# Minimum Bisection is NP-hard on Unit Disk Graphs

Josep Díaz[1] George B. Mertzios[2]

[1]Universitat Politécnica de Catalunya, Spain.

[2]School of Engineering and Computing Sciences, Durham University, UK.

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# Graph partitioning

Graph partitioning:

- appropriately partitioning the vertices of a graph into subsets that fulfill certain conditions

Many practical applications:

- computer vision
- image processing
- VLSI layout design
- parallel computing
  - evenly distribute the computational load to processors, while minimizing processor communication
- sub-routine in many *divide-and-conquer* algorithms

For an overview: [Bichot, Siarry, *Graph Partitioning*, 2011]

# Graph partitioning - terminology

## Definition

Given a graph $G = (V, E)$ and $k \geq 2$, a balanced $k$-partition of $G$ is a partition of $V$ into sets $V_1, V_2, \ldots, V_k$ such that $|V_i| \leq \left\lceil \frac{|V|}{k} \right\rceil$, $1 \leq i \leq k$.

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- For $k = 2$: bisection of $G$

## Problem (Min-Bisection)

*Input: a graph $G$*

*Goal: compute a bisection of $G$ with the minimum possible size*
*(also known as the bisection width of $G$)*

# Minimum bisection

MIN-BISECTION is NP-hard:

- on general graphs [Garey, Johnson, 1979]
- on everywhere dense graphs ($\deg(v) = \Omega(n)$ for every vertex $v$)
- on bounded maximum degree graphs [MacGregor, 1978]
- on $d$-regular graphs [Bui, Chaudhuri, Leighton, Sipser, 1987]

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Due to its wide applications:

- many heuristics and exact algorithms appeared since the 70's
  [Kernighan, Lin, *Bell System Technical Journal*, 1978]
  [Delling et al., *ALENEX*, 2012]

On the positive side (theoretically):

- fixed parameter tractable [Cygan et al., *STOC*, 2014]
- the currently best approximation ratio is $O(\log n)$
  [Räcke, *STOC*, 2008]

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MIN-BISECTION is solvable in polynomial time:

- on trees and hypercubes [MacGregor, *PhD thesis - Berkeley*, 1978] [Díaz, Petit, Serna, *ACM Comp. Surveys*, 2002]
- on graphs with bounded treewidth [Jansen et al., *SIAM Journal on Computing*, 2005]
- on solid grid graphs (no holes)
  - in $O(n^5)$-time [Papadimitriou, Sideri, *Math. Systems Theory*, 1996] (and $O(n^{5+2h})$-time for a grid graph with $h$ holes)
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- unit disk graphs

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The first two problems are equivalent:

- MIN-BISECTION on planar $\leq$ MIN-BISECTION on grid with holes [Papadimitriou, Sideri, *Math. Systems Theory*, 1996]
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Furthermore:

- MIN-BISECTION on planar graphs with max. degree 4
  $\leq$ MIN-BISECTION on unit disk graphs
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Our result:

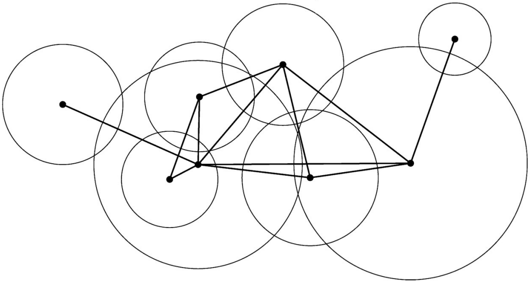
**Theorem**

MIN-BISECTION *is NP-complete on unit disk graphs.*

# Disk graphs and unit disk graphs

## Definition

A graph $G = (V, E)$ is a disk graph if we can assign every vertex $v \in V$ to a disk $D_v$ in the plane such that $uv \in E$ if and only if $D_u \cap D_v \neq \emptyset$.



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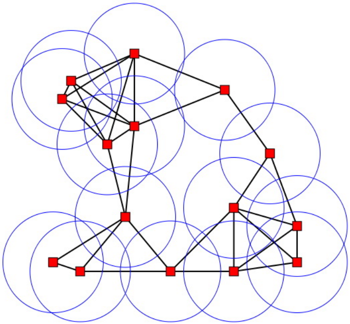
A disk graph $G$ is a unit disk graph if it can be drawn in such a way that every disk $D_v$ has equal radius (e.g. radius 1).



# Disk graphs and unit disk graphs

## Alternative Definition

*A disk graph $G = (V, E)$ is a unit disk graph if we can associate every vertex $v \in V$ to a point $p_v$ in the plane, such that $uv \in E$ if and only if the distance between $p_u$ and $p_v$ is at most a fixed constant $c$ (e.g. $c = 1$).*

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(Unit) disk graphs: appear naturally in wireless communication networks

- center of a disk $\longrightarrow$ the position of a device (phone, antenna, ...)
- radius $\longrightarrow$ the distance that a wireless signal can reach
- the bisection width determines the communication bandwidth

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## Theorem (Kratochvíl, 1996; Breu et al., 1998)

*Given a graph $G$, it is NP-hard to decide whether $G$ is a (unit) disk graph.*

- but (unit) disk graphs usually come along with a given representation

**Theorem**

Min-Bisection *is* *NP-complete* *on* *unit disk* *graphs.*

*Proof:* reduction from a variant of maximum satisfiability

**Problem (monotone Max-XOR(3))**

*Input:*

- *an XOR-formula* $\phi$ *with variables* $x_1, x_2, \ldots, x_n$*, i.e. a boolean formula that is the conjunction of XOR-clauses of the form* $(x_i \oplus x_k)$
- $\phi$ *is monotone, i.e. no variable is negated*
- *every variable* $x_i$ *appears in exactly 3 clauses of* $\phi$

*Goal:*

- *compute a truth assignment that XOR-satisfies most clauses of* $\phi$

- A clause $(x_i \oplus x_k)$ is XOR-satisfied if:
  - either $x_i = 0$ and $x_k = 1$
  - or $x_i = 1$ and $x_k = 0$
- We can decide in polynomial time whether $\phi$ is XOR-satisfiable, but:

**Lemma**

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- Given a monotone XOR(3)-formula $\phi$ with $n$ variables, we construct:
  - an auxiliary unit disk graph $G_n$ from $n$ and
  - a unit disk graph $H_\phi$ from $G_n$ and $\phi$

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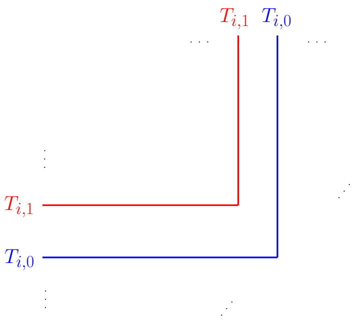
*$\phi$ a truth assignment that XOR-satisfies at least $k$ clauses*
*$\Leftrightarrow$ $H_\phi$ has a bisection with value at most $2n^4(n-1)+3n-2k$.*

# The auxiliary unit disk graph $G_n$

- For every variable $x_i$:
  - we define two (non-straight) lines $T_{i,0}$ and $T_{i,1}$ (called tracks of $x_i$)



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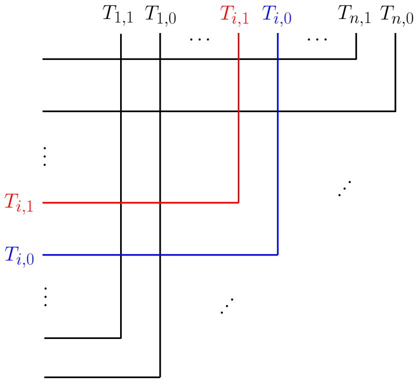

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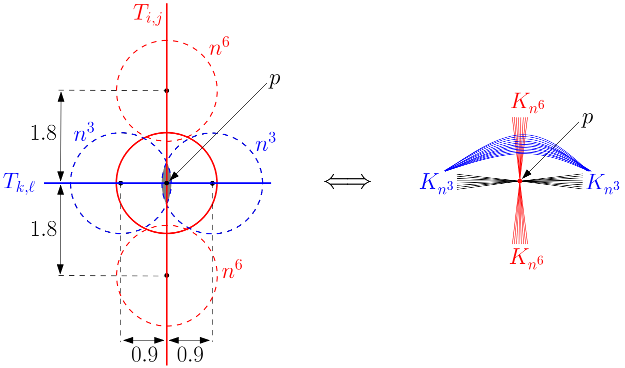
- Main challenge at every intersection point $p$ of two tracks:
  - arrange the disk centers such that every track has the same color on both sides of $p$
  - this should happen regardless of the color of the other track!

# The auxiliary unit disk graph $G_n$

Solution to this challenge:

- use large cliques (i.e. disks centered closely to each other)

1. At the intersection point $p$ of two tracks $T_{i,j}$ and $T_{k,\ell}$:



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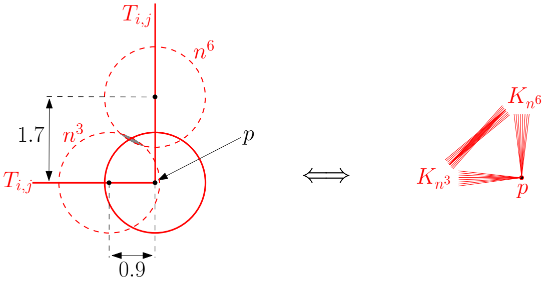

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Therefore:

**Lemma**

*Let $\mathcal{B}$ be a bisection of $G_n$ with size less than $n^6$.*
*Then for every track $T_{i,j}$, all disks on $T_{i,j}$ have the same color in $\mathcal{B}$.*

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Furthermore, we can prove:

**Lemma**

*Let $\mathcal{B}$ be a minimum bisection of $G_n$ with size less than $n^6$. Then, for every $1 \leq i \leq n$, the tracks $T_{i,0}$ and $T_{i,1}$ have different colors in $\mathcal{B}$.*

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Thus, as long as we keep the size of a bisection less than $n^6$:

- truth assignments of $\phi$ $\longleftrightarrow$ minimum bisections of $G_n$

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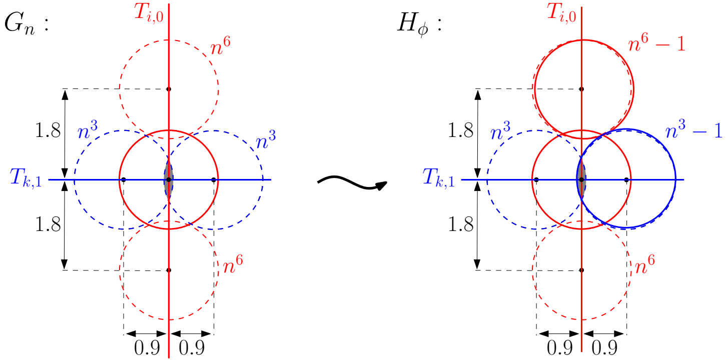

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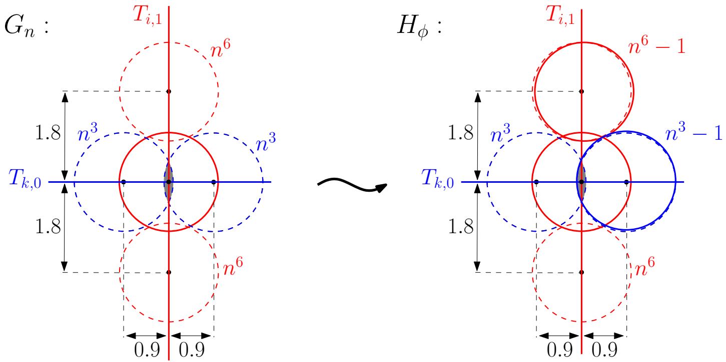

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$\Rightarrow$ in a truth assignment of $\phi$:

- for the values $x_i = x_k$ that do not XOR-satisfy $(x_i \oplus x_k)$,
- we "pay" two more bi-colored edges in the resulting bisection of $H_\phi$

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**Theorem**

*$\phi$ a truth assignment that XOR-satisfies at least $k$ clauses*
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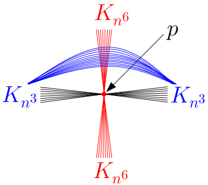


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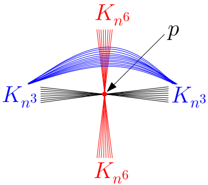


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- If $(x_i \oplus x_k)$ is an XOR-satisfied clause of $\phi$ (i.e. $x_i \neq x_k$):
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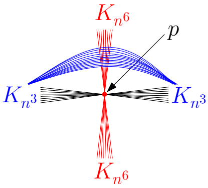


- if $x_i = 0$ then assign colors: $T_{i,0}$ and $T_{i,1}$
- if $x_i = 1$ then assign colors: $T_{i,0}$ and $T_{i,1}$
- If $(x_i \oplus x_k)$ is not a clause of $\phi$:
  - we pay exactly $2n^3 + 2n^3 = 4n^3$ edges in the bisection
- If $(x_i \oplus x_k)$ is an XOR-satisfied clause of $\phi$ (i.e. $x_i \neq x_k$):
  - we pay exactly $2n^3 + 2n^3 = 4n^3$ edges
- If $(x_i \oplus x_k)$ is not an XOR-satisfied clause of $\phi$ (i.e. $x_i = x_k$):
  - we pay exactly $(2n^3+1) + (2n^3+1) = 4n^3 + 2$ edges

# MIN-BISECTION on unit disk graphs

**Theorem**

*$\phi$ a truth assignment that XOR-satisfies at least $k$ clauses*
*$\Leftrightarrow$ $H_\phi$ has a bisection with value at most $2n^4(n-1)+3n-2k$.*

*Proof idea:* $(\Rightarrow)$:

- We have $m = \frac{3n}{2}$ clauses
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$\Rightarrow$ we pay:

$$\left(\binom{n}{2} - m\right) \cdot 4n^3 + k \cdot 4n^3 + (m-k) \cdot (4n^3+2) = 2n^4(n-1)+3n-2k$$

edges in the bisection

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## Lemma

*Let $\mathcal{B}$ be a minimum bisection of $H_\phi$ with size less than $n^6$. Then, for every $1 \leq i \leq n$, the tracks $T_{i,0}$ and $T_{i,1}$ have different colors in $\mathcal{B}$.*

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- if $T_{i,0}$ and $T_{i,1}$, then set $x_i = 0$
- if $T_{i,0}$ and $T_{i,1}$, then set $x_i = 1$
- count the number of intersections that contribute $4n^3$ (resp. $4n^3 + 2$)
  $\Rightarrow$ at least $k$ clauses of $\phi$ are XOR-satisfied $\square$

# Open problems

1. Is MIN-BISECTION tractable on planar graphs?
    - the above approach for unit disk graphs is based on large cliques

    $\Rightarrow$ it is not clear whether it can be extended to planar graphs

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1. Is MIN-BISECTION tractable on planar graphs?
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2. Does MIN-BISECTION have a constant approximation ratio on general graphs?
   - or on planar / unit disk graphs?
   - it is known: no PTAS unless $\text{NP} \not\subseteq \cap_{\varepsilon>0} \text{BPTIME}(2^{n^{\varepsilon}})$
     (i.e. NP does not have randomized algorithms that run in sub-exponential time) [Khot, *FOCS*, 2004]
   - Can we show "no PTAS unless P=NP" ?

Thank you for your attention!