Discrete Mathematics
LECTURE 7
Induction & Recursion

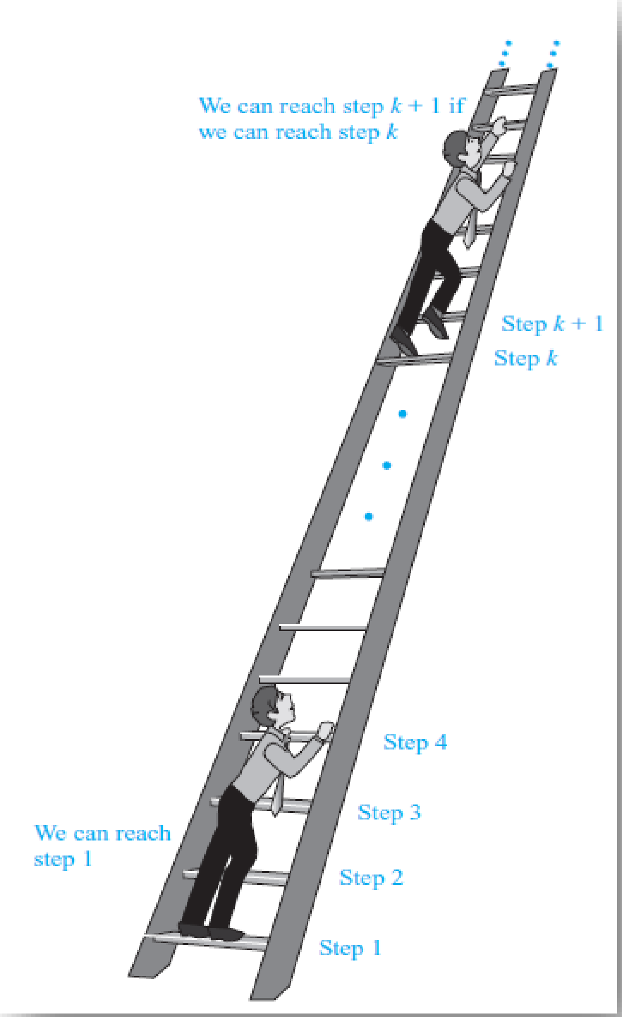
Assistant Professor Gülüzar ÇİT

Outline

- Mathematical Induction
- Recursion
- References

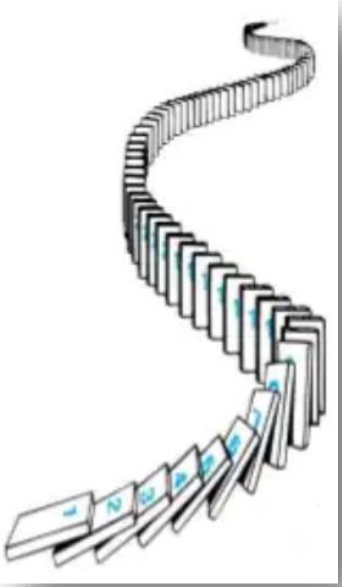
Mathematical Induction

- Suppose that we have an infinite ladder, and we want to know whether we can reach every step on this ladder. We know two things:
  - we can reach the first rung of the ladder.
  - if we can reach a particular rung of the ladder, then we can reach the next rung.



Mathematical Induction…

- To prove that *P(n)* is true for all positive integers *n*, where *P(n)* is a propositional function, complete two steps:
- **BASIS STEP**
  - verify that *P(1)* is true.
- **INDUCTIVE STEP**
  - e show that the conditional statement $P(k) \rightarrow P(k + 1)$ is true for all positive integers *k*.



- $\left(P(1) \wedge \forall k\big(P(k) \rightarrow P(k+1)\big)\right) \rightarrow \forall n P(n)$

Mathematical Induction…

- Example:
  - Show that "if n is a positive integer, then $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$"

Mathematical Induction…

- **Example:**
  - Show that "if n is a positive integer, then $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$"
- **Solution:**

Show that *P(1)* is true and that the conditional statement *P(k)* implies *P(k+1)* is true for *k* = 1*,* 2*,* 3*, ...*

Basis Step:

*P(*1*)* is true, because $1 = \frac{1(1+1)}{2}$

Inductive Step:

We assume that *P(k)* holds for an arbitrary positive integer *k*.

$$1 + 2 + \cdots + \mathrm{k} = \frac{\mathrm{k}(\mathrm{k}+1)}{2}$$

Show that P(k+1) is also true,

$$1 + 2 + \cdots + \mathrm{k} + (\mathrm{k} + 1) = \frac{(\mathrm{k}+1)\big((\mathrm{k}+1)+1\big)}{2} \checkmark$$

Mathematical Induction…

- <u>Example:</u>
  - Show that "if n is a positive odd integer, then $1 + 3 + 5 + \cdots + (2n - 1) = n^2$"

Mathematical Induction…

- **Example:**
  - Show that "if *n* is a positive odd integer, then $1 + 3 + 5 + \cdots + (2n - 1) = n^2$"
- **Solution:**

Show that *P(1)* is true and that the conditional statement *P(k)* implies *P(k+1)* is true for *k* = 1, 2, 3, ...

Basis Step:

P(1) is true, because $1 = 1^2$

Inductive Step:

We assume that *P(k)* holds for an arbitrary positive integer *k*.

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2$$

Show that P(k+1) is also true,

$$1 + 3 + 5 \ldots + (2k - 1) + (2k + 1) = (k + 1)^2$$

$$k^2 + (2k + 1) = (k + 1)^2 \checkmark$$

Mathematical Induction…

- Example:
  - Use mathematical induction to show that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for all nonnegative integers *n*

Mathematical Induction…

Example:

- Use mathematical induction to show that $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ for all nonnegative integers *n*

Solution:

Show that *P(0)* is true and that the conditional statement *P(k)* implies *P(k*+1*)* is true for *k* = 0,1, 2, 3, ...

Basis Step:

P(0*)* is true, because $2^0 = 2^{0+1} - 1 \Rightarrow 1 = 1$

Inductive Step:

We assume that *P(k)* holds for an arbitrary positive integer *k*.

$$1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1$$

Show that P(k+1) is also true,

$$1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{k+2} - 1$$

$$2^{k+1} \cancel{-1} + 2^{k+1} = 2^{k+2} \cancel{-1}$$

$$2.2^{k+1} = 2^{k+2} \checkmark$$

Mathematical Induction…

Example:

- **Sums of Geometric Progression:** Use mathematical induction to prove this formula for the sum of a finite number of terms of a geometric progression with initial term *a* and a common ratio *r*:
- $\sum_{j=0}^{n} ar^j = a + ar + ar^2 + \cdots + ar^n = \frac{ar^{n+1}-a}{r-1}$ when $r \neq 1$ and where *n* is a nonnegative integer

Solution:

Show that *P(0)* is true and the conditional statement *P(k)* implies *P(k*+1*)* is true for *k* = 0,1,2,3, . . .

Basis Step:

P(0*)* is true, because $a = \frac{ar^1-a}{r-1} = \frac{a\cancel{(r-1)}}{\cancel{r-1}}$

Inductive Step:

We assume that *P(k)* holds for an arbitrary positive integer *k*.

$$a + ar + ar^2 + \cdots + ar^k = \frac{ar^{k+1} - a}{r - 1}$$

Show that P(k+1) is also true,

$$a + ar + ar^2 + \cdots + ar^k + ar^{k+1} = \frac{ar^{k+2}-a}{r-1} ✓$$

Mathematical Induction…

- Example:
  - Use mathematical induction to prove the inequality $n < 2^n$ for all positive integer *n*

Mathematical Induction…

Example:

- Use mathematical induction to prove the inequality $\mathrm{n} < 2^n$ for all positive integer *n*

Solution:

Let $P(n)$ be the proposition that $\mathrm{n} < 2^n$

P(1) is true, because $1 < 2^1 \Longrightarrow 1 < 2$

We assume that *P(k)* holds for an arbitrary positive integer *k*.

$$k < 2^k$$

Show that P(k+1) is also true,

$$k + 1 < 2^{k+1} \Longrightarrow k + 1 < 2^k.2 \Longrightarrow k + 1 < 2^k + 2^k \quad \checkmark$$

Mathematical Induction…

- <u>Example:</u>
  - Use mathematical induction to prove that $2^n < n!$ for every integer *n* with $n \geq 4$

Mathematical Induction…

- Example:
  - Use mathematical induction to prove that $2^n < n!$ for every integer *n* with $n \geq 4$
- Solution:

Let $P(n)$ be the proposition that $2^n < n!$

P(1) is true, because $2^4 < 4! \Longrightarrow 16 < 24$

We assume that *P(k)* holds for an arbitrary positive integer *k*.

$$2^k < k!$$

Show that P(k+1) is also true,

$$2^{k+1} < (k+1)! \Longrightarrow 2.2^k < (k+1).k! \quad \checkmark$$

Mathematical Induction…

- Example:
  - **Proving Divisibility Results:** Use mathematical induction to prove that $n^3 - n$ is divisible by 3 whenever n is a positive integer

Mathematical Induction…

- Example:
  - **Proving Divisibility Results:** Use mathematical induction to prove that $n^3 - n$ is divisible by 3 whenever *n* is a positive integer
- Solution:

Let $P(n)$ be the proposition that $(3|n^3 - n)$

P(1) is true, because $(3|1^3 - 1) \Longrightarrow (3|0)$

We assume that *P(k)* holds for an arbitrary positive integer *k*.

$$(3|k^3 - k)$$

Show that P(k+1) is also true,

$$\left(3\middle|\left((k+1)^3 - (k+1)\right)\right) \Longrightarrow (3|(k^3 + 3k^2 + 3k \cancel{+1} - k \cancel{-1}))$$

$$\Longrightarrow \left(3\middle|\left((k^3 - k) + 3(k^2 + k)\right)\right)$$

$$\Longrightarrow (3|(k^3 - k)) + (3|(k^2 + k)) \checkmark$$

Mathematical Induction…

- Example:
  - **Proving Divisibility Results:** Use mathematical induction to prove that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for every nonnegative integer *n*

Mathematical Induction…

Example:

- **Proving Divisibility Results:** Use mathematical induction to prove that $7^{n+2} + 8^{2n+1}$ is divisible by 57 for every nonnegative integer *n*

Solution:

Let $P(n)$ be the proposition that $(57|7^{n+2} + 8^{2n+1})$

P(1) is true, because $(57|7^{0+2} + 8^{2.0+1}) \Longrightarrow (57|57)$

We assume that *P(k)* holds for an arbitrary positive integer *k*.

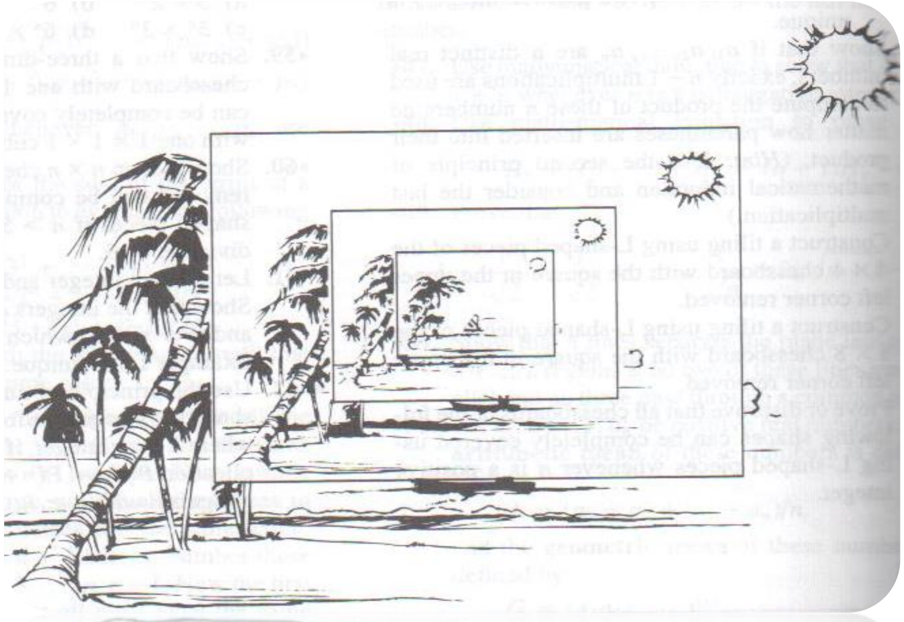
$$\left(57\middle|7^{k+2} + 8^{2k+1}\right)$$

Show that P(k+1) is also true,

$$\begin{aligned}\left(57\middle|7^{k+3} + 8^{2k+3}\right) &\Longrightarrow \left(57\middle|7.7^{k+2} + 8^{2k+1}.64\right)\\ &\Longrightarrow \left(57\middle|7.7^{k+2} + 57.8^{2k+1} + 7.8^{2k+1}\right)\\ &\Longrightarrow \left(57\middle|7\left(7^{k+2} + 8^{2k+1}\right) + 57.8^{2k+1}\right)\\ &\Longrightarrow \left(57\middle|7\left(7^{k+2} + 8^{2k+1}\right)\right) + \left(57\middle|57.8^{2k+1}\right) \checkmark\end{aligned}$$

Mathematical Recursion

- **Recursion**
  - the process of defining an object in terms of itself.
  - can be used to define sequences, functions, and sets



Mathematical Recursion

➢ Example: Redefine recursively the sequence of powers of 2 that is given by $a_n = 2^n$ for n = 0, 1, 2, ...

$$a_n = 2^n$$
$$a_0 = 2^0 \Longrightarrow \boldsymbol{a_0 = 1}$$
$$a_n = 2.2^{n-1} \Longrightarrow \boldsymbol{a_n = 2.a_{n-1}} \checkmark$$

References

- K.H. Rosen, Discrete Mathematics and Its Applications, Seventh Edition, Mc Graw Hill, 2012.
- R.P. Grimaldi, Discrete and Combinatorial Mathematcics, An Applied Introduction, Fifth Edition, Pearson, 2003.
- S.S. Epp, Discrete Mathematics with Applications, Fouth Edition, 2010.
- N. Yurtay, "Ayrık İşlemsel Yapılar" Lecture Notes, Sakarya University.