Fınd K'st Smallest Element / Order Istatistic

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Order Statistics:

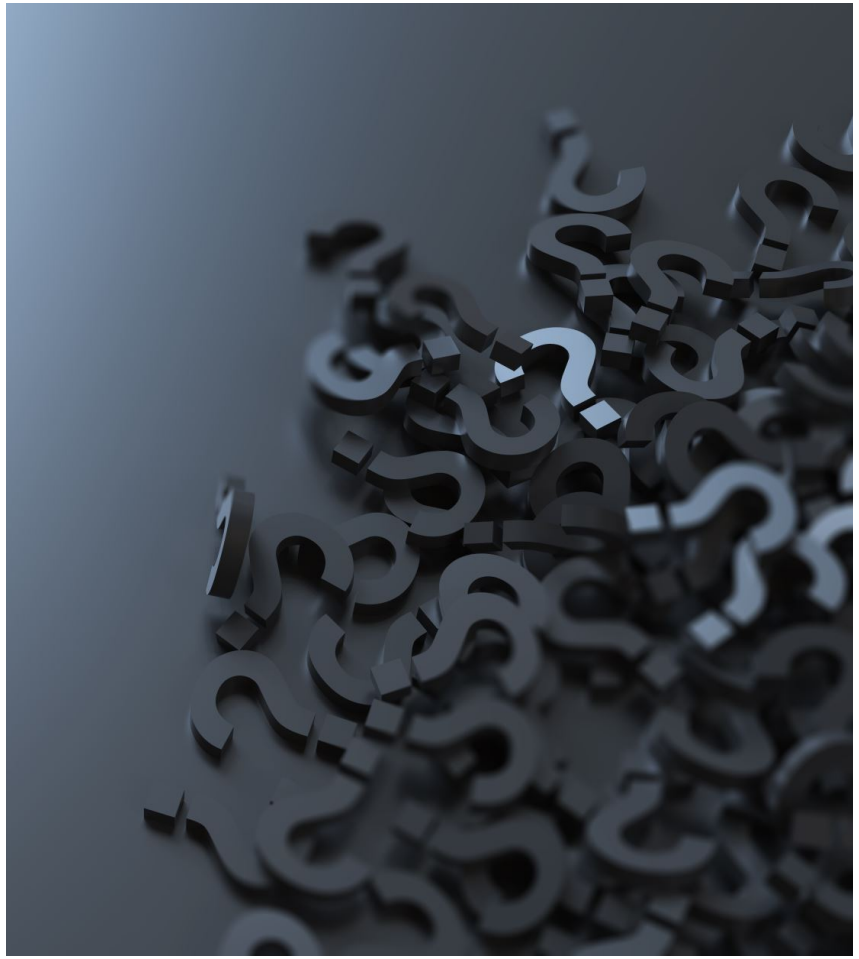
Randomized Divide And Conquer

Analysis Of Expected Time

Worst-case Linear-time Order Statistics

Analysis

Order statistics



- Select the ith smallest of n elements (the element with rank i).
  - i = 1: minimum;
  - i = n: maximum;
  - i = $\lfloor (n+1)/2 \rfloor$ or $\lceil (n+1)/2 \rceil$: median.
- **Naive algorithm:**

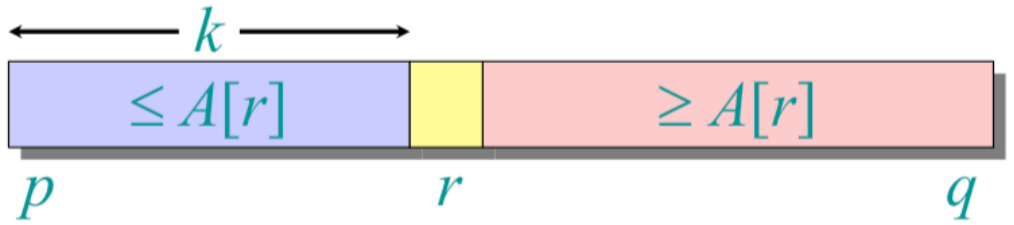
Sort and index ith element.

Worst-case running time = $\Theta(n \lg n) + \Theta(1) =$ **$\Theta(n \lg n)$,**

using merge sort or heapsort (not quicksort).

Randomized Divide-and-Conquer Algorithm

```
RAND-SELECT(A, p, q, i)        ▷ ith smallest of A[p..q]
    if p = q then return A[p]
    r ← RAND-PARTITION(A, p, q)
    k ← r − p + 1              ▷ k = rank(A[r])
    if i = k then return A[r]
    if i < k
        then return RAND-SELECT(A, p, r − 1, i)
        else return RAND-SELECT(A, r + 1, q, i − k)
```



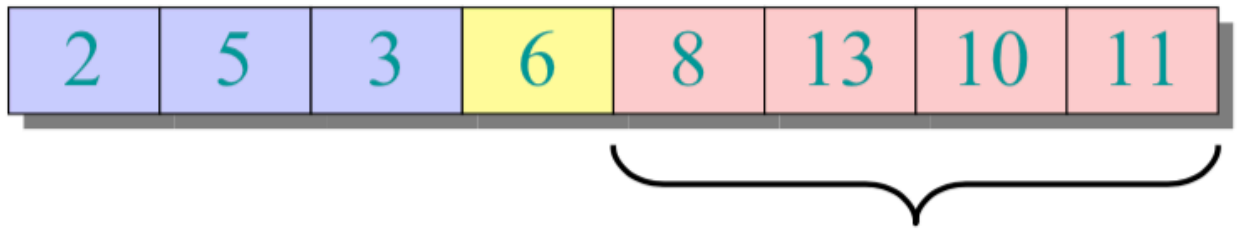
Example:

Select the $i = 7$th smallest:

$i = 7$

pivot

Partition:



$k = 4$

Select the $7 - 4 = 3$rd smallest recursively.

Intuition For Analysis

(All our analyses today assume that all elements are distinct.)

Lucky:

$$T(n) = T(9n/10) + \Theta(n)$$
$$= \Theta(n)$$

$n^{\log_{10/9} 1} = n^0 = 1$

CASE 3

Unlucky:

$$T(n) = T(n-1) + \Theta(n)$$
$$= \Theta(n^2)$$

arithmetic series

Worse than sorting!

Remember:

- **Time Complexity: Q(n^2)**

Sum Of An Arithmetic Series

$$S_n = a + a + d + a + 2d + \ldots + l - 2d + l - d + l$$
$$S_n = l + l - d + l - 2d + \ldots + a + 2d + a + d + a$$
$$2S_n = a + l + a + l + a + l + \ldots + a + l + a + l + a + l$$
$$= n(a + l)$$

$$S_n = \frac{n}{2}(a + l)$$ if we know l

otherwise;

$$S_n = \frac{n}{2}\{a + a + (n-1)d\}$$

$$S_n = \frac{n}{2}\{2a + (n-1)d\}$$

Analysis Of Expected Time

- The analysis follows that of randomized quicksort, but it's a little different.

- Let T(n) = the random variable for the running time of **RAND-SELECT** on an input of size n, assuming random numbers are independent.

- For **k = 0, 1, …, n–1**, define **the indicator random variable**

$$X_k = \begin{cases} 1 & \text{if PARTITION generates a } k : n-k-1 \text{ split,} \\ 0 & \text{otherwise.} \end{cases}$$

Analysis (continued)

- To obtain an upper bound, assume that the ith element always falls in the larger side of the partition:

$$T(n) = \begin{cases} T(\max\{0, n-1\}) + \Theta(n) & \text{if } 0 : n-1 \text{ split}, \\ T(\max\{1, n-2\}) + \Theta(n) & \text{if } 1 : n-2 \text{ split}, \\ \quad\vdots & \\ T(\max\{n-1, 0\}) + \Theta(n) & \text{if } n-1 : 0 \text{ split}, \end{cases}$$

$$= \sum_{k=0}^{n-1} X_k \left(T(\max\{k, n-k-1\}) + \Theta(n)\right).$$

Calculating Expectation

- Take expectations of both sides.

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(\max\{k, n-k-1\}) + \Theta(n)\right)\right]$$

Calculating Expectation

$$E[T(n)] = E\left[\sum_{k=0}^{n-1} X_k \left(T(\max\{k, n-k-1\}) + \Theta(n)\right)\right]$$

$$= \sum_{k=0}^{n-1} E\left[X_k \left(T(\max\{k, n-k-1\}) + \Theta(n)\right)\right]$$

Linearity of expectation.

Calculating Expectation

$$
\begin{aligned}
E[T(n)] &= E\left[\sum_{k=0}^{n-1} X_k \left(T(\max\{k, n-k-1\}) + \Theta(n)\right)\right] \\
&= \sum_{k=0}^{n-1} E\left[X_k \left(T(\max\{k, n-k-1\}) + \Theta(n)\right)\right] \\
&= \sum_{k=0}^{n-1} E[X_k] \cdot E\left[T(\max\{k, n-k-1\}) + \Theta(n)\right]
\end{aligned}
$$

Independence of X_k from other random choices.

Calculating Expectation

$$
\begin{aligned}
E[T(n)] &= E\left[\sum_{k=0}^{n-1} X_k \left(T(\max\{k, n-k-1\}) + \Theta(n)\right)\right] \\
&= \sum_{k=0}^{n-1} E\left[X_k \left(T(\max\{k, n-k-1\}) + \Theta(n)\right)\right] \\
&= \sum_{k=0}^{n-1} E[X_k] \cdot E\left[T(\max\{k, n-k-1\}) + \Theta(n)\right] \\
&= \frac{1}{n} \sum_{k=0}^{n-1} E\left[T(\max\{k, n-k-1\})\right] + \frac{1}{n} \sum_{k=0}^{n-1} \Theta(n)
\end{aligned}
$$

Linearity of expectation; $E[X_k] = 1/n$.

Calculating Expectation

$$
\begin{aligned}
E[T(n)] &= E\left[\sum_{k=0}^{n-1} X_k \left(T(\max\{k, n-k-1\}) + \Theta(n)\right)\right] \\
&= \sum_{k=0}^{n-1} E\left[X_k \left(T(\max\{k, n-k-1\}) + \Theta(n)\right)\right] \\
&= \sum_{k=0}^{n-1} E[X_k] \cdot E\left[T(\max\{k, n-k-1\}) + \Theta(n)\right] \\
&= \frac{1}{n} \sum_{k=0}^{n-1} E\left[T(\max\{k, n-k-1\})\right] + \frac{1}{n} \sum_{k=0}^{n-1} \Theta(n) \\
&\le \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} E[T(k)] + \Theta(n)
\end{aligned}
$$

Upper terms appear twice.

Hairy Recurrence

- (But not quite as hairy as the quicksort one.)

$$E[T(n)] = \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} E[T(k)] + \Theta(n)$$

Prove: $E[T(n)] \le cn$ for constant $c > 0$.

The constant c can be chosen large enough so that E[T(n)] ≤ cn for the base cases.

Substitution Method

Substitution Method

$$E[T(n)] \le \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} ck + \Theta(n)$$

Substitute inductive hypothesis.

Use fact: $\sum_{k=\lfloor n/2 \rfloor}^{n-1} k \le \frac{3}{8} n^2$ (exercise).

Substitution Method

- if c is chosen large enough so that cn/4 dominates the Θ(n)!

$$
\begin{aligned}
E[T(n)] &\le \frac{2}{n} \sum_{k=\lfloor n/2 \rfloor}^{n-1} ck + \Theta(n) \\
&\le \frac{2c}{n}\left(\frac{3}{8}n^2\right) + \Theta(n) \\
&= cn - \left(\frac{cn}{4} - \Theta(n)\right) \\
&\le cn ,
\end{aligned}
$$

SUMMARY OF RANDOMİZED ORDER-STATISTIC SELECTİON

- Works fast: linear expected time.

- Excellent algorithm in practice.

- But, the worst case is very bad: Θ(n^2).

Worst-case Linear-time Order Statistics

SELECT(i, n)

1. Divide the n elements into groups of **5.** Find the median of each 5-element group by rote.

2. Recursively SELECT the median x of the $\lfloor n/5 \rfloor$ group medians to be the pivot.

3. Partition around the pivot x. Let **k = rank(x).**

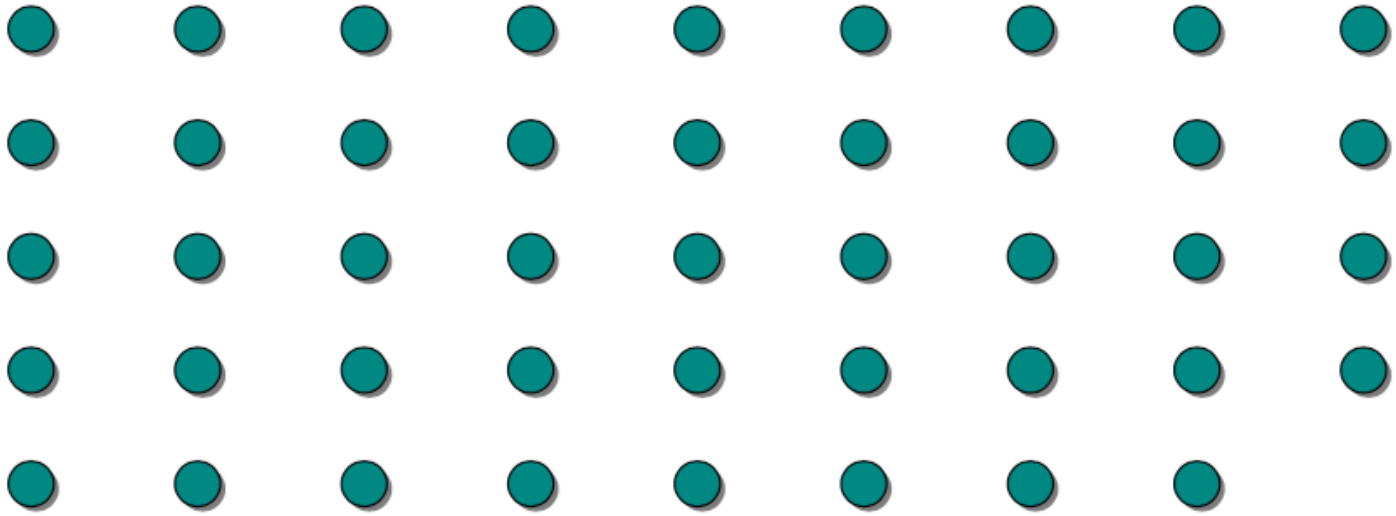
4.if i=k **then return** x

elseif i<k

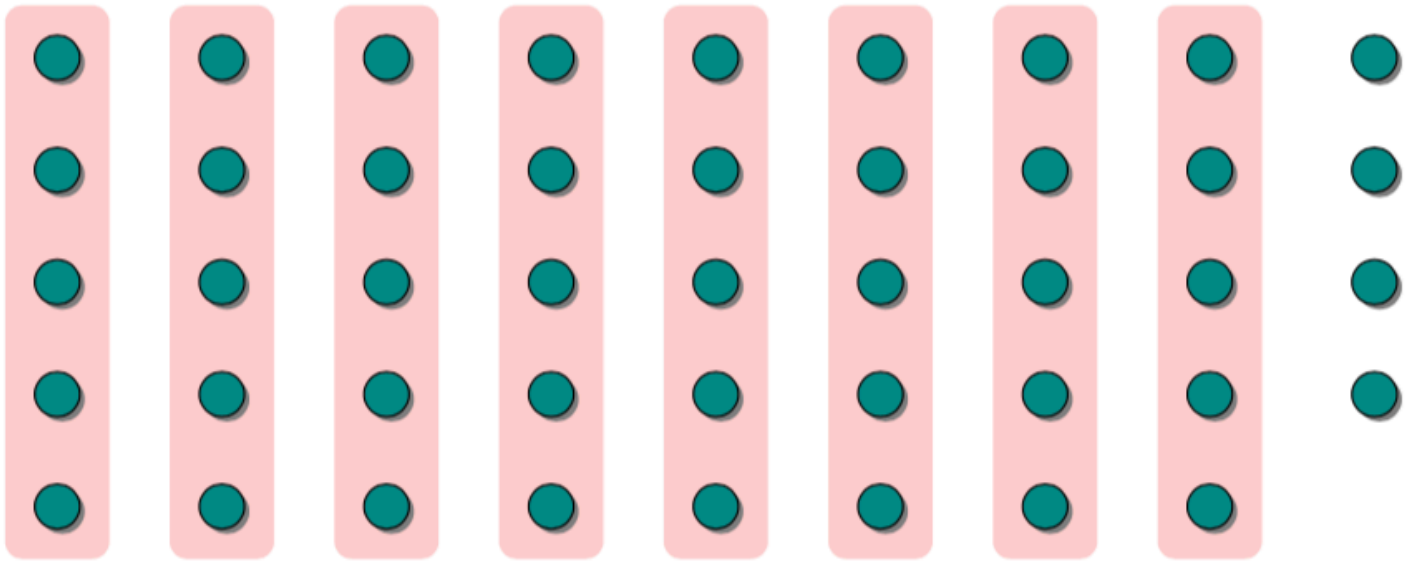
then recursively SELECT the **ith** smallest element in the lower part

else recursively SELECT the **(i–k)th** smallest element in the upper part

Choosing The Pivot

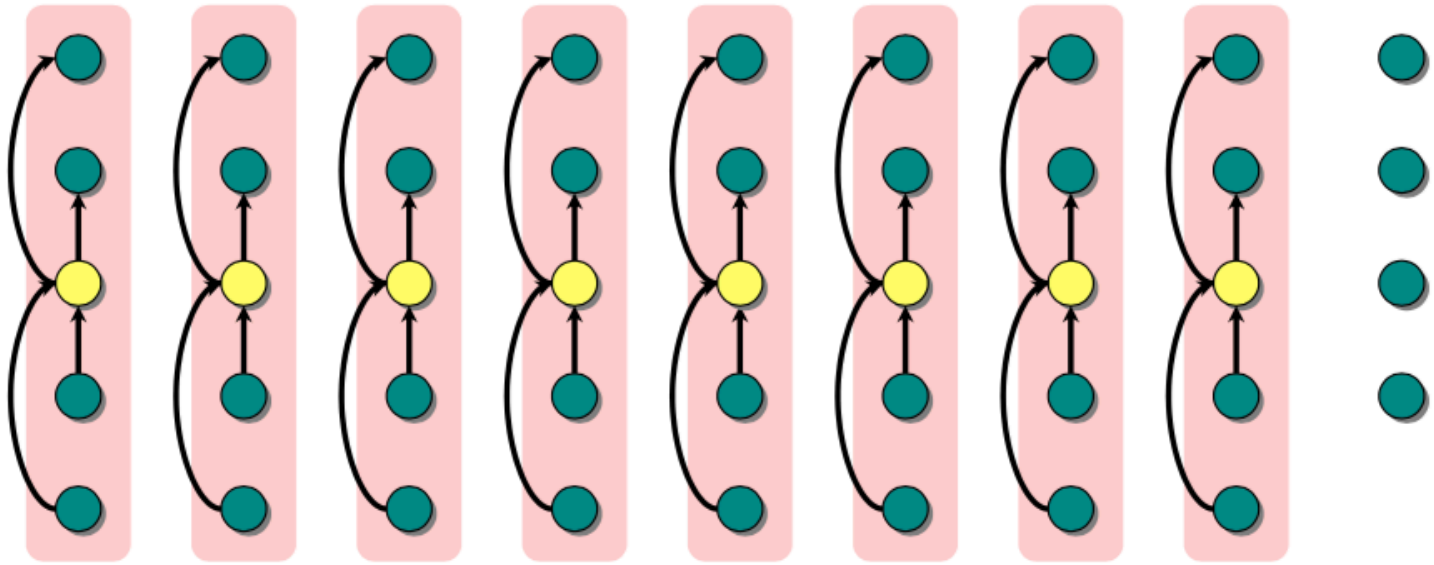


Choosing The Pivot



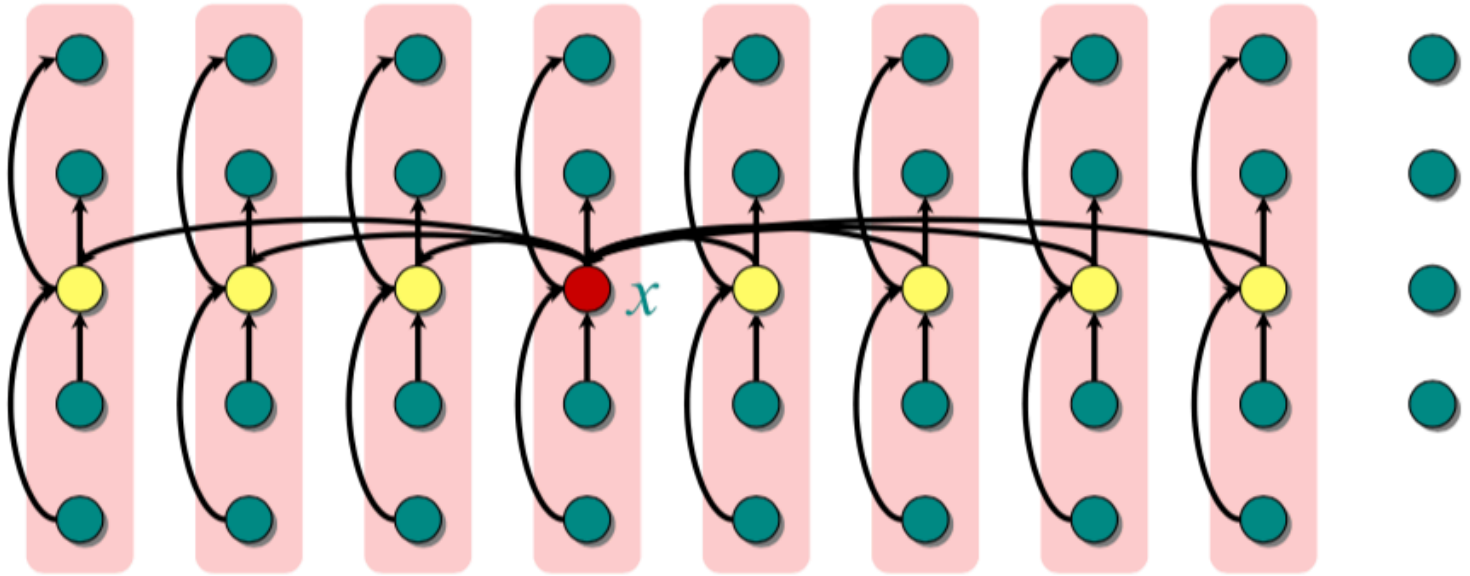
1. Divide the n elements into groups of 5.

Choosing The Pivot



1. Divide the n elements into groups of 5. Find the median of each 5-element group by rote.

Choosing The Pivot

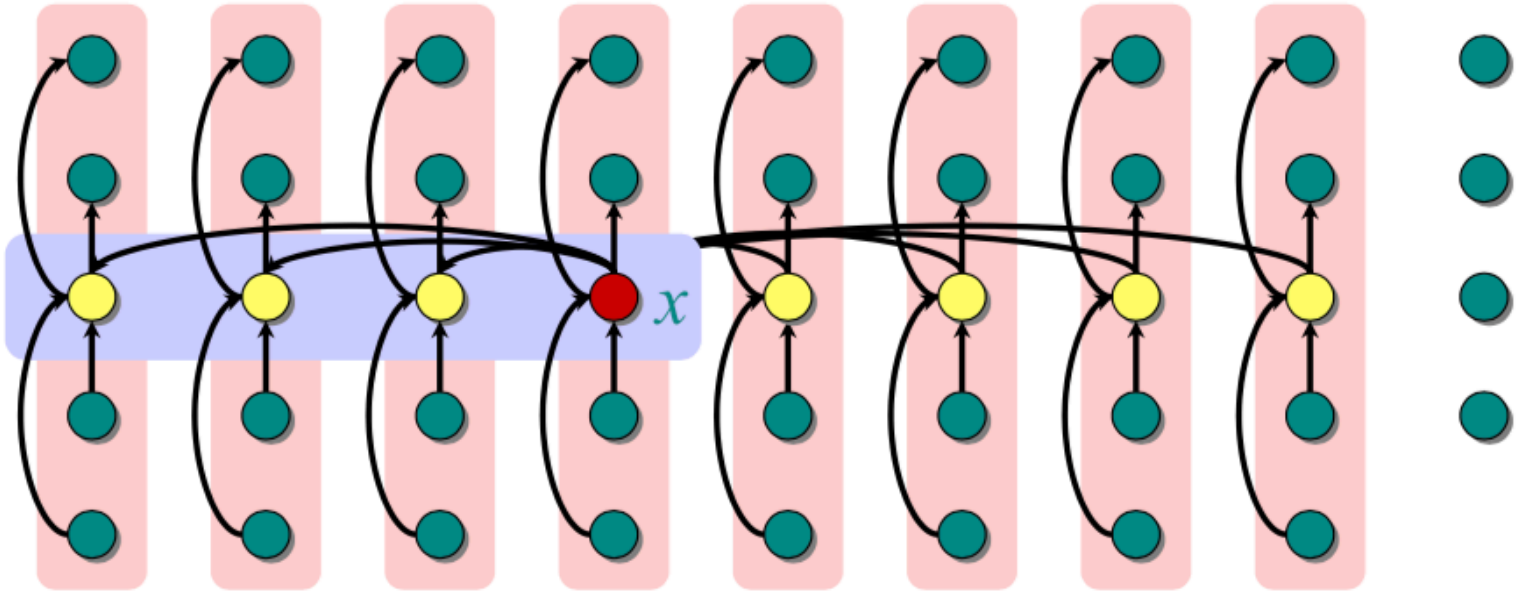


1. Divide the n elements into groups of 5. Find the median of each 5-element group by rote.

2. Recursively SELECT the median x of the $\lfloor n/5 \rfloor$ group medians to be the pivot.

Analysis

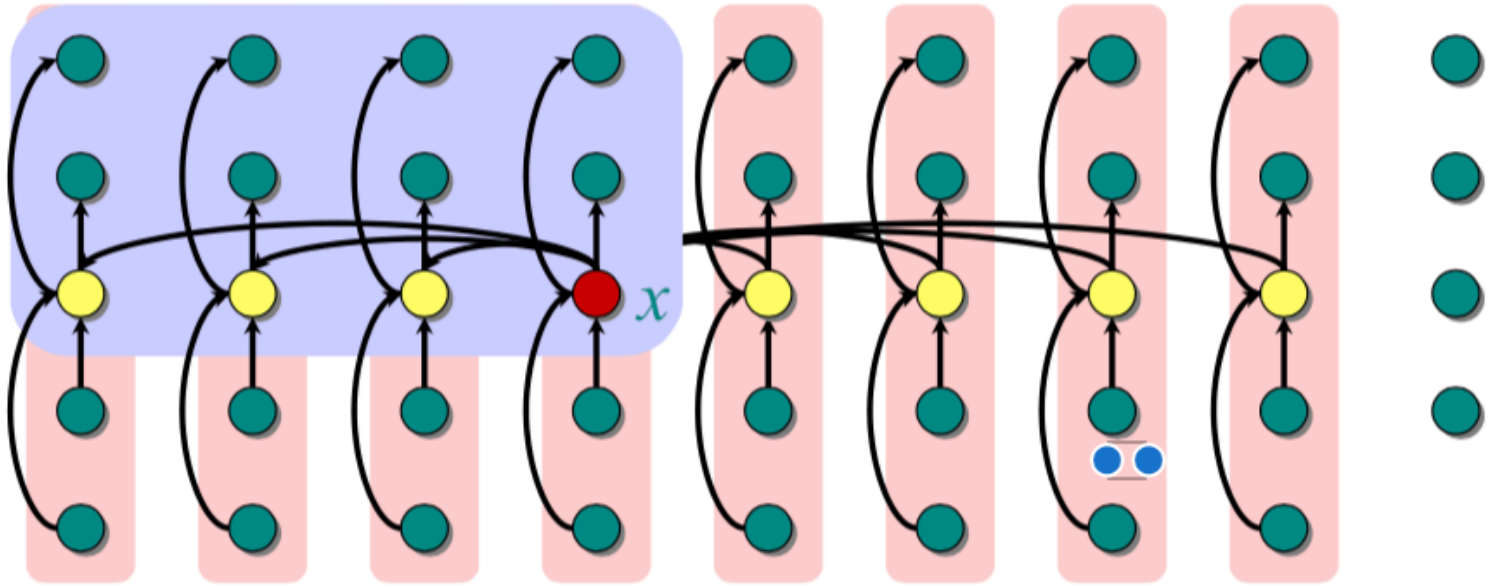
Analysis



At least half the group medians are $\leq x$, which is at least $\lfloor \lfloor n/5 \rfloor /2 \rfloor = \lfloor n/10 \rfloor$ group medians.

Analysis

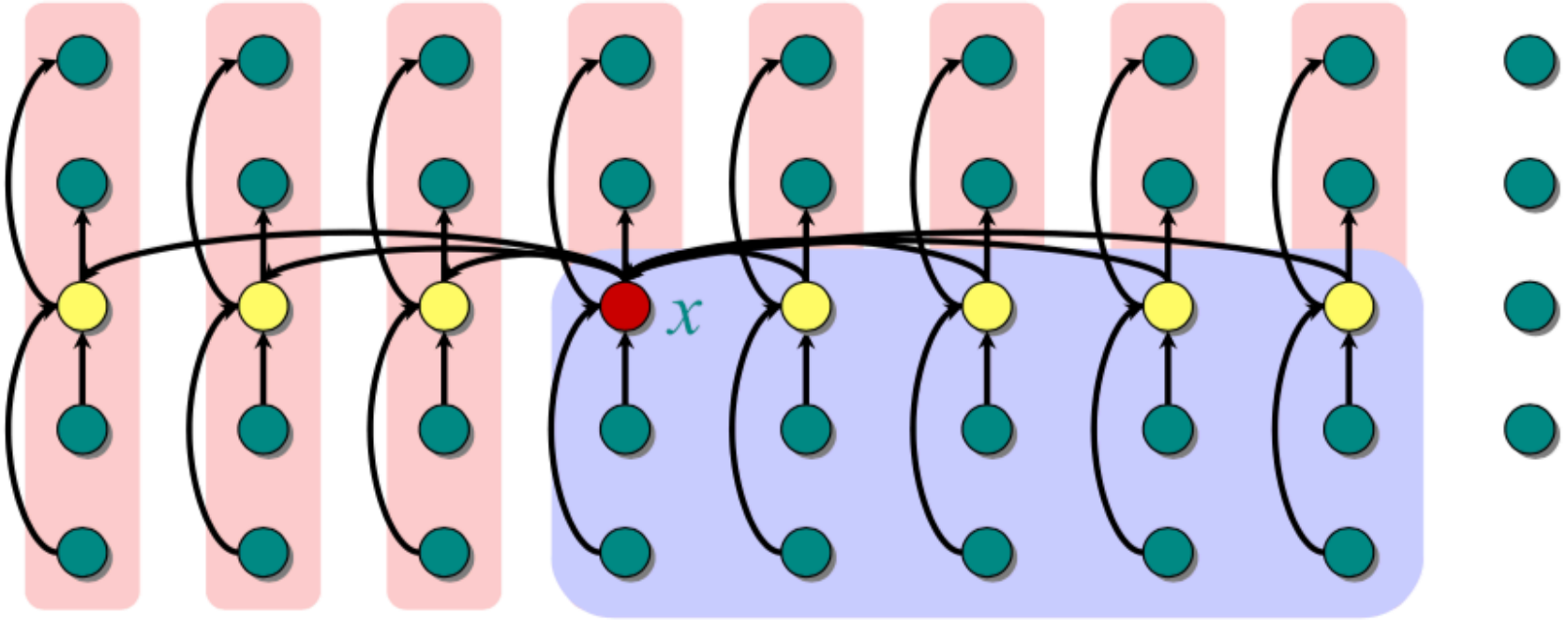
- Therefore, at least 3 $\lfloor n/10 \rfloor$ elements are $\leq x$.



Analysis

- At least half the group medians are ≤ x, which is at least $\lfloor \lfloor n/5 \rfloor /2 \rfloor = \lfloor n/10 \rfloor$ group medians.

• Therefore, at least $3 \lfloor n/10 \rfloor$ elements are ≤ x. • Similarly, at least $3 \lfloor n/10 \rfloor$ elements are ≥ x.



Minor Simplification

- For $n \geq 50$, we have $3\lfloor n/10 \rfloor \geq n/4$.
- Therefore, for $n \geq 50$ the recursive call to SELECT in Step 4 is executed recursively on $\leq 3n/4$ elements.
- Thus, the recurrence for running time can assume that Step 4 takes time $T(3n/4)$ in the worst case.
- For $n < 50$, we know that the worst-case time is $T(n) = \Theta(1)$.

Developing The Recurrence

$T(n)$ SELECT(i, n)

$\Theta(n)$ 1. Divide the n elements into groups of 5. Find the median of each 5-element group by rote.

$T(n/5)$ 2. Recursively SELECT the median x of the $\lfloor n/5 \rfloor$ group medians to be the pivot.

$\Theta(n)$ 3. Partition around the pivot x. Let $k = \text{rank}(x)$.

$T(3n/4)$ 4. **if** $i = k$ **then return** x
elseif $i < k$
then recursively SELECT the ith smallest element in the lower part
else recursively SELECT the $(i–k)$th smallest element in the upper part

Solving The Recurrence

Solving The Recurrence

$$T(n) = T\left(\frac{1}{5}n\right) + T\left(\frac{3}{4}n\right) + \Theta(n)$$

Substitution:

$T(n) \le cn$

$$\begin{aligned} T(n) &\le \frac{1}{5}cn + \frac{3}{4}cn + \Theta(n) \\ &= \frac{19}{20}cn + \Theta(n) \\ &= cn - \left(\frac{1}{20}cn - \Theta(n)\right) \\ &\le cn \quad , \end{aligned}$$

if c is chosen large enough to handle both the $\Theta(n)$ and the initial conditions.

Conclusions

Conclusions

- Since the work at each level of recursion is a constant fraction (19/20) smaller, the work per level is a geometric series dominated by the linear work at the root.
- • In practice, this algorithm runs slowly, because the constant in front of n is large.
- • The randomized algorithm is far more practical.