SAKARYA ÜNİVERSİTESİ

Discrete Mathematics
LECTURE 2
Logic

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Outline

- Logic
  - Propositions
- References

Logic

- **Logic**
  - a formal study of mathematics
  - the study of mathematic reasoning and proofs itself
  - The basis of mathematical logic is **propositional logic**,
    - which was essentially invented by Aristotle.
    - here the model is a collection of **statements** that are either true or false.

- **Propositional logic** is the simplest form of logic. Here the only statements that are considered are **propositions**, which contain no variables. Because propositions contain no variables, they are either always true or always false.
- Examples of propositions: $2 + 2 = 4$. (Always true). $2 + 2 = 5$. (Always false).
- Examples of non-propositions: $x + 2 = 4$. (May be true, may not be true; it depends on the value of x.)

Logic…

Proposition

- In logic, the content of an asserted statement whose value must be either **true** or **false**.
- The basic building blocks of the theory of logic

❑ The capital of TURKIYE is ANKARA	=> a TRUE proposition
❑ 7 = 8	=> a FALSE proposition
❑ Is Istanbul the capital of America?	=> not a proposition
❑ There is an integer number both prime and even	=> a TRUE proposition
❑ What a beautiful day!...	=> not a proposition
❑ Get up and do your homework!	=> not a proposition

Logic...

- Just as we use letters to represent numbers in algebra, we use variables such as p, q, and r to represent propositions.

p : 2+2=5

- In terms of truth value, there are 2 cases for one proposition, 4 cases for two propositions, ..., and 2^n cases for n propositions.

Logic…

- **Operations on Propositions**
  - **Negation**
    - the "negation of *p*" statement
      - false when *p* is true, and true when *p* is false
      - read as "not *p*"
      - denoted as $\neg p$, or sometimes $\sim p$, $-p$, p' or $\bar{p}$.

p	$\neg p$
0	1
1	0

Logic…

- **Operations on Propositions…**
  - **AND**
    - the "*p* and *q*" statement
      - conjunction of the statements *p*,*q*
      - denoted as $p \wedge q$
      - called the intersection of the propositions p and q.
      - true when both p and q statements are true, and false otherwise

p	q	$p \wedge q$
0	0	0
0	1	0
1	0	0
1	1	1

$\mathrm{p}: 3+3=5, \mathrm{q}: 1+6=8 \Rightarrow p \wedge q = 0$

Logic...

- **Operations on Propositions...**
  - **OR**
    - the "*p* or *q*" statement
      - disjunction of the statemens *p*,*q*
      - denoted as $p \lor q$
      - called the union of the propositions p and q.
      - false when both statements p and q are false, and true otherwise

p	q	$p \lor q$
0	0	0
0	1	1
1	0	1
1	1	1

$$\text{p: }3+3=5,\ \text{q: }1+6=7 \Rightarrow p \lor q = 1$$

Logic...

- **Operations on Propositions...**
  - **Exclusive OR**
    - the "p or q but not both" statement
    - denoted as $p \oplus q$
    - logically equivalent to $(p \vee q) \wedge \sim (p \vee q)$
    - true only when one of the statements p and q are true, and false otherwise

p	q	$p \oplus q$
0	0	0
0	1	1
1	0	1
1	1	0

Logic…

- **Operations on Propositions…**
  - **Implication**
    - represents an "**if. . . then**" claim
    - composite and conditional proposition/statement
    - the "***p* implies *q***" statement
      - implication *p* by *q*
      - denoted as $p \to q$ or $p \Rightarrow q$
      - usually rendered as **"if p then q"**
      - the proposition *p* is called the hypothesis of the conditional proposition, and the proposition *q* is called its conclusion
      - means *q* is true whenever *p* is true
      - the only way to be false is for *p* to be true but *q* to be false
      - can also be rewritten as $\neg p \lor q$

p	q	$p \to q$
0	0	1
0	1	1
1	0	0
1	1	1

Logic...

- **Operations on Propositions...**
  - **Implication...**
    - Example:
      - "If I weigh more than 60 kg, then I will go on a diet"
        - p: I weigh more than 60 kg
        - q: I will start a diet
        - $p \rightarrow q$

p	q	$p \rightarrow q$
0	0	1
0	1	1
1	0	0
1	1	1

Logic…

- **Operations on Propositions…**
  - **Biconditional**
    - the "p if and only if q" statement
      - represents the $p \to q$ and $q \to p$
      - denoted as $p \leftrightarrow q$ or $p \Leftrightarrow q$
      - The only way to be false is for one side to be true and other side to be false.
      - can also be rewritten as $(p \to q) \wedge (q \to p)$

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$	$\boldsymbol{p \leftrightarrow q}$
0	0	1	1	1	1
0	1	1	0	0	0
1	0	0	1	0	0
1	1	1	1	1	1

Logic…

- **Operations on Propositions…**
  - **Biconditional…**
    - Example:
      - "If Ali is eligible to vote, then he is at least 18 years old."

Logic…

- **Order of operations Propositions**
  - ❶Negation (~)
  - ❷$\wedge$,$\vee$
  - ❸$\rightarrow$, $\leftrightarrow$

- the order of operations can be overridden through the use of parentheses.
  - $\sim p \wedge q = (\sim p) \wedge q$.

Logic…

- **Tautology**
  - a compound proposition that is always true no matter what the truth-values of the propositions it contains

p	p'	$p \vee p'$
0	1	1
1	0	1

Logic...

- **Contradiction**
  - a compound proposition that is always false no matter what the truth-values of the propositions it contains

p	p'	$p \wedge p'$
0	1	0
1	0	0

- The negation of a tautology is a contradiction and vice versa.

Logic…

Truth Table

Example:

- Create a truth table for the proposition $(p \wedge q')'$

p	q	q'	$p \wedge p'$	$(p \wedge p')'$
0	0	1	0	1
0	1	0	0	1
1	0	1	1	0
1	1	0	0	1

Logic...

- **Logical Equivalence**
  - denoted by using "$\equiv$"
  - can be inferred by using truth tables or algebraic formulas

Logic...

Logical Equivalence...

Example:

- Show $p' \vee q \equiv p \longrightarrow q$

p	q	p'	p'∨q	p⟶q
0	0	1	1	1
0	1	1	1	1
1	0	0	0	0
1	1	0	1	1

Logic...

Logical Equivalence...

Example:

- Show $p \leftrightarrow q \equiv (p \longrightarrow q) \land (q \longrightarrow p)$

p	q	$(p \longrightarrow q)$	$q \longrightarrow p$	$(p \longrightarrow q) \land (q \longrightarrow p)$	$p \leftrightarrow q$
0	0	1	1	1	1
0	1	1	0	0	0
1	0	0	1	0	0
1	1	1	1	1	1

Logical Equivalence…

Example:

- Show $p \oplus q \equiv (p \vee q) \wedge \sim (p \wedge q)$

p	q	$p \oplus q$	$p \vee q$	$p \wedge q$	$\sim (p \wedge q)$	$(p \vee q) \wedge \sim (p \wedge q)$
0	0	0	0	0	1	0
0	1	1	1	0	1	1
1	0	1	1	0	1	1
1	1	0	1	1	0	0

Logic…

Logical Equivalence…

1	$p \wedge q \equiv q \wedge p$ $p \vee q \equiv q \vee p$	Commutative laws
2	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge \mathrm{r}$ $p \vee (q \vee r) \equiv (p \vee q) \vee \mathrm{r}$	Associative laws
3	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	Distributive laws
4	$p \wedge T \equiv p$ $p \vee F \equiv p$	Identity laws
5	$p \wedge p' \equiv F$ $p \vee p' \equiv T$	Negation laws
6	$(p')' \equiv p$	Double negative law
7	$p \wedge p \equiv p$ $p \vee p \equiv p$	Idempotent laws
8	$p \wedge F \equiv F$ $p \vee T \equiv T$	Universal bound laws
9	$(p \wedge q)' \equiv p' \vee q'$ $(p \vee q)' \equiv p' \wedge q'$	De Morgan’s laws
10	$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorbtion laws
11	$T' \equiv F$ $F' \equiv T$	Negations of T and F

Logic...

Logical Equivalence...

Example:

- Show De Morgan's law
  - $(p \vee q)' \equiv (p' \wedge q')$

p	q	$p \vee q$	$(p \vee q)'$	p'	q'	$p' \wedge q'$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

Logical Equivalence...

Example:

- Show distribution of AND on OR
  - $p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$

p	**q**	**r**	$p \lor q$	$p \lor r$	$(p \lor q) \land (p \lor r)$	$q \land r$	$p \lor (q \land r)$
0	0	0	0	0	0	0	0
0	0	1	0	1	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	1	1	1	1
1	0	0	1	1	1	0	1
1	0	1	1	1	1	0	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

- **Logical Equivalence…**
  - **Example:**
    - Show absorbtion law
      - $p \vee (p \wedge q) \equiv p$

p	q	$p \wedge q$	$p \vee (p \wedge q)$
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

Logic…

Logical Equivalence…

Example:

- Verify the logical equivalence $(p' \wedge q)' \wedge (p \vee q) \equiv p$

$$
\begin{aligned}
(p' \wedge q)' \wedge (p \vee q) &\equiv \big((p')' \vee q'\big) \wedge (p \vee q) && \text{de Morgan's law} \\
&\equiv (p \vee q') \wedge (p \vee q) && \text{double negation law} \\
&\equiv p \vee (q' \wedge q) && \text{distributive law} \\
&\equiv p \vee (q \wedge q') && \text{commutative law for } \wedge \\
&\equiv p \vee F && \text{negation law} \\
&\equiv p && \text{identity law}
\end{aligned}
$$

Logical Equivalence…

Example:

- Simplify $\left(\left((p \vee q) \wedge r\right)' \vee q'\right)'$

$$\left(\left((p \vee q) \wedge r\right)' \vee q'\right)' \equiv \left(\left((p \vee q) \wedge r\right)'\right)' \wedge (q')' \qquad \text{de Morgan's law}$$

$$\equiv \left((p \vee q) \wedge r\right) \wedge q \qquad \text{double negation law}$$

$$\equiv (p \vee q) \wedge (r \wedge q) \qquad \text{commutative law for } \wedge$$

$$\equiv (p \vee q) \wedge (q \wedge r) \qquad \text{commutative law for } \wedge$$

$$\equiv (p \vee q \wedge q) \wedge r \qquad \text{associative law}$$

$$\equiv q \wedge r \qquad \text{absorbtion law}$$

Logic…

EXAMPLE:

- Translate this English sentence into a logical expression?
  - "You can access the Internet from campus only if you are a computer science major or you are not a freshman."

- p: "You can Access the internet from campus"
- q: "You are comouter science major"
- r: "You are a freshman"

- $p \to (r \vee s')$

Logic…

EXAMPLE:

- Translate this English sentence into a logical expression?
  - "You cannot ride the roller coaster if you are under 4 feet tall unless you are older than 16 years old."

- p: "You can ride the roller coaster"
- q: "You are under than 4 feet tall"
- r: "You are older than 16 years old"

- $(q \land r') \longrightarrow s'$

Logic...

- Application Areas
  - Programming Languages
    - Boolean operations
  - Digital Logic Circuits
  - Boolean Searches
    - Web page searching
  - Games
    - Sudoku Puzzles (propositional satisfiability)

Logic...

➢Study Question:

➢After baking a pie for the two nieces and two nephews who are visiting her, Aunt Nil leaves the pie on her kitchen table to cool. Then she drives to the mall to shopping. Upon her return she finds that someone has eaten one-quarter of the pie. Since no one was in her house that day – except the four visitors – Aunt Nil questions each niece and nephew about who ate the piece of pie. The four "suspects" tell her the following:

Ahmet:	Cenk ate the piece of pie.
Buse:	I did not eat the piece of pie.
Cenk:	Deniz ate the pie.
Deniz:	Cenk lied when she said I ate the pie.

If only one of these four statements is true and only one of the four did this naughty thing, then who is (s)he?

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