Discrete Mathematics
LECTURE 3
Sets

Assistant Professor Gülüzar ÇİT

Outline

- Sets
  - Describing a set
    - Roster method
    - Set builder notation
    - Interval notation
    - Venn diagrams
  - Equal sets
  - The empty set
  - Subset
  - The size of a set
  - Power sets
  - Cartesian product
  - Truth sets
  - Set Operations
  - Set Identities
- References

Sets...

- an unordered collection of objects (elements/members)
- used to group objects together.
- often, but not always, the objects in a set have similar properties.
  - all the students who are currently enrolled in your school
  - all the students currently taking a course in discrete mathematics at any school
  - those students enrolled in your school who are taking a course in discrete maths
    - can be obtained by taking the elements common to the first two collections

- $a \in A$ denotes that a is an element of the set A.
- $a \notin A$ denotes that a is not an element of the set A.

Sets…

- **Describing a set**
  - **Roster Method**
    - uses a notation where all members of the set are listed between braces
    - the set V of all vowels in the English alphabet $\Rightarrow V = \{a, e, i, o, u\}$.
    - the set O of odd positive integers less than 10 $\Rightarrow O = \{1, 3, 5, 7, 9\}$.

Sets…

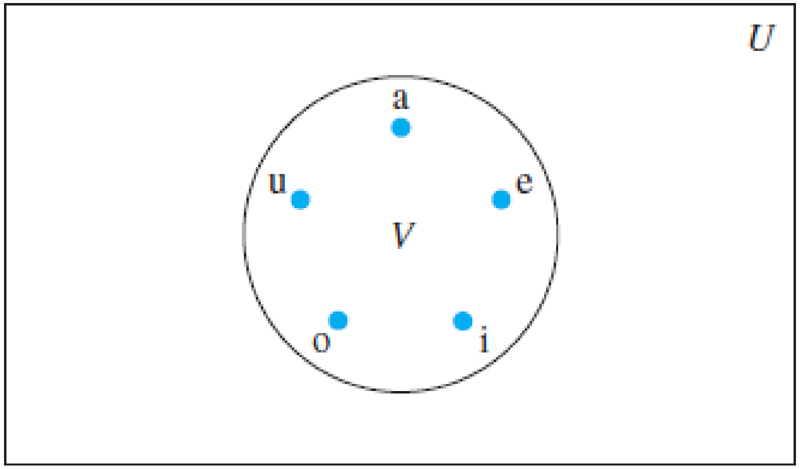
- **Describing a set…**
  - **Set Builder Notation**
    - characterizes all elements in the set by stating the property/properties they must have to be members
    - the set O of all odd positive integers less than 10 can
      - $O = \{x \mid x$ is an odd positive integer less than $10\}$,
      - $O = \{x \in \mathbf{Z}+ \mid x$ is odd and $x < 10\}$.
    - the set $\mathbf{Q}^+$ of all positive rational numbers can be written as
      - $\mathbf{Q}^+ = \{x \in \mathbf{R} \mid x = p/q$, for some positive integers p and $q\}$.

Sets…

- **Describing a set…**
  - **Interval Notation**
    - when a and b are real numbers with $a < b$
      - $[a, b] = \{x \mid a \leq x \leq b\}$
      - $[a, b) = \{x \mid a \leq x < b\}$
      - $(a, b] = \{x \mid a < x \leq b\}$
      - $(a, b) = \{x \mid a < x < b\}$

Sets…

- **Describing a set…**
  - **Venn Diagrams**
    - representing sets graphically
    - set of vovels in English aplhabet



Sets…

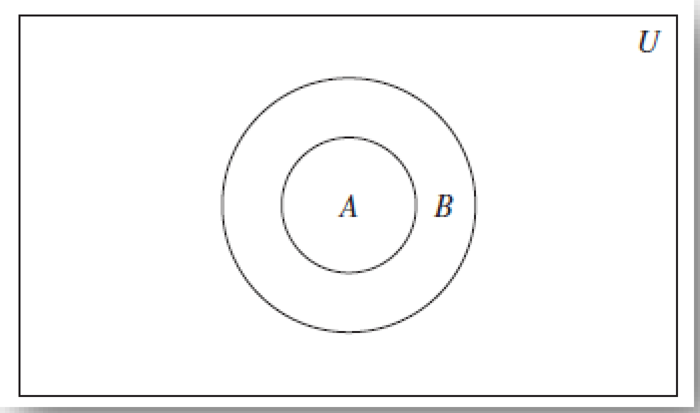
- **Equal sets…**
  - Two sets are *equal* if and only if they have the same elements
  - if A and B are sets, then A and B are equal
    - if and only if $\forall x\ (x \in A \leftrightarrow x \in B)$.
  - we write $A = B$ if A and B are equal sets.

Sets…

- **The empty set**
  - a special set that has no elements.
  - called also as **null set**,
  - denoted by Ø or { }

Sets…

- **Subset**
  - the set A is a **subset** of B if and only if every element of A is also an element of B.
  - the notation $A \subseteq B$ indicates that A is a subset of the set B.
  - $A \subseteq B$ if and only if the quantification $\forall x(x \in A \to x \in B)$ is true.
  - to show that A is not a subset of B we need only find one element $x \in A$ with $x \notin B$. Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.



Sets…

- **The size of a set**
  - If there are exactly n distinct elememts in a set S where n is a nonnegative integer,
    - S is a finite set
    - N is the cardinality of S
  - Cardinality of S is denoted by $|S|$.
  - A set is said to be infinite if it is not finite.

Sets…

- **Power sets**
  - the **power set** of *S* is
    - the set of all subsets of a given set *S*.
    - denoted by $\mathcal{P}(S)$.
  - if a set has n elements, then its power set has 2^n elements.
  - **Example:**
    - What is the power set of the set $\{0,1,2\}$?
    - $\mathcal{P}(\{0,1,2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$

Sets…

- **Cartesian Product**
  - The **ordered n-tuple $(\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_n})$** is
    - the ordered collection that has a_1 as its first element, a_2 as its second element, . . . , and a_n as its nth element.
    - ordered 2-tuples are called **ordered pairs**.

Sets…

- **Cartesian Product…**
  - The **cartesian product** of set *A* and set *B*
    - the set of all ordered pairs *(a, b)*, where *a* ∈ *A* and *b* ∈ *B*
    - denoted by *A* × *B*,
    - $\mathrm{A} \times B = \{(a, b) | a \in A \wedge b \in A\}$
    - $\mathrm{A} \times B$ and $\mathrm{A} \times B$ are not equal, unless *A*=∅ or *B*=∅

    - **Example:**
      - What is the cartesian product of $A = \{1,2\}$ and B= $\{a, b, c\}$?

      - $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$
      - $B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$

Sets…

- **Cartesian Product…**
  - The **cartesian product** of the sets $A_1, A_2, \dots A_n$
    - set of ordered *n*-tuples $(a_1, a_2, \dots a_n)$ where a_i belongs to A_i for $i = 1,2, \dots n$
    - denoted by $A_1 \times A_2 \times \cdots \times A_n$
    - $A_1 \times A_2 \times \cdots \times A_n = \{a_1, a_2, \dots, a_n | a_i \in A_i\ for\ i = 1,2, \dots, n\}$

    - **Example:**
      - What is the cartesian product A$\times B \times C$ where $A = \{0,1\}$, B $= \{1,2\}$ and C$= \{0,1,2\}$?

      - $A \times B \times C =$
$$\left\{\begin{matrix}(0,1,0), (0,1,1), (0,1,2), (0,2,0), (0,2,1), (0,2,2), \\ (1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)\end{matrix}\right\}$$

Sets…

- **Cartesian Product…**
  - The **cartesian product** of the set with itself
    - $A2$ to denote $A \times A$
    - $A3$ to denote $A \times A$, and so on.
    - $An = \{(a_1, a_2, \ldots a_n) | a_i \in A \; for \; i = 1,2, \ldots, n\}$
  - **Example:**
    - What is $A2$ and $A3$ when $A = \{1,2\}$
    - $A2 = \{(1,1), (1,2), (2,1), (2,2)\}$
    - $A3 = \begin{Bmatrix} (1,1,1), (1,1,2), (1,2,1), (1,2,2), \\ (2,1,1), (2,1,2), (2,2,1), (2,2,2) \end{Bmatrix}$

Sets…

- **Cartesian Product…**
  - A subset *R* of the Cartesian product $A \times B$ is called a **relation** from the set *A* to the set *B*.
  - The elements of *R* are ordered pairs, where the first element belongs to *A* and the second to *B*
  - A relation from a set *A* to itself is called a relation on *A*.

  - **Example:**
    - What are the ordered pairs in the less than or equal to, which contains (a, b) if $a \leq b$ on the set $\{0,1,2,3\}$?
    - $$R = \left\{ \begin{matrix} (0,0), (0,1), (0,2), (0,3), (1,1), (1,2), (1,3), \\ (2,2), (2,3), (3,3) \end{matrix} \right\}$$

Sets…

- **Truth Sets**
  - A subset *R* of the Cartesian product $A \times B$ is called a **relation** from the set *A* to the set *B*.
  - The elements of *R* are ordered pairs, where the first element belongs to *A* and the second to *B*
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  - **Example:**
    - What are the ordered pairs in the less than or equal to, which contains (a, b) if $a \leq b$ on the set $\{0,1,2,3\}$?
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Sets…

Truth Sets…

- The **truth set** of a predicate *P* to be the set of elements in domain *D* if the set for whih *P(x)* is true.
- *P(x)* is denoted by $\{x \in D | P(x)\}$

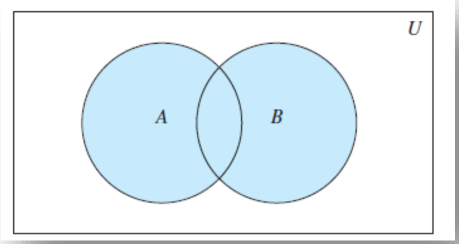
- **Example:**
  - What is the truth set of predicate P(x), where the domain is the set oof integers and P(x) is "$|x| = 1$"

  - The truth set of P $\Longrightarrow \{x \in Z | |x| = 1\} \Rightarrow \{-1,1\}$

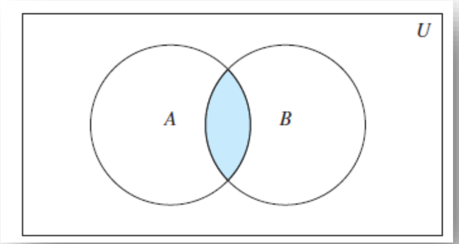
Sets…

Set Operations

- The **union** of the sets *A* and *B*
  - the set that contains those elements that are either in *A* or in *B*, or in both
  - denoted by $A \cup B$
  - $A \cup B = \{x | x \in A \vee x \in B\}$



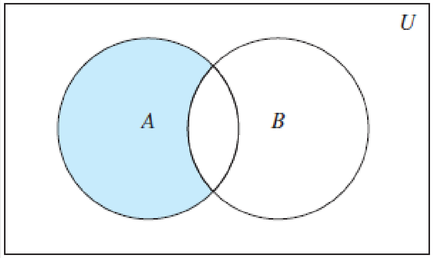
- The **intersection** of the sets *A* and *B*
  - the set that contains those elements that are either in *A* or in *B*, or in both
  - denoted by $A \cap B$
  - $A \cap B = \{x | x \in A \wedge x \in B\}$



Sets…

Set Operations…

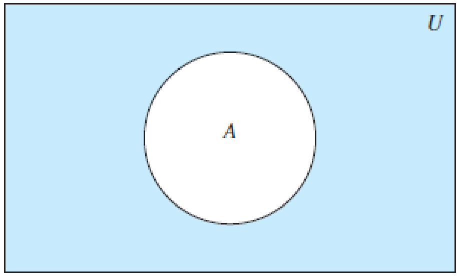
- Two sets are called **disjoint** if their intersection is the empty set
  - A and B are disjoint if $A \cap B = \emptyset$
- The **difference** of sets *A* and *B*
  - the set containing those elements that are in *A* but not in *B*.
  - denoted by *A* – *B*
  - sometimes denoted by *A**B*
  - also called the complement of B with respect to A.
  - an element *x* belongs to the difference of *A* and *B* if and only if $x \in A$ and $x \notin B$
  - $A - B = \{x | x \in A \wedge x \notin B\}$



Sets...

Set Operations..

- The **complement** of the set *A* is the complement of *A* with respect to the universal set *U*.
- denoted by $\bar{A}$
- Therefore, the complement of the set $\bar{A}$ is $U - A$
- An element belongs to $\bar{A}$ if and only if $x \notin A$
- $A = \{x | x \in U \wedge x \notin A\}$



- $A - B = A \cap \bar{B}$

Sets...

➢ Set Identities

$\boldsymbol{A \cup A = A}$ $\boldsymbol{A \cup \emptyset = A}$	**Identity laws**
$\boldsymbol{A \cup U = U}$ $\boldsymbol{A \cap \emptyset = \emptyset}$	**Domination laws**
$\boldsymbol{A \cup A = A}$ $\boldsymbol{A \cap A = A}$	**Idempotent laws**
$\overline{(\overline{\boldsymbol{A}}) =}$	**Complementation law**
$\boldsymbol{A \cup B = \mathrm{B} \cup \mathrm{A}}$ $\boldsymbol{A \cap B = \mathrm{B} \cap \mathrm{A}}$	**Commutative laws**
$\boldsymbol{A \cup (B \cup C) = (A \cup B) \cup C}$ $\boldsymbol{A \cap (B \cap C) = (A \cap B) \cap C}$	**Associative laws**
$\boldsymbol{A \cup (B \cap C) = (A \cup B) \cap (A \cup C)}$ $\boldsymbol{A \cap (B \cup C) = (A \cap B) \cup (A \cap C)}$	**Distributive laws**
$\boldsymbol{\overline{A \cap B} = \overline{A} \cup \overline{B}}$ $\boldsymbol{\overline{A \cup B} = \overline{A} \cap \overline{B}}$	**De Morgan's laws**
$\boldsymbol{A \cup (A \cap B) = \mathrm{A}}$ $\boldsymbol{A \cap (A \cup B) = \mathrm{A}}$	**Absorbtion laws**
$\boldsymbol{A \cup \overline{A} = U}$ $\boldsymbol{A \cap \overline{A} = \emptyset}$	**Complement laws**

Sets…

- **Generalized Unions & Intersections**
  - The **union** of a collection of sets
    - the set that contains those elements that are members at least one set in the collection
    - the union of the sets $A_1, A_2, \ldots, A_n$
      - denoted by $A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$
  - The **intersection** of a collection of sets
    - the set that contains those elements that are members of all the sets in the collection
    - the intersection of the sets $A_1, A_2, \ldots, A_n$
      - denoted by $A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$

Sets…

Set Operations & Identities…

Example:

- Use set builder notation and logical equivalences to establish the first De Morgan law $\overline{A \cap B} = \bar{A} \cup \bar{B}$

$\overline{A \cap B} \equiv \{x \mid x \notin A \cap B\}$	definition of complement
$\equiv \{x \mid \backsim (x \in A \cap B)\}$	definition of does not belong symbol
$\equiv \{x \mid \backsim (x \in A \land x \in B)\}$	definition of intersection
$\equiv \{x \mid \backsim (x \in A) \lor \backsim (x \in B)\}$	De Morgan law for loqical equivalences
$\equiv \{x \mid x \notin A \lor x \notin B\}$	definition of does not belong symbol
$\equiv \{x \mid x \in \bar{A} \lor x \in \bar{B}\}$	definition of complement
$\equiv \{x \mid x \in \bar{A} \cup \bar{B}\}$	definition of union
$\equiv \bar{A} \cup \bar{B}$	set builder notation

Sets…

Set Operations & Identities…

Example:

- Let A, B and C sets. Show that $\overline{A \cup (B \cap C)} = (\bar{C} \cup \bar{B}) \cap \bar{A}$

$$
\begin{aligned}
\overline{A \cup (B \cap C)} &= \bar{A} \cap (\overline{B \cap C}) && \text{De Morgan Law} \\
&= \bar{A} \cap (\bar{B} \cup \bar{C}) && \text{De Morgan Law} \\
&= (\bar{B} \cup \bar{C}) \cap \bar{A} && \text{commutative law for intersection} \\
&= (\bar{C} \cup \bar{B}) \cap \bar{A} && \text{commutative law for union}
\end{aligned}
$$

Sets...

Set Operations & Identities...

Example:

- Let A and B sets. Show that $A - B = A \cap \overline{B}$

$$
\begin{aligned}
A - B &= \{x \mid x \in A \land x \notin B\} && \text{set builder notation} \\
&= \{x \mid x \in A \land x \in \overline{B}\} && \text{definition of complement} \\
&= A \cap \overline{B} && \text{set builder notation}
\end{aligned}
$$

Sets…

Set Operations & Identities…

Example:

- Let A, B and C sets. Show that $A \times (B \cap C) = (A \times B) \cap (A \times C)$

$$
\begin{aligned}
A \times (B \cap C) &= \{(x, y) | x \in A \wedge y \in B \cap C\} \\
&= \{(x, y) | x \in A \wedge y \in B \wedge y \in C\} \\
&= \{(x, y) | x \in A \wedge x \in A \wedge y \in B \wedge y \in C\} \\
&= \{(x, y) | (x \in A \wedge y \in B) \wedge (x \in A \wedge y \in C)\} \\
&= \left\{(x, y) \middle| \big((x, y) \in A \times B\big) \wedge \big((x, y) \in A \times C\big)\right\} \\
&= (A \times B) \cap (A \times C)
\end{aligned}
$$

Sets…

Set Operations & Identities…

Example:

- Use a membership table to show that $A \cap (B \cap C) = (A \cap B) \cup (A \cap C)$

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	0	0	0	0
1	0	0	0	0	0	0	0
1	0	1	1	1	0	1	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

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