of  $N_2$ , such that  $r_0 = r(w_0)$ . Since we move the right line of some parallelograms to the left, i.e. we decrease some parallelograms, no new adjacencies are introduced in  $R''$  in comparison to  $R'$ . Suppose that the adjacency  $vx$  has been removed from  $R'$  in  $R''$ , for some  $v \in V_0(u) \cap V_B$ , where  $r(v) >_{R'} r_0 = r(w_0)$ . Therefore, since we perform parallel movements of lines in  $R'$ , i.e. since every slope  $\phi_z$  in  $R''$  equals the value of  $\phi_z$  in  $R'$  for every vertex  $z$  of  $G$ , it follows that  $P_v \ll_{R''} P_x$ , while  $P_v$  intersects  $P_x$  in  $R'$ .

Since  $w_0 \in N(u)$ , and since the endpoints of  $P_{w_0}$  do not move during Transformation 2, it follows by Lemma 1 that  $r(u) <_{R'} r(w_0)$  and  $r(u) <_{R''} r(w_0)$ . Thus, since  $r(v)$  comes in  $R''$  immediately before  $r_0 = r(w_0)$ , it follows that  $r(u) <_{R''} r(v) <_{R''} r(w_0)$ . Suppose that  $x \in N(u)$ . Then,  $L(x) <_{R'} L(u)$  by Lemma 1, and thus also  $L(x) <_{R''} L(u)$ , since the left lines of all parallelograms do not move during Transformation 2. Therefore,  $R(v) <_{R''} L(x) <_{R''} L(u) = R(u)$ , since  $P_v \ll_{R''} P_x$ . That is,  $r(u) <_{R''} r(v)$  and  $L(v) \leq_{R''} R(v) <_{R''} R(u)$ , and thus  $\phi_v > \phi_u$  in both  $R'$  and  $R''$ . Furthermore,  $L(v) <_{R'} R(u)$  (since also  $L(v) <_{R''} R(u)$ ) and  $r(u) <_{R'} r_0 = r(w_0) <_{R'} r(v)$ , and thus  $P_v$  intersects  $P_u$  in  $R'$ . Therefore, since  $v \in V_B$  and  $\phi_v > \phi_u$  in  $R'$ , it follows that  $v \in N(u)$ , which is a contradiction. Thus,  $x \notin N(u)$ .

Now, since by assumption  $vx \in E$ , and since  $v \in V_0(u)$  and  $x \notin N(u)$ , it follows that  $x \in V_0(u)$ , and thus  $l(x) \leq_R \ell_0$  by definition of  $\ell_0$ . Therefore, since the left lines of all parallelograms do not move during Transformation 1, it follows that also  $l(x) \leq_{R'} \ell_0$ . Note that both  $r_0 = r(w_0)$  and  $l(x)$  do not move by Transformation 2. Therefore, since  $r(v)$  comes by Transformation 2 in  $R''$  immediately before  $r_0$ , and since  $P_v \ll_{R''} P_x$ , it follows that  $r(v) <_{R''} r_0 = r(w_0) <_{R''} l(x)$ . Finally, since both  $r(w_0)$  and  $l(x)$  do not move during Transformation 2, it follows that also  $r(w_0) <_{R'} l(x)$  in  $R'$ . Thus, since  $l(x) \leq_{R'} \ell_0$ , it follows that  $r(w_0) <_{R'} \ell_0$  in  $R'$ , which is a contradiction, since  $w_0 \in N_2$ . Therefore, no adjacency  $vx$  has been removed from  $R'$  in  $R''$ , i.e.  $R''$  is a projection representation of  $G$ . This completes the proof of the lemma. ■

**Lemma 10**  $R'''$  is a projection representation of  $G$ .

**Proof.** The proof is done in two parts. In Part 1 we prove that  $u$  is adjacent in  $R'''$  to all vertices of  $N(u)$ , while in Part 2 we prove that  $u$  is not adjacent in  $R'''$  to any vertex of  $V \setminus N[u]$ .

*Part 1.* In this part we prove that  $u$  is adjacent in  $R'''$  to all vertices of  $N(u)$ . Denote by  $a_u$  and  $b_u$  the coordinates of the upper and lower endpoint of  $P_u$  in the initial projection representation  $R$  on  $L_1$  and on  $L_2$ , respectively. Then, since the endpoints of  $P_u$  do not move by Transformations 1 and 2,  $a_u$  and  $b_u$  remain the endpoints of  $P_u$  also in the representations  $R'$  and  $R''$ ; however, note that  $a_u$  and  $b_u$  are not the endpoints of  $P_u$  in  $R'''$ . Then,  $L(w) <_{R''} a_u$  for every  $w \in N(u)$  by Lemma 1, and thus also  $L(w) <_{R'''} a_u$  for every  $w \in N(u)$ , since only the endpoints of  $P_u$  move during Transformation 3.

Note now that  $a_u <_{R''} L_0$ , since  $L_0 = \min_{R''} \{L(x) \mid x \in (V_B \setminus N(u)) \setminus V_0(u), P_u \ll_{R''} P_x\}$ . Furthermore, recall by Corollary 1 that for all vertices  $w \in N_1 \setminus N_2$ , for which  $R(w) <_{R''} L_0$ , the values  $R(w)$  lie immediately before  $L_0$  in  $R''$ . Therefore, in particular,  $a_u <_{R''} R(w)$  for every

$w \in N_1 \setminus N_2$ , since  $a_u <_{R''} L_0$ , and thus  $L(w) <_{R''} a_u <_{R''} R(w)$  for every  $w \in N_1 \setminus N_2 \subseteq N(u)$  by the previous paragraph. Therefore, since  $a_u <_{R''} L_0$ , and since the upper endpoint  $R(u)$  of the line  $P_u$  lies in  $R'''$  immediately before  $\min_{R''}\{L_0, R(w) \mid w \in N_1 \setminus N_2\}$ , cf. the statement of Transformation 3, it follows that also  $L(w) <_{R'''} a_u <_{R'''} R(u) <_{R'''} R(w)$  for every  $w \in N_1 \setminus N_2$ . That is,  $L(w) <_{R'''} R(u) <_{R'''} R(w)$  for every  $w \in N_1 \setminus N_2$ , and thus  $P_u$  intersects  $P_w$  in  $R'''$  for every  $w \in N_1 \setminus N_2$ . Therefore, since all vertices of  $\{u\} \cup N_1 \setminus N_2$  are bounded in  $R'''$ ,  $u$  is adjacent in  $R'''$  to all vertices of  $N_1 \setminus N_2$ .

Consider now an arbitrary vertex  $w \in N_2$ . Recall that  $r_0 = \min_{R'}\{r(w) \mid w \in N_2\}$ , i.e.  $r_0 \leq_{R'} r(w)$ . Thus, since the endpoint  $r(w)$  does not move by Transformation 2, it follows that also  $r_0 \leq_{R''} r(w)$ . Furthermore, by Transformation 2,  $r(v) <_{R''} r_0 \leq_{R''} r(w)$  for every  $v \in V_0(u) \cap V_B$ . This holds clearly also in  $R'''$ , i.e.  $r(v) <_{R'''} r(w)$  for every  $v \in V_0(u) \cap V_B$  and every  $w \in N_2$ . Since the lower endpoint of the line  $P_u$  comes immediately after  $\max_{R''}\{r(v) \mid V_0(u) \cap V_B\}$  in  $R'''$ , it follows that  $r(v) <_{R'''} l(u) = r(u) <_{R'''} r(w)$  for every  $v \in V_0(u) \cap V_B$  and every  $w \in N_2$ . Thus, since also  $L(w) <_{R'''} a_u <_{R'''} R(u)$  for every  $w \in N(u)$ , it follows that  $P_u$  intersects  $P_w$  in  $R'''$  for every  $w \in N_2$ . Therefore, since all vertices of  $\{u\} \cup N_2$  are bounded in  $R'''$ ,  $u$  is adjacent in  $R'''$  to all vertices of  $N_2$ . Thus, since  $N_2 \cup (N_1 \setminus N_2) = N(u)$ ,  $u$  is adjacent in  $R'''$  to all vertices of  $N(u)$ .

*Part 2.* In this part we prove that  $u$  is not adjacent in  $R'''$  to any vertex of  $V \setminus N[u]$ . To this end, recall first by Lemma 4 that  $u^*$  is a bounded covering vertex of  $u$  in  $G$  (and thus  $u^* \in V_0(u) \cap V_B$ ), such that  $P_u$  intersects  $P_{u^*}$  in  $R$  and  $\phi_{u^*} < \phi_u$  in  $R$ . Therefore,  $l(u^*) <_R l(u) = r(u)$  by Lemma 2, and thus also  $l(u^*) <_{R''} r(u)$ , since the endpoint  $l(u^*)$  remains the same in the representations  $R$ ,  $R'$ , and  $R''$ . Recall now that  $L_0 = \min_{R''}\{L(x) \mid x \in (V_B \setminus N(u)) \setminus V_0(u), P_u \ll_{R''} P_x\}$ . Denote by  $y_0$  the vertex of  $(V_B \setminus N(u)) \setminus V_0(u)$ , such that  $L_0 = L(y_0)$ , and thus  $P_u \ll_{R''} P_{y_0}$ . Therefore, since  $l(u^*) <_{R''} r(u)$ , it follows that  $l(u^*) <_{R''} l(u) <_{R''} l(y_0)$ . Since  $u^* \in V_0(u)$  and  $y_0 \notin N(u) \cup V_0(u)$ , it follows that  $u^*y_0 \notin E$ . Therefore, since both  $u^*$  and  $y_0$  are bounded vertices,  $P_{u^*}$  does not intersect  $P_{y_0}$  in  $R''$ , and thus  $P_{u^*} \ll_{R''} P_{y_0}$ , since  $l(u^*) <_{R''} l(y_0)$ . Moreover, since by Transformation 3 only the line  $P_u$  is moved, it follows that also  $P_{u^*} \ll_{R'''} P_{y_0}$ .

Since by Transformation 1 only some endpoints of vertices  $w \in N_1 \subseteq N(u)$  are moved, the value  $R(u^*)$  remains the same in  $R$  and in  $R'$ . Furthermore,  $r(u) <_{R'} r_0$  by definition of  $r_0$  and by Lemma 1. Suppose that the right line of  $P_{u^*}$  is moved during Transformation 2. Then,  $r(u) <_{R'} r_0 <_{R'} r(u^*)$ , while  $r(u^*)$  comes immediately before  $r_0$  in  $R''$ , i.e.  $r(u) <_{R''} r(u^*) <_{R''} r_0$ , since  $r_0$  does not move during Transformation 2. Therefore, since  $l(u^*) <_R l(u)$  by Lemma 2 (and thus also  $l(u^*) <_{R''} l(u)$ ), it follows that  $P_{u^*}$  still intersects  $P_u$  in  $R''$ .

Denote by  $v_0$  the vertex of  $V_0(u) \cap V_B$ , such that  $r(v_0) = \max_{R''}\{r(v) \mid v \in V_0(u) \cap V_B\}$ , cf. the statement of Transformation 3. Since  $v_0 \in V_0(u)$  and  $y_0 \notin N(u) \cup V_0(u)$ , it follows that  $v_0y_0 \notin E$ . Therefore, since both  $v_0$  and  $y_0$  are bounded vertices, either  $P_{y_0} \ll_{R''} P_{v_0}$  or  $P_{v_0} \ll_{R''} P_{y_0}$ . Suppose that  $P_{y_0} \ll_{R''} P_{v_0}$ , and thus  $P_{u^*} \ll_{R''} P_{y_0} \ll_{R''} P_{v_0}$ . Then, since  $u^*, v_0 \in V_0(u)$  and since  $V_0(u)$  is connected, there exists at least one vertex  $v \in V_0(u)$ , such that  $P_v$  intersects  $P_{y_0}$  in  $R''$ . Similarly, since  $y_0 \notin N(u) \cup V_0(u)$ , it follows that  $vy_0 \notin E$ . Therefore, since  $y_0$  is a bounded vertex,  $v$  must be an unbounded vertex with  $\phi_v > \phi_{y_0}$ , and thus  $N(v) \subseteq N(y_0)$  by Lemma 3. Then,  $N(v)$  includes at least one vertex  $v' \in V_0(u)$ , and thus  $v' \in N(y_0)$ . Therefore,  $y_0 \in V_0(u)$ , which is a contradiction. Thus,  $P_{v_0} \ll_{R''} P_{y_0}$ . Moreover, since by Transformation 3 only the line  $P_u$  is moved, it follows that also  $P_{v_0} \ll_{R'''} P_{y_0}$ .

We will prove in the following that  $u$  is not adjacent in  $R'''$  to any vertex  $x \notin N(u)$ . For the sake of contradiction, suppose that  $P_x$  intersects  $P_u$  in  $R'''$ , for some vertex  $x \notin N(u)$ . We distinguish in the following the cases regarding  $x$ .

*Case 2a.*  $x \in V_B \setminus N(u)$  (i.e.  $x$  is bounded) and  $x \in V_0(u)$ . Then,  $r(x) \leq_{R''} r(v_0)$  and  $r(u^*) \leq_{R''} r(v_0)$  by definition of  $v_0$ , and thus also  $r(x) \leq_{R'''} r(v_0)$  and  $r(u^*) \leq_{R'''} r(v_0)$ . Therefore, by Transformation 3,  $r(x) \leq_{R'''} r(v_0) <_{R'''} l(u)$ , i.e.  $r(x) <_{R'''} l(u)$ , and thus  $L(u) <_{R'''} R(x)$ , since we assumed that  $P_x$  intersects  $P_u$  in  $R'''$ . Furthermore,  $r(x) \leq_{R'''} r(v_0) <_{R'''} l(y_0)$ , i.e.  $r(x) <_{R'''} l(y_0)$ , since  $P_{v_0} \ll_{R'''} P_{y_0}$ . Recall by Corollary 1 that for all vertices  $w \in N_1 \setminus N_2$ , for which  $R(w) <_{R''} L_0 = L(y_0)$ , the values  $R(w)$  lie immediately before  $L_0$  in  $R''$ , and thus also in  $R'''$ .

Thus, since  $L(u) <_{R'''} R(x)$ , and since the upper endpoint  $L(u) = R(u)$  of  $P_u$  comes immediately before  $\min\{L_0, R(w) \mid w \in N_1 \setminus N_2\}$  in  $R'''$ , it follows that  $L(u) <_{R'''} L_0 = L(y_0) <_{R'''} R(x)$ . Therefore, since also  $r(x) <_{R'''} l(y_0)$ ,  $P_x$  intersects  $P_{y_0}$  in  $R'''$ , and thus also in  $R''$ . Then  $xy_0 \in E$ , since both  $x$  and  $y_0$  are bounded, and therefore  $y_0 \in V_0(u)$ , which is a contradiction. It follows that  $P_x$  does not intersect  $P_u$  in  $R'''$  for every  $x \in V_B \setminus N(u)$ , such that  $x \in V_0(u)$ . In particular, since  $u^*, v_0 \in V_B \setminus N(u)$  and  $u^*, v_0 \in V_0(u)$ , it follows that neither  $P_{u^*}$  nor  $P_{v_0}$  intersects  $P_u$  in  $R'''$ . Therefore, since  $r(u^*) \leq_{R'''} r(v_0) <_{R'''} l(u)$  by Transformation 3, it follows that  $P_{u^*} \ll_{R'''} P_u$  and  $P_{v_0} \ll_{R'''} P_u$ .

*Case 2b.*  $x \in V_B \setminus N(u)$  (i.e.  $x$  is bounded) and  $x \notin V_0(u)$ . Then,  $u^*x \notin E$ , since  $u^* \in V_0(u)$ . Furthermore, since both  $x$  and  $u^*$  (resp.  $v_0$ ) are bounded vertices,  $P_{u^*}$  (resp.  $P_{v_0}$ ) does not intersect  $P_x$  in  $R'''$ , i.e. either  $P_x \ll_{R'''} P_{u^*}$  or  $P_{u^*} \ll_{R'''} P_x$  (resp. either  $P_x \ll_{R'''} P_{v_0}$  or  $P_{v_0} \ll_{R'''} P_x$ ). If  $P_x \ll_{R'''} P_{u^*}$  (resp.  $P_x \ll_{R'''} P_{v_0}$ ), then  $P_x \ll_{R'''} P_{u^*} \ll_{R'''} P_u$  (resp.  $P_x \ll_{R'''} P_{v_0} \ll_{R'''} P_u$ ) by the previous paragraph. This is a contradiction to the assumption that  $P_x$  intersects  $P_u$  in  $R'''$ . Therefore  $P_{u^*} \ll_{R'''} P_x$  and  $P_{v_0} \ll_{R'''} P_x$ , and thus also  $P_{u^*} \ll_{R''} P_x$  and  $P_{v_0} \ll_{R''} P_x$ . Thus, in particular  $r(v_0) <_{R'''} l(x)$ . Furthermore, the lower endpoint  $l(u) = r(u)$  of  $P_u$  comes by Transformation 3 immediately after  $r(v_0)$  in  $R'''$ , and thus  $r(v_0) <_{R'''} r(u) <_{R'''} l(x)$ . Then  $L(x) <_{R'''} R(u)$ , since we assumed that  $P_x$  intersects  $P_u$  in  $R'''$ .

We distinguish now the cases according to the relative positions of  $P_u$  and  $P_x$  in  $R''$ . If  $P_x \ll_{R''} P_u$ , then  $P_{u^*} \ll_{R''} P_x \ll_{R''} P_u$  by the previous paragraph, which is a contradiction, since  $P_{u^*}$  intersects  $P_u$  in  $R''$ , as we proved above. If  $P_u \ll_{R''} P_x$ , then  $L_0 \leq_{R''} L(x)$ , since  $x \in (V_B \setminus N(u)) \setminus V_0(u)$  and  $L_0 = \min_{R''}\{L(x) \mid x \in (V_B \setminus N(u)) \setminus V_0(u), P_u \ll_{R''} P_x\}$ . Thus  $R(u) <_{R'''} L_0 \leq_{R''} L(x)$  by Transformation 3, which is a contradiction, since  $L(x) <_{R'''} R(u)$  by the previous paragraph. If  $P_u$  intersects  $P_x$  in  $R''$ , then  $\phi_x < \phi_u$  in  $R''$ , since  $x$  is bounded,  $u$  is unbounded, and  $x \notin N(u)$ . Therefore,  $N(u) \subseteq N(x)$  by Lemma 3, and thus  $x$  is a covering vertex of  $u$ , i.e.  $x \in V_0(u)$ , which is a contradiction to the assumption of Case 2b. Thus,  $P_x$  does not intersect  $P_u$  in  $R'''$ , for every  $x \in V_B \setminus N(u)$ , such that  $x \notin V_0(u)$ .

*Case 2c.*  $x \in V_U$  (i.e.  $x$  is unbounded), such that  $\phi_x < \phi_u$  in  $R'''$ . Then, since both  $P_x$  and  $P_u$  are lines in  $R'''$ , it follows that  $l(x) <_{R'''} l(u)$  and  $R(x) >_{R'''} R(u)$ . Thus, by Transformation 3,  $l(x) <_{R'''} r(v_0) <_{R'''} l(u)$  and  $R(u) <_{R'''} L_0 = L(y_0) <_{R'''} R(x)$ . Since  $P_{v_0} \ll_{R'''} P_{y_0}$ , it follows that  $P_x$  intersects both  $P_{v_0}$  and  $P_{y_0}$  in  $R'''$  (and thus also in  $R''$ ), and that  $\phi_x < \phi_{v_0}$  and  $\phi_x < \phi_{y_0}$ . Therefore, since both  $v_0$  and  $y_0$  are bounded, it follows that  $x \in N(v_0)$  and  $x \in N(y_0)$ . Thus  $x, y_0 \in V_0(u)$ , since  $v_0 \in V_0(u)$ . This is a contradiction, since  $y_0 \notin V_0(u)$  by definition of  $y_0$ . It follows that  $P_x$  does not intersect  $P_u$  in  $R'''$  for every  $x \in V_U$ , for which  $\phi_x < \phi_u$  in  $R'''$ .

Summarizing, due to Part 1 and due to Cases 2a, 2b, and 2c of Part 2, it follows that  $P_u$  intersects in  $R'''$  only the parallelograms  $P_z$ , for every  $z \in N(u)$ , and possibly some trivial parallelograms (lines)  $P_x$ , where  $x \in V_U$  and  $\phi_x > \phi_u$  in  $R'''$ . However, since  $\phi_x > \phi_u$  in  $R'''$  for all these vertices  $x$ , it follows that  $u$  is not adjacent to these vertices in  $R'''$ . Thus  $R'''$  is a projection representation of  $G$ , since  $R''$  is a projection representation of  $G$  by Lemma 9. This completes the proof of the lemma. ■

Thus,  $R^* = R'''$  is a projection representation of  $G$  with  $k - 1$  unbounded vertices. This completes the proof of Theorem 1. ■

### 3.2 The case where $u$ has neither the left nor the right border property

In this section we consider graphs in  $(\text{TOLERANCE} \cap \text{TRAPEZOID}) \setminus \text{BOUNDED TOLERANCE}$  that admit a projection representation, in which there is no unbounded vertex  $u$  with the right or the left border property. The proof of the main Theorem 2 of this section is based on the fact that  $G$  has simultaneously a projection representation  $R$  and a trapezoid representation  $R_T$ . In this theorem we choose a certain unbounded vertex  $u$  of  $G$  and we prove that there is another projection representation  $R^*$  of  $G$ , in which  $u$  has been replaced by a bounded vertex. First, we introduce in the following the notion of *neighborhood maximality* for unbounded vertices in a tolerance graph.

**Definition 7** Let  $G$  be a tolerance graph,  $R$  be a projection representation of  $G$ , and  $u$  be an unbounded vertex in  $R$ . Then,  $u$  is unbounded-maximal if there exists no unbounded vertex  $v$  in  $R$ , such that  $N(u) \subset N(v)$ .

This notion of an unbounded-maximal vertex will be used in Lemma 12, in order to obtain for an arbitrary tolerance graph  $G$  a projection representation with a special property. Before we present Lemma 12, we first present the next auxiliary lemma.

**Lemma 11** Let  $G$  be a tolerance graph,  $R$  be a projection representation of  $G$ , and  $u$  be an unbounded vertex of  $G$  in  $R$ , such that  $u$  is unbounded-maximal. Then, there exists a projection representation  $R^*$  of  $G$  with the same unbounded vertices, such that  $\phi_u < \phi_v$  for every unbounded vertex  $v \neq u$ , for which  $N(v) \subset N(u)$ .

**Proof.** First, recall that we can assume w.l.o.g. that all slopes of the parallelograms in a projection representation are distinct [13, 15, 18]. We will construct the projection representation  $R^*$  of  $G$  as follows. Let  $u$  be an unbounded vertex of  $G$  in  $R$ , such that  $u$  is unbounded-maximal, and let  $v \neq u$  be an arbitrary unbounded vertex of  $G$  in  $R$ , such that  $N(v) \subset N(u)$  and  $\phi_v < \phi_u$ . Suppose first that  $P_u$  intersects  $P_v$  in  $R$ . Then, since  $uv \notin E$  and  $\phi_v < \phi_u$ , it follows that  $N(u) \subseteq N(v)$  by Lemma 3, which is a contradiction.

Suppose now that  $P_v$  does not intersect  $P_u$  in  $R$ . Let  $P_u \ll_R P_v$ , i.e.  $r(u) <_R r(v)$  and  $L(u) <_R L(v)$ . Furthermore, let  $\Delta = r(v) - r(u)$ . Since for every  $w \in N(v)$ , it holds also  $w \in N(u)$ , it follows by Lemma 1 that  $r(u) <_R r(v) <_R r(w)$  and  $L(w) <_R L(u) <_R L(v)$  for every  $w \in N(v) \subset N(u)$ . Furthermore,  $\phi_w > \phi_u > \phi_v$  for every  $w \in N(v) \subset N(u)$ . We can now move the upper endpoint  $L(v)$  of the line  $P_v$  in  $R$  to the point  $L(u) + \Delta - \varepsilon$ , for a sufficiently small positive number  $\varepsilon > 0$ . In the resulting projection representation  $R'$ ,  $\phi_u < \phi_v$ .

We will prove that  $R'$  is a projection representation of the same graph  $G$ . Indeed, consider first a vertex  $w \in N(v)$ . Then,  $r(u) <_{R'} r(v) <_{R'} r(w)$  and  $L(w) <_{R'} L(u) <_{R'} L(v) = L(u) + \Delta - \varepsilon$ . Furthermore,  $\phi_u < \phi_v < \phi_w$ , since  $\varepsilon > 0$  has been chosen to be sufficiently small. Therefore,  $P_v$  still intersects  $P_w$  in  $R'$  and  $\phi_v < \phi_w$  for every  $w \in N(v)$ , i.e.  $v$  remains adjacent in  $R'$  to all vertices  $w \in N(v)$ .

Suppose now that  $v$  obtains a new adjacency with a vertex  $y$  in  $R'$ . Then, due to Lemma 1,  $y$  is bounded in both  $R$  and  $R'$ ,  $r(v) <_{R'} r(y)$  and  $L(y) <_{R'} L(v)$ . Since the lower endpoint  $r(v)$  of  $P_v$  remains the same in both  $R$  and  $R'$ , and since the upper endpoint  $L(v)$  of  $P_v$  in  $R'$  is to the left of the upper endpoint of  $P_v$  in  $R$ , it follows that also  $r(v) <_R r(y)$  and  $L(y) <_R L(v)$ , i.e.  $P_y$  intersects  $P_v$  also in  $R$ . Thus, since the slope  $\phi_v$  in  $R$  is smaller than the corresponding slope  $\phi_v$  in  $R'$ , it follows that  $y$  is adjacent to  $v$  also in  $R$ , i.e.  $y \in N(v)$ , which is a contradiction. Therefore,  $v$  does not obtain any new adjacency in  $R'$ . Thus,  $v$  is adjacent in  $R'$  to exactly the vertices  $w \in N(v)$ , i.e.  $R'$  is a projection representation of the same tolerance graph  $G$ .

The case where  $P_v \ll_R P_u$  is symmetric. Namely, in this case let  $\Delta = L(u) - L(v)$ ; then, construct the projection representation  $R'$  by moving the lower endpoint  $r(v)$  of the line  $P_v$  in  $R$  to the point  $r(u) - \Delta + \varepsilon$ , for a sufficiently small positive number  $\varepsilon > 0$ . Similarly, the resulting projection representation  $R'$  is a projection representation of  $G$ , while  $\phi_u < \phi_v$ . We repeat the above procedure, as long as there exists an unbounded vertex  $v \neq u$  in  $R$ , such that  $N(v) \subset N(u)$  and  $\phi_v < \phi_u$ . The resulting projection representation  $R^*$  of  $G$  satisfies the conditions of the lemma. ■

We are now ready to present Lemma 12.

**Lemma 12** Let  $G$  be a tolerance graph and  $R$  be a projection representation of  $G$  with at least one unbounded vertex. Then, there exists a projection representation  $R^*$  of  $G$  with the same unbounded vertices, such that the unbounded vertex  $u$ , for which  $\phi_u = \min\{\phi_x \mid x \in V_U\}$  in  $R^*$ , is unbounded-maximal.

**Proof.** Recall that  $V_U$  denotes the set of unbounded vertices of  $G$  in  $R$ . Let  $S = \{u \in V_U \mid u \text{ is unbounded-maximal}\}$ . Furthermore, let  $R'$  be the projection representation obtained by applying



for every  $u \in S$  the procedure described in the proof of Lemma 11. Then,  $R'$  has the same unbounded vertices  $V_U$ , while  $\phi_u < \phi_v$  for every  $u \in S$  and every unbounded vertex  $v \neq u$ , for which  $N(v) \subset N(u)$ . We choose now  $u$  to be that unbounded vertex, for which  $\phi_u = \min\{\phi_x \mid x \in S\}$ . Then,  $u$  satisfies the conditions of the lemma. ■

Assume that there exists a graph  $G \in (\text{TOLERANCE} \cap \text{TRAPEZOID}) \setminus \text{BOUNDED TOLERANCE}$ , and let  $G$  have the smallest number of vertices. Furthermore, let  $R$  and  $R_T$  be a canonical projection and a trapezoid representation of  $G$ , respectively, and  $u$  be an arbitrary unbounded vertex of  $G$  in  $R$ . Then  $V_0(u) \neq \emptyset$  by Lemma 4, and thus also  $V_0(u)$  is connected by Lemma 5. Therefore, since  $u$  is not adjacent to any vertex of  $V_0(u)$  by Definition 5, either all trapezoids of  $V_0(u)$  lie to the left, or all to the right of  $T_u$  in  $R_T$ .

Consider first the case where all trapezoids of  $V_0(u)$  lie to the *left* of  $T_u$  in  $R_T$ , i.e.  $T_x \ll_{R_T} T_u$  for every  $x \in V_0(u)$ . Recall by Lemma 6 that  $N(v) \neq N(u)$  for every unbounded vertex  $v \neq u$  in  $R$ . Denote by  $Q_u = \{v \in V_U \mid N(v) \subset N(u)\}$  the set of unbounded vertices  $v$  of  $G$  in  $R$ , whose neighborhood set is strictly included in the neighborhood set of  $u$ . The next lemma follows easily by the definition of  $Q_u$ .

**Lemma 13** *For every  $v \in Q_u$ , every covering vertex  $u^*$  of  $u$  is also a covering vertex of  $v$ . Furthermore,  $Q_u \cap V_0(u) = \emptyset$ .*

**Proof.** Since  $u^*$  is a covering vertex of  $u$  by assumption,  $u^* \notin N(u)$  and  $N(u) \subseteq N(u^*)$  by Definition 5. Let  $v \in Q_u$ . Then, since  $N(v) \subset N(u)$  and  $u^* \notin N(u)$ , it follows that  $u^* \notin N(v)$ . Furthermore,  $N(v) \subset N(u) \subseteq N(u^*)$ , and thus  $u^*$  is a covering vertex of  $v$  by Definition 5. Suppose now that  $v \in V_0(u)$ . Then,  $v$  is an isolated vertex in  $G \setminus N[u]$ , since  $N(v) \subset N(u)$ . Thus, since  $v$  is unbounded and  $u^*$  is bounded, i.e.  $v \neq u^*$ , it follows that  $v$  and  $u^*$  do not lie in the same connected component of  $V_0(u)$ , i.e.  $V_0(u)$  is not connected, which is a contradiction. Thus,  $v \notin V_0(u)$  for every  $v \in Q_u$ , i.e.  $Q_u \cap V_0(u) = \emptyset$ . ■

Since no two unbounded vertices are adjacent, it follows in particular that  $T_v$  does not intersect  $T_u$  in  $R_T$ , for every  $v \in Q_u$ . Therefore, we can partition the set  $Q_u$  into the two subsets  $Q_1(u) = \{v \in Q_u \mid T_v \ll_{R_T} T_u\}$  and  $Q_2(u) = \{v \in Q_u \mid T_u \ll_{R_T} T_v\}$ .

Consider now a vertex  $v \in Q_1(u) \subseteq Q_u$ . Note that for every  $x \in V_0(u)$ ,  $T_v$  does not intersect  $T_x$  in  $R_T$ , since otherwise  $v \in V_0(u)$ , which is a contradiction by Lemma 13. Therefore, since in particular  $V_0(u)$  is connected by Lemma 5, it follows that for every  $x \in V_0(u)$ , either  $T_v \ll_{R_T} T_x$  or  $T_x \ll_{R_T} T_v$ . We will now prove that  $T_v \ll_{R_T} T_x$  for every  $x \in V_0(u)$ . Suppose otherwise that  $T_x \ll_{R_T} T_v$  for every  $x \in V_0(u)$ . Then, since  $v \in Q_1(u)$ , it follows that  $T_x \ll_{R_T} T_v \ll_{R_T} T_u$  for every  $x \in V_0(u)$ . Therefore, since  $V_0(u)$  includes all covering vertices of  $u$  by Definition 5, it follows that  $T_{x_0} \ll_{R_T} T_v \ll_{R_T} T_u$  for every covering vertex  $x_0$  of  $u$ . Thus, since  $N(u) \subseteq N(x_0)$ , it follows that  $T_z$  intersects  $T_v$  in  $R_T$  for every  $z \in N(u) \subseteq N(x_0)$ . Therefore  $N(v) \subseteq N(u)$ , which is a contradiction, since  $v \in Q_1(u) \subseteq Q_u$ . Therefore  $T_v \ll_{R_T} T_x$  for every  $v \in Q_1(u)$  and every  $x \in V_0(u)$ , i.e.  $Q_1(u) = \{v \in Q_u \mid T_v \ll_{R_T} T_x \text{ for every } x \in V_0(u)\}$ .

Consider now the case where all trapezoids of  $V_0(u)$  lie to the *right* of  $T_u$  in  $R_T$ , i.e.  $T_u \ll_{R_T} T_x$  for every  $x \in V_0(u)$ . Then, by performing vertical axis flipping of  $R_T$ , we partition similarly to the above the set  $Q_u$  into the sets  $Q_1(u)$  and  $Q_2(u)$ . That is, in this (symmetric) case the sets  $Q_1(u)$  and  $Q_2(u)$  will be  $Q_1(u) = \{v \in Q_u \mid T_x \ll_{R_T} T_v \text{ for every } x \in V_0(u)\}$  and  $Q_2(u) = \{v \in Q_u \mid T_v \ll_{R_T} T_u\}$ .

In the following we define three conditions on  $G$ , regarding the unbounded vertices of  $G$  in  $R$ ; the third one depends also on the trapezoid representation  $R_T$  of  $G$ . The second condition is weaker than the first one, while the third condition is weaker than the other two, as it is stated in Observation 1. Then, we prove Theorem 2, assuming that the third condition holds.

**Condition 1** *The projection representation  $R$  of  $G$  has exactly one unbounded vertex.*

**Condition 2** *For every unbounded vertex  $u$  of  $G$  in  $R$ ,  $Q_u = \emptyset$ ; namely, all unbounded vertices are unbounded-maximal.*

**Condition 3** For every unbounded vertex  $u$  of  $G$  in  $R$ ,  $Q_2(u) = \emptyset$ , i.e.  $Q_u = Q_1(u)$ .

The next observation, which connects the above conditions, follows easily.

**Observation 1** Condition 1 implies Condition 2, and Condition 2 implies Condition 3.

In the remainder of the section we assume that Condition 3 holds, which is weaker than Conditions 1 and 2. We present now the main theorem of this section.

**Theorem 2** Let  $G = (V, E) \in (\text{TOLERANCE} \cap \text{TRAPEZOID}) \setminus \text{BOUNDED TOLERANCE}$  with the smallest number of vertices. Let  $R_T$  be a trapezoid representation of  $G$  and  $R$  be a projection representation of  $G$  with  $k$  unbounded vertices. Then, assuming that  $G$  satisfies Condition 3, there exists a projection representation  $R^*$  of  $G$  with  $k - 1$  unbounded vertices.

**Proof (sketch).** The full proof of the theorem can be found in the Appendix. The proof is done constructively, by exploiting the fact that  $G$  can be represented by both the projection representation  $R$  and the trapezoid representation  $R_T$ .

If at least one unbounded vertex of  $G$  in  $R$  has the right or the left border property, there exists a projection representation  $R^*$  of  $G$  with  $k - 1$  unbounded vertices by Theorem 1, where all unbounded vertices of  $R^*$  are also unbounded vertices in  $R$ . Suppose that every unbounded vertex of  $G$  in  $R$  has neither the right nor the left border property in  $R$ . Let  $u$  be the unbounded vertex in  $R$ , such that  $\phi_u = \min\{\phi_x \mid x \in V_U\}$  in  $R$ ; then, we may assume by Lemma 12 that  $u$  is an unbounded-maximal vertex of  $G$ . By possibly performing vertical axis flipping of  $R_T$ , we may assume w.l.o.g. that all trapezoids of  $V_0(u)$  lie to the left of  $T_u$  in  $R_T$ , i.e.  $T_x \ll_{R_T} T_u$  for every  $x \in V_0(u)$ .

We now construct a projection representation  $R^*$  of the same graph  $G$ , in which  $u$  is replaced by a bounded vertex, while all other  $k - 1$  unbounded vertices of  $R$  remain also unbounded in  $R^*$ . We start by constructing a subgraph  $G_0$  of  $G$ , such that  $u \in V(G_0)$  and all vertices of  $V(G_0) \setminus \{u\}$  are bounded. Then, we prove that  $G_0 \setminus \{u\}$  is a module in  $G \setminus \{u\}$ , by exploiting the fact that  $G$  can be represented by both  $R$  and  $R_T$ . That is, we prove that  $N(v) \setminus V(G_0) = N(v') \setminus V(G_0)$  for all vertices  $v, v' \in V(G_0) \setminus \{u\}$ . Furthermore, we define in a particular way a line segment  $\ell$  with endpoints on the lines  $L_1$  and  $L_2$ , respectively. Then, we replace the parallelograms of the vertices of  $G_0$  in  $R$  by a particular projection representation  $R_0$  of  $G_0$ , which is  $\varepsilon$ -squeezed with respect to the line segment  $\ell$ . We denote the resulting projection representation by  $R_\ell$ . Then we prove that  $R_\ell \setminus \{u\}$  is a projection representation of the graph  $G \setminus \{u\}$  – although  $R_\ell$  is not necessarily a projection representation of  $G$  – and that  $u$  has the right border property in  $R_\ell$ . Then, similarly to Transformations 1, 2, and 3 in the proof of Theorem 1, we apply three other transformations to  $R_\ell$  (Transformations 4, 5, and 6, respectively), obtaining thus the projection representations  $R'_\ell$ ,  $R''_\ell$ , and  $R'''_\ell$ , respectively. Then we set  $R^* = R'''_\ell$ , and we prove that  $R^*$  is a projection representation of the graph  $G$  itself. Moreover,  $R^*$  has the same unbounded vertices as  $R$  except for  $u$  (which became bounded in  $R^*$ ), and thus  $R^*$  has  $k - 1$  unbounded vertices. This completes the proof of Theorem 2. ■

Note that, within the proof of Theorem 2 (see the Appendix), we mainly use the facts that  $u$  is an unbounded-maximal vertex of  $G$  and that the slope  $\phi_u$  of  $u$  is the smallest among all unbounded vertices in  $R$ . On the contrary, the assumption that  $G$  satisfies Condition 3 is used only for a technical part of the proof, namely that  $G_0 \setminus \{u\}$  is a module in  $G \setminus \{u\}$  (cf. Lemma 35 in the Appendix).

### 3.3 The general case

Recall now that  $\text{TOLERANCE} \cap \text{COCOMPARABILITY} = \text{TOLERANCE} \cap \text{TRAPEZOID}$  (cf. the discussion before Lemma 5). The next main theorem follows by recursive application of Theorem 2.

**Theorem 3** Let  $G = (V, E) \in (\text{TOLERANCE} \cap \text{COCOMPARABILITY})$ ,  $R_T$  be a trapezoid representation of  $G$ , and  $R$  be a projection representation of  $G$ . Then, assuming that  $G$  satisfies one of the Conditions 1, 2, or 3,  $G$  is a bounded tolerance graph.

**Proof.** Since  $G = (V, E) \in (\text{TOLERANCE} \cap \text{COCOMPARABILITY})$ , it follows that  $G$  is also a trapezoid graph [8]. That is,  $G \in (\text{TOLERANCE} \cap \text{TRAPEZOID})$ . Suppose that  $G$  is not a bounded tolerance graph. We can assume w.l.o.g. that  $G$  has the smallest number of vertices among the graphs in  $(\text{TOLERANCE} \cap \text{TRAPEZOID}) \setminus \text{BOUNDED TOLERANCE}$ . Let  $R_0$  be a projection representation of  $G$  with the smallest possible number  $k_0$  of unbounded vertices. Note that  $k_0 \geq 1$ ; indeed, if otherwise  $k_0 = 0$ , then  $G$  is a bounded tolerance graph, which is a contradiction to the assumption on  $G$ . Suppose that the projection representation  $R$  of  $G$  has  $k$  unbounded vertices, where  $k \geq k_0$ . Then, there exists by Theorem 2 a projection representation  $R^*$  of  $G$  with  $k - 1$  unbounded vertices. In particular, due to the proof of Theorem 2,  $R^*$  has the same unbounded vertices as  $R$ , except for  $u$  (which became bounded in  $R^*$ ).

If Condition 1 holds for the projection representation  $R$  of  $G$ , i.e. if  $k = k_0 = 1$ , then  $R^*$  has no unbounded vertex, i.e.  $R^*$  is a parallelogram representation of  $G$ . This is a contradiction to the assumption that  $G$  is not a bounded tolerance (i.e. parallelogram) graph. If Condition 2 holds for  $R$ , then it also holds for  $R^*$ , since all unbounded vertices of  $R^*$  are also unbounded vertices of  $R$ . Similarly, if Condition 3 holds for  $R$  and  $R_T$ , then it follows directly that it holds also for the pair  $R^*$  and  $R_T$  of representations of  $G$  (since for every unbounded vertex  $u$  in  $R^*$ , the set  $Q_2(u)$  depends only on the trapezoid representation  $R_T$ ).

Therefore, we can apply iteratively  $k - k_0 + 1$  times the constructive proof of Theorem 2, obtaining eventually a projection representation  $R^{**}$  of  $G$  with  $k_0 - 1$  unbounded vertices. This is a contradiction to the minimality of  $k_0$ . Therefore,  $G$  is a bounded tolerance graph. This completes the proof of the theorem. ■

As an immediate implication of Theorem 3, we prove in the next corollary that Conjecture 1 is true in particular for every graph  $G$  that has no three independent vertices  $a, b, c$  such that  $N(a) \subset N(b) \subset N(c)$ , since Condition 2 is guaranteed to be true for every such graph  $G$ . Therefore the conjecture is also true for the complements of triangle-free graphs. Thus, since in particular no bipartite graph has a triangle, the next corollary immediately implies the correctness of Conjecture 1 for the complements of trees and of bipartite graphs, which were the only known results until now [1, 21].

**Corollary 2** *Let  $G = (V, E) \in (\text{TOLERANCE} \cap \text{COCOMPARABILITY})$ . Suppose that there do not exist three independent vertices  $a, b, c \in V$  such that  $N(a) \subset N(b) \subset N(c)$ . Then,  $G$  is a bounded tolerance graph.*

**Proof.** Due to Theorem 3, it suffices to prove that Condition 2 is true for  $G$ , with respect to *any possible* canonical (projection) representation  $R$  and *any* trapezoid representation  $R_T$  of  $G$ . Let  $R$  be a canonical representation of  $G$ . Suppose that Condition 2 is not true for  $G$ . Then, there exists an unbounded vertex  $u \in V_U$  such that  $Q_u \neq \emptyset$ . That is, there exists by the definition of the set  $Q_u$  an unbounded vertex  $v \in V_U \setminus \{u\}$  such that  $N(v) \subset N(u)$ . Note that  $v \notin N(u)$ , since no two unbounded vertices are adjacent in  $G$ . Furthermore, there exists at least one covering vertex  $u^*$  of  $u$  in  $G$ , since  $V_0(u) \neq \emptyset$  (cf. Lemma 4), and thus  $u^* \notin N(u)$  and  $N(u) \subset N(u^*)$ . Therefore, since  $N(v) \subset N(u)$  and  $u^* \notin N(u)$ , it follows that also  $u^* \notin N(v)$ , i.e. the vertices  $v, u, u^*$  are independent. Moreover  $N(v) \subset N(u) \subset N(u^*)$ , which comes in contradiction to the assumption of the lemma. Therefore Condition 2 holds for  $G$ , and thus  $G$  is a bounded tolerance graph by Theorem 3. ■

We now formally define the notion of a *minimally unbounded tolerance graph*.

**Definition 8** *Let  $G \in \text{TOLERANCE} \setminus \text{BOUNDED TOLERANCE}$ . If  $G \setminus \{u\}$  is a bounded tolerance graph for every vertex  $u$  of  $G$ , then  $G$  is a minimally unbounded tolerance graph.*

Assume now that Conjecture 1 is not true, and let  $G$  be a counterexample with the smallest number of vertices. Then, in particular,  $G$  is a tolerance but not a bounded tolerance graph; furthermore, since  $G$  has the smallest number of vertices, the removal of any vertex of  $G$  makes it a bounded tolerance graph. That is,  $G$  is a minimally unbounded tolerance graph by Definition 8.

Now, if our Conjecture 2 is true (see Section 1), then  $G$  has a projection representation  $R$  with exactly one unbounded vertex, i.e.  $R$  satisfies Condition 1. Thus,  $G$  is a bounded tolerance graph by Theorem 3, which is a contradiction, since  $G$  has been assumed to be a counterexample to Conjecture 1. Thus, we obtain the following theorem.

**Theorem 4** *Conjecture 2 implies Conjecture 1.*

Therefore, in order to prove Conjecture 1, it suffices to prove Conjecture 2. Moreover, to the best of our knowledge, all known examples of minimally unbounded tolerance graphs have a tolerance representation with exactly one unbounded vertex; for such examples, see e.g. [13].

## 4 Concluding remarks and open problems

In this article we dealt with the over 25 years old conjecture of [11], which states that if a graph  $G$  is both tolerance and cocomparability, then it is also a bounded tolerance graph. Our main result was that this conjecture is true for every graph  $G$  that admits a tolerance representation with exactly one unbounded vertex. Our proofs are constructive, in the sense that, given a tolerance representation  $R$  of a graph  $G$ , we transform  $R$  into a bounded tolerance representation  $R^*$  of  $G$ . Furthermore, we conjectured that any *minimal* graph  $G$  that is a tolerance but not a bounded tolerance graph, has a tolerance representation with exactly one unbounded vertex. Our results imply the non-trivial result that, in order to prove the conjecture of [11], it suffices to prove our conjecture. An interesting problem for further research that we leave open is to prove this new conjecture (which, in contrast to one stated in [11], does not concern any other class of graphs, such as cocomparability or trapezoid graphs). Since cocomparability graphs can be efficiently recognized [24], a positive answer to this conjecture (and thus also to the conjecture of [11]) would enable us to efficiently distinguish between tolerance and bounded tolerance graphs, although it is NP-complete to recognize each of these graph classes separately [19].

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## Appendix: Proof of Theorem 2

**Proof.** First, we may assume w.l.o.g. by the minimality of the number of vertices of  $G$  that  $G$  is connected. If  $R$  is not a canonical representation of  $G$ , then there exists a projection representation of  $G$  with  $k - 1$  unbounded vertices by Definition 2. Suppose for the sequel of the proof that  $R$  is a canonical representation of  $G$ . If at least one unbounded vertex of  $G$  in  $R$  has the right or the left border property, there exists a projection representation of  $G$  with  $k - 1$  unbounded vertices by Theorem 1. Suppose in the sequel that every unbounded vertex of  $G$  in  $R$  has neither the right nor the left border property in  $R$ . Let  $u$  be the unbounded vertex in  $R$ , such that  $\phi_u = \min\{\phi_x \mid x \in V_U\}$  in  $R$ . The proof is done constructively, by exploiting the fact that  $G$  can be represented by both the projection representation  $R$  and the trapezoid representation  $R_T$ . Namely, we will construct a projection representation  $R^*$  of the same graph  $G$ , in which  $u$  is replaced by a bounded vertex, while all other  $k - 1$  unbounded vertices of  $R$  remain also unbounded in  $R^*$ .

By Lemma 4, there exists at least one bounded covering vertex  $u^*$  of  $u$ , such that  $P_{u^*}$  intersects  $P_u$  in  $R$  and  $\phi_{u^*} < \phi_u$ . Therefore,  $V_0(u) \neq \emptyset$ , and thus  $V_0(u)$  is connected by Lemma 5. Since  $V_0(u)$  is connected, and since  $u$  is not adjacent to any vertex of  $V_0(u)$ , it follows that either all trapezoids of  $V_0(u)$  lie to the left, or all to the right of  $T_u$  in  $R_T$ . By possibly performing vertical axis flipping of  $R_T$ , we may assume w.l.o.g. that all trapezoids of  $V_0(u)$  lie to the left of  $T_u$  in  $R_T$ , i.e.  $T_x \ll_{R_T} T_u$  for every  $x \in V_0(u)$ . Moreover, we may assume w.l.o.g. by Lemma 12 that  $u$  is an unbounded-maximal vertex of  $G$ . Recall by Lemma 6 that  $N(v_1) \neq N(v_2)$  for any two unbounded vertices  $v_1, v_2$ . Denote now by  $Q_u = \{v \in V_U \mid N(v) \subset N(u)\}$ . Furthermore, since we assumed that Condition 3 holds,  $Q_u = Q_1(u) = \{v \in Q_u \mid T_v \ll_{R_T} T_x \text{ for every } x \in V_0(u)\}$ .

### The vertex sets $D_1$ , $D_2$ , $S_2$ , and $\tilde{X}_1$ and the vertex $x_2$

Define the sets  $D_1(u, R) = \{v \in V_0(u) \mid P_v \ll_R P_u\}$ ,  $D_2(u, R) = \{v \in V_0(u) \mid P_u \ll_R P_v\}$ , and  $S_2(u, R) = \{v \in V_0(u) \mid P_v \not\ll_R P_u\}$ . Note that  $V_0(u) = D_1(u, R) \cup S_2(u, R)$  and that  $D_2(u, R) \subseteq S_2(u, R)$ . For simplicity reasons, we will refer in the following to the sets  $D_1(u, R)$ ,  $D_2(u, R)$ , and  $S_2(u, R)$  just by  $D_1$ ,  $D_2$ , and  $S_2$ , respectively. Note that  $Q_u \cap D_1 = \emptyset$ ,  $Q_u \cap D_2 = \emptyset$ , and  $Q_u \cap S_2 = \emptyset$ , since  $D_1, D_2, S_2 \subseteq V_0(u)$  and by Lemma 13.

Since  $u$  does not have the right border property in  $R$ , there exist by Definition 6 vertices  $w \in N(u)$  and  $x \in V_0(u)$ , such that  $P_w \ll_R P_x$ . Therefore, in particular,  $r(w) <_R l(x)$ . Since  $u$  is unbounded in  $R$ , and since  $w \in N(u)$ , Lemma 1 implies that  $r(u) <_R r(w)$ , and thus  $r(u) <_R l(x)$ . For the sake of contradiction, suppose that  $L(x) <_R R(u)$ . Then,  $P_x$  intersects  $P_u$  in  $R$  and  $\phi_x > \phi_u$ . Thus,  $x$  is unbounded in  $R$ , since otherwise  $x \in N(u)$ , which is a contradiction. Furthermore,  $N(x) \subseteq N(u)$  by Lemma 3, and thus  $x \in Q_u$ , which is a contradiction by Lemma 13, since  $x \in V_0(u)$ . Therefore,  $R(u) <_R L(x)$ , and thus  $P_u \ll_R P_x$ , since also  $r(u) <_R l(x)$ . That is,  $x \in D_2$ . Since  $u$  has not the left border property in  $R$ , there exist vertices  $w' \in N(u)$  and  $y \in V_0(u)$ , such that  $P_y \ll_R P_{w'}$ . Therefore, in the reverse projection representation  $\hat{R}$  of  $R$ ,  $P_{w'} \ll_{\hat{R}} P_y$ . Then, applying the same arguments as above, it follows that  $P_u \ll_{\hat{R}} P_y$ , and thus  $P_y \ll_R P_u$ . That is,  $y \in D_1$ . Summarizing, both sets  $D_1$  and  $D_2 \subseteq S_2$  are not empty.

Among the vertices of  $D_1 \cup D_2$  let  $x_1$  be such a vertex, that for every other vertex  $x' \in D_1 \cup D_2 \setminus \{x_1\}$ , either  $T_{x'}$  intersects  $T_{x_1}$  in the trapezoid representation  $R_T$ , or  $T_{x_1} \ll_{R_T} T_{x'}$ . That is, there exists no vertex  $x'$  in  $D_1 \cup D_2$ , whose trapezoid lies to the left of  $T_{x_1}$  in  $R_T$ . By possibly building the reverse project representation  $\hat{R}$  of  $R$ , we may assume w.l.o.g. that  $P_{x_1} \ll_R P_u$ , i.e.  $x_1 \in D_1$ .

As already mentioned above, since  $u$  does not have the right border property in  $R$ , there exist vertices  $w \in N(u)$  and  $x \in D_2 \subseteq V_0(u)$ , such that  $P_w \ll_R P_x$ . Among the vertices  $x \in D_2$ , for which  $P_w \ll_R P_x$ , let  $x_2$  be such a vertex, that for every other vertex  $x' \in D_2 \setminus \{x_2\}$  with  $P_w \ll_R P_{x'}$ , either  $T_{x'}$  intersects  $T_{x_2}$  in the trapezoid representation  $R_T$ , or  $T_{x_2} \ll_{R_T} T_{x'}$ . That is, there exists no vertex  $x'$  in  $D_2$  with  $P_w \ll_R P_{x'}$ , whose trapezoid  $T_{x'}$  lies to the left of  $T_{x_2}$  in  $R_T$ .

Furthermore,  $x_1 x_2 \notin E$ , since  $x_1 \in D_1$  and  $x_2 \in D_2$ , i.e.  $P_{x_1} \ll_R P_u \ll_R P_{x_2}$ . Therefore, since  $T_x \ll_{R_T} T_u$  for every  $x \in V_0(u)$ , it follows by the definition of  $x_1$  that  $T_{x_1} \ll_{R_T} T_{x_2} \ll_{R_T} T_u$ . Thus,

since  $wu \in E$  and  $wx_2 \notin E$ , it follows that also  $T_{x_1} \ll_{R_T} T_{x_2} \ll_{R_T} T_w$ , i.e.  $wx_1 \notin E$ . That is,  $x_1$ ,  $x_2$ , and  $w$  are three independent vertices in  $G$ .

We now construct iteratively the vertex set  $\tilde{X}_1 \subseteq D_1$  from the vertex  $x_1$ , as follows. Initially, we set  $\tilde{X}_1 = \{x_1\}$ . If  $N(w) \cap N(\tilde{X}_1) \subset N(\tilde{X}_1)$ , then set  $\tilde{X}_1$  to be equal to  $\tilde{X}_1 \cup N(\tilde{X}_1) \setminus N(w)$ . Iterate, until finally  $N(w) \cap N(\tilde{X}_1) = N(\tilde{X}_1)$ . This process terminates, since every time we strictly augment the current set  $\tilde{X}_1$ . Furthermore, at the end of this procedure,  $N(\tilde{X}_1) \neq \emptyset$ , since otherwise  $G$  is not connected, which is a contradiction. Moreover, the vertices of  $\tilde{X}_1$  at every step of this procedure induce a connected subgraph of  $G$ .

**Lemma 14** *For the constructed set  $\tilde{X}_1$ ,  $\tilde{X}_1 \subseteq D_1$ . Furthermore,  $P_x \ll_R P_w$  and  $T_x \ll_{R_T} T_{x_2}$  for every  $x \in \tilde{X}_1$ .*

**Proof.** The proof of the lemma is done by induction on  $|\tilde{X}_1|$ . Suppose first that  $|\tilde{X}_1| = 1$ , i.e.  $\tilde{X}_1 = \{x_1\}$ . Then,  $\{x_1\} \subseteq D_1$  and  $T_{x_1} \ll_{R_T} T_{x_2}$  by definition of  $x_1$ . We will now prove that also  $P_{x_1} \ll_R P_w$ . Otherwise, suppose first that  $P_w \ll_R P_{x_1}$ . Then, since  $x_1 \in D_1$ , it follows that  $P_w \ll_R P_{x_1} \ll_R P_u$ , and thus  $w \notin N(u)$ , which is a contradiction. Thus, either  $P_{x_1}$  intersects  $P_w$  in  $R$ , or  $P_{x_1} \ll_R P_w$ . Suppose that  $P_{x_1}$  intersects  $P_w$  in  $R$ . Then,  $x_1$  is unbounded and  $\phi_{x_1} > \phi_w > \phi_u$ , since  $w$  is bounded and  $x_1w \notin E$ . Then, Lemma 3 implies that  $N(x_1) \subseteq N(w)$ . Furthermore, since  $T_{x_1} \ll_{R_T} T_{x_2} \ll_{R_T} T_w$ , it follows that  $T_z$  intersects  $T_{x_2}$  in  $R_T$  for every  $z \in N(x_1) \subseteq N(w)$ , and thus also  $N(x_1) \subseteq N(x_2)$ . Therefore, since  $P_{x_1} \ll_R P_u \ll_R P_{x_2}$ , it follows that for every  $z \in N(x_1) \subseteq N(x_2)$ ,  $z$  is bounded in  $R$ ,  $\phi_u < \phi_{x_1} < \phi_z$ , and  $P_z$  intersects  $P_u$  in  $R$ . Thus,  $N(x_1) \subseteq N(u)$ , i.e.  $x_1 \in Q_u$ , which is a contradiction by Lemma 13, since  $x_1 \in V_0(u)$ . It follows that  $P_{x_1}$  does not intersect  $P_w$  in  $R$ , and thus  $P_{x_1} \ll_R P_w$ . This proves the induction basis.

For the induction step, suppose that the statement of the lemma holds for the set  $\tilde{X}_1$  constructed after an iteration of the construction procedure, and let  $v \in N(\tilde{X}_1) \setminus N(w)$ . Suppose first that  $v \in N(u)$ , and thus  $v$  is bounded in  $R$ . Then, since by the induction hypothesis  $T_x \ll_{R_T} T_{x_2} \ll_{R_T} T_u$  for every  $x \in \tilde{X}_1$ , and since  $v \in N(x) \cap N(u)$  for some  $x \in \tilde{X}_1$ , it follows that  $T_v$  intersects  $T_{x_2}$  in  $R_T$ , and thus  $vx_2 \in E$ . On the other hand, since  $P_x \ll_R P_w \ll_R P_{x_2}$  for every  $x \in \tilde{X}_1$  by the induction hypothesis, and since  $v \in N(x) \cap N(x_2)$  for some  $x \in \tilde{X}_1$ , it follows that  $P_v$  intersects  $P_w$  in  $R$ , and thus  $vw \in E$ , since both  $v$  and  $w$  are bounded. This is a contradiction, since  $v \in N(\tilde{X}_1) \setminus N(w)$ . Thus,  $v \notin N(u)$  for every  $v \in N(\tilde{X}_1) \setminus N(w)$ . Therefore, since  $v \in N(\tilde{X}_1)$  and  $\tilde{X}_1 \subseteq V_0(u)$ , it follows that  $v \in V_0(u)$  for every  $v \in N(\tilde{X}_1) \setminus N(w)$ , and thus the updated set  $\tilde{X}_1$  is  $\tilde{X}_1 \cup N(\tilde{X}_1) \setminus N(w) \subseteq V_0(u)$ .

Since  $v \in N(x)$  for some  $x \in \tilde{X}_1$ , and since  $P_x \ll_R P_w$  for every  $x \in \tilde{X}_1$  by the induction hypothesis, it follows that either  $P_v$  intersects  $P_w$  in  $R$ , or  $P_v \ll_R P_w$ . Suppose that  $P_v$  intersects  $P_w$  in  $R$ . Then,  $v$  is unbounded and  $\phi_v > \phi_w$ , since  $v \notin N(w)$  and  $w$  is bounded. Therefore,  $N(v) \subseteq N(w)$  by Lemma 3, and thus in particular  $x \in N(w)$  for some  $x \in \tilde{X}_1$ , which is a contradiction to the induction hypothesis. Therefore,  $P_v$  does not intersect  $P_w$  in  $R$ , and thus  $P_v \ll_R P_w$  for every  $v \in N(\tilde{X}_1) \setminus N(w)$ .

We will prove that also  $P_v \ll_R P_u$  for every  $v \in N(\tilde{X}_1) \setminus N(w)$ . Otherwise, suppose first that  $P_u \ll_R P_v$ . Then, since  $P_v \ll_R P_w$  by the previous paragraph, it follows that  $P_u \ll_R P_v \ll_R P_w$ , and thus  $w \notin N(u)$ , which is a contradiction. Suppose now that  $P_v$  intersects  $P_u$  in  $R$ . Recall that  $v \notin N(u)$ , as we proved above. If  $\phi_u > \phi_v$ , then  $N(u) \subseteq N(v)$  by Lemma 3, and thus also  $w \in N(v)$ , which is a contradiction, since  $v \in N(\tilde{X}_1) \setminus N(w)$ . If  $\phi_u < \phi_v$ , then  $v$  is unbounded, since otherwise  $v \in N(u)$ , which is a contradiction. Furthermore,  $N(v) \subseteq N(u)$  by Lemma 3, and thus  $v \in Q_u$ , which is a contradiction by Lemma 13, since  $v \in V_0(u)$  as we proved above. Therefore,  $P_v \ll_R P_u$ , i.e.  $v \in D_1$ , for every  $v \in N(\tilde{X}_1) \setminus N(w)$ , and thus the updated set  $\tilde{X}_1$  is  $\tilde{X}_1 \cup N(\tilde{X}_1) \setminus N(w) \subseteq D_1$ .

Since the updated set  $\tilde{X}_1 \cup N(\tilde{X}_1) \setminus N(w)$  is a subset of  $D_1$ , i.e.  $x \in V_0(u)$  and  $P_x \ll_R P_u$  for every  $x \in \tilde{X}_1 \cup N(\tilde{X}_1) \setminus N(w)$ , it follows in particular that  $xx_2 \notin E$  for every  $x \in \tilde{X}_1 \cup N(\tilde{X}_1) \setminus N(w)$ , since  $P_u \ll_R P_{x_2}$ . Recall furthermore that the set  $\tilde{X}_1 \cup N(\tilde{X}_1) \setminus N(w)$  induces a connected subgraph of  $G$ . Thus, since  $T_{x_1} \ll_{R_T} T_{x_2}$ , it follows that  $T_x \ll_{R_T} T_{x_2}$  for every  $x \in \tilde{X}_1 \cup N(\tilde{X}_1) \setminus N(w)$ . This completes the induction step, and the lemma follows. ■

**Corollary 3** *For the constructed set  $\tilde{X}_1$ ,  $N(\tilde{X}_1) \setminus N(u) \neq \emptyset$ .*

**Proof.** Suppose for the sake of contradiction that  $N(\tilde{X}_1) \setminus N(u) = \emptyset$ , i.e.  $N(\tilde{X}_1) \subseteq N(u)$ . Since  $\tilde{X}_1 \subseteq D_1 \subseteq V_0(u)$  by Lemma 14, it follows that  $P_x \ll_R P_u$  for every  $x \in \tilde{X}_1$ , and thus in particular  $x \notin N(u)$  for every  $x \in \tilde{X}_1$ . Therefore, since  $\tilde{X}_1$  induces a connected subgraph of  $G$ , it follows that  $\tilde{X}_1$  is a connected component of  $G \setminus N[u]$ . Therefore, since  $V_0(u)$  is connected, it follows that  $V_0(u) = \tilde{X}_1$ . This is a contradiction, since  $\emptyset \neq D_2 \subseteq V_0(u)$ . Therefore,  $N(\tilde{X}_1) \setminus N(u) \neq \emptyset$ . ■

Recall by definition of  $x_2$  that for every vertex  $x' \in D_2 \setminus \{x_2\}$  with  $P_w \ll_R P_{x'}$ , either  $T_{x'}$  intersects  $T_{x_2}$  in the trapezoid representation  $R_T$ , or  $T_{x_2} \ll_{R_T} T_{x'}$ . We will now prove in the following lemma that this property holds actually for all vertices  $x' \in S_2 \setminus \{x_2\}$ .

**Lemma 15** *For every vertex  $x' \in S_2 \setminus \{x_2\}$ , either  $T_{x'}$  intersects  $T_{x_2}$  in the trapezoid representation  $R_T$ , or  $T_{x_2} \ll_{R_T} T_{x'}$ .*

**Proof.** Consider an arbitrary vertex  $x' \in S_2 \setminus \{x_2\}$ . If  $x' \in N(x_2)$ , then clearly  $T_{x'}$  intersects  $T_{x_2}$  in  $R_T$ . Thus, it suffices to consider in the sequel of the proof only the case where  $x' \notin N(x_2)$ , i.e. the case where  $T_{x'}$  does not intersect  $T_{x_2}$  in  $R_T$ . Suppose for the sake of contradiction that  $T_{x'} \ll_{R_T} T_{x_2}$ , i.e.  $T_{x'} \ll_{R_T} T_{x_2} \ll_{R_T} T_w$ . Then, in particular,  $x' \notin N(w)$ . Furthermore, note that  $x' \notin N(u)$ , since  $x' \in S_2 \subseteq V_0(u)$ .

Suppose first that  $x' \in S_2 \setminus D_2$ , i.e.  $P_{x'}$  intersects  $P_u$  in  $R$ . If  $\phi_{x'} > \phi_u$ , then  $x'$  is unbounded, since otherwise  $x' \in N(u)$  which is a contradiction. Furthermore,  $N(x') \subseteq N(u)$  by Lemma 3, and thus  $x' \in Q_u$ , which is a contradiction by Lemma 13, since  $x \in V_0(u)$ . If  $\phi_{x'} < \phi_u$ , then  $N(u) \subseteq N(x')$  by Lemma 3, and thus in particular  $wx' \in E$ , which is a contradiction, since  $x' \notin N(w)$ . Therefore, the lemma holds for every vertex  $x' \in S_2 \setminus D_2$ .

Suppose now that  $x' \in D_2$ , i.e.  $P_u \ll_R P_{x'}$ . If  $P_w \ll_R P_{x'}$ , then the lemma follows by definition of  $x_2$ . If  $P_{x'} \ll_R P_w$ , then  $P_u \ll_R P_{x'} \ll_R P_w$ , and thus  $w \notin N(u)$ , which is a contradiction. Suppose that  $P_{x'}$  intersects  $P_w$  in  $R$ . Then,  $x'$  is unbounded and  $\phi_{x'} > \phi_w > \phi_u$ , since  $w$  is bounded and  $x' \notin N(w)$ . Note that  $P_x \ll_R P_u \ll_R P_{x'}$  for every  $x \in \tilde{X}_1$ , since  $x' \in D_2$  and  $\tilde{X}_1 \subseteq D_1$  by Lemma 14. Therefore,  $x' \notin N(x)$  for every  $x \in \tilde{X}_1$ , and thus in particular  $x' \notin N(x_1)$ , since  $x_1 \in \tilde{X}_1$ . Therefore,  $T_{x'}$  does not intersect  $T_{x_1}$  in  $R_T$ , and thus  $T_{x_1} \ll_{R_T} T_{x'}$  by definition of  $x_1$ . Furthermore, since  $\tilde{X}_1$  induces a connected subgraph of  $G$ , and since  $x' \notin N(x)$  for every  $x \in \tilde{X}_1$ , it follows that  $T_x \ll_{R_T} T_{x'}$  for every  $x \in \tilde{X}_1$ . Recall now that  $T_{x_2} \ll_{R_T} T_w$  and that we assumed that  $T_{x'} \ll_{R_T} T_{x_2}$ . That is,  $T_x \ll_{R_T} T_{x'} \ll_{R_T} T_{x_2} \ll_{R_T} T_w$  for every  $x \in \tilde{X}_1$ .

Recall that  $N(\tilde{X}_1) \subseteq N(w)$  by the construction of the set  $\tilde{X}_1$ . Therefore, since  $T_x \ll_{R_T} T_{x'} \ll_{R_T} T_w$  for every  $x \in \tilde{X}_1$ , it follows that  $T_z$  intersects  $T_{x'}$  in  $R_T$  for every  $z \in N(\tilde{X}_1) \subseteq N(w)$ , and thus  $N(\tilde{X}_1) \subseteq N(x')$ . On the other hand, since  $P_x \ll_R P_u \ll_R P_{x'}$  for every  $x \in \tilde{X}_1$  in the projection representation  $R$ , it follows that  $P_z$  intersects  $P_u$  in  $R$  for every  $z \in N(\tilde{X}_1) \subseteq N(x')$ . Furthermore, since  $x'$  is unbounded and  $\phi_{x'} > \phi_u$  in  $R$ , it follows that  $z$  is bounded in  $R$  and  $\phi_z > \phi_{x'} > \phi_u$  for every  $z \in N(\tilde{X}_1) \subseteq N(x')$ . Therefore,  $z \in N(u)$  for every  $z \in N(\tilde{X}_1)$ , i.e.  $N(\tilde{X}_1) \subseteq N(u)$ , which is a contradiction by Corollary 3. This completes the proof of the lemma. ■

## The vertex sets $C_u$ , $C_2$ , $X_1$ , and $H$

Let  $C_u$  be the connected component of  $G \setminus Q_u \setminus N[\tilde{X}_1, x_2]$ , in which  $u$  belongs. Note that, in particular,  $w$  belongs to  $C_u$ , since  $wu \in E$ ,  $w \notin Q_u$ , and  $wx, wx_2 \notin E$  for every  $x \in \tilde{X}_1$ , and thus  $C_u \setminus \{u\} \neq \emptyset$ . Recall that the trapezoids of all vertices of  $V_0(u)$  lie to the left of the trapezoid of  $u$  in the trapezoid representation  $R_T$ ;  $S_2$  is exactly the subset of vertices of  $V_0(u)$ , whose parallelograms do not lie to the left of the parallelogram  $P_u$  of  $u$  in  $R$ . Let  $\tilde{C}_2$  be the set of connected components of  $G \setminus Q_u \setminus N[\tilde{X}_1]$ , in which the vertices of  $S_2$  belong. Since  $x_2 \in S_2$ , note that  $V(C_u \cup \tilde{C}_2)$  induces the set of connected components of  $G \setminus Q_u \setminus N[\tilde{X}_1]$ , in which the vertices of  $S_2 \cup \{u\}$  belong. Furthermore, let  $\tilde{C}_2 = \tilde{C}_2 \setminus N[u, w] \setminus C_u$ . Finally, let  $\tilde{H}$  be the induced subgraph of  $G \setminus Q_u \setminus N[\tilde{X}_1]$  on the vertices of  $N[u, w] \cap N(x_2)$ . Note now that  $V(C_u \cup \tilde{C}_2) = V(C_u \cup \tilde{C}_2 \cup \tilde{H})$ , i.e.  $V(C_u \cup \tilde{C}_2 \cup \tilde{H})$  also induces the set of connected components of  $G \setminus Q_u \setminus N[\tilde{X}_1]$ , in which the vertices of  $S_2 \cup \{u\}$  belong.



Let  $v$  be a vertex of the set  $\tilde{C}_2$ , and thus  $v \notin N(u)$  by the definition of  $\tilde{C}_2$ . Suppose that  $P_v$  intersects  $P_u$  in  $R$ . If  $\phi_v > \phi_u$ , then  $v$  is unbounded, since otherwise  $v \in N(u)$ , which is a contradiction. Furthermore,  $N(v) \subseteq N(u)$  by Lemma 3, and thus  $v \in Q_u$ , which is a contradiction to the definition of  $\tilde{C}_2$ . If  $\phi_v < \phi_u$ , then  $N(u) \subseteq N(v)$  by Lemma 3, and thus  $w \in N(v)$ , which is again a contradiction to the definition of  $\tilde{C}_2$ . Therefore, there is no vertex  $v$  of  $\tilde{C}_2$ , such that  $P_v$  intersects  $P_u$  in  $R$ . That is, for every  $v \in \tilde{C}_2$  either  $P_v \ll_R P_u$  or  $P_u \ll_R P_v$ . Let now  $A_1, A_2, \dots, A_k, A_{k+1}, \dots, A_\ell$  be the connected components of  $\tilde{C}_2$ , such that  $P_v \ll_R P_u$  for every  $v \in A_i$ ,  $i = 1, 2, \dots, k$ , and  $P_u \ll_R P_v$  for every  $v \in A_j$ ,  $j = k+1, k+2, \dots, \ell$ .

We partition first the set  $\{A_{k+1}, \dots, A_\ell\}$  of components into two possibly empty subsets, namely  $\mathcal{B}_1$  and  $\mathcal{B}_2$ , as follows. A component  $A_j \in \mathcal{B}_2$ ,  $j = k+1, k+2, \dots, \ell$ , if  $A_j \cap S_2 \neq \emptyset$ ; otherwise,  $A_j \in \mathcal{B}_1$ . Then, since any component  $A_j \in \mathcal{B}_2$  is a connected subgraph of  $G \setminus N[u]$ , and since  $A_j$  has at least one vertex of  $S_2 \subseteq V_0(u)$ , it follows that  $v \in V_0(u)$  for every  $v \in A_j$ , where  $A_j \in \mathcal{B}_2$ . Furthermore,  $v \in D_2$  for every  $v \in A_j \in \mathcal{B}_2$ , since  $P_u \ll_R P_v$  for every  $v \in A_j$ . Thus,  $A_j \subseteq D_2$  for every component  $A_j \in \mathcal{B}_2$ , while  $A_j \cap D_2 = \emptyset$  for every component  $A_j \in \mathcal{B}_1$ . That is, in particular the next observation follows.

**Observation 2**  $V(\mathcal{B}_1) \subseteq V \setminus Q_u \setminus N[u] \setminus V_0(u)$ , where  $V(\mathcal{B}_1) = \bigcup_{A_j \in \mathcal{B}_1} A_j$ .

We partition now the set  $\{A_1, A_2, \dots, A_k\}$  of components into two possibly empty subsets, namely  $\mathcal{A}_1$  and  $\mathcal{A}_2$ , as follows. A component  $A_i \in \mathcal{A}_2$ ,  $i = 1, 2, \dots, k$ , if  $\tilde{H} \subseteq N(x)$  for all vertices  $x \in A_i$ ; otherwise,  $A_i \in \mathcal{A}_1$ . That is,  $\mathcal{A}_2$  includes exactly those components  $A_i$ ,  $i = 1, 2, \dots, k$ , for which all vertices of  $A_i$  are adjacent to all vertices of  $\tilde{H}$ .

We now extend the vertex set  $\tilde{X}_1$  to the set  $X_1 = \tilde{X}_1 \cup V(\mathcal{A}_1)$ , where  $V(\mathcal{A}_1) = \bigcup_{A_i \in \mathcal{A}_1} A_i$ , and define  $C_2 = \mathcal{A}_2 \cup \mathcal{B}_2$ . Furthermore, similarly to the definition of  $\tilde{H}$ , let  $H$  be the induced subgraph of  $G \setminus Q_u \setminus N[X_1]$  on the vertices of  $N[u, w] \cap N(x_2)$ . Note that  $H \subseteq \tilde{H}$ , since  $\tilde{X}_1 \subseteq X_1$ , and thus for every component  $A_i \in \mathcal{A}_2$ , all vertices of  $A_i$  are also adjacent to all vertices of  $H$ . Furthermore, since  $X_1 = \tilde{X}_1 \cup V(\mathcal{A}_1)$ , and since no vertex of  $\mathcal{A}_1$  is adjacent to any vertex of  $\tilde{X}_1$ , note that  $N(X_1) = N(\tilde{X}_1) \cup N(V(\mathcal{A}_1))$  and that  $N[X_1] = N[\tilde{X}_1] \cup N[V(\mathcal{A}_1)]$ , i.e. in particular  $N(\tilde{X}_1) \subseteq N(X_1)$ . Moreover,  $N(X_1) \neq \emptyset$ , since  $N(\tilde{X}_1) \neq \emptyset$ .

Recall that  $V(C_u \cup \tilde{C}_2 \cup \tilde{H})$  induces the set of connected components of  $G \setminus Q_u \setminus N[\tilde{X}_1]$ , in which the vertices of  $S_2 \cup \{u\}$  belong. The next lemma follows by the definitions of  $C_u$ ,  $C_2$ , and  $H$ .

**Lemma 16**  $V(C_u \cup C_2 \cup H)$  induces a subgraph of  $G \setminus Q_u \setminus N[X_1] \setminus \mathcal{B}_1$  that includes all connected components of  $G \setminus Q_u \setminus N[X_1] \setminus \mathcal{B}_1$ , in which the vertices of  $S_2 \cup \{u\}$  belong. Furthermore,  $N(V(C_u \cup C_2 \cup H)) \subseteq Q_u \cup N(X_1) \cup V(\mathcal{B}_1)$ .

**Proof.** Consider a vertex  $v \in N(V(\mathcal{A}_1))$ . That is,  $v \in N(v')$  and  $v \notin V(\mathcal{A}_1)$ , for some vertex  $v' \in V(\mathcal{A}_1)$ , i.e.  $v' \in A_i$  for some  $A_i \in \mathcal{A}_1$ . First note that  $v' \notin N(x_2)$ , since  $P_{v'} \ll_R P_u \ll_R P_{x_2}$  for every  $v' \in A_i$  by definition of  $\mathcal{A}_1$ . If  $v \in Q_u$ , then  $N(v) \subset N(u)$  by definition of  $Q_u$ , and thus  $v' \in N(u)$ , which is a contradiction due to the definition of  $\tilde{C}_2$ , and since  $v' \in V(\mathcal{A}_1) \subseteq \tilde{C}_2$ . Therefore  $v \notin Q_u$ . We will now prove that  $v \in N(\tilde{X}_1)$  or  $v \in \tilde{H}$ . To this end, suppose that  $v \notin N(\tilde{X}_1)$ . If  $v \in \tilde{C}_2$ , then  $v$  is a vertex of the connected component  $A_i$  of  $\tilde{C}_2$ , since  $v \in N(v')$  and  $v' \in A_i$ . This is a contradiction, since  $v \notin V(\mathcal{A}_1)$ ; thus  $v \notin \tilde{C}_2$ . That is,  $v' \in \tilde{C}_2 \subseteq \tilde{\tilde{C}}_2$  and  $v \notin \tilde{C}_2$ . Therefore, since  $v \in N(v')$  and  $v \notin Q_u \cup N(\tilde{X}_1)$ , it follows by definitions of  $\tilde{C}_2$  and  $\tilde{\tilde{C}}_2$  that  $v \in C_u$  or  $v \in N[u, w]$ . Let  $v \in C_u$ . Then, since  $v' \in N(v)$  and  $v' \notin N(x_2)$ , it follows that also  $v' \in C_u$ , which is a contradiction by definition of  $\tilde{C}_2$ . Let  $v \in N[u, w]$ . If  $v \notin N(x_2)$ , then  $v \in C_u$  and  $v' \in C_u$ , which is again a contradiction. If  $v \in N(x_2)$ , then  $v \in \tilde{H}$  by definition of  $\tilde{H}$ . Summarizing, if  $v \notin N(\tilde{X}_1)$ , then  $v \in \tilde{H}$ . That is, for an arbitrary vertex  $v \in N(V(\mathcal{A}_1))$ , either  $v \in N(\tilde{X}_1)$  or  $v \in \tilde{H}$ , i.e.  $N(V(\mathcal{A}_1)) \subseteq N(\tilde{X}_1) \cup \tilde{H}$ .

Note by definition of  $C_u$  and of  $\tilde{C}_2$  that  $V(C_u) \cap V(\tilde{H}) = \emptyset$  and that  $V(\tilde{C}_2) \cap V(\tilde{H}) = \emptyset$ . Therefore, it follows by the previous paragraph that  $V(C_u) \cap N(V(\mathcal{A}_1)) \subseteq V(C_u) \cap (N(\tilde{X}_1) \cup \tilde{H}) = \emptyset$

and that  $V(\tilde{C}_2) \cap N(V(\mathcal{A}_1)) \subseteq V(\tilde{C}_2) \cap (N(\tilde{X}_1) \cup \tilde{H}) = \emptyset$ . Thus,

$$V(C_u) \setminus N(V(\mathcal{A}_1)) = V(C_u) \quad (1)$$

$$V(\tilde{C}_2) \setminus N(V(\mathcal{A}_1)) = V(\tilde{C}_2) \quad (2)$$

Recall now that  $N(X_1) = N(\tilde{X}_1) \cup N(V(\mathcal{A}_1))$ . Therefore, it follows by definition of  $H$  that

$$\begin{aligned} V(\tilde{H}) &= V(\tilde{H} \setminus N(V(\mathcal{A}_1))) \cup V(\tilde{H} \cap N(V(\mathcal{A}_1))) \\ &= V(H) \cup V(\tilde{H} \cap N(V(\mathcal{A}_1))) \end{aligned} \quad (3)$$

Furthermore, recall that  $V(\tilde{C}_2) = V(C_2) \cup V(\mathcal{A}_1) \cup V(\mathcal{B}_1)$  by definition of  $C_2$ , and thus it follows by (3) that

$$\begin{aligned} V(C_u \cup \tilde{C}_2 \cup \tilde{H}) &= V(C_u) \cup V(C_2) \cup V(\mathcal{A}_1) \cup V(\mathcal{B}_1) \\ &\quad \cup V(H) \cup V(\tilde{H} \cap N(V(\mathcal{A}_1))) \end{aligned} \quad (4)$$

Therefore, it follows by (1), (2), and (4) that

$$V(C_u \cup \tilde{C}_2 \cup \tilde{H}) \setminus N[V(\mathcal{A}_1)] \setminus V(\mathcal{B}_1) = V(C_u) \cup V(C_2) \cup V(H) \quad (5)$$

Thus, since  $N[X_1] = N[\tilde{X}_1] \cup N[V(\mathcal{A}_1)]$ , it follows that also

$$V(C_u \cup \tilde{C}_2 \cup \tilde{H}) \setminus N[X_1] \setminus V(\mathcal{B}_1) = V(C_u \cup C_2 \cup H) \quad (6)$$

Therefore, since  $V(C_u \cup \tilde{C}_2 \cup \tilde{H})$  induces the set of connected components of  $G \setminus Q_u \setminus N[\tilde{X}_1]$ , in which the vertices of  $S_2 \cup \{u\}$  belong, it follows in particular by (6) that  $V(C_u \cup C_2 \cup H)$  induces a subgraph of  $G \setminus Q_u \setminus N[X_1] \setminus \mathcal{B}_1$ ; moreover, this subgraph includes all connected components of  $G \setminus Q_u \setminus N[X_1] \setminus \mathcal{B}_1$ , in which the vertices of  $S_2 \cup \{u\}$  belong. On the other hand, since  $V(C_u \cup \tilde{C}_2 \cup \tilde{H})$  induces a set of connected components of  $G \setminus Q_u \setminus N[\tilde{X}_1]$ , it follows that  $N(V(C_u \cup \tilde{C}_2 \cup \tilde{H})) \subseteq Q_u \cup N(\tilde{X}_1)$ . Therefore, it follows by (6) that  $N(V(C_u \cup C_2 \cup H)) \subseteq Q_u \cup N(X_1) \cup V(\mathcal{B}_1)$ . This completes the proof of the lemma. ■

For the sequel of the proof, denote for simplicity  $N_1(v) = N(v) \cap N(X_1)$  for every vertex  $v \in V \setminus X_1$ . Moreover,  $C_u$  is also the connected component of  $G \setminus Q_u \setminus N[X_1, x_2]$  (and not only of  $G \setminus Q_u \setminus N[\tilde{X}_1, x_2]$ ), in which  $u$  belongs, as we prove in the next lemma. The next two lemmas extend Lemma 14.

**Lemma 17** *For the constructed sets  $X_1$  and  $C_2$ ,  $N_1(w) = N(X_1)$ ,  $X_1 \subseteq D_1$ , and  $C_2 \subseteq V_0(u)$ . Furthermore,  $C_u$  is the connected component of  $G \setminus Q_u \setminus N[X_1, x_2]$ , in which  $u$  belongs.*

**Proof.** Recall first that  $N(\tilde{X}_1) \subseteq N(w)$  by the construction of the set  $\tilde{X}_1$ . Consider an arbitrary component  $A_i \in \mathcal{A}_1 \cup \mathcal{A}_2 = \{A_1, A_2, \dots, A_k\}$ . Recall that  $v \notin N(x_2)$  for every  $v \in A_i$ , since  $P_v \ll_R P_u \ll_R P_{x_2}$ . We will prove now that  $N(A_i) \setminus N[\tilde{X}_1] \subseteq N(x_2)$ . Suppose otherwise that there exists a vertex  $v \in A_i$  and a vertex  $v' \in N(v) \setminus N[\tilde{X}_1]$ , such that  $v' \notin A_i$  and  $v' \notin N(x_2)$ . By definition of  $\tilde{C}_2$  it follows that either  $v' \in Q_u$ , or  $v' \in N[u, w]$ , or  $v' \in C_u$ . Suppose that  $v' \in Q_u$ . Then,  $N(v') \subset N(u)$ , and thus  $v \in N(u)$ , since  $vv' \in E$ . This is a contradiction, since  $P_v \ll_R P_u$  for every  $v \in A_i$ , where  $A_i \in \mathcal{A}_1 \cup \mathcal{A}_2$ . Therefore, either  $v' \in N[u, w]$  or  $v' \in C_u$ . Then, since  $u, w \in C_u$  and  $v' \notin N(x_2)$ , it follows by the definition of  $C_u$  that always  $v' \in C_u$ . Thus,  $v \in C_u$ , since  $v \in N(v')$  and  $v \notin N(x_2)$ , which is a contradiction to definition of  $\tilde{C}_2$ . Therefore,  $N(A_i) \setminus N[\tilde{X}_1] \subseteq N(x_2)$  for every  $A_i \in \mathcal{A}_1 \cup \mathcal{A}_2$ . Therefore, in particular  $N(V(\mathcal{A}_1)) \setminus N[\tilde{X}_1] \subseteq N(x_2)$ , and thus  $(N(V(\mathcal{A}_1)) \setminus N[\tilde{X}_1]) \cap N(x_2) = N(V(\mathcal{A}_1)) \setminus N[\tilde{X}_1]$ .

Recall that if a vertex  $v \in N[\tilde{X}_1]$ , then  $v \notin C_u$  by definition of  $C_u$ . Moreover, as we have proved in the previous paragraph, if a vertex  $v \in N(V(\mathcal{A}_1)) \setminus N[\tilde{X}_1]$ , then  $v \in N(x_2)$ , and thus again  $v \notin C_u$  by definition of  $C_u$ . Therefore, since  $X_1 = \tilde{X}_1 \cup V(\mathcal{A}_1)$ , it follows that if a vertex  $v \in N[X_1]$ , then  $v \notin C_u$ . That is,  $C_u$  is the connected component of  $G \setminus Q_u \setminus N[X_1, x_2]$ , in which  $u$  belongs.

Let  $A_i \in \mathcal{A}_1$ . Note that no vertex  $v \in A_i$  is adjacent to any vertex of  $\tilde{X}_1$ . Indeed, otherwise  $v \in N(w)$  by definition of  $\tilde{X}_1$ , which is a contradiction to the definition of  $\tilde{C}_2$ . Since  $A_i \subseteq \tilde{C}_2$  includes no vertex of  $C_u$ , it follows in particular that  $v \notin N(w)$  for every  $v \in A_i$ . Indeed, otherwise  $v \in C_u$ , since also  $v \notin N(x_2)$ , which is a contradiction. Consider now a vertex  $z \in (N(A_i) \setminus N[\tilde{X}_1]) \cap N(x_2)$ , i.e.  $z \in (N(v) \setminus N[\tilde{X}_1]) \cap N(x_2)$  and  $z \notin A_i$ , for some  $v \in A_i$ . Suppose first that  $P_v$  intersects  $P_w$  in  $R$ . Then,  $v$  is unbounded and  $\phi_v > \phi_w$ , since  $w$  is bounded, and thus  $N(v) \subseteq N(w)$  by Lemma 3. Therefore, in particular,  $z \in N(w)$ . Suppose now that  $P_v$  does not intersect  $P_w$  in  $R$ . Then,  $P_v \ll_R P_u \ll_R P_{x_2}$  and  $P_v \ll_R P_w \ll_R P_{x_2}$ , since  $wu \in E$ . Thus,  $P_z$  intersects  $P_w$  and  $P_u$  in  $R$ , since  $z \in N(v) \cap N(x_2)$ . If  $z$  is unbounded, then  $\phi_z > \phi_u$ , since  $\phi_u = \min\{\phi_x \mid x \in V_U\}$  in  $R$  by assumption. Therefore,  $N(z) \subseteq N(u)$  by Lemma 3, and thus  $x_2 \in N(u)$ , which is a contradiction. Therefore,  $z$  is bounded, and thus  $z \in N(w)$ , since  $P_z$  intersects  $P_w$  in  $R$  and both  $z$  and  $w$  are bounded. Summarizing,  $z \in N(w)$  for every  $z \in (N(A_i) \setminus N[\tilde{X}_1]) \cap N(x_2)$ . That is,  $(N(A_i) \setminus N[\tilde{X}_1]) \cap N(x_2) \subseteq N(w)$  for every  $A_i \in \mathcal{A}_1$ , i.e.  $(N(V(\mathcal{A}_1)) \setminus N[\tilde{X}_1]) \cap N(x_2) \subseteq N(w)$ . Therefore, since  $X_1 = \tilde{X}_1 \cup V(\mathcal{A}_1)$ , and since no vertex of  $\mathcal{A}_1$  is adjacent to any vertex of  $\tilde{X}_1$ , it follows that

$$\begin{aligned} N(X_1) &= N(\tilde{X}_1) \cup (N(V(\mathcal{A}_1)) \setminus N[\tilde{X}_1]) \\ &= N(\tilde{X}_1) \cup ((N(V(\mathcal{A}_1)) \setminus N[\tilde{X}_1]) \cap N(x_2)) \subseteq N(w) \end{aligned} \quad (7)$$

since  $(N(V(\mathcal{A}_1)) \setminus N[\tilde{X}_1]) \cap N(x_2) = N(V(\mathcal{A}_1)) \setminus N[\tilde{X}_1]$  and  $N(\tilde{X}_1) \subseteq N(w)$ . That is,  $N(X_1) \subseteq N(w)$ , i.e.  $N_1(w) = N(X_1)$ .

Let now  $A_i \in \mathcal{A}_1 \cup \mathcal{A}_2$ , and let  $v \in A_i$ . Suppose first that  $P_x \ll_R P_v$  for some  $x \in \tilde{X}_1$ , i.e.  $P_x \ll_R P_v \ll_R P_u \ll_R P_{x_2}$ . Then, since  $x, x_2 \in V_0(u)$ , and since  $V_0(u)$  is connected, there exists a vertex  $z \in V_0(u)$ , such that  $P_z$  intersects  $P_v$  in  $R$ . If  $zv \in E$ , then  $v \in V_0(u)$ , and thus  $A_i \subseteq V_0(u)$ . Let now  $zv \notin E$ . If  $\phi_z > \phi_v$  then  $N(z) \subseteq N(v)$  by Lemma 3. Then, since  $z \in V_0(u)$ , and since  $V_0(u)$  is connected with at least two vertices,  $z$  has at least one neighbor  $z' \in V_0(u)$ , and thus  $z' \in N(v)$ . Then,  $v \in V_0(u)$ , and thus  $A_i \subseteq V_0(u)$ . On the other hand, if  $\phi_v > \phi_z$ , then  $N(v) \subseteq N(z)$  by Lemma 3. Furthermore,  $v$  is unbounded, since otherwise  $zv \in E$ , which is a contradiction. If  $N(v) \subseteq N(u)$ , then  $v \in Q_u$ , which is a contradiction to the definition of  $\tilde{C}_2$ . Suppose now that  $N(v) \not\subseteq N(u)$ , i.e.  $v$  has at least one neighbor  $v' \notin N(u)$ . Then,  $v' \in N(z)$ , since  $N(v) \subseteq N(z)$ . Therefore,  $v' \in V_0(u)$  and  $v \in V_0(u)$ , and thus  $A_i \subseteq V_0(u)$ . Summarizing, if  $P_x \ll_R P_v$  for some  $x \in \tilde{X}_1$ , then  $A_i \subseteq V_0(u)$ .

Suppose now that  $P_v$  intersects  $P_x$  in  $R$ , for some  $x \in \tilde{X}_1$ . Recall that  $\tilde{X}_1 \subseteq V_0(u)$  by Lemma 14, and thus  $x \in V_0(u)$ . If  $vx \in E$ , then  $v \in V_0(u)$ , and thus  $A_i \subseteq V_0(u)$ . Let now  $vx \notin E$ . Then, it follows similarly to the previous paragraph that  $A_i \subseteq V_0(u)$ .

Suppose finally that  $P_v \ll_R P_x$ , i.e.  $P_v \ll_R P_x \ll_R P_u \ll_R P_{x_2}$ , for some  $x \in \tilde{X}_1$ . Recall that  $N(A_i) \setminus N[\tilde{X}_1] \subseteq N(x_2)$ , and thus for every vertex  $v' \in N(v) \setminus N[\tilde{X}_1]$ , such that  $v' \notin A_i$ , it follows that  $v' \in N(x_2)$ . Consider such a vertex  $v'$ . Then,  $P_{v'}$  intersects  $P_u$  and  $P_x$  in  $R$ , since  $P_v \ll_R P_x \ll_R P_u \ll_R P_{x_2}$ . Note that  $v' \notin N(x)$ , since otherwise  $v' \in N(\tilde{X}_1)$ , which is a contradiction to the assumption that  $v' \in N(v) \setminus N[\tilde{X}_1]$ . Suppose that  $v' \in N(u)$ , and thus  $v'$  is bounded in  $R$  and  $\phi_{v'} > \phi_u$ . Then, since  $v' \notin N(x)$ , it follows that  $x$  is unbounded and  $\phi_x > \phi_{v'} > \phi_u$ . Thus,  $N(x) \subseteq N(v')$  by Lemma 3. If  $\tilde{X}_1 \neq \{x\}$ , then  $x$  has at least one neighbor  $x'$  in  $\tilde{X}_1$  and  $x' \in N(v')$ , since  $N(x) \subseteq N(v')$ . Thus,  $v' \in N(\tilde{X}_1)$ , which is a contradiction to the assumption that  $v' \in N(v) \setminus N[\tilde{X}_1]$ . Let  $\tilde{X}_1 = \{x\}$  and  $z \in N(x)$ . Then,  $N(x) \subseteq N(w)$  by definition of  $\tilde{X}_1$ , i.e.  $z \in N(w)$ . Thus, since  $T_x \ll_{R_T} T_{x_2} \ll_{R_T} T_w$ , it follows that  $T_z$  intersects  $T_{x_2}$  in  $R_T$ , i.e.  $z \in N(x_2)$ . Thus,  $P_z$  intersects  $P_u$  in  $R$ , since  $P_x \ll_R P_u \ll_R P_{x_2}$  and  $z \in N(x) \cap N(x_2)$ . However,  $z$  is bounded and  $\phi_z > \phi_x > \phi_u$ , since  $x$  is unbounded. Thus,  $zu \in E$ , i.e.  $z \in N(u)$ . Since this holds for an arbitrary  $z \in N(x)$ , it follows that  $N(x) \subseteq N(u)$ , and thus  $x \in Q_u$ , which is a contradiction by Lemma 13, since  $\tilde{X}_1 = \{x\} \subseteq V_0(u)$ . Thus,  $v' \notin N(u)$  for every vertex  $v' \in N(v) \setminus N[\tilde{X}_1]$ , such that  $v' \notin A_i$ . Therefore, since  $v' \in N(x_2)$  for all such vertices  $v'$ , and since  $x_2 \in V_0(u)$ , it follows that  $v', v \in V_0(u)$ , and thus  $A_i \subseteq V_0(u)$ .

Summarizing,  $A_i \subseteq V_0(u)$  in every case, and thus  $A_i \subseteq D_1$  for every component  $A_i \in \mathcal{A}_1 \cup \mathcal{A}_2$ . Furthermore, recall that  $\tilde{X}_1 \subseteq D_1$  by Lemma 14. Thus, since also  $A_i \subseteq D_1$  for every component

$A_i \in \mathcal{A}_1$ , it follows that  $X_1 = \tilde{X}_1 \cup V(\mathcal{A}_1) \subseteq D_1$ .

Recall now that  $A_j \subseteq D_2$  for every component  $A_j \in \mathcal{B}_2$ , where  $k+1 \leq j \leq \ell$ , and thus  $A_j \subseteq V_0(u)$  for every  $A_j \in \mathcal{B}_2$ . Therefore, since also  $A_i \subseteq V_0(u)$  for every  $A_i \in \mathcal{A}_2$ , and since  $C_2 = \mathcal{A}_2 \cup \mathcal{B}_2$ , it follows that  $C_2 \subseteq V_0(u)$ . This completes the proof of the lemma. ■

**Lemma 18** *For every  $x \in X_1$ ,  $T_x \ll_{R_T} T_{x_2}$  and  $P_x \ll_R P_w$ .*

**Proof.** Consider a component  $A_i \in \mathcal{A}_1$ . Recall that  $v \notin N(x_2)$  for every  $v \in A_i$ , since  $P_v \ll_R P_u \ll_R P_{x_2}$ . Thus, since  $A_i$  is connected, either  $T_{x_2} \ll_{R_T} T_v$  or  $T_v \ll_{R_T} T_{x_2}$  for every vertex  $v \in A_i$ . Suppose that  $T_{x_2} \ll_{R_T} T_v$  for every  $v \in A_i$ ; let  $v \in A_i$  be such a vertex. Since  $v \in X_1 \subseteq V_0(u)$  by Lemma 17, it follows that  $T_v \ll_{R_T} T_u$ . Recall that  $v \notin N(u) \cup N(w)$  by definition of  $\tilde{C}_2$ . Therefore, since  $w \in N(u)$ , it follows that also  $T_v \ll_{R_T} T_w$ . Consider now a vertex  $z \in \tilde{H} = N[u, w] \cap N(x_2) \setminus Q_u \setminus N[\tilde{X}_1]$ . Then, since  $T_{x_2} \ll_{R_T} T_v \ll_{R_T} T_u$  and  $T_{x_2} \ll_{R_T} T_v \ll_{R_T} T_w$ , it follows that  $T_z$  intersects  $T_v$  in  $R_T$ , and thus  $vz \in E$ . Since this holds for every vertex  $v \in A_i$  and every vertex  $z \in \tilde{H}$ , it follows that  $A_i \in \mathcal{A}_2$ , which is a contradiction. Thus,  $T_v \ll_{R_T} T_{x_2}$  for every vertex  $v \in A_i$ , where  $A_i \in \mathcal{A}_1$ . Therefore, since also  $T_x \ll_{R_T} T_{x_2}$  for every vertex  $x \in \tilde{X}_1$  by Lemma 14, it follows that  $T_x \ll_{R_T} T_{x_2}$  for every vertex  $x \in X_1$ .

We will prove now that  $P_v \ll_R P_w$  for every  $v \in A_i$ , where  $A_i \in \mathcal{A}_1$ . Otherwise, suppose first that  $P_w \ll_R P_v$  for some  $v \in A_i$ . Then, since  $P_v \ll_R P_u$  for every  $v \in A_i$ , it follows that  $P_w \ll_R P_v \ll_R P_u$ , and thus  $w \notin N(u)$ , which is a contradiction. Suppose now that  $P_v$  intersects  $P_w$  in  $R$ , for some  $v \in A_i$ . Then, since  $v \notin N(w)$  by definition of  $\tilde{C}_2$ , and since  $w$  is bounded, it follows that  $v$  is unbounded and  $\phi_v > \phi_w > \phi_u$ . Thus,  $N(v) \subseteq N(w)$  by Lemma 3. Let now  $z \in N(v) \subseteq N(w)$ . Then, since  $T_v \ll_{R_T} T_{x_2} \ll_{R_T} T_w$  (cf. the previous paragraph), it follows that  $T_z$  intersects  $T_{x_2}$  in  $R_T$ , i.e.  $z \in N(x_2)$ . Since this holds for an arbitrary  $z \in N(v)$ , it follows that also  $N(v) \subseteq N(x_2)$ . Therefore, since  $P_v \ll_R P_u \ll_R P_{x_2}$ , it follows that  $P_z$  intersects  $P_u$  in  $R$  for every  $z \in N(v) \subseteq N(x_2)$ . Furthermore, since  $v$  is unbounded, it follows that  $z$  is bounded and  $\phi_z > \phi_v > \phi_u$  for every  $z \in N(v)$ , and thus  $N(v) \subseteq N(u)$ . That is,  $v \in Q_u$ , which is a contradiction by Lemma 13, since  $v \in A_i \subseteq X_1 \subseteq V_0(u)$ . It follows that  $P_v \ll_R P_w$  for every  $v \in A_i$ , where  $A_i \in \mathcal{A}_1$ . Therefore, since also  $P_x \ll_R P_w$  for every vertex  $x \in \tilde{X}_1$  by Lemma 14, it follows that  $P_x \ll_R P_w$  for every vertex  $x \in X_1$ . This completes the lemma. ■

**Lemma 19** *Denote  $N = N(X_1) = N_1(w)$ . Then,  $N_1(u) \subset N$  and  $N_1(x_2) = N_1(v) = N$  for every bounded vertex  $v \in C_u \setminus \{u\}$  in  $R$ .*

**Proof.** First note that  $N_1(u) \subseteq N$ , since  $N = N(X_1)$  and  $N_1(u) = N(u) \cap N(X_1)$  by definition. Recall that  $N(\tilde{X}_1) \subseteq N = N(X_1)$  and that  $N(\tilde{X}_1) \setminus N(u) \neq \emptyset$  by Corollary 3. Therefore also  $N \setminus N(u) \neq \emptyset$ , and thus  $N_1(u) \subset N$ .

Consider a vertex  $z \in N$ , i.e.  $z \in N(x) \cap N(w)$  for some  $x \in X_1$  by Lemma 17. Then, since  $T_x \ll_{R_T} T_{x_2} \ll_{R_T} T_w$  by Lemma 18, it follows that  $T_z$  intersects  $T_{x_2}$  in  $R_T$ . Therefore,  $z \in N(x_2)$ , and thus  $z \in N_1(x_2)$ . Since this holds for every  $z \in N$ , it follows that  $N \subseteq N_1(x_2)$ . Thus, since by definition  $N_1(x_2) \subseteq N$ , it follows that  $N_1(x_2) = N$ .

Consider now a bounded vertex  $v \in C_u$  in  $R$  and a vertex  $z \in N$ . Then,  $z \in N(x) \cap N(x_2)$  for some  $x \in X_1$ , since  $N_1(x_2) = N$  by the previous paragraph. Recall that  $C_u$  is connected and that no vertex of  $C_u$  is adjacent to  $x_2$  by the definition of  $C_u$ . Thus, since  $w \in C_u$  and  $T_{x_2} \ll_{R_T} T_w$ , it follows that  $T_{x_2}$  lies in  $R_T$  to the left of all trapezoids of the vertices of  $C_u$ ; in particular, Lemma 18 implies that  $T_x \ll_{R_T} T_{x_2} \ll_{R_T} T_v$  for every  $x \in X_1$ .

Suppose first that  $P_x \ll_R P_v \ll_R P_{x_2}$ . Then,  $P_z$  intersects  $P_v$  in  $R$ . Suppose that  $z \notin N(v)$ . Then, since  $v$  is bounded, it follows that  $z$  is unbounded and  $\phi_z > \phi_v$ , and thus  $N(z) \subseteq N(v)$  by Lemma 3. Therefore, since  $x \in N(z)$ , it follows that  $x \in N(v)$ , i.e.  $v \in N(X_1)$ , which is a contradiction by Lemma 17. Thus,  $z \in N(v)$ .

Suppose now that  $P_v$  intersects  $P_x$  (resp.  $P_{x_2}$ ) in  $R$ . Recall that, since  $v \in C_u$ ,  $v \notin N(x)$  by Lemma 17 (resp.  $v \notin N(x_2)$  by definition of  $C_u$ ). Thus, either  $N(v) \subseteq N(x)$  or  $N(x) \subseteq N(v)$  (resp.  $N(v) \subseteq N(x_2)$  or  $N(x_2) \subseteq N(v)$ ) by Lemma 3. If  $N(v) \subseteq N(x)$  (resp.  $N(v) \subseteq N(x_2)$ ), then  $v$

is an isolated vertex in  $G \setminus Q_u \setminus N[X_1, x_2]$ , and thus  $v \notin C_u$ , since  $v \neq u$ , which is a contradiction. If  $N(x) \subseteq N(v)$  (resp.  $N(x_2) \subseteq N(v)$ ), then  $z \in N(v)$ , since in particular  $z \in N(x)$  (resp.  $z \in N(x_2)$ ). Note here that this paragraph holds for both cases, where  $v$  is a bounded or an unbounded vertex in  $R$ .

Suppose that  $P_{x_2} \ll_R P_v$ . Then,  $v \notin N(u)$  and  $v \notin N(w)$ , since  $P_u \ll_R P_{x_2}$  and  $P_w \ll_R P_{x_2}$ . Furthermore, since  $C_u$  is connected, there must exist a vertex  $v'$  of  $C_u$ , such that  $P_{v'}$  intersects  $P_{x_2}$  in  $R$ , and a path  $P$  from  $v'$  to  $v$ , where all intermediate vertices are  $v'' \in C_u$ , such that  $P_{x_2} \ll_R P_{v''}$ , i.e.  $v'' \notin N(u)$  and  $v'' \notin N(w)$ . Recall that  $v' \notin N(x_2)$  by definition of  $C_u$ , since  $v' \in C_u$ . Then, since  $P_{v'}$  intersects  $P_{x_2}$  in  $R$ , it follows by the previous paragraph that  $z \in N(v')$ .

Let  $v' \in N(u)$ , and thus  $v'$  is bounded and  $\phi_{v'} > \phi_u$ . Then,  $x_2$  is unbounded and  $\phi_{x_2} > \phi_{v'} > \phi_u$ , since  $v'$  is bounded and  $v' \notin N(x_2)$ . Consider now an arbitrary  $z' \in N$ . Recall that  $z' \in N(x') \cap N(x_2)$  for some  $x' \in X_1$ , and thus  $P_{z'}$  intersects  $P_u$  in  $R$ , since  $P_{x'} \ll_R P_u \ll_R P_{x_2}$ . Furthermore,  $z'$  is bounded and  $\phi_{z'} > \phi_{x_2} > \phi_u$ , since  $x_2$  is unbounded. Thus,  $z' \in N(u)$ . Since this holds for an arbitrary  $z' \in N$ , it follows that  $N_1(u) = N$ , which is a contradiction.

Let  $v' \notin N(u)$ . Since  $v, v' \notin N(u)$ , and since  $v'' \notin N(u)$  for all intermediate vertices  $v''$  of the path  $P$ , it follows that either  $T_u \ll_{R_T} T_{v'}$  and  $T_u \ll_{R_T} T_v$ , or  $T_{v'} \ll_{R_T} T_u$  and  $T_v \ll_{R_T} T_u$ . Recall that  $z \in N(v')$ . Therefore, if  $T_u \ll_{R_T} T_{v'}$ , then  $T_z$  intersects  $T_u$  in  $R_T$ , i.e.  $z \in N(u)$ , since in this case  $T_{x_2} \ll_{R_T} T_u \ll_{R_T} T_{v'}$  and  $z \in N(v') \cap N(x_2)$ . Since this holds for an arbitrary  $z \in N$ , it follows that  $N_1(u) = N$ , which is a contradiction. Thus,  $T_{v'} \ll_{R_T} T_u$  and  $T_v \ll_{R_T} T_u$ . Since  $v \notin N(w)$ ,  $T_w$  does not intersect  $T_v$  in  $R_T$ , i.e. either  $T_w \ll_{R_T} T_v$  or  $T_v \ll_{R_T} T_w$ . If  $T_w \ll_{R_T} T_v$ , then  $T_w \ll_{R_T} T_v \ll_{R_T} T_u$ , and thus  $w \notin N(u)$ , which is a contradiction. Therefore,  $T_v \ll_{R_T} T_w$ , i.e.  $T_{x_2} \ll_{R_T} T_v \ll_{R_T} T_w$ . Thus,  $T_z$  intersects  $T_v$  in  $R_T$ , i.e.  $z \in N(v)$ , since  $z \in N(x_2) \cap N(w)$ .

Suppose finally that  $P_v \ll_R P_x$ . Then,  $v \notin N(u)$  and  $v \notin N(w)$ , since  $P_x \ll_R P_u$  and  $P_x \ll_R P_w$ . Furthermore, since  $C_u$  is connected, there must exist a vertex  $v'$  of  $C_u$ , such that  $P_{v'}$  intersects  $P_x$  in  $R$ , and a path  $P$  from  $v'$  to  $v$ , where all intermediate vertices are  $v'' \in C_u$ , such that  $P_{v''} \ll_R P_{x_1}$ , i.e.  $v'' \notin N(u)$  and  $v'' \notin N(w)$ . Recall that  $v' \notin N(x)$  by Lemma 17, since  $v' \in C_u$ . Then, since  $P_{v'}$  intersects  $P_x$  in  $R$ , it follows (similarly to the above case where  $P_v$  intersects  $P_x$  in  $R$ ) that  $z \in N(v')$ .

Let  $v' \in N(u)$ , and thus  $v'$  is bounded and  $\phi_{v'} > \phi_u$ . Then,  $x$  is unbounded and  $\phi_x > \phi_{v'} > \phi_u$ , since  $v'$  is bounded and  $v' \notin N(x)$ . Thus  $N(x) \subseteq N(v')$  by Lemma 3. Since  $x \in X_1$ , either  $x \in \tilde{X}_1$  or  $x \in A_i$  for some  $A_i \in \mathcal{A}_1$ . Let  $x \in \tilde{X}_1$  (resp.  $x \in A_i$  for some  $A_i \in \mathcal{A}_1$ ). If  $\tilde{X}_1 \neq \{x\}$  (resp.  $A_i \neq \{x\}$ ), then  $x$  has at least one neighbor  $x'$  in  $\tilde{X}_1$  (resp. in  $A_i$ ) and  $x' \in N(v')$ , since  $N(x) \subseteq N(v')$ . Thus,  $v' \in N(X_1)$ , which is a contradiction by Lemma 17, since  $v' \in C_u$ . If  $\tilde{X}_1 = \{x\}$  (resp.  $A_i = \{x\}$ ), then  $\{x\}$  is a connected component of  $X_1$ . Therefore,  $z' \notin X_1$  for every neighbor  $z' \in N(x)$ , and thus  $N(x) \subseteq N(x_2)$ , since  $N_1(x_2) = N(X_1)$ , as we proved above. That is,  $P_{z'}$  intersects  $P_u$  for every  $z' \in N(x)$ , since in this case  $P_x \ll_R P_u \ll_R P_{x_2}$  and  $z' \in N(x) \cap N(x_2)$ . However,  $z'$  is bounded and  $\phi_{z'} > \phi_x > \phi_u$ , since  $x$  is unbounded. Thus,  $z' \in N(u)$  for every  $z' \in N(x)$ . That is,  $N(x) \subseteq N(u)$ , and thus  $x \in Q_u$ , which is a contradiction by Lemma 13, since  $x \in X_1 \subseteq V_0(u)$ .

Let  $v' \notin N(u)$ . Since  $v, v' \notin N(u)$ , and since  $v'' \notin N(u)$  for all intermediate vertices  $v''$  of the path  $P$ , it follows that either  $T_u \ll_{R_T} T_{v'}$  and  $T_u \ll_{R_T} T_v$ , or  $T_{v'} \ll_{R_T} T_u$  and  $T_v \ll_{R_T} T_u$ . Recall that  $z \in N(v')$ . Therefore, if  $T_u \ll_{R_T} T_{v'}$ , then  $T_z$  intersects  $T_u$  in  $R_T$ , i.e.  $z \in N(u)$ , since in this case  $T_x \ll_{R_T} T_u \ll_{R_T} T_{v'}$  and  $z \in N(x) \cap N(v')$ . Since this holds for an arbitrary  $z \in N$ , it follows that  $N_1(u) = N$ , which is a contradiction. Thus,  $T_{v'} \ll_{R_T} T_u$  and  $T_v \ll_{R_T} T_u$ . Since  $v \notin N(w)$ ,  $T_w$  does not intersect  $T_v$  in  $R_T$ , i.e. either  $T_w \ll_{R_T} T_v$  or  $T_v \ll_{R_T} T_w$ . If  $T_w \ll_{R_T} T_v$ , then  $T_w \ll_{R_T} T_v \ll_{R_T} T_u$ , and thus  $w \notin N(u)$ , which is a contradiction. Therefore,  $T_v \ll_{R_T} T_w$ , i.e.  $T_x \ll_{R_T} T_v \ll_{R_T} T_w$ . Thus,  $T_z$  intersects  $T_v$  in  $R_T$ , i.e.  $z \in N(v)$ , since  $z \in N(x) \cap N(w)$ .

Summarizing,  $z \in N(v)$  for any  $z \in N$  and any bounded vertex  $v$  of  $C_u$  in  $R$ , i.e.  $N \subseteq N_1(v)$ . Then, since  $N_1(v) \subseteq N(X_1) = N$ , it follows that  $N_1(v) = N$  for every bounded vertex  $v$  of  $C_u$  in  $R$ . This completes the proof of the lemma. ■

The next two lemmas follow easily and will be used in the sequel.

**Lemma 20** *Let  $v \in V \setminus Q_u \setminus N[u] \setminus V_0(u)$ . Then, either  $P_{x_2} \ll_R P_v$  or  $P_v \ll_R P_x$  for every  $x \in X_1$ .*

**Proof.** Let  $v \in V \setminus Q_u \setminus N[u] \setminus V_0(u)$ . Recall that  $X_1 \subseteq V_0(u)$  by Lemma 17 and that  $x_2 \in V_0(u)$  by definition of  $x_2$ . Suppose first that  $P_v$  intersects  $P_x$ , for some  $x \in X_1$  (resp.  $P_v$  intersects  $P_{x_2}$ ). If  $v \in N(x)$  (resp.  $v \in N(x_2)$ ), then  $v \in V_0(u)$ , since also  $v \notin N(u)$ , which is a contradiction. Therefore,  $v \notin N(x)$  (resp.  $v \notin N(x_2)$ ). If  $\phi_x > \phi_v$  (resp.  $\phi_{x_2} > \phi_v$ ), then  $N(x) \subseteq N(v)$  (resp.  $N(x_2) \subseteq N(v)$ ) by Lemma 3. Then, since  $x$  (resp.  $x_2$ ) is not the only vertex of  $V_0(u)$ , and since  $V_0(u)$  is connected, it follows that  $x$  (resp.  $x_2$ ) is adjacent to another vertex  $q \in V_0(u)$ . Therefore  $q \in N(v)$ , since  $N(x) \subseteq N(v)$  (resp.  $N(x_2) \subseteq N(v)$ ), and thus also  $v \in V_0(u)$ , which is a contradiction. If  $\phi_x < \phi_v$  (resp.  $\phi_{x_2} < \phi_v$ ), then  $N(v) \subseteq N(x)$  (resp.  $N(v) \subseteq N(x_2)$ ) by Lemma 3. Then, in particular,  $v$  is unbounded, since otherwise  $v \in N(x)$  (resp.  $v \in N(x_2)$ ), which is a contradiction. Since  $v \notin Q_u$  by the assumption on  $v$ , there exists at least one vertex  $z \in N(v) \setminus N(u)$ . Therefore,  $z \in N(x)$  (resp.  $z \in N(x_2)$ ), since  $N(v) \subseteq N(x)$  (resp.  $N(v) \subseteq N(x_2)$ ), and thus  $z \in V_0(u)$  and  $v \in V_0(u)$ , which is a contradiction. Thus,  $P_v$  does not intersect  $P_{x_2}$  or  $P_x$ , for any  $x \in X_1$ .

Suppose now that  $P_x \ll_R P_v \ll_R P_{x_2}$  for some  $x \in X_1$ . Then, since  $x_2 \in V_0(u)$  and  $x \in X_1 \subseteq V_0(u)$ , and since  $V_0(u)$  is connected, there exists a vertex  $y \in V_0(u)$ , such that  $P_y$  intersects  $P_v$  in  $R$ . Then  $v \notin N(y)$ , since otherwise  $v \in V_0(u)$ , which is a contradiction. If  $\phi_y > \phi_v$ , then  $N(y) \subseteq N(v)$  by Lemma 3. Since  $V_0(u)$  is connected with at least two vertices, there exists at least one neighbor  $q \in V_0(u)$  of  $y$ . Then  $q \in N(v)$ , since  $N(y) \subseteq N(v)$ , and thus  $v \in V_0(u)$ , which is a contradiction. If  $\phi_y < \phi_v$ , then  $N(v) \subseteq N(y)$  by Lemma 3. Then, in particular,  $v$  is unbounded, since otherwise  $v \in N(y)$ , which is a contradiction. Since  $v \notin Q_u$  by the assumption on  $v$ , there exists at least one vertex  $z \in N(v) \setminus N(u)$ . Therefore,  $z \in N(y)$ , since  $N(v) \subseteq N(y)$ , and thus  $z \in V_0(u)$  and  $v \in V_0(u)$ , which is again a contradiction.

Therefore, if  $v \in V \setminus Q_u \setminus N[u] \setminus V_0(u)$ , then either  $P_{x_2} \ll_R P_v$  or  $P_v \ll_R P_x$  for every  $x \in X_1$ . This completes the proof of the lemma. ■

**Lemma 21** *For every  $v \in V \setminus N[u] \setminus V_0(u)$ , either  $T_{x_2} \ll_{R_T} T_v$  or  $T_v \ll_{R_T} T_x$  for every  $x \in X_1$ .*

**Proof.** Let  $v \in V \setminus N[u] \setminus V_0(u)$ . Recall first that  $X_1 \subseteq V_0(u)$  by Lemma 17 and that  $x_2 \in V_0(u)$  by definition of  $x_2$ . If  $T_v$  intersects  $T_{x_2}$  or  $T_x$  for some  $x \in X_1$  in  $R_T$ , then  $v \in V_0(u)$ , since  $v \notin N[u]$ , which is a contradiction. Thus,  $T_v$  does not intersect  $T_{x_2}$  or  $T_x$  in  $R_T$ , for any  $x \in X_1$ . Suppose that  $T_x \ll_{R_T} T_v \ll_{R_T} T_{x_2}$  for some  $x \in X_1$ . Then, since  $V_0(u)$  is connected, it follows that  $T_z$  intersects  $T_v$  in  $R_T$  for at least one vertex  $z \in V_0(u)$ , and thus also  $v \in V_0(u)$ , which is again a contradiction. Thus, either  $T_{x_2} \ll_{R_T} T_v$  or  $T_v \ll_{R_T} T_x$  for every  $x \in X_1$ . ■

## Some properties of the sets $C_u$ and $C_2$

In the next three lemmas we prove some basic properties of the vertex sets  $C_u$  and  $C_u$ , which will be mainly used in the sequel of the proof of Theorem 2.

**Lemma 22** *For every vertex  $v \in C_u \setminus \{u\}$ ,  $v \in V_0(u) \cup N(u)$ .*

**Proof.** Consider a vertex  $v \in C_u \setminus \{u\}$ . Then,  $v \notin Q_u$  by definition of  $C_u$ . Suppose that  $v \notin V_0(u) \cup N(u)$ , i.e.  $v \in V \setminus Q_u \setminus N[u] \setminus V_0(u)$ . Then, either  $P_{x_2} \ll_R P_v$  or  $P_v \ll_R P_x$  for every  $x \in X_1$  by Lemma 20.

Suppose first that  $P_{x_2} \ll_R P_v$ . Then, since  $C_u$  is connected, and since  $P_u \ll_R P_{x_2}$ , there must exist a vertex  $v'$  of  $C_u$ , such that  $P_{v'}$  intersects  $P_{x_2}$  in  $R$ , and a path  $P$  from  $v'$  to  $v$ , where all intermediate vertices are  $v'' \in C_u$ , such that  $P_{x_2} \ll_R P_{v''}$ . Therefore, since  $P_u \ll_R P_{x_2} \ll_R P_{v''}$ , it follows that  $v'' \notin N(u)$  for all these intermediate vertices. Furthermore,  $v' \notin N(x_2)$  by definition of  $C_u$ . If  $\phi_{x_2} < \phi_{v'}$ , then  $N(v') \subseteq N(x_2)$  by Lemma 3. Therefore,  $v'$  is an isolated vertex of  $G \setminus Q_u \setminus N[X_1, x_2]$ , and thus  $v' \notin C_u$ , which is a contradiction. If  $\phi_{x_2} > \phi_{v'}$ , then  $N(x_2) \subseteq N(v')$  by Lemma 3. Then, in particular,  $x_2$  is unbounded, since otherwise  $v' \in N(x_2)$ , which is a contradiction. Thus,  $\phi_{x_2} > \phi_u$ , since  $\phi_u = \min\{\phi_x \mid x \in V_U\}$ . Furthermore, since  $N_1(x_2) = N$  by Lemma 19, and since  $P_x \ll_R P_u \ll_R P_{x_2}$  for every  $x \in X_1$ , it follows that  $P_z$  intersects  $P_u$  in  $R$  for every  $z \in N$ . Moreover, since  $x_2$  is unbounded, and since  $z \in N(x_2)$  for every  $z \in N$ , it follows

that  $z$  is bounded and  $\phi_z > \phi_{x_2} > \phi_u$  for every  $z \in N$ . Therefore,  $N \subseteq N(u)$ , i.e.  $N_1(u) = N$ , which is a contradiction by Lemma 19.

Suppose now that  $P_v \ll_R P_x$  for every  $x \in X_1$ . Then, since  $C_u$  is connected, and since  $P_x \ll_R P_u$  for every  $x \in X_1$ , there must exist a vertex  $v'$  of  $C_u$ , such that  $P_{v'}$  intersects  $P_{x_0}$  in  $R$  for some  $x_0 \in X_1$ , and a path  $P$  from  $v'$  to  $v$ , where all intermediate vertices are  $v'' \in C_u$ , such that  $P_{v''} \ll_R P_x$  for every  $x \in X_1$ . Therefore, since  $P_{v''} \ll_R P_x \ll_R P_u$  for every  $x \in X_1$ , it follows that  $v'' \notin N(u)$  for all these intermediate vertices. Furthermore,  $v' \notin N(x_0)$  by Lemma 17, since  $v' \in C_u$ .

Let first  $v' \notin N(u)$ . If  $\phi_{x_0} < \phi_{v'}$ , then  $N(v') \subseteq N(x_0)$  by Lemma 3. Therefore,  $v'$  is an isolated vertex of  $G \setminus Q_u \setminus N[X_1, x_2]$ , and thus  $v' \notin C_u$ , which is a contradiction. If  $\phi_{x_0} > \phi_{v'}$ , then  $N(x_0) \subseteq N(v')$  by Lemma 3. Then, in particular,  $x_0$  is unbounded, since otherwise  $v' \in N(x_0)$ , which is a contradiction. Since  $x_0 \in X_1 \subseteq V_0(u)$ , and since  $x_0$  is not the only vertex of  $V_0(u)$ , it follows that  $x_0$  has at least one neighbor  $z \in V_0(u)$ . Thus,  $z \in N(v')$ , since  $N(x_0) \subseteq N(v')$ . Therefore, since  $v' \notin N(u)$ , it follows that also  $v' \in V_0(u)$ . Thus, since  $v \notin N(u)$  and  $v'' \notin N(u)$  for all intermediate vertices  $v''$  of the path  $P$ , it follows that  $v \in V_0(u)$  and  $v'' \in V_0(u)$  for all these vertices  $v''$ . This is a contradiction to the assumption that  $v \notin V_0(u) \cup N(u)$ .

Let now  $v' \in N(u)$ . Then,  $P_{v'}$  intersects  $P_x$  for every  $x \in X_1$ , since  $P_{v''} \ll_R P_x \ll_R P_u$  for every  $x \in X_1$  and for every intermediate vertex  $v''$  of the path  $P$ . If  $\phi_x < \phi_{v'}$  for at least one  $x \in X_1$ , then  $N(v') \subseteq N(x)$  by Lemma 3. Therefore,  $v'$  is an isolated vertex of  $G \setminus Q_u \setminus N[X_1, x_2]$ , and thus  $v' \notin C_u$ , which is a contradiction. Otherwise, if  $\phi_x > \phi_{v'}$  for every  $x \in X_1$ , then  $N(x) \subseteq N(v')$  for every  $x \in X_1$  by Lemma 3. Then, in particular, every  $x \in X_1$  is unbounded, since otherwise  $v' \in N(x)$ , which is a contradiction. Thus,  $\phi_x > \phi_u$  for every  $x \in X_1$ , since  $\phi_u = \min\{\phi_x \mid x \in V_U\}$ . Furthermore, since  $N_1(x_2) = N = N(X_1)$  by Lemma 19, and since  $P_x \ll_R P_u \ll_R P_{x_2}$  for every  $x \in X_1$ , it follows that  $P_z$  intersects  $P_u$  in  $R$  for every  $z \in N$ . Moreover, since every  $x \in X_1$  is unbounded, it follows that for every  $z \in N$ ,  $z$  is bounded and  $\phi_z > \phi_x > \phi_u$  for at least one  $x \in X_1$ . Therefore,  $N \subseteq N(u)$ , i.e.  $N_1(u) = N$ , which is a contradiction by Lemma 19. Summarizing,  $v \in V_0(u) \cup N(u)$  for every  $v \in C_u \setminus \{u\}$ . ■

**Lemma 23** *For every vertex  $v \in C_u \setminus \{u\}$ ,  $N_1(v) = N$ .*

**Proof.** If  $v$  is a bounded vertex in  $R$ , then the lemma follows by Lemma 19. Suppose now that  $v$  is unbounded. Then, since  $v \notin Q_u$  by definition of  $C_u$ , it follows that there exists at least one vertex  $y_v \in N(v) \setminus N(u)$ . Furthermore, there exists at least one vertex  $y_u \in N(u) \setminus N(v)$ . Indeed, otherwise  $N(u) \subseteq N(v)$ , and thus  $N(u) \subset N(v)$  by Lemma 6, i.e.  $u$  is not unbounded maximal, which is a contradiction. Then, both  $y_u$  and  $y_v$  are bounded vertices in  $R$ , since  $u$  and  $v$  are unbounded. Furthermore, since  $uv \notin E$ , either  $T_u \ll_{R_T} T_v$  or  $T_v \ll_{R_T} T_u$ .

Let first  $T_u \ll_{R_T} T_v$ . Since  $y_v \notin N(u)$ ,  $T_{y_v}$  does not intersect  $T_u$  in  $R_T$ , i.e. either  $T_{y_v} \ll_{R_T} T_u$  or  $T_u \ll_{R_T} T_{y_v}$ . If  $T_{y_v} \ll_{R_T} T_u$ , then  $T_{y_v} \ll_{R_T} T_u \ll_{R_T} T_v$ , and thus  $y_v \notin N(v)$ , which is a contradiction. Therefore,  $T_u \ll_{R_T} T_{y_v}$ . Moreover,  $T_x \ll_{R_T} T_{x_2} \ll_{R_T} T_u \ll_{R_T} T_{y_v}$  for every  $x \in X_1$  by Lemma 18, and thus in particular  $y_v \notin N(X_1)$  and  $y_v \notin N(x_2)$ . Suppose that  $N_1(y_v) \neq N$ . Then,  $y_v \notin C_u$  by Lemma 19, since  $y_v$  is bounded. Thus, since  $v \in C_u$ ,  $y_v \in N(v)$ , and  $y_v \notin Q_u$ , it follows by Lemma 17 that either  $y_v \in N(X_1)$  or  $y_v \in N(x_2)$ , which is a contradiction. Therefore,  $N_1(y_v) = N$ . Thus, for every  $z \in N$ ,  $T_z$  intersects  $T_u$  in  $R_T$ , i.e.  $z \in N(u)$ , since  $T_{x_2} \ll_{R_T} T_u \ll_{R_T} T_{y_v}$  and  $z \in N(x_2) \cap N(y_v)$ . Therefore,  $N_1(u) = N$ , which is a contradiction by Lemma 19.

Let now  $T_v \ll_{R_T} T_u$ . Since  $y_u \notin N(v)$ ,  $T_{y_u}$  does not intersect  $T_v$  in  $R_T$ , i.e. either  $T_{y_u} \ll_{R_T} T_v$  or  $T_v \ll_{R_T} T_{y_u}$ . If  $T_{y_u} \ll_{R_T} T_v$ , then  $T_{y_u} \ll_{R_T} T_v \ll_{R_T} T_u$ , and thus  $y_u \notin N(u)$ , which is a contradiction. Therefore,  $T_v \ll_{R_T} T_{y_u}$ . Recall that  $C_u$  is connected and that no vertex of  $C_u$  is adjacent to  $x_2$  by the definition of  $C_u$ . Thus, since  $u \in C_u$  and  $T_{x_2} \ll_{R_T} T_u$ , it follows that  $T_{x_2}$  lies in  $R_T$  to the left of all trapezoids of the vertices of  $C_u$ ; in particular, Lemma 18 implies that  $T_x \ll_{R_T} T_{x_2} \ll_{R_T} T_v \ll_{R_T} T_{y_u}$  for every  $x \in X_1$ . Thus, in particular,  $y_u \notin N(X_1)$  and  $y_u \notin N(x_2)$ . Suppose that  $N_1(y_u) \neq N$ . Then,  $y_u \notin C_u$  by Lemma 19, since  $y_u$  is bounded. Thus, since  $u \in C_u$ ,  $y_u \in N(u)$ , and  $y_u \notin Q_u$ , it follows by Lemma 17 that either  $y_u \in N(X_1)$  or  $y_u \in N(x_2)$ , which is a contradiction. Thus,  $N_1(y_u) = N$ . Therefore, for every  $z \in N$ ,  $T_z$  intersects  $T_v$  in  $R_T$ , i.e.  $z \in N(v)$ ,

since  $T_{x_2} \ll_{R_T} T_v \ll_{R_T} T_{y_u}$  and  $z \in N(x_2) \cap N(y_u)$ . Thus,  $N_1(v) = N$ . This completes the proof of the lemma. ■

**Lemma 24** *For every vertex  $v \in C_2$ ,  $N_1(v) = N$ .*

**Proof.** Recall first that  $N_1(w) = N$  by Lemma 17. Let  $v \in C_2$  and  $x \in X_1$ . Recall that  $v \notin N(w)$  by definition of  $\tilde{C}_2$ , and that  $v \notin N(x)$  by definition of  $\tilde{C}_2$ , and thus either  $T_v \ll_{R_T} T_x$  or  $T_x \ll_{R_T} T_v$ . We will first prove that  $T_x \ll_{R_T} T_v$ . Recall that  $X_1 = \tilde{X}_1 \cup V(\mathcal{A}_1)$ .

Consider first the case where  $x \in \tilde{X}_1$ . Note that  $T_{x_1} \ll_{R_T} T_v$  for every vertex  $v$  of  $C_2$ , due to the definition of  $x_1$ , and since  $v \notin N(x_1)$  and  $C_2 \subseteq D_1 \cup D_2 \setminus \{x_1\}$ . Recall also that  $\tilde{X}_1$  induces a connected subgraph of  $G$  and that  $v \notin N[\tilde{X}_1]$  for every vertex  $v$  of  $C_2$  by definition of  $C_2$ . Thus, in this case  $T_x \ll_{R_T} T_v$  for every  $x \in \tilde{X}_1$ .

Consider now the case where  $x \in A_i$ , for some  $A_i \in \mathcal{A}_1$ , where  $1 \leq i \leq k$ . Recall that  $C_2 = \mathcal{A}_2 \cup \mathcal{B}_2$ . Suppose first that  $v \in A_j$  for some  $A_j \in \mathcal{B}_2$ , where  $k+1 \leq j \leq \ell$ . Then,  $v \in D_2$ , since  $A_j \subseteq D_2$ , as we proved above. If  $T_v \ll_{R_T} T_x$ , then  $T_v \ll_{R_T} T_x \ll_{R_T} T_{x_2}$  by Lemma 18, which is a contradiction by Lemma 15, since  $v \in D_2 \subseteq S_2$ . Thus,  $T_x \ll_{R_T} T_v$ . Suppose now that  $v \in A_p$ , for some  $A_p \in \mathcal{A}_2$ , where  $1 \leq p \leq k$ . For the sake of contradiction, suppose that  $T_v \ll_{R_T} T_x$ , i.e.  $T_v \ll_{R_T} T_x \ll_{R_T} T_{x_2}$ . Thus, since  $x \in A_i$  and  $A_i \neq A_p$ , it follows that  $T_v \ll_{R_T} T_y \ll_{R_T} T_{x_2}$  for every  $y \in A_i$ . Recall by definition of  $\mathcal{A}_2$  that  $v$  is adjacent to all vertices  $v' \in \tilde{H}$ . Thus, since  $v' \in N(v) \cap N(x_2)$  for every  $v' \in \tilde{H}$ , it follows that  $T_{v'}$  intersects  $T_y$  in  $R_T$ , i.e.  $y \in N(v')$ , for every  $y \in A_i$  and every  $v' \in \tilde{H}$ . This is a contradiction by the definition of  $\mathcal{A}_1$ , and thus again  $T_x \ll_{R_T} T_v$ .

Summarizing,  $T_x \ll_{R_T} T_v$  for every  $v \in C_2$  and every  $x \in X_1$ . Since  $v \in V_0(u)$  for every  $v \in C_2$  by Lemma 17, it follows that  $T_v \ll_{R_T} T_u$ . Since  $v \notin N(w)$  by definition of  $C_2$ ,  $T_v$  does not intersect  $T_w$  in  $R_T$ , i.e. either  $T_w \ll_{R_T} T_v$  or  $T_v \ll_{R_T} T_w$ . If  $T_w \ll_{R_T} T_v$ , then  $T_w \ll_{R_T} T_v \ll_{R_T} T_u$ , and thus  $w \notin N(u)$ , which is a contradiction. Therefore  $T_v \ll_{R_T} T_w$ , and thus  $T_x \ll_{R_T} T_v \ll_{R_T} T_w$  for every  $x \in X_1$ . Consider now a vertex  $z \in N = N(X_1)$ . Then,  $z \in N(x) \cap N(w)$  for some  $x \in X_1$ , since  $N_1(w) = N = N(X_1)$  by Lemma 17. Therefore,  $T_z$  intersects  $T_v$  in  $R_T$ , i.e.  $z \in N(v)$ , since  $T_x \ll_{R_T} T_v \ll_{R_T} T_w$ . Since this holds for every  $z \in N$ , it follows that  $N_1(v) = N$ . This completes the proof of the lemma. ■

## The recursive definition of the vertex subsets $H_i$ , $i \geq 1$ , of $H$

In the following, we define a partition of the set  $H$  into the subsets  $H_1, H_2, \dots$

**Definition 9** *Denote  $H_0 = N$ . Then,  $H_i = \{x \in H \setminus \bigcup_{j=1}^{i-1} H_j \mid H_{i-1} \not\subseteq N(x)\}$  for every  $i \geq 1$ .*

It is now easy to see by Definition 9 that either  $H_i = \emptyset$  for every  $i \in \mathbb{N}$ , or there exists some  $p \in \mathbb{N}$ , such that  $H_p \neq \emptyset$  and  $H_i = \emptyset$  for every  $i > p$ . That is, either  $\bigcup_{i=1}^{\infty} H_i = \emptyset$ , or  $\bigcup_{i=1}^{\infty} H_i = \bigcup_{i=1}^p H_i$ , for some  $p \in \mathbb{N}$ . Furthermore,  $\bigcup_{i=1}^{\infty} H_i \subseteq H$  by Definition 9.

**Definition 10** *Let  $v_i \in H_i$ , for some  $i \geq 1$ . Then, a sequence  $(v_0, v_1, \dots, v_{i-1}, v_i)$  of vertices, such that  $v_j \in H_j$ ,  $j = 0, 1, \dots, i-1$ , and  $v_{j-1}v_j \notin E$ ,  $j = 1, 2, \dots, i$ , is an  $H_i$ -chain of  $v_i$ .*

It is easy to see by Definition 9 that for every set  $H_i \neq \emptyset$ ,  $i \geq 1$ , and for every vertex  $v_i \in H_i$ , there exists at least one  $H_i$ -chain of  $v_i$ . The next two lemmas will be used in the sequel of the proof of Theorem 2.

**Lemma 25** *Let  $v_1 \in H_1$  and  $(v_0, v_1)$  be an  $H_1$ -chain of  $v_1$ . Then,  $v_1$  is a bounded vertex,  $P_{v_0} \ll_R P_{v_1}$  and  $T_{v_0} \ll_{R_T} T_{v_1}$ .*

**Proof.** First, we will prove that  $v_1$  is a bounded vertex in  $R$ . Suppose otherwise that  $v_1$  is unbounded, and thus  $v_1 \notin N(u)$ . Suppose that  $P_{v_1}$  intersects  $P_u$  in  $R$ . Then,  $\phi_{v_1} > \phi_u$ , since  $\phi_u = \min\{\phi_x \mid x \in V_U\}$ , and thus  $N(v_1) \subseteq N(u)$  by Lemma 3. Recall that  $x_2 \in N(v_1)$ , since



$v_1 \in H_1 \subseteq H$ , and thus also  $x_2 \in N(u)$ . Then,  $x_2 \in N(u)$ , which is a contradiction. Therefore,  $P_{v_1}$  does not intersect  $P_u$  in  $R$ . If  $P_{v_1} \ll_R P_u$ , then  $P_{v_1} \ll_R P_u \ll_R P_{x_2}$ , and thus  $v_1 \notin N(x_2)$ , which is a contradiction by definition of  $H$ . Therefore,  $P_u \ll_R P_{v_1}$ . Furthermore,  $x_2$  is bounded and  $\phi_{x_2} > \phi_{v_1}$ , since  $v_1$  is assumed to be unbounded and  $v_1 \in N(x_2)$  by definition of  $H$ . Recall that  $T_x \ll_{R_T} T_{x_2} \ll_{R_T} T_u$  for every  $x \in X_1$  by Lemma 18. Thus, since  $v_1 \in N(x_2)$ ,  $v_1 \notin N(u)$ , and  $v_1 \notin N(x)$  for every  $x \in X_1$ , it follows that also  $T_x \ll_{R_T} T_{v_1} \ll_{R_T} T_u$  for every  $x \in X_1$ . Moreover,  $N(u) \not\subseteq N(v_1)$ , since  $u$  is unbounded-maximal and by Lemma 6. Let  $y \in N(u) \setminus N(v_1)$ , and thus  $y$  is bounded. Then,  $T_{v_1} \ll_{R_T} T_y$ , since  $T_{v_1} \ll_{R_T} T_u$ , and since  $y \in N(u)$  and  $y \notin N(v_1)$ . Therefore,  $T_x \ll_{R_T} T_{v_1} \ll_{R_T} T_y$  for every  $x \in X_1$ , and thus, in particular  $y \notin N(X_1)$ .

Suppose that  $N_1(y) \neq N$ . Then,  $y \notin C_u$  by Lemma 23. Thus, since  $u \in C_u$ ,  $y \in N(u)$ , and  $y \notin Q_u$ , it follows by Lemma 17 that either  $y \in N(X_1)$  or  $y \in N(x_2)$ . Therefore,  $y \in N(x_2)$ , since  $y \notin N(X_1)$ , as we have proved above. Let  $z \in N \setminus N_1(y)$ . Then,  $z \in N(x) \cap N(x_2)$  for some  $x \in X_1$ . Thus, since  $P_x \ll_R P_u \ll_R P_{x_2}$ , it follows that  $P_z$  intersects  $P_u$  in  $R$ . Suppose that  $z$  is unbounded. Then,  $\phi_z > \phi_u$ , since  $\phi_u = \min\{\phi_x \mid x \in V_U\}$ , and thus  $N(z) \subseteq N(u)$  by Lemma 3. Then,  $x_2 \in N(u)$ , which is a contradiction. Therefore,  $z$  is bounded, and thus  $P_y$  does not intersect  $P_z$ , since  $y$  is also bounded and  $z \notin N(y)$ . That is, either  $P_y \ll_R P_z$  or  $P_z \ll_R P_y$ .

Suppose first that  $P_y \ll_R P_z$ . If  $P_y \ll_R P_x$ , then  $P_y \ll_R P_x \ll_R P_u$ , and thus  $y \notin N(u)$ , which is a contradiction. If  $P_x \ll_R P_y$ , then  $P_x \ll_R P_y \ll_R P_z$ , and thus  $z \notin N(x)$ , which is again a contradiction. Thus,  $P_y$  intersects  $P_x$  in  $R$ . Recall that  $y \notin N(x)$ , since  $y \notin N(X_1)$ . Thus, since  $y$  is bounded, it follows that  $x$  is unbounded and  $\phi_x > \phi_y$ . Then,  $N(x) \subseteq N(y)$  by Lemma 3, and thus  $z \in N(y)$ , which is a contradiction.

Suppose now that  $P_z \ll_R P_y$ . Recall that  $L(y) <_R L(u)$  by Lemma 1, since  $y \in N(u)$ , and thus  $R(z) <_R L(y) <_R L(u) <_R L(x_2)$ . Therefore,  $r(u) <_R l(x_2) <_R r(z) <_R l(y)$ , since  $z \in N(x_2)$ . That is,  $L(y) <_R L(x_2)$  and  $l(x_2) <_R l(y)$ , and thus  $\phi_y > \phi_{x_2} > \phi_{v_1}$  (since  $\phi_{x_2} > \phi_{v_1}$ , as we proved above). If  $P_y$  intersects  $P_{v_1}$  in  $R$ , then  $y \in N(v_1)$ , since  $y$  is bounded, which is a contradiction. Therefore,  $P_y$  does not intersect  $P_{v_1}$  in  $R$ , i.e. either  $P_{v_1} \ll_R P_y$  or  $P_y \ll_R P_{v_1}$ . If  $P_{v_1} \ll_R P_y$ , then  $P_u \ll_R P_{v_1} \ll_R P_y$ , and thus  $y \notin N(u)$ , which is a contradiction. Therefore,  $P_y \ll_R P_{v_1}$ .

Summarizing,  $P_z \ll_R P_y \ll_R P_{v_1}$ , and thus  $r(z) <_R r(y) <_R r(v_1)$ . Recall that  $v_1 \in N[u, w] = N(u) \cup N(w)$  by definition of  $H$ . Therefore,  $v_1 \in N(w)$ , since  $v_1 \notin N(u)$ , and thus  $r(v_1) <_R r(w)$  by Lemma 1. Recall that  $r(w) <_R l(x_2)$ , since  $P_w \ll_R P_{x_2}$ . That is,  $r(z) <_R r(y) <_R r(v_1) <_R r(w) <_R l(x_2)$ , i.e.  $r(z) <_R l(x_2)$ . On the other hand,  $R(z) <_R L(y)$ , since  $P_z \ll_R P_y$ . Furthermore,  $L(y) <_R L(u)$  by Lemma 1 and since  $y \in N(u)$ , and  $L(u) <_R L(x_2)$ , since  $P_u \ll_R P_{x_2}$ . That is,  $R(z) <_R L(y) <_R L(u) <_R L(x_2)$ , i.e.  $R(z) <_R L(x_2)$ . Therefore, since also  $r(z) <_R l(x_2)$ , it follows that  $P_z \ll_R P_{x_2}$ . This is a contradiction, since  $z \in N = N_1(x_2)$  by Lemma 19. Therefore,  $N_1(y) = N$ .

Since  $N_1(y) = N$ , and since  $T_x \ll_{R_T} T_{v_1} \ll_{R_T} T_y$  for every  $x \in X_1$ , it follows that  $T_z$  intersects  $T_{v_1}$  in  $R_T$ , i.e.  $z \in N(v_1)$ , for every  $z \in N$ . Thus  $N_1(v_1) = N$ , i.e.  $N = H_0 \subseteq N(v_1)$ , which is a contradiction by Definition 9, since  $v_1 \in H_1$ . Therefore,  $v_1$  is a bounded vertex in  $R$ .

Recall now that  $v_0 \in N(x_0) \cap N(x_2)$  for some  $x_0 \in X_1$ , since  $v_0 \in N = N_1(x_2)$  by Lemma 19. Furthermore,  $v_1 \notin N(x_0)$  by definition of  $H$ , since otherwise  $v_1 \in N(X_1)$ , which is a contradiction. Suppose that  $P_{v_1}$  intersects  $P_{x_0}$  in  $R$ . If  $\phi_{v_1} > \phi_{x_0}$ , then  $v_1 \notin N(x_0)$ , since  $v_1$  is bounded, which is a contradiction. Thus,  $\phi_{v_1} < \phi_{x_0}$ . Then,  $N(x_0) \subseteq N(v_1)$  by Lemma 3, and thus  $v_0 \in N(v_1)$ , which is a contradiction. Therefore,  $P_{v_1}$  does not intersect  $P_{x_0}$  in  $R$ . If  $P_{v_1} \ll_R P_{x_0}$ , then  $P_{v_1} \ll_R P_{x_0} \ll_R P_u \ll_R P_{x_2}$ , and thus  $v_1 \notin N(x_2)$ , which is a contradiction. Thus,  $P_{x_0} \ll_R P_{v_1}$ .

Furthermore,  $P_{v_0}$  intersects  $P_u$  in  $R$ , since  $P_{x_0} \ll_R P_u \ll_R P_{x_2}$  and  $v_0 \in N(x_0) \cap N(x_2)$ . If  $v_0$  is unbounded, then  $\phi_{v_0} > \phi_u$ , since  $\phi_u = \min\{\phi_x \mid x \in V_U\}$ , and thus  $N(v_0) \subseteq N(u)$  by Lemma 3. Then,  $x_2 \in N(u)$ , which is a contradiction. Therefore,  $v_0$  is bounded, and thus  $P_{v_0}$  does not intersect  $P_{v_1}$  in  $R$ , since  $v_1$  is also bounded and  $v_0 \notin N(v_1)$ . That is, either  $P_{v_1} \ll_R P_{v_0}$  or  $P_{v_0} \ll_R P_{v_1}$ . If  $P_{v_1} \ll_R P_{v_0}$ , then  $P_{x_0} \ll_R P_{v_1} \ll_R P_{v_0}$ , and thus  $v_0 \notin N(x_0)$ , which is a contradiction. Thus,  $P_{v_0} \ll_R P_{v_1}$ .

Finally, recall that  $T_x \ll_{R_T} T_{x_2}$  for every  $x \in X_1$  by Lemma 18. Therefore,  $T_x \ll_{R_T} T_{v_1}$  for every  $x \in X_1$ , since  $v_1 \in N(x_2)$  and  $v_1 \notin N(x)$  for every  $x \in X_1$ . Moreover,  $T_{v_1}$  does not intersect

$T_{v_0}$  in  $R_T$ , since  $v_0 \notin N(v_1)$ . Thus, either  $T_{v_1} \ll_{R_T} T_{v_0}$  or  $T_{v_0} \ll_{R_T} T_{v_1}$ . If  $T_{v_1} \ll_{R_T} T_{v_0}$ , then  $T_x \ll_{R_T} T_{v_1} \ll_{R_T} T_{v_0}$  for every  $x \in X_1$ , and thus  $v_0 \notin N = N(X_1)$ , which is a contradiction. Thus,  $T_{v_1} \ll_{R_T} T_{v_0}$ . This completes the proof of the lemma. ■

**Lemma 26** *Let  $v_i \in H_i$ , for some  $i \geq 2$ , and  $(v_0, v_1, \dots, v_i)$  be an  $H_i$ -chain of  $v_i$ . Then, for every  $j = 1, 2, \dots, i-1$ ,*

1.  $P_{v_{j-1}} \ll_R P_{v_j}$  and  $T_{v_{j-1}} \ll_{R_T} T_{v_j}$ , if  $j$  is odd,
2.  $P_{v_j} \ll_R P_{v_{j-1}}$  and  $T_{v_j} \ll_{R_T} T_{v_{j-1}}$ , if  $j$  is even.

**Proof.** The proof will be done by induction on  $j$ . For  $j = 1$ , the induction basis follows by Lemma 25. For the induction step, let  $2 \leq j < i-1$ . Note that  $v_{j-2} \in N(v_j) \setminus N(v_{j-1})$  and  $v_{j+1} \in N(v_{j-1}) \setminus N(v_j)$ . Therefore,  $N(v_j) \not\subseteq N(v_{j-1})$  and  $N(v_{j-1}) \not\subseteq N(v_j)$ , and thus  $P_{v_j}$  does not intersect  $P_{v_{j-1}}$  in  $R$  by Lemma 3, since  $v_{j-1}v_j \notin E$ . Thus, either  $P_{v_{j-1}} \ll_R P_{v_j}$  or  $P_{v_j} \ll_R P_{v_{j-1}}$ . Furthermore, either  $T_{v_{j-1}} \ll_{R_T} T_{v_j}$  or  $T_{v_j} \ll_{R_T} T_{v_{j-1}}$ , since  $v_{j-1}v_j \notin E$ .

Let  $j$  be odd, i.e.  $j-1$  is even, and suppose by induction hypothesis that  $P_{v_{j-1}} \ll_R P_{v_{j-2}}$  and  $T_{v_{j-1}} \ll_{R_T} T_{v_{j-2}}$ . If  $P_{v_j} \ll_R P_{v_{j-1}}$  (resp.  $T_{v_j} \ll_{R_T} T_{v_{j-1}}$ ), then  $P_{v_j} \ll_R P_{v_{j-2}}$  (resp.  $T_{v_j} \ll_{R_T} T_{v_{j-2}}$ ). Thus,  $v_jv_{j-2} \notin E$ , i.e.  $v_j \in H_{j-1}$  by Definition 9, which is a contradiction. Therefore,  $P_{v_{j-1}} \ll_R P_{v_j}$  and  $T_{v_{j-1}} \ll_{R_T} T_{v_j}$ , if  $j$  is odd.

Let now  $j$  be even, i.e.  $j-1$  is odd, and suppose by induction hypothesis that  $P_{v_{j-2}} \ll_R P_{v_{j-1}}$  and  $T_{v_{j-2}} \ll_{R_T} T_{v_{j-1}}$ . If  $P_{v_{j-1}} \ll_R P_{v_j}$  (resp.  $T_{v_{j-1}} \ll_{R_T} T_{v_j}$ ), then  $P_{v_{j-2}} \ll_R P_{v_j}$  (resp.  $T_{v_{j-2}} \ll_{R_T} T_{v_j}$ ), and thus  $v_jv_{j-2} \notin E$ , which is again a contradiction. Therefore,  $P_{v_j} \ll_R P_{v_{j-1}}$  and  $T_{v_j} \ll_{R_T} T_{v_{j-1}}$ , if  $j$  is even. This completes the induction step, and thus the lemma follows. ■

The next lemma, which follows now easily by Lemmas 23, 24, 25, and 26, will be mainly used in the sequel.

**Lemma 27** *All vertices of  $N \cup H \cup C_2 \cup C_u \setminus \{u\}$  are bounded.*

**Proof.** Consider first a vertex  $v \in N$ . Then,  $v \in N(x) \cap N(x_2)$  for some  $x \in X_1$  by Lemma 24. Thus,  $P_v$  intersects  $P_u$  in  $R$ , since  $P_x \ll_R P_u \ll_R P_{x_2}$ . Suppose that  $v$  is unbounded. Then,  $\phi_v > \phi_u$ , since  $\phi_u = \min\{\phi_x \mid x \in V_U\}$ , and thus  $N(v) \subseteq N(u)$  by Lemma 3. Then,  $x_2 \in N(u)$ , which is a contradiction. Thus, every  $v \in N$  is bounded.

Consider now a vertex  $v \in H$ . If  $v \in H_1$ , then  $v$  is bounded by Lemma 25. Suppose that  $v \in H \setminus H_1$  and that  $v$  is unbounded. Then,  $\phi_v > \phi_u$ , since  $\phi_u = \min\{\phi_x \mid x \in V_U\}$ . Furthermore,  $H_0 = N \subseteq N(v)$  by Definition 9, and thus  $N_1(v) = N$ . If  $P_v \ll_R P_u$ , then  $P_v \ll_R P_u \ll_R P_{x_2}$ , and thus  $v \notin N(x_2)$ , which is a contradiction to the definition of  $H$ . If  $P_v$  intersects  $P_u$  in  $R$ , then  $N(v) \subseteq N(u)$  by Lemma 3, since  $\phi_v > \phi_u$ , and thus  $x_2 \in N(u)$ , which is again a contradiction. Therefore,  $P_u \ll_R P_v$ , i.e.  $P_x \ll_R P_u \ll_R P_v$  for every  $x \in X_1$ , and thus  $P_z$  intersects  $P_u$  in  $R$  for every  $z \in N_1(v) = N = N(X_1)$ . However,  $z$  is bounded and  $\phi_z > \phi_v > \phi_u$  for every  $z \in N_1(v)$ , since  $v$  is unbounded. Therefore,  $N_1(v) \subseteq N(u)$ , and thus  $N_1(u) = N$ , which is a contradiction by Lemma 19. Thus, every  $v \in H \setminus H_1$  is bounded.

Consider finally a vertex  $v \in C_2 \cup C_u \setminus \{u\}$  and suppose that  $v$  is unbounded. Then, similarly to the above,  $\phi_v > \phi_u$ , since  $\phi_u = \min\{\phi_x \mid x \in V_U\}$ . Furthermore,  $N_1(v) = N$  by Lemmas 23 and 24, while also  $N_1(x_2) = N$  by Lemma 19. Suppose that  $P_v \ll_R P_u$ , i.e.  $P_v \ll_R P_u \ll_R P_{x_2}$ . Then, since  $N_1(v) = N_1(x_2) = N$ ,  $P_z$  intersects  $P_u$  in  $R$  for every  $z \in N$ . Furthermore,  $z$  is bounded and  $\phi_z > \phi_v > \phi_u$  for every  $z \in N_1(v)$ , since  $v$  is unbounded. Therefore,  $N_1(v) \subseteq N(u)$ , and thus  $N_1(u) = N$ , which is a contradiction by Lemma 19. Suppose that  $P_v$  intersects  $P_u$  in  $R$ . Then,  $N(v) \subseteq N(u)$  by Lemma 3, since  $\phi_v > \phi_u$ . Therefore,  $N(v) \subset N(u)$  by Lemma 6, and thus  $v \in Q_u$ , which is a contradiction to the definitions of  $C_u$  and  $C_2$ . Suppose that  $P_u \ll_R P_v$ , i.e.  $P_x \ll_R P_u \ll_R P_v$  for every  $x \in X_1$ . Then, since  $N_1(v) = N = N(X_1)$ ,  $P_z$  intersects  $P_u$  in  $R$  for every  $z \in N$ . Furthermore,  $z$  is bounded and  $\phi_z > \phi_v > \phi_u$  for every  $z \in N_1(v)$ , since  $v$  is unbounded. Therefore,  $N_1(v) \subseteq N(u)$ , and thus  $N_1(u) = N$ , which is a contradiction by Lemma 19. Thus, every  $v \in C_2 \cup C_u \setminus \{u\}$  is bounded. This completes the lemma. ■

**Lemma 28** For every vertex  $v \in C_u \setminus \{u\}$ , it holds  $H_i \subseteq N(v)$  for every  $i \geq 1$ .

**Proof.** Let  $v$  be a vertex of  $C_u \setminus \{u\}$ . Recall that  $N_1(v) = N$  by Lemma 23. Consider first the case where  $v \in N[u, w] = N(u) \cup N(w)$ . The proof will be done by induction on  $i$ . For  $i = 1$ , consider a vertex  $v_1 \in H_1$  and an  $H_1$ -chain  $(v_0, v_1)$  of  $v_1$ , where  $v_0 \in H_0 = N = N(X_1)$ . Since  $v_0 v_1 \notin E$ , either  $T_{v_1} \ll_{R_T} T_{v_0}$  or  $T_{v_0} \ll_{R_T} T_{v_1}$ . Suppose that  $T_{v_1} \ll_{R_T} T_{v_0}$ . Then, since  $T_x \ll_{R_T} T_{x_2}$  for every  $x \in X_1$  by Lemma 18, and since  $v_1 \in N(x_2) \setminus N(x)$  for every  $x \in X_1$  by definition of  $H$ , it follows that  $T_x \ll_{R_T} T_{v_1}$  for every  $x \in X_1$ . That is,  $T_x \ll_{R_T} T_{v_1} \ll_{R_T} T_{v_0}$  for every  $x \in X_1$ , and thus  $v_0 \notin N(x)$  for every  $x \in X_1$ , which is a contradiction. Thus,  $T_{v_0} \ll_{R_T} T_{v_1}$ . Furthermore,  $T_{x_2} \ll_{R_T} T_v$ , since  $T_{x_2} \ll_{R_T} T_u$  and  $C_u$  is connected. Suppose that  $v_1 \notin N(v)$ . Then,  $T_{v_1} \ll_{R_T} T_v$ , since  $T_{x_2} \ll_{R_T} T_v$  and  $v_1 \in N(x_2) \setminus N(v)$ . That is,  $T_{v_0} \ll_{R_T} T_{v_1} \ll_{R_T} T_v$ , and thus  $v_0 \notin N(v)$ , which is a contradiction, since  $N_1(v) = N$  and  $v_0 \in N$ . Thus,  $v_1 \in N(v)$  for every  $v_1 \in H_1$ . This proves the induction basis.

For the induction step, let  $i \geq 2$ , and suppose that  $v' \in N(v)$  for every  $v' \in H_j$ , where  $0 \leq j \leq i - 1$ . Let  $v_i \in H_i$  and  $(v_0, v_1, \dots, v_{i-2}, v_{i-1}, v_i)$  be an  $H_i$ -chain of  $v_i$ . Note that  $v_{i-2}$  exists, since  $i \geq 2$ , and thus  $v_{i-1} v_{i-2} \notin E$  and  $v_i v_{i-2} \in E$  by Definition 9. For the sake of contradiction, suppose that  $v_i \notin N(v)$ . We will now prove that  $P_v \ll_R P_{x_2}$ . Otherwise, suppose first that  $P_{x_2} \ll_R P_v$ . Then,  $P_u \ll_R P_{x_2} \ll_R P_v$  and  $P_w \ll_R P_{x_2} \ll_R P_v$ , and thus  $v \notin N[u, w] = N(u) \cup N(w)$ , which is a contradiction to the assumption on  $v$ . Suppose now that  $P_v$  intersects  $P_{x_2}$  in  $R$ . Then, either  $N(x_2) \subseteq N(v)$  or  $N(v) \subseteq N(x_2)$  by Lemma 3, since  $v \notin N(x_2)$  by the definition of  $C_u$ . If  $N(x_2) \subseteq N(v)$ , then  $v_i \in N(v)$ , since  $v_i \in N(x_2)$ , which is a contradiction. Let  $N(v) \subseteq N(x_2)$ . Then, since  $C_u$  is connected and  $v \neq u$ ,  $v$  is adjacent to at least one vertex  $z \in C_u$ , and thus  $z \in N(x_2)$ , which is a contradiction to the definition of  $C_u$ . Thus,  $P_v \ll_R P_{x_2}$ .

Recall that  $v_{i-1} \in N(v)$  by the induction hypothesis. Since  $v \in N(v_{i-1}) \setminus N(v_i)$  and  $v_{i-2} \in N(v_i) \setminus N(v_{i-1})$ , it follows that  $P_{v_i}$  does not intersect  $P_{v_{i-1}}$  in  $R$  by Lemma 3. Similarly,  $P_{v_i}$  does not intersect  $P_v$  in  $R$ , since  $x_2 \in N(v_i) \setminus N(v)$  and  $v_{i-1} \in N(v) \setminus N(v_i)$ . Thus, since  $v_{i-1} \in N(v)$ , either  $P_{v_i} \ll_R P_{v_{i-1}}$  and  $P_{v_i} \ll_R P_v$ , or  $P_{v_{i-1}} \ll_R P_{v_i}$  and  $P_v \ll_R P_{v_i}$ . Suppose that  $P_{v_i} \ll_R P_{v_{i-1}}$  and  $P_{v_i} \ll_R P_v$ . Then,  $P_{v_i} \ll_R P_v \ll_R P_{x_2}$ , and thus  $v_i \notin N(x_2)$ , which is a contradiction.

Thus,  $P_{v_{i-1}} \ll_R P_{v_i}$  and  $P_v \ll_R P_{v_i}$ . Recall now by Lemmas 25 and 26 that either  $P_{v_{i-2}} \ll_R P_{v_{i-1}}$  or  $P_{v_{i-1}} \ll_R P_{v_{i-2}}$ . If  $P_{v_{i-2}} \ll_R P_{v_{i-1}}$ , then  $P_{v_{i-2}} \ll_R P_{v_{i-1}} \ll_R P_{v_i}$ , and thus  $v_i v_{i-2} \notin E$ , which is a contradiction. Therefore,  $P_{v_{i-1}} \ll_R P_{v_{i-2}}$ . Thus, also  $T_{v_{i-1}} \ll_{R_T} T_{v_{i-2}}$  and  $i$  is odd, by Lemmas 25 and 26. Since  $v_{i-1} v_i \notin E$ , either  $T_{v_i} \ll_{R_T} T_{v_{i-1}}$  or  $T_{v_{i-1}} \ll_{R_T} T_{v_i}$ . If  $T_{v_i} \ll_{R_T} T_{v_{i-1}}$ , then  $T_{v_i} \ll_{R_T} T_{v_{i-1}} \ll_{R_T} T_{v_{i-2}}$ , and thus  $v_i v_{i-2} \notin E$ , which is a contradiction. Therefore,  $T_{v_{i-1}} \ll_{R_T} T_{v_i}$ , and thus  $T_v \ll_{R_T} T_{v_i}$ , since  $v \in N(v_{i-1}) \setminus N(v_i)$ . Recall also that  $T_{x_2} \ll_{R_T} T_v$ , since  $T_{x_2} \ll_{R_T} T_u$  and  $C_u$  is connected. That is,  $T_{x_2} \ll_{R_T} T_v \ll_{R_T} T_{v_i}$ , and thus  $v_i \notin N(x_2)$ , which is a contradiction. Thus,  $v_i \in N(v)$ . This completes the induction step.

Summarizing, we have proved that  $H_i \subseteq N(v)$  for every  $i \geq 1$  and for every vertex  $v \in C_u \setminus \{u\}$ , such that  $v \in N[u, w]$ . This holds in particular for  $w$ , i.e.  $H_i \subseteq N(w)$  for every  $i \geq 1$ , since  $w \in N(u)$  is a vertex of  $C_u \setminus \{u\}$ . Consider now the case where  $v \notin N[u, w]$ . Then, since  $w \in N(u)$ , either  $T_u \ll_{R_T} T_v$  and  $T_w \ll_{R_T} T_v$ , or  $T_v \ll_{R_T} T_u$  and  $T_v \ll_{R_T} T_w$ . Suppose that  $T_u \ll_{R_T} T_v$ , i.e.  $T_x \ll_{R_T} T_{x_2} \ll_{R_T} T_u \ll_{R_T} T_v$  for every  $x \in X_1$  by Lemma 18. Recall that  $N_1(v) = N$  by Lemma 23. That is,  $T_z$  intersects  $T_u$  in  $R_T$ , i.e.  $z \in N(u)$ , for every  $z \in N_1(v) = N$ , and thus  $N_1(u) = N$ , which is a contradiction by Lemma 19. Thus,  $T_v \ll_{R_T} T_u$  and  $T_v \ll_{R_T} T_w$ . Then,  $T_{x_2} \ll_{R_T} T_v$ , since  $T_{x_2} \ll_{R_T} T_u$  and  $C_u$  is connected. That is,  $T_{x_2} \ll_{R_T} T_v \ll_{R_T} T_w$ . Then, since every  $z \in H_i$ ,  $i \geq 1$ , is adjacent to both  $x_2$  and  $w$ , it follows that  $T_z$  intersects  $T_v$  in  $R_T$ , i.e.  $z \in N(v)$ , for every  $z \in H_i$ , where  $i \geq 1$ . Thus,  $H_i \subseteq N(v)$  for every  $i \geq 1$  and for every vertex  $v \in C_u \setminus \{u\}$ , such that  $v \notin N[u, w]$ . This completes the proof of the lemma. ■

**Lemma 29** For every vertex  $v \in C_2$ , it holds  $H_i \subseteq N(v)$  for every  $i \geq 1$ .

**Proof.** Recall that  $C_2 = \mathcal{A}_2 \cup \mathcal{B}_2$ , where  $A_j \subseteq D_2$  for every  $A_j \in \mathcal{B}_2$ ,  $k+1 \leq j \leq \ell$ , and  $\mathcal{A}_2$  includes exactly those components  $A_i$ ,  $1 \leq i \leq k$ , for which all vertices of  $A_i$  are adjacent to all vertices of  $\tilde{H}$ . Therefore, if  $v \in A_i$  for some component  $A_i \in \mathcal{A}_2$ , then  $H \subseteq \tilde{H} \subseteq N(v)$  by definition, and thus  $H_i \subseteq N(v)$  for every  $i \geq 1$ .

Let now  $v \in A_j$ , for some  $A_j \in \mathcal{B}_2$ , and suppose first that  $v \notin N(x_2)$ . Then, since  $v \in D_2 \subseteq S_2 \subseteq V_0(u)$ , it follows that  $T_v \ll_{R_T} T_u$  and that  $T_{x_2} \ll_{R_T} T_v$  by Lemma 15 (since  $v \notin N(x_2)$ ), i.e.  $T_{x_2} \ll_{R_T} T_v \ll_{R_T} T_u$ . Moreover,  $v \notin N(w)$  by definition of  $\tilde{C}_2$ . Thus,  $T_v \ll_{R_T} T_w$ , since  $T_v \ll_{R_T} T_u$  and  $w \in N(u) \setminus N(v)$ . That is,  $T_{x_2} \ll_{R_T} T_v \ll_{R_T} T_w$ . Let now  $z \in H_i$ , for some  $i \geq 1$ . Then,  $z \in N(x_2)$  and  $z \in N(w)$  by Lemma 28, and thus  $T_z$  intersects  $T_v$  in  $R_T$ , i.e.  $z \in N(v)$ . Therefore,  $H_i \subseteq N(v)$  for every  $i \geq 1$ , where  $v \notin N(x_2)$ .

Suppose now that  $v \in N(x_2)$ . We will prove by contradiction that  $H_i \subseteq N(v)$  for every  $i \geq 1$ . Suppose otherwise that there exists an index  $i \geq 1$ , such that  $v_i \notin N(v)$ , for some vertex  $v_i \in H_i$ . W.l.o.g. let  $i$  be the smallest such index, i.e.  $v' \in N(v)$  for every  $v' \in H_j$ , where  $0 \leq j \leq i-1$  (recall that  $H_0 = N$ , and thus  $v' \in N(v)$  for every  $v' \in H_0$  by Lemma 24).

Let  $(v_0, v_1, \dots, v_{i-1}, v_i)$  be an  $H_i$ -chain of  $v_i$ . If  $i = 1$ , then  $P_{v_1}$  does not intersect  $P_{v_0}$  in  $R$  by Lemma 25. If  $i \geq 2$ , then  $v_{i-2} \in N(v_i) \setminus N(v_{i-1})$  and  $v \in N(v_{i-1}) \setminus N(v_i)$ ; therefore  $N(v_{i-1}) \not\subseteq N(v_i)$  and  $N(v_i) \not\subseteq N(v_{i-1})$ , and thus  $P_{v_i}$  does not intersect  $P_{v_{i-1}}$  in  $R$  by Lemma 3. That is,  $P_{v_i}$  does not intersect  $P_{v_{i-1}}$  in  $R$  for every  $i \geq 1$ . Recall now that  $v_i \in N[u, w]$  by definition of  $H$ , and that  $v \notin N[u, w]$  by definition of  $\tilde{C}_2$ . If  $v_i \in N(u)$  (resp.  $v_i \in N(w)$ ), then  $u \in N(v_i) \setminus N(v)$  (resp.  $w \in N(v_i) \setminus N(v)$ ). Furthermore,  $v_{i-1} \in N(v) \setminus N(v_i)$ , i.e.  $N(v_i) \not\subseteq N(v)$  and  $N(v) \not\subseteq N(v_i)$ , and thus  $P_{v_i}$  does not intersect  $P_v$  in  $R$  by Lemma 3. Therefore, since  $vv_{i-1} \in E$ , it follows that either  $P_{v_{i-1}} \ll_R P_{v_i}$  and  $P_v \ll_R P_{v_i}$ , or  $P_{v_i} \ll_R P_{v_{i-1}}$  and  $P_{v_i} \ll_R P_v$ .

Suppose first that  $P_{v_{i-1}} \ll_R P_{v_i}$  and  $P_v \ll_R P_{v_i}$ . Recall that  $v_i \in N[u, w]$  and that  $v \notin N[u, w]$ . Let  $v_i \in N(u)$  (resp.  $v_i \in N(w)$ ). Then,  $P_v$  does not intersect  $P_u$  (resp.  $P_w$ ) in  $R$  by Lemma 3, since  $x_2 \in N(v) \setminus N[u, w]$  and  $v_i \in N(u) \setminus N(v)$  (resp.  $v_i \in N(w) \setminus N(v)$ ). Thus, since  $P_u \ll_R P_{x_2}$  (resp.  $P_w \ll_R P_{x_2}$ ) and  $v \in N(x_2) \setminus N(u)$  (resp.  $v \in N(x_2) \setminus N(w)$ ), it follows that  $P_u \ll_R P_v$  (resp.  $P_w \ll_R P_v$ ). That is,  $P_u \ll_R P_v \ll_R P_{v_i}$  (resp.  $P_w \ll_R P_v \ll_R P_{v_i}$ ), i.e.  $v_i \notin N(u)$  (resp.  $v_i \notin N(w)$ ), which is a contradiction.

Suppose now that  $P_{v_i} \ll_R P_{v_{i-1}}$  and  $P_{v_i} \ll_R P_v$ . Then,  $i \neq 1$  by Lemma 25. That is,  $i \geq 2$ , i.e.  $v_{i-2}$  exists. Recall by Lemmas 25 and 26 that either  $P_{v_{i-1}} \ll_R P_{v_{i-2}}$  or  $P_{v_{i-2}} \ll_R P_{v_{i-1}}$ . If  $P_{v_{i-1}} \ll_R P_{v_{i-2}}$ , then  $P_{v_i} \ll_R P_{v_{i-1}} \ll_R P_{v_{i-2}}$ , and thus  $v_i v_{i-2} \notin E$ , which is a contradiction. Therefore,  $P_{v_{i-2}} \ll_R P_{v_{i-1}}$ , and thus also  $T_{v_{i-2}} \ll_{R_T} T_{v_{i-1}}$  and  $i$  is even by Lemmas 25 and 26. Since  $v_{i-1} v_i \notin E$ , either  $T_{v_{i-1}} \ll_{R_T} T_{v_i}$  or  $T_{v_i} \ll_{R_T} T_{v_{i-1}}$ . If  $T_{v_{i-1}} \ll_{R_T} T_{v_i}$ , then  $T_{v_{i-2}} \ll_{R_T} T_{v_{i-1}} \ll_{R_T} T_{v_i}$ , and thus  $v_i v_{i-2} \notin E$ , which is a contradiction. Therefore,  $T_{v_i} \ll_{R_T} T_{v_{i-1}}$ , and thus also  $T_{v_i} \ll_{R_T} T_v$ , since  $v \in N(v_{i-1}) \setminus N(v_i)$ . Recall also that  $T_{x_2} \ll_{R_T} T_u$  and  $T_{x_2} \ll_{R_T} T_w$ . Thus, also  $T_v \ll_{R_T} T_u$  and  $T_v \ll_{R_T} T_w$ , since  $v \in N(x_2) \setminus N[u, w]$ . That is,  $T_{v_i} \ll_{R_T} T_v \ll_{R_T} T_u$  and  $T_{v_i} \ll_{R_T} T_v \ll_{R_T} T_w$ , i.e.  $v_i \notin N[u, w]$ , which is a contradiction. Thus,  $H_i \subseteq N(v)$  for every  $i \geq 1$ . This completes the proof of the lemma. ■

## The recursive definition of the vertex subsets $H'_i$ , $i \geq 0$ , of $H$

Similarly to Definitions 9 and 10, we partition in the following the set  $H \setminus \bigcup_{i=1}^{\infty} H_i$  into the subsets  $H'_0, H'_1, \dots$

**Definition 11** Let  $H' = H \setminus \bigcup_{i=1}^{\infty} H_i$  and  $H'_0 = \{x \in H' \mid xv \in E \text{ for some } v \in V \setminus Q_u \setminus N[u] \setminus V_0(u)\}$ . Furthermore,  $H'_i = \{x \in H' \setminus \bigcup_{j=0}^{i-1} H'_j \mid H'_{i-1} \not\subseteq N(x)\}$  for every  $i \geq 1$ .

It is now easy to see by Definition 11 that either  $H'_i = \emptyset$  for every  $i \in \mathbb{N} \cup \{0\}$ , or there exists some  $p \in \mathbb{N} \cup \{0\}$ , such that  $H'_p \neq \emptyset$  and  $H'_i = \emptyset$  for every  $i > p$ . That is, either  $\bigcup_{i=0}^{\infty} H'_i = \emptyset$ , or  $\bigcup_{i=0}^{\infty} H'_i = \bigcup_{i=0}^p H'_i$ , for some  $p \in \mathbb{N} \cup \{0\}$ , while  $\bigcup_{i=0}^{\infty} H'_i \subseteq H'$  by Definition 11. Furthermore, it is easy to observe by Definitions 9 and 11 that every vertex of  $H \setminus \bigcup_{i=1}^{\infty} H_i \setminus \bigcup_{i=0}^{\infty} H'_i$  is adjacent to every vertex of  $N(X_1) \cup \bigcup_{i=1}^{\infty} H_i \cup \bigcup_{i=0}^{\infty} H'_i$ , and to no vertex of  $V \setminus Q_u \setminus N[u] \setminus V_0(u)$ .

**Definition 12** Let  $v_i \in H'_i$ , for some  $i \geq 1$ . Then, a sequence  $(v_0, v_1, \dots, v_{i-1}, v_i)$  of vertices, such that  $v_j \in H'_j$ ,  $j = 0, 1, \dots, i-1$ , and  $v_{j-1} v_j \notin E$ ,  $j = 1, 2, \dots, i$ , is an  $H'_i$ -chain of  $v_i$ .

It is easy to see by Definition 11 that for every set  $H'_i \neq \emptyset$ ,  $i \geq 1$ , and for every vertex  $v_i \in H'_i$ , there exists at least one  $H'_i$ -chain of  $v_i$ . Now, similarly to Lemmas 25 and 26, we state the following two lemmas.

**Lemma 30** *Let  $v_1 \in H'_1$  and  $(v_0, v_1)$  be an  $H'_1$ -chain of  $v_1$ . Then,  $v_0, v_1 \in N(u)$ ,  $P_{v_1} \ll_R P_{v_0}$  and  $T_{v_1} \ll_{R_T} T_{v_0}$ .*

**Proof.** First, recall that there exists a bounded covering vertex  $u^*$  of  $u$  by Lemma 4, and thus  $w \in N(u) \subseteq N(u^*)$ . Let  $y \in V \setminus Q_u \setminus N[u] \setminus V_0(u)$  be a vertex, such that  $yv_0 \in E$ ; such a vertex  $y$  exists by Definition 11. Then,  $y \notin N(w)$ , since either  $P_w \ll_R P_{x_2} \ll_R P_y$  or  $P_y \ll_R P_x \ll_R P_w$  for every  $x \in X_1$  by Lemma 20. Consider the trapezoid representation  $R_T$ . Then, either  $T_{x_2} \ll_{R_T} T_y$  or  $T_y \ll_{R_T} T_x$  for every  $x \in X_1$  by Lemma 21. Suppose that  $T_y \ll_{R_T} T_x$  for every  $x \in X_1$ , i.e.  $T_y \ll_{R_T} T_x \ll_{R_T} T_{x_2}$  for every  $x \in X_1$ . Then, since  $v_0 \in N(y)$  and  $v_0 \in N(x_2)$ ,  $T_{v_0}$  intersects  $T_x$  in  $R_T$  for every  $x \in X_1$ , and thus  $v_0 \in N(X_1)$ . This is a contradiction, since  $v_0 \in H'_0 \subseteq H$ , and since  $H$  is an induced subgraph of  $G \setminus Q_u \setminus N[X_1]$ . Thus,  $T_{x_2} \ll_{R_T} T_y$ .

Since  $y \notin N(u)$  by the assumption on  $y$ , either  $T_y \ll_{R_T} T_u$  or  $T_u \ll_{R_T} T_y$ . Suppose that  $T_y \ll_{R_T} T_u$ , i.e.  $T_{x_2} \ll_{R_T} T_y \ll_{R_T} T_u$ . Then, also  $T_{x_2} \ll_{R_T} T_y \ll_{R_T} T_w$ , since  $w \in N(u)$  and  $w \notin N(y)$ . Note that  $y \notin N(u^*)$ , since otherwise  $y \in V_0(u)$ , which is a contradiction. Thus, since also  $w \in N(u^*)$ , it follows that  $T_{x_2} \ll_{R_T} T_y \ll_{R_T} T_{u^*}$ . Then, since  $x_2, u^* \in V_0(u)$ , and since  $V_0(u)$  is connected,  $T_y$  intersects  $T_z$  for some  $z \in V_0(u)$ , and thus  $y \in V_0(u)$ , which is a contradiction. Therefore,  $T_u \ll_{R_T} T_y$ , i.e.  $T_{x_2} \ll_{R_T} T_u \ll_{R_T} T_y$ . Thus, since  $v_0 \in N(x_2)$  and  $v_0 \in N(y)$ ,  $T_{v_0}$  intersects  $T_u$  in  $R_T$ , i.e.  $v_0 \in N(u)$ ; in particular,  $v_0$  is bounded.

Since  $v_1 v_0 \notin E$ , either  $T_{v_0} \ll_{R_T} T_{v_1}$  or  $T_{v_1} \ll_{R_T} T_{v_0}$ . Suppose that  $T_{v_0} \ll_{R_T} T_{v_1}$ . Recall that  $y v_1 \notin E$  by Definition 11, since  $y \in V \setminus Q_u \setminus N[u] \setminus V_0(u)$  and  $v_1 \in H_1$ . That is, either  $T_{v_1} \ll_{R_T} T_y$  or  $T_y \ll_{R_T} T_{v_1}$ . If  $T_{v_1} \ll_{R_T} T_y$ , then  $T_{v_0} \ll_{R_T} T_{v_1} \ll_{R_T} T_y$ , i.e.  $y v_0 \notin E$ , which is a contradiction. If  $T_y \ll_{R_T} T_{v_1}$ , then  $T_{x_2} \ll_{R_T} T_y \ll_{R_T} T_{v_1}$ , i.e.  $v_1 \notin N(x_2)$ , which is a contradiction. Thus,  $T_{v_1} \ll_{R_T} T_{v_0}$ .

Consider now the projection representation  $R$ , and recall that  $v_1 v_0, v_1 y \notin E$ . Furthermore, recall that  $v_0 \notin N(\tilde{X}_1)$  by definition of  $H$ , and that either  $P_{x_2} \ll_R P_y$  or  $P_y \ll_R P_x$  for every  $x \in X_1$  by Lemma 20. Suppose that  $P_y \ll_R P_x$  for every  $x \in X_1$ , and thus  $P_y \ll_R P_x \ll_R P_u \ll_R P_{x_2}$  for every  $x \in \tilde{X}_1 \subseteq X_1$ . Then,  $P_{v_0}$  intersects  $P_x$  in  $R$  for every  $x \in \tilde{X}_1$ , since  $v_0 \in N(y) \cap N(x_2)$ . Furthermore,  $v_0 x \notin E$  for every  $x \in \tilde{X}_1$ , since  $v_0 \notin N(\tilde{X}_1)$ . Thus, every  $x \in \tilde{X}_1$  is unbounded and  $\phi_x > \phi_{v_0} > \phi_u$ , since  $v_0$  is bounded and  $v_0 \in N(u)$ , as we proved above. Moreover, since  $\tilde{X}_1$  is connected, and since no two unbounded vertices are adjacent, it follows that  $\tilde{X}_1$  has one vertex, i.e.  $\tilde{X}_1 = \{x_1\}$ . Thus,  $N(x_1) = N(\tilde{X}_1) \subseteq N(x_2)$  by Lemma 19, since  $\tilde{X}_1 \subseteq X_1$ . Therefore,  $P_z$  intersects  $P_u$  in  $R$ , for every  $z \in N(x_1)$ , since  $P_{x_1} \ll_R P_u \ll_R P_{x_2}$ . Furthermore,  $z$  is bounded and  $\phi_z > \phi_{x_1} > \phi_u$  for every  $z \in N(x_1)$ , since  $x_1$  is unbounded. That is,  $z \in N(u)$  for every  $z \in N(x_1)$ , i.e.  $N(x_1) \subseteq N(u)$ , and thus  $x_1$  is an isolated vertex of  $G \setminus N[u]$ . Therefore, since  $x_1$  is unbounded and  $u^*$  is bounded in  $R$ , it follows that  $x_1$  and  $u^*$  do not lie in the same connected component of  $G \setminus N[u]$ . That is,  $V_0(u)$  is not connected, which is a contradiction. Thus,  $P_{x_2} \ll_R P_y$ , i.e.  $P_u \ll_R P_{x_2} \ll_R P_y$ .

Suppose that  $P_{v_1}$  intersects  $P_y$  in  $R$ . Then, either  $N(v_1) \subseteq N(y)$  or  $N(y) \subseteq N(v_1)$  by Lemma 3, since  $v_1 y \notin E$ . If  $N(v_1) \subseteq N(y)$ , then  $x_2 \in N(y)$ , which is a contradiction, since  $P_{x_2} \ll_R P_y$ . On the other hand, if  $N(y) \subseteq N(v_1)$ , then  $v_0 \in N(v_1)$ , since  $y v_0 \in E$ , which is a contradiction. Thus,  $P_{v_1}$  does not intersect  $P_y$  in  $R$ , i.e. either  $P_y \ll_R P_{v_1}$  or  $P_{v_1} \ll_R P_y$ . If  $P_y \ll_R P_{v_1}$ , then  $P_{x_2} \ll_R P_y \ll_R P_{v_1}$ , i.e.  $v_1 \notin N(x_2)$ , which is a contradiction. Thus,  $P_{v_1} \ll_R P_y$ .

Suppose that  $P_{v_1}$  intersects  $P_{v_0}$  in  $R$ . Then,  $v_1$  is unbounded and  $\phi_{v_1} > \phi_{v_0} > \phi_u$ , since  $v_0$  is bounded and  $v_0 \in N(u)$ . Furthermore, note that  $N_1(v_1) = N$ , since otherwise  $v_1 \in H_1$  by Definition 9, and thus  $v_1 \notin H' = H \setminus \bigcup_{i=1}^{\infty} H_i$ , which is a contradiction. Consider now a vertex  $z \in N$ . Then,  $z \in N(x) \cap N(x_2)$ , for some  $x \in X_1$ . Furthermore,  $z \in N(v_1)$ , since  $N_1(v_1) = N$ ; thus,  $z$  is bounded and  $\phi_z > \phi_{v_1} > \phi_u$ , since  $v_1$  is unbounded. On the other hand,  $P_z$  intersects  $P_u$  in  $R$ , since  $P_x \ll_R P_u \ll_R P_{x_2}$  and  $z \in N(x) \cap N(x_2)$ . Thus  $z \in N(u)$ , since  $z$  is bounded and  $\phi_z > \phi_u$ . Since this holds for an arbitrary  $z \in N$ , it follows that  $N_1(u) = N$ , which is a contradiction by Lemma 19. Thus,  $P_{v_1}$  does not intersect  $P_{v_0}$  in  $R$ , i.e. either  $P_{v_0} \ll_R P_{v_1}$  or  $P_{v_1} \ll_R P_{v_0}$ . If  $P_{v_0} \ll_R P_{v_1}$ , then  $P_{v_0} \ll_R P_{v_1} \ll_R P_y$ , i.e.  $y \notin N(v_0)$ , which is a contradiction. Thus,  $P_{v_1} \ll_R P_{v_0}$ .

Recall that  $v_0 \in N(u)$  as we have proved above, and thus  $L(v_0) <_R L(u)$  by Lemma 1. Furthermore,  $R(v_1) <_R L(v_0)$ , since  $P_{v_1} \ll_R P_{v_0}$ , and thus  $R(v_1) <_R L(u)$ . On the other hand,

since  $v_1 \in N(x_2)$ , and since  $R(v_1) <_R L(u) <_R L(x_2)$ , it follows that  $l(x_2) <_R r(v_1)$ , and thus  $l(u) <_R r(v_1)$ , since  $P_u \ll_R P_{x_2}$ . Therefore, since also  $R(v_1) <_R L(u)$ ,  $P_{v_1}$  intersects  $P_u$  in  $R$  and  $\phi_{v_1} > \phi_u$ . If  $v_1 \notin N(u)$ , then  $N(v_1) \subseteq N(u)$  by Lemma 3, and thus  $x_2 \in N(u)$ , since  $x_2 \in N(v_1)$  by definition of  $H$ , which is a contradiction. Therefore,  $v_1 \in N(u)$ . This completes the proof of the lemma. ■

**Lemma 31** *Let  $v_i \in H'_i$ , for some  $i \geq 2$ , and  $(v_0, v_1, \dots, v_i)$  be an  $H'_i$ -chain of  $v_i$ . Then, for every  $j = 1, 2, \dots, i-1$ :*

1.  $P_{v_{j-1}} \ll_R P_{v_j}$  and  $T_{v_{j-1}} \ll_{R_T} T_{v_j}$ , if  $j$  is even,
2.  $P_{v_j} \ll_R P_{v_{j-1}}$  and  $T_{v_j} \ll_{R_T} T_{v_{j-1}}$ , if  $j$  is odd.

**Proof.** The proof will be done by induction on  $j$ . For  $j = 1$ , the induction basis follows by Lemma 30. For the induction step, let  $2 \leq j < i$ . Note that  $v_{j-2} \in N(v_j) \setminus N(v_{j-1})$  and  $v_{j+1} \in N(v_{j-1}) \setminus N(v_j)$ . Therefore,  $N(v_j) \not\subseteq N(v_{j-1})$  and  $v_{j+1} \in N(v_{j-1}) \not\subseteq N(v_j)$ , and thus  $P_{v_j}$  does not intersect  $P_{v_{j-1}}$  in  $R$  by Lemma 3, since  $v_{j-1}v_j \notin E$ . Thus, either  $P_{v_{j-1}} \ll_R P_{v_j}$  or  $P_{v_j} \ll_R P_{v_{j-1}}$ . Furthermore, clearly either  $T_{v_{j-1}} \ll_{R_T} T_{v_j}$  or  $T_{v_j} \ll_{R_T} T_{v_{j-1}}$ , since  $v_{j-1}v_j \notin E$ .

Let  $j$  be even, i.e.  $j-1$  is odd, and suppose by induction hypothesis that  $P_{v_{j-1}} \ll_R P_{v_{j-2}}$  and  $T_{v_{j-1}} \ll_{R_T} T_{v_{j-2}}$ . If  $P_{v_j} \ll_R P_{v_{j-1}}$  (resp.  $T_{v_j} \ll_{R_T} T_{v_{j-1}}$ ), then  $P_{v_j} \ll_R P_{v_{j-2}}$  (resp.  $T_{v_j} \ll_{R_T} T_{v_{j-2}}$ ). Thus,  $v_jv_{j-2} \notin E$ , i.e.  $v_j \in H'_{j-1}$  by Definition 11, which is a contradiction. Therefore,  $P_{v_{j-1}} \ll_R P_{v_j}$  and  $T_{v_{j-1}} \ll_{R_T} T_{v_j}$ , if  $j$  is even.

Let now  $j$  be odd, i.e.  $j-1$  is even, and suppose by induction hypothesis that  $P_{v_{j-2}} \ll_R P_{v_{j-1}}$  and  $T_{v_{j-2}} \ll_{R_T} T_{v_{j-1}}$ . If  $P_{v_{j-1}} \ll_R P_{v_j}$  (resp.  $T_{v_{j-1}} \ll_{R_T} T_{v_j}$ ), then  $P_{v_{j-2}} \ll_R P_{v_j}$  (resp.  $T_{v_{j-2}} \ll_{R_T} T_{v_j}$ ), and thus  $v_jv_{j-2} \notin E$ , which is again a contradiction. Therefore,  $P_{v_j} \ll_R P_{v_{j-1}}$  and  $T_{v_j} \ll_{R_T} T_{v_{j-1}}$ , if  $j$  is odd. This completes the induction step, and thus the lemma follows. ■

**Lemma 32**  $H'_i \subseteq N(u)$ , for every  $i \geq 0$ .

**Proof.** The proof will be done by induction on  $i$ . For  $i = 0$  and  $i = 1$ , the lemma follows by Lemma 30. This proves the induction basis. For the induction step, let  $i \geq 2$ . Suppose that  $v_i \notin N(u)$ , and let  $(v_0, v_1, \dots, v_{i-2}, v_{i-1}, v_i)$  be an  $H'_i$ -chain of  $v_i$ . By the induction hypothesis,  $v_j \in N(u)$  for every  $j = 0, 1, \dots, i-1$ . Then, in particular,  $r(u) <_R r(v_{i-1})$  and  $L(v_{i-1}) <_R L(u)$  by Lemma 1. Furthermore,  $v_{i-2} \in N(v_i) \setminus N(v_{i-1})$  and  $u \in N(v_{i-1}) \setminus N(v_i)$ , i.e.  $N(v_i) \not\subseteq N(v_{i-1})$  and  $N(v_{i-1}) \not\subseteq N(v_i)$ , and thus Lemma 3 implies that  $P_{v_i}$  does not intersect  $P_{v_{i-1}}$  in  $R$ , since  $v_iv_{i-1} \notin E$ .

Suppose first that  $i$  is odd. Then,  $P_{v_{i-2}} \ll_R P_{v_{i-1}}$  by Lemma 31. Thus, since  $v_i \in N(v_{i-2})$ , and since  $P_{v_i}$  does not intersect  $P_{v_{i-1}}$  in  $R$  by the previous paragraph, it follows that  $P_{v_i} \ll_R P_{v_{i-1}}$ . Therefore, in particular,  $R(v_i) <_R L(v_{i-1}) <_R L(u)$ , i.e.  $R(v_i) <_R L(u)$ . On the other hand,  $v_i \in N(x_2)$ , and thus  $T_{v_i}$  intersects  $T_{x_2}$  in  $R_T$ . Therefore, since  $R(v_i) <_R L(u) <_R L(x_2)$ , it follows that  $l(x_2) <_R r(v_i)$ . Furthermore, since  $P_u \ll_R P_{x_2}$ , it follows that  $l(u) <_R l(x_2) <_R r(v_i)$ . That is,  $R(v_i) <_R L(u)$  and  $l(u) <_R r(v_i)$ , i.e.  $P_{v_i}$  intersects  $P_u$  in  $R$  and  $\phi_{v_i} > \phi_u$ . If  $v_i \notin N(u)$ , then  $N(v_i) \subseteq N(u)$  by Lemma 3, and thus  $x_2 \in N(u)$ , which is a contradiction. Therefore,  $v_i \in N(u)$  if  $i$  is odd.

Suppose now that  $i$  is even. Then,  $T_{v_{i-1}} \ll_{R_T} T_{v_{i-2}}$  by Lemma 31. Thus, since  $v_i \in N(v_{i-2})$  and  $v_i \notin N(v_{i-1})$ , it follows that  $T_{v_{i-1}} \ll_{R_T} T_{v_i}$ . Recall that  $T_{x_2} \ll_{R_T} T_u$ . Since we assumed that  $v_i \notin N(u)$ , either  $T_{v_i} \ll_{R_T} T_u$  or  $T_u \ll_{R_T} T_{v_i}$ . If  $T_{v_i} \ll_{R_T} T_u$ , then  $T_{v_{i-1}} \ll_{R_T} T_{v_i} \ll_{R_T} T_u$ , i.e.  $v_{i-1} \notin N(u)$ , which is a contradiction by the induction hypothesis. If  $T_u \ll_{R_T} T_{v_i}$ , then  $T_{x_2} \ll_{R_T} T_u \ll_{R_T} T_{v_i}$ , i.e.  $v_i \notin N(x_2)$ , which is a contradiction. Thus,  $v_i \in N(u)$  if  $i$  is even. This completes the induction step and the lemma follows. ■

Now, similarly to Lemmas 28, and 29, we state the following two lemmas.

**Lemma 33** *For every vertex  $v \in C_u \setminus \{u\}$ , it holds  $H'_i \subseteq N(v)$  for every  $i \geq 0$ .*

**Proof.** Let  $v$  be a vertex of  $C_u \setminus \{u\}$ . Recall that  $N_1(v) = N$  by Lemma 23. Consider first the case where  $v \in N(u) \cup N(w)$ . The proof will be done by induction on  $i$ . For  $i = 0$ , consider a vertex  $v_0 \in H'_0$  and a vertex  $y \in V \setminus Q_u \setminus N[u] \setminus V_0(u)$ , such that  $yv_0 \in E$ ; such a vertex  $y$  exists by Definition 11. Recall that  $T_{x_2} \ll_{R_T} T_u \ll_{R_T} T_y$  and that  $P_u \ll_R P_{x_2} \ll_R P_y$  by the proof of Lemma 30.

Let first  $v \notin N(u)$  (and thus  $v \in N(w)$ ). If  $T_u \ll_{R_T} T_v$ , i.e.  $T_x \ll_{R_T} T_u \ll_{R_T} T_v$  for every  $x \in X_1$ , then  $T_z$  intersects  $T_u$  in  $R_T$  for every  $z \in N_1(v) = N$ . Thus,  $N_1(u) = N$ , which is a contradiction by Lemma 19. Therefore,  $T_v \ll_{R_T} T_u$ . Furthermore,  $T_{x_2} \ll_{R_T} T_v \ll_{R_T} T_u$ , since  $T_{x_2} \ll_{R_T} T_u$ , and since  $v \in C_u$  and  $C_u$  is connected. That is,  $T_{x_2} \ll_{R_T} T_v \ll_{R_T} T_u \ll_{R_T} T_y$ . Then,  $T_{v_0}$  intersects  $T_v$  in  $R_T$ , since  $v_0 \in N(x_2) \cap N(y)$ , i.e.  $v_0 \in N(v)$ .

Let now  $v \in N(u)$ , and thus  $v$  is bounded and  $\phi_v > \phi_u$  in the projection representation  $R$ . Suppose that  $v \in N(y)$ . Then,  $P_v$  intersects  $P_{x_2}$  in  $R$ , since  $P_u \ll_R P_{x_2} \ll_R P_y$ , and since  $v \in N(u)$  and  $v \in N(y)$ . Recall that  $v \notin N(x_2)$ , since  $v \in C_u$ . Thus, since  $v$  is bounded, it follows that  $x_2$  is unbounded and  $\phi_{x_2} > \phi_v > \phi_u$ . Recall that  $N_1(x_2) = N$  by Lemma 19. Consider now a vertex  $z \in N$ , i.e.  $z \in N(x) \cap N(x_2)$  for some  $x \in X_1$ . Then,  $z$  is bounded and  $\phi_z > \phi_{x_2} > \phi_u$ , since  $x_2$  is unbounded. Furthermore,  $P_z$  intersects  $P_u$  in  $R$ , since  $P_x \ll_R P_u \ll_R P_{x_2}$  and  $z \in N(x) \cap N(x_2)$ , and thus  $z \in N(u)$ . Since this holds for an arbitrary  $z \in N$ , it follows that  $N_1(u) = N$ , which is a contradiction by Lemma 19. Thus,  $v \notin N(y)$ . Then,  $T_v \ll_{R_T} T_y$ , since  $T_u \ll_{R_T} T_y$ , and since  $v \in N(u)$  and  $v \notin N(y)$ . Furthermore,  $T_{x_2} \ll_{R_T} T_v$ , since  $T_{x_2} \ll_{R_T} T_u$ , and since  $v \in N(u)$  and  $v \notin N(x_2)$ . Therefore,  $T_{x_2} \ll_{R_T} T_v \ll_{R_T} T_y$ , and thus  $T_{v_0}$  intersects  $T_v$  in  $R_T$ , i.e.  $v_0 \in N(v)$ , since  $v_0 \in N(x_2) \cap N(y)$ . Summarizing,  $v_0 \in N(v)$  for every vertex  $v_0 \in H'_0$  and for every vertex  $v \in C_u \setminus \{u\}$ , such that  $v \in N(u) \cup N(w)$ , i.e.  $H'_0 \subseteq N(v)$  for all these vertices  $v$ . This proves the induction basis.

For the induction step, let  $i \geq 1$ , and suppose that  $v' \in N(v)$  for every  $v' \in H_j$ , where  $0 \leq j \leq i - 1$ . Let  $v_i \in H_i$  and  $(v_0, v_1, \dots, v_{i-2}, v_{i-1}, v_i)$  be an  $H_i$ -chain of  $v_i$ . For the sake of contradiction, suppose that  $v_i \notin N(v)$ . We will first prove that  $P_v \ll_R P_{x_2}$ . Otherwise, suppose first that  $P_{x_2} \ll_R P_v$ . Then,  $P_u \ll_R P_{x_2} \ll_R P_v$  and  $P_w \ll_R P_{x_2} \ll_R P_v$ , and thus  $v \notin N(u) \cup N(w)$ , which is a contradiction to the assumption on  $v$ . Suppose now that  $P_v$  intersects  $P_{x_2}$  in  $R$ . Then, either  $N(x_2) \subseteq N(v)$  or  $N(v) \subseteq N(x_2)$  by Lemma 3, since  $v \notin N(x_2)$  by definition of  $\tilde{C}_2$ . If  $N(x_2) \subseteq N(v)$ , then  $v_i \in N(v)$ , since  $v_i \in N(x_2)$ , which is a contradiction. Let  $N(v) \subseteq N(x_2)$ . Then, since  $C_u$  is connected with at least two vertices,  $v$  is adjacent to at least one vertex  $z \in C_u$ , and thus  $z \in N(x_2)$ , which is a contradiction. Thus,  $P_v \ll_R P_{x_2}$ .

Recall that  $v_{i-1} \in N(v)$  by the induction hypothesis. If  $i = 1$ ,  $P_{v_1}$  does not intersect  $P_{v_0}$  in  $R$  by Lemma 30. If  $i \geq 2$ , i.e. if  $v_{i-2}$  exists, then  $P_{v_i}$  does not intersect  $P_{v_{i-1}}$  in  $R$  by Lemma 3, since  $v_{i-2} \in N(v_i) \setminus N(v_{i-1})$  and  $v \in N(v_{i-1}) \setminus N(v_i)$ . Thus,  $P_{v_i}$  does not intersect  $P_{v_{i-1}}$  in  $R$  for every  $i \geq 1$ . Similarly,  $P_{v_i}$  does not intersect  $P_v$  in  $R$ , since  $x_2 \in N(v_i) \setminus N(v)$  and  $v_{i-1} \in N(v) \setminus N(v_i)$ . Therefore, since  $v_{i-1} \in N(v)$ , it follows that either  $P_{v_i} \ll_R P_{v_{i-1}}$  and  $P_{v_i} \ll_R P_v$ , or  $P_{v_{i-1}} \ll_R P_{v_i}$  and  $P_v \ll_R P_{v_i}$ . Suppose that  $P_{v_i} \ll_R P_{v_{i-1}}$  and  $P_{v_i} \ll_R P_v$ . Then,  $P_{v_i} \ll_R P_v \ll_R P_{x_2}$ , and thus  $v_i \notin N(x_2)$ , which is a contradiction.

Therefore,  $P_{v_{i-1}} \ll_R P_{v_i}$  and  $P_v \ll_R P_{v_i}$ , and thus  $i \neq 1$  by Lemma 30. That is,  $i \geq 2$ , i.e.  $v_{i-2}$  exists. Furthermore, either  $P_{v_{i-2}} \ll_R P_{v_{i-1}}$  or  $P_{v_{i-1}} \ll_R P_{v_{i-2}}$  by Lemma 31. If  $P_{v_{i-2}} \ll_R P_{v_{i-1}}$ , then  $P_{v_{i-2}} \ll_R P_{v_{i-1}} \ll_R P_{v_i}$ , and thus  $v_i v_{i-2} \notin E$ , which is a contradiction. Therefore  $P_{v_{i-1}} \ll_R P_{v_{i-2}}$ , and thus also  $T_{v_{i-1}} \ll_{R_T} T_{v_{i-2}}$  and  $i$  is even, by Lemma 31. Furthermore,  $T_{v_{i-1}} \ll_{R_T} T_{v_i}$ , since  $v_i \in N(v_{i-2})$  and  $v_i \notin N(v_{i-1})$ . Moreover,  $T_v \ll_{R_T} T_{v_i}$ , since  $T_{v_{i-1}} \ll_{R_T} T_{v_i}$ , and since  $v \in N(v_{i-1})$  and  $v \notin N(v_i)$ . Recall also that  $T_{x_2} \ll_{R_T} T_v$ , since  $T_{x_2} \ll_{R_T} T_u$ , and since  $v \in C_u$  and  $C_u$  is connected. That is,  $T_{x_2} \ll_{R_T} T_v \ll_{R_T} T_{v_i}$ , and thus  $v_i \notin N(x_2)$ , which is a contradiction. Thus,  $v_i \in N(v)$  in the case where  $v \in N(u) \cup N(w)$ . This completes the induction step.

Summarizing, we have proved that  $H'_i \subseteq N(v)$  for every  $i \geq 0$  and for every vertex  $v \in C_u \setminus \{u\}$ , such that  $v \in N(u) \cup N(w)$ . This holds in particular for  $w$ , i.e.  $H'_i \subseteq N(w)$  for every  $i \geq 0$ , since  $w$  is a vertex of  $C_u \setminus \{u\}$  and  $w \in N(u) \subseteq N(u) \cup N(w)$ . Consider now the case where  $v \notin N(u) \cup N(w)$ . Then, since  $w \in N(u)$ , either  $T_u \ll_{R_T} T_v$  and  $T_w \ll_{R_T} T_v$ , or  $T_v \ll_{R_T} T_u$  and  $T_v \ll_{R_T} T_w$ . Suppose first that  $T_u \ll_{R_T} T_v$ , i.e.  $T_x \ll_{R_T} T_{x_2} \ll_{R_T} T_u \ll_{R_T} T_v$  for every  $x \in X_1$

by Lemma 18. Recall that  $N_1(v) = N$  by Lemma 23. Then,  $T_z$  intersects  $T_u$  in  $R_T$ , i.e.  $z \in N(u)$ , for every  $z \in N_1(v) = N$ , and thus  $N_1(u) = N$ , which is a contradiction by Lemma 19. Therefore,  $T_v \ll_{R_T} T_u$  and  $T_v \ll_{R_T} T_w$ . Furthermore,  $T_{x_2} \ll_{R_T} T_v$ , since  $T_{x_2} \ll_{R_T} T_u$ , and since  $v \in C_u$  and  $C_u$  is connected. That is,  $T_{x_2} \ll_{R_T} T_v \ll_{R_T} T_w$ . Then, since every  $z \in H'_i$ ,  $i \geq 0$ , is adjacent to both  $x_2$  and  $w$ , as we proved above, it follows that  $T_z$  intersects  $T_v$  in  $R_T$ , i.e.  $z \in N(v)$ , for every  $z \in H'_i$ , where  $i \geq 0$ . Thus,  $H'_i \subseteq N(v)$  for every  $i \geq 0$  and for every vertex  $v \in C_u \setminus \{u\}$ , such that  $v \notin N(u) \cup N(w)$ . This completes the proof of the lemma. ■

**Lemma 34** *For every vertex  $v \in C_2$ , it holds  $H'_i \subseteq N(v)$  for every  $i \geq 0$ .*

**Proof.** Recall that  $C_2 = \mathcal{A}_2 \cup \mathcal{B}_2$ , where  $A_j \subseteq D_2$  for every  $A_j \in \mathcal{B}_2$ ,  $k+1 \leq j \leq \ell$ , and  $\mathcal{A}_2$  includes exactly those components  $A_i$ ,  $1 \leq i \leq k$ , for which all vertices of  $A_i$  are adjacent to all vertices of  $\tilde{H}$ . Therefore, if  $v \in A_i$  for some component  $A_i \in \mathcal{A}_2$ , then  $H' \subseteq H \subseteq \tilde{H} \subseteq N(v)$  by definition, and thus  $H'_i \subseteq N(v)$  for every  $i \geq 0$ .

Let now  $v \in A_j$ , for some  $A_j \in \mathcal{B}_2$ , and thus  $v \in D_2$ . Suppose first that  $v \notin N(x_2)$ . Then,  $T_{x_2} \ll_{R_T} T_v$  by Lemma 15, and  $T_v \ll_{R_T} T_u$ , since  $v \in D_2 \subseteq S_2 \subseteq V_0(u)$ . Moreover,  $v \notin N(w)$ , since otherwise  $v \in C_u$ , which is a contradiction to the definition of  $C_2$ . Thus,  $T_v \ll_{R_T} T_w$ , since  $T_v \ll_{R_T} T_u$ , and since  $w \in N(u)$  and  $w \notin N(v)$ . That is,  $T_{x_2} \ll_{R_T} T_v \ll_{R_T} T_w$ . Let now  $z \in H'_i$ , for some  $i \geq 0$ . Then,  $z \in N(x_2)$  by definition of  $H'$  and  $z \in N(w)$  by Lemma 33, and thus  $T_z$  intersects  $T_v$  in  $R_T$ , i.e.  $z \in N(v)$ . Therefore,  $H'_i \subseteq N(v)$  for every  $i \geq 0$ , in the case where  $v \notin N(x_2)$ .

Suppose now that  $v \in N(x_2)$ . We will prove by induction on  $i$  that  $H'_i \subseteq N(v)$  for every  $i \geq 0$ . For  $i = 0$ , let first  $v_0 \in H'_0$  and  $y \in V \setminus Q_u \setminus N[u] \setminus V_0(u)$  be a vertex, such that  $yv_0 \in E$ ; such a vertex  $y$  exists by Definition 11. For the sake of contradiction, assume that  $v_0 \notin N(v)$ . Recall that  $v_0 \in N(u)$  by Lemma 32, and thus  $v_0$  is bounded and  $\phi_{v_0} > \phi_u$ . Suppose that  $P_{v_0}$  intersects  $P_v$  in  $R$ . Then,  $v$  is unbounded and  $\phi_v > \phi_{v_0} > \phi_u$ , since  $v_0$  is bounded and  $v_0 \notin N(v)$ . Recall that  $N_1(v) = N$  by Lemma 24. Consider now a vertex  $z \in N$ , i.e.  $z \in N(x) \cap N(x_2)$  for some  $x \in X_1$ . Then,  $z \in N(v)$ , since  $N_1(v) = N$ , and thus  $z$  is bounded and  $\phi_z > \phi_v > \phi_u$ , since  $v$  is unbounded. On the other hand,  $P_z$  intersects  $P_u$  in  $R$ , since  $P_x \ll_R P_u \ll_R P_{x_2}$  and  $z \in N(x) \cap N(x_2)$ . Thus,  $z \in N(u)$ , since  $z$  is bounded and  $\phi_z > \phi_u$ . Since this holds for an arbitrary  $z \in N$ , it follows that  $N_1(u) = N$ , which is a contradiction by Lemma 19. Thus,  $P_{v_0}$  does not intersect  $P_v$  in  $R$ , i.e. either  $P_v \ll_R P_{v_0}$  or  $P_{v_0} \ll_R P_v$ .

Let first  $P_v \ll_R P_{v_0}$ . Suppose that  $P_v$  intersects  $P_u$  in  $R$ . Recall that  $v \notin N(u)$ , since  $v \in C_2$ , and thus either  $N(u) \subseteq N(v)$  or  $N(v) \subseteq N(u)$  by Lemma 3. If  $N(u) \subseteq N(v)$ , then  $v_0 \in N(v)$ , which is a contradiction. If  $N(v) \subseteq N(u)$ , then  $x_2 \in N(u)$ , which is again a contradiction. Thus,  $P_v$  does not intersect  $P_u$  in  $R$ , i.e. either  $P_v \ll_R P_u$  or  $P_u \ll_R P_v$ . If  $P_v \ll_R P_u$ , then  $P_v \ll_R P_u \ll_R P_{x_2}$ , i.e.  $v \notin N(x_2)$ , which is a contradiction to the assumption on  $v$ . Thus,  $P_u \ll_R P_v$ . Moreover, since we assumed that  $P_v \ll_R P_{v_0}$ , it follows that  $P_u \ll_R P_v \ll_R P_{v_0}$ , and thus  $v_0 \notin N(u)$ , which is a contradiction by Lemma 32.

Let now  $P_{v_0} \ll_R P_v$ . Suppose that  $P_v$  intersects  $P_y$  in  $R$ . Recall that  $v \in V_0(u)$  by Lemma 17, and thus  $vy \notin E$ , since otherwise  $y \in V_0(u)$ , which is a contradiction. Thus, either  $N(y) \subseteq N(v)$  or  $N(v) \subseteq N(y)$  by Lemma 3. If  $N(y) \subseteq N(v)$ , then  $v_0 \in N(v)$ , which is a contradiction. If  $N(v) \subseteq N(y)$ , then  $x_2 \in N(y)$  (since we assumed that  $x_2 \in N(v)$ ), and thus  $y \in V_0(u)$ , which is a contradiction. Thus,  $P_v$  does not intersect  $P_y$  in  $R$ , i.e. either  $P_v \ll_R P_y$  or  $P_y \ll_R P_v$ . If  $P_v \ll_R P_y$ , then  $P_{v_0} \ll_R P_v \ll_R P_y$ , i.e.  $yv_0 \notin E$ , which is a contradiction. Suppose that  $P_y \ll_R P_v$ . Recall that  $P_{x_2} \ll_R P_y$  by the proof of Lemma 30. Thus  $P_{x_2} \ll_R P_y \ll_R P_v$ , i.e.  $v \notin N(x_2)$ , which is a contradiction to the assumption on  $v$ . Therefore,  $v_0 \in N(v)$ , and thus  $H'_0 \subseteq N(v)$ . This proves the induction basis.

For the induction step, let  $i \geq 1$ , and suppose that  $v' \in N(v)$  for every  $v' \in H'_j$ , where  $0 \leq j \leq i-1$ . For the sake of contradiction, assume that  $v_i \notin N(v)$ . Let  $(v_0, v_1, \dots, v_{i-1}, v_i)$  be an  $H_i$ -chain of  $v_i$ . If  $i = 1$ ,  $P_{v_1}$  does not intersect  $P_{v_0}$  in  $R$  by Lemma 30. If  $i \geq 2$ , i.e. if  $v_{i-2}$  exists, then  $P_{v_i}$  does not intersect  $P_{v_{i-1}}$  in  $R$  by Lemma 3, since  $v_{i-2} \in N(v_i) \setminus N(v_{i-1})$  and  $v \in N(v_{i-1}) \setminus N(v_i)$ . Thus,  $P_{v_i}$  does not intersect  $P_{v_{i-1}}$  in  $R$  for every  $i \geq 1$ . Recall now that



$v_i \in N[u, w] = N(u) \cup N(w)$ , since  $v_i \in H$ , and that  $v \notin N[u, w] = N(u) \cup N(w)$  by definition of  $C_2$ . If  $v_i \in N(u)$  (resp.  $v_i \in N(w)$ ), then  $u \in N(v_i) \setminus N(v)$  (resp.  $w \in N(v_i) \setminus N(v)$ ). Furthermore,  $v_{i-1} \in N(v) \setminus N(v_i)$ , i.e.  $N(v_i) \not\subseteq N(v)$  and  $N(v) \not\subseteq N(v_i)$ , and thus  $P_{v_i}$  does not intersect  $P_v$  in  $R$  by Lemma 3. Therefore, since  $v_{i-1} \in N(v)$ , it follows that either  $P_{v_{i-1}} \ll_R P_{v_i}$  and  $P_v \ll_R P_{v_i}$  or  $P_{v_i} \ll_R P_{v_{i-1}}$  and  $P_{v_i} \ll_R P_v$ .

Suppose first that  $P_{v_{i-1}} \ll_R P_{v_i}$  and  $P_v \ll_R P_{v_i}$ . Recall that  $v_i \in N(u)$  or  $v_i \in N(w)$ . Furthermore, recall that  $v \in N(x_2)$  by our assumption on  $v$ . Let  $v_i \in N(u)$  (resp.  $v_i \in N(w)$ ). Then,  $P_v$  does not intersect  $P_u$  (resp.  $P_w$ ) in  $R$  by Lemma 3, since  $x_2 \in N(v) \setminus N(u)$  (resp.  $x_2 \in N(v) \setminus N(w)$ ) and  $v_i \in N(u) \setminus N(v)$  (resp.  $v_i \in N(w) \setminus N(v)$ ). Therefore, since  $P_u \ll_R P_{x_2}$  (resp.  $P_w \ll_R P_{x_2}$ ) and  $v \in N(x_2)$ , it follows that  $P_u \ll_R P_v$  (resp.  $P_w \ll_R P_v$ ). That is,  $P_u \ll_R P_v \ll_R P_{v_i}$  (resp.  $P_w \ll_R P_v \ll_R P_{v_i}$ ), i.e.  $v_i \notin N(u)$  (resp.  $v_i \notin N(w)$ ), which is a contradiction.

Suppose now that  $P_{v_i} \ll_R P_{v_{i-1}}$  and  $P_{v_i} \ll_R P_v$ . If  $i = 1$ , then  $T_{v_1} \ll_{R_T} T_{v_0}$  by Lemma 30. If  $i \geq 2$ , i.e. if  $v_{i-2}$  exists, then  $P_{v_{i-2}} \ll_R P_{v_{i-1}}$ . Indeed, otherwise  $P_{v_i} \ll_R P_{v_{i-1}} \ll_R P_{v_{i-2}}$ , i.e.  $v_i v_{i-2} \notin E$ , which is a contradiction. Thus, also  $T_{v_{i-2}} \ll_{R_T} T_{v_{i-1}}$  and  $i$  is odd by Lemma 31. Therefore,  $T_{v_i} \ll_{R_T} T_{v_{i-1}}$  if  $i \geq 2$ , since otherwise  $T_{v_{i-2}} \ll_{R_T} T_{v_{i-1}} \ll_{R_T} T_{v_i}$ , i.e.  $v_i v_{i-2} \notin E$ , which is a contradiction. That is,  $T_{v_i} \ll_{R_T} T_{v_{i-1}}$  for all  $i \geq 1$ . Therefore, since  $v \in N(v_{i-1})$  and  $v \notin N(v_i)$ , it follows that  $T_{v_i} \ll_{R_T} T_v$ . Recall also that  $T_{x_2} \ll_{R_T} T_u$  and  $T_{x_2} \ll_{R_T} T_w$ . Thus,  $T_v \ll_{R_T} T_u$  and  $T_v \ll_{R_T} T_w$ , since we assumed that  $v \in N(x_2)$ , and since  $v \notin N(u) \cup N(w)$  by definition of  $C_2$ . That is,  $T_{v_i} \ll_{R_T} T_v \ll_{R_T} T_u$  and  $T_{v_i} \ll_{R_T} T_v \ll_{R_T} T_w$ , i.e.  $v_i \notin N(u) \cup N(w)$ , which is a contradiction. Therefore,  $v_i \in N(v)$ , and thus  $H'_i \subseteq N(v)$ . This completes the induction step, and the lemma follows. ■

## The subgraph $G_0$ of $G$

Let  $G_0$  be the graph induced in  $G$  by the vertices of  $C_u \cup C_2 \cup (H \setminus \bigcup_{i=1}^{\infty} H_i \setminus \bigcup_{i=0}^{\infty} H'_i)$ . Note that  $G_0$  is an induced subgraph also of  $G \setminus Q_u \setminus N[X_1]$ . Furthermore, note that every vertex of  $G_0 \setminus \{u\}$  is bounded by to Lemma 27. Recall that  $C_2 \subseteq V_0(u)$  by Lemma 17 and that  $C_u \setminus \{u\} \subseteq N(u) \cup V_0(u)$  by Lemma 22. Consider now a vertex  $v \in H \setminus \bigcup_{i=1}^{\infty} H_i \setminus \bigcup_{i=0}^{\infty} H'_i$ . If  $v \notin N(u)$ , then  $v \in V_0(u)$ , since  $x_2 \in V_0(u)$  and  $v \in N(x_2)$  by definition of  $H$ . Thus, the next observation follows.

**Observation 3** *Every vertex of  $G_0 \setminus \{u\}$  is bounded. Furthermore,  $V(G_0) \subseteq N[u] \cup V_0(u)$ .*

**Lemma 35**  *$G_0 \setminus \{u\}$  is a module in  $G \setminus \{u\}$ . In particular,  $N(v) \setminus V(G_0) = N(X_1) \cup \bigcup_{i=1}^{\infty} H_i \cup \bigcup_{i=0}^{\infty} H'_i$  for every vertex  $v \in V(G_0) \setminus \{u\}$ .*

**Proof.** First recall by Lemma 16 that  $N(V(C_u \cup C_2 \cup H)) \subseteq Q_u \cup N(X_1) \cup V(\mathcal{B}_1)$ , where  $V(\mathcal{B}_1) = \bigcup_{A_j \in \mathcal{B}_1} A_j$ . Consider a vertex  $q \in Q_u$ . Then, since we assumed in the statement of Theorem 2 that Condition 3 holds, and since  $X_1 \subseteq D_1 \subseteq V_0(u)$  by Lemma 17, it follows that  $T_q \ll_{R_T} T_x \ll_{R_T} T_u$  for every  $x \in X_1$ . Thus, since  $N(q) \subset N(u)$  by definition of  $Q_u$ , it follows that  $T_z$  intersects  $T_x$  in  $R_T$  for every  $z \in N(q) \subset N(u)$  and every  $x \in X_1$ . Therefore, in particular,  $N(q) \subseteq N(X_1)$  for every  $q \in Q_u$ . Thus, no vertex  $q \in Q_u$  is adjacent to any vertex of  $V(C_u \cup C_2 \cup H)$ , since  $V(C_u \cup C_2 \cup H)$  induces a subgraph of  $G \setminus Q_u \setminus N[X_1] \setminus \mathcal{B}_1$  by Lemma 16. Thus,  $N(V(C_u \cup C_2 \cup H)) \cap Q_u = \emptyset$ , i.e.  $N(V(C_u \cup C_2 \cup H)) \subseteq N(X_1) \cup V(\mathcal{B}_1)$ .

Recall that  $V(G_0) = C_u \cup C_2 \cup (H \setminus \bigcup_{i=1}^{\infty} H_i \setminus \bigcup_{i=0}^{\infty} H'_i)$  by definition of  $G_0$ . Consider now an arbitrary vertex  $v \in V(G_0) \setminus \{u\}$ . Then, it follows by the previous paragraph that

$$N(v) \setminus V(G_0) \subseteq N(X_1) \cup V(\mathcal{B}_1) \cup \left( \bigcup_{i=1}^{\infty} H_i \cup \bigcup_{i=0}^{\infty} H'_i \right) \quad (8)$$

We will prove that  $N(v) \setminus V(G_0) = N(X_1) \cup (\bigcup_{i=1}^{\infty} H_i \cup \bigcup_{i=0}^{\infty} H'_i)$ . If  $v \in C_u \setminus \{u\}$ , then  $N(X_1) \subseteq N(v)$ , since  $N_1(v) = N = N(X_1)$  by Lemma 23. Similarly, if  $v \in C_2$ , then  $N(X_1) \subseteq N(v)$ , since  $N_1(v) = N = N(X_1)$  by Lemma 24. If  $v \in H \setminus \bigcup_{i=1}^{\infty} H_i \setminus \bigcup_{i=0}^{\infty} H'_i$ , then  $N = H_0 \subseteq N(v)$  by Definition 9 (where  $N = N(X_1)$ ), since otherwise  $v \in H_1$ , which is a contradiction. That is,  $N(X_1) \subseteq N(v)$  for every vertex  $v \in V(G_0) \setminus \{u\}$ .

If  $v \in C_u \setminus \{u\}$ , then  $\bigcup_{i=1}^{\infty} H_i \cup \bigcup_{i=0}^{\infty} H'_i \subseteq N(v)$  by Lemmas 28 and 33. Similarly, if  $v \in C_2$ , then  $\bigcup_{i=1}^{\infty} H_i \cup \bigcup_{i=0}^{\infty} H'_i \subseteq N(v)$  by Lemmas 29 and 34. If  $v \in H \setminus \bigcup_{i=1}^{\infty} H_i \setminus \bigcup_{i=0}^{\infty} H'_i$ , then  $\bigcup_{i=1}^{\infty} H_i \cup \bigcup_{i=0}^{\infty} H'_i \subseteq N(v)$  by Definitions 9 and 11. Indeed, otherwise  $v \in H_i$  for some  $i \geq 1$ , or  $v \in H'_i$  for some  $i \geq 0$ , which is a contradiction. That is,  $\bigcup_{i=1}^{\infty} H_i \cup \bigcup_{i=0}^{\infty} H'_i \subseteq N(v)$  for every vertex  $v \in V(G_0) \setminus \{u\}$ .

We will now prove that  $N(v) \cap V(\mathcal{B}_1) = \emptyset$ . Suppose for the sake of contradiction that  $v' \in N(v)$ , for some  $v' \in V(\mathcal{B}_1)$ . Note that  $v' \notin N(u)$  by definition of  $\tilde{C}_2$ . Let first  $v \in C_u \setminus \{u\}$ . Then, either  $v \in V_0(u)$  or  $v \in N(u)$  by Lemma 22. If  $v \in V_0(u)$ , then also  $v' \in V_0(u)$ , which is a contradiction by definition of  $\mathcal{B}_1$ . Suppose that  $v \in N(u)$ . Recall that  $v' \in V(\mathcal{B}_1) \subseteq V \setminus Q_u \setminus N[u] \setminus V_0(u)$  by our assumption on  $v'$  and by Observation 2. Thus, either  $P_u \ll_R P_{x_2} \ll_R P_{v'}$  or  $P_{v'} \ll_R P_x \ll_R P_u$  for every  $x \in X_1$  by Lemma 20. Therefore  $P_u \ll_R P_{x_2} \ll_R P_{v'}$ , since  $P_u \ll_R P_{v'}$  for every  $v' \in V(\mathcal{B}_1)$  by definition of  $\mathcal{B}_1$ . Then, since we assumed that  $v \in N(u)$  and  $v \in N(v')$ , it follows that  $P_v$  intersects  $P_{x_2}$  in  $R$ . Furthermore,  $x_2 \in C_2$  is a bounded vertex by Lemma 27;  $v$  is also a bounded vertex, since  $v \in N(u)$ . Therefore  $v \in N(x_2)$ , which is a contradiction by definition of  $C_u$ . Thus,  $N(v) \cap V(\mathcal{B}_1) = \emptyset$  for every  $v \in C_u \setminus \{u\}$ .

Let now  $v \in C_2$ . Then  $v \in V_0(u)$ , since  $C_2 \subseteq V_0(u)$  by Lemma 17, and thus also  $v' \in V_0(u)$ , since  $v' \notin N(u)$ . This which is a contradiction by definition of  $\mathcal{B}_1$ . Therefore,  $N(v) \cap V(\mathcal{B}_1) = \emptyset$  for every  $v \in C_2$ . Let finally  $v \in H \setminus \bigcup_{i=1}^{\infty} H_i \setminus \bigcup_{i=0}^{\infty} H'_i$ . Recall that  $v' \in V(\mathcal{B}_1) \subseteq V \setminus Q_u \setminus N[u] \setminus V_0(u)$ . Thus, since  $v \in H \setminus \bigcup_{i=1}^{\infty} H_i$ , and since  $vv' \in E$ , it follows by Definition 11 that  $v \in H'_0$ . This is a contradiction to the assumption that  $v \in H \setminus \bigcup_{i=1}^{\infty} H_i \setminus \bigcup_{i=0}^{\infty} H'_i$ . Therefore,  $N(v) \cap V(\mathcal{B}_1) = \emptyset$  for every  $v \in H \setminus \bigcup_{i=1}^{\infty} H_i \setminus \bigcup_{i=0}^{\infty} H'_i$ . That is,  $N(v) \cap V(\mathcal{B}_1) = \emptyset$  for every vertex  $v \in V(G_0) \setminus \{u\}$ .

Summarizing,  $N(X_1) \cup (\bigcup_{i=1}^{\infty} H_i \cup \bigcup_{i=0}^{\infty} H'_i) \subseteq N(v)$  and  $N(v) \cap V(\mathcal{B}_1) = \emptyset$  for every vertex  $v \in V(G_0) \setminus \{u\}$ . Therefore, it follows by (8) that

$$N(v) \setminus V(G_0) = N(X_1) \cup (\bigcup_{i=1}^{\infty} H_i \cup \bigcup_{i=0}^{\infty} H'_i) \quad (9)$$

for every vertex  $v \in V(G_0) \setminus \{u\}$ . Thus, in particular,  $G_0 \setminus \{u\}$  is a module in  $G \setminus \{u\}$ , since every vertex of  $G_0 \setminus \{u\}$  has the same neighbors in  $G \setminus G_0$ . This completes the proof of the lemma. ■

Now let  $G'_0 = G[V(G_0) \cup \{u^*\}]$ . Then, since  $u^* \in V_0(u)$  and  $V(G_0) \subseteq N[u] \cup V_0(u)$  by Observation 3, it follows that also  $V(G'_0) \subseteq N[u] \cup V_0(u)$ . Furthermore, Observation 3 implies that the set  $V(G'_0) \setminus \{u\}$  has only bounded vertices, since  $u^*$  is also bounded. Furthermore, since  $N_1(u) \neq N$  by Lemma 19 (where  $N = N(X_1)$ ), there exists at least one vertex  $q \in N \setminus N(u)$ , which is bounded by Lemma 27. Moreover  $q \in N(x_2)$ , since  $N = N(X_1) \subseteq N(x_2)$  by Lemma 19. Therefore,  $P_q$  intersects  $P_u$  in  $R$ , since  $q \in N(X_1) \cap N(x_2)$  and  $P_x \ll_R P_u \ll_R P_{x_2}$  for every  $x \in X_1$ . Furthermore,  $\phi_q < \phi_u$  in  $R$ , since otherwise  $q \in N(u)$ , which is a contradiction. Thus,  $N(u) \subseteq N(q)$  by Lemma 3, i.e.  $q$  is a covering vertex of  $u$ . Furthermore  $q \notin V(G_0)$ , since  $q \in N = N(X_1)$ . Then,  $q$  is adjacent to all vertices of  $C_2 \cup C_u \setminus \{u\}$  by Lemmas 23 and 24. Furthermore,  $q \in N$  is adjacent to all vertices of  $H \setminus \bigcup_{i=1}^{\infty} H_i \setminus \bigcup_{i=0}^{\infty} H'_i$  by Definition 9, since no vertex of  $H_1$  is included in  $H \setminus \bigcup_{i=1}^{\infty} H_i \setminus \bigcup_{i=0}^{\infty} H'_i$ . Summarizing,  $q$  is a bounded covering vertex of  $u$ ,  $P_q$  intersects  $P_u$  in  $R$ , and  $\phi_q < \phi_u$  in  $R$ , and thus we may assume w.l.o.g. that  $u^* = q$ , as the next observation states.

**Observation 4** *Without loss of generality, we may assume that  $u^* \in N = N(X_1)$ , i.e.  $u^* \notin V(G_0)$ , and that  $u^*$  is adjacent to every vertex of  $V(G_0) \setminus \{u\}$ ; thus, in particular,  $G'_0$  is connected.*

Moreover,  $G'_0 = G[V(G_0) \cup \{u^*\}]$  has strictly less vertices than  $G$ , since no vertex of  $X_1 \neq \emptyset$  is included in  $G'_0$ . We assume now that the following condition holds. Its correctness will be proved later, in Lemma 45.

**Condition 4** *Let  $G = (V, E)$  be a connected graph in  $\text{TOLERANCE} \cap \text{TRAPEZOID}$ ,  $R$  be a projection representation of  $G$  with  $u$  as the only unbounded vertex, such that  $V_0(u) \neq \emptyset$  is connected and  $V = N[u] \cup V_0(u)$ . Then, there exists a projection representation  $R^{**}$  of  $G$  with  $u$  as the only unbounded vertex, such that  $u$  has the right border property in  $R^{**}$ .*

## The projection representation $R_\ell$

We define now the line segment  $\ell$  with one endpoint  $a_\ell$  on  $L_1$  and the other endpoint  $b_\ell$  on  $L_2$  as follows. First recall that  $r(w) >_R r(u)$  by Lemma 1, since  $w \in N(u)$ . Let  $\Delta = r(w) - r(u) >_R 0$  be the distance on  $L_2$  between the lower right endpoints of  $P_w$  and  $P_u$  in  $R$ . Define in  $R$  the values  $a_\ell = \min\{L(x_2), L(u) + \Delta\}$  and  $b_\ell = r(w)$  as the endpoints of the line segment  $\ell$  on  $L_1$  and  $L_2$ , respectively. Note that  $\phi_\ell \geq \phi_u$  in  $R$ , where  $\phi_\ell$  denotes the slope of the line segment  $\ell$ . Recall that  $\phi_w > \phi_u$  in  $R$  (since  $w \in N(u)$ ), and thus in particular  $R(w) <_R L(u) + \Delta$ . Therefore, since  $P_u \ll_R P_{x_2}$  and  $P_w \ll_R P_{x_2}$ , it follows that the line segment  $\ell$  lies between  $P_u$  and  $P_{x_2}$  in  $R$ , as well as between  $P_w$  and  $P_{x_2}$  in  $R$ . Denote by  $a_u$  and  $b_u$  the upper and the lower endpoint of  $P_u$  in  $R$ , respectively. Then, always  $a_\ell > a_u$  and  $b_\ell > b_u$  by definition of the line segment  $\ell$ .

Note that  $G'_0$  satisfies the requirements of Condition 4. Thus, since we assumed that Condition 4 holds, there exists a representation  $R'_0$  of  $G'_0$  with  $u$  as the only unbounded vertex, where  $u$  has the right border property in  $R'_0$ . Let  $R''_0$  be the projection representation of  $G_0$  that is obtained if we remove from  $R'_0$  the parallelogram that corresponds to  $u^*$ . Let  $\varepsilon > 0$  be a sufficiently small positive number. Consider now the  $\varepsilon$ -squeezed projection representation  $R_0$  of  $G_0$  with respect to the line segment  $\ell$ , which is obtained from  $R''_0$ . Then, replace the parallelograms of the vertices of  $G_0$  in  $R$  by the projection representation  $R_0$ , and denote the resulting projection representation by  $R_\ell$ .

**Remark 1** Recall that w.l.o.g. all slopes of the parallelograms in the projection representation  $R$  are distinct [13, 15, 18]. Therefore, since  $\varepsilon > 0$  is assumed to be sufficiently small, we can assume w.l.o.g. that, for every vertex  $x \in V(G_0)$ , the slopes  $\phi_x$  are arbitrarily “close” to  $\phi_\ell$  (and to each other) in  $R_\ell$ . That is, we can assume w.l.o.g. that for every vertex  $v \notin V(G_0)$ , if  $\phi_v > \phi_\ell$  (resp.  $\phi_v < \phi_\ell$ ) in  $R_\ell$ , then also  $\phi_v > \phi_x$  (resp.  $\phi_v < \phi_x$ ) in  $R_\ell$  for every vertex  $x \in V(G_0)$ .

**Remark 2** Recall that the vertices of  $G_0$  in  $R_\ell$  lie on an  $\varepsilon$ -squeezed projection representation  $R_0$  with respect to the line segment  $\ell$ , where  $\varepsilon > 0$  is a sufficiently (very) small positive number. Therefore, in particular  $b_\ell - \varepsilon <_{R_\ell} l(v) \leq_{R_\ell} r(v) <_{R_\ell} b_\ell + \varepsilon$  and  $a_\ell - \varepsilon <_{R_\ell} L(v) \leq_{R_\ell} R(v) <_{R_\ell} a_\ell + \varepsilon$  for every vertex  $v \in V(G_0)$ . On the other hand, since  $\varepsilon$  has been chosen to be sufficiently small, we may assume w.l.o.g. that for every vertex  $z \notin V(G_0)$ , the lower right endpoint  $r(z)$  (resp. the lower left endpoint  $l(z)$ ) of  $P_z$  in  $R_\ell$  does not lie between  $b_\ell - \varepsilon$  and  $b_\ell + \varepsilon$ , i.e. either  $r(z) <_{R_\ell} b_\ell - \varepsilon$  or  $r(z) >_{R_\ell} b_\ell + \varepsilon$  (resp. either  $l(z) <_{R_\ell} b_\ell - \varepsilon$  or  $l(z) >_{R_\ell} b_\ell + \varepsilon$ ). Similarly, for every vertex  $z \notin V(G_0)$ , the upper right endpoint  $R(z)$  (resp. the upper left endpoint  $L(z)$ ) of  $P_z$  in  $R_\ell$  does not lie between  $a_\ell - \varepsilon$  and  $a_\ell + \varepsilon$ , i.e. either  $R(z) <_{R_\ell} a_\ell - \varepsilon$  or  $R(z) >_{R_\ell} a_\ell + \varepsilon$  (resp. either  $L(z) <_{R_\ell} a_\ell - \varepsilon$  or  $L(z) >_{R_\ell} a_\ell + \varepsilon$ ).

## Properties of $R_\ell$

**Lemma 36**  $R_\ell \setminus \{u\}$  is a projection representation of  $G \setminus \{u\}$ .

**Proof.** Recall that all vertices of  $G_0 \setminus \{u\}$  are bounded by Observation 3 and that  $N(v) \setminus V(G_0) = N(X_1) \cup \bigcup_{i=1}^\infty H_i \cup \bigcup_{i=0}^\infty H'_i$  for every vertex  $v \in V(G_0) \setminus \{u\}$  by Lemma 35. We will prove that for a vertex  $z \in V(G \setminus G_0)$  and a vertex  $v \in V(G_0) \setminus \{u\}$ ,  $z$  is adjacent to  $v$  in  $R_\ell$  if and only if  $z \in N(X_1) \cup \bigcup_{i=1}^\infty H_i \cup \bigcup_{i=0}^\infty H'_i$ .

Consider a vertex  $z \in N(X_1) \cup \bigcup_{i=1}^\infty H_i \cup \bigcup_{i=0}^\infty H'_i$ . Then  $z$  is a vertex of  $G \setminus G_0$  by definition of  $G_0$ . Furthermore,  $z$  is bounded by Lemma 27. If  $z \in \bigcup_{i=1}^\infty H_i \cup \bigcup_{i=0}^\infty H'_i$ , then  $z \in N(w) \cap N(x_2)$  by the definition of  $H$ . Let  $z \in N(X_1)$ . Then again  $z \in N(x_2)$ , since  $N_1(x_2) = N = N(X_1)$  by Lemma 19. Furthermore  $z \in N(w)$ , since  $N_1(w) = N(X_1)$  by Lemma 17. That is,  $z \in N(w) \cap N(x_2)$  for every case regarding  $z$ , and thus  $P_z$  intersects both  $P_w$  and  $P_{x_2}$  in  $R$ . Recall now by definition of the line segment  $\ell$  that  $\ell$  lies between  $P_w$  and  $P_{x_2}$  in  $R$ . Therefore, since  $P_z$  intersects both  $P_w$  and  $P_{x_2}$  in  $R$ , it follows that also  $P_z$  intersects  $\ell$  in  $R$ . Thus,  $z$  is adjacent in  $R_\ell$  to every vertex  $v \in V(G_0) \setminus \{u\}$ , since both  $z$  and  $v$  are bounded.

Conversely, consider a vertex  $z \in V(G \setminus G_0)$  and a vertex  $v \in V(G_0) \setminus \{u\}$ , such that  $z$  is adjacent to  $v$  in  $R_\ell$ . Then, in particular  $P_z$  intersects  $\ell$  in  $R$ . Recall that  $v$  is bounded by Observation 3.

Therefore, either  $z$  is bounded or  $z$  is unbounded and  $\phi_z < \phi_\ell$  (in both  $R$  and  $R_\ell$ ). Furthermore, observe that  $z \notin X_1$ , since  $P_x \ll_R P_u$  for every  $x \in X_1$ , and since  $P_z$  intersects  $\ell$  in  $R$ . Suppose that  $z \in V(\mathcal{B}_1)$ , and thus  $z \in V \setminus Q_u \setminus N[u] \setminus V_0(u)$  by Observation 2. Then, either  $P_u \ll_R P_{x_2} \ll_R P_z$  or  $P_z \ll_R P_x \ll_R P_u \ll_R P_{x_2}$  for every  $x \in X_1$  by Lemma 20. Thus,  $P_z$  does not intersect the line segment  $\ell$  in  $R$ , since  $\ell$  lies between  $P_u$  and  $P_{x_2}$  in  $R$  by definition of  $\ell$ , which is a contradiction. Thus,  $z \notin V(\mathcal{B}_1)$ .

Suppose first that  $z$  is bounded, and thus also  $z \notin Q_u$ . We will prove that  $z \in N(X_1) \cup \bigcup_{i=1}^{\infty} H_i \cup \bigcup_{i=0}^{\infty} H'_i$ . To this end, we distinguish the cases where  $z \in V_0(u)$ ,  $z \in N(u)$ , and  $z \in V \setminus N[u] \setminus V_0(u)$ . Recall by Lemma 16 that  $V(C_u \cup C_2 \cup H)$  induces a subgraph of  $G \setminus Q_u \setminus N[X_1] \setminus \mathcal{B}_1$  that includes all connected components of  $G \setminus Q_u \setminus N[X_1] \setminus \mathcal{B}_1$ , in which the vertices of  $S_2 \cup \{u\}$  belong. Let first  $z \in V \setminus N[u] \setminus V_0(u)$ , i.e.  $z \in V \setminus Q_u \setminus N[u] \setminus V_0(u)$ . Then either  $P_u \ll_R P_{x_2} \ll_R P_z$  or  $P_z \ll_R P_x \ll_R P_u \ll_R P_{x_2}$  for every  $x \in X_1$  by Lemma 20, and thus  $P_z$  does not intersect  $\ell$  in  $R$ , which is a contradiction. Let now  $z \in V_0(u)$ ; then  $z \in S_2$ , since  $P_z$  intersects  $\ell$  in  $R$  (i.e.  $P_z \not\ll_R P_u$ ). Then, since  $z \notin X_1 \cup Q_u \cup V(\mathcal{B}_1)$ , it follows that either  $z \in N(X_1)$  or  $z \in V(C_u \cup C_2 \cup H)$ . Therefore, since we assumed that  $z \notin V(G_0)$ , it follows that either  $z \in N(X_1)$  or  $z \in \bigcup_{i=1}^{\infty} H_i \cup \bigcup_{i=0}^{\infty} H'_i$ , i.e.  $z \in N(X_1) \cup \bigcup_{i=1}^{\infty} H_i \cup \bigcup_{i=0}^{\infty} H'_i$ . Let finally  $z \in N(u)$ . If  $z \notin N(X_1)$ , then  $z \in V(C_u \cup H)$  by the definition of  $H$  and by Lemma 17. That is, either  $z \in N(X_1)$  or  $z \in V(C_u \cup H)$ . Thus, since we assumed that  $z \notin V(G_0)$ , it follows again that either  $z \in N(X_1)$  or  $z \in \bigcup_{i=1}^{\infty} H_i \cup \bigcup_{i=0}^{\infty} H'_i$ , i.e.  $z \in N(X_1) \cup \bigcup_{i=1}^{\infty} H_i \cup \bigcup_{i=0}^{\infty} H'_i$ . Summarizing, if  $z$  is bounded, then  $z \in N(X_1) \cup \bigcup_{i=1}^{\infty} H_i \cup \bigcup_{i=0}^{\infty} H'_i$ .

Suppose now that  $z$  is unbounded and  $\phi_z < \phi_\ell$  (in both  $R$  and  $R_\ell$ ). Then,  $a_\ell <_R L(z)$  and  $l(z) <_R b_\ell$ . Recall that  $z \notin X_1$ ; furthermore also  $z \notin N(X_1)$ , since  $z$  is unbounded and every vertex of  $N = N(X_1)$  is bounded by Lemma 27. Therefore,  $z \notin N[X_1]$ . We distinguish now in the definition of the line segment  $\ell$ , the cases where  $a_\ell <_R L(x_2)$  and  $a_\ell =_R L(x_2)$  in  $R$ .

*Case 1.*  $a_\ell <_R L(x_2)$ . Then  $a_\ell =_R L(u) + \Delta$  in  $R$ , and thus  $\phi_\ell = \phi_u$  in  $R$  by definition of the line segment  $\ell$ . Therefore,  $\phi_z < \phi_\ell = \phi_u$  in  $R$  for some unbounded vertex  $z$ , since we assumed that  $\phi_z < \phi_\ell$  in  $R$ . This is a contradiction, since  $\phi_u = \min\{\phi_x \text{ in } R \mid x \in V_U\}$  by our initial assumption on  $u$ .

*Case 2.*  $a_\ell =_R L(x_2)$ . Recall that  $P_w \ll_R P_{x_2}$ . Then,  $R(w) <_R L(x_2) =_R a_\ell <_R L(z)$  and  $l(z) <_R b_\ell =_R r(w) <_R l(x_2)$ , since we assumed that  $\phi_z < \phi_\ell$ . Therefore,  $P_z$  intersects both  $P_w$  and  $P_{x_2}$  in  $R$ , while also  $\phi_z < \phi_w$  and  $\phi_z < \phi_{x_2}$  in  $R$ . Thus  $z \in N(w) \cap N(x_2)$ , since both  $w$  and  $x_2$  are bounded. Therefore, since also  $z \notin N[X_1]$ , it follows that  $z \in H$  by definition of  $H$ . If  $z \in H \setminus \bigcup_{i=1}^{\infty} H_i \setminus \bigcup_{i=0}^{\infty} H'_i$ , then  $z \in V(G_0)$ , which is a contradiction. Therefore,  $z \in \bigcup_{i=1}^{\infty} H_i \setminus \bigcup_{i=0}^{\infty} H'_i$ .

Summarizing, if  $z$  is adjacent to  $v$  in  $R_\ell$  for a vertex  $z \in V(G \setminus G_0)$  and a vertex  $v \in V(G_0) \setminus \{u\}$ , then  $z \in N(X_1) \cup \bigcup_{i=1}^{\infty} H_i \cup \bigcup_{i=0}^{\infty} H'_i$ . This completes the proof of the lemma. ■

**Corollary 4** *For every  $z \in N(u)$ ,  $P_z$  intersects  $P_u$  in  $R_\ell$ .*

**Proof.** If  $z \in V(G_0)$ , then  $P_z$  intersects  $P_u$  in  $R_0$ , since  $R_0$  is a projection representation of  $G_0$ . Therefore,  $P_z$  intersects  $P_u$  also in  $R_\ell$ , since  $R_0$  is a sub-representation of  $R_\ell$ . Suppose now that  $z \notin V(G_0)$ . Then, either  $z \in N(X_1)$  or  $z \in V(C_u \cup H)$ , since we assumed that  $z \in N(u)$ . Thus, either  $z \in N(X_1)$  or  $z \in \bigcup_{i=1}^{\infty} H_i \cup \bigcup_{i=0}^{\infty} H'_i$ , since  $z \notin V(G_0)$ , and thus  $z$  is adjacent to every vertex  $v$  of  $G_0 \setminus \{u\}$  by Lemma 35. Therefore,  $P_z$  intersects the line segment  $\ell$  in both  $R$  and  $R_\ell$  (cf. the proof of Lemma 36), and thus in particular  $P_z$  intersects also  $P_u$  in  $R_\ell$ . ■

Note that, since the position and the slope of  $P_u$  is not the same in  $R$  and in  $R_\ell$ , the projection representation  $R_\ell$  may be *not* a projection representation of  $G$ . Similarly to the Transformations 1, 2, and 3 in the proof of Theorem 1, we define in the sequel the Transformations 4, 5, and 6. After applying these transformations to  $R_\ell$ , we obtain eventually a projection representation  $R^*$  of  $G$  with  $k - 1$  unbounded vertices. The following lemma will be mainly used in the remaining part of the proof of Theorem 2.

**Lemma 37**  *$u$  has the right border property in  $R_\ell$ .*

**Proof.** Recall first that  $u$  has the right border property in  $R_0$ . Suppose for the sake of contradiction that  $u$  has not the right border property in  $R_\ell$ . Then, there exist vertices  $z \in N(u)$  and  $y \in V_0(u)$ , such that  $P_z \ll_{R_\ell} P_y$ . We will now prove that  $b_u <_{R_\ell} r(z)$  for the lower right endpoint  $r(z)$  of every  $z \in N(u)$ . If  $z \in V(G_0)$ , then clearly  $b_u <_{R_\ell} r(z)$ , since  $b_u < b_\ell$  and  $R_0$  is an  $\varepsilon$ -squeezed projection representation of  $G_0$  with respect to  $\ell$ , where  $\varepsilon > 0$  is sufficiently small. If  $z \notin V(G_0)$ , then  $b_u = r(u) <_R r(z)$  in  $R$  by Lemma 1, and thus also  $b_u <_{R_\ell} r(z)$ , since the endpoints of  $P_z$  remain the same in both  $R$  and  $R_\ell$ . That is,  $b_u <_{R_\ell} r(z)$  for every  $z \in N(u)$ .

*Case 1.* Let first  $z \in V(G_0)$ . Then,  $y \notin V(G_0)$ , since  $u$  has the right border property in  $R_0$ . Furthermore  $b_u <_{R_\ell} r(z) <_{R_\ell} r(y)$ , since  $P_z \ll_{R_\ell} P_y$ . Therefore, since  $y \notin V(G_0)$ , i.e. since the endpoints of  $P_y$  remain the same in both  $R$  and  $R_\ell$ , it follows that also  $b_u <_R r(y)$ . Thus  $y \in S_2$ , since we assumed that  $y \in V_0(u)$ ; therefore in particular  $y \notin X_1$ , since  $X_1 \subseteq D_1$  by Lemma 17. Furthermore,  $y \notin Q_u$  by Lemma 13 and  $y \notin V(\mathcal{B}_1)$  by definition of  $\mathcal{B}_1$ , since  $y \in V_0(u)$ . Recall now by Lemma 16 that  $V(C_u \cup C_2 \cup H)$  induces a subgraph of  $G \setminus Q_u \setminus N[X_1] \setminus \mathcal{B}_1$  that includes all connected components of  $G \setminus Q_u \setminus N[X_1] \setminus \mathcal{B}_1$ , in which the vertices of  $S_2 \cup \{u\}$  belong. Therefore, since  $y \in S_2$  and  $y \notin Q_u \cup X_1 \cup V(\mathcal{B}_1)$ , it follows that  $y \in N(X_1) \cup V(C_u \cup C_2 \cup H)$ . Thus  $y \in N(X_1) \cup \bigcup_{i=1}^\infty H_i \cup \bigcup_{i=0}^\infty H'_i$ , since otherwise  $y \in V(G_0)$ , which is a contradiction. Therefore,  $y$  is adjacent to every vertex  $v \in V(G_0) \setminus \{u\}$  by Lemma 35. Thus, in particular,  $P_y$  intersects  $P_z$  in  $R_\ell$ , since  $z \in V(G_0) \setminus \{u\}$  and  $R_\ell \setminus \{u\}$  is a projection representation of  $G \setminus \{u\}$  by Lemma 36. This is a contradiction, since we assumed that  $P_z \ll_{R_\ell} P_y$ .

*Case 2.* Let now  $z \notin V(G_0)$ . Since we assumed that  $z \in N(u)$ , it follows that either  $z \in N(X_1)$  or  $z \in V(C_u \cup H)$ . Therefore, either  $z \in N(X_1)$  or  $z \in \bigcup_{i=1}^\infty H_i \cup \bigcup_{i=0}^\infty H'_i$ , since  $z \notin V(G_0)$ , and thus  $z$  is adjacent to every vertex  $v \in V(G_0) \setminus \{u\}$  by Lemma 35. Then, in particular,  $P_z$  intersects  $P_v$  in  $R_\ell$ , for every vertex  $v \in V(G_0) \setminus \{u\}$ , and thus  $y \notin V(G_0)$ , since we assumed that  $P_z \ll_{R_\ell} P_y$ . Therefore, since both  $y, z \notin V(G_0)$  and  $P_z \ll_{R_\ell} P_y$ , it follows that also  $P_z \ll_R P_y$ , and thus in particular  $b_u <_R r(z) <_R r(y)$  by Lemma 1. Thus  $y \in S_2$ , since we assumed that  $y \in V_0(u)$ ; therefore in particular  $y \notin X_1$ , since  $X_1 \subseteq D_1$  by Lemma 17. Furthermore,  $y \notin Q_u$  by Lemma 13 and  $y \notin V(\mathcal{B}_1)$  by definition of  $\mathcal{B}_1$ , since  $y \in V_0(u)$ . Therefore, since  $y \in S_2$  and  $y \notin Q_u \cup X_1 \cup V(\mathcal{B}_1)$ , it follows (similarly to the previous paragraph) that  $y \in N(X_1) \cup V(C_u \cup C_2 \cup H)$ . Thus  $y \in N(X_1) \cup \bigcup_{i=1}^\infty H_i \cup \bigcup_{i=0}^\infty H'_i$ , since otherwise  $y \in V(G_0)$ , which is a contradiction.

Suppose that  $y \in N(X_1)$ , i.e.  $y \in N(x)$  for some  $x \in X_1$ . Recall that  $P_x \ll_R P_u$ , since  $X_1 \subseteq D_1$  by Lemma 17. If  $P_u \ll_R P_y$ , then  $P_x \ll_R P_u \ll_R P_y$ , i.e.  $y \notin N(x)$ , which is a contradiction. Thus  $P_u \not\ll_R P_y$ , i.e. either  $P_y$  intersects  $P_u$  in  $R$  or  $P_y \ll_R P_u$ . Suppose that  $P_y$  intersects  $P_u$  in  $R$ , and thus either  $N(y) \subseteq N(u)$  or  $N(u) \subseteq N(y)$  by Lemma 3, since  $y \notin N(u)$ . If  $N(y) \subseteq N(u)$ , then  $x \in N(u)$ , where  $x \in X_1$ , which is a contradiction. If  $N(u) \subseteq N(y)$ , then  $z \in N(y)$ , which is a contradiction, since we assumed that  $P_z \ll_{R_\ell} P_y$ . Therefore,  $P_y$  does not intersect  $P_u$  in  $R$ , and thus  $P_y \ll_R P_u$ , i.e.  $P_z \ll_R P_y \ll_R P_u$ . Then  $z \notin N(u)$ , which is a contradiction. Therefore,  $y \notin N(X_1)$ , and thus  $y \in \bigcup_{i=1}^\infty H_i \cup \bigcup_{i=0}^\infty H'_i$ . On the other hand  $y \notin \bigcup_{i=0}^\infty H'_i$ , since otherwise  $y \in N(u)$  by Lemma 32, which is a contradiction. Thus  $y \in \bigcup_{i=1}^\infty H_i$ . Summarizing,  $z \notin V(G_0)$  and  $y = v_i \in H_i$  for some  $i \geq 1$ .

We will now prove by induction on  $i$  that  $v_i \in N(u)$  or  $P_z \not\ll_R P_{v_i}$ , for every vertex  $v_i \in H_i$ ,  $i \geq 1$ . This then completes the proof of the lemma, since  $v_i = y \notin N(u)$  (by the assumption that  $y \in V_0(u)$ ), and thus  $P_z \not\ll_R P_{v_i} = P_y$ , which is a contradiction (since we assumed that  $P_z \ll_{R_\ell} P_y$ , and thus also  $P_z \ll_R P_y$ ).

For the sake of contradiction, suppose that  $v_i \notin N(u)$  and  $P_z \ll_R P_{v_i}$  for some  $i \geq 1$ . Then, note that  $z \notin N(v_i)$ . Recall that  $v_i \in N(x_2)$  due to the definition of  $H$ , and since  $v_i \in H$ . Therefore, since  $v_i \notin N(u)$  and  $x_2 \in V_0(u)$ , it follows that  $v_i \in V_0(u)$ , and thus  $T_{v_i} \ll_{R_T} T_u$  in the trapezoid representation  $R_T$ . Therefore, also  $T_{v_i} \ll_{R_T} T_z$ , since  $z \in N(u) \setminus N(v_i)$ . Recall now that  $T_x \ll_{R_T} T_{x_2}$  for every  $x \in X_1$  by Lemma 18. Thus, since  $v_i \in N(x_2)$  and  $v_i \notin N(X_1)$  by definition of  $H$ , it follows that  $T_x \ll_{R_T} T_{v_i}$  for every  $x \in X_1$ , i.e.  $T_x \ll_{R_T} T_{v_i} \ll_{R_T} T_z$  for every  $x \in X_1$ . Thus, in particular,  $z \notin N(X_1)$ .

For the induction basis, let  $i = 1$ . Suppose that  $N_1(z) = N$ . Then, for every  $v \in N$ ,  $T_v$  intersects  $T_{v_1}$  in  $R_T$ , i.e.  $v \in N(v_1)$ , since  $v \in N(X_1) \cap N(z)$  and  $T_x \ll_{R_T} T_{v_1} \ll_{R_T} T_z$  for every

$x \in X_1$ . Thus,  $N_1(v_1) = N$ , i.e.  $N = H_0 \subseteq N(v_1)$ , which is a contradiction by Definition 9, since  $v_1 \in H_1$ .

Therefore  $N_1(z) \neq N$ , and thus there exists a vertex  $v \in N \setminus N(z)$ , i.e.  $v \in N(x) \setminus N(z)$  for some  $x \in X_1$ . Then  $v \in N(x_2)$ , since  $N_1(x_2) = N = N(X_1)$  by Lemma 19. Thus, since  $v \in N(x) \cap N(x_2)$  and  $P_x \ll_R P_u \ll_R P_{x_2}$ , it follows that  $P_v$  intersects  $P_u$  in  $R$ . If  $v \notin N(u)$ , then either  $N(v) \subseteq N(u)$  or  $N(u) \subseteq N(v)$  by Lemma 3. If  $N(v) \subseteq N(u)$ , then  $x_2 \in N(u)$ , which is a contradiction. If  $N(u) \subseteq N(v)$ , then  $z \in N(v)$ , which is again a contradiction. Therefore,  $v \in N(u)$  for all vertices  $v \in N \setminus N(z)$ .

Consider now the trapezoid representation  $R_T$ . Recall that  $T_x \ll_{R_T} T_{v_1} \ll_{R_T} T_u$  and  $T_x \ll_{R_T} T_{v_1} \ll_{R_T} T_z$  for every  $x \in X_1$ . Consider an arbitrary vertex  $v \in N = N(X_1)$ . If  $v \in N(z)$ , then  $T_v$  intersects  $T_{v_1}$  in  $R_T$ , since  $v \in N(X_1) \cap N(z)$  and  $T_x \ll_{R_T} T_{v_1} \ll_{R_T} T_z$  for every  $x \in X_1$ ; therefore  $v \in N(v_1)$ . Otherwise, if  $v \notin N(z)$ , then  $v \in N(u)$ , as we proved in the previous paragraph. Then,  $T_v$  intersects  $T_{v_1}$  in  $R_T$ , since  $v \in N(X_1) \cap N(u)$  and  $T_x \ll_{R_T} T_{v_1} \ll_{R_T} T_u$  for every  $x \in X_1$ ; therefore again  $v \in N(v_1)$ . Thus,  $v \in N(v_1)$  for every  $v \in N$ , i.e.  $N = H_0 \subseteq N(v_1)$ , which is a contradiction by Definition 9, since  $v_1 \in H_1$ . Therefore,  $v_1 \in N(u)$  or  $P_z \not\ll_R P_{v_1}$  for every vertex  $v_1 \in H_1$ . This proves the induction basis.

For the induction step, let  $i \geq 2$ . Let  $(v_0, v_1, \dots, v_{i-2}, v_{i-1}, v_i)$  be an  $H_i$ -chain of  $v_i$ . By the induction hypothesis,  $v_{i-1} \in N(u)$  or  $P_z \not\ll_R P_{v_{i-1}}$ . Recall that  $T_{v_i} \ll_{R_T} T_z$ , as we proved above. Assume that  $z \in N(v_{i-1})$ . Then, since  $z \in N(v_{i-1}) \setminus N(v_i)$  and  $v_{i-2} \in N(v_i) \setminus N(v_{i-1})$ ,  $P_{v_i}$  does not intersect  $P_{v_{i-1}}$  in  $R$  by Lemma 3. Suppose first that  $i$  is even. Then,  $P_{v_{i-2}} \ll_R P_{v_{i-1}}$  by Lemmas 25 and 26. Thus, since  $v_i \in N(v_{i-2})$  and  $P_{v_i}$  does not intersect  $P_{v_{i-1}}$  in  $R$ , it follows that  $P_{v_i} \ll_R P_{v_{i-1}}$ . Then, since we assumed that  $P_z \ll_R P_{v_i}$ , it follows that  $P_z \ll_R P_{v_i} \ll_R P_{v_{i-1}}$ , i.e.  $z \notin N(v_{i-1})$ . This is a contradiction to the assumption that  $z \in N(v_{i-1})$ . Suppose now that  $i$  is odd, i.e.  $i \geq 3$ . Then,  $T_{v_{i-1}} \ll_{R_T} T_{v_{i-2}}$  by Lemma 26. Thus, since  $v_i \in N(v_{i-2}) \setminus N(v_{i-1})$ , it follows that  $T_{v_{i-1}} \ll_{R_T} T_{v_i}$ . Then, since  $T_{v_i} \ll_{R_T} T_z$ , it follows that  $T_{v_{i-1}} \ll_{R_T} T_{v_i} \ll_{R_T} T_z$ , i.e.  $z \notin N(v_{i-1})$ . This is again a contradiction to the assumption that  $z \in N(v_{i-1})$ .

Therefore  $z \notin N(v_{i-1})$ . Recall that  $v_{i-1}$  is a bounded vertex by Lemma 27. Furthermore,  $z$  is a bounded vertex, since  $z \in N(u)$ . Therefore, since  $z \notin N(v_{i-1})$ , it follows that  $P_{v_{i-1}}$  does not intersect  $P_z$  in  $R$ , i.e. either  $P_{v_{i-1}} \ll_R P_z$  or  $P_z \ll_R P_{v_{i-1}}$ .

*Case 2a.*  $P_{v_{i-1}} \ll_R P_z$ . Then, since  $z \in N(u)$  and  $P_u \ll_R P_{x_2}$ , it follows by Lemma 1 that  $R(v_{i-1}) <_R L(z) <_R L(u) <_R L(x_2)$ , i.e.  $R(v_{i-1}) <_R L(x_2)$ . Thus, since  $v_{i-1} \in N(x_2)$  and  $P_u \ll_R P_{x_2}$ , it follows that  $r(u) <_R l(x_2) <_R r(v_{i-1})$ . That is,  $R(v_{i-1}) <_R L(u) = R(u)$  and  $r(u) <_R r(v_{i-1})$ , i.e.  $P_{v_{i-1}}$  intersects  $P_u$  in  $R$  and  $\phi_{v_{i-1}} > \phi_u$ . If  $v_{i-1} \notin N(u)$ , then  $N(v_{i-1}) \subseteq N(u)$  by Lemma 3, and thus  $x_2 \in N(u)$ , which is a contradiction. Thus,  $v_{i-1} \in N(u)$ .

Since  $P_{v_{i-1}} \ll_R P_z$  and  $P_z \ll_R P_{v_i}$  by assumption, it follows that  $P_{v_{i-1}} \ll_R P_{v_i}$ . Recall by Lemmas 25 and 26 that either  $P_{v_{i-2}} \ll_R P_{v_{i-1}}$  or  $P_{v_{i-1}} \ll_R P_{v_{i-2}}$ . If  $P_{v_{i-2}} \ll_R P_{v_{i-1}}$ , then  $P_{v_{i-2}} \ll_R P_{v_{i-1}} \ll_R P_{v_i}$ , i.e.  $v_{i-2}v_i \notin E$ , which is a contradiction. Therefore,  $P_{v_{i-1}} \ll_R P_{v_{i-2}}$  and  $i$  is odd, and thus  $T_{v_{i-1}} \ll_{R_T} T_{v_{i-2}}$  by Lemmas 25 and 26. Thus, since  $v_i \in N(v_{i-2}) \setminus N(v_{i-1})$ , it follows that also  $T_{v_{i-1}} \ll_{R_T} T_{v_i}$ . Recall now that  $T_{v_i} \ll_{R_T} T_u$ , as we proved above. Therefore, it follows that  $T_{v_{i-1}} \ll_{R_T} T_{v_i} \ll_{R_T} T_u$ , and thus  $v_{i-1} \notin N(u)$ , which is a contradiction by the previous paragraph.

*Case 2b.*  $P_z \ll_R P_{v_{i-1}}$ . Then,  $v_{i-1} \in N(u)$  by the induction hypothesis, and thus  $v_{i-1}$  is bounded. Furthermore,  $v_i$  is also bounded by Lemma 27, since  $v_i \in H$ . Therefore,  $P_{v_i}$  does not intersect  $P_{v_{i-1}}$  in  $R$ , since  $v_{i-1}v_i \notin E$ , and thus either  $P_{v_i} \ll_R P_{v_{i-1}}$  or  $P_{v_{i-1}} \ll_R P_{v_i}$ . Recall that  $v_i \notin N(u)$  and  $P_z \ll_R P_{v_i}$  by assumption. Suppose first that  $P_{v_i} \ll_R P_{v_{i-1}}$ , that is,  $P_z \ll_R P_{v_i} \ll_R P_{v_{i-1}}$ . Then, since  $z \in N(u)$  and  $v_{i-1} \in N(u)$ , it follows that  $P_u$  intersects  $P_{v_i}$  in  $R$ . Since  $v_i \notin N(u)$ , either  $N(v_i) \subseteq N(u)$  or  $N(u) \subseteq N(v_i)$  by Lemma 3. If  $N(v_i) \subseteq N(u)$ , then  $x_2 \in N(u)$ , which is a contradiction. If  $N(u) \subseteq N(v_i)$ , then  $v_{i-1} \in N(v_i)$ , which is again a contradiction.

Suppose now that  $P_{v_{i-1}} \ll_R P_{v_i}$ . Recall by Lemmas 25 and 26 that either  $P_{v_{i-2}} \ll_R P_{v_{i-1}}$  or  $P_{v_{i-1}} \ll_R P_{v_{i-2}}$ . If  $P_{v_{i-2}} \ll_R P_{v_{i-1}}$ , then  $P_{v_{i-2}} \ll_R P_{v_{i-1}} \ll_R P_{v_i}$ , i.e.  $v_{i-2}v_i \notin E$ , which is a contradiction. Therefore,  $P_{v_{i-1}} \ll_R P_{v_{i-2}}$  and  $i$  is odd, and thus  $T_{v_{i-1}} \ll_{R_T} T_{v_{i-2}}$  by Lemmas 25 and 26. Thus, since  $v_i \in N(v_{i-2}) \setminus N(v_{i-1})$ , it follows that also  $T_{v_{i-1}} \ll_{R_T} T_{v_i}$ . Recall now that

$T_{v_i} \ll_{R_T} T_u$ , as we proved above. Therefore,  $T_{v_{i-1}} \ll_{R_T} T_{v_i} \ll_{R_T} T_u$ , and thus  $v_{i-1} \notin N(u)$ , which is a contradiction. This completes the induction step and the lemma follows. ■

### The projection representations $R'_\ell$ , $R''_\ell$ , and $R'''_\ell$

**Notation 1** In the following, whenever we refer to  $N(u)$ , we will mean  $N_G(u)$ , i.e. the neighborhood set of vertex  $u$  in  $G$ . Note that, since  $R_\ell$  may be not a projection representation of  $G$  (although  $R_\ell \setminus \{u\}$  is a projection representation of  $G \setminus \{u\}$  by Lemma 36), the set  $N_G(u)$  does not coincide necessarily with the set of adjacent vertices of  $u$  in the graph induced by  $R_\ell$ .

Similarly to the proof of Theorem 1, we add to  $G$  an isolated bounded vertex  $t$ . This isolated vertex  $t$  corresponds to a parallelogram  $P_t$ , such that  $P_v \ll_R P_t$  and  $P_v \ll_{R_\ell} P_t$  for every other vertex  $v$  of  $G$ . Denote by  $V_B$  and  $V_U$  the set of bounded and unbounded vertices of  $G$  in  $R_\ell$ , after the addition of the auxiliary vertex  $t$  to  $G$  (note that  $t \in V_B$ ).

Now, we define for every  $z \in N(u)$  the value  $L_0(z) = \min_{R_\ell} \{L(x) \mid x \in V_B \setminus N(u), P_z \ll_{R_\ell} P_x\}$ . For every vertex  $x \in V_B \setminus N(u)$ , such that  $P_z \ll_{R_\ell} P_x$  for some  $z \in N(u)$ , it follows that  $x \notin V_0(u)$ , since  $u$  has the right border property in  $R_\ell$  by Lemma 37. Thus, for every  $z \in N(u)$ ,  $L_0(z) = \min_{R_\ell} \{L(x) \mid x \in V_B \setminus N(u) \setminus V_0(u), P_z \ll_{R_\ell} P_x\}$ . Note that the value  $L_0(z)$  is well defined for every  $z \in N(u)$ , since in particular  $t \in V_B \setminus N(u)$  and  $P_z \ll_{R_\ell} P_t$ . Furthermore, note that for every every  $z \in N(u)$ , the endpoint  $L_0(z)$  does not correspond to any vertex of  $G_0$ , since  $V(G_0) \subseteq N[u] \cup V_0(u)$  by Observation 3. Define now the value  $\ell_0 = \max_{R_\ell} \{l(x) \mid x \in V_0(u)\}$  and the subset  $N_1 = \{z \in N(u) \mid r(z) <_{R_\ell} \ell_0\}$  of neighbors of  $u$  (in  $G$ , and not in  $R_\ell$ ). Similarly to Transformation 1 in the proof of Theorem 1, we construct now the projection representation  $R'_\ell$  from  $R_\ell$  as follows.

**Transformation 4** For every  $z \in N_1$ , move the right line of  $P_z$  parallel to the right, until either  $r(z)$  comes immediately after  $\ell_0$  on  $L_2$ , or  $R(z)$  comes immediately before  $L_0(z)$  on  $L_1$ . Denote the resulting projection representation by  $R'_\ell$ .

**Remark 3** Suppose now that the endpoint  $\ell_0$  corresponds to a vertex of  $V(G_0)$ , i.e.  $b_\ell - \varepsilon <_{R_\ell} \ell_0 <_{R_\ell} b_\ell + \varepsilon$  by Remark 2. Then, since  $\varepsilon$  has been chosen to be sufficiently small, we make w.l.o.g. the following convention in the statement of Transformation 4: for every vertex  $z \in N_1$ , such that  $z \notin V(G_0)$ , either  $r(z) <_{R'_\ell} b_\ell - \varepsilon$  (in the case where  $r(z) <_{R'_\ell} \ell_0$ ) or  $r(z)$  comes immediately after  $b_\ell + \varepsilon$  on  $L_2$ , i.e.  $r(z) >_{R'_\ell} b_\ell + \varepsilon$  (in the case where  $r(z) >_{R'_\ell} \ell_0$ ). Summarizing, similarly to  $R_\ell$ , we may assume in  $R'_\ell$  w.l.o.g. that for every vertex  $z \in N(u)$ , such that  $z \notin V(G_0)$ , either  $r(z) <_{R'_\ell} b_\ell - \varepsilon$  or  $r(z) >_{R'_\ell} b_\ell + \varepsilon$ .

Note that the left lines of all parallelograms do not move during Transformation 4. Thus, in particular, the value of  $\ell_0$  is the same in  $R_\ell$  and in  $R'_\ell$ , i.e.  $\ell_0 = \max_{R'_\ell} \{l(x) \mid x \in V_0(u)\}$ . As we will prove in Lemma 40, the representation  $R'_\ell \setminus \{u\}$  is a projection representation of the graph  $G \setminus \{u\}$ , and thus the parallelograms of two bounded vertices intersect in  $R_\ell$  if and only if they intersect also in  $R'_\ell$ . Therefore, for every  $z \in N(u)$ , the value  $L_0(z)$  remains the same in  $R_\ell$  and in  $R'_\ell$ , i.e.  $L_0(z) = \min_{R'_\ell} \{L(x) \mid x \in V_B \setminus N(u) \setminus V_0(u), P_z \ll_{R'_\ell} P_x\}$  for every  $z \in N(u)$ . Similarly to the proof of Theorem 1, we define now the subset  $N_2 = \{z \in N(u) \mid \ell_0 <_{R'_\ell} r(z)\}$  of neighbors of  $u$ . Since the lower right endpoint  $r(z)$  of all parallelograms  $P_z$  in  $R'_\ell$  is greater than or equal to the corresponding value  $r(z)$  in  $R_\ell$ , it follows that  $N(u) \setminus N_1 = \{z \in N(u) \mid \ell_0 <_{R_\ell} r(z)\} \subseteq \{z \in N(u) \mid \ell_0 <_{R'_\ell} r(z)\} = N_2$ . Thus,  $N(u) \setminus N_2 \subseteq N_1$  and  $N_2 \cup (N_1 \setminus N_2) = N(u)$ . If  $N_2 \neq \emptyset$ , we define the value  $r_0 = \min_{R'_\ell} \{r(z) \mid z \in N_2\}$ .

**Lemma 38** If  $N_2 \neq \emptyset$ , i.e. if the value  $r_0$  can be defined, then  $r(u) <_{R'_\ell} r_0$ .

**Proof.** Denote by  $z_0$  the vertex of  $N_2$ , such that  $r_0 = r(z_0)$ . Let first  $z_0 \in V(G_0)$ . Then  $r(z_0) >_{R_0} r(u)$  by Lemma 1, since  $N_2 \subseteq N(u)$ , and since  $R_0$  is a projection representation of  $G_0$ . Thus, also  $r(z_0) >_{R_\ell} r(u)$ , since  $R_0$  is a sub-representation of  $R_\ell$ . Furthermore,  $r_0 = r(z_0) >_{R'_\ell} r(u)$ ,

since the lower right endpoints  $r(z)$  do not decrease by Transformation 4. Let now  $z_0 \notin V(G_0)$ . Then, either  $r(z_0) <_{R'_\ell} b_\ell - \varepsilon$  or  $r(z_0) >_{R'_\ell} b_\ell + \varepsilon$  by Remark 3. Recall that  $x_2 \in V(G_0)$ , and thus  $b_\ell - \varepsilon <_{R_\ell} l(x_2) <_{R_\ell} b_\ell + \varepsilon$  by Remark 2. Thus, since also  $x_2 \in V_0(u)$ , it follows by definition of  $\ell_0$  that  $b_\ell - \varepsilon <_{R_\ell} l(x_2) \leq_{R_\ell} \ell_0$ . Therefore  $b_\ell - \varepsilon <_{R'_\ell} \ell_0 <_{R'_\ell} r(z_0)$ , since  $z_0 \in N_2$ . Thus  $r(z_0) >_{R'_\ell} b_\ell + \varepsilon$  by Remark 3 (since  $z_0 \notin V(G_0)$ ), i.e.  $r(z_0) >_{R'_\ell} b_\ell + \varepsilon >_{R'_\ell} r(u)$ . Summarizing,  $r_0 = r(z_0) >_{R'_\ell} r(u)$  in all cases. ■

Define now the value  $L_0 = \min_{R_\ell} \{L(x) \mid x \in V_B \setminus N(u) \setminus V_0(u), P_u \ll_{R_\ell} P_x\}$ ; again,  $L_0$  is well defined, since in particular  $t \in V_B \setminus N(u) \setminus V_0(u)$  and  $P_u \ll_{R_\ell} P_t$ . Then, since by Transformation 4 only some endpoints of vertices  $z \in N(u)$  are moved, it follows that the value  $L_0$  does not change in  $R'_\ell$ , i.e.  $L_0 = \min_{R'_\ell} \{L(x) \mid x \in V_B \setminus N(u) \setminus V_0(u), P_u \ll_{R'_\ell} P_x\}$ . The following property of the projection representation  $R'_\ell$  can be obtained easily by Transformation 4.

**Lemma 39** *For all vertices  $z \in N_1 \setminus N_2$ , for which  $R(z) <_{R'_\ell} L_0$ , the values  $R(z)$  lie immediately before  $L_0$  in  $R'_\ell$ .*

**Proof.** Let  $z \in N_1 \setminus N_2$ . By definition of the sets  $N_1$  and  $N_2$ , it follows that  $r(z) <_{R_\ell} \ell_0$  and  $r(z) <_{R'_\ell} \ell_0$  in both  $R_\ell$  and  $R'_\ell$ . Thus,  $R(z)$  comes immediately before  $L_0(z)$  in  $R'_\ell$  during Transformation 4. We will now prove that  $L_0 \leq_{R_\ell} L_0(z)$ . Consider a vertex  $x \in V_B \setminus N(u) \setminus V_0(u)$ , such that  $P_z \ll_{R_\ell} P_x$ , i.e.  $r(z) <_{R_\ell} l(x)$  and  $R(z) <_{R_\ell} L(x)$ . Then, in particular  $x \notin V(G_0)$ , since  $x \notin N(u) \cup V_0(u)$  and  $V(G_0) \subseteq N[u] \cup V_0(u)$  by Observation 3. Suppose that  $P_x$  intersects  $P_u$  in  $R_\ell$ , i.e.  $P_x$  intersects the line segment  $\ell$  in  $R_\ell$ . Then, in particular  $P_x$  intersects also  $P_{x_2}$  in  $R_\ell$ , since  $x_2 \in V(G_0)$ , and thus  $x \in N(x_2)$ , since both  $x$  and  $x_2$  are bounded in  $R_\ell$ . Therefore  $x \in V_0(u)$ , since  $x_2 \in V_0(u)$  and  $x \notin N(u)$ , which is a contradiction. Thus,  $P_x$  does not intersect  $P_u$  in  $R_\ell$ , i.e. either  $P_x \ll_{R_\ell} P_u$  or  $P_u \ll_{R_\ell} P_x$ . If  $P_x \ll_{R_\ell} P_u$ , then  $P_z \ll_{R_\ell} P_x \ll_{R_\ell} P_u$ , which is a contradiction, since  $P_z$  intersects  $P_u$  in  $R_\ell$  by Corollary 4. Therefore,  $P_u \ll_{R_\ell} P_x$ . That is, for every  $x \in V_B \setminus N(u) \setminus V_0(u)$ , for which  $P_z \ll_{R_\ell} P_x$ , it follows that also  $P_u \ll_{R_\ell} P_x$ . Thus, it follows by the definitions of  $L_0$  and of  $L_0(z)$  that  $L_0 \leq_{R_\ell} L_0(z)$ .

Furthermore, also  $L_0 \leq_{R'_\ell} L_0(z)$  in  $R'_\ell$ , since by Transformation 4 only some endpoints of vertices  $z \in N(u)$  are moved. Therefore, since  $R(z)$  comes immediately before  $L_0(z)$  in  $R'_\ell$  during Transformation 4, it follows that either  $R(z)$  comes immediately before  $L_0$  in  $R'_\ell$  during Transformation 4 (in the case where  $L_0 =_{R'_\ell} L_0(z)$ ) or  $R(z) >_{R'_\ell} L_0$  (in the case where  $L_0 <_{R'_\ell} L_0(z)$ ). ■

If  $N_2 = \emptyset$ , then we set  $R''_\ell = R'_\ell$ ; otherwise, if  $N_2 \neq \emptyset$ , we construct the projection representation  $R''_\ell$  from  $R'_\ell$  as follows.

**Transformation 5** *For every  $v \in V_0(u) \cap V_B$ , such that  $r(v) >_{R'_\ell} r_0$ , we move the right line of  $P_v$  in  $R'_\ell$  to the left, such that  $r(v)$  comes immediately before  $r_0$  in  $L_2$ . Denote the resulting projection representation by  $R''_\ell$ .*

Since by Transformation 5 only some endpoints of vertices  $v \in V_0(u) \cap V_B$  are moved, it follows that the value  $L_0$  does not change in  $R''_\ell$ , i.e.  $L_0 = \min_{R''_\ell} \{L(x) \mid x \in V_B \setminus N(u) \setminus V_0(u), P_u \ll_{R''_\ell} P_x\}$ . The next property of the projection representation  $R''_\ell$  follows by Lemma 39.

**Corollary 5** *For all vertices  $z \in N_1 \setminus N_2$ , for which  $R(z) <_{R''_\ell} L_0$ , the values  $R(z)$  lie immediately before  $L_0$  in  $R''_\ell$ .*

**Proof.** Let  $x_0$  be the vertex of  $V_B \setminus N(u) \setminus V_0(u)$ , such that  $L_0 = L(x_0)$ . Recall by Lemma 39 that for all vertices  $z \in N_1 \setminus N_2$ , for which  $R(z) <_{R'_\ell} L_0$ , the values  $R(z)$  lie immediately before  $L_0$  in  $R'_\ell$ . Furthermore, note that the parallelograms of all neighbors  $z \in N(u)$  of  $u$  do not move by Transformation 5. Therefore, since also the value  $L_0$  is the same in both  $R'_\ell$  and  $R''_\ell$ , it suffices to prove that there do not exist vertices  $v \in V_0(u) \cap V_B$  and  $z \in N_1 \setminus N_2$ , such that  $R(z) <_{R''_\ell} R(v) <_{R''_\ell} L_0$  in  $R''_\ell$ . Suppose otherwise that  $R(z) <_{R''_\ell} R(v) <_{R''_\ell} L_0 = L(x_0)$  for two vertices  $v \in V_0(u) \cap V_B$  and  $z \in N_1 \setminus N_2$ . Thus, since only the right lines of some parallelograms  $P_v$ , where  $v \in V_0(u) \cap V_B$ ,



are moved to the left by Transformation 5, it follows that  $R(z) <_{R'_\ell} L_0 = L(x_0) <_{R'_\ell} R(v)$  in  $R'_\ell$ . Therefore, in particular  $P_v$  intersects  $P_{x_0}$  in  $R'_\ell$ , and thus  $v \in N(x_0)$ , since both  $v$  and  $x_0$  are bounded. Thus  $x_0 \in V_0(u)$ , since also  $v \in V_0(u)$ . This is a contradiction, since  $x_0 \in V_B \setminus N(u) \setminus V_0(u)$ . This completes the proof. ■

We construct now the projection representation  $R''_\ell$  from  $R'_\ell$  as follows.

**Transformation 6** Move the line  $P_u$  in  $R'_\ell$ , such that its upper endpoint  $L(u) = R(u)$  comes immediately before  $\min_{R'_\ell}\{L_0, R(z) \mid z \in N_1 \setminus N_2\}$  and its lower endpoint  $l(u) = r(u)$  comes immediately after  $\max_{R'_\ell}\{r(v) \mid v \in V_0(u) \cap V_B\}$ . Finally, make  $u$  a bounded vertex. Denote the resulting projection representation by  $R''_\ell$ .

Note by the statement of Transformation 6 that  $R''_\ell$  is a projection representation with  $k - 1$  unbounded vertices, since  $u$  is a bounded vertex in  $R''_\ell$ .

### Properties of $R'_\ell$ , $R''_\ell$ , and $R'''_\ell$

In the following (in Lemmas 40, 41), we prove that the projection representations  $R'_\ell \setminus \{u\}$  and  $R''_\ell \setminus \{u\}$  (constructed by Transformations 4 and 5, respectively) are both projection representations of  $G \setminus \{u\}$ . Furthermore, we prove in Lemma 42 that  $R'''_\ell$  is a projection representation of  $G$ ; that is,  $R^* = R'''_\ell$  is a projection representation of  $G$  with  $k - 1$  unbounded vertices, as Theorem 2 states.

**Lemma 40**  $R'_\ell \setminus \{u\}$  is a projection representation of  $G \setminus \{u\}$ .

**Proof.** Denote by  $x_0$  the vertex of  $V_0(u)$ , such that  $\ell_0 = l(x_0)$ . Since we move the right line of some parallelograms to the right, i.e. we increase some parallelograms, all adjacencies of  $R_\ell$  are kept in  $R'_\ell$ . Suppose that  $R'_\ell$  has the new adjacency  $zv$  that is not an adjacency in  $R_\ell$ , for some  $z \in N_1$ . Therefore, since perform parallel movements of lines, i.e. since every slope  $\phi_x$  in  $R'_\ell$  equals the value of  $\phi_x$  in  $R_\ell$  for every vertex  $x$  of  $G$ , it follows that  $P_z \ll_{R_\ell} P_v$  and  $P_z$  intersects  $P_v$  in  $R'_\ell$ . Thus,  $v \notin V_0(u)$ , since  $u$  has the right border property in  $R_\ell$  by Lemma 37. Furthermore,  $r(z) <_{R_\ell} \ell_0 = l(x_0)$ , since  $z \in N_1$ . However, since  $x_0 \in V_0(u)$ , and since  $u$  has the right border property in  $R_\ell$ , it follows that  $P_z$  intersects  $P_{x_0}$  in  $R_\ell$ , and thus  $L(x_0) <_{R_\ell} R(z)$ . We distinguish in the following the cases where  $v \notin N(u)$  and  $v \in N(u)$ .

*Case 1.*  $v \notin N(u)$ . Then, since also  $v \notin V_0(u)$ , it follows by Observation 3 that  $v \notin V(G_0)$ . We will derive a contradiction to the assumption that  $R'_\ell$  has the new adjacency  $zv$  that is not an adjacency in  $R_\ell$ , for some  $z \in N_1$ . Recall that every slope  $\phi_x$  in  $R'_\ell$  equals the value of  $\phi_x$  in  $R_\ell$  for every vertex  $x$  of  $G$ . Suppose first that  $r(z) <_{R'_\ell} l(v)$ . Then, since  $P_z$  intersects  $P_v$  in  $R'_\ell$ , it follows that  $L(v) <_{R'_\ell} R(z)$ , and thus  $\phi_v > \phi_z$  in  $R'_\ell$ . If  $v$  is unbounded, then  $z$  is not adjacent to  $v$  in  $R'_\ell$ , which is a contradiction to the assumption. Thus  $v$  is bounded, i.e.  $v \in V_B \setminus N(u)$  and  $P_z \ll_{R_\ell} P_v$ , and thus  $L_0(z) \leq_{R_\ell} L(v)$  by definition of  $L_0(z)$ . Furthermore, since all left lines of the parallelograms in  $R_\ell$  do not move during Transformation 4, it follows that also  $L_0(z) \leq_{R'_\ell} L(v)$ . Thus,  $R(z) <_{R'_\ell} L_0(z) \leq_{R'_\ell} L(v)$  by the statement of Transformation 4, which is a contradiction, since  $L(v) <_{R'_\ell} R(z)$ .

Suppose now that  $l(v) <_{R'_\ell} r(z)$ . We will first prove that in this case  $l(v) <_{R_\ell} l(x_0)$ . Suppose otherwise that  $l(x_0) <_{R_\ell} l(v)$ . Let  $x_0 \notin V(G_0)$ . Then, since  $r(z)$  comes in  $R'_\ell$  at most immediately after  $\ell_0 = l(x_0)$  on  $L_2$ , it follows that  $l(x_0) <_{R'_\ell} r(z) <_{R'_\ell} l(v)$ . This is a contradiction to the assumption that  $l(v) <_{R'_\ell} r(z)$ . Let  $x_0 \in V(G_0)$ . Then,  $b_\ell - \varepsilon <_{R_\ell} l(x_0) <_{R_\ell} b_\ell + \varepsilon$  by Remark 2. Furthermore, since  $v \notin V(G_0)$ , and since we assumed that  $l(x_0) <_{R_\ell} l(v)$ , it follows that  $l(x_0) <_{R_\ell} b_\ell + \varepsilon <_{R_\ell} l(v)$  by Remark 2. If  $z \in V(G_0)$ , then  $r(z)$  comes in  $R'_\ell$  (due to the statement of Transformation 4) at most immediately after  $\ell_0 = l(x_0)$  on  $L_2$ , and thus in this case  $l(x_0) <_{R'_\ell} r(z) <_{R'_\ell} b_\ell + \varepsilon <_{R'_\ell} l(v)$ . This is a contradiction to the assumption that  $l(v) <_{R'_\ell} r(z)$ . Otherwise, if  $z \notin V(G_0)$ , then  $r(z)$  comes in  $R'_\ell$  (due to Remark 3) immediately after  $b_\ell + \varepsilon$  on  $L_2$ , and thus in this case  $l(x_0) <_{R'_\ell} b_\ell + \varepsilon <_{R'_\ell} r(z) <_{R'_\ell} l(v)$ . This is again a contradiction to the assumption that  $l(v) <_{R'_\ell} r(z)$ . Therefore  $l(v) <_{R_\ell} l(x_0)$ .

Recall that  $L(x_0) <_{R_\ell} R(z)$ , and thus also  $L(x_0) <_{R_\ell} R(z) <_{R_\ell} L(v)$ , since  $P_z \ll_{R_\ell} P_v$ . Therefore, since also  $l(v) <_{R_\ell} l(x_0)$  by the previous paragraph, it follows that  $P_{x_0}$  intersects  $P_v$  in  $R_\ell$  and  $\phi_{x_0} > \phi_v$  in  $R_\ell$ . If  $x_0$  is bounded, then  $x_0v \in E$ , and thus  $v \in V_0(u)$ , since  $x_0 \in V_0(u)$  and  $v \notin N(u)$ , which is a contradiction. Therefore,  $x_0$  is unbounded, and thus  $x_0v \notin E$ . Therefore,  $N(x_0) \subseteq N(v)$  by Lemma 3. Recall now that there exists a bounded covering vertex  $u^*$  of  $u$  in  $G$ , and thus  $u^*, x_0 \in V_0(u)$ . Furthermore,  $u^* \neq x_0$ , since  $x_0$  is unbounded. Therefore, since  $V_0(u)$  is connected with at least two vertices,  $x_0$  is adjacent to at least one other vertex  $y \in V_0(u)$ , and thus  $y \in N(v)$ , since  $N(x_0) \subseteq N(v)$ . Thus  $v \in V_0(u)$ , since  $v \notin N(u)$ , which is again a contradiction. Summarizing,  $R'_\ell$  has no new adjacency  $zv$  that is not an adjacency in  $R_\ell$ , for any  $v \notin N(u)$  and any  $z \in N_1$ .

*Case 2.*  $v \in N(u)$ . We distinguish in the following the cases where  $z \notin V(G_0)$  and  $z \in V(G_0)$ .

*Case 2a.*  $z \notin V(G_0)$ . Since  $z \in N(u)$ , it follows that  $P_z$  intersects  $P_u$  in  $R_\ell$  by Corollary 4, and thus  $P_z$  intersects the line segment  $\ell$  in  $R_\ell$ . If  $v \in V(G_0)$ , then  $P_z$  intersects  $P_v$  in  $R_\ell$  (since  $v \in N(u)$ ), which is a contradiction. Thus,  $v \notin V(G_0)$ . Therefore, since both  $z, v \notin V(G_0)$ , and since  $P_z \ll_{R_\ell} P_v$ , it follows that also  $P_z \ll_R P_v$ . Therefore, since  $v \in N(u)$ , it follows that  $R(z) <_R L(v) <_R a_u =_R L(u)$  by Lemma 1, and thus  $L(x_0) <_{R_\ell} R(z) <_{R_\ell} L(v) <_{R_\ell} a_u$ , since the endpoints of  $P_z$  and  $P_v$  remain the same in both  $R$  and  $R_\ell$ . Therefore  $x_0 \notin V(G_0)$ , since otherwise  $L(x_0) >_{R_\ell} a_\ell - \varepsilon >_{R_\ell} a_u$  (by definition of the line segment  $\ell$ ). Thus, also  $L(x_0) <_R R(z) <_R L(v) <_R a_u$ . Furthermore  $b_u =_R r(u) <_R r(z) <_R \ell_0 = l(x_0)$  due to Lemma 1, since  $z \in N_1$ . Then,  $P_{x_0}$  intersects  $P_u$  in  $R$  and  $\phi_{x_0} > \phi_u$ , since  $L(x_0) <_R a_u$  and  $b_u <_R l(x_0)$ . If  $x_0 \notin N(u)$ , then  $N(x_0) \subseteq N(u)$  by Lemma 3, and thus  $x_0 \in Q_u$ . This is a contradiction by Lemma 13, since  $x_0 \in V_0(u)$  by assumption. Thus  $x_0 \in N(u)$ , which is again a contradiction, since  $x_0 \in V_0(u)$ .

*Case 2b.*  $z \in V(G_0)$ . Then, note that  $r(u) <_{R_0} r(z)$  by Lemma 1, and thus also  $b_u <_{R_\ell} r(u) <_{R_\ell} r(z)$ , since  $R_0$  is a projection representation of  $G_0$  (and a sub-representation of  $R_\ell$ ). Suppose that  $v \notin V(G_0)$ . Then, since we assumed that  $v \in N(u)$ , it follows by Corollary 4 that  $P_v$  intersects  $P_u$  in  $R_\ell$ . That is,  $P_v$  intersects the line segment  $\ell$  in  $R_\ell$ , and thus  $P_v$  intersects  $P_z$  in  $R_\ell$ , which is a contradiction, since  $P_z \ll_{R_\ell} P_v$ . Therefore,  $v \in V(G_0)$ .

Consider the projection representation  $R_0$  of  $G_0$  (which is a sub-representation of  $R_\ell$ ) and suppose that  $x_0 \in V(G_0)$ . Then,  $r(u) <_{R_0} r(z) <_{R_0} \ell_0 = l(x_0)$  and  $L(z) <_{R_0} L(u) = R(u)$  by Lemma 1. If  $L(x_0) <_{R_0} R(u)$ , then  $P_u$  intersects  $P_{x_0}$  in  $R_0$  and  $\phi_{x_0} > \phi_u$  in  $R_0$ . Thus, since  $x_0 \in V(G_0) \setminus \{u\}$  and every vertex of  $G_0 \setminus \{u\}$  is bounded by Lemma 27, it follows that  $x_0 \in N(u)$ . This is a contradiction, since  $x_0 \in V_0(u)$  by definition of  $x_0$ . Therefore  $R(u) <_{R_0} L(x_0)$ . Recall now that  $L(x_0) <_{R_\ell} R(z)$  and  $P_z \ll_{R_\ell} P_v$ ; thus, also  $L(x_0) <_{R_0} R(z)$  and  $P_z \ll_{R_0} P_v$ , since  $R_0$  is a sub-representation of  $R_\ell$ . Therefore,  $R(u) <_{R_0} L(x_0) <_{R_0} R(z) <_{R_0} L(v)$  and  $r(u) <_{R_0} r(z) <_{R_0} l(v)$ . That is,  $R(u) <_{R_0} L(v)$  and  $r(u) <_{R_0} l(v)$ , i.e.  $P_u \ll_{R_0} P_v$ , and thus  $v \notin N(u)$ , which is a contradiction to the assumption of Case 2. Therefore,  $x_0 \notin V(G_0)$ .

Since  $x_0 \notin V(G_0)$ , i.e. the endpoints of  $P_{x_0}$  remain the same in both  $R$  and  $R_\ell$ , and since  $b_u <_{R_\ell} r(z) <_{R_\ell} \ell_0 = l(x_0)$ , it follows that also  $b_u <_R l(x_0)$ . Suppose that  $L(x_0) <_R a_u$ . Then,  $P_{x_0}$  intersects  $P_u$  in  $R$  and  $\phi_{x_0} > \phi_u$ . Thus,  $x_0$  is unbounded, since otherwise  $x_0 \in N(u)$ , which is a contradiction. Furthermore,  $N(x_0) \subseteq N(u)$  by Lemma 3, and thus  $x_0 \in Q_u$ , which is a contradiction by Lemma 13, since  $x_0 \in V_0(u)$  by assumption. Therefore  $a_u <_R L(x_0)$ , i.e.  $P_u \ll_R P_{x_0}$ , since also  $b_u <_R l(x_0)$ . Thus  $x_0 \in D_2 \subseteq S_2$ , since  $x_0 \in V_0(u)$ . Furthermore  $x_0 \notin N[X_1]$ , since  $P_x \ll_R P_u \ll_R P_{x_0}$  for every  $x \in X_1$ . Moreover,  $x_0 \notin Q_u$  by Lemma 13 and  $x_0 \notin V(\mathcal{B}_1)$  by definition of  $\mathcal{B}_1$ , since  $x_0 \in V_0(u)$ . Recall now by Lemma 16 that  $V(C_u \cup C_2 \cup H)$  induces a subgraph of  $G \setminus Q_u \setminus N[X_1] \setminus \mathcal{B}_1$  that includes all connected components of  $G \setminus Q_u \setminus N[X_1] \setminus \mathcal{B}_1$ , in which the vertices of  $S_2 \cup \{u\}$  belong. Therefore, since  $x_0 \in S_2$  and  $x_0 \notin Q_u \cup N[X_1] \cup V(\mathcal{B}_1)$ , it follows that  $x_0 \in V(C_u \cup C_2 \cup H)$ . Thus  $x_0 \in \bigcup_{i=1}^{\infty} H_i \cup \bigcup_{i=0}^{\infty} H'_i$ , since otherwise  $x_0 \in V(G_0)$ , which is a contradiction. If  $x_0 \in \bigcup_{i=0}^{\infty} H'_i$ , then  $x_0 \in N(u)$  by Lemma 32, which is a contradiction, since  $x_0 \in V_0(u)$ . Therefore  $x_0 \in \bigcup_{i=1}^{\infty} H_i$ .

Let  $x_0 = v_i \in H_i$ , for some  $i \geq 1$ , and let  $(v_0, v_1, \dots, v_i)$  be an  $H_i$ -chain of  $v_i$ . Note that  $v_j \in N(u) \cup V_0(u)$  for every vertex  $v_j$ , where  $0 \leq j \leq i$ ; indeed, if  $v_j \notin N(u)$ , then  $v_j \in V_0(u)$ , since  $x_2 \in V_0(u)$  and  $v_j \in N(x_2)$  by definition of  $H$ . Furthermore, recall that every vertex  $v_j$ ,

where  $0 \leq j \leq i$ , is a bounded vertex by Lemma 27. Therefore, since  $v_i v_{i-1} \notin E$ , it follows that  $P_{v_i}$  does not intersect  $P_{v_{i-1}}$  in  $R_\ell$ , i.e. either  $P_{v_i} \ll_{R_\ell} P_{v_{i-1}}$  or  $P_{v_{i-1}} \ll_{R_\ell} P_{v_i}$ . Moreover, either  $P_{v_j} \ll_{R_\ell} P_{v_{j-1}}$  or  $P_{v_{j-1}} \ll_{R_\ell} P_{v_j}$  for every  $j \in \{1, 2, \dots, i-1\}$  by Lemma 26. Thus, either  $P_{v_{j-1}} \ll_{R_\ell} P_{v_j}$  or  $P_{v_j} \ll_{R_\ell} P_{v_{j-1}}$  for every  $j \in \{1, 2, \dots, i\}$ .

We will prove by induction on  $j$  that  $v_j \in V_0(u)$ ,  $b_\ell - \varepsilon <_{R_\ell} r(v_j)$ , and  $L(v_j) <_{R_\ell} a_\ell - \varepsilon$ , for every  $j \in \{0, 1, \dots, i\}$ . Recall first that every  $v_j$ , where  $0 \leq j \leq i$ , is adjacent to every vertex of  $G_0 \setminus \{u\}$  by Lemma 35. Thus, in particular every  $P_{v_j}$ , where  $0 \leq j \leq i$ , intersects the line segment  $\ell$  in  $R_\ell$ , since  $R_\ell \setminus \{u\}$  is a projection representation of  $G \setminus \{u\}$  by Lemma 36. Furthermore, recall that  $v_j \notin V(G_0)$  by definition of  $G_0$ , for every  $j \in \{0, 1, \dots, i\}$ , and thus the endpoints of every  $P_{v_j}$ ,  $j \in \{0, 1, \dots, i\}$ , remain the same in both  $R$  and  $R_\ell$ . Furthermore, since  $v_j \notin V(G_0)$ , either  $l(v_j) <_{R_\ell} b_\ell - \varepsilon$  or  $l(v_j) >_{R_\ell} b_\ell + \varepsilon$  by Remark 2, for every  $v_j$ , where  $0 \leq j \leq i$ .

For the induction basis, let  $j = i$ . Then,  $x_0 = v_i \in V_0(u)$  by definition of  $x_0$ . If  $l(x_0) <_{R_\ell} b_\ell - \varepsilon$ , then  $l(x_0) <_{R_\ell} b_\ell - \varepsilon <_{R_\ell} r(z) <_{R_\ell} b_\ell + \varepsilon$ , since  $x_0 \notin V(G_0)$  and  $z \in V(G_0)$  (cf. Remark 2). This is a contradiction, since  $r(z) <_{R_\ell} \ell_0 = l(x_0)$  by definition of  $N_1$ . Therefore  $b_\ell + \varepsilon <_{R_\ell} l(x_0) \leq_{R_\ell} r(x_0)$ . Thus, since  $P_{x_0} = P_{v_i}$  intersects the line segment  $\ell$  in  $R_\ell$ , it follows that  $L(x_0) <_{R_\ell} a_\ell - \varepsilon$ . That is,  $v_i \in V_0(u)$ ,  $b_\ell + \varepsilon <_{R_\ell} r(v_i)$ , and  $L(v_i) <_{R_\ell} a_\ell - \varepsilon$ . This completes the induction basis.

For the induction step, assume that  $v_j \in V_0(u)$ ,  $b_\ell + \varepsilon <_{R_\ell} r(v_j)$ , and  $L(v_j) <_{R_\ell} a_\ell - \varepsilon$ , for some  $j \in \{1, 2, \dots, i\}$ . We will prove that also  $v_{j-1} \in V_0(u)$ ,  $b_\ell + \varepsilon <_{R_\ell} r(v_{j-1})$ , and  $L(v_{j-1}) <_{R_\ell} a_\ell - \varepsilon$ . Let first  $P_{v_{j-1}} \ll_{R_\ell} P_{v_j}$ . Suppose that  $v_{j-1} \notin V_0(u)$ . Then, since  $v_{j-1} \in N(u) \cup V_0(u)$ , it follows that  $v_{j-1} \in N(u)$ . That is,  $P_{v_{j-1}} \ll_{R_\ell} P_{v_j}$ , where  $v_{j-1} \in N(u)$  and  $v_j \in V_0(u)$ . This is a contradiction, since  $u$  has the right border property in  $R_\ell$  by Lemma 37. Therefore  $v_{j-1} \in V_0(u)$ . Furthermore, since we assumed that  $P_{v_{j-1}} \ll_{R_\ell} P_{v_j}$ , and since  $L(v_j) <_{R_\ell} a_\ell - \varepsilon$  by the induction hypothesis, it follows that  $R(v_{j-1}) <_{R_\ell} L(v_j) <_{R_\ell} a_\ell - \varepsilon$ . Thus, also  $L(v_{j-1}) <_{R_\ell} a_\ell - \varepsilon$ , since  $L(v_{j-1}) \leq_{R_\ell} R(v_{j-1})$ . Furthermore, since  $P_{v_{j-1}}$  intersects the line segment  $\ell$  in  $R_\ell$ , it follows that  $b_\ell + \varepsilon <_{R_\ell} r(v_{j-1})$ . That is,  $v_{j-1} \in V_0(u)$ ,  $b_\ell + \varepsilon <_{R_\ell} r(v_{j-1})$ , and  $L(v_{j-1}) <_{R_\ell} a_\ell - \varepsilon$ .

Let now  $P_{v_j} \ll_{R_\ell} P_{v_{j-1}}$ , and thus also  $P_{v_j} \ll_R P_{v_{j-1}}$ , since  $v_{j-1}, v_j \notin V(G_0)$ . Then, since  $b_\ell + \varepsilon <_{R_\ell} r(v_j)$  (and thus also  $b_\ell + \varepsilon <_R r(v_j)$ ) by the induction hypothesis, it follows that  $b_\ell + \varepsilon <_{R_\ell} r(v_j) <_{R_\ell} l(v_{j-1})$ . Therefore  $b_\ell + \varepsilon <_{R_\ell} r(v_{j-1})$ , since  $l(v_{j-1}) \leq_{R_\ell} r(v_{j-1})$ . Furthermore, since  $b_\ell + \varepsilon <_{R_\ell} l(v_{j-1})$ , and since  $P_{v_{j-1}}$  intersects the line segment  $\ell$  in  $R_\ell$ , it follows that  $R(v_{j-1}) <_{R_\ell} a_\ell - \varepsilon$ . Therefore  $L(v_{j-1}) <_{R_\ell} a_\ell - \varepsilon$ , since  $L(v_{j-1}) \leq_{R_\ell} R(v_{j-1})$ . That is,  $b_\ell + \varepsilon <_{R_\ell} r(v_{j-1})$  and  $L(v_{j-1}) <_{R_\ell} a_\ell - \varepsilon$ . Recall that also  $b_\ell + \varepsilon <_{R_\ell} l(v_{j-1})$ . Thus  $b_u <_R b_\ell + \varepsilon <_R l(v_{j-1})$ , since  $b_u <_R b_\ell$  (by definition of the line segment  $\ell$ ), and since the endpoints of  $P_{v_{j-1}}$  remain the same in both  $R$  and  $R_\ell$ . Suppose now that  $v_{j-1} \notin V_0(u)$ . Then, since  $v_{j-1} \in N(u) \cup V_0(u)$ , it follows that  $v_{j-1} \in N(u)$ , i.e. in particular  $P_{v_{j-1}}$  intersects  $P_u$  in  $R$ . Thus, since  $b_u =_R r(u) <_R l(v_{j-1})$ , it follows that  $L(v_{j-1}) <_R a_u =_R L(u)$ . Therefore  $R(v_j) <_R L(v_{j-1}) <_R a_u$ , since we assumed that  $P_{v_j} \ll_R P_{v_{j-1}}$ . Then, since  $R(v_j) <_R a_u$  and  $b_u <_R b_\ell + \varepsilon <_R r(v_j)$ , it follows that  $P_{v_j}$  intersects  $P_u$  in  $R$  and  $\phi_{v_j} > \phi_u$ . Thus  $v_j \in N(u)$ , since  $v_j$  is bounded in  $R$ , which is a contradiction to the induction hypothesis that  $v_j \in V_0(u)$ . Therefore,  $v_{j-1} \in V_0(u)$ . This completes the induction step, and thus  $v_j \in V_0(u)$ ,  $b_\ell - \varepsilon <_{R_\ell} r(v_j)$ , and  $L(v_j) <_{R_\ell} a_\ell - \varepsilon$ , for every  $j \in \{0, 1, \dots, i\}$ .

Consider now the vertex  $v_0 \in H_0 = N$ . Then  $P_{v_0}$  intersects  $P_u$  in  $R$ , since  $v_0 \in N(X_1) \cap N(x_2)$  by Lemma 19, and since  $P_x \ll_R P_u \ll_R P_{x_2}$  for every  $x \in X_1$ . Recall that  $x_0 = v_i \in H_i$ , for some  $i \geq 1$ , and that  $(v_0, v_1, \dots, v_i)$  is an  $H_i$ -chain of  $v_i$ . Thus, in particular,  $v_1$  exists, since  $i \geq 1$ . Furthermore,  $L(v_1) <_{R_\ell} a_\ell - \varepsilon$  by the previous paragraph. Thus also  $L(v_1) <_R a_\ell - \varepsilon$ , since the endpoints of  $P_{v_1}$  remain the same in both  $R$  and  $R_\ell$ . Therefore, since  $P_{v_0} \ll_R P_{v_1}$  by Lemma 25, it follows that  $R(v_0) <_R L(v_1) <_R a_\ell - \varepsilon$ . On the other hand,  $b_\ell - \varepsilon <_{R_\ell} r(v_0)$  by the previous paragraph, and thus also  $b_\ell - \varepsilon <_R r(v_0)$ . That is,  $R(v_0) <_R a_\ell - \varepsilon$  and  $b_\ell - \varepsilon <_R r(v_0)$ , and thus in particular  $\phi_{v_0} > \phi_\ell$  in  $R$ . Therefore  $\phi_{v_0} > \phi_\ell \geq \phi_u$  in  $R$ , since  $\phi_\ell \geq \phi_u$  in  $R$  by the definition of the line segment  $\ell$ . Thus, since  $P_{v_0}$  intersects  $P_u$  in  $R$ , it follows that  $v_0 \in N(u)$ . This is a contradiction, since  $v_0 \in V_0(u)$  by the previous paragraph.

This completes Case 2b, and thus also due to Cases 1 and 2a, it follows that  $R'_\ell$  has no new adjacency  $zv$  that is not an adjacency in  $R_\ell$ , for any  $z \in N_1$ , i.e.  $R'_\ell \setminus \{u\}$  is a projection representation of  $G \setminus \{u\}$ . This completes the proof of the lemma. ■

**Lemma 41**  $R'_\ell \setminus \{u\}$  is a projection representation of  $G \setminus \{u\}$ .

**Proof.** Denote by  $z_0$  the vertex of  $N_2$ , such that  $r_0 = r(z_0)$ . Since during Transformation 5 we move the right line of some parallelograms to the left, i.e. we decrease some parallelograms, no new adjacencies are introduced in  $R'_\ell$  in comparison to  $R'_\ell$ . Suppose that  $vx \in E$  and that the adjacency  $vx$  has been removed from  $R'_\ell$  in  $R'_\ell$ , for some  $v \in V_0(u) \cap V_B$ , such that  $r(v) >_{R'_\ell} r_0 = r(z_0)$ . Therefore, since we perform parallel movements of lines in  $R'_\ell$ , i.e. since every slope  $\phi_y$  in  $R'_\ell$  equals the value of  $\phi_y$  in  $R'_\ell$  for every vertex  $y$  of  $G$ , it follows that  $P_v \ll_{R'_\ell} P_x$  and that  $P_v$  intersects  $P_x$  in  $R'_\ell$ . Note that  $l(v) \leq_{R'_\ell} \ell_0$ , since  $v \in V_0(u)$  and  $\ell_0 = \max_{R'_\ell} \{l(x) \mid x \in V_0(u)\}$ .

We first assume that  $x \notin N(u)$ . Since  $r(v)$  comes in  $R'_\ell$  immediately before  $r_0$ , and since  $P_v \ll_{R'_\ell} P_x$ , it follows that  $r(v) <_{R'_\ell} r_0 <_{R'_\ell} l(x)$ , and thus also  $r_0 <_{R'_\ell} l(x)$ . Furthermore, since  $vx \in E$  by assumption, and since  $v \in V_0(u)$ , it follows that  $x \in V_0(u)$ . Therefore  $l(x) \leq_{R'_\ell} \ell_0$ , since  $\ell_0 = \max_{R'_\ell} \{l(x) \mid x \in V_0(u)\}$ , and thus  $r_0 = r(z_0) <_{R'_\ell} l(x) \leq_{R'_\ell} \ell_0$ , i.e.  $r(z_0) <_{R'_\ell} \ell_0$ . This is a contradiction, since  $z_0 \in N_2$ . Therefore, no adjacency  $vx$  has been removed from  $R'_\ell$  in  $R'_\ell$  in the case where  $x \notin N(u)$ .

Assume now that  $x \in N(u)$ , and thus the endpoints of  $P_x$  in  $R'_\ell$  remain the same also in  $R''_\ell$ .

*Case 1.*  $v \in V(G_0)$ . Then, since the endpoints of  $P_v$  do not move during Transformation 4, it follows by Remark 2 that  $b_\ell - \varepsilon <_{R'_\ell} l(v) \leq_{R'_\ell} r(v) <_{R'_\ell} b_\ell + \varepsilon$  and  $a_\ell - \varepsilon <_{R'_\ell} L(v) \leq_{R'_\ell} R(v) <_{R'_\ell} a_\ell + \varepsilon$  in  $R'_\ell$ . Thus, in particular also  $b_\ell - \varepsilon <_{R'_\ell} l(v)$  and  $a_\ell - \varepsilon <_{R'_\ell} L(v)$  in  $R''_\ell$ , since the left lines of all parallelograms do not move during Transformation 5. Therefore  $b_\ell - \varepsilon <_{R''_\ell} l(v) <_{R''_\ell} l(x)$  and  $a_\ell - \varepsilon <_{R''_\ell} L(v) <_{R''_\ell} L(x)$ , since  $P_v \ll_{R''_\ell} P_x$ . Furthermore, also  $b_\ell - \varepsilon <_{R_\ell} l(x)$  and  $a_\ell - \varepsilon <_{R_\ell} L(x)$  in  $R_\ell$ , since left lines of all parallelograms do not move during Transformations 4 and 5. We distinguish in the following the cases where  $x \notin V(G_0)$  and  $x \in V(G_0)$ .

*Case 1a.*  $x \notin V(G_0)$ . Then, either  $l(x) <_{R_\ell} b_\ell - \varepsilon$  or  $l(x) >_{R_\ell} b_\ell + \varepsilon$  (resp. either  $L(x) <_{R_\ell} a_\ell - \varepsilon$  or  $L(x) >_{R_\ell} a_\ell + \varepsilon$ ) by Remark 2. Thus, since  $b_\ell - \varepsilon <_{R_\ell} l(x)$  and  $a_\ell - \varepsilon <_{R_\ell} L(x)$  by the previous paragraph, it follows that  $l(x) >_{R_\ell} b_\ell + \varepsilon$  and  $L(x) >_{R_\ell} a_\ell + \varepsilon$ . Therefore  $r(v) <_{R_\ell} b_\ell + \varepsilon <_{R_\ell} l(x)$  and  $R(v) <_{R_\ell} a_\ell + \varepsilon <_{R_\ell} L(x)$  by Remark 2, i.e.  $P_v \ll_{R_\ell} P_x$  in  $R_\ell$ , and thus  $vx \notin E$ . This is a contradiction, since we assumed that  $vx \in E$ .

*Case 1b.*  $x \in V(G_0)$ . Recall by Lemma 38 that  $r(u) <_{R'_\ell} r_0 = r(z_0)$ , and thus  $r(u) <_{R'_\ell} r_0 <_{R'_\ell} r(v)$ . Therefore, since  $r(v)$  comes immediately before  $r_0$  in  $R'_\ell$  during Transformation 5, it follows that  $r(u) <_{R'_\ell} r(v) <_{R'_\ell} r_0$ . Therefore,  $r(u) <_{R'_\ell} r(v) <_{R'_\ell} l(x)$ , since  $P_v \ll_{R'_\ell} P_x$ . Suppose that  $P_x$  intersects  $P_u$  in  $R'_\ell$ . Then, since  $r(u) <_{R'_\ell} l(x)$ , it follows that  $L(x) <_{R'_\ell} R(u)$ ; thus  $R(v) <_{R'_\ell} L(x) <_{R'_\ell} R(u)$ , since  $P_v \ll_{R'_\ell} P_x$ . That is,  $r(u) <_{R'_\ell} r(v)$  and  $R(v) <_{R'_\ell} R(u)$ , i.e.  $P_v$  intersects  $P_u$  in  $R'_\ell$  and  $\phi_v > \phi_u$  in  $R'_\ell$ . Therefore,  $P_v$  intersects  $P_u$  and  $\phi_v > \phi_u$  also in  $R'_\ell$  and in  $R_\ell$ . Thus, since  $v \in V(G_0)$ , and since  $R_0$  is a sub-representation of  $R_\ell$ ,  $P_v$  intersects  $P_u$  in  $R_0$  and  $\phi_v > \phi_u$  in  $R_0$ . Therefore, since  $v$  is bounded (recall that  $v \in V_0(u) \cap V_B$  by our initial assumption on  $v$ ), it follows that  $v \in N(u)$ , which is a contradiction. Therefore,  $P_x$  does not intersect  $P_u$  in  $R'_\ell$ , and thus  $P_u \ll_{R'_\ell} P_x$ , since  $r(u) <_{R'_\ell} l(x)$ . Thus also  $P_u \ll_{R'_\ell} P_x$  and  $P_u \ll_{R_\ell} P_x$ , since the left line of  $P_x$  does not move by Transformations 4 and 5. Therefore  $P_u \ll_{R_0} P_x$ , since  $x \in V(G_0)$  and  $R_0$  is a sub-representation of  $R_\ell$ . Thus  $x \notin N(u)$ , which is a contradiction to our assumption on  $x$ .

*Case 2.*  $v \notin V(G_0)$ .

*Case 2a.*  $x \notin V(G_0)$ . We will now prove that  $b_u <_{R'_\ell} r(v) <_{R'_\ell} l(x)$ . Recall that  $z_0 \in N(u)$ . Thus, if  $z_0 \in V(G_0)$ , then  $r(u) <_{R_0} r(z_0)$  by Lemma 1, and thus also  $r(u) <_{R_\ell} r(z_0)$ , since  $R_0$  is a sub-representation of  $R_\ell$ . Furthermore  $b_u <_{R'_\ell} r(u) <_{R'_\ell} r(z_0)$ , since the right endpoint  $r(z_0)$  of  $P_{z_0}$  does not decrease by Transformation 4. On the other hand, let  $z_0 \notin V(G_0)$ . Then  $b_u <_{R_\ell} r(z_0)$  by Lemma 1, and thus also  $b_u <_{R_\ell} r(z_0)$ , since  $z_0 \notin V(G_0)$  (i.e. the endpoints of  $P_{z_0}$  are the same in both  $R$  and  $R_\ell$ ). Furthermore  $b_u <_{R'_\ell} r(z_0)$ , since  $r(z_0)$  does not decrease by Transformation 4. That is,  $b_u <_{R'_\ell} r(z_0) = r_0 <_{R'_\ell} r(v)$  in both cases where  $z_0 \in V(G_0)$  and  $z_0 \notin V(G_0)$ . Therefore, since  $r(v)$  comes immediately before  $r_0 = r(z_0)$  in  $R'_\ell$  by Transformation 5, it follows that  $b_u <_{R'_\ell} r(v) <_{R'_\ell} r_0$ . Thus,  $b_u <_{R'_\ell} r(v) <_{R'_\ell} l(x)$ , since  $P_v \ll_{R'_\ell} P_x$ .

Furthermore, since the left lines of the parallelograms do not move by Transformations 4 and 5, it follows that also  $b_u <_{R_\ell} l(x)$ . Therefore  $r(u) =_R b_u <_R l(x)$ , since  $x \notin V(G_0)$  (i.e. the endpoints of  $P_x$  are the same in both  $R$  and  $R_\ell$ ). Thus, since we assumed that  $x \in N(u)$ , it follows that  $L(x) <_R$

$a_u =_R L(u)$ . Similarly, since the left lines of the parallelograms do not move by Transformations 4 and 5, and since  $x \notin V(G_0)$ , it follows that also  $L(x) <_{R_\ell} a_u$  and  $L(x) <_{R'_\ell} a_u$ . Thus,  $R(v) <_{R'_\ell} L(x) <_{R'_\ell} a_u$ , since  $P_v \ll_{R'_\ell} P_x$ . That is,  $b_u <_{R'_\ell} r(v)$  (by the previous paragraph) and  $L(v) \leq_{R'_\ell} R(v) <_{R'_\ell} a_u$ . Therefore, since the slope  $\phi_v$  of  $P_v$  (where  $v \notin V(G_0)$ ) remains the same in the representations  $R$ ,  $R_\ell$ ,  $R'_\ell$ , and  $R''_\ell$ , and since the lower right endpoint  $r(v)$  in  $R$  is greater than or equal to the corresponding value  $r(v)$  in  $R''_\ell$ , it follows that  $P_v$  intersects  $P_u$  in  $R$  and  $\phi_v > \phi_u$  in  $R$ . Thus  $v \in N(u)$ , since  $v$  is bounded (recall that  $v \in V_0(u) \cap V_B$ ), which is a contradiction to the assumption that  $v \in V_0(u)$ .

*Case 2b.*  $x \in V(G_0)$ . Recall that  $v \notin V(G_0)$  by the assumption of Case 2. Therefore, since  $vx \notin E$ , it follows by Lemma 35 that  $v \in N(X_1) \cup \bigcup_{i=1}^\infty H_i \cup \bigcup_{i=0}^\infty H'_i$ . Recall that  $v \in V_0(u) \cap V_B$ , and thus in particular  $v \notin N(u)$ . Therefore  $v \notin \bigcup_{i=0}^\infty H'_i$  by Lemma 32, and thus  $v \in N(X_1) \cup \bigcup_{i=1}^\infty H_i$ . We distinguish in the following the cases where  $v \in N(X_1)$  and  $v \in \bigcup_{i=1}^\infty H_i$ .

*Case 2b-i.*  $v \in N = N(X_1)$ . Then,  $P_v$  intersects  $P_u$  in  $R$ , since  $v \in N(X_1) \cap N(x_2)$  by Lemma 19, and since  $P_x \ll_R P_u \ll_R P_{x_2}$  for every  $x \in X_1$ . Recall that  $v$  is bounded and  $v \notin N(u)$ , since  $v \in V_0(u) \cap V_B$  by our initial assumption on  $v$ , and thus  $\phi_v < \phi_u \leq \phi_\ell$  in  $R$ . Therefore,  $\phi_v < \phi_\ell$  also in  $R_\ell$ , since  $v \notin V(G_0)$  (i.e. the endpoints of  $P_v$  remain the same in both  $R$  and  $R_\ell$ ). On the other hand, since  $z_0 \in N(u)$ , it follows that  $\phi_{z_0} > \phi_u$  in  $R$ , and thus  $\phi_v < \phi_u < \phi_{z_0}$  in  $R$ . Furthermore, recall by Remark 2 that  $b_\ell - \varepsilon <_{R_\ell} l(x) <_{R_\ell} b_\ell + \varepsilon$  in  $R_\ell$ , since  $x \in V(G_0)$  by the assumption of Case 2b. Therefore, since the left lines of the parallelograms do not move by Transformations 4 and 5, it follows that also  $b_\ell - \varepsilon <_{R'_\ell} l(x) <_{R'_\ell} b_\ell + \varepsilon$  in  $R'_\ell$ . Similarly, it follows by Remark 2 that  $a_\ell - \varepsilon <_{R''_\ell} L(x) <_{R''_\ell} a_\ell + \varepsilon$  in  $R''_\ell$ .

Let first  $z_0 \notin V(G_0)$ . Then, either  $r(z_0) >_{R'_\ell} b_\ell + \varepsilon$  or  $r(z_0) <_{R'_\ell} b_\ell - \varepsilon$  by Remark 3. Suppose that  $r(z_0) >_{R'_\ell} b_\ell + \varepsilon$ . Then, since  $r(v)$  comes by Transformation 5 immediately before  $r_0 = r(z_0)$  in  $R'_\ell$ , it follows that  $b_\ell + \varepsilon <_{R'_\ell} r(v) <_{R'_\ell} r(z_0)$ . Thus  $b_\ell + \varepsilon <_{R'_\ell} r(v) <_{R'_\ell} l(x)$ , since  $P_v \ll_{R'_\ell} P_x$ . This is a contradiction, since  $b_\ell - \varepsilon <_{R'_\ell} l(x) <_{R'_\ell} b_\ell + \varepsilon$ . Therefore  $r(z_0) <_{R'_\ell} b_\ell - \varepsilon$ .

Recall now by Corollary 4 that  $P_{z_0}$  intersects  $P_u$  in  $R_\ell$ , since  $z_0 \in N(u)$ . Therefore, since  $P_{z_0}$  does not decrease during Transformation 4,  $P_{z_0}$  intersects  $P_u$  also in  $R'_\ell$ , i.e.  $P_{z_0}$  intersects the line segment  $\ell$  in  $R'_\ell$ . Furthermore, since  $z_0 \notin V(G_0)$ , either  $R(z_0) >_{R'_\ell} a_\ell + \varepsilon$  or  $R(z_0) <_{R'_\ell} a_\ell - \varepsilon$  by Remark 3. Therefore, since  $r(z_0) <_{R'_\ell} b_\ell - \varepsilon$  and  $P_{z_0}$  intersects the line segment  $\ell$  in  $R'_\ell$ , it follows that  $R(z_0) >_{R'_\ell} a_\ell + \varepsilon$ ; thus also  $R(z_0) >_{R'_\ell} a_\ell + \varepsilon$ , since the endpoints of  $P_{z_0}$  do not change by Transformation 5. Recall now that  $\phi_v < \phi_{z_0}$  in  $R$ . Therefore also  $\phi_v < \phi_{z_0}$  in  $R'_\ell$ , since  $v, z_0 \notin V(G_0)$  (i.e. the slopes  $\phi_{z_0}$  and  $\phi_v$  remain the same in both  $R$  and  $R'_\ell$ ). Furthermore, recall that  $r(v)$  comes by Transformation 5 immediately before  $r(z_0)$  (i.e. sufficiently close to  $r(z_0)$ ) in  $R'_\ell$ . Therefore, since  $a_\ell + \varepsilon <_{R'_\ell} R(z_0)$  and  $\phi_v < \phi_{z_0}$  in  $R'_\ell$ , it follows that  $a_\ell + \varepsilon <_{R'_\ell} R(z_0) <_{R'_\ell} R(v)$ . Thus  $a_\ell + \varepsilon <_{R'_\ell} R(v) <_{R'_\ell} L(x)$ , since  $P_v \ll_{R'_\ell} P_x$ . This is a contradiction, since  $a_\ell - \varepsilon <_{R'_\ell} L(x) <_{R'_\ell} a_\ell + \varepsilon$  in  $R'_\ell$ .

Let now  $z_0 \in V(G_0)$ . Then  $r(u) <_{R_0} r(z_0)$  by Lemma 1, since  $z_0 \in N(u)$ . Thus, also  $r(u) <_{R_\ell} r(z_0)$ , since  $R_0$  is a sub-representation of  $R_\ell$ . Furthermore  $r(u) <_{R'_\ell} r(z_0)$ , since the value  $r(z_0)$  does not decrease by Transformations 4 and 5. Therefore, since  $r(v)$  comes by Transformation 5 immediately before  $r(z_0)$ , it follows that  $r(u) <_{R'_\ell} r(v) <_{R'_\ell} r(z_0)$ . Similarly,  $L(x) <_{R_0} L(u)$  by Lemma 1, since  $x \in N(u)$ , and thus also  $L(x) <_{R_\ell} L(u)$ . Furthermore  $L(x) <_{R'_\ell} L(u)$ , since the left lines of the parallelograms do not move by Transformations 4 and 5. Therefore  $R(v) <_{R'_\ell} L(x) <_{R'_\ell} L(u)$ , since  $P_v \ll_{R'_\ell} P_x$ . That is,  $r(u) <_{R'_\ell} r(v)$  and  $R(v) <_{R'_\ell} L(u) = R(u)$ , and thus  $\phi_v > \phi_u$  in  $R'_\ell$ . Therefore,  $\phi_v > \phi_u$  also in  $R_\ell$ , since all the slopes are the same in both  $R_\ell$  and  $R'_\ell$ . However, recall that  $\phi_v < \phi_\ell$  in  $R_\ell$  (as we proved in the beginning of Case 2b-i), and thus  $\phi_v < \phi_u$  in  $R_\ell$  by Remark 1, since  $u \in V(G_0)$ . This is a contradiction, since  $\phi_v > \phi_u$  in  $R_\ell$ .

*Case 2b-ii.*  $v \in \bigcup_{i=1}^\infty H_i$ . Let  $v = v_i \in H_i$  for some  $i \geq 1$  and let  $(v_0, v_1, \dots, v_i)$  be an  $H_i$ -chain of  $v_i$ . Recall that  $P_v \ll_{R'_\ell} P_x$  and that  $P_v$  intersects  $P_x$  in  $R'_\ell$  by our initial assumption on  $v$  and on  $x$ . Assume w.l.o.g. that  $i \geq 1$  is the smallest index, such that  $P_v = P_{v_i}$  does not intersect  $P_x$  in  $R'_\ell$ , i.e. in particular  $P_{v_{i-1}}$  intersects  $P_x$  in  $R'_\ell$ . Recall that both  $v_i$  and  $v_{i-1}$  are bounded by Lemma 27, and thus  $P_{v_i}$  does not intersect  $P_{v_{i-1}}$  in  $R'_\ell$ , i.e. either  $P_{v_{i-1}} \ll_{R'_\ell} P_{v_i}$  or  $P_{v_i} \ll_{R'_\ell} P_{v_{i-1}}$ . Let first  $P_{v_{i-1}} \ll_{R'_\ell} P_{v_i}$ . Recall that the left line of  $P_{v_i}$  does not move by Transformation 5 and that

the right line of  $P_{v_{i-1}}$  is possibly moved to the left by Transformation 5. Thus, also  $P_{v_{i-1}} \ll_{R'_\ell} P_{v_i}$  in  $R'_\ell$ . Furthermore, since  $P_{v_i} = P_v \ll_{R''_\ell} P_x$  by our assumption on  $v$ , it follows that  $P_{v_{i-1}} \ll_{R'_\ell} P_x$ . This is a contradiction, since  $P_{v_{i-1}}$  intersects  $P_x$  in  $R'_\ell$ .

Let now  $P_{v_i} \ll_{R'_\ell} P_{v_{i-1}}$ , and thus in particular  $l(v_i) <_{R'_\ell} l(v_{i-1})$ . Thus also  $l(v_i) <_{R_\ell} l(v_{i-1})$ , since the left lines of  $P_{v_i}$  and  $P_{v_{i-1}}$  do not move by Transformation 4. Furthermore  $l(v_i) <_R l(v_{i-1})$ , since  $v_i, v_{i-1} \notin V(G_0)$  (i.e.  $P_{v_i}$  and  $P_{v_{i-1}}$  remain the same in both  $R$  and  $R_\ell$ ). Recall now that  $v_i$  and  $v_{i-1}$  are bounded by Lemma 27, and thus  $P_{v_i}$  does not intersect  $P_{v_{i-1}}$  in  $R$ , i.e. either  $P_{v_{i-1}} \ll_R P_{v_i}$  or  $P_{v_i} \ll_R P_{v_{i-1}}$ . Therefore, since  $l(v_i) <_R l(v_{i-1})$ , it follows that  $P_{v_i} \ll_R P_{v_{i-1}}$ .

We will now prove that  $b_u <_R r(v_i) <_R l(v_{i-1})$ . Recall that  $z_0 \in N(u)$ . Thus, if  $z_0 \in V(G_0)$ , then  $r(u) <_{R_0} r(z_0)$  by Lemma 1, and thus also  $r(u) <_{R_\ell} r(z_0)$ , since  $R_0$  is a sub-representation of  $R_\ell$ . Furthermore  $b_u <_{R'_\ell} r(u) <_{R'_\ell} r(z_0)$ , since the right endpoint  $r(z_0)$  of  $P_{z_0}$  does not decrease by Transformation 4. On the other hand, let  $z_0 \notin V(G_0)$ . Then  $b_u <_R r(z_0)$  by Lemma 1, and thus also  $b_u <_{R_\ell} r(z_0)$ , since  $z_0 \notin V(G_0)$  (i.e. the endpoints of  $P_{z_0}$  are the same in both  $R$  and  $R_\ell$ ). Furthermore  $b_u <_{R'_\ell} r(z_0)$ , since  $r(z_0)$  does not decrease by Transformation 4. That is, in both cases where  $z_0 \in V(G_0)$  and  $z_0 \notin V(G_0)$ , it follows that  $b_u <_{R'_\ell} r(z_0) = r_0 <_{R'_\ell} r(v)$  (since  $r_0 <_{R'_\ell} r(v)$  by our initial assumption on  $v$ ), and thus  $b_u <_{R'_\ell} r(v) = r(v_i)$ . Furthermore,  $b_u <_{R'_\ell} r(v_i) <_{R'_\ell} l(v_{i-1})$ , since we assumed that  $P_{v_i} \ll_{R'_\ell} P_{v_{i-1}}$ . Recall now that the value  $r(v_i)$  remains the same in both  $R_\ell$  and  $R'_\ell$ , since  $v_i \notin N(u)$  and by Transformation 4 only some endpoints of vertices of  $N(u)$  are moved. Furthermore, the value  $l(v_{i-1})$  remains the same in both  $R_\ell$  and  $R'_\ell$ , since the left lines of the parallelograms do not move by Transformation 4. Therefore  $b_u <_{R_\ell} r(v_i) <_{R_\ell} l(v_{i-1})$ , since also  $b_u <_{R'_\ell} r(v_i) <_{R'_\ell} l(v_{i-1})$ . Moreover, since  $v_i, v_{i-1} \notin V(G_0)$  (i.e. the endpoints of  $P_{v_i}$  and  $P_{v_{i-1}}$  remain the same in both  $R$  and  $R_\ell$ ), it follows that  $b_u <_R r(v_i) <_R l(v_{i-1})$ .

Suppose that  $v_{i-1} \in N(u)$ . Then  $L(v_{i-1}) <_R L(u) = a_u$  by Lemma 1, and thus  $R(v_i) <_R L(v_{i-1}) <_R a_u$ , since  $P_{v_i} \ll_R P_{v_{i-1}}$ . That is,  $R(v_i) <_R a_u$  and  $b_u <_R r(v_i)$  (by the previous paragraph). Therefore,  $P_{v_i}$  intersects  $P_u$  in  $R$  and  $\phi_{v_i} > \phi_u$  in  $R$ . Thus, since  $v_i$  is bounded, it follows that  $v_i \in N(u)$ . This is a contradiction to the assumption that  $v_i = v \in V_0(u)$ . Therefore  $v_{i-1} \notin N(u)$ . Thus, since  $v_{i-1} \in N(x_2)$  (by definition of  $H$ ) and  $x_2 \in V_0(u)$ , it follows that  $v_{i-1} \in V_0(u)$ . Therefore, in particular  $l(v_{i-1}) \leq_{R'_\ell} \ell_0$ , since  $\ell_0 = \max_{R'_\ell} \{l(x) \mid x \in V_0(u)\}$ .

Recall now that  $P_{v_i} \ll_{R'_\ell} P_{v_{i-1}}$  (as we assumed) and that  $r_0 = r(z_0) <_{R'_\ell} r(v) = r(v_i)$  (by our initial assumption on  $v$ ). Therefore  $r(z_0) <_{R'_\ell} r(v_i) <_{R'_\ell} l(v_{i-1}) \leq_{R'_\ell} \ell_0$ , i.e.  $r(z_0) <_{R'_\ell} \ell_0$ . This is a contradiction, since  $z_0 \in N_2$ .

Summarizing Cases 1 and 2, it follows that no adjacency  $vx$  has been removed from  $R'_\ell$  in  $R''_\ell$  in the case where  $x \in N(u)$ . This completes the proof of the lemma. ■

**Lemma 42**  $R'''_\ell$  is a projection representation of  $G$ .

**Proof.** The proof is done in two parts. In Part 1 we prove that  $u$  is adjacent in  $R'''_\ell$  to all vertices of  $N(u)$ , while in Part 2 we prove that  $u$  is not adjacent in  $R'''_\ell$  to any vertex of  $V \setminus N[u]$ .

*Part 1.* In this part we prove that  $u$  is adjacent in  $R'''_\ell$  to all vertices of  $N(u)$ . Denote by  $\hat{a}_u$  and  $\hat{b}_u$  the coordinates of the upper and lower endpoint of  $P_u$  in the projection representation  $R_\ell$  on  $L_1$  and on  $L_2$ , respectively. Then, since the endpoints of  $P_u$  do not move by Transformations 4 and 5,  $\hat{a}_u$  and  $\hat{b}_u$  remain the endpoints of  $P_u$  also in the representations  $R'_\ell$  and  $R''_\ell$ . Let  $z \in N(u)$  be arbitrary. Suppose that  $z \notin V(G_0)$ . Then, the left line of  $P_z$  remains the same in the representations  $R$ ,  $R_\ell$ ,  $R'_\ell$ , and  $R''_\ell$ . Therefore, since  $L(z) <_R a_u =_R L(u)$  by Lemma 1, it follows that also  $L(z) <_{R'_\ell} a_u <_{R'_\ell} L(u) = \hat{a}_u$ . Suppose that  $z \in V(G_0)$ . Then,  $L(z) <_{R_0} L(u)$  by Lemma 1, since  $R_0$  is a projection representation of  $G_0$ , and thus also  $L(z) <_{R_\ell} L(u) = \hat{a}_u$ , since  $R_0$  is a sub-representation of  $R_\ell$ . Furthermore  $L(z) <_{R'_\ell} L(u) = \hat{a}_u$ , since the left line of  $P_z$  remains the same in the representations  $R_\ell$ ,  $R'_\ell$ , and  $R''_\ell$ . Summarizing,  $L(z) <_{R'_\ell} \hat{a}_u$  for every vertex  $z \in N(u)$ . Therefore, since the endpoint  $L(z)$  does not move by Transformation 6, it follows that also  $L(z) <_{R''_\ell} \hat{a}_u$  for every vertex  $z \in N(u)$ .

Note now that  $\hat{a}_u <_{R''_\ell} L_0$ , since  $L_0 = \min_{R''_\ell} \{L(x) \mid x \in V_B \setminus N(u) \setminus V_0(u), P_u \ll_{R''_\ell} P_x\}$ . Furthermore, recall by Corollary 5 that for all vertices  $z \in N_1 \setminus N_2$ , for which  $R(z) <_{R''_\ell} L_0$ , the values  $R(z)$  lie immediately before  $L_0$  in  $R''_\ell$ . Therefore, since  $\hat{a}_u <_{R''_\ell} L_0$ , it follows in particular that

$\hat{a}_u <_{R'_\ell} R(z)$  for every  $z \in N_1 \setminus N_2$ , and thus  $L(z) <_{R'_\ell} \hat{a}_u <_{R'_\ell} R(z)$  for every  $z \in N_1 \setminus N_2 \subseteq N(u)$  by the previous paragraph. Therefore, since  $\hat{a}_u <_{R'_\ell} L_0$ , and since the upper endpoint  $R(u)$  of the line  $P_u$  lies in  $R'_\ell$  immediately before  $\min_{R'_\ell}\{L_0, R(z) \mid z \in N_1 \setminus N_2\}$ , cf. the statement of Transformation 6, it follows that also  $L(z) <_{R'_\ell} \hat{a}_u <_{R'_\ell} R(u) <_{R'_\ell} R(z)$  for every  $z \in N_1 \setminus N_2$ . That is,  $L(z) <_{R'_\ell} R(u) <_{R'_\ell} R(z)$  for every  $z \in N_1 \setminus N_2$ , and thus  $P_u$  intersects  $P_z$  in  $R'_\ell$  for every  $z \in N_1 \setminus N_2$ . Therefore, since all vertices of  $\{u\} \cup N_1 \setminus N_2$  are bounded in  $R'_\ell$ ,  $u$  is adjacent in  $R'_\ell$  to all vertices of  $N_1 \setminus N_2$ .

Consider now an arbitrary vertex  $z \in N_2$ . Recall that  $r_0 = \min_{R'_\ell}\{r(z) \mid z \in N_2\}$ , i.e.  $r_0 \leq_{R'_\ell} r(z)$ . Thus, since the endpoint  $r(z)$  does not move by Transformation 5, it follows that also  $r_0 \leq_{R'_\ell} r(z)$ . Furthermore, by Transformation 5,  $r(v) <_{R'_\ell} r_0 \leq_{R'_\ell} r(z)$  for every  $v \in V_0(u) \cap V_B$ . This holds clearly also in  $R'_\ell$ , i.e.  $r(v) <_{R'_\ell} r(z)$  for every  $v \in V_0(u) \cap V_B$ . Since the lower endpoint of the line  $P_u$  comes immediately after  $\max_{R'_\ell}\{r(v) \mid v \in V_0(u) \cap V_B\}$ , it follows that  $r(v) <_{R'_\ell} l(u) = r(u) <_{R'_\ell} r(z)$  for every  $v \in V_0(u) \cap V_B$  and every  $z \in N_2$ . Thus, since also  $L(z) <_{R'_\ell} \hat{a}_u <_{R'_\ell} R(u)$  for every  $z \in N(u)$ , it follows that  $P_u$  intersects  $P_z$  in  $R'_\ell$  for every  $z \in N_2$ . Therefore, since all vertices of  $\{u\} \cup N_2$  are bounded in  $R'_\ell$ ,  $u$  is adjacent in  $R'_\ell$  to all vertices of  $N_2$ . Thus, since  $N_2 \cup (N_1 \setminus N_2) = N(u)$ ,  $u$  is adjacent in  $R'_\ell$  to all vertices of  $N(u)$ .

*Part 2.* In this part we prove that  $u$  is not adjacent in  $R'_\ell$  to any vertex of  $V \setminus N[u]$ . To this end, recall first by Lemma 4 that  $u^*$  is a bounded covering vertex of  $u$  in  $G$  (and thus  $u^* \in V_0(u) \cap V_B$ ), such that  $P_u$  intersects  $P_{u^*}$  in the initial projection representation  $R$  and  $\phi_{u^*} < \phi_u$  in  $R$ . Therefore,  $l(u^*) <_R b_u =_R r(u)$  by Lemma 2. Furthermore,  $u^* \notin V(G_0)$  by Observation 4. Therefore, the endpoint  $l(u^*)$  remains the same in the representations  $R, R_\ell, R'_\ell$ , and  $R''_\ell$ , and thus  $l(u^*) <_{R'_\ell} b_u$ , since also  $l(u^*) <_R b_u$ . Therefore, since  $b_u <_{R'_\ell} \hat{b}_u =_{R'_\ell} r(u)$ , it follows that also  $l(u^*) <_{R'_\ell} \hat{b}_u =_{R'_\ell} r(u)$ . Recall now that  $L_0 = \min_{R'_\ell}\{L(x) \mid x \in V_B \setminus N(u) \setminus V_0(u), P_u \ll_{R'_\ell} P_x\}$ . Denote by  $y_0$  the vertex of  $V_B \setminus N(u) \setminus V_0(u)$ , such that  $L_0 = L(y_0)$  in  $R'_\ell$ , and thus  $P_u \ll_{R'_\ell} P_{y_0}$ . Therefore, since  $l(u^*) <_{R'_\ell} r(u)$ , it follows that  $l(u^*) <_{R'_\ell} r(u) <_{R'_\ell} l(y_0)$ . Now, since  $u^* \in V_0(u)$  and  $y_0 \notin N(u) \cup V_0(u)$ , it follows that  $u^*y_0 \notin E$ . Thus,  $P_{u^*} \ll_{R'_\ell} P_{y_0}$ , since both  $u^*$  and  $y_0$  are bounded vertices and  $l(u^*) <_{R'_\ell} l(y_0)$ . Moreover, since by Transformation 6 only the line  $P_u$  is moved, it follows that also  $P_{u^*} \ll_{R'_\ell} P_{y_0}$ .

Recall that  $u^* \notin V(G_0)$  and that  $u^*$  is adjacent to every vertex of  $V(G_0) \setminus \{u\}$  by Observation 4. Therefore  $u^* \in N(x_2)$ , since  $x_2 \in V(G_0) \setminus \{u\}$ , and thus  $P_{u^*}$  intersects the line segment  $\ell$  in  $R_\ell$ ; in particular,  $P_{u^*}$  intersects  $P_u$  in  $R_\ell$ . Moreover, since by Transformation 4 the parallelogram  $P_{u^*}$  is not modified,  $P_{u^*}$  intersects  $P_u$  also in  $R'_\ell$ . Denote by  $z_0$  the vertex of  $N_2$ , such that  $r_0 = r(z_0)$ . We will now prove that  $r(u) <_{R'_\ell} r_0 = r(z_0)$ . Suppose first that  $z_0 \notin V(G_0)$ . Then, in particular, either  $r(z_0) <_{R'_\ell} b_\ell - \varepsilon <_{R'_\ell} l(x_2)$  or  $r(x_2) <_{R'_\ell} b_\ell + \varepsilon <_{R'_\ell} r(z_0)$  by Remarks 2 and 3. Recall that  $\ell_0 = \max_{R'_\ell}\{l(x) \mid x \in V_0(u)\}$  and that  $z_0 \in N_2$ , and thus  $l(x_2) \leq_{R'_\ell} \ell_0 <_{R'_\ell} r(z_0)$ . Therefore  $r(x_2) <_{R'_\ell} b_\ell + \varepsilon <_{R'_\ell} r(z_0)$ . Thus, since  $u \in V(G_0)$ , also  $r(u) <_{R'_\ell} b_\ell + \varepsilon <_{R'_\ell} r(z_0)$  in the case where  $z_0 \notin V(G_0)$ . Suppose now that  $z_0 \in V(G_0)$ ; then  $r(u) <_{R_0} r(z_0)$  by Lemma 1. Thus, since  $R_0$  is a sub-representation of  $R'_\ell$ , and since  $r(z_0)$  does not decrease by Transformation 4, it follows that  $r(u) <_{R'_\ell} r(z_0) = r_0$  in the case where  $z_0 \in V(G_0)$ . That is,  $r(u) <_{R'_\ell} r_0 = r(z_0)$  in both cases, where  $z_0 \in V(G_0)$  and  $z_0 \notin V(G_0)$ .

We will now prove that  $P_{u^*}$  intersects  $P_u$  also in  $R'_\ell$ . This holds clearly in the case where the right line of  $P_{u^*}$  is not moved during Transformation 5, since  $P_{u^*}$  intersects  $P_u$  in  $R'_\ell$  by the previous paragraph. Suppose now that the right line of  $P_{u^*}$  is moved during Transformation 5. Then,  $r(u) <_{R'_\ell} r_0 <_{R'_\ell} r(u^*)$ , while  $r(u^*)$  comes immediately before  $r_0$  in  $R'_\ell$ , i.e.  $r(u) <_{R'_\ell} r(u^*) <_{R'_\ell} r_0$ , since  $r_0 = r(z_0)$  does not move during Transformation 5. Therefore, since the left line of  $P_{u^*}$  does not move during Transformation 5, and since  $P_{u^*}$  intersects  $P_u$  in  $R'_\ell$ , it follows that  $P_{u^*}$  intersects  $P_u$  also in  $R'_\ell$ .

Denote by  $v_0$  the vertex of  $V_0(u) \cap V_B$ , such that  $r(v_0) = \max_{R'_\ell}\{r(v) \mid v \in V_0(u) \cap V_B\}$ , cf. the statement of Transformation 6. Since  $v_0 \in V_0(u)$  and  $y_0 \notin N(u) \cup V_0(u)$ , it follows that  $v_0y_0 \notin E$ . Therefore, since both  $v_0$  and  $y_0$  are bounded vertices, either  $P_{y_0} \ll_{R'_\ell} P_{v_0}$  or  $P_{v_0} \ll_{R'_\ell} P_{y_0}$ . Suppose that  $P_{y_0} \ll_{R'_\ell} P_{v_0}$ , and thus  $P_{u^*} \ll_{R'_\ell} P_{y_0} \ll_{R'_\ell} P_{v_0}$ . Then, since  $u^*, v_0 \in V_0(u)$  and since  $V_0(u)$  is connected, there exists at least one vertex  $v \in V_0(u)$ , such that  $P_v$  intersects  $P_{y_0}$  in  $R'_\ell$ . Similarly

$vy_0 \notin E$ , since  $y_0 \notin N(u) \cup V_0(u)$ . Therefore, since  $y_0$  is a bounded vertex,  $v$  must be an unbounded vertex with  $\phi_v > \phi_{y_0}$  in  $R_\ell''$ , and thus  $N(v) \subseteq N(y_0)$  by Lemma 3. Then,  $N(v)$  includes at least one vertex  $v' \in V_0(u)$ , and thus  $v' \in N(y_0)$ . Therefore,  $y_0 \in V_0(u)$ , which is a contradiction. Thus,  $P_{v_0} \ll_{R_\ell''} P_{y_0}$ . Moreover, since by Transformation 6 only the line  $P_u$  is moved, it follows that also  $P_{v_0} \ll_{R_\ell'''} P_{y_0}$ .

We will prove in the following that  $u$  is not adjacent in  $R_\ell'''$  to any vertex  $x \notin N(u)$ . For the sake of contradiction, suppose that  $P_x$  intersects  $P_u$  in  $R_\ell'''$ . We distinguish in the following the cases regarding  $x$ .

*Case 2a.*  $x \in V_B \setminus N(u)$  (i.e.  $x$  is bounded) and  $x \in V_0(u)$ . Then,  $r(x) \leq_{R_\ell''} r(v_0)$  and  $r(u^*) \leq_{R_\ell''} r(v_0)$  by definition of  $v_0$ , and thus also  $r(x) \leq_{R_\ell'''} r(v_0)$  and  $r(u^*) \leq_{R_\ell'''} r(v_0)$ . Therefore, by Transformation 6,  $r(x) \leq_{R_\ell'''} r(v_0) <_{R_\ell'''} l(u)$ , i.e.  $r(x) <_{R_\ell'''} l(u)$ . Thus  $L(u) <_{R_\ell'''} R(x)$ , since we assumed that  $P_x$  intersects  $P_u$  in  $R_\ell'''$ . Furthermore,  $r(x) \leq_{R_\ell'''} r(v_0) <_{R_\ell'''} l(y_0)$ , i.e.  $r(x) <_{R_\ell'''} l(y_0)$ , since  $P_{v_0} \ll_{R_\ell'''} P_{y_0}$ . Recall by Corollary 5 that for all vertices  $z \in N_1 \setminus N_2$ , for which  $R(z) <_{R_\ell''} L_0 = L(y_0)$ , the values  $R(z)$  lie immediately before  $L_0$  in  $R_\ell''$ , and thus also in  $R_\ell'''$ . Thus, since  $L(u) <_{R_\ell'''} R(x)$ , and since the upper point  $L(u) = R(u)$  lies immediately before  $\min\{L_0, R(z) \mid z \in N_1 \setminus N_2\}$  in  $R_\ell'''$ , it follows that  $L(u) <_{R_\ell'''} L_0 = L(y_0) <_{R_\ell'''} R(x)$ . Therefore, since also  $r(x) <_{R_\ell'''} l(y_0)$ ,  $P_x$  intersects  $P_{y_0}$  in  $R_\ell'''$ , and thus also in  $R_\ell''$ . Thus  $xy_0 \in E$ , since both  $x$  and  $y_0$  are bounded, and therefore  $y_0 \in V_0(u)$ , which is a contradiction. Therefore,  $P_x$  does not intersect  $P_u$  in  $R_\ell'''$ , for every  $x \in V_B \setminus N(u)$ , such that  $x \in V_0(u)$ . In particular, since  $u^*, v_0 \in V_B \setminus N(u)$  and  $u^*, v_0 \in V_0(u)$ , it follows that neither  $P_{u^*}$  nor  $P_{v_0}$  intersects  $P_u$  in  $R_\ell'''$ . Therefore, since  $r(u^*) \leq_{R_\ell''} r(v_0) <_{R_\ell'''} l(u)$  by Transformation 6, it follows that  $P_{u^*} \ll_{R_\ell'''} P_u$  and  $P_{v_0} \ll_{R_\ell'''} P_u$ .

*Case 2b.*  $x \in V_B \setminus N(u)$  (i.e.  $x$  is bounded) and  $x \notin V_0(u)$ . Then  $u^*x \notin E$ , since  $u^* \in V_0(u)$ . Furthermore, since both  $x$  and  $u^*$  (resp.  $v_0$ ) are bounded vertices, either  $P_x \ll_{R_\ell'''} P_{u^*}$  or  $P_{u^*} \ll_{R_\ell'''} P_x$  (resp. either  $P_x \ll_{R_\ell'''} P_{v_0}$  or  $P_{v_0} \ll_{R_\ell'''} P_x$ ). If  $P_x \ll_{R_\ell'''} P_{u^*}$  (resp.  $P_x \ll_{R_\ell'''} P_{v_0}$ ), then  $P_x \ll_{R_\ell'''} P_{u^*} \ll_{R_\ell'''} P_u$  (resp.  $P_x \ll_{R_\ell'''} P_{v_0} \ll_{R_\ell'''} P_u$ ) by the previous paragraph. This is a contradiction to the assumption that  $P_x$  intersects  $P_u$  in  $R_\ell'''$ . Therefore  $P_{u^*} \ll_{R_\ell'''} P_x$  and  $P_{v_0} \ll_{R_\ell'''} P_x$ , and thus also  $P_{u^*} \ll_{R_\ell''} P_x$  and  $P_{v_0} \ll_{R_\ell''} P_x$ . Thus, in particular  $r(v_0) <_{R_\ell''} l(x)$ . Furthermore, the lower endpoint  $l(u) = r(u)$  of  $P_u$  comes by Transformation 6 immediately after  $r(v_0)$  in  $R_\ell'''$ , and thus  $r(v_0) <_{R_\ell'''} r(u) <_{R_\ell'''} l(x)$ . Then,  $L(x) <_{R_\ell'''} R(u)$ , since we assumed that  $P_x$  intersects  $P_u$  in  $R_\ell'''$ .

We distinguish now the cases according to the relative positions of  $P_u$  and  $P_x$  in  $R_\ell''$ . If  $P_x \ll_{R_\ell''} P_u$ , then  $P_{u^*} \ll_{R_\ell''} P_x \ll_{R_\ell''} P_u$  by the previous paragraph, which is a contradiction, since  $P_{u^*}$  intersects  $P_u$  in  $R_\ell''$ , as we proved above. If  $P_u \ll_{R_\ell''} P_x$ , then  $L_0 \leq_{R_\ell''} L(x)$ , since  $x \in V_B \setminus N(u) \setminus V_0(u)$  and  $L_0 = \min_{R_\ell''}\{L(x) \mid x \in V_B \setminus N(u) \setminus V_0(u), P_u \ll_{R_\ell''} P_x\}$ . Thus  $R(u) <_{R_\ell'''} L_0 \leq_{R_\ell'''} L(x)$  by Transformation 3, which is a contradiction, since  $L(x) <_{R_\ell'''} R(u)$  by the previous paragraph. Suppose that  $P_x$  intersects  $P_u$  in  $R_\ell''$ . Note that  $x \notin V(G_0)$ , since  $x \notin N(u) \cup V_0(u)$  and  $V(G_0) \subseteq N[u] \cup V_0(u)$  by Observation 3. Thus, since we assumed that  $P_x$  intersects  $P_u$  in  $R_\ell''$ , i.e.  $P_x$  intersects the line segment  $\ell$  in  $R_\ell''$ , it follows that  $P_x$  intersects also  $P_{x_2}$  in  $R_\ell''$ . Therefore  $x \in N(x_2)$ , since both  $x$  and  $x_2$  are bounded, and thus  $x \in V_0(u)$ , since also  $x_2 \in V_0(u)$ . This is a contradiction, since  $x \notin V_0(u)$  by the assumption of Case 2b. Therefore,  $P_x$  does not intersect  $P_u$  in  $R_\ell''$ , for every  $x \in V_B \setminus N(u)$ , such that  $x \notin V_0(u)$ .

*Case 2c.*  $x \in V_U$  (i.e.  $x$  is unbounded), such that  $\phi_x < \phi_u$  in  $R_\ell'''$ . Then, since both  $P_x$  and  $P_u$  are lines in  $R_\ell'''$ , it follows that  $l(x) <_{R_\ell'''} l(u)$  and  $R(x) >_{R_\ell'''} R(u)$ . Thus, by Transformation 6,  $l(x) <_{R_\ell'''} r(v_0) <_{R_\ell'''} l(u)$  and  $R(u) <_{R_\ell'''} L_0 = L(y_0) <_{R_\ell'''} R(x)$ . Since  $P_{v_0} \ll_{R_\ell'''} P_{y_0}$  (as we proved above), it follows that  $P_x$  intersects both  $P_{v_0}$  and  $P_{y_0}$  in  $R_\ell'''$  (and thus also in  $R_\ell''$ ), and that  $\phi_x < \phi_{v_0}$  and  $\phi_x < \phi_{y_0}$  in both  $R_\ell''$  and  $R_\ell'''$ . Therefore, since both  $v_0$  and  $y_0$  are bounded, it follows that  $x \in N(v_0)$  and  $x \in N(y_0)$ . Thus  $x, y_0 \in V_0(u)$ , since  $v_0 \in V_0(u)$ . This is a contradiction, since  $y_0 \notin V_0(u)$  by definition of  $y_0$ . Therefore,  $P_x$  does not intersect  $P_u$  in  $R_\ell'''$ , for every  $x \in V_U$ , for which  $\phi_x < \phi_u$  in  $R_\ell'''$ .

Summarizing, due to Part 1 and due to Cases 2a, 2b, and 2c of Part 2, it follows that  $P_u$  intersects in  $R_\ell'''$  only the parallelograms  $P_z$ , for every  $z \in N(u)$ , and possibly some trivial parallelograms



(lines)  $P_x$ , where  $x \in V_U$  and  $\phi_x > \phi_u$  in  $R_\ell'''$ . However, since  $\phi_x > \phi_u$  in  $R_\ell'''$  for all these vertices  $x$ , it follows that  $u$  is not adjacent to these vertices in  $R_\ell'''$ . Thus  $R_\ell'''$  is a projection representation of  $G$ , since  $R_\ell'' \setminus \{u\}$  is a projection representation of  $G \setminus \{u\}$  by Lemma 41. This completes the proof of the lemma. ■

The next lemma follows now easily by Lemma 42 and by the fact that  $V_0(u)$  induces a connected subgraph of  $G$ .

**Lemma 43** *The (bounded) vertex  $u$  has the right border property in  $R_\ell'''$ , i.e. there exists no pair of vertices  $z \in N(u)$  and  $v \in V_0(u)$ , such that  $P_z \ll_{R_\ell'''} P_v$ .*

**Proof.** Recall first that  $u_0^* \in V_0(u) \cap V_B$  by Lemma 4, i.e.  $V_0(u) \cap V_B \neq \emptyset$ . Furthermore, recall that by Transformation 6 the lower endpoint  $l(u) = r(u)$  of  $P_u$  comes immediately after  $\max\{r(v) \mid v \in V_0(u) \cap V_B\}$  in  $R_\ell'''$ , and thus  $r(v) <_{R_\ell'''} r(u)$  for every  $v \in V_0(u) \cap V_B$ . Since  $u$  is a bounded vertex in  $R_\ell'''$ , and since  $R_\ell'''$  is a projection representation of  $G$  by Lemma 42,  $P_u$  does not intersect  $P_v$  in  $R_\ell'''$ , for any  $v \in V_0(u) \cap V_B$ . Therefore, for every  $v \in V_0(u) \cap V_B$ , either  $P_u \ll_{R_\ell'''} P_v$  or  $P_v \ll_{R_\ell'''} P_u$ . If  $P_u \ll_{R_\ell'''} P_v$  for a vertex  $v \in V_0(u) \cap V_B$ , then in particular  $r(u) <_{R_\ell'''} r(v)$ , which is a contradiction. Therefore,  $P_v \ll_{R_\ell'''} P_u$  for every  $v \in V_0(u) \cap V_B$ .

Suppose now for the sake of contradiction that  $P_z \ll_{R_\ell'''} P_v$  for two vertices  $z \in N(u)$  and  $v \in V_0(u)$ . Suppose first that  $v$  is a bounded vertex, i.e.  $v \in V_0(u) \cap V_B$ . Then, since  $P_v \ll_{R_\ell'''} P_u$  by the previous paragraph, it follows that  $P_z \ll_{R_\ell'''} P_v \ll_{R_\ell'''} P_u$ , and thus  $z \notin N(u)$ , which is a contradiction.

Suppose now that  $v$  is an unbounded vertex. Then, since  $V_0(u)$  is connected and  $V_0(u) \cap V_B \neq \emptyset$ , there exists at least one bounded vertex  $v' \in V_0(u) \cap V_B$ , such that  $v' \in N(v)$ . Then  $P_{v'} \ll_{R_\ell'''} P_u$ , as we proved above. We distinguish now the cases according to the relative positions of  $P_v$  and  $P_u$  in  $R_\ell'''$ . If  $P_v \ll_{R_\ell'''} P_u$ , then  $P_z \ll_{R_\ell'''} P_v \ll_{R_\ell'''} P_u$  by the assumption on  $z$  and  $v$ , and thus  $z \notin N(u)$ , which is a contradiction. If  $P_u \ll_{R_\ell'''} P_v$ , then  $P_{v'} \ll_{R_\ell'''} P_u \ll_{R_\ell'''} P_v$ , and thus  $v' \notin N(v)$ , which is again a contradiction. Suppose that  $P_v$  intersects  $P_u$  in  $R_\ell'''$ . Then,  $\phi_v > \phi_u$  in  $R_\ell'''$ , since  $u$  is bounded in  $R_\ell'''$  and  $v \notin N(u)$ . Therefore, in particular  $r(u) <_{R_\ell'''} r(v)$ . Furthermore, since  $v$  is unbounded and  $v' \in N(u)$ , it follows that  $r(v) <_{R_\ell'''} r(v')$  by Lemma 1, and thus  $r(u) <_{R_\ell'''} r(v) <_{R_\ell'''} r(v')$ , i.e.  $r(u) <_{R_\ell'''} r(v')$ . This is a contradiction, since  $P_{v'} \ll_{R_\ell'''} P_u$  for every  $v' \in V_0(u) \cap V_B$ , as we proved above. Summarizing, there exist no vertices  $z \in N(u)$  and  $v \in V_0(u)$ , such that  $P_z \ll_{R_\ell'''} P_v$ . This completes the proof of the lemma. ■

## The correctness of Condition 4

Note now that the projection representation  $R_\ell'''$  of  $G$  (cf. Lemma 42) has  $k - 1$  unbounded vertices, since the input graph  $G$  has  $k$  unbounded vertices, and since  $u$  is bounded in  $R_\ell'''$ . Therefore, the projection representation  $R^* = R_\ell'''$  satisfies the conditions of Theorem 2. However, in order to complete the proof of Theorem 2, we have to prove the correctness of Condition 4 (cf. Lemma 45). To this end, we first prove Lemma 44.

Recall that for simplicity reasons, before applying Transformations 4, 5, and 6, we have added to  $G$  an isolated bounded vertex  $t$ , and thus also  $t \in V_B \setminus N(u) \setminus V_0(u)$ . This isolated vertex  $t$  corresponds to a parallelogram  $P_t$ , such that  $P_v \ll_R P_t$  and  $P_v \ll_{R_\ell} P_t$  for every other vertex  $v$  of  $G$ ; thus also  $P_v \ll_{R_\ell'} P_t$ ,  $P_v \ll_{R_\ell''} P_t$ , and  $P_v \ll_{R_\ell'''} P_t$  for every vertex  $v \neq t$  of  $G$ . The next lemma follows now easily by Transformation 6 and Lemma 42.

**Lemma 44** *If  $V_B \setminus N(u) \setminus V_0(u) = \{t\}$ , then there exists a projection representation  $R^\#$  of  $G$  with the same unbounded vertices as in  $R$ , where  $u$  has the right border property in  $R^\#$ .*

**Proof.** Suppose that  $V_B \setminus N(u) \setminus V_0(u) = \{t\}$ , i.e. the set  $V_B \setminus N(u) \setminus V_0(u)$  is empty in  $G$  before the addition of the isolated bounded vertex  $t$ . Then, the values  $L_0$  and  $L_0(z)$  for every  $z \in N(u)$  are all equal to  $L(t)$ . Therefore, since we can place the parallelogram  $P_t$  that corresponds to  $t$  arbitrarily much to the right of every other parallelogram in the projection representation  $R_\ell$ , these

values can become arbitrarily big in  $R_\ell$ . Recall that  $N_1 = \{z \in N(u) \mid r(z) <_{R_\ell} \ell_0\}$  by definition. Then, during Transformation 4,  $r(z)$  comes immediately after  $\ell_0$  on  $L_2$  for every  $z \in N_1$  (i.e.  $R(z)$  does not come immediately before  $L_0(z)$  on  $L_1$ , since  $L_0(z) = L(t)$  is arbitrarily big). Therefore,  $\ell_0 <_{R'_\ell} r(z)$  for every  $z \in N_1$ , and thus  $\ell_0 <_{R'_\ell} r(z)$  for every  $z \in N(u)$ . That is,  $N_2 = N(u)$ , since by definition  $N_2 = \{z \in N(u) \mid \ell_0 <_{R'_\ell} r(z)\}$ . Thus, in particular  $N_1 \setminus N_2 = N_1 \setminus N(u) = \emptyset$ , since  $N_1 \subseteq N(u)$  by definition.

Consider now the projection representation  $R_\ell'''$ , which is obtained by applying Transformation 6 to  $R_\ell''$ . Recall that by Transformation 6 the upper endpoint  $L(u) = R(u)$  of the line  $P_u$  comes immediately before  $\min\{L_0, R(z) \mid z \in N_1 \setminus N_2\} = L_0$  in  $R_\ell'''$  (since  $N_1 \setminus N_2 = \emptyset$  by the previous paragraph). Then, since the value  $L_0 = L(t)$  has been chosen arbitrarily big, the slope  $\phi_u$  of  $P_u$  becomes arbitrarily small in  $R_\ell'''$ , i.e. in particular smaller than all other slopes in  $R_\ell'''$ . Furthermore, since  $R_\ell'''$  is a projection representation of  $G$  by Lemma 42, it follows that  $P_u$  intersects in  $R_\ell'''$  only the parallelograms  $P_z$ , for every  $z \in N(u)$ , and possibly some trivial parallelograms (lines)  $P_x$ , where  $x$  is an unbounded vertex and  $\phi_x > \phi_u$  in  $R_\ell'''$ . Denote now by  $R^\#$  the projection representation that is obtained from  $R_\ell'''$  if we make  $u$  again an unbounded vertex. Then, since the slope  $\phi_u$  is smaller than all other slopes in both  $R_\ell'''$  and  $R^\#$ , it follows in particular that  $\phi_u < \phi_z$  in  $R^\#$  for every  $z \in N(u)$ . Therefore,  $u$  remains adjacent to all vertices  $z \in N(u)$  in the graph induced by  $R^\#$ , and thus  $R^\#$  is a projection representation of  $G$ , in which  $u$  is an unbounded vertex.

Finally, recall by Lemma 43 that there exists no pair of vertices  $z \in N(u)$  and  $v \in V_0(u)$ , such that  $P_z \ll_{R_\ell'''} P_v$  in  $R_\ell'''$ . Therefore, since the only difference between  $R_\ell'''$  and  $R^\#$  is that  $u$  is made bounded in  $R^\#$ , there exists also in  $R^\#$  no pair of vertices  $z \in N(u)$  and  $v \in V_0(u)$ , such that  $P_z \ll_{R^\#} P_v$  in  $R^\#$ . That is,  $u$  has the right border property in  $R^\#$ . This completes the proof of the lemma. ■

Now we can prove the correctness of Condition 4.

**Lemma 45** *Condition 4 is true.*

**Proof.** Let  $G = (V, E)$  be a connected graph in  $\text{TOLERANCE} \cap \text{TRAPEZOID}$  and  $R$  be a projection representation of  $G$  with  $u$  as the only unbounded vertex. Let furthermore  $V_0(u) \neq \emptyset$  be connected and  $V = N[u] \cup V_0(u)$ . If  $u$  has the right (resp. the left) border property in  $R$ , then  $R$  (resp. the reverse representation  $\hat{R}$  of  $R$ ) satisfies Condition 4. Suppose now that  $u$  has neither the left nor the right border property in  $R$ , and suppose w.l.o.g. that  $G$  has the smallest number of vertices among the graphs that satisfy the above conditions. Then, since  $V_0(u) \neq \emptyset$  is connected, the whole proof of Theorem 2 above applies to  $G$ . In particular, we can construct similarly to the above the induced subgraphs  $G_0$  and  $G'_0 = G[V(G_0) \cup \{u^*\}]$  of  $G$ . Then,  $V(G_0) \subseteq N[u] \cup V_0(u)$  by Observation 3, and thus also  $V(G'_0) \subseteq N[u] \cup V_0(u)$ , since  $u^* \in V_0(u)$ . Furthermore,  $u$  is the only unbounded vertex of  $G'_0$ .

Recall that  $G'_0$  is a connected subgraph of  $G$  by Observation 4. Furthermore,  $G'_0$  has strictly smaller vertices than  $G$ , and thus Condition 4 applies to  $G'_0$ , i.e. we can construct the projection representations  $R_\ell$ ,  $R'_\ell$ ,  $R''_\ell$ , and  $R'''_\ell$ , as above. Moreover, since  $V = V(G) = N[u] \cup V_0(u)$  by assumption, it follows that  $V_B \setminus N(u) \setminus V_0(u) = \{t\}$  after adding an isolated bounded vertex  $t$  to  $R_\ell$ . Thus, there exists by Lemma 44 a projection representation  $R^{**} = R^\#$  of  $G$  with the same unbounded vertices as in  $R$  (i.e. with  $u$  as the only unbounded vertex), such that  $u$  has the right border property in  $R^{**}$ . This completes the proof of the lemma. ■

Summarizing, since also the correctness of Condition 4 has been proved in Lemma 45, the projection representation  $R^* = R_\ell'''$  of  $G$ , cf. Lemma 42, has  $k - 1$  unbounded vertices, since the input graph  $G$  has  $k$  unbounded vertices, and since  $u$  is bounded in  $R_\ell'''$ . This completes the proof of Theorem 2. ■