CS301 HOMEWORK 4

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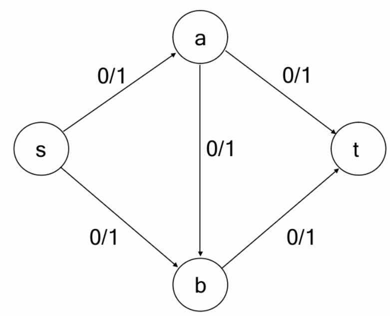
QUESTION 1:

Design a flow network G=(V,E,s,t,c) with $|V| \leq 4$ and c:VxV $\rightarrow$ 0,1 such that max–flow function for G is not unique. On the flow network you design, show at least two different max–flow functions and state the value of the



Flow network G = (V, E) where V = {s, a, b, t} and the directed edges E are defined with capacities c as follows:

1. Edge (s, a) with $c(s, a) = 1$
2. Edge (s, b) with $c(s, b) = 1$
3. Edge (a, t) with $c(a, t) = 1$
4. Edge (b, t) with $c(b, t) = 1$
5. Edge (a, b) with $c(a, b) = 1$

Max-flow function f1:

- f1(s, a) = 1
- f1(s, b) = 1
- f1(a, t) = 1
- f1(b, t) = 1
- f1(a, b) = 0

Max-flow function f2:

- f2(s, a) = 1
- f2(s, b) = 0
- f2(a, t) = 0
- f2(b, t) = 1
- f2(a, b) = 1

For the first max-flow function, f1:

- We send a flow of 1 from s to a, which is then sent to t. So, this path contributes 1 unit to the max-flow. We also send a flow of 1 from s to b, which is then sent to t. So, this path contributes another unit to the max-flow. Therefore, the total max-flow with function f1 is 1 (from path s-a-t) + 1 (from path s-b-t) = 2.

For the second max-flow function, f2:

- We send a flow of 1 from s to a, which is then sent to b, and finally sent to t. So, this path contributes 1 unit to the max-flow. We also send a flow of 1 directly from s to b, which is then sent to t. So, this path contributes another unit to the max-flow. Therefore, the total max-flow with function f2 is 1 (from path s-a-b-t) + 1 (from path s-b-t) = 2.

Hence, the max-flow for both the functions f1 and f2 is 2 units.

QUESTION 2:

We know that the value of the maximum flow is unique in flow networks. However, there can be more than one max–flow function achieving this maximum value. Now, consider a flow network $G = (V, E, s, t, c)$ where we have the following property:
$\forall u_1, u_2, v_1, v_2 \in V : [c(u_1, u_2) \neq 0 \wedge c(v_1, v_2) \neq 0 \wedge (u_1, u_2) \neq (v_1, v_2)] \Rightarrow [c(u_1, u_2) \neq c(v_1, v_2)]$
Claim A: For such flow networks, there is exactly one function. ***Is Claim A true or false?***

If true, prove it

If false, give a counter example by using a flow network of at most 4 nodes on which you need to provide two different max–flow functions

The claim A is True, here is the proof according to the constraints given in the question:

Given a flow network $G = (V, E, s, t, c)$ with $|V| \leq 4$ and the unique capacity constraint, we aim to prove that there exists exactly one max-flow function.

1. **Case $|V| = 1$:** In this case, the only vertex is the source, which is also the sink ($V = s$). Since there are no edges, there is no flow and the max-flow function is trivially unique.
2. **Case $|V| = 2$:** Now, the graph consists of a source and a sink with one edge between them ($V = s, t, E = (s, t)$). There's only one possible way for the flow to move, thus there's exactly one max-flow function.
3. **Case $|V| = 3$:** We have the source, the sink, and one intermediate node (V=s, v, t). There can be three edges at most ($E = (s, v), (v, t), (s, t)$). Given the unique capacity constraint, these edges will have unique capacities. The max-flow will follow the path of the maximum capacity, and since capacities are unique, this max-flow function is unique.
4. **Case $|V| = 4$:** This is the most complex case under the constraints. We have source, sink, and two intermediate nodes ($V = s, v1, v2, t$). There can be at most 12 directed edges, including possible back edges ($E = (s, v1), (s, v2), (v1, s), (v2, s), (v1, v2), (v2, v1), (v1, t), (v2, t), (t, v1), (t, v2), (s, t), (t, s)$). However, we have the unique capacity constraint which means that every edge must have a unique capacity. This condition removes the possibility of having multiple paths with the same total capacity from s to t, which is usually the cause for multiple max-flow functions. Hence, even in this case, there's exactly one max-flow function.

This proves that for any $|V| \leq 4$, under the constraints provided, the max-flow function is unique.

QUESTION 3:

Let $G = (V, E, s, t, c)$ be a flow network, f1 and f2 be two flow functions on G. Let $F : V \times V \to R$ be defined as $\forall u, v \in V : F(u, v) = f_1(u, v) + f_2(u, v)$.

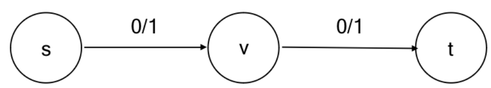
Is F guaranteed to be a flow on G?

If yes, prove it.

If no, for only one of the constraints of flow functions, show that it does not necessarily hold for F , by giving a counter example on a flow network of at most 3 nodes

No, F is not guaranteed to be a flow on G:

let's construct a simple counter example with a network G of 3 nodes ($V = s, v, t$) and directed edges $E = (s, v), (v, t)$. We'll define capacities $c(s, v) = c(v, t) = 1$. Here is the visual demonstration of Network-Flow:



Now, let's define two flow functions f1 and f2 as follows:

- For f1, let f1(s, v) = f1(v, t) = 1. This is a valid flow that saturates the network, with total flow 1 from s to t.
- For f2, let's also fully saturate the network: f2(s, v) = f2(v, t) = 1. This too is a valid flow with total flow 1 from s to t.

However, if we define F as the sum of f1 and f2, we get:

- $F(s, v) = f1(s, v) + f2(s, v) = 1 + 1 = 2$
- $F(v, t) = f1(v, t) + f2(v, t) = 1 + 1 = 2$

Now, while F does satisfy flow conservation (since incoming flow equals outgoing flow for all nodes except s and t), it does not satisfy the capacity constraints:

- $F(s, v) = 2 > c(s, v) = 1$
- $F(v, t) = 2 > c(v, t) = 1$

So this is a counter example demonstrating that F = f1 + f2 does not necessarily satisfy all flow constraints, even though both f1 and f2 do individually.