Discrete Mathematics

LECTURE 9

Relation

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Outline

- Relations
  - Binary Relations
  - Properties on Relations
  - n-ary Relations
  - Representing Relations
  - Equivalence Relations
  - Partial Orderings
- References

Relations

- The most direct way to express a relationship between elements of two sets is to use ordered pairs made up of two related elements.
- sets of ordered pairs are called **binary relations**.

- A binary relation from a set A *to* a set B is a subset of $A \times B$.
- Generalley denoted with R

Relations

- $R \subset A \times B \longrightarrow R = \{(x, y) | (x, y) \in A \times B\}$

- $R^{-1} \subset B \times A \longrightarrow R^{-1} = \{(y, x) | (y, x) \in B \times A\}$
  $= \{(y, x) | (x, y) \in R\}$

- if $s(A) = m$ and $s(A) = n$, then $2^{m.n}$ different relations will be declared

Relations

- **Binary Relations**
  - represent relationships between the elements of two sets.
  - A binary relation from a set *A to* a set *B*
    - a subset of $A \times B$.
    - a set *R* of ordered pairs where the first element of each ordered pair comes from *A* and the second element comes from *B*
  - $a\mathcal{R}b$ to denote $(a, b) \in R$
  - $a\not{\mathcal{R}}b$ to denote $(a, b) \notin R$
  - when *(a, b)* belongs to *R*, *a* is said to be **related to** *b* by *R*.

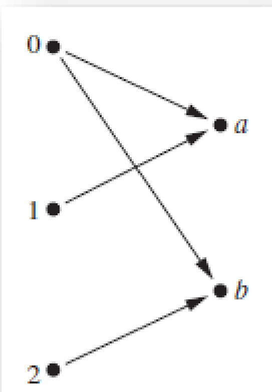
Relations

- **Binary Relations…**
  - Example: Let *A* = {0, 1, 2} and *B* = {*a*, *b*}. Write an example binary relation from *A* to *B*. Then {*(0, a), (0, b), (1, a), (2, b)*} is a relation from *A* to *B*.
  - Solution:

$$R_1 = \{(0, a), (0, b), (1, a), (2, b)\} \checkmark$$

Relations

- **Binary Relations…**
  - Example: Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Represent $\{(0,a),(0,b),(1,a),(2,b)\}$ graphically.
  - Solution:



Relations

- **Binary Relations…**
  - Example: Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$. Represent $\{(0,a),(0,b),(1,a),(2,b)\}$ with a table.

  - Solution:

R	a	b
0	×	×
1	×	
2		×

Relations

- **Binary Relations…**
  - A relation on a set A
    - a relation from A to A.
    - a subset of AxA

Relations

- **Binary Relations…**
  - Example: Let A be the set $\{1,2,3,4\}$. Which ordered pairs are in the relation $R = \{(a,b) | a \text{ divides } b\}$?
  - Solution:

$$R = \{(1,1), (1,2), (1,3), (1,4), (2,2), (2,4), (3,3), (4,4)\} \checkmark$$

Relations

- **Binary Relations…**
  - Example: Consider these relations on the set of integers:

$$R_1 = \{(a,b) | a \leq b\}$$
$$R_2 = \{(a,b) | a = b \; or \; a = -b\}$$
$$R_3 = \{(a,b) | a + b \leq 3\}$$

  - Solution:

$$R_1 = \{\ldots, (-1,0), (0,0), (0,1), (0,2), \ldots, (1,1), (1,2), \ldots\}$$
$$R_2 = \{\ldots, (-1,-1), (-1,1), (0,0), (1,1), (2,2), \ldots\}$$
$$R_3 = \{\ldots, (-1,1), (-1,0), (-1,0), \ldots, (1,1), (2,0), (2,1)\} \checkmark$$

Relations

- **Properties on Relations**
  - There are several properties that are used to classify relations on a set.
  - most important of these are
    - reflective
    - symmetric
    - antisymmetric
    - transitive

Relations

- **Properties on Relations…**
  - A relation R on a set A is called **reflexive** if $\forall a \in A\ (a, a) \in R$.
  - In other words, a relation on A is reflexive if every element of A is related to itself.

Relations

Properties on Relations…

Example: Consider these relations on the set of integers:

$$R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$$
$$R_2 = \{(1,1),(1,2),(2,1)\}$$
$$R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$$
$$R_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$$
$$R_5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$$
$$R_6 = \{(3,4)\}$$

Which of these relations are reflexive?

Solution:

relations containing $(a, a) \Rightarrow R_3$ and R_5 ✓

Relations

- **Properties on Relations…**
  - Example: Is the “divides” relation on the set of positive integers reflexive?

  - Solution:

    $a|a$ whenever a is a positive integer ⇒ YES ✓

Relations

- **Properties on Relations…**
  - A relation R on a set A is called **symmetric** if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.
  - if R is symmetric then $R = R^{-1}$.

  - A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$ then $a = b$ is called **antisymmetric**.
  - in other words, if $\forall(a, b) \in R$ then $\forall(b, a) \notin R$ whenever $a \neq b$.

  - The terms symmetric and antisymmetric are not opposites

Relations

Properties on Relations…

Example: Consider these relations on the set of integers:

$R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\}$
$R_2 = \{(1,1),(1,2),(2,1)\}$
$R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$
$R_4 = \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}$
$R_5 = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}$
$R_6 = \{(3,4)\}$

Which of these relations are symmetric and which are antisymmetric?

Solution:

- R_2 and R_3 are symmetric
- R_4, R_5 and R_6 are antisymmetric ✓

Relations

- **Properties on Relations…**
  - Example: Is the "divides" relation on the set of positive integers symmetric? Is it antisymmetric?
  - Solution:

$1|2$ but $2|1 \Rightarrow$ symmetric
$a|b$ and $b|a$ then $a = b \Rightarrow$ antisymmetric ✓

Relations

- **Properties on Relations…**
  - A relation R on a set A is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$, for all $a, b, c \in A$.

Relations

- **Properties on Relations…**
  - Example: Is the “divides” relation on the set of positive integers transitive?
  - Solution:

If $a|b$ and $b|c$ then $a|c \Rightarrow$ transitive ✓

Relations

- **n-ary Relations**
  - Relationships among elements of more than two sets
  - Let $A_1, A_2, \ldots, A_n$ be sets. An n-ary relation on these sets is a subset of $A_1 \times A_2 \times \cdots \times A_n$.
    - sets $A_1, A_2, \ldots, A_n$ are called the **domains** of the relation
    - *n* is called its **degree**

Relations

- **n-ary Relations…**
  - Example: Let *R* be the relation on $Z \times Z \times Z^{+}$ consisting of triples (a, b, m) where a, b and m are integers with $m \geq 1$ and $a \equiv b \pmod{m}$. Give some example triples that belongs to R.

  - Solution:

$(8,2,3) \Longrightarrow 8 \equiv 2 \pmod{3}$ ✓
$(-1,4,5) \Longrightarrow -1 + 5 * 1 \equiv 4 \pmod{5}$ ✓
$(14,0,7) \Longrightarrow 14 \equiv 0 \pmod{7}$ ✓

Relations

- **n-ary Relations…**
  - Example: Let R be the relation consisting of 5-tuples (A,N, S,D, T) representing airplane flights, where A is the airline, N is the flight number, S is the starting point, D is the destination, and T is the departure time. For instance, if Nadir Express Airlines has flight 963 from Newark to Bangor at 15:00, then (Nadir, 963, Newark, Bangor, 15:00) belongs to R. The degree of this relation is 5, and its domains are the set of all airlines, the set of flight numbers, the set of cities, the set of cities (again), and the set of times.

Relations

- **Representing Relations**
  - a relation between finite sets can be represented using a zero–one matrix.
  - suppose that *R* is a relation from $A = \{a_1, a_2, \ldots, a_m\}$ to $B = \{b_1, b_2, \ldots, b_n\}$.
  - when $A = B$ we use the same ordering for *A* and *B*.
  - the relation *R* can be represented by the matrix $M_R = [m_{ij}]$,

    where $m_{ij} = \begin{cases} 1 \; if \; (a_i, b_j) \in R \\ 0 \; if \; (a_i, b_j) \notin R \end{cases}$

Representing Relations…

- **Example:** Suppose that $A = \{1,2,3\}$ and $B = \{1,2\}$. Let R be the relation from A to B containing (a, b) if $a \in A, b \in B$ and $a > b$. What is the matrix representing R?

- **Solution:**

$$R = \{(2,1), (3,1), (3,2)\}$$

$$M = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \checkmark$$

Relations

Representing Relations…

- Example: Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3, b_4, b_5\}$. Which ordered pair are in the relation R represented by the matrix
- $M = \begin{bmatrix} 01000 \\ 10110 \\ 10101 \end{bmatrix}$?

- Solution:

$R = \{(a_1, b_2), (a_2, b_1), (a_2, b_3), (a_2, b_4), (a_3, b_1), (a_3, b_3), (a_3, b_5)\}$ ✓

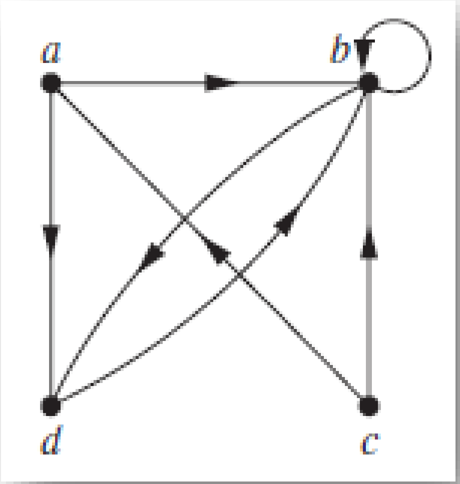
Relations

- **Representing Relations**
  - Another way of representing relations on a finite set
  - Each element of the set by a point
  - Each ordered pair is represented using an arc/line with its direction indicated by an arrow
  - This type of representation is called **directed graphs** of **digraphs**
    - consists of a set *V* of *vertices* (or *nodes*) together with a set *E* of ordered pairs of elements of *V* called *edges* (or *arcs*).
    - the vertex *a* is called the *initial vertex* of the edge *(a, b)*, and the vertex *b* is called the *terminal vertex* of this edge.
    - an edge of the form *(a, a)* is represented using an arc from the vertex *a* back to itself. such an edge is called a **loop**.

Relations

- **Representing Relations…**
  - Example: Draw a directed graph with vertices a, b, c, d and edges $(a, b), (a, d)$, $(b, b), (b, d)$, $(c, a), (c, b)$ and (d, b).

  - Solution:



Relations

- **Equivalence Relations**
  - a relation on a set A is called an **equivalence relation** if it is **reflexive**, **symmetric**, and **transitive**.
  - two elements *a* and *b* that are related by an equivalence relation are called **equivalent**.
  - the notation $a \sim b$ is often used to denote that *a* and *b* are equivalent elements with respect to a particular equivalence relation.

Relations

- **Equivalence Relations…**
  - Example: Let R be the relation on the set of real numbers such that aRb if and only if $a - b$ is an integer. Is R an equivalence relation?
  - Solution:

- $a - a = 0$ is an integer $\forall a \in R \Longrightarrow$ reflective
- if $a - b$ is an integer then $b - a$ is also an integer $\Longrightarrow$ symmetric
- if $a - b$ and $b - c$ are integers then $a - c = a - b - (b - c)$ is also an integer $\Longrightarrow$ transitive

reflective + symmetric + transitive $\Longrightarrow$ Equivalence relation ✓

Relations

- **Partial Orderings**
  - a relation R on a set S is called a **partial ordering** if it is reflexive, antisymmetric, and transitive.
  - members of S are called **elements** of the partially ordered set

Relations

➢ Partial Orderings…

➢ Example: Show that the "greater than or equal" relation (≥) is a partial ordering on the set of integers.

➢ Solution:

- $a \geq a$ is an integer $\forall a \in Z \implies$ reflective
- if $a \geq b$ and $b \geq a$ then $a = b \implies$ antisymmetric
- if $a \geq b$ and $b \geq c$ then $a \geq b \geq c \Longrightarrow a \geq c \implies$ transitive

reflective + antisymmetric + transitive ⟹ ≥ is a partial relation on Z ✓

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