Algèbre de chemins

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1 Montrer que $(R^+, \max, \min, 0, \infty)$ est un semi-anneau idempotent

On veut démontrer que $(R^+, \max, \min, 0, \infty)$ est un semi-anneau idempotent. Pour ce faire, on doit prouver que :

1. $(R^+, \max, 0)$ est un monoïde commutatif
2. $(R^+, \min, \infty)$ est un monoïde
3. l'opération min est distributive par rapport à max
4. l'élément 0 est absorbant pour l'opération min

Pour chacune des quatre démonstrations qui suivent, on prend $a, b, c \in R^+$.

1.1 $(R^+, \max, 0)$ est un monoïde commutatif

$$\max(a, b) = \max(b, a)$$

et

$$\max(a, 0) = \max(0, a) = a$$

$(R^+, \max, 0)$ est donc un monoïde commutatif.

1.2 $(R^+, \min, \infty)$ est un monoïde

$$\min(a, \infty) = \min(\infty, a) = a$$

$(R^+, \min, \infty)$ est donc un monoïde.

1.3 l'opération $\min$ est distributive par rapport à $\max$

Pour que cette opération soit distributive, il faut que :

$$\min(a, \max(b, c)) = \max(\min(a, b), \min(a, c))$$

et

$$\min(\max(a, b), c) = \max(\min(a, c), \min(a, b))$$

On étudie les six cas de figure possibles.

1.3.1 $a \geq b \geq c$

$$\begin{aligned} \min(a, \max(b, c)) &= \min(a, b) = b \\ \max(\min(a, b), \min(a, c)) &= \max(b, c) = b \end{aligned}$$

1.3.2 $a \geq c \geq b$

$$\min(a, \max(b, c)) = \min(a, c) = c$$

$$\max(\min(a, b), \min(a, c)) = \max(b, c) = c$$

1.3.3 $b \geq a \geq c$

$$\min(a, \max(b, c)) = \min(a, b) = a$$

$$\max(\min(a, b), \min(a, c)) = \max(a, c) = a$$

1.3.4 $b \geq c \geq a$

$$\min(a, \max(b, c)) = \min(a, b) = a$$

$$\max(\min(a, b), \min(a, c)) = \max(a, a) = a$$

1.3.5 $c \geq a \geq b$

$$\min(a, \max(b, c)) = \min(a, c) = a$$

$$\max(\min(a, b), \min(a, c)) = \max(b, a) = a$$

1.3.6 $c \geq b \geq a$

$$\min(a, \max(b, c)) = \min(a, c) = a$$
$$\max(\min(a, b), \min(a, c)) = \max(a, a) = a$$

1.4 l'élément 0 est absorbant pour l'opération min

$$\min(a, 0) = \min(0, a) = 0$$

L'élément 0 est donc absorbant pour min.

Les quatres conditions précedemment citées sont validées, $(R^+, \max, \min, 0, \infty)$ est donc un semi-anneau idempotent.

1.5 Matrice d'adjacence

La matrice d'adjacence est la suivante. On considère que pour tout $i \in \{1, \ldots, 8\}$, $A_{ii} = \infty$.

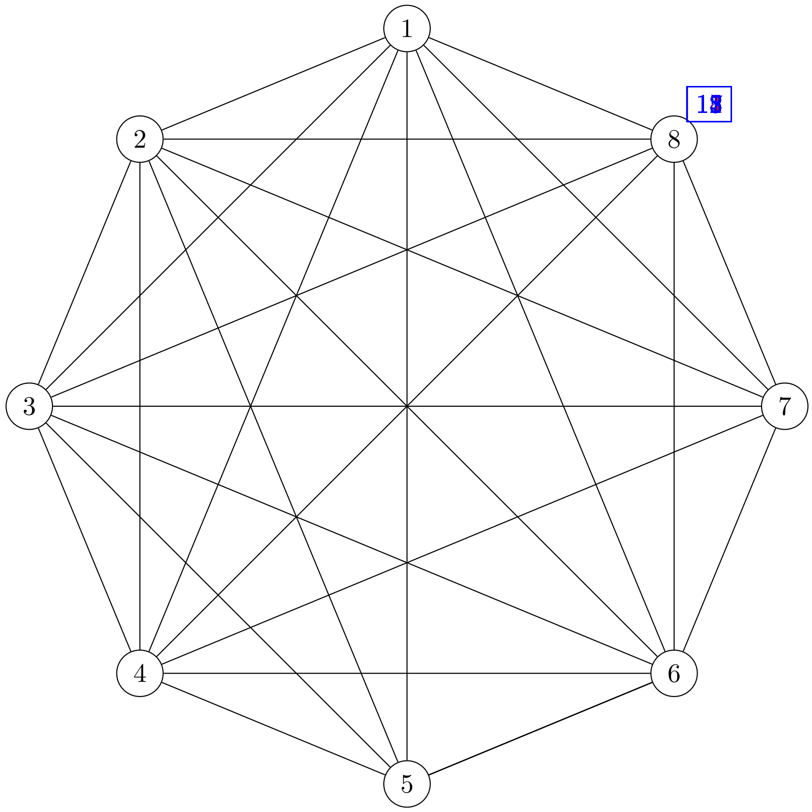
$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} \infty & 12 & 14 & 10 & 0 & 0 & 0 & 0 \\ 12 & \infty & 0 & 17 & 8 & 0 & 0 & 0 \\ 14 & 0 & \infty & 5 & 0 & 3 & 0 & 0 \\ 10 & 17 & 5 & \infty & 11 & 6 & 15 & 0 \\ 0 & 8 & 0 & 11 & \infty & 0 & 18 & 11 \\ 0 & 0 & 3 & 6 & 0 & \infty & 4 & 15 \\ 0 & 0 & 0 & 15 & 18 & 4 & \infty & 9 \\ 0 & 0 & 0 & 0 & 11 & 15 & 9 & \infty \end{pmatrix} \end{array}$$

2 Appliquer l'algorithme de Warshall pour calculer A^*

Après avoir adapté l'algorithme fourni, on obtient la matrice A^* suivante.

$$A^* = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} \infty & 12 & 14 & 12 & 12 & 11 & 12 & 11 \\ 12 & \infty & 12 & 17 & 15 & 11 & 15 & 11 \\ 14 & 12 & \infty & 12 & 12 & 11 & 12 & 11 \\ 12 & 17 & 12 & \infty & 15 & 11 & 15 & 11 \\ 12 & 15 & 12 & 15 & \infty & 11 & 18 & 11 \\ 11 & 11 & 11 & 11 & 11 & \infty & 11 & 15 \\ 12 & 15 & 12 & 15 & 18 & 11 & \infty & 11 \\ 11 & 11 & 11 & 11 & 11 & 15 & 11 & \infty \end{pmatrix} \end{array}$$

3 Graphe de A^*



4 Appliquer l'algorithme de Dijkstra

On déroule manuellement l'algorithme de Dijkstra.

<u>Initialisation</u>

$$\begin{aligned} &\pi(1) = \infty \\ &\pi(2) = \cdots = \pi(8) = 0 \\ &T = \{1, 2, 3, 4, 5, 6, 7, 8\} \end{aligned}$$

<u>Itération 1</u>

$$\begin{aligned} \forall j \in T, \quad & i = \mathrm{MAX}(\pi_j) = 1 \\ & T = \{2, 3, 4, 5, 6, 7, 8\} \end{aligned}$$

$$\pi(2) = \max(\pi(2), \min(\pi(1), A_{12})) = \max(0, \min(\infty, 12)) = 12$$
$$\pi(3) = \max(\pi(3), \min(\pi(1), A_{13})) = \max(0, \min(\infty, 14)) = 14$$
$$\pi(4) = \max(\pi(4), \min(\pi(1), A_{14})) = \max(0, \min(\infty, 10)) = 10$$
$$\pi(5) = \max(\pi(5), \min(\pi(1), A_{15})) = \max(0, \min(\infty, 0)) = 0$$
$$\pi(6) = \max(\pi(6), \min(\pi(1), A_{16})) = \max(0, \min(\infty, 0)) = 0$$
$$\pi(7) = \max(\pi(7), \min(\pi(1), A_{17})) = \max(0, \min(\infty, 0)) = 0$$
$$\pi(8) = \max(\pi(8), \min(\pi(1), A_{18})) = \max(0, \min(\infty, 0)) = 0$$

<u>Itération 2</u>

$$\forall j \in T, \quad i = \mathrm{MAX}(\pi_j) = 3$$
$$T = \{2, 4, 5, 6, 7, 8\}$$

$$\pi(2) = \max(\pi(2), \min(\pi(3), A_{32})) = \max(12, \min(14, 0)) = 12$$
$$\pi(4) = \max(\pi(4), \min(\pi(3), A_{34})) = \max(10, \min(14, 5)) = 10$$
$$\pi(5) = \max(\pi(5), \min(\pi(3), A_{35})) = \max(0, \min(14, 0)) = 0$$
$$\pi(6) = \max(\pi(6), \min(\pi(3), A_{36})) = \max(0, \min(14, 3)) = 3$$
$$\pi(7) = \max(\pi(7), \min(\pi(3), A_{37})) = \max(0, \min(14, 0)) = 0$$
$$\pi(8) = \max(\pi(8), \min(\pi(3), A_{38})) = \max(0, \min(14, 0)) = 0$$

<u>Itération 3</u>

$$\forall j \in T, \quad i = \mathrm{MAX}(\pi_j) = 2$$
$$T = \{4, 5, 6, 7, 8\}$$

$$\pi(4) = \max(\pi(4), \min(\pi(2), A_{24})) = \max(10, \min(12, 17)) = 12$$
$$\pi(5) = \max(\pi(5), \min(\pi(2), A_{25})) = \max(0, \min(12, 8)) = 8$$
$$\pi(6) = \max(\pi(6), \min(\pi(2), A_{26})) = \max(3, \min(12, 0)) = 3$$
$$\pi(7) = \max(\pi(7), \min(\pi(2), A_{27})) = \max(0, \min(12, 0)) = 0$$
$$\pi(8) = \max(\pi(8), \min(\pi(2), A_{28})) = \max(0, \min(12, 0)) = 0$$

<u>Itération 4</u>

$$\forall j \in T, \quad i = \mathrm{MAX}(\pi_j) = 4$$
$$T = \{5, 6, 7, 8\}$$

$$
\begin{aligned}
\pi(5) &= \max(\pi(5), \min(\pi(4), A_{45})) = \max(8, \min(12, 11)) = 11 \\
\pi(6) &= \max(\pi(6), \min(\pi(4), A_{46})) = \max(3, \min(12, 6)) = 6 \\
\pi(7) &= \max(\pi(7), \min(\pi(4), A_{47})) = \max(0, \min(12, 15)) = 12 \\
\pi(8) &= \max(\pi(8), \min(\pi(4), A_{48})) = \max(0, \min(12, 0)) = 0
\end{aligned}
$$

Itération 5

$$
\begin{aligned}
\forall j \in T, \quad & i = \text{MAX}(\pi_j) = 7 \\
& T = \{5, 6, 8\}
\end{aligned}
$$

$$
\begin{aligned}
\pi(5) &= \max(\pi(5), \min(\pi(7), A_{75})) = \max(11, \min(12, 18)) = 12 \\
\pi(6) &= \max(\pi(6), \min(\pi(7), A_{76})) = \max(6, \min(12, 4)) = 6 \\
\pi(8) &= \max(\pi(8), \min(\pi(7), A_{78})) = \max(0, \min(12, 9)) = 9
\end{aligned}
$$

Itération 6

$$
\begin{aligned}
\forall j \in T, \quad & i = \text{MAX}(\pi_j) = 5 \\
& T = \{6, 8\}
\end{aligned}
$$

$$
\begin{aligned}
\pi(6) &= \max(\pi(6), \min(\pi(5), A_{56})) = \max(6, \min(12, 0)) = 6 \\
\pi(8) &= \max(\pi(8), \min(\pi(5), A_{58})) = \max(9, \min(12, 11)) = 11
\end{aligned}
$$

Itération 7

$$
\begin{aligned}
\forall j \in T, \quad & i = \text{MAX}(\pi_j) = 8 \\
& T = \{6\}
\end{aligned}
$$

$$
\pi(6) = \max(\pi(6), \min(\pi(8), A_{86})) = \max(6, \min(11, 15)) = 11
$$

Itération 8

$$
\begin{aligned}
\forall j \in T, \quad & i = \text{MAX}(\pi_j) = 6 \\
& T = \emptyset
\end{aligned}
$$

$T = \emptyset$ alors on a terminé BLABLA

5 Jacobi

6 Jacobi amélioré