The Recognition of Triangle Graphs*

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Abstract

Trapezoid graphs are the intersection graphs of trapezoids, where every trapezoid has a pair of opposite sides lying on two parallel lines L_1 and L_2 of the plane. This subclass of perfect graphs has received considerable attention as it generalizes in a natural way both interval and permutation graphs. In particular, trapezoid graphs have been introduced in order to generalize some well known applications of these graphs on channel routing in integrated circuits. Strictly between permutation and trapezoid graphs lie the *triangle* graphs – also known as *PI** graphs (for Point-Interval) – where the intersecting objects are triangles with one point of the triangle on the one line and the other two points (i.e. interval) of the triangle on the other line. Note that there is no restriction on which line between L_1 and L_2 contains one point of the triangle and which line contains the other two. Due to both their interesting structure and their practical applications, several efficient algorithms for optimization problems that are NP-hard in general graphs have been designed for trapezoid graphs – which also apply to triangle graphs. In spite of this, the complexity status of the triangle graph recognition problem (namely, the problem of deciding whether a given graph is a triangle graph) has been the most fundamental open problem on this class of graphs since its introduction two decades ago. Moreover, since triangle graphs lie naturally between permutation and trapezoid graphs, and since they share a very similar structure with them, it was expected that the recognition of triangle graphs is polynomial, as it is also the case for permutation and trapezoid graphs. In this article we surprisingly prove that the recognition of triangle graphs is NP-complete, even in the case where the input graph is known to be a trapezoid graph.

Keywords: Intersection graphs, trapezoid graphs, PI graphs, PI* graphs, recognition problem, NP-complete.

1 Introduction

A graph $G = (V, E)$ with n vertices is the *intersection graph* of a family $F = \{S_1, \ldots, S_n\}$ of subsets of a set S if there exists a bijection $\mu : V \to F$ such that for any two distinct vertices $u, v \in V$, $uv \in E$ if and only if $\mu(u) \cap \mu(v) \neq \emptyset$. Then, F is called an *intersection model* of G. Note that every graph has a trivial intersection model based on adjacency relations [18]. However, some intersection models provide a natural and intuitive understanding of the structure of a class of graphs, and turn out to be very helpful to obtain structural results, as well as to find efficient algorithms to solve optimization problems [18]. Many important graph classes can be described as intersection graphs of set families that are derived from some kind of geometric configuration.

Consider two parallel horizontal lines on the plane, L_1 (the upper line) and L_2 (the lower line). Various intersection graphs can be defined on objects formed with respect to these two

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lines. In particular, for *permutation* graphs, the objects are line segments that have one endpoint on L_1 and the other one on L_2. Generalizing to objects that are trapezoids with one interval on L_1 and the opposite interval on L_2, *trapezoid* graphs have been introduced independently in [5] and [6]. Given a trapezoid graph G, an intersection model of G with trapezoids between L_1 and L_2 is called a *trapezoid representation* of G. Trapezoid graphs are perfect graphs [3, 9] and generalize in a natural way both interval graphs (when the trapezoids are rectangles) and permutation graphs (when the trapezoids are trivial, i.e. lines). In particular, the main motivation for the introduction of trapezoid graphs was to generalize some well known applications of interval and permutation graphs on channel routing in integrated circuits [6].

Moreover, two interesting subclasses of trapezoid graphs have been introduced in [5]. A trapezoid graph G is a *simple-triangle* graph if it admits a trapezoid representation, in which every trapezoid is a triangle with one point on L_1 and the other two points (i.e. interval) on L_2. Similarly, G is a *triangle* graph if it admits a trapezoid representation, in which every trapezoid is a triangle, but now there is no restriction on which line between L_1 and L_2 contains one point of the triangle and which one contains the other two points (i.e. the interval) of the triangle. Such an intersection model of a simple-triangle (resp. triangle) graph G with triangles between L_1 and L_2 is called a *simple-triangle* (resp. *triangle* representation of G). Simple-triangle and triangle graphs are also known as *PI* and *PI** graphs, respectively [3–5, 15], where PI stands for "Point-Interval"; note that, using this notation, permutation graphs are *PP* (for "Point-Point") graphs, while trapezoid graphs are *II* (for "Interval-Interval") graphs [5]. In particular, both interval and permutation graphs are strictly contained in simple-triangle graphs, which are strictly contained in triangle graphs, which are strictly contained in trapezoid graphs [3, 5]. For instance, it is easy to see that every interval graph G is also a simple-triangle graph: given an interval representation of G, replace every interval I_v in this representation by an isosceles triangle T_v of unit height, which has the interval I_v as its base. The resulting representation is a simple-triangle representation of G, since for any two vertices u and v of G, the intervals I_u and I_v intersect if and only if T_u and T_v intersect.

Due to both their interesting structure and their practical applications, trapezoid graphs have attracted many research efforts. In particular, efficient algorithms for several optimization problems that are NP-hard in general graphs have been designed for trapezoid graphs [2,7,11–13, 16, 24], which also apply to triangle and simple-triangle graphs. Furthermore, several efficient algorithms appeared for the recognition problems of both permutation [9, 17] and trapezoid graphs [14, 16, 20]; see [25] for an overview.

In spite of this, the complexity status of both triangle and simple-triangle recognition problems have been the most fundamental open problems on these classes of graphs since their introduction two decades ago [3]. Since, on the one hand, very few subclasses of perfect graphs are known to be NP-hard to recognize (for instance, *perfectly orderable* graphs [22], *EPT* graphs [10], and recently *tolerance* and *bounded tolerance* graphs [21]) and, on the other hand, triangle and simple-triangle graphs lie naturally between permutation and trapezoid graphs, while they share a very similar structure with them, it was plausible that the recognition of triangle and simple-triangle graphs was polynomial.

Our contribution. In this article we establish the complexity of recognizing triangle graphs. Namely, we prove that this problem is surprisingly NP-hard, by providing a reduction from the 3SAT problem. Specifically, given a boolean formula formula ϕ in conjunctive normal form with three literals in every clause (3-CNF), we construct a trapezoid graph G_ϕ, which is a triangle graph if and only if ϕ is satisfiable. Therefore, as the recognition problems for both triangle and simple-triangle graphs are in the complexity class NP, it follows in particular that the triangle graph recognition problem is NP-complete. This complements the recent surprising result that the recognition of *parallelogram* graphs (i.e. the intersection graphs of parallelograms between two parallel lines L_1 and L_2), which coincides with *bounded tolerance* graphs, is NP-complete [21].

Organization of the paper. Background definitions and properties of trapezoid graphs and their representations are presented in Section 2. In Section 3 we introduce the notion of a *standard trapezoid representation*, the existence of which is a sufficient condition for a trapezoid graph to be a triangle graph. In Sections 4 and 5, we investigate the structure of some specific trapezoid and triangle graphs, respectively, and prove special properties of them. We use these graphs as parts of the gadgets in our reduction of 3SAT to the recognition problem of triangle graphs, which we present in Section 6. Finally, we discuss the presented results and further research in Section 7.

2 Triangle and simple-triangle graphs

In this section we provide some notation and properties of trapezoid graphs and their representations, which will be mainly applied in the sequel to triangle and simple-triangle graphs.

Notation. We consider in this article simple undirected and directed graphs with no loops or multiple edges. In an undirected graph G, the edge between vertices u and v is denoted by uv, and in this case u and v are said to be *adjacent* in G. Given a graph $G = (V, E)$ and a subset $S \subseteq V$, $G[S]$ denotes the induced subgraph of G on the vertices in S. Furthermore, we denote for simplicity by $G - S$ the induced subgraph $G[V \setminus S]$ of G. Moreover, given a graph G, we denote its vertex set by $V(G)$. A connected graph $G = (V, E)$ is called *k-connected*, where $k \geq 1$, if k is the smallest number of vertices that have to be removed from G such that the resulting graph is disconnected. Furthermore, a vertex v of a 1-connected graph G is called a *cut vertex* of G, if $G - \{v\}$ is disconnected. By possibly performing a small shift of the endpoints, we assume throughout the article without loss of generality that all endpoints of the trapezoids (resp. triangles) in a trapezoid (resp. triangle or simple-triangle) representation are distinct [8, 11, 12]. Given a trapezoid (resp. triangle or simple-triangle) graph G along with a trapezoid (resp. triangle or simple-triangle) representation R, we may not distinguish in the following between a vertex of G and the corresponding trapezoid (resp. triangle) in R, whenever it is clear from the context. Moreover, given an induced subgraph H of G, we denote by $R[H]$ the restriction of the representation R on the trapezoids (resp. triangles) of H.

Consider a trapezoid graph $G = (V, E)$ and a trapezoid representation R of G, where for any vertex $u \in V$ the trapezoid corresponding to u in R is denoted by T_u. Since trapezoid graphs are also cocomparability graphs (there is a transitive orientation of the complement) [9], we can define the partial order $(V, \ll_R)$, such that $u \ll_R v$, or equivalently $T_u \ll_R T_v$, if and only if T_u lies completely to the left of T_v in R (and thus also $uv \notin E$). Otherwise, if neither $T_u \ll_R T_v$ nor $T_v \ll_R T_u$, we will say that T_u *intersects* T_v in R (and thus also $uv \in E$). Furthermore, we define the total order $<_R$ on the lines L_1 and L_2 in R as follows. For two points a and b on L_1 (resp. on L_2), if a lies to the left of b on L_1 (resp. on L_2), then we will write $a <_R b$.

There are several trapezoid representations of a particular trapezoid graph G. For instance, given one such representation R, we can obtain another one R' by *vertical axis flipping* of R, i.e. R' is the mirror image of R along an imaginary line perpendicular to L_1 and L_2. Moreover, we can obtain another representation R'' of G by *horizontal axis flipping* of R, i.e. R'' is the mirror image of R along an imaginary line parallel to L_1 and L_2. We will use extensively these two basic operations throughout the article. For every trapezoid T_u in R, where $u \in V$, we define by $l(u)$ and $r(u)$ (resp. $L(u)$ and $R(u)$) the lower (resp. upper) left and right endpoint of T_u, respectively (cf. the trapezoid T_v in Figure 2). Since every triangle and simple-triangle

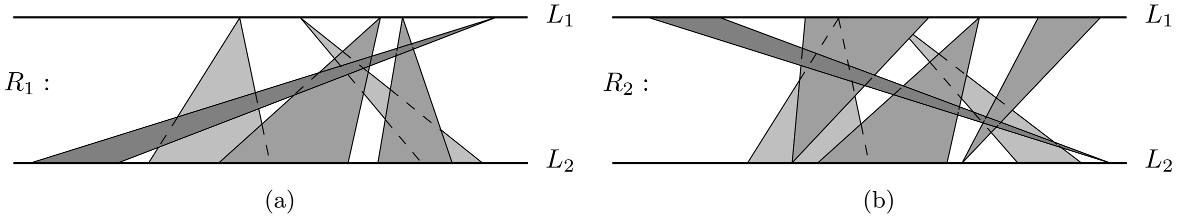

Figure 1: (a) A simple-triangle representation R_1 and (b) a triangle representation R_2.

representation is a special type of a trapezoid representation, all the above notions can be also applied to triangle and simple-triangle graphs. Note here that, if R is a simple-triangle representation of $G = (V, E)$, then $L(u) = R(u)$ for every $u \in V$; similarly, if R is a triangle representation of G, then $L(u) = R(u)$ or $l(u) = r(u)$ for every $u \in V$. An example of a simple-triangle and a triangle representation is shown in Figure 1.

It can be easily seen that every triangle (resp. single-triangle) graph G with n vertices has a triangle (resp. single-triangle) representation of G, in which the endpoints of the triangles in both lines L_1 and L_2 are integers between 1 and $2n$. That is, every triangle (resp. single-triangle) graph G with n vertices has a representation with size polynomial on n, and thus the recognition problems of both both triangle and simple-triangle graphs are in NP, as the next observation states.

Observation 1 *The triangle and simple-triangle graph recognition problems are in the complexity class NP.*

3 Standard trapezoid representations

In this section we investigate several properties of trapezoid and triangle graphs and their representations. In particular, we introduce the notion of a *standard trapezoid representation.* We prove that a sufficient condition for a trapezoid graph G to be a triangle graph is that G admits such a standard representation. These properties of trapezoid and triangle graphs, as well as the notion of a standard trapezoid representation will then be used in our reduction for the triangle graph recognition problem. In order to define the notion of a standard trapezoid representation (cf. Definition 3), we first provide the following two definitions regarding an arbitrary trapezoid T_v in a trapezoid representation.

Definition 1 *Let R be a trapezoid representation of a trapezoid graph $G = (V, E)$ and T_v be a trapezoid in R, where $v \in V$. Let R' and R'' be the representations obtained by vertical axis flipping and by horizontal axis flipping of R, respectively. Then,*

- T_v *is* upper-right-closed in R *if there exist two vertices $u, w \in N(v)$, such that $T_u \ll_R T_w$, $L(w) <_R R(v)$, and $r(v) <_R l(w)$; otherwise* T_v *is* upper-right-open in R,
- T_v *is* upper-left-closed in R *if T_v is upper-right-closed in R'; otherwise* T_v *is* upper-left-open in R,
- T_v *is* lower-right-closed in R *if T_v is upper-right-closed in R''; otherwise* T_v *is* lower-right-open in R,
- T_v *is* lower-left-closed in R *if T_v is lower-right-closed in R'; otherwise* T_v *is* lower-left-open in R.

Intuitively, if the trapezoid T_v is *upper-right-closed* in the trapezoid representation R (cf. Definition 1), then there exists another trapezoid T_w in R that "invades" in T_v only at its upper right corner (cf. the trapezoid T_{v_3} in Figure 2). In addition, according to Definition 1, there exists another vertex $u \in N(v)$, such that $T_u \ll_R T_w$. Intuitively, the existence of such a trapezoid T_u in R means that, if we move the left endpoints $L(w)$ and $l(w)$ of T_w to the left to cover the whole trapezoid T_v, then we will change the graph G, since in this case the trapezoid T_w will intersect the trapezoid T_u in the resulting representation.

Definition 2 *Let R be a trapezoid representation of a trapezoid graph $G = (V, E)$ and T_v be a trapezoid in R, where $v \in V$. Then,*

- *T_v is* right-closed *in R if T_v is both upper-right-closed and lower-right-closed in R; otherwise T_v is* right-open *in R,*
- *T_v is* left-closed *in R if T_v is both upper-left-closed and lower-left-closed in R; otherwise T_v is* left-open *in R,*
- *T_v is* closed *in R if T_v is both right-closed and left-closed in R; otherwise T_v is* open *in R.*

As an example for Definitions 1 and 2, consider the trapezoid representation R in Figure 2. In this figure, the trapezoid T_v is upper-left-closed and lower-left-closed, as well as upper-right-closed and lower-right-open. Therefore, T_v is left-closed and right-open in R, i.e. T_v is open in R. For better visibility, we place in Figure 2 three bold bullets on the upper right, upper left, and lower left endpoints of the trapezoid T_v, in order to indicate that T_v is upper-right-closed, upper-left-closed, and lower-left-closed, respectively.

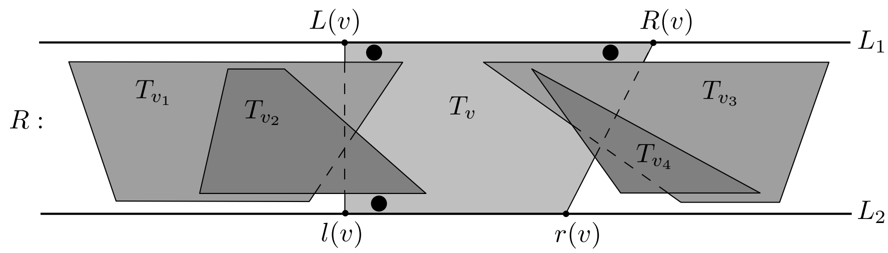

Figure 2: A standard trapezoid representation R, in which the trapezoid T_v is left-closed, upper-right-closed, and lower-right-open.

We are now ready to define the notion of a standard trapezoid representation.

Definition 3 *Let $G = (V, E)$ be a trapezoid graph and R be a trapezoid representation of G. If, for every $v \in V$, the trapezoid T_v is open in R or T_v is a triangle in R, then R is a* standard trapezoid representation*.*

For example, the trapezoid representation R in Figure 2 is a standard representation. Indeed, none of the trapezoids $T_{v_1}, T_{v_2}, T_{v_3}$ is right-closed or left-closed, while T_v is lower-right-open (and therefore also right-open by Definition 2). Thus, each of the trapezoids T_v, T_{v_1}, T_{v_2}, and T_{v_3} is open in R. Moreover, T_{v_4} is a triangle in R.

Note that every triangle representation is a standard trapezoid representation by Definition 3. We now provide the main theorem of this section, which states a sufficient condition for a trapezoid graph to be a triangle graph.

Theorem 1 *Let $G = (V, E)$ be a trapezoid graph. If there exists a standard trapezoid representation of G, then G is a triangle graph.*

Proof. Let R be a standard trapezoid representation of G. If R is a triangle representation, then G is clearly a triangle graph. Suppose otherwise that R has a trapezoid T_v, where $v \in V$, that is not a triangle in R. We will construct a triangle representation R^* of G. Since R is standard by assumption, T_v is right-open or left-open in R by Definition 2. By possibly performing a vertical axis flipping, we may assume without loss of generality that T_v is right-open in R. That is, T_v is upper-right-open or lower-right-open in R by Definition 1. Similarly, by possibly performing a horizontal axis flipping, we may assume without loss of generality that T_v is upper-right-open in R.

We construct now from R a new trapezoid representation of G, as follows. First, for every vertex $w \in V$ with $L(v) <_R L(w) <_R R(v)$ and $r(v) <_R l(w)$, we move the upper left endpoint $L(w)$ of T_w directly before $L(v)$ on the line L_1. Note that $w \in N(v)$ for every such vertex w. Moreover, in the case where the upper left endpoint $L(w)$ of T_w coincides with its upper right endpoint $R(w)$ in R, i.e. if T_w is a triangle in R with one point on L_1, we also move the upper right endpoint $R(w)$ of T_w to the same position as its upper left endpoint $L(w)$ in R'. That is, if T_w is a triangle in R, it remains a triangle also in R'. During the movement of all these endpoints, we keep the same relative positions among them on L_1 as in the initial trapezoid representation R. Then, we reduce the trapezoid T_v to a triangle, by moving the upper right endpoint $R(v)$ of T_v to the left until it coincides with its upper left endpoint $L(v)$. Let R' be the resulting trapezoid representation. An example of the construction of R' is illustrated in Figure 3.

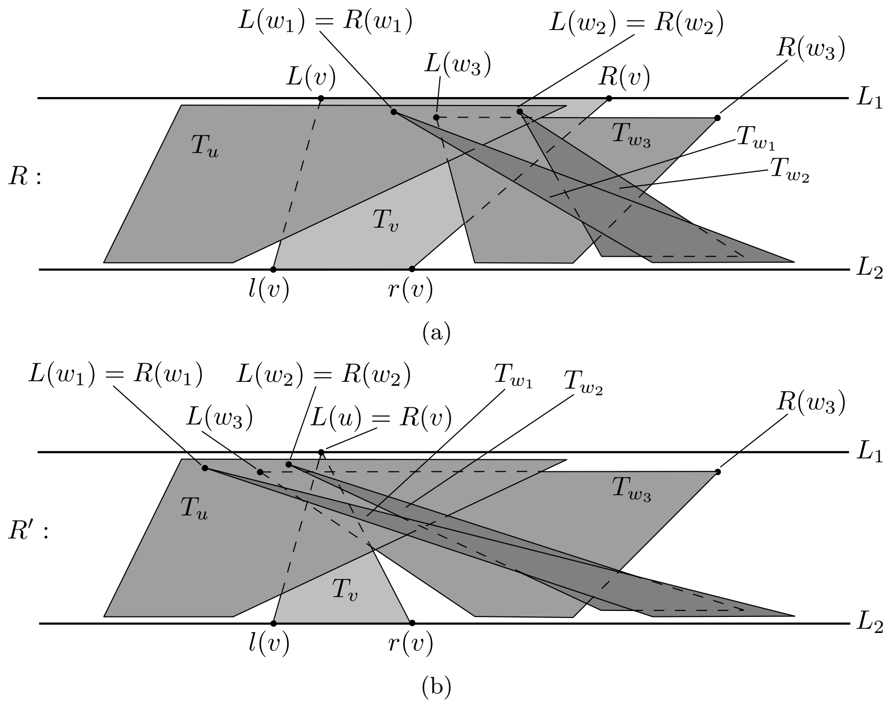

Figure 3: (a) A standard trapezoid representation R of a trapezoid graph G and (b) the transformation of R to a trapezoid representation R' of G with one triangle more.

We will prove that R' is a representation of the same graph G. First recall that, during the transformation of R to R', we moved the endpoints $L(w)$ of the trapezoids T_w for every vertex w, for which $L(v) <_R L(w) <_R R(v)$ and $r(v) <_R l(w)$ (cf. w_1, w_2, and w_3 in Figure 3). Suppose such a trapezoid T_w intersects a new trapezoid T_u in R', while T_w did not intersect T_u in R. That is, $T_u \ll_R T_w$. Then, since $L(w)$ came directly before $L(v)$ on the line L_1, it follows that

$L(v) <_R R(u) <_R L(w) <_R R(v)$, and thus T_u intersects T_v in R, i.e. $u \in N(v)$. That is, there exist two vertices $u, w \in N(v)$, such that $T_u \ll_R T_w$, $L(w) <_R R(v)$, and $r(v) <_R l(w)$, and thus T_v is upper-right-closed in R by Definition 1, which is a contradiction to the assumption. Therefore, T_w does not intersect any new trapezoid in R'.

Let $L(w) \neq R(w)$ in R, i.e. T_w is not a triangle in R with one point on L_1 (cf. w_3 in Figure 3). In this case, the upper right endpoint $R(w)$ of T_w remains the same in both R and R', and thus T_w increases during the transformation of R to R'. Therefore, if $L(w) \neq R(w)$ in R, then T_w keeps in R' all its intersections with other trapezoids.

Let $L(w) = R(w)$ in R, i.e. T_w is a triangle in R with one point on L_1 (cf. w_1 and w_2 in Figure 3). Recall that in this case, we also move during the transformation of R to R' the upper right endpoint $R(w)$ of T_w to the same position as its upper left endpoint $L(w)$ in R'. Suppose that, after this movement, T_w misses in R' its intersection with a trapezoid T_x in R. That is, T_w intersects T_x in R, while $T_w \ll_{R'} T_x$. Therefore, $L(x) <_R L(w) = R(w) <_R R(v)$ and $r(v) <_R l(w) \leq_R r(w) <_R l(x)$, i.e. $L(x) <_R R(v)$ and $r(v) <_R l(x)$. We distinguish now the two cases regarding the relative position of the endpoints $L(v)$ and $L(x)$ in the initial representation R. Let first $L(v) <_R L(x)$. In this case, $L(v) <_R L(x) <_R R(v)$ and $r(v) <_R l(x)$. Therefore, the endpoint $L(x)$ of T_x is moved directly before $L(v)$ on the line L_1, while the relative position of $L(x)$ and $L(w)$ remains the same in both R and R'. That is, $L(x) <_{R'} L(w)$, which is a contradiction, since $T_w \ll_{R'} T_x$ by assumption. Let now $L(x) <_R L(v)$. Then, $L(x)$ remains the same in both R and R', while $L(w)$ is moved directly before $L(v)$ in R'. That is, $L(x) <_{R'} L(w) <_{R'} L(v)$, which is again a contradiction, since $T_w \ll_{R'} T_x$ by assumption. Therefore, if $L(w) = R(w)$ in R, then T_w keeps in R' all its intersections with other trapezoids.

Recall now that we reduced the trapezoid T_v to a triangle (cf. Figure 3(b)). Suppose that, after this operation, T_v misses in R' its intersection with a trapezoid T_x. That is, T_v intersects a trapezoid T_x in R, while T_v does not intersect T_x in R', i.e. $T_v \ll_{R'} T_x$. Therefore, $L(v) <_R L(x) <_R R(v)$ and $r(v) <_R l(x)$. Thus, the upper left endpoint $L(x)$ is moved directly before $L(v)$ on the line L_1 during the transformation of R to R', which is a contradiction, since $T_v \ll_{R'} T_x$. Therefore, T_v keeps all its intersections in R'. Thus, R' is a trapezoid representation of the same graph G, in which T_v is a triangle, while all triangles in R remain also triangles in R' (cf. w_1 and w_2 in Figure 3).

After applying iteratively the above construction for every trapezoid T_v that is not a triangle in R, we obtain a triangle representation R^* of G, i.e. G is a triangle graph. This completes the proof of the theorem. ■

4 Basic constructions of trapezoid graphs

In this section we investigate some small trapezoid graphs and prove special properties of them. These graphs will then be used as parts of the gadgets in our reduction of 3SAT to the recognition problem of triangle graphs in Section 6. For simplicity of the presentation, we do not distinguish in the sequel of the article between a vertex v of a trapezoid graph G and the trapezoid T_v of v in a trapezoid representation of G.

Lemma 1 *Let $G = (V, E)$ be the trapezoid graph induced by the trapezoid representation of Figure 4(a). Then, in* any *trapezoid representation R of G, such that $v \ll_R v'$,*

- *v is upper-right-closed in R and v' is lower-left-closed in R, or*
- *v is lower-right-closed in R and v' is upper-left-closed in R.*

Proof. Consider a trapezoid representation R of G, such that $v \ll_R v'$. Since the vertices v_1 and v_2 are indistinguishable, as $N(v_1) = N(v_2)$, we may assume without loss of generality that $v_1 \ll_R v_2$. Furthermore, note that if both $R(v) <_R L(v_2)$ and $r(v) <_R l(v_2)$, then $v \ll_R v_2$, which is a contradiction, since v_2 intersects v in R. Therefore, $L(v_2) <_R R(v)$ or $l(v_2) <_R r(v)$.

Suppose that $L(v_2) <_R R(v)$. An example of such a trapezoid representation R is the representation R_1 in Figure 4(a). Then, since $v_1 \ll_R v_2$ and $v \ll_R v'$ by assumption, it follows that $R(v_1) <_R L(v_2) <_R R(v) <_R L(v')$, i.e. $R(v_1) <_R L(v')$. Now, if $r(v_1) <_R l(v')$, then $v_1 \ll_R v'$, which is a contradiction, since v_1 intersects v' in R. Thus $l(v') <_R r(v_1)$. Therefore, since $v \ll_R v'$ and $v_1 \ll_R v_2$ by assumption, it follows that $r(v) <_R l(v') <_R r(v_1) <_R l(v_2)$, i.e. $r(v) <_R l(v_2)$. Summarizing, there exist two vertices $v_1, v_2 \in N(v)$, such that $v_1 \ll_R v_2$, $L(v_2) <_R R(v)$, and $r(v) <_R l(v_2)$, and thus v is upper-right-closed in R by Definition 1. Moreover, $R(v_1) <_R L(v')$ and $l(v') <_R r(v_1)$, and thus v' is lower-left-closed in R by Definition 1.

Suppose now that $l(v_2) <_R r(v)$. Consider the trapezoid representation R' of G that is obtained by performing a horizontal axis flipping of R. Examples of these trapezoid representations R and R' are the representations R_2 and R_1 in Figures 4(b) and 4(a), respectively. Note that $L(v_2) <_{R'} R(v)$, since $l(v_2) <_R r(v)$. Moreover, it remains $v_1 \ll_{R'} v_2$, since also $v_1 \ll_R v_2$. Therefore, it follows by the previous paragraph that v is upper-right-closed in R' and that v' is lower-left-closed in R'. Thus, v is lower-right-closed in R and v' is upper-left-closed in R. This completes the proof of the lemma. ■

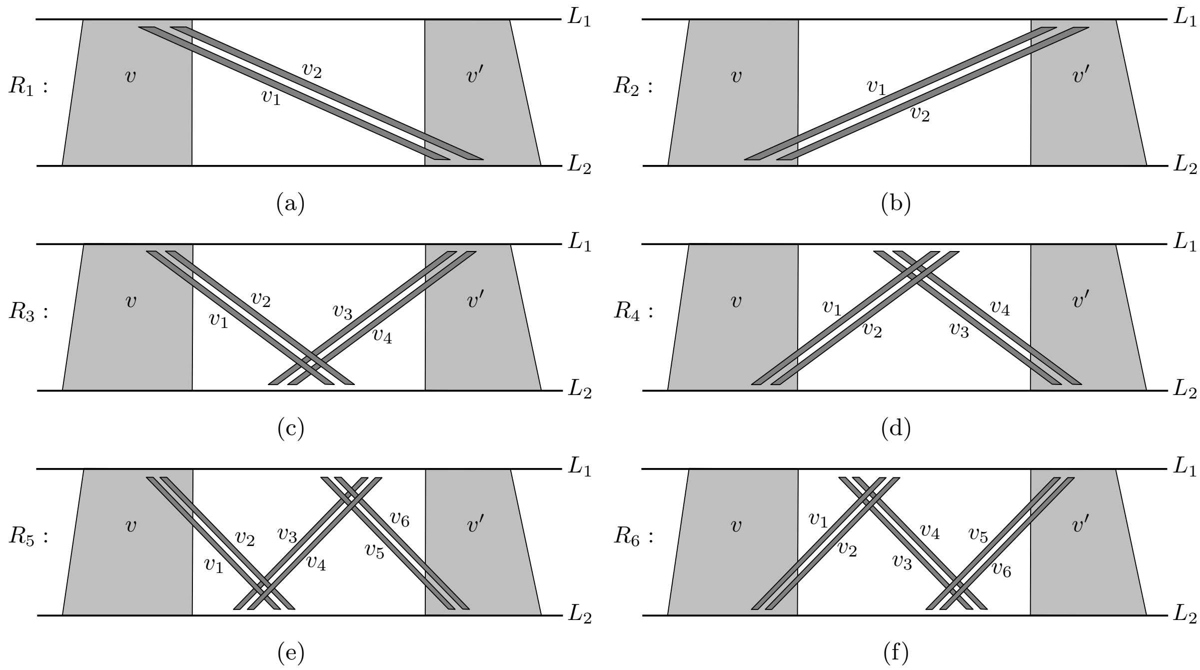

Figure 4: Six basic trapezoid representations.

The next two lemmas concern similar properties of the graphs induced by the trapezoid representations of Figures 4(c) and 4(e), respectively.

Lemma 2 *Let $G = (V, E)$ be the trapezoid graph induced by the trapezoid representation of Figure 4(c). Then, in* any *trapezoid representation R of G, such that $v \ll_R v'$,*

- *v is upper-right-closed in R and v' is upper-left-closed in R, or*
- *v is lower-right-closed in R and v' is lower-left-closed in R.*

Proof. Consider a trapezoid representation R of G, such that $v \ll_R v'$. Since the vertices v_3 and v_4 are indistinguishable, as $N(v_3) = N(v_4)$, we may assume without loss of generality that $v_3 \ll_R v_4$. Furthermore, since $v \ll_R v'$ and $v_1 \in N(v) \setminus N(v')$, it follows that $v_1 \ll_R v'$. Moreover, similarly to the proof of Lemma 1, note that if both $R(v) <_R L(v_2)$ and $r(v) <_R l(v_2)$, then $v \ll_R v_2$, which is a contradiction, since v_2 intersects v in R. Therefore, $L(v_2) <_R R(v)$ or $l(v_2) <_R r(v)$.

Suppose that $L(v_2) <_R R(v)$. An example of such a trapezoid representation R is the representation R_3 in Figure 4(c). Consider the induced subgraph $G_1 = G[\{v, v_1, v_2, v_4\}]$ of G; note that G_1 is the graph investigated in Lemma 1, where vertex v_4 corresponds to vertex v' of Lemma 1. Similarly to the proof of Lemma 1 in the corresponding case, it follows that v is upper-right-closed in R and that $l(v_4) <_R r(v_1)$. Therefore, since $v_3 \ll_R v_4$ and $v_1 \ll_R v'$, it follows that $r(v_3) <_R l(v_4) <_R r(v_1) <_R l(v')$, i.e. $r(v_3) <_R l(v')$. Now, if $R(v_3) <_R L(v')$, then $v_3 \ll_R v'$, which is a contradiction, since v_3 intersects v' in R. Thus $L(v') <_R R(v_3)$. Summarizing, there exist two vertices $v_3, v_4 \in N(v')$, such that $v_3 \ll_R v_4$, $r(v_3) <_R l(v')$, and $L(v') <_R R(v_3)$, and thus v' is upper-left-closed in R by Definition 1.

Suppose now that $l(v_2) <_R r(v)$. Consider the trapezoid representation R' of G that is obtained by performing a horizontal axis flipping of R. Examples of these trapezoid representations R and R' are the representations R_4 and R_3 in Figures 4(d) and 4(c), respectively. Note that $L(v_2) <_{R'} R(v)$, since $l(v_2) <_R r(v)$. Moreover, it remains $v_3 \ll_{R'} v_4$, since also $v_3 \ll_R v_4$. Therefore, it follows by the previous paragraph that v is upper-right-closed in R' and that v' is upper-left-closed in R'. Thus, v is lower-right-closed in R and v' is lower-left-closed in R. This completes the proof of the lemma. ■

Lemma 3 *Let $G = (V, E)$ be the trapezoid graph induced by the trapezoid representation of Figure 4(e). Then, in* any *trapezoid representation R of G, such that $v \ll_R v'$,*

- *v is upper-right-closed in R and v' is lower-left-closed in R, or*
- *v is lower-right-closed in R and v' is upper-left-closed in R.*

Proof. Consider a trapezoid representation R of G, such that $v \ll_R v'$. Since the vertices v_5 and v_6 are indistinguishable, as $N(v_5) = N(v_6)$, we may assume without loss of generality that $v_5 \ll_R v_6$. Furthermore, since $v \ll_R v'$ and $v_1 \in N(v) \setminus N(v')$, it follows that $v_1 \ll_R v'$. Therefore, since $v_1 \ll_R v'$ and $v_3 \in N(v_1) \setminus N(v')$, it follows that $v_3 \ll_R v'$. Moreover, similarly to the proof of Lemma 1, note that if both $R(v) <_R L(v_2)$ and $r(v) <_R l(v_2)$, then $v \ll_R v_2$, which is a contradiction, since v_2 intersects v in R. Therefore, $L(v_2) <_R R(v)$ or $l(v_2) <_R r(v)$.

Suppose that $L(v_2) <_R R(v)$. An example of such a trapezoid representation R is the representation R_5 in Figure 4(e). Consider the induced subgraph $G_1 = G[\{v, v_1, v_2, v_3, v_4, v_6\}]$ of G; note that G_1 is the graph investigated in Lemma 2, where vertex v_6 corresponds to vertex v' of Lemma 2. Similarly to the proof of Lemma 2 in the corresponding case, it follows that v is upper-right-closed in R and that $L(v_6) <_R R(v_3)$. Therefore, since $v_5 \ll_R v_6$ and $v_3 \ll_R v'$, it follows that $R(v_5) <_R L(v_6) <_R R(v_3) <_R L(v')$, i.e. $R(v_5) <_R L(v')$. Now, if $r(v_5) <_R l(v')$, then $v_5 \ll_R v'$, which is a contradiction, since v_5 intersects v' in R. Thus $l(v') <_R r(v_5)$. Summarizing, there exist two vertices $v_5, v_6 \in N(v')$, such that $v_5 \ll_R v_6$, $R(v_5) <_R L(v')$, and $l(v') <_R r(v_5)$, and thus v' is lower-left-closed in R by Definition 1.

Suppose now that $l(v_2) <_R r(v)$. Consider the trapezoid representation R' of G that is obtained by performing a horizontal axis flipping of R. Examples of these trapezoid representations R and R' are the representations R_6 and R_5 in Figures 4(f) and 4(e), respectively. Note that $L(v_2) <_{R'} R(v)$, since $l(v_2) <_R r(v)$. Moreover, it remains $v_5 \ll_{R'} v_6$, since also $v_5 \ll_R v_6$. Therefore, it follows by the previous paragraph that v is upper-right-closed in R' and that v' is lower-left-closed in R'. Thus, v is lower-right-closed in R and v' is upper-left-closed in R. This completes the proof of the lemma. ■

5 Basic constructions of triangle graphs

In this section we investigate the structure of some specific triangle graphs and devise special properties of them. As triangle graphs are also trapezoid graphs, in order to prove these properties, we use some of the results provided in Section 4. Similarly to the trapezoid graphs investigated in Section 4, also the investigated graphs of the present section will then be used as gadgets in our reduction for the triangle graph recognition problem in Section 6. Before investigating any specific triangle graph, we first provide in the next theorem a generic result that concerns the triangle representations of the 1-connected triangle graphs. An example of a 1-connected graph G with a cut vertex v is illustrated in Figure 5(a).

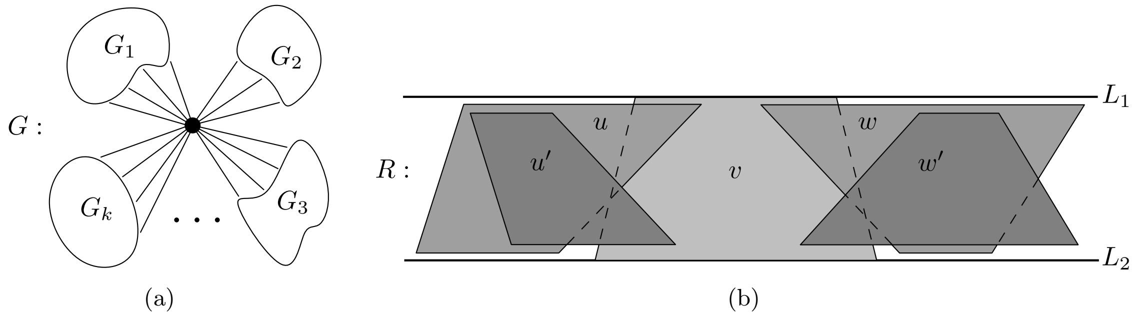

Figure 5: (a) A 1-connected graph G, where $G - \{v\}$ has $k \geq 2$ connected components $G_1, G_2, \ldots, G_k$ and (b) a trapezoid representation for such a graph G.

Theorem 2 *Let $G = (V, E)$ be a 1-connected triangle graph and $v \in V$ be a cut vertex of G. Then, in* any *triangle representation R of G, the trapezoid of v is open in R.*

Proof. Let R be any triangle representation of G. For the sake of contradiction, suppose that v is closed in R, i.e. v is both left-closed and right-closed in R. We will prove that, in this case, the trapezoid T_v has four distinct endpoints, and thus T_v is not a triangle in R, which comes in contradiction to the assumption that R is a triangle representation. Let $G_1, G_2, \ldots, G_k$ be the connected components of $G - \{v\}$, where $k \geq 2$. Then, for any $i \neq j$, either all trapezoids of G_i lie completely to the left or to the right of all trapezoids of G_j in R.

Note that v is upper-right-closed in R (as v is right-closed in R by assumption). Therefore, in particular, there exists by Definition 1 a vertex $w \in N(v)$, such that $L(w) <_R R(v)$ and $r(v) <_R l(w)$ (cf. Figure 5(b)). Since v is also lower-right-closed in R, there exists a vertex $w' \in N(v)$, such that $l(w') <_R r(v)$ and $R(v) <_R L(w')$ (cf. Figure 5(b)). Summarizing, $l(w') <_R r(v) <_R l(w)$ and $L(w) <_R R(v) <_R L(w')$. Therefore, in particular, the trapezoids of w and of w' intersect in R, i.e. $ww' \in E$. Therefore, both $w, w' \in V(G_i)$, for some $i = 1, 2, \ldots, k$.

Similarly, since v is upper-left-closed and lower-left-closed in R (as v is left-closed in R by assumption), there exist two vertices $u, u' \in N(v)$, such that $R(u') <_R L(v) <_R R(u)$ and $r(u) <_R l(v) <_R r(u')$, cf. Figure 5(b). Therefore, in particular, the trapezoids of u and of u' intersect in R, i.e. $uu' \in E$. Thus, both $u, u' \in V(G_j)$, for some $j = 1, 2, \ldots, k$.

Suppose that $j = i$, i.e. the vertices w, w', u, u' belong to the same connected component of $G - \{v\}$. Consider now another connected component G_ℓ of $G - \{v\}$, where $\ell \neq i$. Note that G_ℓ exists, since $G - \{v\}$ has at least two connected components. Recall that either all trapezoids of G_ℓ lie to the left or to the right of all trapezoids of G_i in R. Suppose that the trapezoids of G_ℓ lie to the left of the trapezoids of G_i in R, i.e. $x \ll_R y$ for every $x \in V(G_\ell)$ and $y \in V(G_i)$. Then, since $r(u) <_R l(v)$ and $R(u') <_R L(v)$, and since $u, u' \in V(G_i)$, it follows that $x \ll_R v$ for every $x \in V(G_\ell)$. Thus no vertex of G_ℓ is adjacent to v, i.e. G is not connected, which is a contradiction by the assumption on G. Similarly, suppose that the trapezoids of G_ℓ lie to the right of the trapezoids of G_i in R, i.e. $y \ll_R x$ for every $x \in V(G_\ell)$ and $y \in V(G_i)$. Then, since $r(v) <_R l(w)$ and $R(v) <_R L(w')$, and since $w, w' \in V(G_i)$, it follows that $v \ll_R x$ for every $x \in V(G_\ell)$. Thus no vertex of G_ℓ is adjacent to v, i.e. G is not connected, which is again a contradiction. Therefore $j \neq i$.

Recall that $r(u) <_R l(v) \leq_R r(v) <_R l(w)$, and thus $r(u) <_R l(w)$. Furthermore, recall that $R(u') <_R L(v) \leq_R R(v) <_R L(w')$, and thus $R(u') <_R L(w')$. Moreover, since $u, u' \in V(G_j)$ and $w, w' \in V(G_i)$, where $j \neq i$, it follows that $uw, u'w' \notin E$, and thus $u \ll_R w$ and $u' \ll_R w'$. Summarizing, there exist four distinct vertices $u, u', w, w' \in N(v)$, such that $L(v) <_R R(u) <_R L(w) <_R R(v)$ and $l(v) <_R r(u') <_R l(w') <_R r(v)$. Therefore $L(v) <_R R(v)$ and $l(v) <_R r(v)$, i.e. $L(v) \neq R(v)$ and $l(v) \neq r(v)$, which contradicts the fact that R is a triangle representation. An example of such a forbidden representation R, where $V(G_1) = \{u, u'\}$ and $V(G_2) = \{w, w'\}$, is illustrated in Figure 5(b). Therefore, v is open in R. ■

We now use the generic Theorem 2, as well as the results of Section 4, in order to prove some properties of the trapezoid representations of Figure 6. Note that, although the representations of Figure 6 are not triangle representations, they are standard trapezoid representations, and thus the graphs induced by these representations are triangle graphs by Theorem 1.

Lemma 4 *Let $G = (V, E)$ be the triangle graph induced by the trapezoid representation of Figure 6(a). Then, in* any *triangle representation R of G, such that $a_7 \ll_R u$, u is left-open in R if and only if w is right-open in R.*

Proof. Let R be a triangle representation of G, such that $a_7 \ll_R u$. Note that $G - \{u, w\}$ has the two connected components $G_1 = G[a_1, a_2, a_3, a_4, a_5, a_6, a_7]$ and $G_2 = G[v, b_1, b_2, b_3, b_4, b_5, b_6]$, and thus one of these two induced subgraphs of G lies completely to the left of the other in R. If $v \ll_R a_7 \ll_R u$, then a_7 would intersect with a triangle of G_2, which is a contradiction, since $a_7 \in V(G_1)$. Furthermore, if $a_7 \ll_R v \ll_R u$, then v would intersect with a triangle of G_1, which is a contradiction, since $v \in V(G_2)$. Therefore $a_7 \ll_R u \ll_R v$; similarly, $a_7 \ll_R w \ll_R v$. Therefore, every triangle of G_1 must lie completely to the left of every triangle of G_2 in R.

($\Rightarrow$) Suppose that u is left-open in R, i.e. u is upper-left-open or lower-left-open in R. By possibly performing a horizontal axis flipping of R, we may assume without loss of generality that u is lower-left-open in R. Consider the induced subgraphs $H_1 = G[\{a_7, a_1, a_2, u\}]$ and $H_2 = G[\{a_7, a_1, a_2, w\}]$ of G. Note that both H_1 and H_2 are isomorphic to the graph investigated in Lemma 1. Since u is assumed to be lower-left-open in R (and thus also in the restriction $R[H_1]$ of the triangle representation R), Lemma 1 implies that u is upper-left-closed and a_7 is lower-right-closed in $R[H_1]$. Therefore, a_7 is lower-right-closed also in the restriction $R[H_1 - \{u\}] = R[H_2 - \{w\}]$ of R. Thus, Lemma 1 implies that a_7 is lower-right-closed and w is upper-left-closed in the restriction $R[H_2]$ of R, and thus w is upper-left-closed in R.

Consider now the induced subgraphs $H_3 = G[\{a_7, a_3, a_4, u\}]$ and $H_4 = G[\{a_7, a_3, a_4, a_5, a_6, w\}]$ of G. Note that H_3 is isomorphic to the graph investigated in Lemma 1, while H_4 is isomorphic to the graph investigated in Lemma 2. Since u is assumed to be lower-left-open in R (and thus also in $R[H_3]$), Lemma 1 implies that u is upper-left-closed and a_7 is lower-right-closed in $R[H_3]$. Therefore, a_7 is lower-right-closed also in the restriction $R[H_3 - \{u\}] = R[H_4 - \{a_5, a_6, w\}]$ of the triangle representation R. Thus, Lemma 2 implies that a_7 is lower-right-closed and w is lower-left-closed in the restriction $R[H_4]$ of R, and thus w is lower-left-closed in R. Therefore, since w is also upper-left-closed in R by the previous paragraph, it follows that w is left-closed in R.

Recall that R is a triangle representation by assumption, and thus the restriction $R[G - \{u\}]$ is also a triangle representation. Moreover, since w is left-closed in R, it follows that w is also left-closed in $R[G - \{u\}]$. Note now that the connected graph $G - \{u\}$ satisfies the conditions of Theorem 2. Indeed, w is a cut vertex of $G - \{u\}$ and $(G - \{u\}) - \{w\}$ has the two connected components $G_1 = G[a_1, a_2, a_3, a_4, a_5, a_6, a_7]$ and $G_2 = G[v, b_1, b_2, b_3, b_4, b_5, b_6]$. Therefore, since w is left-closed in $R[G - \{u\}]$, Theorem 2 implies that w is right-open in $R[G - \{u\}]$, and thus also w is right-open in R.

($\Leftarrow$) Consider the triangle representation R' of G that is obtained by performing a vertical axis flipping of R. Note that $v \ll_{R'} w$, since $w \ll_R v$. Furthermore, note that there is a trivial automorphism of G, which maps vertex u to w, vertex a_7 to v, and the vertices $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ to the vertices $\{b_1, b_2, b_3, b_4, b_5, b_6\}$, in this order. That is, the relation $a_7 \ll_R u$ in the representation R is mapped by this automorphism to the relation $v \ll_{R'} w$ in the representation R'. It follows now directly by the necessity part ($\Rightarrow$) that, if w is left-open in R', then u is right-open in R'. That is, if w is right-open in R, then u is left-open in R. ■

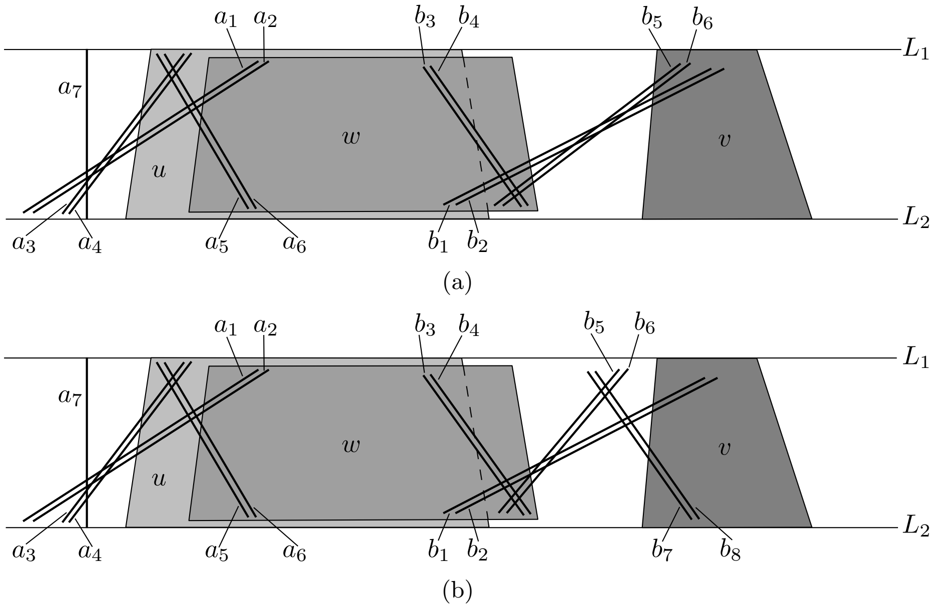

Figure 6: Two basic trapezoid representations.

Now, using Lemma 4, we can prove the next two lemmas.

Lemma 5 *Let $G = (V, E)$ be the triangle graph induced by the trapezoid representation of Figure 6(a). Then, in* any *triangle representation R of G, such that $a_7 \ll_R u$, u is left-open in R if and only if v is left-open in R.*

Proof. Let R be a triangle representation of G, such that $a_7 \ll_R u$. Recall by the proof of Lemma 4 that $w \ll_R v$.

($\Rightarrow$) Suppose that u is left-open in R. Then, w is right-open in R by Lemma 4, i.e. w is upper-right-open or lower-right-open in R. By possibly performing a horizontal axis flipping of R, we may assume without loss of generality that w is upper-right-open in R. Consider the induced subgraphs $H_1 = G[\{w, b_1, b_2, v\}]$ and $H_2 = G[\{w, b_5, b_6, v\}]$ of G. Note that both H_1 and H_2 are isomorphic to the graph investigated in Lemma 1. Since w is assumed to be upper-right-open in R (and thus also in both restrictions $R[H_1]$ and $R[H_2]$ of R), Lemma 1 implies that w is lower-right-closed and v is upper-left-closed in both $R[H_1]$ and $R[H_2]$, and thus v is upper-left-closed in R. Therefore, since b_1, b_2, b_5, b_6 are the only neighbors of v in G, it follows that v is lower-left-open in R, and thus v is left-open in R.

($\Leftarrow$) Suppose that v is left-open in R, i.e. v is upper-left-open or lower-left-open in R. By possibly performing a horizontal axis flipping of R, we may assume without loss of generality that v is lower-left-open in R. Consider the induced subgraphs $H_3 = G[\{u, b_1, b_2, v\}]$ and $H_4 = G[\{u, b_3, b_4, b_5, b_6, v\}]$ of G. Note that H_3 is isomorphic to the graph investigated in Lemma 1, while H_4 is isomorphic to the graph investigated in Lemma 2. Since v is assumed to be lower-left-open in R, it follows that v is lower-left-open also in the restrictions $R[H_3]$ and $R[H_4]$ of R. Therefore, Lemma 1 implies that v is upper-left-closed and u is lower-right-closed in $R[H_3]$, and thus also in R. Similarly, Lemma 2 implies that v is upper-left-closed and u is upper-right-closed in $R[H_4]$, and thus also in R. Summarizing, u is both lower-right-closed and upper-right-closed in R, and thus u is right-closed in R.

Recall that R is a triangle representation by assumption, and thus the restriction $R[G-\{w\}]$ is also a triangle representation. Moreover, since u is right-closed in R, it follows that u is also right-closed in $R[G-\{w\}]$. Note now that the connected graph $G-\{w\}$ satisfies the conditions of Theorem 2. Indeed, u is a cut vertex of $G-\{w\}$ and $(G-\{w\})-\{u\}$ has the two connected components $G_1 = G[a_1, a_2, a_3, a_4, a_5, a_6, a_7]$ and $G_2 = G[v, b_1, b_2, b_3, b_4, b_5, b_6]$. Therefore, since u is right-closed in $R[G-\{w\}]$, Theorem 2 implies that u is left-open in $R[G-\{w\}]$, and thus also u is left-open in R. ∎

Lemma 6 *Let $G = (V, E)$ be the triangle graph induced by the trapezoid representation of Figure 6(b). Then, in* any *triangle representation R of G, such that $a_7 \ll_R u$, u is left-open in R if and only if v is left-closed in R.*

Proof. Let R be a triangle representation of G, such that $a_7 \ll_R u$. Note that the induced subgraph $H = G - \{b_8, v\}$ is isomorphic to the graph investigated in Lemmas 4 and 5. That is, H is isomorphic to the graph induced by the trapezoid representation of Figure 6(a).

($\Rightarrow$) Suppose that u is left-open in R. Then, u is also left-open in the restriction $R[H]$ of R. Therefore, w is right-open in $R[H]$ by Lemma 4, and thus w is also right-open in R. That is, w is upper-right-open or lower-right-open in R. By possibly performing a horizontal axis flipping of R, we may assume without loss of generality that w is upper-right-open in R. Consider the induced subgraphs $H_1 = G[\{w, b_1, b_2, v\}]$ and $H_2 = G[\{w, b_5, b_6, b_7, b_8, v\}]$ of G. Note that H_1 is isomorphic to the graph investigated in Lemma 1, while H_2 is isomorphic to the graph investigated in Lemma 2. Since w is assumed to be upper-right-open in R, it follows that w is upper-right-open also in the restrictions $R[H_1]$ and $R[H_2]$ of R. Therefore, Lemma 1 implies that w is lower-right-closed and v is upper-left-closed in $R[H_1]$, and thus also in R. Similarly, Lemma 2 implies that w is lower-right-closed and v is lower-left-closed in $R[H_2]$, and thus also in R. Summarizing, v is both upper-left-closed and lower-left-closed in R, and thus v is left-closed in R.

($\Leftarrow$) Suppose that u is left-closed in R. Recall that R is a triangle representation by assumption, and thus the restriction $R[G-\{w\}]$ is also a triangle representation. Moreover, since u is left-closed in R, it follows that u is also left-closed in $R[G-\{w\}]$. Note now that the connected graph $G-\{w\}$ satisfies the conditions of Theorem 2. Indeed, u is a cut vertex of $G-\{w\}$ and $(G-\{w\})-\{u\}$ has the two connected components $G_1 = G[a_1, a_2, a_3, a_4, a_5, a_6, a_7]$ and $G_2 = G[v, b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8]$. Therefore, since u is left-closed in $R[G-\{w\}]$, Theorem 2 implies that u is right-open in $R[G-\{w\}]$, and thus also u is right-open in R. That is, u is upper-right-open or lower-right-open in R. By possibly performing a horizontal axis flipping of R, we may assume without loss of generality that u is upper-right-open in R.

Consider the induced subgraphs $H_3 = G[\{u, b_1, b_2, v\}]$ and $H_4 = G[\{u, b_3, b_4, b_5, b_6, b_7, b_8, v\}]$ of G. Note that H_3 is isomorphic to the graph investigated in Lemma 1, while H_4 is isomorphic to the graph investigated in Lemma 3. Since u is assumed to be upper-right-open in R, it follows that u is upper-right-open also in the restrictions $R[H_3]$ and $R[H_4]$ of R. Therefore, Lemma 1 implies that u is lower-right-closed and v is upper-left-closed in $R[H_3]$, and thus also in R. Similarly, Lemma 3 implies that u is lower-right-closed and v is upper-left-closed in $R[H_4]$. That is, v is upper-left-closed in both $R[H_3]$ and $R[H_4]$, and thus v is upper-left-closed in R. Therefore, since b_1, b_2, b_7, b_8 are the only neighbors of v in G, it follows that v is lower-left-open in R, and thus v is left-open in R. This completes the proof of the lemma. ∎

6 The recognition of triangle graphs

In this section we provide a reduction from the *three-satisfiability (3SAT)* problem to the problem of recognizing whether a given graph is a triangle graph. Given a boolean formula ϕ in conjunctive normal form with three literals in each clause (3-CNF), ϕ is *satisfiable* if there is a truth assignment of ϕ, such that every clause contains at least one true literal. The problem of deciding whether a given 3-CNF formula ϕ is satisfiable is one of the most known NP-complete problems. We can assume without loss of generality that each clause has literals that correspond to three distinct variables. Given the formula ϕ, we construct in polynomial time a trapezoid graph G_ϕ, such that G_ϕ is a triangle graph if and only if ϕ is satisfiable. Before constructing the whole trapezoid graph G_ϕ, we construct first some smaller trapezoid graphs for each clause and each variable that appears in the given formula ϕ.

6.1 The construction for each clause

Consider a 3-CNF formula $\phi = \alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_k$ with k clauses $\alpha_1, \alpha_2, \ldots, \alpha_k$ and n boolean variables $x_1, x_2, \ldots, x_n$, such that $\alpha_i = (\ell_{i,1} \vee \ell_{i,2} \vee \ell_{i,3})$ for $i = 1, 2, \ldots, k$. For the literals $\ell_{i,1}, \ell_{i,2}, \ell_{i,3}$ of the clause α_i, let $\ell_{i,1} \in \{x_{r_{i,1}}, \overline{x_{r_{i,1}}}\}$, $\ell_{i,2} \in \{x_{r_{i,2}}, \overline{x_{r_{i,2}}}\}$, and $\ell_{i,3} \in \{x_{r_{i,3}}, \overline{x_{r_{i,3}}}\}$, where $1 \leq r_{i,1} < r_{i,2} < r_{i,3} \leq n$. Let L_1 and L_2 be two parallel lines in the plane. To every clause α_i, where $i = 1, 2, \ldots, k$, we associate the trapezoid representation R_{α_i} with 7 trapezoids that is illustrated in Figure 7. Note that the trapezoid of the vertex z_i in R_{α_i} is trivial, i.e. a line. In this construction, the trapezoids of the vertices $v_{i,1}$, $v_{i,2}$, and $v_{i,3}$ correspond to the literals $\ell_{i,1}$, $\ell_{i,2}$, and $\ell_{i,3}$, respectively. Furthermore, by the construction of R_{α_i}, the left line of $v_{i,j}$ lies completely to the left of the left line of $v_{i,j+1}$ in R_{α_i} for $j \in \{1, 2\}$.

R_{α_i} :

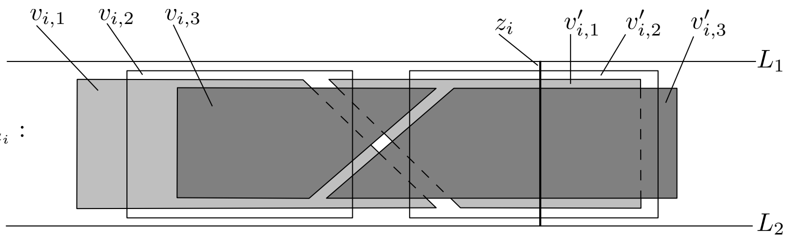

Figure 7: The construction R_{α_i} that is associated to the clause α_i of the formula ϕ, for $i \in \{1, 2, \ldots, k\}$.

We prove now two basic properties of the construction R_{α_i} in Figure 7 for the clause α_i that will be then used in the proof of correctness of our reduction.

Lemma 7 *Let G_{α_i} be the trapezoid graph induced by the trapezoid representation R_{α_i} of Figure 7. Then, in* any *trapezoid representation R of G_{α_i}, such that $v_{i,1} \ll_R z_i$, one of $v_{i,1}, v_{i,2}, v_{i,3}$ is right-closed in R.*

Proof. Let R be a trapezoid representation of G_{α_i}, such that $v_{i,1} \ll_R z_i$. Note that $v_{i,2}, v_{i,3} \in N(v_{i,1}) \setminus N(z_i)$. Thus, since $v_{i,1} \ll_R z_i$ by assumption, it follows that also

$v_{i,2} \ll_R z_i$ and $v_{i,3} \ll_R z_i$. Furthermore, note that $v'_{i,1} \in N(z_i) \setminus N(v_{i,1})$, $v'_{i,2} \in N(z_i) \setminus N(v_{i,2})$, and $v'_{i,3} \in N(z_i) \setminus N(v_{i,3})$. Therefore, since $v_{i,1} \ll_R z_i$, $v_{i,2} \ll_R z_i$, and $v_{i,3} \ll_R z_i$, it follows that $v_{i,1} \ll_R v'_{i,1}$, $v_{i,2} \ll_R v'_{i,2}$, and $v_{i,3} \ll_R v'_{i,3}$. Moreover, note that we can locally change appropriately in R the right lines of $v_{i,1}, v_{i,2}, v_{i,3}$ and the left lines of $v'_{i,1}, v'_{i,2}, v'_{i,3}$, such that the relative position of the endpoints $R(v_{i,1}), R(v_{i,2}), R(v_{i,3})$ on the line L_1 is arbitrary. Therefore, we assume throughout the proof without loss of generality that $R(v_{i,1}) <_R R(v_{i,2}) <_R R(v_{i,3})$ (cf. Figure 7).

We will now prove that $r(v_{i,3}) <_R r(v_{i,2}) <_R r(v_{i,1})$. Suppose otherwise that $r(v_{i,1}) <_R r(v_{i,2})$. Then, since $R(v_{i,1}) <_R R(v_{i,2})$ and $v_{i,2} \ll_R v'_{i,2}$ by the previous paragraph, it follows that also $v_{i,1} \ll_R v'_{i,2}$. This is a contradiction, since $v_{i,1}v'_{i,2} \in E(G_{\alpha_i})$ (cf. Figure 7). Therefore $r(v_{i,2}) <_R r(v_{i,1})$. Now suppose that $r(v_{i,2}) <_R r(v_{i,3})$. Then, similarly, since $R(v_{i,2}) <_R R(v_{i,3})$ and $v_{i,3} \ll_R v'_{i,3}$ by the previous paragraph, it follows that also $v_{i,2} \ll_R v'_{i,3}$. This is again a contradiction, since $v_{i,2}v'_{i,3} \in E(G_{\alpha_i})$. Summarizing, $r(v_{i,3}) <_R r(v_{i,2}) <_R r(v_{i,1})$ (cf. Figure 7).

Recall that $v_{i,1} \ll_R v'_{i,1}$. Therefore, since $r(v_{i,2}) <_R r(v_{i,1})$ by the previous paragraph, it follows that $r(v_{i,2}) <_R r(v_{i,1}) <_R r(v'_{i,1})$, i.e. $r(v_{i,2}) <_R l(v'_{i,1})$. Now, if $R(v_{i,2}) <_R L(v'_{i,1})$, then $v_{i,2} \ll_R v'_{i,1}$, which is a contradiction, since $v_{i,2}v'_{i,1} \in E(G_{\alpha_i})$ (cf. Figure 7). Thus $L(v'_{i,1}) <_R R(v_{i,2})$. Summarizing, there exist two vertices $v_{i,1}, v'_{i,1} \in N(v_{i,2})$, such that $v_{i,1} \ll_R v'_{i,1}$, $L(v'_{i,1}) <_R R(v_{i,2})$, and $r(v_{i,2}) <_R l(v'_{i,1})$, and thus $v_{i,2}$ is upper-right-closed in R by Definition 1. Therefore, since also $R(v_{i,2}) <_R R(v_{i,3})$ and $r(v_{i,3}) <_R r(v_{i,2})$, it follows that $L(v'_{i,1}) <_R R(v_{i,3})$ and $r(v_{i,3}) <_R l(v'_{i,1})$. Thus, since $v_{i,1}, v'_{i,1} \in N(v_{i,3})$, it follows by Definition 1 that also $v_{i,3}$ is upper-right-closed in R.

Similarly, since $v_{i,3} \ll_R v'_{i,3}$ and $R(v_{i,2}) <_R R(v_{i,3})$, it follows that $R(v_{i,2}) <_R R(v_{i,3}) <_R L(v'_{i,3})$, i.e. $R(v_{i,2}) <_R L(v'_{i,3})$. Now, if $r(v_{i,2}) <_R l(v'_{i,3})$, then $v_{i,2} \ll_R v'_{i,3}$, which is a contradiction, since $v_{i,2}v'_{i,3} \in E(G_{\alpha_i})$. Thus $l(v'_{i,3}) <_R r(v_{i,2})$. Summarizing, there exist two vertices $v_{i,3}, v'_{i,3} \in N(v_{i,2})$, such that $v_{i,3} \ll_R v'_{i,3}$, $l(v'_{i,3}) <_R r(v_{i,2})$, and $R(v_{i,2}) <_R L(v'_{i,3})$, and thus $v_{i,2}$ is lower-right-closed in R by Definition 1. Therefore, since also $r(v_{i,2}) <_R r(v_{i,1})$ and $R(v_{i,1}) <_R R(v_{i,2})$, it follows that $l(v'_{i,3}) <_R r(v_{i,1})$ and $R(v_{i,1}) <_R L(v'_{i,3})$. Thus, since $v_{i,3}, v'_{i,3} \in N(v_{i,1})$, it follows by Definition 1 that also $v_{i,1}$ is lower-right-closed in R.

Summarizing, $v_{i,3}$ is upper-right-closed in R and $v_{i,1}$ is lower-right-closed in R, while $v_{i,2}$ is both upper-right-closed and lower-right-closed in R, i.e. $v_{i,2}$ is right-closed in R by Definition 2. This completes the proof of the lemma. ∎

Corollary 1 *Consider the trapezoid representation R_{α_i} of Figure 7. For every $p \in \{1,2,3\}$, we can locally change appropriately in R_{α_i} the right lines of $v_{i,1}, v_{i,2}, v_{i,3}$ and the left lines of $v'_{i,1}, v'_{i,2}, v'_{i,3}$, such that $v_{i,p}$ is right-closed and $v_{i,p'}$ is right-open, for every $p' \in \{1,2,3\} \setminus \{p\}$.*

Proof. Note that in the representation R_{α_i} of Figure 7, the relative position of the endpoints $R(v_{i,1}), R(v_{i,2}), R(v_{i,3})$ on the line L_1 is $R(v_{i,1}) <_{R_{\alpha_i}} R(v_{i,2}) <_{R_{\alpha_i}} R(v_{i,3})$. Then, it follows by the proof of Lemma 7 that $v_{i,2}$ is right-closed in R_{α_i}. Moreover, it is straightforward to see that the other two trapezoids $v_{i,1}$ and $v_{i,3}$ are right-open in R_{α_i} (in particular, $v_{i,1}$ is upper-right-open in R_{α_i} and $v_{i,3}$ is lower-right-open in R_{α_i}).

Furthermore, recall by the proof of Lemma 7 that we can locally change appropriately in R_{α_i} the right lines of $v_{i,1}, v_{i,2}, v_{i,3}$ and the left lines of $v'_{i,1}, v'_{i,2}, v'_{i,3}$, such that the relative position of the endpoints $R(v_{i,1}), R(v_{i,2}), R(v_{i,3})$ on the line L_1 is arbitrary. For an arbitrary $p \in \{1,2,3\}$, consider now the trapezoid representation R that is obtained by changing locally these lines in R_{α_i}, such that the endpoint $R(v_{i,p})$ lies in the middle of $R(v_{i,1}), R(v_{i,2}), R(v_{i,3})$ on L_1. Then, similarly to the above, $v_{i,p}$ is right-closed in this representation R, while the other two trapezoids $v_{i,p'}$ are right-open in R, for every $p' \in \{1,2,3\} \setminus \{p\}$. This completes the proof of the corollary. ∎

6.2 The construction for each variable

Let x_j be a variable of the formula ϕ, where $1 \leq j \leq n$. Let x_j appear in ϕ (either as x_j or negated as $\overline{x_j}$) in the m_j clauses $\alpha_{i_{j,1}}, \alpha_{i_{j,2}}, \ldots, \alpha_{i_{j,m_j}}$, where $1 \leq i_{j,1} < i_{j,2} < \ldots < i_{j,m_j} \leq k$. Then, we associate to the variable x_j the trapezoid representation R_{x_j} with $2m_j + 7$ trapezoids that is illustrated in Figure 8. In this construction, the trapezoids of the vertices $u_{j,t}$ and $w_{j,t}$, where $1 \leq t \leq m_j$, correspond to the appearance of the variable x_j (either as x_j or negated as $\overline{x_j}$) in the clause $\alpha_{i_{j,t}}$ in ϕ. Note that the trapezoids of the vertices $a_j^1, a_j^2, \ldots, a_j^7$ are trivial, i.e. lines. By the construction of R_{x_j}, the right line of $u_{j,t}$ lies completely to the left of the right line of $w_{j,t}$ for all values of $j = 1, 2, \ldots, n$ and $t = 1, 2, \ldots, m_j$. Furthermore, the right line of each of $\{u_{j,t}, w_{j,t}\}$ lies completely to the left of the right line of each of $\{u_{j,t'}, w_{j,t'}\}$ in R_{x_j}, whenever $t < t'$.

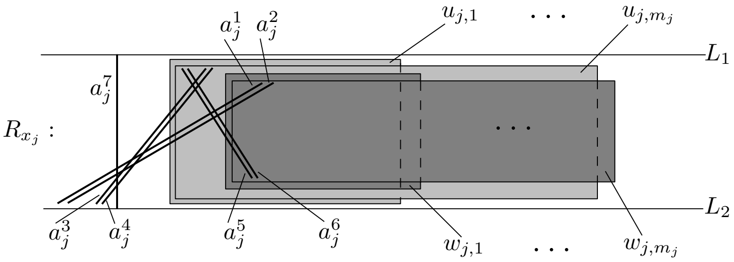

Figure 8: The construction R_{x_j} that is associated to the variable x_j of the formula ϕ, where $j = 1, 2, \ldots, n$.

6.3 The construction the trapezoid graph G_ϕ

We construct now a trapezoid representation R_ϕ of the whole trapezoid graph G_ϕ, by composing the constructions R_{α_i} and R_{x_j} presented in Sections 6.1 and 6.2, as follows. First, we place in R_ϕ the k trapezoid representations R_{α_i}, where $i = 1, 2, \ldots, k$, between the lines L_1 and L_2 such that, whenever $i < i'$, every trapezoid of R_{α_i} lies completely to the left of every trapezoid of $R_{\alpha_{i'}}$. Then, we place in R_ϕ the n trapezoid representations R_{x_j}, where $j = 1, 2, \ldots, n$, between the lines L_1 and L_2 such that, whenever $j < j'$, the lines of $a_j^1, a_j^2, \ldots, a_j^7$ and the left lines of all $u_{j,t}, w_{j,t}$, lie completely to the left of the lines of $a_{j'}^1, a_{j'}^2, \ldots, a_{j'}^7$ and the left lines of all $u_{j',t'}, w_{j',t'}$. Moreover, for every $j, j' = 1, 2, \ldots, n$, the lines of $a_j^1, a_j^2, \ldots, a_j^7$ and the left lines of all $u_{j,t}, w_{j,t}$, lie in R_ϕ completely to the left of the right lines of all $u_{j',t'}, w_{j',t'}$. Thus, note in particular that every $u_{j,t}$ intersects every other $u_{j',t'}$ and every $w_{j',t'}$ in R_ϕ.

Let $j \in \{1, 2, \ldots, n\}$ and $t \in \{1, 2, \ldots, m_j\}$. Recall that, by the construction of R_{x_j} in Section 6.2, the pair of trapezoids $\{u_{j,t}, w_{j,t}\}$ corresponds to the appearance of the variable x_j in a clause α_i of ϕ, where $i = i_{j,t} \in \{1, 2, \ldots, k\}$. That is, either $\ell_{i,p} = x_j$ or $\ell_{i,p} = \overline{x_j}$ for some $p \in \{1, 2, 3\}$, where $\alpha_i = (\ell_{i,1} \vee \ell_{i,2} \vee \ell_{i,3})$. Then, we place in R_ϕ the right lines of the trapezoids $u_{j,t}$ and $w_{j,t}$ directly before the left line of $v_{i,p}$ (i.e. no line of any other trapezoid intersects with or lies between the right lines of $u_{j,t}$ and $w_{j,t}$ and the left line of $v_{i,p}$).

In order to finalize the construction of R_ϕ, we distinguish now the two cases regarding the literal $\ell_{i,p}$ of the clause α_i, in which the variable x_j appears. If $\ell_{i,p} = x_j$, then we add to R_ϕ six trivial trapezoids (i.e. lines) $\{b_{j,t}^1, b_{j,t}^2, \ldots, b_{j,t}^6\}$, as it is shown in Figure 9(a). On the other hand, if $\ell_{i,p} = \overline{x_j}$, then we add to R_ϕ eight trivial trapezoids (i.e. lines) $\{b_{j,t}^1, b_{j,t}^2, \ldots, b_{j,t}^8\}$, as it is shown in Figure 9(b). In particular, we place these six (resp. eight) new lines in R_ϕ such that they intersect only the right lines of $u_{j,t}$ and $w_{j,t}$ and the left line of $v_{i,p}$ in R_ϕ. Note that the trapezoid graphs induced by the representations in Figures 9(a) and 9(b) are isomorphic to the graphs investigated in Lemmas 5 and 6, respectively. This completes the construction of

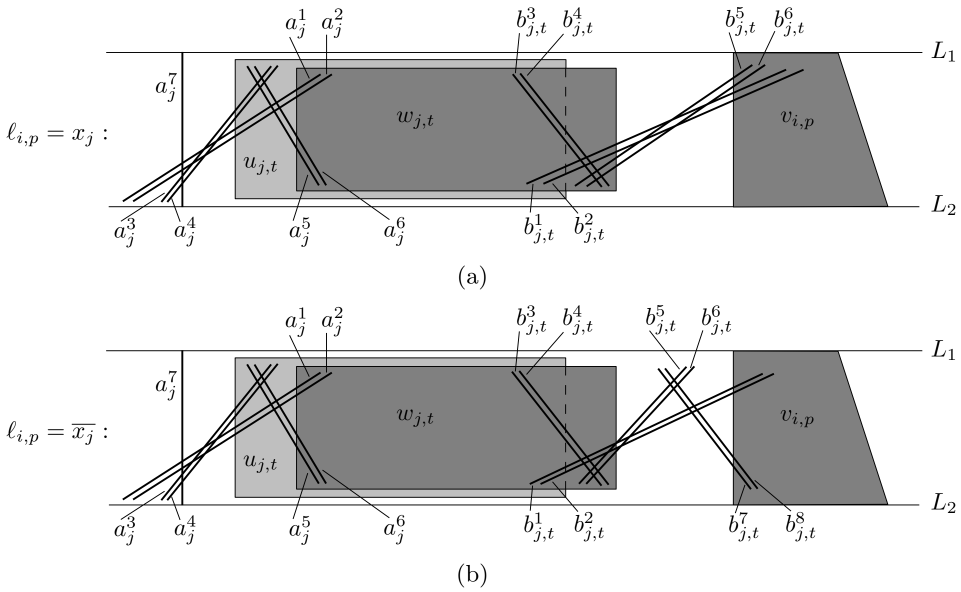

Figure 9: The composition of the trapezoids of R_{x_j} with the trapezoid $v_{i,p}$ of R_{α_i}, in the cases where (a) $\ell_{i,p} = x_j$ and (b) $\ell_{i,p} = \overline{x_j}$.

the trapezoid representation R_ϕ, while G_ϕ is the trapezoid graph induced by R_ϕ.

It is now easy to verify that, by the construction of R_ϕ, all the trapezoids $u_{j,t}$ are upper-left-closed and right-closed in R_ϕ, while all the trapezoids $w_{j,t}$ are lower-right-closed and left-closed in R_ϕ. Furthermore, all the trapezoids $u_{j,t}$ are lower-left-open in R_ϕ and all the trapezoids $w_{j,t}$ are upper-right-open in R_ϕ. Consider now a trapezoid $v_{i,p}$ in R_ϕ. If $v_{i,p}$ corresponds to a positive literal $\ell_{i,p} = x_j$ (for some variable x_j), then $v_{i,p}$ is upper-left-closed and lower-left-open in R_ϕ (cf. Figure 9(a)). On the other hand, if $v_{i,p}$ corresponds to a negative literal $\ell_{i,p} = \overline{x_j}$, then $v_{i,p}$ is left-closed in R_ϕ (cf. Figure 9(b)).

In order to prove the correctness of our reduction (cf. Theorem 3), we prove separately the necessary and sufficient conditions in the next two lemmas.

Lemma 8 *If the formula ϕ is satisfiable, then G_ϕ is a triangle graph.*

Proof. Suppose that ϕ has a satisfying truth assignment τ. Starting from R_ϕ, we will construct a standard trapezoid representation R_0 of G_ϕ. This will then imply that G_ϕ is a triangle graph by Theorem 1.

First consider an index j that corresponds to a variable $x_j = 0$ in the truth assignment τ. Furthermore consider an imaginary line L_3 that is parallel to L_1 and L_2 and has the same distance from both L_1 and L_2. We replace in R_ϕ the lines $\{a_j^1, a_j^2\}$ (resp. the lines $\{b_{j,t}^1, b_{j,t}^2\}$ for every index $t = 1, 2, \ldots, m_j$) by their mirror image along L_3, such that, in the resulting representation, these flipped lines intersect with the same trapezoids as the lines $\{a_j^1, a_j^2\}$ (resp. the lines $\{b_{j,t}^1, b_{j,t}^2\}$) intersect in R_ϕ. In the case where the corresponding literal $\ell_{i,p}$ equals a variable x_j, i.e. if $\ell_{i,p} = x_j = 0$, these flipping operations are illustrated in Figure 10(a). Otherwise, in the case where the corresponding literal $\ell_{i,p}$ equals a negated variable $\overline{x_j}$, i.e. if $\ell_{i,p} = \overline{x_j} = 1$, these flipping operations are illustrated in Figure 10(b). For better visibility, the flipped lines are drawn dashed in Figure 10. For every other index j that corresponds to a variable $x_j = 1$ in τ, we leave the lines $\{a_j^1, a_j^2\}$, as well as all the lines $\{b_{j,t}^1, b_{j,t}^2\}$, at the same position in R_0 as in R_ϕ.

Note that, after performing these flipping operations, for every index j that corresponds to a variable $x_j = 0$ in τ, all the trapezoids $u_{j,t}$ are left-closed and right-open, while all the trape-

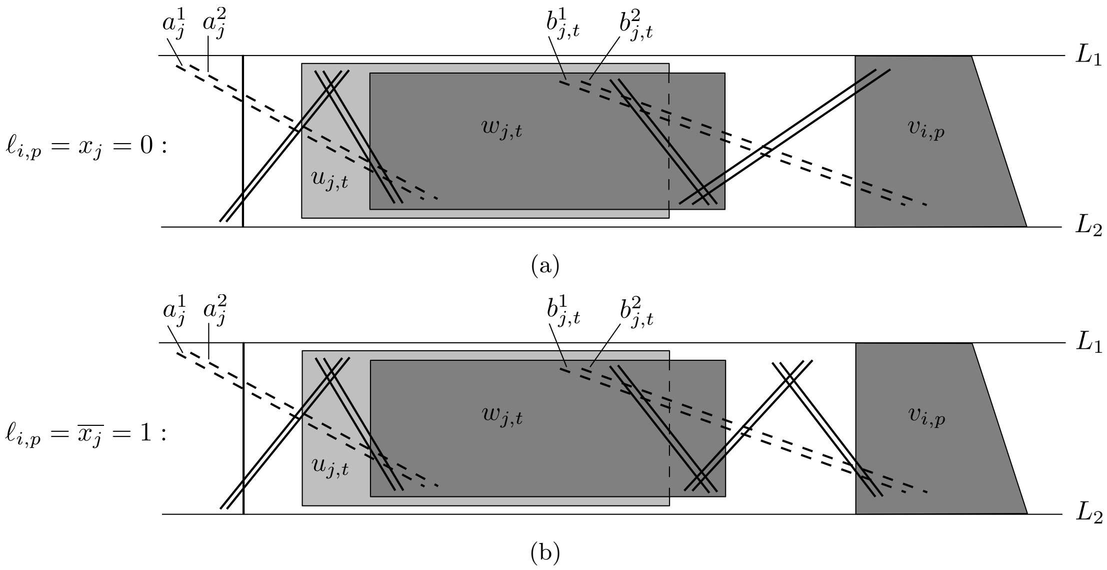

Figure 10: The case where $x_j = 0$ in the truth assignment τ: the horizontal axis flipping operations of the lines $\{a_j^2, a_j^2\}$ and $\{b_{j,t}^2, b_{j,t}^2\}$, $t = 1, 2, \ldots, m_j$, where (a) $\ell_{i,p} = x_j = 0$, (b) $\ell_{i,p} = \overline{x_j} = 1$.

zoids $w_{j,t}$ are left-open and right-closed. On the contrary, for every index j that corresponds to a variable $x_j = 1$ in τ, all the trapezoids $u_{j,t}$ are left-open and right-closed, while all the trapezoids $w_{j,t}$ are left-closed and right-open (cf. Figure 9). That is, after performing the above flipping operations, for every variable x_j of ϕ, all the trapezoids $u_{j,t}$ and $w_{j,t}$ are open in the resulting trapezoid representation. Moreover, for all indices i, p, the trapezoid $v_{i,p}$ is left-open in the resulting trapezoid representation if and only if the literal $\ell_{i,p}$ is satisfied in τ, i.e. if and only if $\ell_{i,p} = 1$ in τ (cf. Figures 9 and 10).

Let us now complete the construction of R_0 from R_ϕ. Consider first a clause α_i of ϕ, where $i = 1, 2, \ldots, k$. Then, α_i has at least one satisfied literal $\ell_{i,p} = 1$ in the truth assignment τ, where $p \in \{1, 2, 3\}$, since τ is assumed to be a satisfying assignment of ϕ. Therefore, after performing the above flipping operations, there exists by the previous paragraph at least one trapezoid $v_{i,p}$, where $p \in \{1, 2, 3\}$, which is left-open in the resulting trapezoid representation. Recall now by Corollary 1 that we can locally change in R_{α_i} the right lines of $v_{i,1}, v_{i,2}, v_{i,3}$ and the left lines of $v'_{i,1}, v'_{i,2}, v'_{i,3}$, such that $v_{i,p}$ is right-closed and $v_{i,p'}$ is right-open, for every $p' \in \{1, 2, 3\} \setminus \{p\}$. Therefore, after changing appropriately these lines of the trapezoids, $v_{i,p}$ is right-closed and left-open in the resulting representation, while $v_{i,p'}$ is right-open, for every $p' \in \{1, 2, 3\} \setminus \{p\}$. That is, all $v_{i,1}, v_{i,2}, v_{i,3}$ are open in the resulting trapezoid representation.

Denote by R_0 the trapezoid representation that is obtained if we perform all the above local changes. Note that all trapezoids $u_{j,t}$, $w_{j,t}$, and $v_{i,p}$ are open in R_0 for all pairs of indices j, t and i, p. Furthermore, it is easy to see that also the trapezoids $v'_{i,p}$ are right-open in R_0 (in particular, they are also right-open in the initial trapezoid representation R_{α_i}, cf. Figure 7). All the remaining trapezoids of R_0 are trivial, i.e. lines, and thus also trivial triangles. Therefore, R_0 is a standard trapezoid representation of G_ϕ by Definition 3, and thus G_ϕ is a triangle graph by Theorem 1. This completes the proof of the lemma. ∎

Lemma 9 *If G_ϕ is a triangle graph, then the formula ϕ is satisfiable.*

Proof. Suppose that G_ϕ is a triangle graph and let R be a triangle representation of G_ϕ. We construct a truth assignment τ of the variables $x_1, x_2, \ldots, x_n$ that satisfies the formula ϕ, as follows. For any $j = 1, 2, \ldots, n$ such that $a_j^7 \ll_R u_{j,1}$, we define $x_j = 1$ if and only if $u_{j,1}$ is left-open in R. Similarly, for any $j = 1, 2, \ldots, n$ such that $u_{j,1} \ll_R a_j^7$, we define $x_j = 1$ if and only if $u_{j,1}$ is right-open in R. We will prove that the truth assignment τ satisfies ϕ.

Let $i \in \{1, 2, \ldots, k\}$. By possibly performing a vertical axis flipping of R, we may assume without loss of generality that $v_{j,1} \ll_R z_i$. Therefore, since $v_{i,2}, v_{i,3} \in N(v_{i,1}) \setminus N(z_i)$, it follows that also $v_{i,2} \ll_R z_i$ and $v_{i,3} \ll_R z_i$. Consider now the subgraph H_0 of G_ϕ induced by the vertices $\{v_{i,1}, v_{i,2}, v_{i,3}, v'_{i,1}, v'_{i,2}, v'_{i,3}, z_i\}$. Note that H_0 is isomorphic to the graph induced by the trapezoid representation R_{α_i} of Figure 7. Then, Lemma 7 implies that one of $v_{i,1}, v_{i,2}, v_{i,3}$ is right-closed in the restriction $R[H_0]$ of R. Let in the following $v_{i,p}$ be right-closed in $R[H_0]$, and thus also right-closed in R, for some $p \in \{1, 2, 3\}$. Thus $L(v'_{i,p'}) <_R R(v_{i,p}) <_R L(v'_{i,p''})$ and $l(v'_{i,p''}) <_R r(v_{i,p}) <_R l(v'_{i,p'})$, for appropriate values of the indices $p', p'' \in \{1, 2, 3\} \setminus \{p\}$, cf. Figure 7.

Furthermore, let x_j be the variable that appears in the literal $\ell_{i,p}$ of the clause α_i, i.e. either $\ell_{i,p} = x_j$ or $\ell_{i,p} = \overline{x_j}$. Moreover, let the trapezoids $\{u_{j,t}, w_{j,t}\}$ correspond to the appearance of the variable x_j in $\ell_{i,p}$, for some index $t \in \{1, 2, \ldots, m_j\}$. Note that, since the trapezoids of vertices $u_{j,t}$ and $v_{i,p}$ do not intersect in R, it follows that either $u_{j,t} \ll_R v_{i,p}$ or $v_{i,p} \ll_R u_{j,t}$. We will prove that $u_{j,t} \ll_R v_{i,p}$. Suppose otherwise that $v_{i,p} \ll_R u_{j,t}$. Then, since $u_{j,t} \notin N(z_i)$, it follows that either $v_{i,p} \ll_R z_i \ll_R u_{j,t}$ or $v_{i,p} \ll_R u_{j,t} \ll_R z_i$. Let first $v_{i,p} \ll_R z_i \ll_R u_{j,t}$. Then, since $b_{j,t}^1 \in N(v_{i,p})$ and $b_{j,t}^1 \in N(u_{j,t})$ by the construction of G_ϕ (cf. Figures 9(a) and 9(b)), it follows that the line of $b_{j,t}^1$ intersects the line of z_i in R. This is a contradiction, since $b_{j,t}^1 \notin N(z_i)$ by the construction of G_ϕ. Let now $v_{i,p} \ll_R u_{j,t} \ll_R z_i$ and let $q \in \{1, 2, 3\} \setminus \{p\}$. Then, since $v'_{i,q} \in N(v_{i,p})$ and $v'_{i,q} \in N(z_i)$ by the construction of G_ϕ (cf. Figure 7), it follows that the trapezoid of $v'_{i,q}$ intersects the trapezoid of $u_{j,t}$ in R. This is again a contradiction, since $v'_{i,q} \notin N(u_{j,t})$ by the construction of G_ϕ. Therefore $u_{j,t} \ll_R v_{i,p}$.

Now, we will now prove that $a_j^7 \ll_R u_{j,t}$. Suppose otherwise that $u_{j,t} \ll_R a_j^7$. Then, since $a_j^7 \notin N(v_{i,p})$, it follows that either $u_{j,t} \ll_R v_{i,p} \ll_R a_j^7$ or $u_{j,t} \ll_R a_j^7 \ll_R v_{i,p}$. Let first $u_{j,t} \ll_R v_{i,p} \ll_R a_j^7$. Then, since $a_j^1 \in N(u_{j,t})$ and $a_j^1 \in N(a_j^7)$ by the construction of G_ϕ (cf. Figures 9(a) and 9(b)), it follows that the line of a_j^1 intersects the trapezoid of $v_{i,p}$ in R. This is a contradiction, since $a_j^1 \notin N(v_{i,p})$ by the construction of G_ϕ. Let now $u_{j,t} \ll_R a_j^7 \ll_R v_{i,p}$. Then, since $b_{j,t}^1 \in N(u_{j,t})$ and $b_{j,t}^1 \in N(v_{i,p})$ by the construction of G_ϕ (cf. Figure 7), it follows that the line of $b_{j,t}^1$ intersects the line of a_j^7 in R. This is again a contradiction, since $b_{j,t}^1 \notin N(a_j^7)$ by the construction of G_ϕ. Therefore $a_j^7 \ll_R u_{j,t}$.

That is, $a_j^7 \ll_R u_{j,t} \ll_R v_{i,p} \ll_R z_i$. Note here by the construction of G_ϕ, that the existence of the lines $\{a_j^1, a_j^2, \ldots, a_j^7\}$ guarantees that $u_{j,1}$ is left-open in R if and only if $u_{j,t}$ is left-open in R, for any $t = 2, \ldots, m_j$ (cf. Figure 8). Therefore, due to the truth assignment τ of the variables $x_1, x_2, \ldots, x_n$ that we defined above, it follows that $x_j = 1$ if and only if $u_{j,t}$ is left-open in R, for any $t = 1, 2, \ldots, m_j$. We distinguish in the following the two cases regarding the literal $\ell_{i,p}$ of the clause α_i.

Let first $\ell_{i,p} = x_j$. Consider the subgraph H_1 of G_ϕ induced by the vertices $\{u_{j,t}, w_{j,t}, v_{i,p}\} \cup \{a_j^1, \ldots, a_j^7\} \cup \{b_{j,t}^1, b_{j,t}^2, \ldots, b_{j,t}^6\}$. Note that H_1 is isomorphic to the graph induced by the trapezoid representation of Figure 6(a). Furthermore, consider the subgraph H_2 of G_ϕ induced by the vertices $V(H_1) \cup \{v'_{i,p'}, v'_{i,p''}\}$, where $\{p', p''\} = \{1, 2, 3\} \setminus \{p\}$. Note now by the construction of G_ϕ that the existence of the lines $\{a_j^1, a_j^2, \ldots, a_j^7\}$ guarantees that the trapezoid $u_{j,t}$ is left-open in R if and only if $u_{j,t}$ is left-open in the restriction $R[H_1]$ of R.

Moreover, the connected graph H_2 satisfies the conditions of Theorem 2. Indeed, $v_{i,p}$ is a cut vertex of H_2 and $H_2 - \{v_{i,p}\}$ has the two connected components H_1 and $G_\phi[v'_{i,p'}, v'_{i,p''}]$. Therefore, since $R[H_2]$ is a triangle representation, Theorem 2 implies that $v_{i,p}$ is open in $R[H_2]$. Recall now that $L(v'_{i,p'}) <_R R(v_{i,p}) <_R L(v'_{i,p''})$ and $l(v'_{i,p''}) <_R r(v_{i,p}) <_R l(v'_{i,p'})$, for appropriate values of the indices $p', p'' \in \{1, 2, 3\} \setminus \{p\}$. Therefore, since $b^1_{j,t} \ll_R v'_{i,p'}$ and $b^1_{j,t} \ll_R v'_{i,p''}$, it follows by Definitions 1 and 2 that $v_{i,p}$ is right-closed in $R[H_2]$. Thus, since $v_{i,p}$ is open in $R[H_2]$, it follows that $v_{i,p}$ is left-open in $R[H_2]$. Therefore, $v_{i,p}$ is also left-open in $R[H_1]$, since H_1 is an induced subgraph of H_2. Now, Lemma 5 implies that $u_{j,t}$ is left-open in $R[H_1]$, since $a^7_j \ll_R u_{j,t}$ and $v_{i,p}$ is left-open in $R[H_1]$. Therefore $u_{j,t}$ is also left-open in R, and thus it follows by the definition of the truth assignment τ that $\ell_{i,p} = x_j = 1$.

Let now $\ell_{i,p} = \overline{x_j}$. Consider the subgraph H_3 of G_ϕ induced by the vertices $\{u_{j,t}, w_{j,t}, v_{i,p}\} \cup \{a^1_j, \ldots, a^7_j\} \cup \{b^1_{j,t}, b^2_{j,t}, \ldots, b^8_{j,t}\}$. Note that H_3 is isomorphic to the graph induced by the trapezoid representation of Figure 6(b). Furthermore, consider the subgraph H_4 of G_ϕ induced by the vertices $V(H_3) \cup \{v'_{i,p'}, v'_{i,p''}\}$, where $\{p', p''\} = \{1, 2, 3\} \setminus \{p\}$.

Moreover, the connected graph H_4 satisfies the conditions of Theorem 2. Indeed, $v_{i,p}$ is a cut vertex of H_4 and $H_4 - \{v_{i,p}\}$ has the two connected components H_3 and $G_\phi[v'_{i,p'}, v'_{i,p''}]$. Therefore, since $R[H_4]$ is a triangle representation, Theorem 2 implies that $v_{i,p}$ is open in $R[H_4]$. Recall that $L(v'_{i,p'}) <_R R(v_{i,p}) <_R L(v'_{i,p''})$ and $l(v'_{i,p''}) <_R r(v_{i,p}) <_R l(v'_{i,p'})$, for appropriate values of the indices p' and p'', where $\{p', p''\} = \{1, 2, 3\} \setminus \{p\}$. Therefore, since $b^1_{j,t} \ll_R v'_{i,p'}$ and $b^1_{j,t} \ll_R v'_{i,p''}$, it follows by Definitions 1 and 2 that $v_{i,p}$ is right-closed in $R[H_4]$. Thus, since $v_{i,p}$ is open in $R[H_4]$, it follows that $v_{i,p}$ is left-open in $R[H_4]$. Therefore, $v_{i,p}$ is also left-open in $R[H_3]$, since H_3 is an induced subgraph of H_4. Now, Lemma 6 implies that $u_{j,t}$ is left-closed in $R[H_3]$, since $a^7_j \ll_R u_{j,t}$ and $v_{i,p}$ is left-open in $R[H_3]$. Therefore $u_{j,t}$ is also left-closed in R, and thus, it follows by the definition of the truth assignment τ that $x_j = 0$, i.e. $\ell_{i,p} = \overline{x_j} = 1$.

Summarizing, for an arbitrary index $i \in \{1, 2, \ldots, k\}$, we proved that there exists an index $p \in \{1, 2, 3\}$, such that the literal $\ell_{i,p}$ is satisfied by the truth assignment τ, i.e. $\ell_{i,p} = 1$. Therefore, every clause α_i, where $i \in \{1, 2, \ldots, k\}$, is satisfied by τ, and thus the whole formula ϕ is satisfied by τ. This completes the proof of the lemma. ∎

The next theorem follows now directly by Lemmas 8 and 9.

Theorem 3 *The formula ϕ is satisfiable if and only if G_ϕ is a triangle graph.*

Therefore, since 3SAT is NP-complete, Theorem 3 implies that the recognition of triangle graphs is NP-hard. Moreover, since the recognition of triangle graphs lies in NP by Observation 1, and since G_ϕ is a trapezoid graph, we can summarize our main result in the next theorem.

Theorem 4 *Given a graph G, it is NP-complete to decide whether G is a triangle graph. The problem remains NP-complete even if the given graph G is known to be a trapezoid graph.*

7 Concluding remarks

In this article we proved that the triangle graph (known also as PI* graph) recognition problem is NP-complete, by providing a reduction from the 3SAT problem, thus answering a longstanding open question. Our reduction implies that this problem remains NP-complete even in the case where the input graph is a trapezoid graph. The recognition of *simple-triangle* graphs [3], as well as the recognition of the related classes of *unit* and *proper tolerance* graphs [1,11] (these are subclasses of bounded tolerance, i.e. parallelogram, graphs [1]), *proper bitolerance* graphs [2,11] (they coincide with *unit bitolerance* graphs [2]), and *multitolerance* graphs [19] (they naturally generalize trapezoid graphs [19,23]) remain interesting open problems for further research.

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