Algèbre de chemins

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1 Montrer que $(R^+, \max, \min, 0, \infty)$ est un semi-anneau idempotent

On veut démontrer que $(R^+, \max, \min, 0, \infty)$ est un semi-anneau idempotent. Pour ce faire, on doit prouver que :

1. $(R^+, \max, 0)$ est un monoïde commutatif
2. $(R^+, \min, \infty)$ est un monoïde
3. l'opération min est distributive par rapport à max
4. l'élément 0 est absorbant pour l'opération min

Pour chacune des quatre démonstrations qui suivent, on prend $a, b, c \in R^+$.

1.1 $(R^+, \max, 0)$ est un monoïde commutatif

$$\max(a, b) = \max(b, a)$$

et

$$\max(a, 0) = \max(0, a) = a$$

$(R^+, \max, 0)$ est donc un monoïde commutatif.

1.2 $(R^+, \min, \infty)$ est un monoïde

$$\min(a, \infty) = \min(\infty, a) = a$$

$(R^+, \min, \infty)$ est donc un monoïde.

1.3 l'opération $\min$ est distributive par rapport à $\max$

Pour que cette opération soit distributive, il faut que :

$$\min(a, \max(b, c)) = \max(\min(a, b), \min(a, c))$$

et

$$\min(\max(a, b), c) = \max(\min(a, c), \min(a, b))$$

On étudie les six cas de figure possibles.

1.3.1 $a \geq b \geq c$

$$\min(a, \max(b, c)) = \min(a, b) = b = \max(b, c) = \max(\min(a, b), \min(a, c))$$

et

$$\min(\max(a, b), c) = \min(a, c) = c = \max(c, c) = \max(\min(a, c), \min(b, c))$$

1.3.2 $a \geq c \geq b$

$$\min(a, \max(b, c)) = \min(a, c) = c = \max(b, c) = \max(\min(a, b), \min(a, c))$$

et

$$\min(\max(a, b), c) = \min(a, c) = c = \max(c, c) = \max(\min(a, c), \min(b, c))$$

1.3.3 $b \geq a \geq c$

$$\min(a, \max(b, c)) = \min(a, b) = a = \max(a, c) = \max(\min(a, b), \min(a, c))$$

et

$$\min(\max(a, b), c) = \min(b, c) = c = \max(c, c) = \max(\min(a, c), \min(b, c))$$

1.3.4 $b \geq c \geq a$

$$\min(a, \max(b, c)) = \min(a, b) = a = \max(a, a) = \max(\min(a, b), \min(a, c))$$

et

$$\min(\max(a, b), c) = \min(b, c) = c = \max(a, c) = \max(\min(a, c), \min(b, c))$$

1.3.5 $c \geq a \geq b$

$$\min(a, \max(b,c)) = \min(a,c) = a = \max(b,a) = \max(\min(a,b), \min(a,c))$$
et
$$\min(\max(a,b),c) = \min(a,c) = a = \max(a,b) = \max(\min(a,c), \min(b,c))$$

1.3.6 $c \geq b \geq a$

$$\min(a, \max(b,c)) = \min(a,c) = a = \max(a,a) = \max(\min(a,b), \min(a,c))$$
et
$$\min(\max(a,b),c) = \min(b,c) = b = \max(a,b) = \max(\min(a,c), \min(b,c))$$

Chaque cas de figure est vérifié, min est donc distributive par rapport à max.

1.4 l'élément 0 est absorbant pour l'opération min

$$\min(a,0) = \min(0,a) = 0$$

L'élément 0 est donc absorbant pour min.

Les quatres conditions précedemment citées sont validées, $(R^+, \max, \min, 0, \infty)$ est donc un semi-anneau idempotent.

1.5 Matrice d'adjacence

La matrice d'adjacence est la suivante. On considère dans le cadre de ce problème que pour tout $i \in \{1, \ldots, 8\}$, $A_{ii} = \infty$.

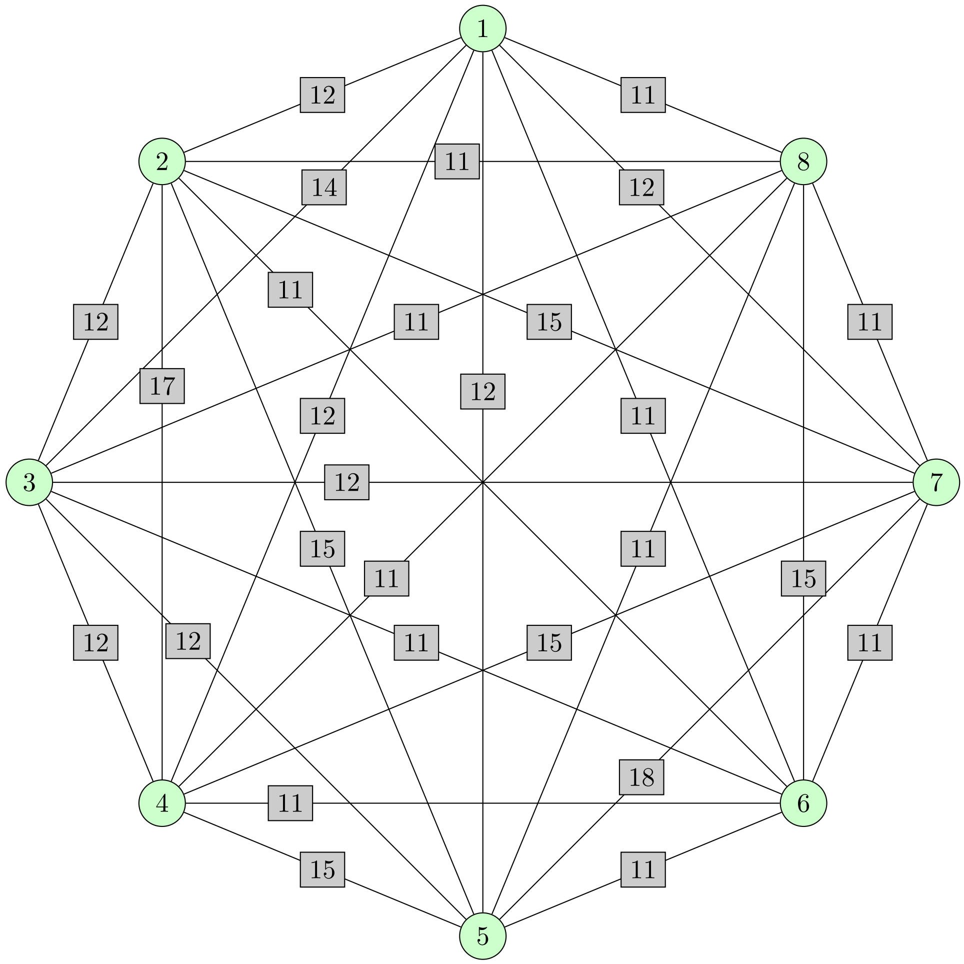
$$A = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} \infty & 12 & 14 & 10 & 0 & 0 & 0 & 0 \\ 12 & \infty & 0 & 17 & 8 & 0 & 0 & 0 \\ 14 & 0 & \infty & 5 & 0 & 3 & 0 & 0 \\ 10 & 17 & 5 & \infty & 11 & 6 & 15 & 0 \\ 0 & 8 & 0 & 11 & \infty & 0 & 18 & 11 \\ 0 & 0 & 3 & 6 & 0 & \infty & 4 & 15 \\ 0 & 0 & 0 & 15 & 18 & 4 & \infty & 9 \\ 0 & 0 & 0 & 0 & 11 & 15 & 9 & \infty \end{pmatrix} \end{array}$$

2 Appliquer l'algorithme de Warshall pour calculer A^*

Après avoir adapté l'algorithme fourni, on obtient la matrice A^* suivante.

$$A^* = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} \infty & 12 & 14 & 12 & 12 & 11 & 12 & 11 \\ 12 & \infty & 12 & 17 & 15 & 11 & 15 & 11 \\ 14 & 12 & \infty & 12 & 12 & 11 & 12 & 11 \\ 12 & 17 & 12 & \infty & 15 & 11 & 15 & 11 \\ 12 & 15 & 12 & 15 & \infty & 11 & 18 & 11 \\ 11 & 11 & 11 & 11 & 11 & \infty & 11 & 15 \\ 12 & 15 & 12 & 15 & 18 & 11 & \infty & 11 \\ 11 & 11 & 11 & 11 & 11 & 15 & 11 & \infty \end{pmatrix} \end{array}$$

3 Graphe de A^*



4 Appliquer l'algorithme de Dijkstra

On déroule manuellement l'algorithme de Dijkstra. On utilise ici sa version améliorée afin de bénéficier d'une liste contenant le prédecesseur de chaque nœud dans le meilleur parcours. On s'en servira typiquement pour reconstruire les chemins.

Initialisation

$$\begin{aligned}
T &= \{1,2,3,4,5,6,7,8\} \\
\pi &= \{\infty,0,0,0,0,0,0,0\} \\
P &= \{\varepsilon,\varepsilon,\varepsilon,\varepsilon,\varepsilon,\varepsilon,\varepsilon,\varepsilon\}
\end{aligned}$$

Itération 1

$$\begin{aligned}
i &= \mathrm{MAX}_{\forall j\in T}(\pi_j) = 1 \\
T &= \{2,3,4,5,6,7,8\}
\end{aligned}$$

$$\begin{aligned}
&\pi(2) = \max(\pi(2), \min(\pi(1), A_{12})) = \max(0, \min(\infty, 12)) = 12, \quad P(2) = 1 \\
&\pi(3) = \max(\pi(3), \min(\pi(1), A_{13})) = \max(0, \min(\infty, 14)) = 14, \quad P(3) = 1 \\
&\pi(4) = \max(\pi(4), \min(\pi(1), A_{14})) = \max(0, \min(\infty, 10)) = 10, \quad P(4) = 1 \\
&\pi(5) = \max(\pi(5), \min(\pi(1), A_{15})) = \max(0, \min(\infty, 0)) = 0 \\
&\pi(6) = \max(\pi(6), \min(\pi(1), A_{16})) = \max(0, \min(\infty, 0)) = 0 \\
&\pi(7) = \max(\pi(7), \min(\pi(1), A_{17})) = \max(0, \min(\infty, 0)) = 0 \\
&\pi(8) = \max(\pi(8), \min(\pi(1), A_{18})) = \max(0, \min(\infty, 0)) = 0
\end{aligned}$$

$$\begin{aligned}
\pi &= \{\infty,12,14,10,0,0,0,0\} \\
P &= \{\varepsilon,1,1,1,\varepsilon,\varepsilon,\varepsilon,\varepsilon\}
\end{aligned}$$

Itération 2

$$\begin{aligned}
i &= \mathrm{MAX}_{\forall j\in T}(\pi_j) = 3 \\
T &= \{2,4,5,6,7,8\}
\end{aligned}$$

$$\begin{aligned}
&\pi(2) = \max(\pi(2), \min(\pi(3), A_{32})) = \max(12, \min(14, 0)) = 12 \\
&\pi(4) = \max(\pi(4), \min(\pi(3), A_{34})) = \max(10, \min(14, 5)) = 10 \\
&\pi(5) = \max(\pi(5), \min(\pi(3), A_{35})) = \max(0, \min(14, 0)) = 0 \\
&\pi(6) = \max(\pi(6), \min(\pi(3), A_{36})) = \max(0, \min(14, 3)) = 3, \quad P(6) = 3 \\
&\pi(7) = \max(\pi(7), \min(\pi(3), A_{37})) = \max(0, \min(14, 0)) = 0 \\
&\pi(8) = \max(\pi(8), \min(\pi(3), A_{38})) = \max(0, \min(14, 0)) = 0
\end{aligned}$$

$$\pi = \{\infty, 12, 14, 10, 0, 3, 0, 0\}$$
$$P = \{\varepsilon, 1, 1, 1, \varepsilon, 3, \varepsilon, \varepsilon\}$$

<u>Itération 3</u>

$$i = \mathrm{MAX}_{\forall j \in T}(\pi_j) = 2$$
$$T = \{4, 5, 6, 7, 8\}$$

$$\pi(4) = \max(\pi(4), \min(\pi(2), A_{24})) = \max(10, \min(12, 17)) = 12, \quad P(4) = 2$$
$$\pi(5) = \max(\pi(5), \min(\pi(2), A_{25})) = \max(0, \min(12, 8)) = 8, \quad P(5) = 2$$
$$\pi(6) = \max(\pi(6), \min(\pi(2), A_{26})) = \max(3, \min(12, 0)) = 3$$
$$\pi(7) = \max(\pi(7), \min(\pi(2), A_{27})) = \max(0, \min(12, 0)) = 0$$
$$\pi(8) = \max(\pi(8), \min(\pi(2), A_{28})) = \max(0, \min(12, 0)) = 0$$

$$\pi = \{\infty, 12, 14, 12, 8, 3, 0, 0\}$$
$$P = \{\varepsilon, 1, 1, 2, 2, 3, \varepsilon, \varepsilon\}$$

<u>Itération 4</u>

$$i = \mathrm{MAX}_{\forall j \in T}(\pi_j) = 4$$
$$T = \{5, 6, 7, 8\}$$

$$\pi(5) = \max(\pi(5), \min(\pi(4), A_{45})) = \max(8, \min(12, 11)) = 11, \quad P(5) = 4$$
$$\pi(6) = \max(\pi(6), \min(\pi(4), A_{46})) = \max(3, \min(12, 6)) = 6, \quad P(6) = 4$$
$$\pi(7) = \max(\pi(7), \min(\pi(4), A_{47})) = \max(0, \min(12, 15)) = 12, \quad P(7) = 4$$
$$\pi(8) = \max(\pi(8), \min(\pi(4), A_{48})) = \max(0, \min(12, 0)) = 0$$

$$\pi = \{\infty, 12, 14, 12, 11, 6, 12, 0\}$$
$$P = \{\varepsilon, 1, 1, 2, 4, 4, 4, \varepsilon\}$$

<u>Itération 5</u>

$$i = \mathrm{MAX}_{\forall j \in T}(\pi_j) = 7$$
$$T = \{5, 6, 8\}$$

$$\pi(5) = \max(\pi(5), \min(\pi(7), A_{75})) = \max(11, \min(12, 18)) = 12, \quad P(5) = 7$$
$$\pi(6) = \max(\pi(6), \min(\pi(7), A_{76})) = \max(6, \min(12, 4)) = 6$$
$$\pi(8) = \max(\pi(8), \min(\pi(7), A_{78})) = \max(0, \min(12, 9)) = 9, \qquad P(8) = 7$$

$$\pi = \{\infty, 12, 14, 12, 12, 6, 12, 9\}$$
$$P = \{\varepsilon, 1, 1, 2, 7, 4, 4, 7\}$$

Itération 6

$$i = \text{MAX}_{\forall j \in T}(\pi_j) = 5$$
$$T = \{6, 8\}$$

$$\pi(6) = \max(\pi(6), \min(\pi(5), A_{56})) = \max(6, \min(12, 0)) = 6$$
$$\pi(8) = \max(\pi(8), \min(\pi(5), A_{58})) = \max(9, \min(12, 11)) = 11, \quad P(8) = 5$$

$$\pi = \{\infty, 12, 14, 12, 12, 6, 12, 11\}$$
$$P = \{\varepsilon, 1, 1, 2, 7, 4, 4, 5\}$$

Itération 7

$$i = \text{MAX}_{\forall j \in T}(\pi_j) = 8$$
$$T = \{6\}$$

$$\pi(6) = \max(\pi(6), \min(\pi(8), A_{86})) = \max(6, \min(11, 15)) = 11, \quad P(6) = 8$$

$$\pi = \{\infty, 12, 14, 12, 12, 11, 12, 11\}$$
$$P = \{\varepsilon, 1, 1, 2, 7, 8, 4, 5\}$$

Itération 8

$$i = \text{MAX}_{\forall j \in T}(\pi_j) = 6$$
$$T = \emptyset$$

$T = \emptyset$ alors l'algorithme est arrivé à la fin de son exécution et on obtient :

	1	2	3	4	5	6	7	8
π	∞	12	14	12	12	11	12	11
P	ε	1	1	2	7	8	4	5

Prenons l'exemple du chemin de capacité maximum reliant 1 à 8. On sait que sa capacité vaut 11 et on peut le reconstruire à l'aide de la liste P des prédecesseurs. Il s'agit de :

$$8 \to P(8) \to P(P(8)) \to \cdots \to 1$$

$$8 \to 5 \to 7 \to 4 \to 2 \to 1$$

5 Appliquer l'algorithme de Jacobi

Nous allons maintenant utiliser l'algorithme de Jacobi pour déterminer les chemins de capacité maximale partant de 1.

<u>Initialisation</u>

$$y^0 = b^T = (\infty, 0, 0, 0, 0, 0, 0, 0)$$

<u>Itération 1</u>

$$y^1 = y^0 \otimes A \oplus b^T = \max(\min(y^0, A), b^T)$$

$$\begin{aligned}\min(y^0, A) &= \min((\infty, 0, 0, 0, 0, 0, 0, 0), \begin{pmatrix} \infty & 12 & 14 & 10 & 0 & 0 & 0 & 0 \\ 12 & \infty & 0 & 17 & 8 & 0 & 0 & 0 \\ 14 & 0 & \infty & 5 & 0 & 3 & 0 & 0 \\ 10 & 17 & 5 & \infty & 11 & 6 & 15 & 0 \\ 0 & 8 & 0 & 11 & \infty & 0 & 18 & 11 \\ 0 & 0 & 3 & 6 & 0 & \infty & 4 & 15 \\ 0 & 0 & 0 & 15 & 18 & 4 & \infty & 9 \\ 0 & 0 & 0 & 0 & 11 & 15 & 9 & \infty \end{pmatrix}) \\ &= (\infty, 12, 14, 10, 0, 0, 0, 0)\end{aligned}$$

$$\begin{aligned}y^1 &= \max(\min(y^0, A), b^T) \\ &= \max((\infty, 12, 14, 10, 0, 0, 0, 0), (\infty, 0, 0, 0, 0, 0, 0, 0)) \\ &= (\infty, 12, 14, 10, 0, 0, 0, 0)\end{aligned}$$

Itération 2

$$y^2 = y^1 \otimes A \oplus b^T = \max(\min(y^1, A), b^T)$$

$$\min(y^1, A) = \min((\infty, 12, 14, 10, 0, 0, 0, 0), \begin{pmatrix} \infty & 12 & 14 & 10 & 0 & 0 & 0 & 0 \\ 12 & \infty & 0 & 17 & 8 & 0 & 0 & 0 \\ 14 & 0 & \infty & 5 & 0 & 3 & 0 & 0 \\ 10 & 17 & 5 & \infty & 11 & 6 & 15 & 0 \\ 0 & 8 & 0 & 11 & \infty & 0 & 18 & 11 \\ 0 & 0 & 3 & 6 & 0 & \infty & 4 & 15 \\ 0 & 0 & 0 & 15 & 18 & 4 & \infty & 9 \\ 0 & 0 & 0 & 0 & 11 & 15 & 9 & \infty \end{pmatrix})$$
$$= (\infty, 12, 14, 12, 10, 6, 10, 0)$$

$$\begin{aligned} y^2 &= \max(\min(y^0, A), b^T) \\ &= \max((\infty, 12, 14, 12, 10, 6, 10, 0), (\infty, 0, 0, 0, 0, 0, 0, 0)) \\ &= (\infty, 12, 14, 12, 10, 6, 10, 0) \end{aligned}$$

Itération 3

On procède la même façon pour les itérations suivantes.

$$y^3 = (\infty, 12, 14, 12, 11, 10, 12, 10)$$

Itération 4

$$y^4 = (\infty, 12, 14, 12, 12, 10, 12, 11)$$

Itération 5

$$y^5 = (\infty, 12, 14, 12, 12, 11, 12, 11)$$

Itération 6

$$y^6 = (\infty, 12, 14, 12, 12, 11, 12, 11)$$

On a $y^5 = y^6$ donc on est arrivé à la fin de l'algorithme. Les résultats fournis par l'algorithme de Jacobi correspondent à ceux déterminés par l'algorithme de Disjkstra.

6 Améliorer l'algorithme de Jacobi