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Discrete Mathematics
LECTURE 8
Counting

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Outline

- Counting
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Counting

- we must count objects to solve many different types of problems.
- counting problems arise throughout mathematics and computer science.

Counting…

- **The Product Rule**
  - one of the basic counting principles
  - used when a procedure can be broken down into a sequence of two tasks.
    - if there are n_1 ways to do the first task and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 . n_2$ ways to do the procedure.
  - an extended version
    - suppose that a procedure is carried out by performing the tasks $T_1, T_2, \dots, T_m$in sequence.
    - if each task $T_i, i = 1,2, \dots, n$ can be done in n_i ways, regardless of how the previous tasks were done, then there are $n_1 . n_2 \dots . n_m$ ways to carry out the procedure

Counting…

- **The Product Rule…**
  - Example: A new company with just two employees, Ahmet and Berk, rents a floor of a building with 12 offices. How many ways are there to assign different offices to these two employees?

  - Solution:

    12*11=132 different ways ✓

Counting…

- **The Product Rule…**
  - Example: The chairs of an auditorium are to be labeled with an uppercase English letter followed by a positive integer not exceeding 100. What is the largest number of chairs that can be labeled differently?

  - Solution:

    26*100=2600 different ways

Counting...

- **The Product Rule...**
  - Example: There are 32 microcomputers in a computer center. Each microcomputer has 24 ports. How many different ports to a microcomputer in the center are there?
  - Solution:

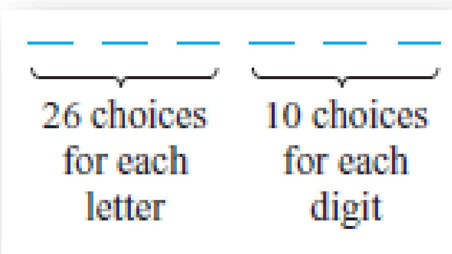
    32*24=768 ports

Counting...

- **The Product Rule...**
  - Example: How many different license plates can be made if each plate contains a sequence of three uppercase English letters followed by three digits?

  - Solution:

    26*26*26*10*10*10=17.576.000 possible license plates



Counting…

- **The Sum Rule**
  - one of the basic counting principles
  - used when a task can be done either in one of n_1 ways or in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task
  - an extended version
    - suppose that a task can be done in one of n_1 ways, in one of n_2 ways, . . . , or in one of n_m ways, where none of the set of n_i ways of doing the task is the same as any of the set of n_j ways, for all pairs *i* and *j* with $1 \leq i < j \leq m$.
    - then the number of ways to do the task is $n_1 + n_2 + \cdots n_m$.

Counting…

The Sum Rule…

- **Example:** A student can choose a computer project from one of three lists. The three lists contain 23, 15, and 19 possible projects, respectively. No project is on more than one list. How many possible projects are there to choose from?

- **Solution:**

  $23 + 15 + 19 = 57$ ways to choose a project.

Counting…

- **Example:** Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?

- **Solution:**

  - total number of digits and lettes $\Longrightarrow$ 36
  - $\mathrm{P}_6 \Longrightarrow$ number of possible passwords with 6 digits
  - $\mathrm{P}_7 \Longrightarrow$ number of possible passwords with 6 digits
  - $\mathrm{P}_8 \Longrightarrow$ number of possible passwords with 6 digits

$$P = P_6 + P_7 + P_8 = (36^6 - 26^6) + (36^7 - 26^7) + (36^8 - 26^8) \checkmark$$

Counting…

- **The Subtraction Rule**
  - If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.
  - Principle of inclusion–exclusion
    - $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

Counting…

- **The Subtraction Rule…**
  - Example: How many bit strings of length eight either start with a 1 bit or end with the two 0 bits?

  - Solution:

- $1XXXXXXX \Longrightarrow$ a byte string starting with a 1 bit $\Longrightarrow 2^7 = 128$
- $XXXXXX00 \Longrightarrow$ a byte string ending with 0 bits $\Longrightarrow 2^6 = 64$
- $1XXXXX00 \Longrightarrow$ a byte string starting with a 1 bit and ending with two 0 bits $\Longrightarrow 2^5 = 32$

$\Longrightarrow 128 + 64 - 32 = 160$ possible bit strings ✓

Counting…

- **The Pigeonhole Principle**
  - a general principle which states that if there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it
  - if k is a positive integer and $k + 1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.
  - Generalized Principle
    - if N objects are placed into k boxes, then there is at least one box containing at least $[N/k]$ objects

Counting…

- **The Pigeonhole Principle…**
  - Example: How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

  - Solution:

    There are 101 possible scores on the final

    $\Longrightarrow 101 + 1 = 102$ students ✓

Counting...

- **The Pigeonhole Principle...**
  - Example: Among 100 people, at least how many of them will born in the same month?
  - Solution:

$$\Longrightarrow \frac{100}{12} = 8,\overline{3} \Longrightarrow 9 \text{ people} \checkmark$$

Counting...

- **The Pigeonhole Principle...**
  - Example: What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D and F?

  - Solution:

$$\lceil N/5 \rceil = 6 \Longrightarrow 5x5 + 1 = 26 \text{ students} ✓$$

Counting...

The Pigeonhole Principle...

- Example: What is the least number of area codes needed to guarantee that the 25 million phones in a state can be assigned distinct 10-digit telephone numbers?

  (Assume that telephone numbers are of the form $NXX - NXX - XXXX$, where the first three digits form the area code, N represents a digit from 2 to 9 inclusive, and X represents any digit.)

- Solution:

  $NXX - XXXX \Longrightarrow 8x10^6 = 8000000$
  $[25000000/8000000] = 3{,}125 \Longrightarrow 4$ phone numbers
  At least 4 different area codes needed ✓

Counting…

- **Permutations**
  - A permutation of a set of distinct objects is an ordered arrangemenet of these objects
  - if *n* and *r* integers with $0 \leq r \leq n$, the r-permutation of *P*
    - $P(n,r) = \frac{n!}{(n-r)!}$
  - The number of r-permutations of a set of n objects with **repetition** allowed
    - $P(n,r) = n^r$

Counting…

- **Permutations…**
  - Example: Let $S = \{a, b, c\}$. What are the 2-permuations of S?
  - Solution:

$P(3,2) = 6 \Longrightarrow$a,b; a,c; b,a; b,c; c,a; c,b ✓

Counting…

- **Permutations…**
  - Example: How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different people who have entered a contest?

  - Solution:

$$P(100,3) = 100.99.98 = 970200 \checkmark$$

Counting…

Permutations…

- Example: How many permutations of the letters *ABCDEFGH* contain the string *ABC*?

- Solution:

  ABC is a block, *D*, *E*, *F*, *G* and *H* are individual letters $\Longrightarrow$ 6 elements
  $6! = 720$ permutations ✓

Counting…

- **Permutations…**
  - Example: How many strings of length r can be formed from the uppercase letters of the English alphabet?
  - Solution:

    26^r strings ✓

Counting…

- **Combinations**
  - A **combination** of elements of a set in an unordered selection of r elements from the set
  - if *n* and *r* integers with $0 \leq r \leq n$, the r-combination
    - $\mathrm{C}(n, r) = \frac{n!}{r!.(n-r)!}$
    - a subset of the set n with r elements
    - also denoted by $\binom{n}{r}$ and called a binomial coefficient

  - The number of r-combinations from a set with n elements with **repetition** of elements is allowed
    - $C(n+r-1, r) = C(n+r-1, n-1)$

Counting...

- **Combinations...**
  - Example: How many ways are there to select five players from a 10-member tennis team to make a trip to a match at another school?
  - Solution:

$$C(10,5) = \frac{10!}{5!5!} = 252 \checkmark$$

Counting...

- **Combinations...**
  - Example: Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

  - Solution:

$$C(9,3).C(11,4) = \frac{9!}{3!6!}.\frac{11!}{4!7!} = 27720 \checkmark$$

Counting…

- **Combinations…**
  - Example: How many ways are there to choose six cookies of a set with four different types of cookies?
  - Solution:

$$C(9,6) = C(9,3) = \frac{9.8.7.6!}{6!.3.2.1} = 84 \checkmark$$

Counting…

- **Combinations…**
  - Example: How many solutions does the equation
    $x_1 + x_2 + x_3 = 11$
    have, where x_1, x_2, and x_3 are nonnegative integers?

  - Solution:

    - 11 items from a set with three elements
    - x_1 items of type one, x_2 items of type two, and x_3 items of type three

$C(3+11-1,11) = C(13,11) = \frac{13.12.11!}{11!.2.1} = 78$ ✓

Counting...

Combinations...

Example: How many solutions does the equation

$x_1 + x_2 + x_3 = 11$

have, where $x_1 \geq 1$, $x_2 \geq 2$, and $x_3 \geq 3$?

Solution:

- a set with three elements
- at least one item of type one, two items of type two and three items of type three

$C(3+5-1,5) = C(7,5) = \frac{7.6.5!}{2!.5!} = 21$ ✓

Counting…

- **Combinations…**
  - Example: How many different strings can be made by reordering the letters of the word SUCCESS?

  - Solution:

- The word consists of 7 letters $\Longrightarrow$ 3 S, 1 U, 2 C and 1 E

$$\binom{7}{3}\cdot\binom{4}{1}\cdot\binom{3}{2}\cdot\binom{1}{1} = \frac{7!}{4!.3!}\cdot\frac{4!}{3!.1!}\cdot\frac{3!}{2!.1!} = \frac{7!}{4!.1!.2!.1!} \checkmark$$

Counting…

- **Binomial Theorem**
  - an expression such as $(x + y)^n$
  - $(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$

    $$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$
  - $\binom{n}{r}$ are binomial coefficients

Counting…

- **Binomial Theorem…**
  - Example: What is the expansion of $(x + y)^4$?
  - Solution:

$$(x+y)^4 = \sum_{j=0}^{4} \binom{4}{j} x^{4-j} y^j$$
$$= \binom{4}{0} x^4 + \binom{4}{1} x^3 y + \binom{4}{2} x^2 y^2 + \binom{4}{3} x y^3 + \binom{4}{4} y^4$$
$$= x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + y^4 \checkmark$$

Counting…

Binomial Theorem…

- **Example:** What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

- **Solution:**

$$(x+y)^{25} = \sum_{j=0}^{25} \binom{25}{j} x^{25-j} y^{j}$$

$$= \binom{25}{0} x^{25} + \binom{25}{1} x^{24} y + \cdots + \binom{25}{r} x^{25-r} y^{r} + \cdots$$

$$\Longrightarrow \binom{25}{13} x^{12} y^{13} \Longrightarrow \frac{25!}{12!13!} = 5200300 \checkmark$$

Counting...

Binomial Theorem...

- **Example:** What is the coefficient of $x^{12}y^{13}$ in the expansion of $(2x-3y)^{25}$?

- **Solution:**

$$(2x-3y)^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j}(-3y)^j$$

$$= \binom{25}{0}(2x)^{25} + \cdots + \binom{25}{r}(2x)^{25-r}(-3y)^r + \cdots$$

$$\Longrightarrow \binom{25}{13}(2x)^{12}(-3y)^{13} \Longrightarrow -\frac{25!}{12!13!}.\,2^{12}.\,3^{13} \checkmark$$

Counting…

Binomial Theorem…

- Example: What is the coefficient of $a^3b^2c^6d^4$ in the expansion of $(a+b+c+d)^{15}$?

- Solution:

$$(a+b+c+d)^{15} = \big((a+b)+(c+d)\big)^{15}$$

$$= \sum_{j=0}^{15}\binom{15}{j}(a+b)^{15-j}(c+d)^{j}$$

$$= \binom{15}{0}(a+b)^{15} + \cdots + \binom{15}{10}(a+b)^{5}(c+d)^{10} + \cdots$$

$$\Rightarrow \binom{15}{10}(a+b)^{5}(c+d)^{10} = \binom{15}{10}\left(\ldots + \binom{5}{2}a^3b^2 + \cdots\right)\left(\ldots + \binom{10}{4}c^6d^4 + \cdots\right)$$

$$\Rightarrow \binom{15}{10}.\binom{5}{2}.\binom{10}{4} = \frac{15!}{10!5!}.\frac{5!}{2!3!}.\frac{10!}{6!4!} = \frac{15!}{2!3!6!4!} \checkmark$$

Counting…

- **Pascal's Identity**
  - one of the most important identities that binomial coefficients satisfy
  - Let *n* and *k* be positive integers with $n \geq k$
    - $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

Counting...

Pascal's Triangle

n=0 $\binom{0}{0}$

n=1 $\binom{1}{0}$ $\binom{1}{1}$

n=2 $\binom{2}{0}$ $\binom{2}{1}$ $\binom{2}{2}$

n=3 $\binom{3}{0}$ $\binom{3}{1}$ $\binom{3}{2}$ $\binom{3}{3}$

n=4 $\binom{4}{0}$ $\binom{4}{1}$ $\binom{4}{2}$ $\binom{4}{3}$ $\binom{4}{4}$

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1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

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