Discrete Mathematics

LECTURE 15
Matching

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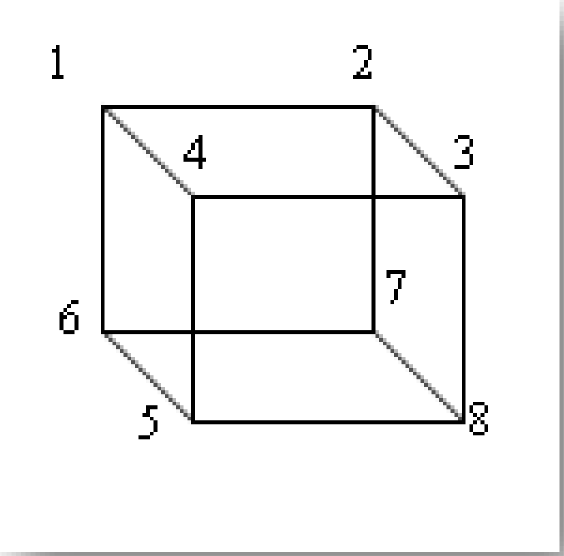
Outline

- Matching
- References

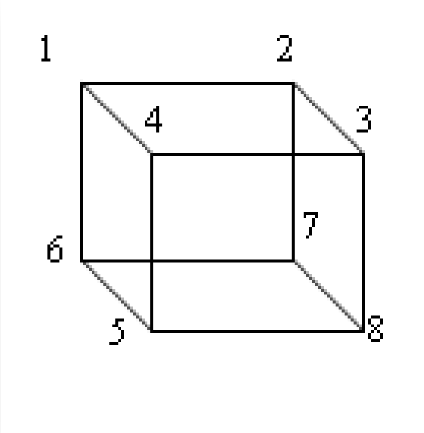
Matching...

- **Bipartite Graph**
  - A graph $G = (V, E)$ with a node set V and a number of edges E if the set V can be separated into two disjoint sets, such as V_1 and V_2, with all the sides connected from one element of V_1 to one element of V_2.



Matching...

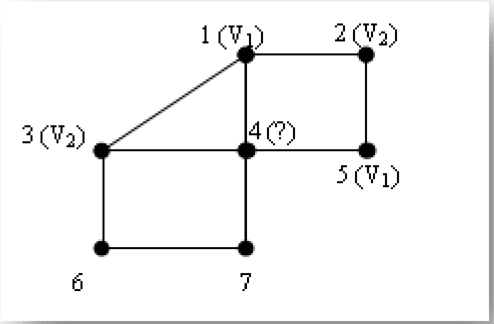
- **Example:** Determine whether the graph G on the below is bipartite or not.



- **Solution:**
  - The set V can be separated into two disjoint sets, such as V_1 and V_2, with all the sides connected from one element of V_1 to one element of V_2
  - V_1 is 1,2,3 and 4
  - V_2 is 5,6,7 and 8
  - The graph is **bepartite** ✓

Matching…

➢ Example: Determine whether the graph G on the below is bipartite or not.



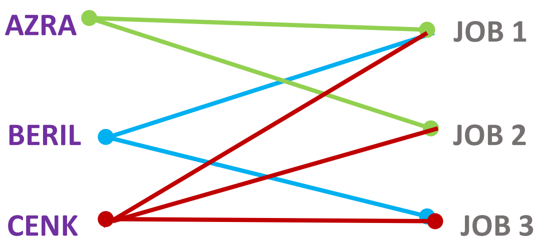
➢ Solution:

- The set V can not be separated into two disjoint sets, such as V_1 and V_2, with all the sides connected from one element of V_1 to one element of V_2
- The graph is **not bepartite** ✓

Matching

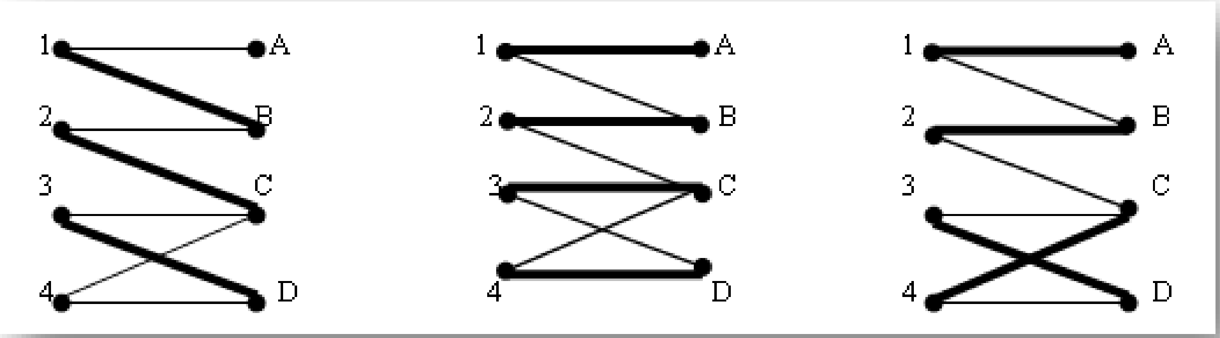
- a **matching** *M* in a simple (undirected) graph $G = (V, E)$ is a subset of the set *E* of edges of the graph such that no two edges are incident with the same vertex.
- in other words, a **matching** is a subset of edges such that if {*s, t*} and {*u, v*} are distinct edges of the matching
- a **matching** is a set of edges, such that no two edges share the same vertex for an undirected graph.

	Azra	Beril	Cenk
Work 1	7,5	6	6,5
Work 2	8	8,5	7
Work 3	5	6,5	5,5



Matching...

- The purpose of mapping is to find a subset e from the edge set E for the graph G.
- Subset e, which can be obtained in such a way that no node of the graph belongs to any edge other than one of the edges in e, is called a **matching** of the graph.
- The **maximum matching** is the matching with the largest number of edges.

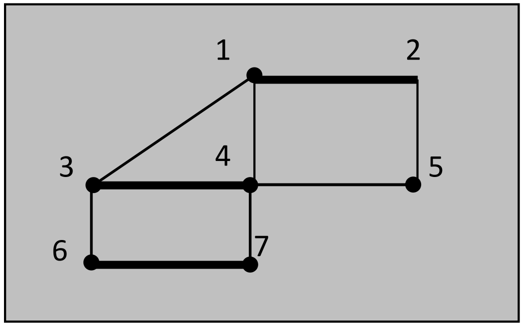


Matching...

- **Example:** Draw of a graph depending on the languages whick the seven people speak. Determine whether the graph is bipartite or not and determine one matching of the graph.

People	**Languages**
1	French, German, English
2	Spanish, French
3	German, Italian
4	Greek, German, Russian, Arabic
5	Spanish, Russian
6	Chineese, Japaneese, Italian
7	Greek, Chineese

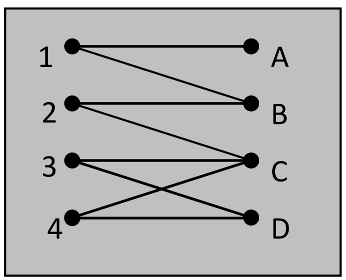
- **Solution:**



Matching…

Maximum Independence Set

- A set of elements that are not on the same line in a bipartite graph matrix A is called **independent**.
- For example, the set with the most elements among the independent sets of elements 1 in the bipartite graph matrix A is called the **maximum independence set 1’s**.

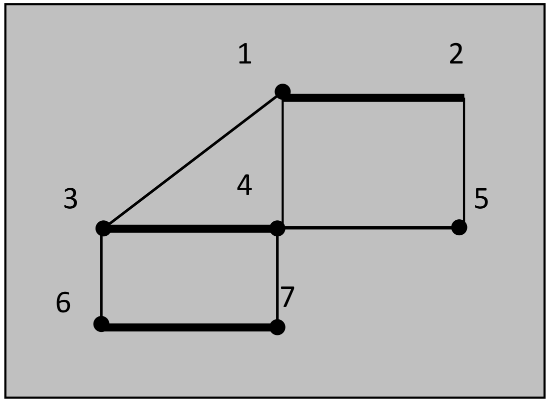


$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} \text{A} & \text{B} & \text{C} & \text{D} \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

$$\begin{bmatrix} 1 & 1^* & 0 & 0 \\ 0 & 1 & 1^* & 0 \\ 0 & 0 & 1 & 1^* \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1^* & 1 & 0 & 0 \\ 0 & 1^* & 1 & 0 \\ 0 & 0 & 1^* & 1 \\ 0 & 0 & 1 & 1^* \end{bmatrix} \quad \begin{bmatrix} 1^* & 1 & 0 & 0 \\ 0 & 1^* & 1 & 0 \\ 0 & 0 & 1 & 1^* \\ 0 & 0 & 1^* & 1 \end{bmatrix}$$

Matching…

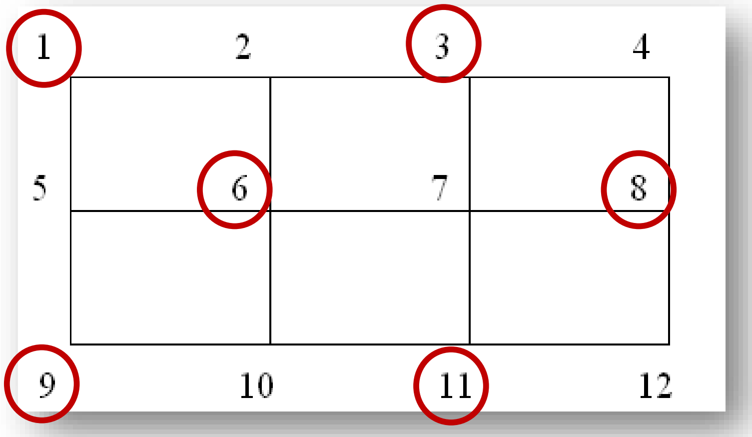
- **Covering**
  - A set of C nodes of a graph G in which each edge has at least one node is called a **covering** C of the graph.
  - C is said to be **minimal covering** of the graph if no covering other than C contains fewer nodes.
  - Example: For the graph in the figure {2,3,4,5,6} is a coverage, but not minimum. Because the coverage {1,3,5,7} has less elements.



Matching...

➢Covering...

➢Example: Let the nodes in the graph be the intersection points of the streets in a city area. A company wants to open some buffet(s) in some areas. However, it is desirable that the buffets be as small as possible and placed in such a way that no one can reach the buffet without going further than a block.

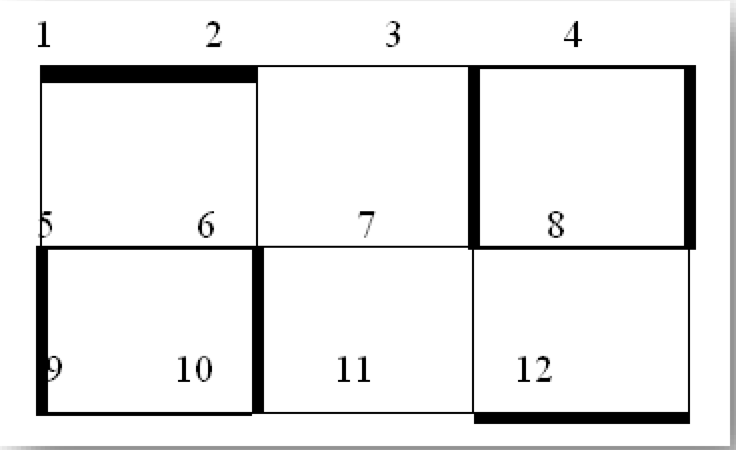


➢Solution:

- A covering is {1,3,6,8,9,11}
- Therefore, the number of buffets is six.✓

Matching…

- **Covering…**
  - Let a graph has M mapping and C covering. In this case, $|M| \leq |C|$. If $|M| = |C|$ then M is a maximum matching and C is a minimum covering.



Matching…

One Matching Algorithm

- Algorithm finds the maximum independence set of 1’s for an bipartite graph shown as an nxn matrix
- There are two basic operations: labeling & scanning
- If any line(row/column) has been labeled once, this line can not be relabeled
- Before scanning a line, it must be labeled

```
STEP 1 Find the maximum matching set of 1’s
STEP 2 (Expand matching)
   DO
      STEP 2.1 Label the columns containing unlabeled 1’s using the # symbol.
      DO
         a) IF (There is not such a column)
            Current independent set will already be the maximum independent set
            END
         b) Label the corresponding 1’s row as the relevant column name
         c) Scan each labeled row for unlabeled 1‘s
         d) IF (There is not such a row)
               (Enpand matching)
               Circle the 1 with the label of row, circle this rown’s relevant column’s labeled 1
               and circle this columns’s relevant row’s unlabeled 1
               Label the unlabeld 1’s and vice versa.
               BREAK
            ELSE
               Label the columns containing unlabeled 1’s as the relevant row name
      WHILE
   WHILE (number of labeled ones < n)
```

Matching…

One Matching Algorithm…

- **Example:** Apply one-matching algorithm on the bipartite matrix on the left

$$\begin{array}{c|cccc} & A & B & C & D \\ \hline 1 & 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 & 0 \end{array}$$

- **Solution:**

$$\begin{array}{c|cccc} & A & B & C & D \\ \hline 1 & 1^* & 0 & 1 & 1 \\ 2 & 0 & 1^* & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 & 0 \end{array} \Longrightarrow \begin{array}{c|cccc} & A & B & C & D \\ \hline 1 & 1^* & 0 & 1 & 1 \\ 2 & 0 & 1^* & 0 & 0 \\ 3 & 1 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 & 0 \\ & & & \# & \# \end{array}$$

Matching…

- **One Matching Algorithm…**
  - Solution:…

$$\Rightarrow \begin{array}{ccc} & \begin{array}{cccc} A & B & C & D \end{array} & \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1^* & 0 & 1 & 1 \\ 0 & 1^* & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} & \begin{array}{c} C \\ \\ \\ \\ \end{array} \\ & \begin{array}{cccc} & & \# & \# \\ & & \checkmark & \checkmark \end{array} & \end{array} \qquad \Rightarrow \begin{array}{ccc} & \begin{array}{cccc} A & B & C & D \end{array} & \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1^* & 0 & 1 & 1 \\ 0 & 1^* & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} & \begin{array}{c} C \checkmark \\ \\ \\ \\ \end{array} \\ & \begin{array}{cccc} 1 & & \# & \# \\ & & \checkmark & \checkmark \end{array} & \end{array}$$

$$\Rightarrow \begin{array}{ccc} & \begin{array}{cccc} A & B & C & D \end{array} & \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1^* & 0 & 1 & 1 \\ 0 & 1^* & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} & \begin{array}{c} C \checkmark \\ \\ A \\ \\ \end{array} \\ & \begin{array}{cccc} 1 & & \# & \# \\ \checkmark & & \checkmark & \checkmark \end{array} & \end{array} \qquad \Rightarrow \begin{array}{ccc} & \begin{array}{cccc} A & B & C & D \end{array} & \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1^* & 0 & 1 & 1 \\ 0 & 1^* & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} & \begin{array}{c} C \checkmark \\ \\ A\ ! \\ \\ \end{array} \\ & \begin{array}{cccc} 1 & & \# & \# \\ \checkmark & & \checkmark & \checkmark \end{array} & \end{array}$$

Matching...

- **One Matching Algorithm...**
  - Solution:...

$$\Rightarrow \begin{array}{c c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1^* & 0 & 1 & 1 \\ 0 & 1^* & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\ & \begin{array}{cccc} 1 & & \# & \# \\ \checkmark & & \checkmark & \checkmark \end{array} \end{array} \quad \begin{array}{c} C \checkmark \\ \\ A\ ! \end{array} \Rightarrow \begin{array}{c c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 1^* & 1 \\ 0 & 1^* & 0 & 0 \\ 1^* & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$$\Rightarrow \begin{array}{c c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 1^* & 1 \\ 0 & 1^* & 0 & 0 \\ 1^* & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\ & \begin{array}{cccc} & & & \# \\ & & & \checkmark \end{array} \end{array} \Rightarrow \begin{array}{c c} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 1^* & 1 \\ 0 & 1^* & 0 & 0 \\ 1^* & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\ & \begin{array}{cccc} & & & \# \\ & & & \checkmark \end{array} \end{array} \quad D$$

Matching…

One Matching Algorithm…

Solution:…

$$\Longrightarrow \begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 1^* & 1 \\ 0 & 1^* & 0 & 0 \\ 1^* & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \\ & \begin{array}{cccc} & & 1 & \# \\ & & \checkmark & \checkmark \end{array} \end{array} \quad D \checkmark$$

$$\Longrightarrow \begin{array}{cc} & \begin{array}{cccc} A & B & C & D \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 0 & 1^* & 1 \\ 0 & 1^* & 0 & 0 \\ 1^* & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

$\Longrightarrow$ The maximum matching is (3,A), (2,B) and (1,C) ✓

Matching…

One Matching Algorithm…

- **Example:** In a business meeting, 6 speaking hours for 6 different speakers are desired to be programmed as 9,10,11,1,2 and 3 o'clock. The time limitations of these six speakers are as follows:
  - $S_1 \Longrightarrow$ "I will be able to speak only before noon"
  - $S_2 \Longrightarrow$ "I will be able to speak only at 9 and 2 o'clock"
  - $S_3 \Longrightarrow$ "I will not be able to speak only 9,11 and 2 o'clock "
  - $S_4 \Longrightarrow$ "I will not be able to speak until 2 o'clock"
  - $S_5 \Longrightarrow$ "I will not be able to speak from 10 to 3 o'clock"
  - $S_6 \Longrightarrow$ "I will not be able to speak from 10 to 2 o'clock "

  What is the most appropriate program that suits to the speakers?

Matching…

One Matching Algorithm…

Solution:

$$\begin{array}{c|c}
 & \begin{array}{cccccc} 9 & 10 & 11 & 1 & 2 & 3 \end{array} \\
\begin{array}{c} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}
\end{array}
\Longrightarrow
\begin{array}{ccc}
 & \begin{array}{cccccc} 9 & 10 & 11 & 1 & 2 & 3 \end{array} & \\
\begin{array}{c} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{array} & \begin{bmatrix} 1^* & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1^* & 0 \\ 0 & 1^* & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1^* \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} & \begin{array}{cc} 11 & \checkmark \\ 9 & \checkmark \\ 1 & \checkmark \\ & \\ 9 & ! \\ 9 & \end{array} \\
 & \begin{array}{cccccc} S_1 & S_3 & \# & \# & S_2 & \\ \checkmark & \checkmark & \checkmark & \checkmark & & \end{array} &
\end{array}$$

Matching…

➢ One Matching Algorithm…

➢ Solution:

$$
\begin{array}{rcl}
 & \begin{array}{cccccc} 9 & 10 & 11 & 1 & 2 & 3 \end{array} & \\
\begin{array}{r} S_1 \\ S_2 \\ S_3 \\ \Longrightarrow S_4 \\ S_5 \\ S_6 \end{array} &
\begin{bmatrix}
1 & 1 & 1^* & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1^* & 0 \\
0 & 1^* & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1^* \\
1^* & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 1
\end{bmatrix} &
\begin{array}{ll} 10 & \checkmark \\ 9 & \checkmark \\ 1 & \checkmark \\ & \\ & \\ & \end{array} \\
 & \begin{array}{cccccc} & S_3 & S_3 & \# & S_2 & \\ & \checkmark & & \checkmark & & \end{array} &
\end{array}
$$

$\Longrightarrow$ The most possible matching between speakers and hours is $(S_1 \to 11.00), (S_2 \to 14.00), (S_3 \to 10.00), (S_4 \to 15.00), (S_5 \to$

Matching…

Macar Algorithm

```
▪ Find the minimum-sum imdependent set with n elements for a matrix with nxn integers

STEP 1 ( Reduce the matrix )
   STEP 1.1 Subtract the smallest element value of each row from all elemants of that row.
   STEP 1.2 Subtract the smallest element value of each column from all elemants of that column
STEP 2 ( Find the maximum set of 0’s)
   STEP 2.1 Find the maximum independent set (S) for 0 elements of the matrix
STEP 3 ( Expand the independent set if |S|<n)
WHILE (|S|<n )
   STEP 3.1 Find the minimum covering of 0’s of the matrix
   STEP 3.2 Let k be the smallest of all these elements not in these covering
   STEP 3.3 Subtract k from all elemants that are not in the covering
   STEP 3.4 Add k to elemants located at the intersection of the covered rows and columns
   STEP 3.5 Place the new recalculated maximum independent set of 0’s in S
END WHILE
```

Matching…

Macar Algorithm…

- **Example:** Let's examine the most appropriate solution by considering the table given below for the working hours of four workers and four jobs.

$$\begin{array}{c|c} & \begin{matrix} I_1 & I_2 & I_3 & I_4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 3 & 6 & 3 & 5 \\ 7 & 3 & 5 & 8 \\ 5 & 2 & 8 & 6 \\ 8 & 3 & 6 & 4 \end{bmatrix} \end{array}$$

- **Solution:**

$$\begin{array}{cc} & \begin{matrix} I_1 & I_2 & I_3 & I_4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 3 & 6 & 3 & 5 \\ 7 & 3 & 5 & 8 \\ 5 & 2 & 8 & 6 \\ 8 & 3 & 6 & 4 \end{bmatrix} \end{array} \Longrightarrow \begin{array}{cc} & \begin{matrix} I_1 & I_2 & I_3 & I_4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 3 & 0 & 2 \\ 4 & 0 & 2 & 5 \\ 3 & 0 & 6 & 4 \\ 5 & 0 & 3 & 1 \end{bmatrix} \end{array} \Longrightarrow \begin{array}{cc} & \begin{matrix} I_1 & I_2 & I_3 & I_4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 3 & 0 & 1 \\ 4 & 0 & 2 & 4 \\ 3 & 0 & 6 & 3 \\ 5 & 0 & 3 & 0 \end{bmatrix} \end{array}$$

Matching...

- **Macar Algorithm...**
  - Solution:

$$\Rightarrow \begin{array}{ccc}
 & \begin{array}{cccc} I_1 & I_2 & I_3 & I_4 \end{array} & \\
\begin{array}{c}1\\2\\3\\4\end{array} & \begin{bmatrix} 0^* & 3 & 0 & 1 \\ 4 & 0^* & 2 & 4 \\ 3 & 0 & 6 & 3 \\ 5 & 0 & 3 & 0^* \end{bmatrix} & \begin{array}{c} I_3 \quad \checkmark \\ \\ \\ \\ \end{array} \\
 & \begin{array}{cccc} 1 & & \# & \\ \checkmark & & \checkmark & \end{array} &
\end{array}$$

$$|S| = 3 < 4 \quad \Rightarrow \quad \begin{array}{cc}
 & \begin{array}{cccc} I_1 & I_2 & I_3 & I_4 \end{array} \\
\begin{array}{c}1\\2\\3\\4\end{array} & \begin{bmatrix} 0^* & 3 & 0 & 1 \\ 4 & 0^* & 2 & 4 \\ 3 & 0 & 6 & 3 \\ 5 & 0 & 3 & 0^* \end{bmatrix}
\end{array}$$

Matching...

➢ Macar Algorithm...

➢ Solution:

$$k = 2 \quad \Longrightarrow \quad \begin{array}{c|c} & \begin{array}{cccc} I_1 & I_2 & I_3 & I_4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0^* & 5 & 0 & 3 \\ 2 & 0^* & 0 & 4 \\ 1 & 0 & 4 & 3 \\ 3 & 0 & 1 & 0^* \end{bmatrix} \\ & \begin{array}{cccc} 1 & 2 & \# & \\ \checkmark & \checkmark & \checkmark & \end{array} \end{array} \quad \begin{array}{cc} I_3 & \checkmark \\ I_3 & \checkmark \\ B & ! \\ B & \end{array} \Longrightarrow \begin{array}{c|c} & \begin{array}{cccc} I_1 & I_2 & I_3 & I_4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0^* & 5 & 0 & 3 \\ 2 & 0 & 0^* & 4 \\ 1 & 0^* & 4 & 3 \\ 3 & 0 & 1 & 0^* \end{bmatrix} \end{array}$$

Matching...

Macar Algorithm...

Solution:

$$|S| = 4 \Longrightarrow \begin{array}{c|cccc} & I_1 & I_2 & I_3 & I_4 \\ \hline 1 & 0^* & 5 & 0 & 3 \\ 2 & 2 & 0 & 0^* & 4 \\ 3 & 1 & 0^* & 4 & 3 \\ 4 & 3 & 0 & 1 & 0^* \end{array} \Longrightarrow \begin{array}{c|cccc} & I_1 & I_2 & I_3 & I_4 \\ \hline 1 & 3^* & 6 & 3 & 5 \\ 2 & 7 & 3 & 5^* & 8 \\ 3 & 5 & 2^* & 8 & 6 \\ 4 & 8 & 3 & 6 & 4 \end{array}$$

$\Longrightarrow$The total of minimum working hours is $3 + 2 + 5 + 4 = 14$ hours ✓

Matching...

Macar Algorithm...

- **Example:** Let's examine the most appropriate solution by considering the table given below for the working hours of five workers and four jobs.

$$\begin{array}{c c} & \begin{array}{ccccc} I_1 & I_2 & I_3 & I_4 & I_5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 3 & 6 & 3 & 5 & 3 \\ 7 & 3 & 5 & 8 & 5 \\ 5 & 2 & 8 & 6 & 2 \\ 8 & 3 & 6 & 4 & 4 \end{bmatrix} \end{array}$$

- **Solution:**

$$\begin{array}{c c} & \begin{array}{ccccc} I_1 & I_2 & I_3 & I_4 & I_5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 3 & 6 & 3 & 5 & 3 \\ 7 & 3 & 5 & 8 & 5 \\ 5 & 2 & 8 & 6 & 2 \\ 8 & 3 & 6 & 4 & 4 \end{bmatrix} \end{array} \Rightarrow \begin{array}{c c} & \begin{array}{ccccc} I_1 & I_2 & I_3 & I_4 & I_5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 3 & 6 & 3 & 5 & 3 \\ 7 & 3 & 5 & 8 & 5 \\ 5 & 2 & 8 & 6 & 2 \\ 8 & 3 & 6 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Matching…

- **Macar Algorithm…**
  - Solution:

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c} \begin{array}{ccccc} I_1 & I_2 & I_3 & I_4 & I_5 \end{array} \\ \begin{bmatrix} 3 & 6 & 3 & 5 & 3 \\ 7 & 3 & 5 & 8 & 5 \\ 5 & 2 & 8 & 6 & 2 \\ 8 & 3 & 6 & 4 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}
\Longrightarrow
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c} \begin{array}{ccccc} I_1 & I_2 & I_3 & I_4 & I_5 \end{array} \\ \begin{bmatrix} 0 & 3 & 0 & 2 & 0 \\ 4 & 0 & 2 & 5 & 2 \\ 3 & 0 & 6 & 4 & 0 \\ 5 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}
\Longrightarrow
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}
\begin{array}{c} \begin{array}{ccccc} I_1 & I_2 & I_3 & I_4 & I_5 \end{array} \\ \begin{bmatrix} 0^* & 3 & 0 & 2 & 0 \\ 4 & 0^* & 2 & 5 & 2 \\ 3 & 0 & 6 & 4 & 0^* \\ 5 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0^* & 0 & 0 \end{bmatrix} \end{array}$$

Matching...

- **Macar Algorithm...**
  - Solution:

$$\Rightarrow \begin{array}{c c} & \begin{array}{ccccc} I_1 & I_2 & I_3 & I_4 & I_5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0^* & 3 & 0 & 2 & 0 \\ 4 & 0^* & 2 & 5 & 2 \\ 3 & 0 & 6 & 4 & 0^* \\ 5 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0^* & 0 & 0 \end{bmatrix} \\ & \begin{array}{ccccc} 1 & & 1 & \# & \\ \checkmark & & \checkmark & \checkmark & \end{array} \end{array} \begin{array}{c} I_3 \ \checkmark \\ \\ \\ I_4 \ \checkmark \end{array}$$

$$|S| = 4 < 5 \Rightarrow \begin{array}{c c} & \begin{array}{ccccc} I_1 & I_2 & I_3 & I_4 & I_5 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} & \begin{bmatrix} 0^* & 3 & 0 & 2 & 0 \\ 4 & 0^* & 2 & 5 & 2 \\ 3 & 0 & 6 & 4 & 0^* \\ 5 & 0 & 3 & 1 & 1 \\ 0 & 0 & 0^* & 0 & 0 \end{bmatrix} \end{array}$$

Matching…

- **Macar Algorithm…**
  - Solution:

$$\Longrightarrow \quad k = 1 \quad \begin{array}{c|c} & \begin{matrix} I_1 & I_2 & I_3 & I_4 & I_5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0^* & 4 & 0 & 2 & 1 \\ 3 & 0^* & 1 & 4 & 2 \\ 2 & 0 & 5 & 3 & 0^* \\ 4 & 0 & 2 & 0 & 1 \\ 0 & 1 & 0^* & 0 & 1 \end{bmatrix} \\ & \begin{matrix} & & & \# & \\ & & & \checkmark & \end{matrix} \end{array} \begin{matrix} \\ \\ \\ I_4 \\ I_4 \end{matrix} \quad ! \quad \Longrightarrow \quad \begin{array}{c|c} & \begin{matrix} I_1 & I_2 & I_3 & I_4 & I_5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0^* & 4 & 0 & 2 & 1 \\ 3 & 0^* & 1 & 4 & 2 \\ 2 & 0 & 5 & 3 & 0^* \\ 4 & 0 & 2 & 0^* & 1 \\ 0 & 1 & 0^* & 0 & 1 \end{bmatrix} \end{array}$$

Matching…

Macar Algorithm…

Solution:

$$\Rightarrow |S| = 5 \quad \begin{array}{c|ccccc} & I_1 & I_2 & I_3 & I_4 & I_5 \\ \hline 1 & 0^* & 4 & 0 & 2 & 1 \\ 2 & 3 & 0^* & 1 & 4 & 2 \\ 3 & 2 & 0 & 5 & 3 & 0^* \\ 4 & 4 & 0 & 2 & 0^* & 1 \\ 5 & 0 & 1 & 0^* & 0 & 1 \end{array} \Rightarrow \begin{array}{c|ccccc} & I_1 & I_2 & I_3 & I_4 & I_5 \\ \hline 1 & 3^* & 6 & 3 & 5 & 3 \\ 2 & 7 & 3^* & 5 & 8 & 5 \\ 3 & 5 & 2 & 8 & 6 & 2^* \\ 4 & 8 & 3 & 6 & 4^* & 4 \end{array}$$

$\Rightarrow$The total of minimum working hours is $3 + 3 + 4 + 2 = 12$ hours ✓

References

- K.H. Rosen, Discrete Mathematics and Its Applications, Seventh Edition, Mc Graw Hill, 2012.
- R.P. Grimaldi, Discrete and Combinatorial Mathematcics, An Applied Introduction, Fifth Edition, Pearson, 2003.
- S.S. Epp, Discrete Mathematics with Applications, Fouth Edition, 2010.
- N. Yurtay, "Ayrık İşlemsel Yapılar" Lecture Notes, Sakarya University.