SAKARYA ÜNİVERSİTESİ

Discrete Mathematics
LECTURE 11
Euler Circuits & Paths

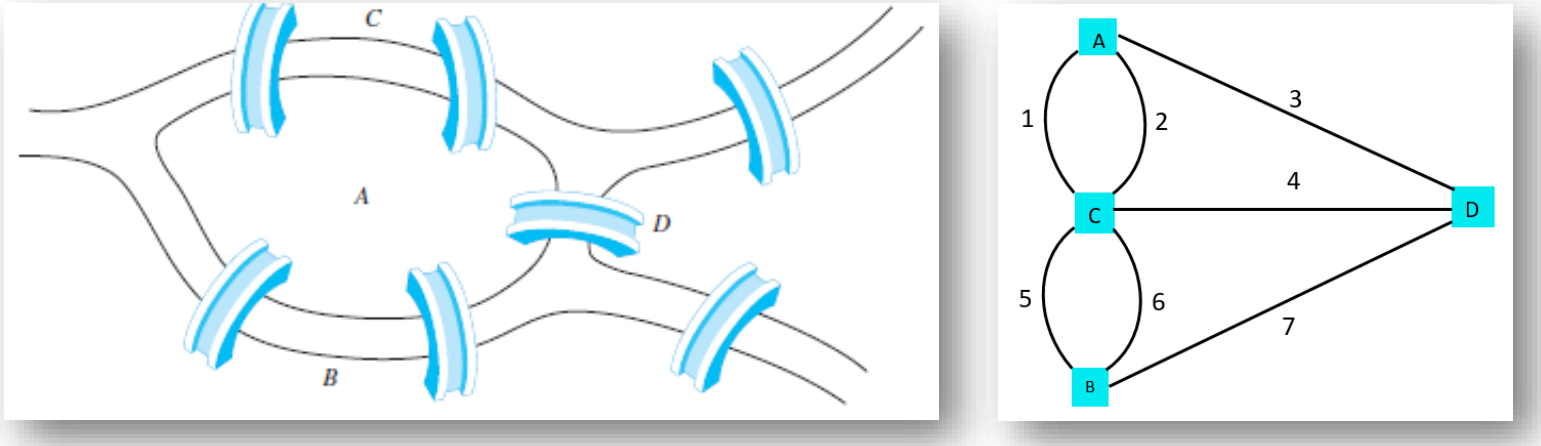
Assistant Professor Gülüzar ÇİT

Outline

- Euler Circuits & Paths
- References

Euler Circuits & Paths

- An **Euler circuit** in a graph *G* is a simple circuit containing every edge of *G*.
  - The degrees of all the nodes of all the graphs are even



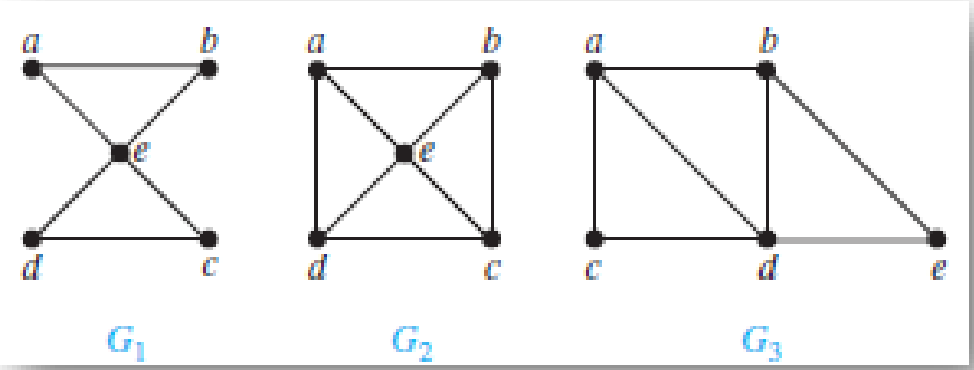
Koningsberg bridges and related graph

Euler Circuits & Paths…

- An **Euler path** in a graph *G* is a simple path containing every edge of *G*.
  - The degrees of two nodes of the graph are odd, the others are even
  - The path is from one of the even degreed node to the other

Euler Circuits & Paths…

➢ **Example:** Which of the undirected graphs have an Euler circuit? Of those that do not, which have an Euler path?

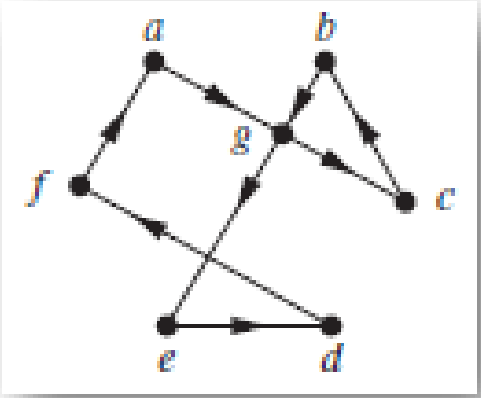


➢ **Solution:**

- The graph G_1 has an Euler circuit, because the degree of all nodes are even
- The graph G_2 has neither an Euler circuit not path
- The graph G_3 has an Euler path, because the degrees of node a and b are odd, the others are even ✓

Euler Circuits & Paths…

➢ **Example:** Is the directed graph on the right have an Euler circuit or an Euler path?



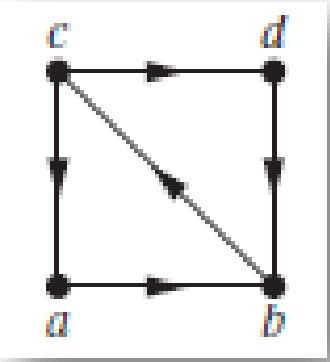
➢ **Solution:**

$deg^-(a) = 1$ $deg^+(a) = 1$
$deg^-(b) = 1$ $deg^+(b) = 1$
$deg^-(c) = 1$ $deg^+(c) = 1$
$deg^-(d) = 1$ $deg^+(d) = 1$
$deg^-(e) = 1$ $deg^+(e) = 1$
$deg^-(f) = 1$ $deg^+(f) = 1$
$deg^-(g) = 2$ $deg^+(g) = 2$

The graph has an Euler circuit, because the total of all nodes are even ✓

Euler Circuits & Paths…

➢ **Example:** Is the directed graph on the right have an Euler circuit or an Euler path?



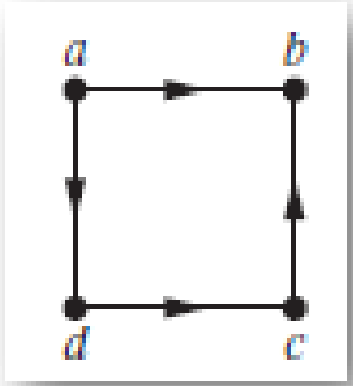
➢ **Solution:**

$deg^-(a) = 1$ $\quad deg^+(a) = 1$
$deg^-(b) = 2$ $\quad deg^+(b) = 1$
$deg^-(c) = 1$ $\quad deg^+(c) = 2$
$deg^-(d) = 1$ $\quad deg^+(d) = 1$

The graph has an Euler path from c to b, because the total of nodes b and c are odd and the out degree of c is one greater than the in degree of c, and it's vice verse for b ✓

Euler Circuits & Paths…

➢ **Example:** Is the directed graph on the right have an Euler circuit or an Euler path?



➢ **Solution:**

$deg^-(a) = 0$ $\quad deg^+(a) = 2$
$deg^-(b) = 2$ $\quad deg^+(b) = 0$
$deg^-(c) = 1$ $\quad deg^+(c) = 1$
$deg^-(d) = 1$ $\quad deg^+(d) = 1$

The graph has neither an Euler circuit nor a path, because the total of all nodes are even but the in and out degrees of nodes a and b either even or 0, but they should have at least one in/out degree✓

Euler Circuits & Paths…

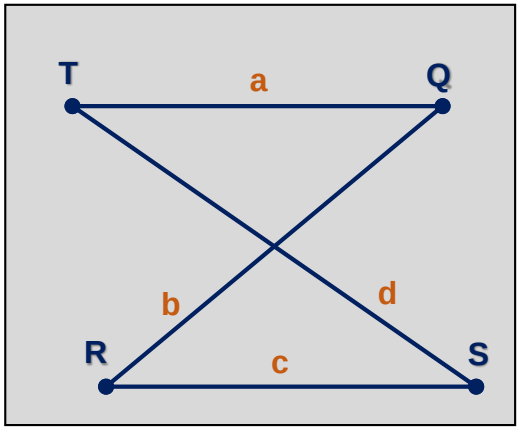
Euler Circuit Algorithm

```
STEP 1 ( starting path )
   a) Make E the set of edges of the graph g.
   b) Choose a node and assign C as the path containing this single node
STEP 2 ( expand path)
   WHILE (E is not empty)
      STEP 2.1 ( choose a starting point to expand)
          a) Set A to a node in C that is connected to an edge in E
          b) Assign P as the only path containing A
      STEP 2.2 (expand P as a path from A to A)
         a) set B = A
         b) WHILE ( E has an edge e connected to B)
            a) Subtract e from E
            b) Replace B with the other node of e
            c) Add edge e and node B to P, respectively
         END WHILE
       STEP 2.3 (expand C)
          Place P in place of A in C
   END WHILE
ADIM 3 Path C is the Euler Circuit
```

Euler Circuits & Paths…

Euler Circuit Algorithm…

Example: Is there an Euler circuit in the graph given on the right? If yes, find it by applying Euler's algorithm.



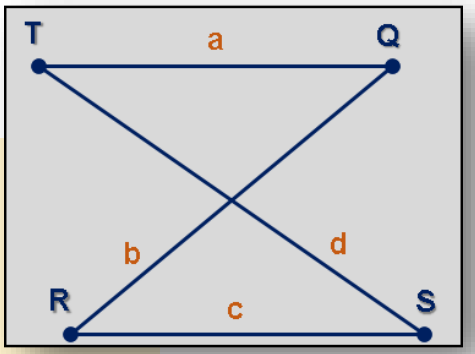
Solution:

The degrees of T,Q,R and S are all even, so there is an Euler circuit in this graph ✓

Euler Circuits & Paths…

Euler Circuit Algorithm…

Solution:…

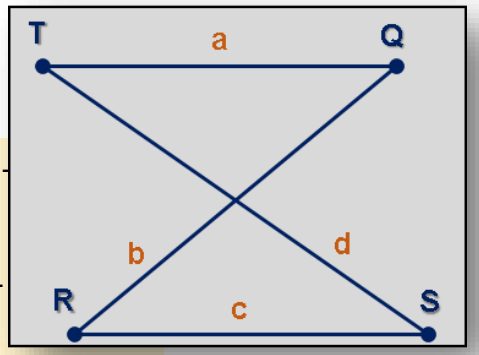



```
STEP 1 Set E as the set of all edges of the graph
          E = {a,b,c,d}
       Choose a node and assign C as the path
       containig this single node
          C ← T
STEP 2 (expand path)
    WHILE (1) (E is not empty)
      STEP 2.1 (choose a starting point to expand)
            set A to a node in C that is connected to an edge in E
               A ← T
            assign P as the only path containing A
               P ← T
      STEP 2.2 (expand P as the path from A to A)
         set B to A
            B ← T
         WHILE (1) (E has an edge connected to B, namely a and d, select a)
            E = {b,c,d}          (subtract e from E)
            B ← Q                (replace B with the other node of e)
            P ← TaQ              (add edge e and node B to P)
         ------------------------------------------------------------
```

Euler Circuits & Paths…

Euler Circuit Algorithm…

Solution:…

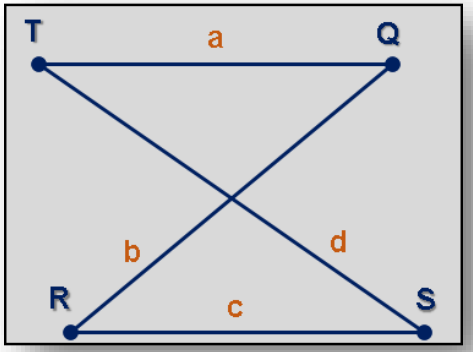



```
        ------------------------------------------------------------
        WHILE (2) (E has an edge connected to B, namely b)
          E = {c,d}, B ← R, P ← TaQbR
        ------------------------------------------------------------
        WHILE (3) (E has an edge connected to B, namely c)
          E = {d}, B ← S, P ← TaQbRcS
        ------------------------------------------------------------
        WHILE (4) (E has an edge connected to B, namely d)
          E = Ø, B ← T, P ← TaQbRcSdT
        ------------------------------------------------------------
         WHILE (5) (E has not got an edge connected to B)
         END WHILE
        ------------------------------------------------------------
      STEP 2.3 (place P in place of A in C)
         C ← TaQbRcSdT
    ------------------------------------------------------------
    WHILE (2) (E is empty)
    END WHILE
STEP 3 path C is the Euler circuit
    C = TaQbRcSdT
```

Euler Circuits & Paths…

- **Euler Circuit Algorithm…**
  - Solution:…

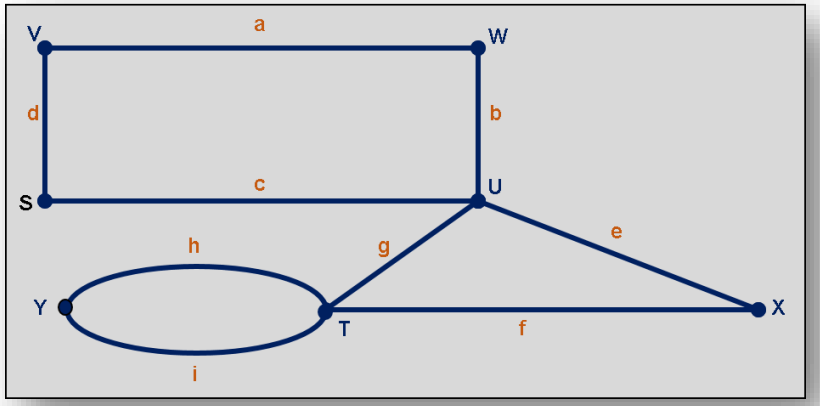
E	A	B	P	C
$\{a, b, c, d\}$	T	T	T	T
$\{b, c, d\}$	T	Q	TaQ	T
$\{c, d\}$	T	R	TaQbR	T
$\{d\}$	T	S	TaQbRcS	T
$\emptyset$	T	T	TaQbRcSdT	TaQbRcSdT



Euler Circuits & Paths…

Euler Circuit Algorithm…

Example: Is there an Euler circuit in the graph given on the right? If yes, find it by applying Euler's algorithm.



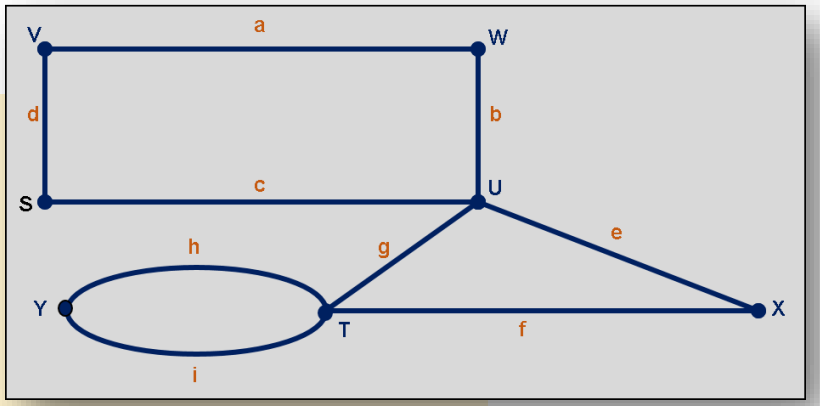
Solution:

The degrees of all nodes are even, so there is an Euler circuit in this graph ✓

Euler Circuits & Paths…

Euler Circuit Algorithm…

Solution:…

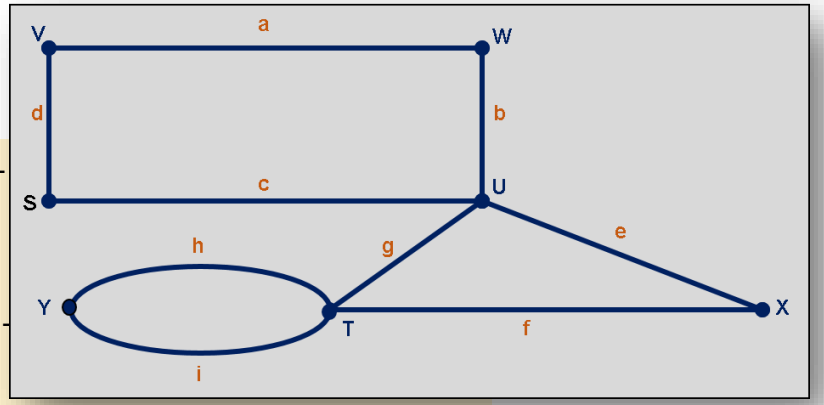



```
STEP 1 Set E as the set of all edges of the graph
          E = {a,b,c,d,e,f,g,h,i}
       Choose a node and assign C as the path
       containig this single node
          C ← V
STEP 2 (expand path)
   WHILE (1) (E is not empty)
     STEP 2.1 (choose a starting point to expand)
           set A to a node in C that is connected to an edge in E
              A ← V
           assign P as the only path containing A
              P ← V
     STEP 2.2 (expand P as the path from A to A)
        set B to A
           B ← V
        WHILE (1) (E has an edge connected to B, namely a and d, select a)
           E = {b,c,d,e,f,g,h,i} (subtract e from E)
           B ← W                 (replace B with the other node of e)
           P ← VaW               (add edge e and node B to P)
        ------------------------------------------------------------------
```

Euler Circuits & Paths…

Euler Circuit Algorithm…

Solution:…

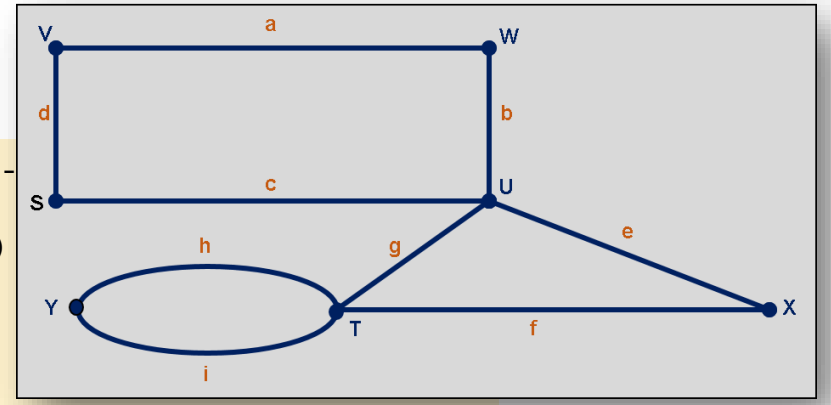



```
-----------------------------------------
WHILE (2) (E has an edge connected to B,
           namely b)
  E = {c,d,e,f,g,h,i}, B ← R, P ← VaWbU
-----------------------------------------
WHILE (3) (E has an edge connected to B,
           namely c,e and g, select c)
  E = {d,e,f,g,h,i}, B ← S, P ← VaWbUcS
-----------------------------------------------------------------
WHILE (4) (E has an edge connected to B, namely d)
  E = {e,f,g,h,i}, B ← T, P ← VaWbUdV
-----------------------------------------------------------------
 WHILE (5) (E has not got an edge connected to B)
 END WHILE
-----------------------------------------------------------------
STEP 2.3 (place P in place of A in C)
  C ← VaWbUdV
-----------------------------------------------------------------
```

Euler Circuits & Paths…

Euler Circuit Algorithm…

Solution:…

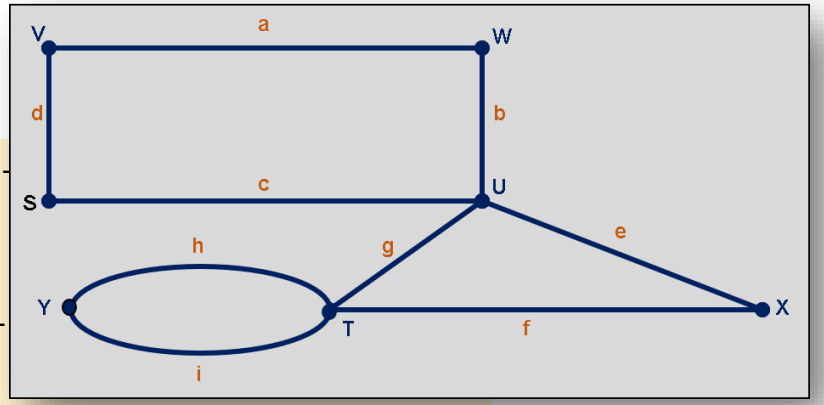



```
-------------------------------------------------
WHILE (2) (E is not empty)
  STEP 2.1 (choose a starting point to expand)
      set A to a node in C
      that is connected to an edge in E
         A ← U
      assign P as the only path containing A
         P ← U
  STEP 2.2 (expand P as the path from A to A)
    set B to A
       B ← U
    WHILE (1) (E has an edge connected to B, namely g and e, select e)
       E = {f,g,h,i}, B ← X, P ← UeX
    ---------------------------------------------------------------
    WHILE (2) (E has an edge connected to B, namely f)
       E = {g,h,i}, B ← T, P ← UeXfT
    ---------------------------------------------------------------
    WHILE (2) (E has an edge connected to B, namely g,h and i, select g)
       E = {h,i}, B ← U, P ← UeXfTgU
    ---------------------------------------------------------------
```

Euler Circuits & Paths…

Euler Circuit Algorithm…

Solution:…

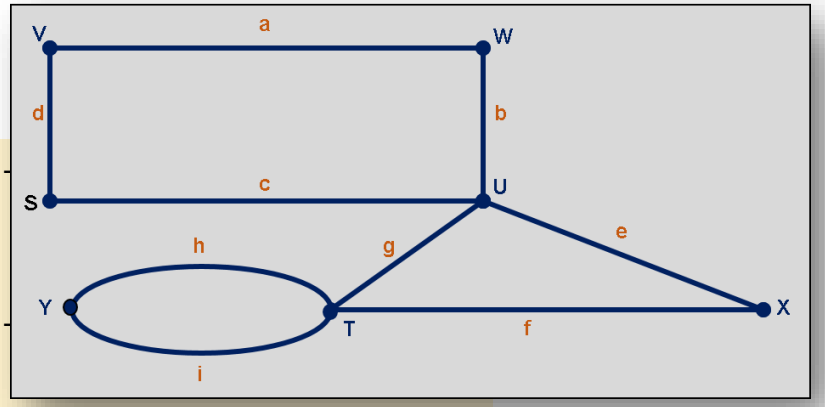



```
      ----------------------------------------
      WHILE (4) (E has not got an edge
                 connected to B)
      END WHILE
      ----------------------------------------
   STEP 2.3 (place P in place of A in C)
      C ← VaWbUeXfTgUdV
 ------------------------------------------------------------------------
WHILE (3) (E is not empty)
   STEP 2.1 (choose a starting point to expand)
       set A to a node in C
       that is connected to an edge in E
          A ← T
       assign P as the only path containing A
          P ← T
   STEP 2.2 (expand P as the path from A to A)
     set B to A
        B ← T
     WHILE (1) (E has an edge connected to B, namely h and i, select h)
        E = {i}, B ← Y, P ← ThY
      ----------------------------------------------------------------------
```

Euler Circuits & Paths…

Euler Circuit Algorithm…

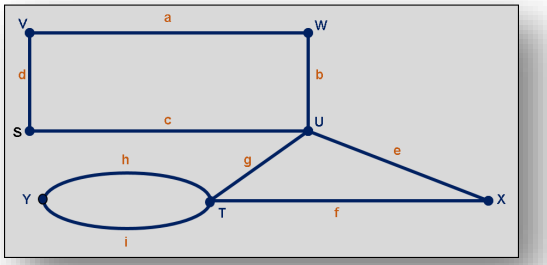
Solution:…




```
        ----------------------------------------
        WHILE (2) (E has an edge connected to B,
                   namely i)
          E = Ø, B ← T, P ← ThYiT
        ----------------------------------------
        WHILE (3) (E has not got an edge
                   connected to B)
        END WHILE
        ------------------------------------------------------------------
      STEP 2.3 (place P in place of A in C)
         C ← VaWbUeXfThYiTgUdV
    ----------------------------------------------------------------------
    WHILE (2) (E is empty)
    END WHILE
STEP 3 path C is the Euler circuit
    C = VaWbUeXfThYiTgUdV
```

Euler Circuits & Paths…

- **Euler Circuit Algorithm…**
  - Solution:…



E	A	B	P	C
$\{a, b, c, d, e, f, g, h, i\}$	V	V	V	V
$\{b, c, d, e, f, g, h, i\}$	V	W	TaW	V
$\{c, d, e, f, g, h, i\}$	V	U	TaWbU	V
$\{d, e, f, g, h, i\}$	V	S	TaWbUcS	V
$\{e, f, g, h, i\}$	V	T	TaWbUcSdT	VaWbUcSdV
$\{e, f, g, h, i\}$	U	U	U	VaWbUcSdV
$\{f, g, h, i\}$	U	X	UeX	VaWbUcSdV
$\{g, h, i\}$	U	T	UeXfT	VaWbUcSdV
$\{h, i\}$	U	U	UeXfTfU	VaWbUeXfTfUcSdV
$\{h, i\}$	T	T	T	VaWbUeXfTfUcSdV
$\{i\}$	T	Y	ThY	VaWbUeXfTfUcSdV
$\emptyset$	T	T	ThYiT	VaWbUeXfThYiTfUcSdV

References

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