Discrete Mathematics
LECTURE 5
Sequences & Summations

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Outline

- Sequences
  - Geometric Progression
  - Arithmetic Progression
  - Recurrence Relations
- Summations
- References

Sequence

- ordered lists of elements
- discrete structure used to represent an ordered list.
  - 1, 2, 3, 5, 8 is a sequence with five terms
  - 1, 3, 9, 27, 81 , . . . , 3*n*, . . . is an **infinite** sequence.
- the notation $\{a_n\}$ is used to describe the sequence.
  - a_n represents an individual term of the sequence $\{a_n\}$
- described by listing the terms of the sequence in order of increasing subscripts

- Example: Consider the sequence $\{a_n\}$, where $a_n = \frac{1}{n}$ for *n*=1,2,3,…

- The list of the terms of this sequence, beginning with a_1
  - $a_1 = 1,\ a_2 = {}^1/_2,\ a_3 = {}^1/_3,\ a_4 = {}^1/_4, \dots$

Sequence…

- **Geometric Progression**
  - a sequence of the form $a, ar, ar^2, \ldots, ar^n, \ldots$
    - a is the initial term
    - r is the common ratio
    - a and r are real numbers
  - discrete analogue of the exponential function $f(x) = ar^x$

- Example: Consider the sequence $\{b_n\}$, where $b_n = (-1)^n$ for n=0,1,2,3,…

  - a geometric progression with initial term of 1 and common ratio of -1
  - the list of the terms of this sequence, beginning with b_0
    - $b_n = 1.(-1)^n \Longrightarrow \; b_0 = 1,\; b_1 = -1,\; b_2 = 1,\; b_3 = -1, \ldots$

Sequence…

- **Arithmetic Progression**
  - a sequence of the form $a, a+d, a+2d, \ldots, a+nd, \ldots$
    - a is the initial term
    - d is the common difference
    - a and d are real numbers
  - discrete analogue of the exponential function $f(x) = dx + a$

  - Example: Consider the sequence $\{s_n\}$, where $s_n = -1 + 4n$ for *n*=0,1,2,3,…

  - a arithmetic progression with initial term of -1 and common difference of 4
  - the list of the terms of this sequence, beginning with s_0
    - $s_n = -1 + 4n \Longrightarrow \ s_0 = -1,\ s_1 = 3,\ s_2 = 7,\ s_3 = 11, \ldots$

Sequence…

- **Recurrence Relation**
  - an equation that express a_n in terms of one of more of the previous terms of the sequence for the sequence $\{a_n\}$
    - $\{a_n\}$ is $a_0, a_1, \dots, a_{n-1}$ for all integers n with $n > n_0$ where n_0 is a nonnegative integer
  - said to **recursively** define a sequence

  - **Example:** Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} + 3$ for *n*=1,2,3,… and suppose that $a_0 = 2$. What are a_1, a_2, a_3?

  - $a_1 = a_0 + 3 = 2 + 3 \Longrightarrow a_1 = 5$
  - $a_2 = a_1 + 3 = 5 + 3 \Longrightarrow a_2 = 8$
  - $a_3 = a_2 + 3 = 8 + 3 \Longrightarrow a_3 = 11$

Sequence…

Recurrence Relation…

- The **initial conditions** for a recursively defined sequence specify the terms that precede the first term where the recurrence relation takes effect

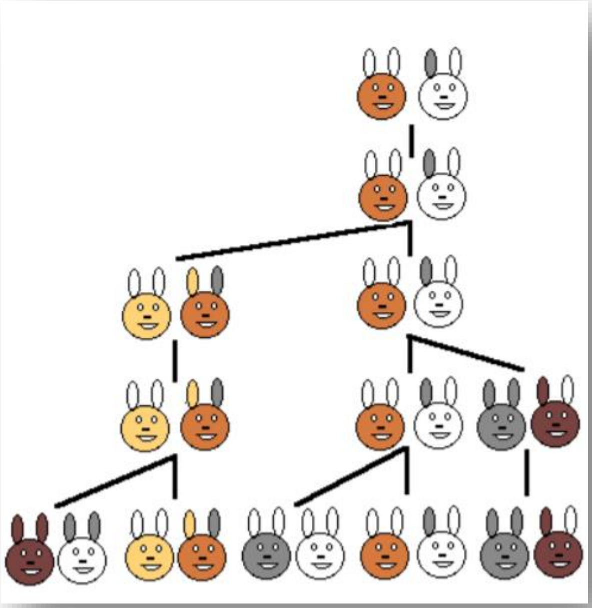
- **Example:** The Fibonacci sequence, $f_0, f_1, f_2, \ldots$ is defined by the initial conditions $f_0 = 0$, $f_1 = 1$ and the recurrence relation $f_n = f_{n-1} + f_{n-2}$ for *n*=2,3,4… Find the Fibonacci numbers $f_2, f_3, f_4, f_5\ and\, f_6.$

$$f_2 = f_1 + f_0 = 1 + 0 \Longrightarrow f_2 = 1$$
$$f_3 = f_2 + f_1 = 1 + 1 \Longrightarrow f_3 = 2$$
$$f_4 = f_3 + f_2 = 2 + 1 \Longrightarrow f_4 = 3$$
$$f_5 = f_4 + f_3 = 3 + 2 \Longrightarrow f_5 = 5$$
$$f_6 = f_5 + f_4 = 5 + 3 \Longrightarrow f_6 = 8$$

Sequence...

➢Recurrence Relation...

➢Example: A couple of rabbits were placed in a cage surrounded by walls. Assuming that each pair of rabbit gives birth to a new pait of rabbits after a month, each new pair takes a month to mature , ant the rabbits do not die, how many pairs of rabbits will there be within the walls at the end of 100 months?



Sequence…

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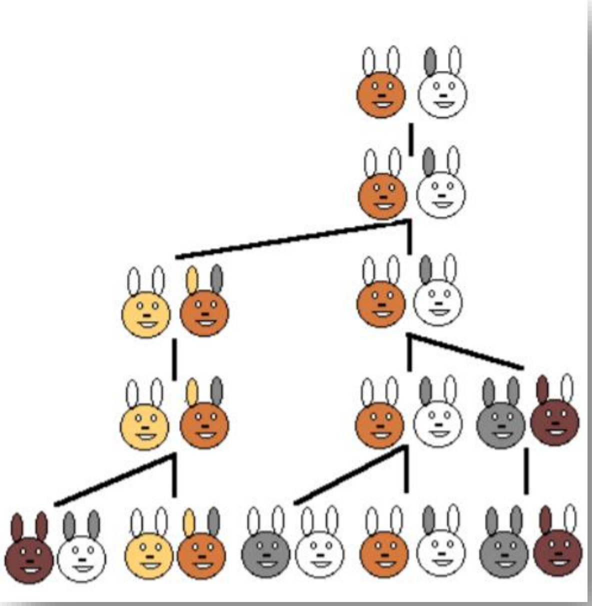

month	number of mature couples	number of young couples	number of total couples
0	0	1	1
1	1	0	1
2	1	1	2
3	1	2	3
4	2	3	5
5	3	5	8

Sequence…

➢Recurrence Relation…

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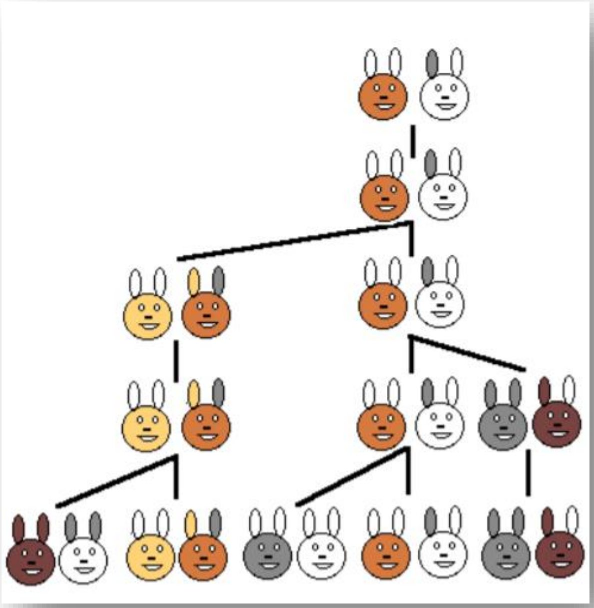


$R_0 = 1$
$R_1 = 1$
…
$R_n = R_{n-1} + R_{n-2}$

Sequences

- **Recurrence Relation…**
  - **closed formula**
    - an explicit formula together within th inital conditions for the terms of the sequence

  - Example: Determine whether the sequence $\{a_n\}$, where $a_n = 3n$ for every nonnegative integer n, is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$ for *n*=2,3,4,…

  - $\{a_n\} \Longrightarrow a_n = 3n \Longrightarrow 2a_{n-1} - a_{n-2} = 2\big(3(n-1)\big) - 3(n-2) = 3n$
  - Therefore, $\{a_n\}$, where $a_n = 3n$ is a solution of the recurrence relation

Sequences

- **Recurrence Relation…**
  - **iteration**
    - A strightforward method for solving recurrence relations
    - iterate, or repeatedly use recurrence relation
    - Two basic approaches
      - forward substitution
        - Finding the successive terms beginning with the inital consition and ending with a_n
      - bacward substitution
        - beginning with an a_n and iterated to express it in terms of falling terms of the sequence until finding it in terms of the first terms of the sequence

Sequences

Recurrence Relation…

- **Example:** Solve the recurrence relation for $\{a_n\}$, where $a_n = a_{n-1} + 1$ with an initial condition of 2 for every nonnegative integer

- with forward substition

$$
\begin{aligned}
&a_1 = 2 \\
&a_2 = 2 + 3 \\
&a_3 = (2 + 3) + 3 = 2 + 3 * 2 \\
&a_4 = (2 + 3 * 2) + 3 = 2 + 3 * 3 \\
&\ldots \\
&a_n = a_{n-1} + 3 = \big(2 + 3 * (n - 2)\big) + 3 = 2 + 3(n - 1)
\end{aligned}
$$

Sequences

Recurrence Relation…

- **Example:** Solve the recurrence relation for $\{a_n\}$, where $a_n = a_{n-1} + 1$ with an initial condition of 2 for every nonnegative integer…

- with backward substition

$$
\begin{aligned}
a_n &= a_{n-1} + 3 \\
&= (a_{n-2} + 3) + 3 \\
&= (a_{n-3} + 3) + 3 * 2 \\
&\dots \\
&= a_2 + 3(n-2) \\
&= (a_1 + 3) + 3(n-2) \\
&= a_1 + 3(n-1) \\
&= 2 + 3(n-1)
\end{aligned}
$$

Sequences

Recurrence Relation...

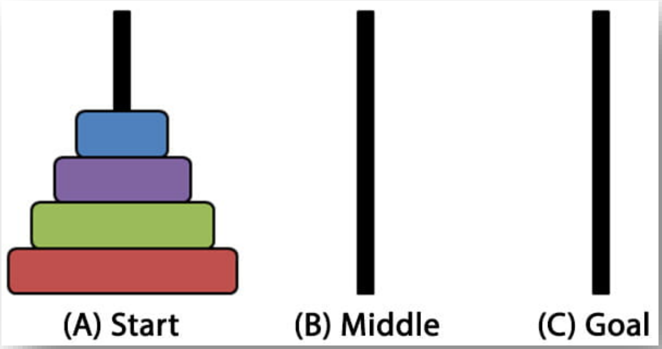
- **Example:** Suppose that a person deposits \$10.000 in a saving account at a bank yielding 11% per year with interest compunded annually. How much will be in the account after 30 years?

$$
\begin{aligned}
&P_1 = P_0 * (1.11) \\
&P_2 = P_1 * (1.11) = P_1 * (1.11)^2 \\
&P_3 = P_2 * (1.11) = P_0 * (1.11)^3 \\
&\ldots \\
&P_n = P_0 * (1.11)^n
\end{aligned}
$$

$$P_{30} = 10000 * (1.11)^{30} \Longrightarrow \$228.922{,}97$$

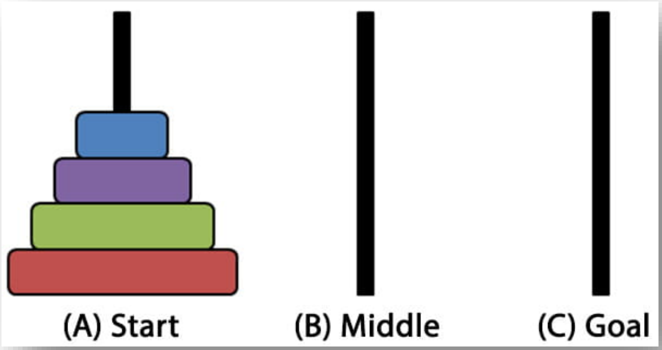
Sequences

- **Recurrence Relation…**
  - Example:
    - Hanoi Towers
      - a maths game/puzzle
      - was invented and released in 1883 by the French mathematician Edouard Lucas
      - consists of three poles and *n* discs of different sizes
      - you can transfer these discs to any pole
      - begins with the smallest disc on top of the first pole whose discs are lined up from smallest to largest



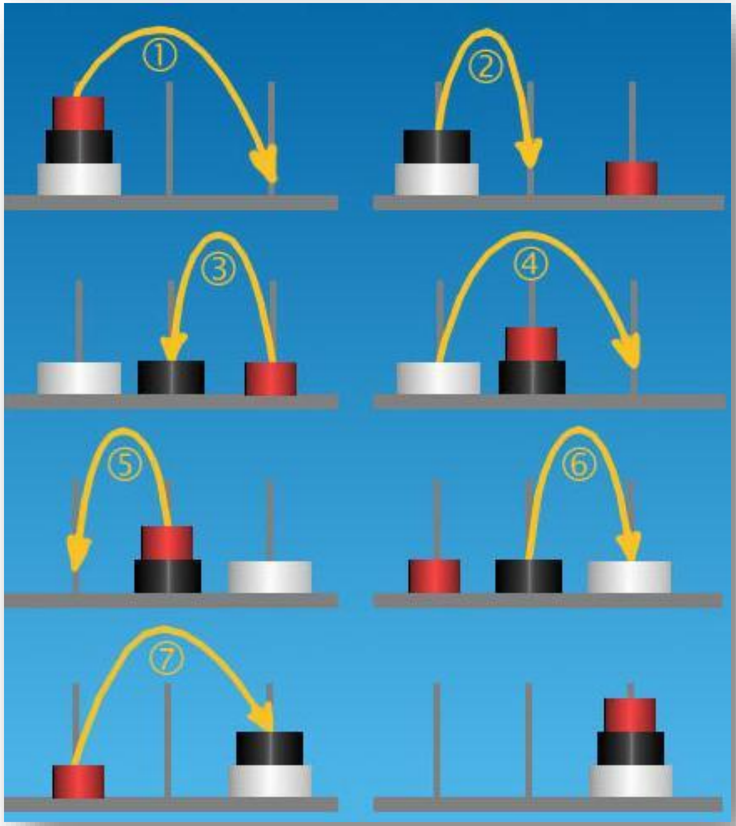
Sequences

- **Recurrence Relation…**
  - Example: …
    - Hanoi Towers…
      - the goal of the game is to move all the discs from the first pole to the last according to the following rules:
        - only one disc can be moved per move.
        - each move consists of taking the top disc from the pole and moving it to another pole.
        - no disc can be placed on a smaller disc.



Sequences

- **Recurrence Relation…**
  - Example: …
    - Hanoi Towers…
      - minimum number of steps required for three discs



Sequences

Recurrence Relation…

Example: …

Hanoi Towers…

$$
\begin{aligned}
&H_1 = 1\\
&H_n = 2(2H_{n-2} + 1) + 1 = 2^2 H_{n-2} + 2 + 1\\
&H_n = 2^2(2H_{n-3} + 1) + 2 + 1 = 2^3 H_{n-3} + 2^2 + 2 + 1\\
&\ldots\\
&H_n = 2^{n-1} H_1 + 2^{n-2} + 2^{n-3} + \cdots + 2 + 1\\
&H_n = 2^{n-1} + 2^{n-2} + 2^{n-3} + \cdots + 2 + 1\\
&H_n = 2H_{n-1} + 1
\end{aligned}
$$

$$
\begin{aligned}
&1\ disc \Rightarrow (2) - 1 = 1\ step\\
&2\ discs \Rightarrow (2 * 2) - 1 = 3\ steps\\
&3\ discs \Rightarrow (2 * 2 * 2) - 1 = 7\ steps\\
&4\ discs \Rightarrow (2 * 2 * 2 * 2) - 1 = 15\ steps\\
&5\ discs \Rightarrow (2 * 2 * 2 * 2 * 2) - 1 = 31\ steps\\
&\ldots\\
&n\ discs \Rightarrow 2^n - 1
\end{aligned}
$$

Summation

- the addition of the terms of a sequence
- begins by describing the notation used to express the sum of the terms from the sequence $\{a_n\}$
- the notation is: $\sum_{j=m}^{n} a_j$ or $\sum_{m \le j \le n} a_j$
  - Σ denotes the summation
  - j is the index of the summation
  - m is the lower limit of the summation
  - n is the upper limit of the summation
  - read as the sum from $j = m$ to $j = n$ of a_j
  - represents $a_m + a_{m+1} + \cdots + a_n$

Summation…

- **Example:** Use summation notation to Express the sum of the first 100 items of the sequence $\{a_j\}$ where $a_j = {}^1/_j$ for $j = 1,2,3, \ldots$

- The lower limir for the index of summation is 1, and the upper limit is 100. So, the sum is $\sum_{i=1}^{100} {}^1/_j$

- **Example:** What is the value of $\sum_{j=1}^{5} j^2$?

- $\sum_{j=1}^{5} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2 = 1 + 4 + 9 + 16 + 25 = 55$

Summation…

- **Shifting a summation**
  - Example: $\sum_{i=1}^{5} j^2 = \sum_{k=0}^{4} (k+1)^2$

- **Double summation**
  - Example: What is the value of $\sum_{i=1}^{4} \sum_{j=1}^{3} ij$?

$$\sum_{i=1}^{4} \sum_{j=1}^{3} ij = \sum_{i=1}^{4} (i + 2i + 3i)$$

$$= \sum_{i=1}^{4} 6i = 6 * \sum_{i=1}^{4} i$$

$$= 6(1 + 2 + 3 + 4) = 60$$

Summation…

- **Example:** What is the value of $\sum_{k=50}^{100} k^2$?

$$\sum_{k=1}^{100} k^2 = \sum_{k=1}^{49} k^2 + \sum_{k=50}^{100} k^2 \quad \text{ve} \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\Rightarrow \sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 + \sum_{k=1}^{49} k^2$$

$$\sum_{k=50}^{100} k^2 = \frac{100*101*201}{6} - \frac{49*50*99}{6} = 338350 - 40425 = 297925$$

References

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- S.S. Epp, Discrete Mathemtics with Applications, Fouth Edition, 2010.
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