Time Series Stationarity & Unit Root Tests

Concepts, Tests, and Applications

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Outline

Stationarity: Definition

Strict Stationarity

A time series $\{Y_t\}$ is strictly stationary if the joint distribution of $(Y_{t_1}, Y_{t_2}, \ldots, Y_{t_k})$ is identical to the joint distribution of $(Y_{t_1+h}, Y_{t_2+h}, \ldots, Y_{t_k+h})$ for all $t_1, t_2, \ldots, t_k$ and for all h.

Weak (Covariance) Stationarity

A time series $\{Y_t\}$ is weakly stationary if:

- $E[Y_t] = \mu$ for all t (constant mean)
- $Var[Y_t] = \sigma^2 < \infty$ for all t (constant variance)
- $Cov[Y_t, Y_{t+h}] = \gamma_h$ depends only on h (autocovariance depends only on lag)

Importance of Stationarity

- Statistical properties remain constant over time
- Predictable behavior allows for reliable forecasting
- Required for many statistical techniques and tests
- Simplifies theoretical analysis
- Ensures meaningful sample statistics (mean, variance, correlations)

Key Point

Most statistical forecasting methods are based on the assumption that the time series can be rendered approximately stationary through transformations.

Non-Stationary Series: Characteristics

Common Non-Stationary Patterns:

- Trend (deterministic or stochastic)
- Seasonality
- Cyclical patterns
- Structural breaks
- Changing variance (heteroskedasticity)

Consequences:

- Spurious regressions
- Inconsistent estimators
- Non-standard distributions
- Unreliable forecasts
- Misleading test statistics

TS vs. DS Processes

Trend Stationary (TS) Processes

A process that is stationary around a deterministic trend:

$$Y_t = f(t) + \varepsilon_t \tag{1}$$

where $f(t)$ is a deterministic function (often linear: $\alpha + \beta t$) and ε_t is a stationary process.

Difference Stationary (DS) Processes

A process that becomes stationary after differencing:

$$\Delta Y_t = Y_t - Y_{t-1} = \mu + \varepsilon_t \tag{2}$$

where ε_t is a stationary process. Example: Random walk with drift.

Distinguishing TS vs. DS

Importance of Correct Identification

- Different detrending methods are required
- Misspecification leads to inefficient forecasts
- Statistical tests have different powers against each

Comparison

Feature	TS Process	DS Process
Shocks	Temporary effect	Permanent effect
Variance	Bounded	Increases with time
Detrending	Subtract trend	Take differences
Forecasting	Returns to trend	No mean reversion

Unit Root Concept

Autoregressive Process

Consider an AR(1) process:

$$Y_t = \phi Y_{t-1} + \varepsilon_t \tag{3}$$

Unit Root

A unit root exists when $\phi = 1$, making the process non-stationary:

$$Y_t = Y_{t-1} + \varepsilon_t \quad \Rightarrow \quad Y_t = Y_0 + \sum_{i=1}^{t} \varepsilon_i \tag{4}$$

Stationarity Condition

For stationarity: $|\phi| < 1$ (all roots of the characteristic equation lie outside the unit circle)

Augmented Dickey-Fuller (ADF) Test

Basic Principle

The ADF test examines the null hypothesis that a time series has a unit root (is non-stationary) against the alternative that it is stationary.

General Equation

$$\Delta Y_t = \alpha + \beta t + \gamma Y_{t-1} + \sum_{i=1}^{p} \delta_i \Delta Y_{t-i} + \varepsilon_t \tag{5}$$

where:

- α is a constant (drift)
- βt is a time trend
- $\gamma = \phi - 1$ (we test $H_0 : \gamma = 0$ vs $H_1 : \gamma < 0$)
- p lags of ΔY_t to control for autocorrelation

ADF Test: Different Specifications

Case 1: No Constant, No Trend

$$\Delta Y_t = \gamma Y_{t-1} + \sum_{i=1}^{p} \delta_i \Delta Y_{t-i} + \varepsilon_t \tag{6}$$

Use when: series clearly has zero mean with no trend

Case 2: With Constant, No Trend

$$\Delta Y_t = \alpha + \gamma Y_{t-1} + \sum_{i=1}^{p} \delta_i \Delta Y_{t-i} + \varepsilon_t \tag{7}$$

Use when: series has non-zero mean but no visible trend

Case 3: With Constant and Trend

$$\Delta Y_t = \alpha + \beta t + \gamma Y_{t-1} + \sum_{i=1}^{p} \delta_i \Delta Y_{t-i} + \varepsilon_t \tag{8}$$

ADF Test: Implementation

Test Statistic

$$\text{ADF} = \frac{\hat{\gamma}}{\text{SE}(\hat{\gamma})} \tag{9}$$

Decision Rule

- $H_0 : \gamma = 0$ (unit root exists, series is non-stationary)
- $H_1 : \gamma < 0$ (no unit root, series is stationary)
- Reject H_0 if ADF statistic ¡ critical value

Critical Values

Critical values are non-standard and depend on:

- Sample size
- Whether constant and/or trend are included
- Significance level (typically 1%, 5%, 10%)

Other Unit Root Tests

Phillips-Perron (PP) Test

- Non-parametric adjustment to control for autocorrelation
- More robust to heteroskedasticity and serial correlation
- Same null and alternative hypotheses as ADF

KPSS Test (Kwiatkowski-Phillips-Schmidt-Shin)

- Reverse null hypothesis: H_0 : Series is stationary
- H_1 : Series has a unit root
- Useful for confirmatory analysis alongside ADF or PP tests

Others

- Elliott-Rothenberg-Stock Test (DF-GLS)
- Ng-Perron Test
- Breakpoint Unit Root Tests

Integrated Series

Definition

A time series Y_t is said to be integrated of order d, denoted as $Y_t \sim I(d)$, if it becomes stationary after differencing d times.

Common Orders of Integration

- $I(0)$: Series is already stationary (no differencing needed)
- $I(1)$: First differences are stationary (most common)
- $I(2)$: Second differences are stationary (less common)

Examples of Different Orders

$I(0)$ Process

White noise, stationary AR, MA, ARMA processes

$$Y_t = \mu + \varepsilon_t \quad \text{or} \quad Y_t = \phi Y_{t-1} + \varepsilon_t \quad \text{where} \quad |\phi| < 1 \tag{10}$$

$I(1)$ Process

Random walk, random walk with drift

$$Y_t = Y_{t-1} + \varepsilon_t \quad \text{or} \quad Y_t = \mu + Y_{t-1} + \varepsilon_t \tag{11}$$

$I(2)$ Process

Double integrated series, common in some economic data

$$\Delta^2 Y_t = \varepsilon_t \quad \text{where} \quad \Delta^2 Y_t = \Delta(\Delta Y_t) = Y_t - 2Y_{t-1} + Y_{t-2} \tag{12}$$

Practical Implications of Integration Order

- $I(0)$ series: Standard OLS, ARMA modeling applies directly
- $I(1)$ series: Risk of spurious regressions, need differencing or cointegration analysis
- $I(2)$ series: Require more complex transformations, often indicate structural issues in data

Determining Integration Order

1. Test original series for stationarity (ADF, PP, KPSS)
2. If non-stationary, difference once and test again
3. Repeat until achieving stationarity
4. The number of differences needed = order of integration

Methods to Achieve Stationarity

For Deterministic Trends:

- Detrending (linear, polynomial)
- Seasonal adjustment
- Dummy variables for structural breaks

For Stochastic Trends:

- First differencing $\Delta Y_t = Y_t - Y_{t-1}$
- Second differencing $\Delta^2 Y_t$
- Seasonal differencing $\Delta_s Y_t = Y_t - Y_{t-s}$

For Variance Instability:

- Logarithmic transformation: $\ln(Y_t)$
- Power transformations: Y_t^{λ} (Box-Cox)
- Square root: $\sqrt{Y_t}$

Transformation Decision Process

Step-by-Step Approach

1. Stabilize variance first (if needed)
2. Test for unit roots to determine order of integration
3. For $I(d)$ series, apply d-th order differencing
4. For trend-stationary series, remove deterministic components
5. Verify stationarity of transformed series

Important Considerations

- Each transformation changes the interpretation of the series
- Over-differencing introduces unnecessary complexity (MA component)
- Some properties might be lost during transformation
- Consider domain knowledge in selecting transformations

Optimal Lag Selection

Importance of Lag Selection

- Too few lags: residual autocorrelation remains
- Too many lags: reduced power and efficiency
- Impacts test results and model performance

Information Criteria Approach

Common information criteria for selecting optimal lag length:

$$\text{AIC} = -2\ln(L) + 2k \tag{13}$$

$$\text{BIC} = -2\ln(L) + k\ln(n) \tag{14}$$

$$\text{HQIC} = -2\ln(L) + 2k\ln(\ln(n)) \tag{15}$$

where L is the likelihood, k is the number of parameters, and n is the sample size.

Information Criteria: Comparison

Characteristics

Criterion	Penalty	Tendency
AIC	$2k$	Often selects larger models
BIC	$k\ln(n)$	More parsimonious models
HQIC	$2k\ln(\ln(n))$	Intermediate between AIC and BIC

Selection Procedure

1. Estimate models with different lag lengths (from 0 to a maximum)
2. Calculate information criteria for each model
3. Select the model with the lowest criterion value
4. Different criteria may suggest different lag lengths

User-Defined Lag Selection

Alternative Approaches

- Fixed lag based on data frequency (e.g., 4 for quarterly, 12 for monthly)
- Sequential testing procedure (t-tests on highest lag)
- Rule of thumb: $\text{Int}(T^{1/4})$ where T is sample size
- Cross-validation for forecasting models

Residual Diagnostics

Always check residuals for:

- Autocorrelation (Ljung-Box, Breusch-Godfrey tests)
- Normality (Jarque-Bera test)
- Heteroskedasticity (ARCH effects)

Testing Strategy Workflow

1. Visualize the data (time plots, ACF, PACF)
2. Identify potential patterns (trend, seasonality, breaks)
3. Select appropriate ADF equation based on visual patterns
4. Determine optimal lag length using information criteria
5. Conduct unit root test (ADF, PP, KPSS for confirmation)
6. If non-stationary, transform and repeat testing
7. Determine order of integration
8. Proceed with appropriate modeling approach

Common Pitfalls and Considerations

- Low power of unit root tests in small samples
- Structural breaks can bias unit root tests toward non-rejection
- Different tests may give conflicting results
- Seasonal unit roots require specialized tests
- Integration order may differ across frequencies
- Economic theory should guide transformation choices
- Consider both statistical and practical significance

Cointegration

Definition

If $X_t \sim I(d)$ and $Y_t \sim I(d)$, they are cointegrated if there exists a linear combination $Z_t = X_t - \beta Y_t$ such that $Z_t \sim I(d-b)$ where $b > 0$.

Implications

- Long-run equilibrium relationship exists
- Error Correction Models (ECM) can be used
- Preserves long-run information lost in differencing
- Avoids spurious regression problems

Structural Breaks and Unit Roots

Problem

Standard unit root tests have low power when structural breaks are present

Solutions

- Perron's modified test with known break date
- Zivot-Andrews test with endogenous break detection
- Clemente-Montañés-Reyes test for multiple breaks
- Bai-Perron test for multiple breaks

Fractional Integration

ARFIMA Models

- Allow for non-integer orders of integration (d)
- Long memory processes: $0 < d < 0.5$ (stationary with long memory)
- Intermediate between $I(0)$ and $I(1)$ processes
- More flexible modeling of persistence

Estimation Methods

- Maximum likelihood
- Semi-parametric approaches
- Wavelet-based estimators

Conclusion

- Stationarity is fundamental for time series analysis
- Unit root tests help determine stationarity properties
- Integration order guides appropriate transformations
- TS vs. DS distinction influences modeling strategy
- Appropriate lag selection is critical for reliable results
- Cointegration extends analysis to non-stationary systems
- Advanced techniques handle complex dynamics in real data

References

[1] Dickey, D.A. and Fuller, W.A. (1979). Distribution of the Estimators for Autoregressive Time Series with a Unit Root. Journal of the American Statistical Association, 74, 427-431.

[2] Phillips, P.C.B. and Perron, P. (1988). Testing for a Unit Root in Time Series Regression. Biometrika, 75, 335-346.

[3] Kwiatkowski, D., Phillips, P.C.B., Schmidt, P. and Shin, Y. (1992). Testing the Null Hypothesis of Stationarity against the Alternative of a Unit Root. Journal of Econometrics, 54, 159-178.

[4] Johansen, S. (1988). Statistical Analysis of Cointegration Vectors. Journal of Economic Dynamics and Control, 12, 231-254.

[5] Engle, R.F. and Granger, C.W.J. (1987). Co-integration and Error Correction: Representation, Estimation, and Testing. Econometrica, 55, 251-276.