Discrete Mathematics
LECTURE 10
Graphs

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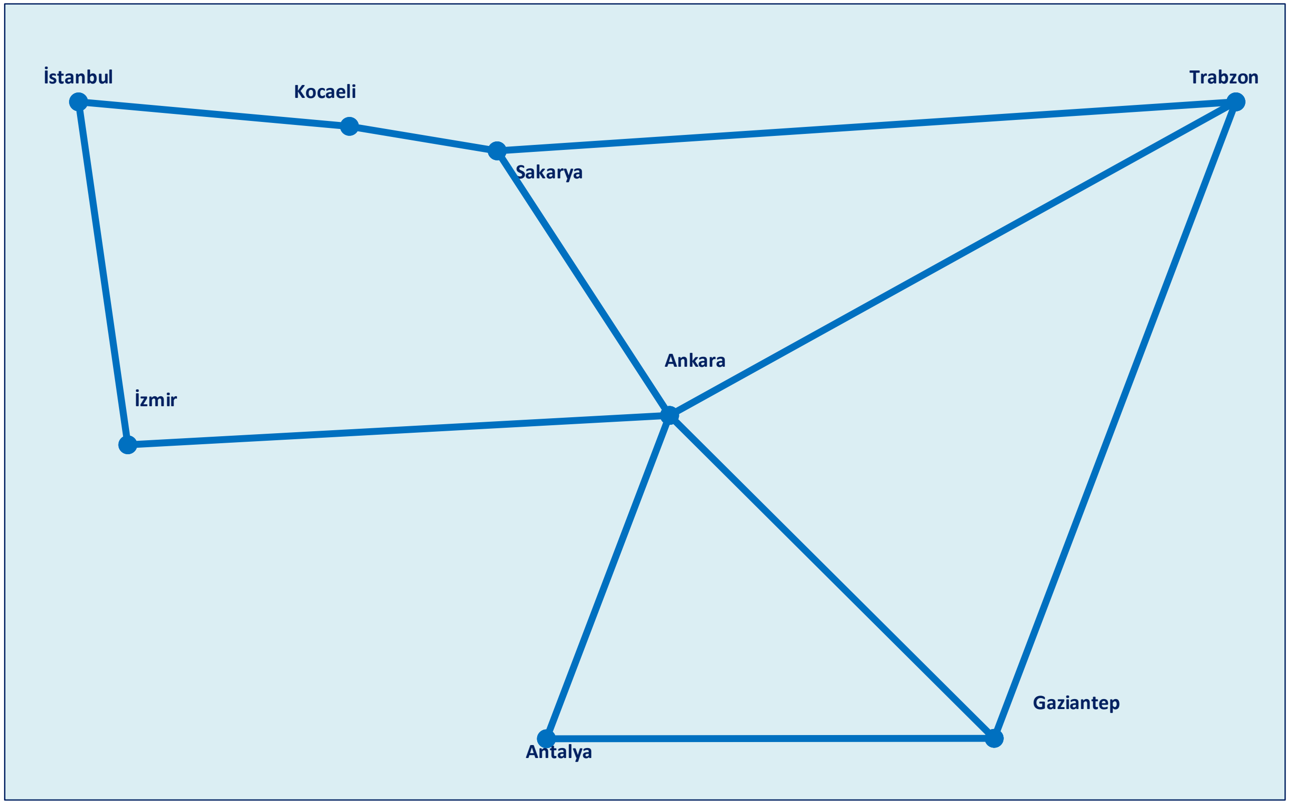
Outline

- Graphs
  - Definition of graphs
  - Some special simple graphs
  - Bipartite graphs
  - Representing graphs
  - Isomorphism on graphs
  - Paths on graphs
- References

Graphs



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Ankara
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Gaziantep
Antalya

Graphs…

- Let R be a binary relation on a set V, let $E \subseteq V \times V$.The pictural representation of E is called a **graph**.
  - the dots are called **vertices/nodes** and denoted as *V*
  - the line segments joining vertices are called **edges** and denoted as *E*
- in general, a graph consists of a set of vertices and a set of edges connecting various pairs of vertices.
- denoted as $G = (V, E)$.
- when an edge connects a vertex to itself, it is called a **loop**.
- two vertices that are connected by an edge are called **adjacent**.

Graphs…

- will be either undirected or directed.
  - edges of undirected graphs are also undirected
  - A **directed graph** (or **digraph**) (V ,E)
    - consists of a nonempty set of vertices V and a set of directed edges (or arcs) E.
    - each directed edge is associated with an ordered pair of vertices.
    - the directed edge associated with the ordered pair (*u, v*) is said to *start* at *u* and *end* at *v*.

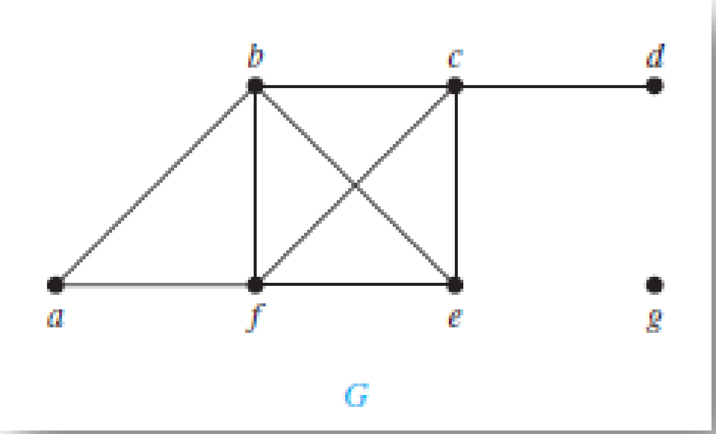
Graphs…

- The **degree** of a vertex in an undirected graph
  - the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex.
  - denoted by $deg(v)$.

  - Let $G = (V, E)$ be an undirected graph with m edges. $\Longrightarrow 2m = \sum_{v \in V} \deg(v)$.

Graphs…

- **Example:** What are the degrees and what are the neighborhoods of the vertices in the graph *G*?



- **Solution:**

$\deg(a) = 2,\ \deg(b) = 4,\ \deg(c) = \deg(f) = 4,$
$\deg(d) = 1, \deg(e) = 3, \deg(g) = 0$
$N(a) = \{b, f\}, N(b) = \{a, c, e, f\}, N(c) = \{b, d, e, f\},$
$N(d) = \{c\}, N(e) = \{b, c, f\}, N(f) = \{a, b, c, e\}, N(g) = \emptyset$ ✓

Graphs…

- In a graph with directed edges
  - the **in-degree** of a vertex v,
    - is the number of edges with v as their terminal vertex.
    - denoted by $\deg^{-}(v)$.
  - The **out-degree** of a vertex v,
    - is the number of edges with v as their initial vertex.
    - denoted by $\deg^{+}(v)$.

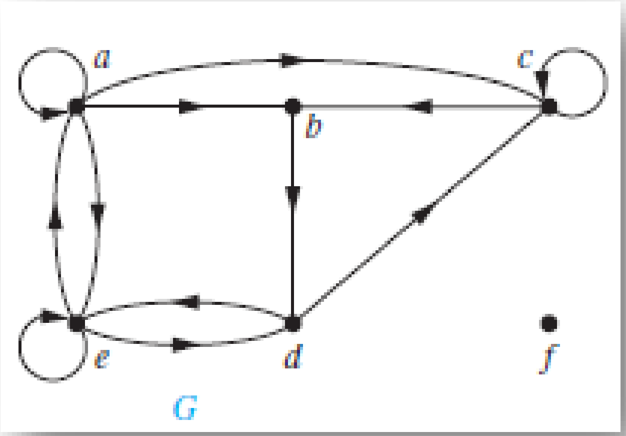
  - A loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.

- Let $G = (V, E)$ be a graph with directed edges.

$$\sum_{v \in V} \deg^{-}(v) + \sum_{v \in V} \deg^{+}(v) = |E|$$

Graphs…

➢ Example: Find the in-degree and out-degree of each vertex in the graph *G* with directed edges



➢ Solution:

$\deg^-(a) = 2, \deg^-(b) = 2, \deg^-(c) = 3,$
$\deg^-(d) = 2, \deg^-(e) = 3, \deg^-(f) = 0$

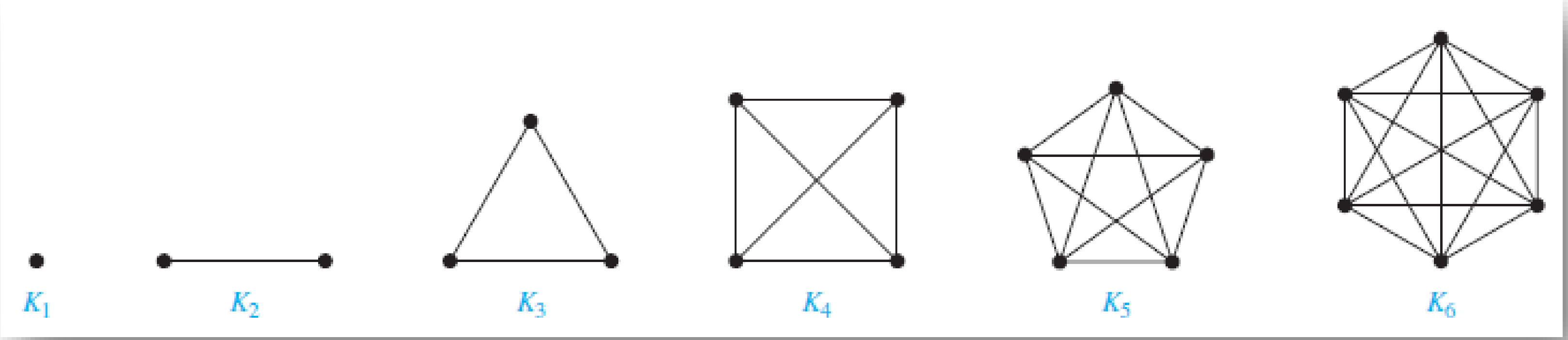
$\deg^+(a) = 4, \deg^+(b) = 1, \deg^+(c) = 2,$
$\deg^+(d) = 2, \deg^+(e) = 3, \deg^+(f) = 0$ ✓

Graphs…

- **Some Special Simple Graphs**
  - Complete Graphs
  - Cycles
  - Wheels
  - N-Cubes

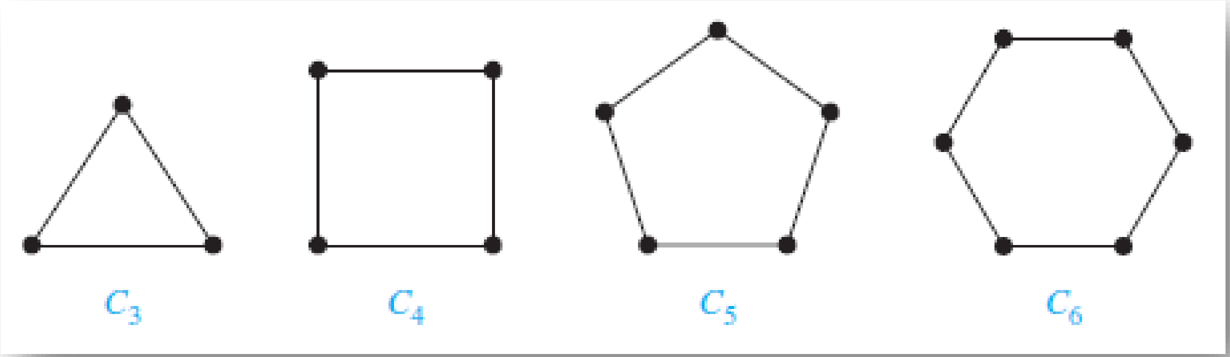
Graphs…

- **Complete Graphs**
  - A complete graph on *n* vertices,
    - a simple graph that contains exactly one edge between each pair of distinct vertices.
    - denoted by K_n.



Graphs…

- **Cycle**
  - A cycle C_n,
    - consists of *n* vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.
    - $n \geq 3$,



Graphs…

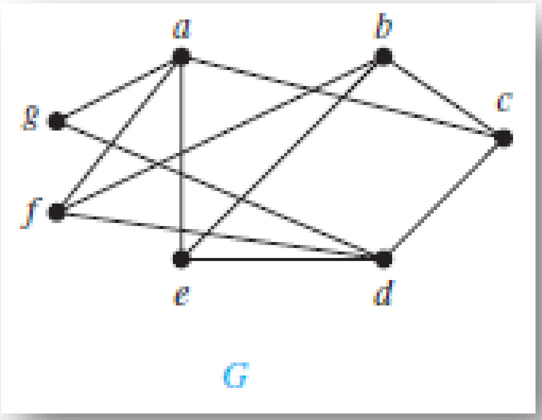
- **Bipartite Graphs**
  - A simple graph G is called **bipartite** if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2
  - so that no edge in G connects either two vertices in V_1 or two vertices in V_2.
  - when this condition holds, we call the pair (V_1, V_2) a bipartition of the vertex set V of G.

  - A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Graphs…

Bipartite Graphs…

- **Example:** Is the graph below bipartite?



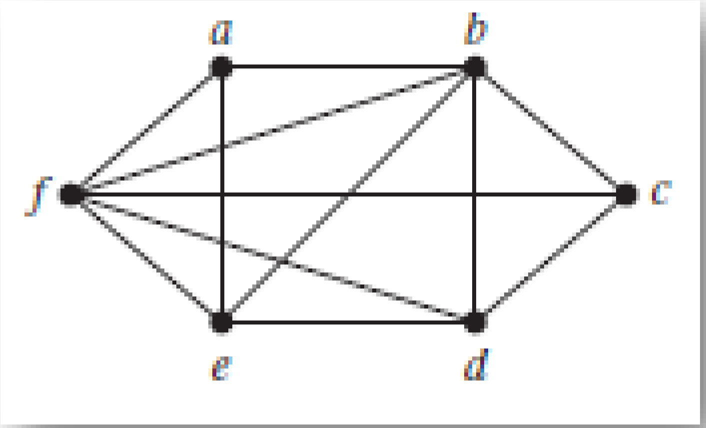
- **Solution:**

its vertex set is the union of two disjoint sets, {*a, b, d*} and {*c, e, f, g*}
$\Longrightarrow$ BIPARTITE ✓

Graphs…

➢ Bipartite Graphs…

➢ Example: Is the graph below bipartite?



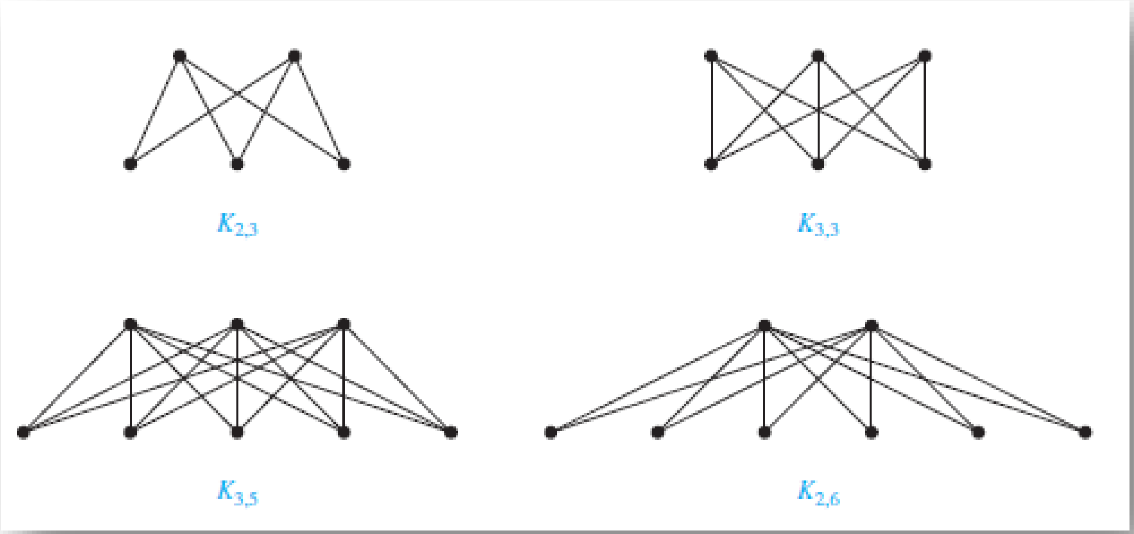
➢ Solution:

its vertex set cannot be partitioned into two subsets ⟹ NOT BIPARTITE ✓

Graphs…

➢ Bipartite Graphs

- ➢ a **complete bipartite graph** is a graph that has its vertex set partitioned into two subsets of *m* and *n* vertices, respectively with an edge between two vertices if and only if one vertex is in the first subset and the other vertex is in the second subset.
- ➢ denoted by $K_{m,n}$



Graphs…

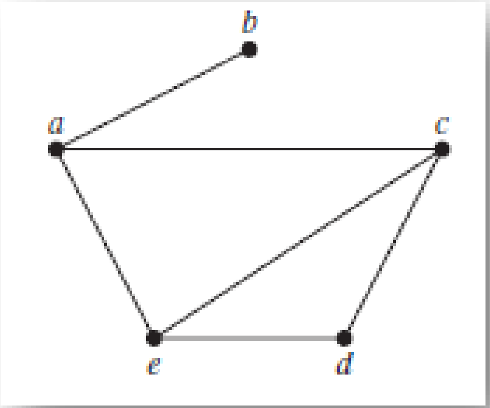
- **Representing Graphs**
  - There are many useful ways to represent graphs
    - Listing all the edges for a simple graph without multple edges
    - Using adjacency lists
    - Using adjacency matrices
    - Using incidence matrices

Graphs…

- **Representing Graphs…**
  - **Adjacency Lists**
    - a way to represent a graph with no multiple edges
    - specify the vertices that are adjacent to each vertex of the graph.

Graphs…

- **Representing Graphs…**
  - **Adjacency Lists…**
    - Example: Use adjacency lists to describe the simple graph.

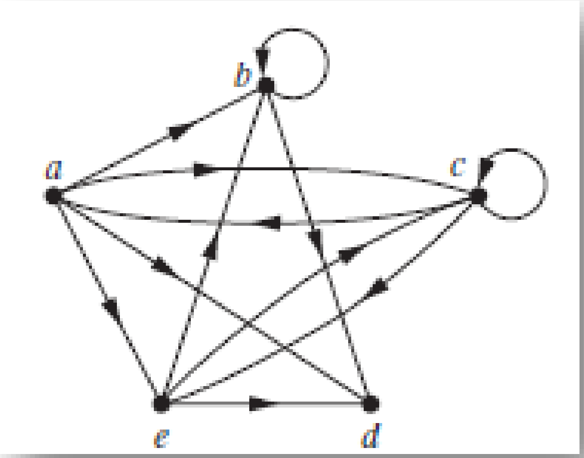


- Solution:

Vertex	Adjacent Vertices
a	b,c,e
b	a
c	a,d,e
d	c,e
e	a,c,d

Graphs…

- **Representing Graphs…**
  - **Adjacency Lists…**
    - Example: Use adjacency lists to describe the directed graph.



- Solution:

Initial Vertex	Terminal Vertices
a	b,c,d,e
b	b,d
c	a,c,e
d	-
e	b,c,d

Graphs…

- **Representing Graphs…**
  - **Adjacency Matrices**
    - suppose that $G = (V, E)$ is a simple graph where $|V| = n$
    - suppose that the vertices of G are listed arbitrarily as $v_1, v_2, \ldots, v_n$.
    - the adjacency matrix A (or A_G) of G, with respect to this listing of the vertices, is the *n* x *n* zero–one matrix with 1 as its (i, j)th entry when v_i and v_j are adjacent, and 0 as its (i, j)th entry when they are not adjacent.
    - in other words, if its adjacency matrix is $A = [a_{ij}]$, then

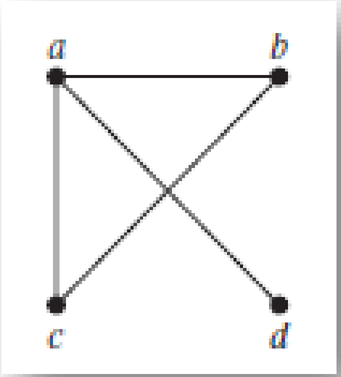
$$a_{ij} = \begin{cases} 1 \; if \; \{v_i, v_j\} \; is \; an \; edge \; of \; G \\ 0 \; otherwise \end{cases}$$

Graphs…

Representing Graphs…

Adjacency Matrices…

- Example: Use an adjacency matrix to represent the graph shown on the right



- Solution:

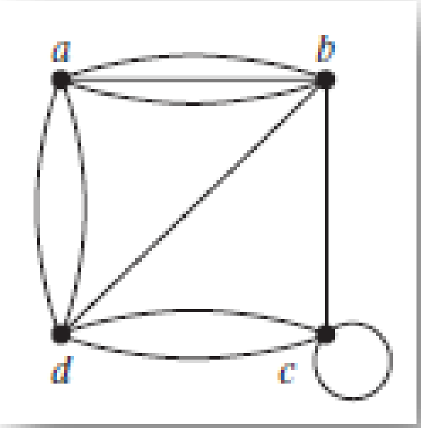
$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \checkmark$$

Graphs…

Representing Graphs…

Adjacency Matrices…

- <u>Example:</u> Use an adjacency matrix to represent the graph shown on the right



- <u>Solution:</u>

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \checkmark$$

Graphs…

- **Representing Graphs…**
  - **Incidence Matrices**
    - Let $G = (V, E)$ be an undirected graph.
    - Suppose that $v_1, v_2, \ldots, v_n$ are the vertices and $e_1, e_2, \ldots, e_m$ are the edges of G.
    - Then the incidence matrix with respect to this ordering of V and E is the *n* × *m* matrix M= $[m_{ij}]$, where

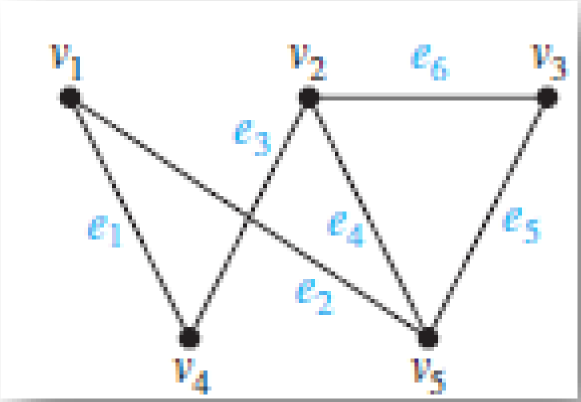
$$m_{ij} = \begin{cases} 1\ when\ e_i\ is\ incident\ with\ v_i \\ 0\ otherwise \end{cases}$$

Graphs…

Representing Graphs…

Incidence Matrices…

- <u>Example:</u> Represent the graph with an incidence matrix shown on the right



- <u>Solution:</u>

$$\begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array} \checkmark$$

Graphs…

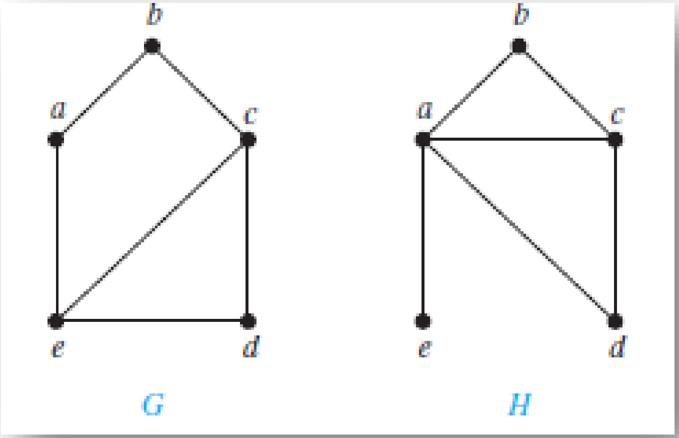
- **Isomorphism on Graphs…**
  - The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are **isomorphic** if there is a one-to-one correspondence between vertices of the two graphs that preserves the adjacency relationship
  - Isomorphism of simple graphs is an equivalence relation.
  - Two simple graphs that are not isomorphic are called **nonisomorphic**.

  - isomorphic simple graphs must have the same number of vertices, same number of edges and degrees of the vertices must also be the same.

Graphs…

➢Isomorphism on Graphs…

➢Example: Show that the graphs displayed on the left are not isomorphic.



➢Solution:

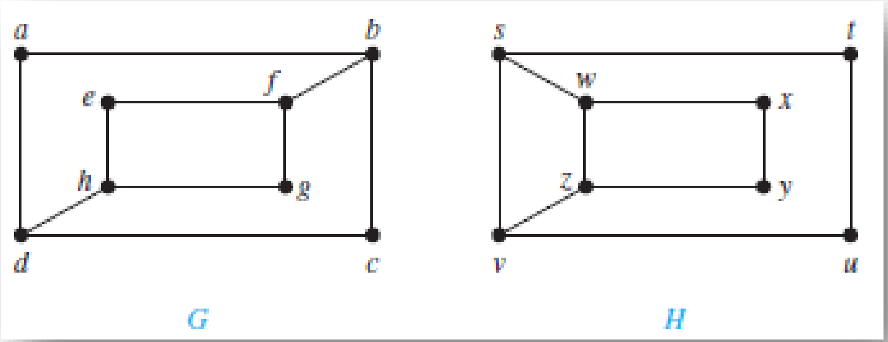
- both G and H have five vertices and six edges.
- H has a vertex of degree one (e), whereas G has no vertices of degree one.

$\Longrightarrow$ G and H are not isomorphic ✓

Graphs…

Isomorphism on Graphs…

- **Example:** Determine whether the graphs shown on the left are isomorphic.



- **Solution:**

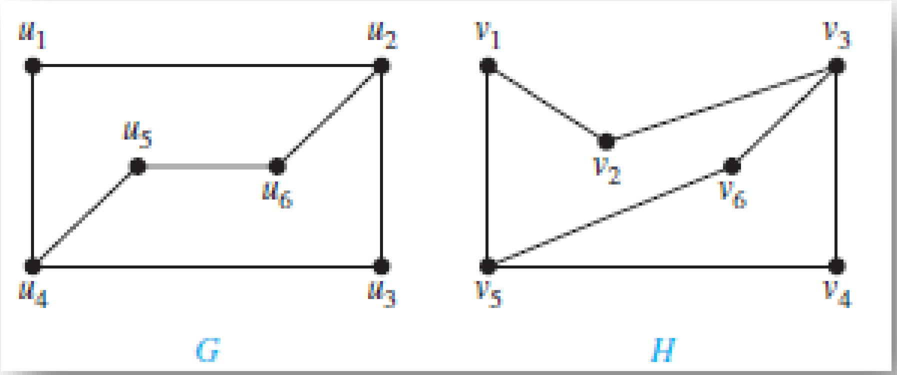
  - both G and H have eight vertices and ten edges.
  - they also both have four vertices of degree two and four of degree three
  - $\deg(a) = 2$ in *G*, *a* must correspond to either *t*,*u*,*x* or *y* in H. However, each of these four vertices in *H* is adjacent to another vertex of degree two in *H*, which is not true for *a* in *G*.

$\Longrightarrow$ *G* and *H* are not isomorphic ✓

Graphs…

Isomorphism on Graphs…

- **Example:** Determine whether the graphs shown on the left are isomorphic.



- **Solution:**

  - both G and H have siz vertices and seven edges.
  - both have four vertices of degree two and two vertices of degree three
  - the subgraphs of *G* and *H* consisting of all vertices of degree two and the edges connecting them are isomorphic

$\Longrightarrow$ *G* and *H* are isomorphic ✓

Graphs…

- **Paths on Graphs…**
  - a path is a sequence of edges that begins at a vertex of a graph and travels from vertex to vertex along edges of the graph.
  - the path is a ***circuit/cycle*** if it begins and ends at the same vertex,
  - a path or circuit/cycle is ***simple*** if it does not contain the same edge more than once.

Graphs…

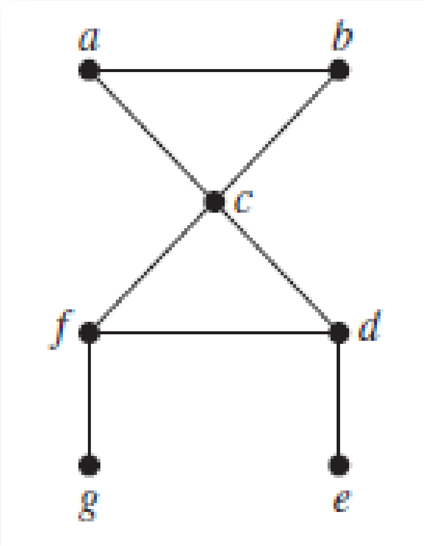
- **Paths on Graphs…**
  - An undirected graph is called **connected** if there is a path between every pair of distinct vertices of the graph.
  - An undirected graph that is not connected is called **disconnected**.
  - We say that we disconnect a graph when we remove vertices or edges, or both, to produce a disconnected subgraph.

  - There is a simple path between every pair of distinct vertices of a connected undirected graph.

Graphs…

➢ Paths on Graphs…

➢ <u>Example:</u> Determine whether the graphs shown on the left are connected or not.



➢ <u>Solution:</u>

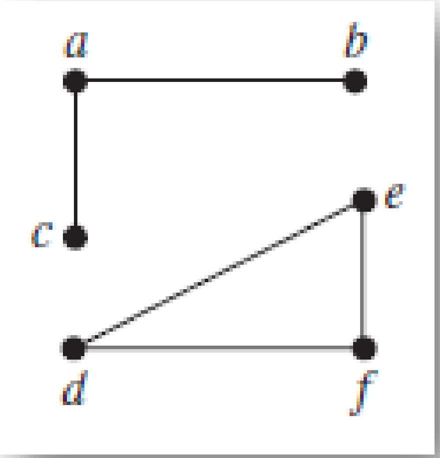
for every pair of distinct vertices there is a path between them

$\Longrightarrow$ CONNECTED ✓

Graphs…

Paths on Graphs…

- **Example:** Determine whether the graphs shown on the left are connected or not.



- **Solution:**

there is no path in $G2$ between vertices a and d $\Longrightarrow$ DISCONNECTED ✓

Graphs…

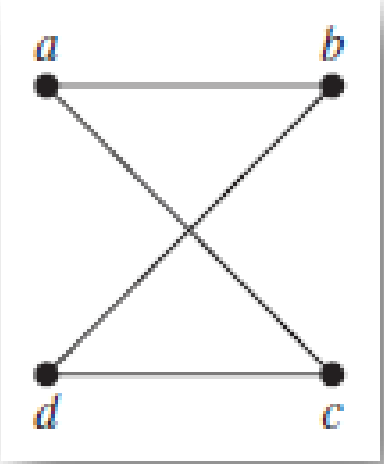
- **Paths on Graphs…**
  - The number of paths between two vertices in a graph can be determined using its adjacency matrix

  - Let *G* be a graph with adjacency matrix *A* with respect to the ordering $v_1, v_2, \dots, v_n$ of the vertices of the graph (with directed or undirected edges, with multiple edges and loops allowed).
  - The number of different paths of length *r* from v_i to v_j , where *r* is a positive integer, equals the (i, j)th entry of A^r.

Graphs…

Paths on Graphs…

- **Example:** How many paths of length four are there from *a* to *d* in the simple graph on the left



- **Solution:**

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{bmatrix} \Longrightarrow 8 \checkmark$$

References

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- R.P. Grimaldi, Discrete and Combinatorial Mathematcics, An Applied Introduction, Fifth Edition, Pearson, 2003.
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