Dependable Model-Based Hybrid Systems Design via Event-B

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Review

- An overview of model-based hybrid system design
  - Requirement engineering
  - System modeling
  - System design
  - System evaluation
- Apply model-based design in Event-B to design a simple hybrid system
  - High-level hybrid automata that schedules low-level sub-systems
  - Requirements of low-level sub-systems

Goals

The Laplace Transform

The Inverse Laplace Transform

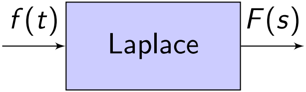
Block Diagram Algebra

Analysis

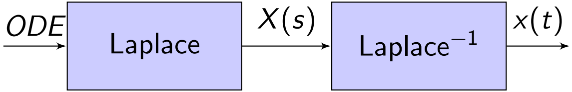
The Laplace Transform

Figure: Pierre Simon Laplace (1749-1827)

What Laplace Transform Does?



Solving ODEs with Laplace Transform



Definition

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st}\,dt$$

- $f(t)$ is an input time-domain function such that $f(t) = 0$ for $t < 0$
- s is a complex variable
- $\mathcal{L}\{\}$ is the Laplace transform operator
- $F(s)$ is the Laplace transform output of $f(t)$

Conditions to Apply Laplace Transform

The Laplace transform exists if $f(t)$ is of exponential order:

$$\exists \sigma \; such \; that : \lim_{t \to \infty} |f(t)e^{-\sigma t}| = 0$$

Exercise, is the following functions of exponential order?

- $f(t) = Ae^{-\alpha t}$
- $f(t) = e^{t^2}$

Laplace Transform of Exponential Function

$$f(t) = \begin{cases} 0 & t < 0 \\ Ae^{-\alpha t} & t \geq 0 \end{cases}$$

$$\begin{aligned} F(s) &= \mathcal{L}\{f(t)\} \\ &= \int_0^\infty f(t)e^{-st}\,dt \\ &= \int_0^\infty Ae^{-\alpha t}e^{-st}\,dt \\ &= A\int_0^\infty e^{-(s+\alpha)t}\,dt \\ &= A\left[\frac{-1}{s+\alpha}e^{-(s+\alpha)t}\Big|_{t=\infty} - \frac{-1}{s+\alpha}e^{-(s+\alpha)t}\Big|_{t=0}\right] \\ &= \frac{A}{s+\alpha} \end{aligned}$$

Laplace Transform of Step Function

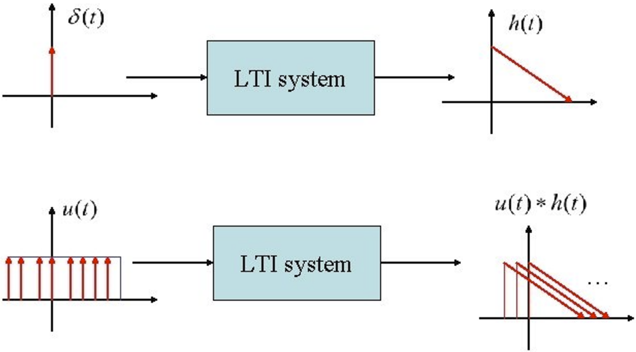
$$f(t) = \begin{cases} 0 & t < 0 \\ A & t \geq 0 \end{cases}$$

$$\begin{aligned}
F(s) &= \mathcal{L}\{f(t)\} \\
&= \int_0^\infty f(t) e^{-st}\, dt \\
&= \int_0^\infty A e^{-st}\, dt \\
&= A \int_0^\infty e^{-st}\, dt \\
&= A \left[\frac{-1}{s} e^{-st}|_{t=\infty} - \frac{-1}{s} e^{-st}|_{t=0} \right] \\
&= \frac{A}{s}
\end{aligned}$$

Important Properties of Laplace Transform

- Convolution: $\mathcal{L}\{f_1(t) * f_2(t)\} = F_1(s)F_2(s)$
- Superposition: $\mathcal{L}\{f_1(t) + f_2(t)\} = F_1(s) + F_2(s)$
- Scaling: $\mathcal{L}\{\alpha f_1(t)\} = \alpha F_1(s)$
- Differentiation:
  $\mathcal{L}\{f^m(t)\} = s^m F(s) - s^{m-1} f(0) - s^{m-2}\dot{f}(0) \ldots - f^{m-1}(0)$
  - $\mathcal{L}\{\dot{f}(t)\} = sF(s) - f(0)$
  - $\mathcal{L}\{\ddot{f}(t)\} = s^2 F(s) - sf(0) - \dot{f}(0)$

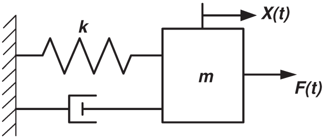
Convolution



- $u(t) * h(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau$
- $\mathcal{L}\{u(t) * h(t)\} = U(s)H(s)$

Example: the Spring Mass Damper

- A mass (of weight m) is attached to the wall via spring (of stiffness k) and a damper (of coefficient b)
- If our system is subjected to an input function $F(t)$, what is the position function $X(t)$ looks like?
  - Using Newton's 2nd low of motion, we can derive an ODE for the system ($\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{1}{m}F$)
  - The solution of this ODE is our targeted $X(t)$



Laplace Method for Solving ODE

1. Start with ODE $\dot{x} = f(x, u)$
2. Given the input $u(t)$
3. Apply Laplace transform on the ODE
4. Solve for $X(s)$ using algebra
5. Applying inverse Laplace transform to obtain $x(t)$

Laplace Method for Solving the Mass Damper

1. Start with ODE $\dot{x} = f(x, u)$:

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{1}{m}F$$

Laplace Method for Solving the Mass Damper

2. Given the input $u(t)$:

$$F = 1(t)$$

Laplace Method for Solving the Mass Damper

3. Apply Laplace transform the ODE:

$$
\begin{aligned}
\mathcal{L}\{\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x\} &= \frac{1}{m}\mathcal{L}\{1(t)\} \\
\mathcal{L}\{\ddot{x}\} + \frac{b}{m}\mathcal{L}\{\dot{x}\} + \frac{k}{m}\mathcal{L}\{x\} &= \frac{1}{m}\mathcal{L}\{1(t)\} \\
\left(s^2 X(s) - sx(0) - \dot{x}(0)\right) + \frac{b}{m}\left(sX(s) - sx(0)\right) + \frac{k}{m}\left(X(s)\right) &= \frac{1}{m}\frac{1}{s}
\end{aligned}
$$

Laplace Method for Solving the Mass Damper

4. Solve for $X(s)$ using algebra:

$$X(s) = \frac{\frac{1}{ms} + sx(0) + \dot{x}(0) + \frac{b}{m}x(0)}{s^2 + \frac{b}{m}s + \frac{k}{m}}$$

Laplace Method for Solving the Mass Damper

5. Applying inverse Laplace transform to obtain $x(t)$: coming next

Goals

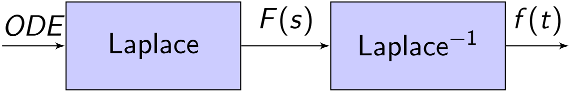
The Laplace Transform

The Inverse Laplace Transform

Block Diagram Algebra

Analysis

Solving ODEs with Laplace Transform



- By the Lerch's theorem, if a function $F(s)$ has the inverse Laplace transformation $f(t)$, then $f(t)$ is uniquely determined.

Definition

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{+st}\,dt$$

- $F(s)$ is an input s-domain function
- s is a complex variable
- $\mathcal{L}^{-\infty}\{\}$ is the inverse Laplace transform operator
- $f(t)$ is the inverse Laplace transform output of $F(s)$

Common Method for Inverse Laplace Transform

- Don't use the definition!
- Remember common inverse Laplace transform pairs
- Break down $F(s)$ by partial fraction expansion:

$$
\begin{aligned}
F(s) &= \frac{b_1 s^m + b2 s^{m-1} + \ldots + b_{m+1}}{s^n + a_1 s^{n-1} + \ldots + a_n} \\
&= \sum_{i=1}^{n} \frac{C_i}{s - p_i}
\end{aligned}
$$

- Map and reduce:

$$
f(t) = \mathcal{L}^{-1}\{F(s)\} = \sum_{i=1}^{n} \mathcal{L}^{-1}\{\frac{C_i}{s - p_i}\}
$$

Common Inverse Laplace Transform Pairs

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t+\theta)$	$\frac{s \sin\theta + \omega \cos\theta}{s^2+\omega^2}$
$\cos(\omega t+\theta)$	$\frac{s \cos\theta - \omega \sin\theta}{s^2+\omega^2}$
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

*Defined for $t \geq 0$; $f(t) = 0$, for $t < 0$.

Example:

$$
\begin{aligned}
F(s) &= \frac{s^2+8s+15}{s^3+3s^2+2s} \\
&= (7.5)\frac{1}{s} + (-8)\frac{1}{s+1} + (1.5)\frac{1}{s+2} \\
\mathcal{L}^{-1}\{F(s)\} &= (7.5)\mathcal{L}^{-1}\{\frac{1}{s}\} + (-8)\mathcal{L}^{-1}\{\frac{1}{s+1}\} + (1.5)\mathcal{L}^{-1}\{\frac{1}{s+2}\} \\
&= 7.5e^{0t} - 8e^{-t} + 1.5e^{-2t}
\end{aligned}
$$

Exercise 1

Find the time-domain function corresponding to each of the following Laplace transforms using partial fraction expansion:

- $\frac{s^2+2s+3}{(s+1)^3}$
- $\frac{s-1}{s^2+2s+2}$
- $\frac{2}{s(s+2)}$
- $\frac{10}{s(s+1)(s+10)}$
- $\frac{3s+2}{s^2+4s+20}$

Exercise 2

Find the time-domain function for the spring mass damper under $x(0) = -2$, $\dot{x}(0) = 3$, and $m = 1, k = 2, b = 3$:

$$X(s) = \frac{\frac{1}{ms} + sx(0) + \dot{x}(0) + \frac{b}{m}x(0)}{s^2 + \frac{b}{m}s + \frac{k}{m}}$$

Exercise 3

Solve the following ODEs using Laplace method:

- $\ddot{y} + \dot{y} + 3y = 0; y(0) = 1, \dot{y}(0) = 2$
- $\ddot{y} + 2\dot{y} = e^t; y(0) = 1, \dot{y}(0) = 2$

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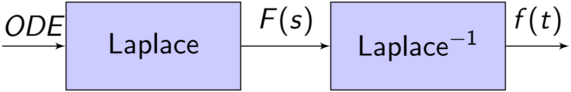
The Laplace Transform

The Inverse Laplace Transform

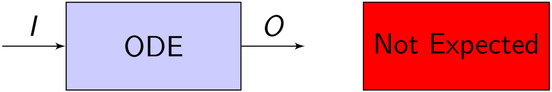
Block Diagram Algebra

Analysis

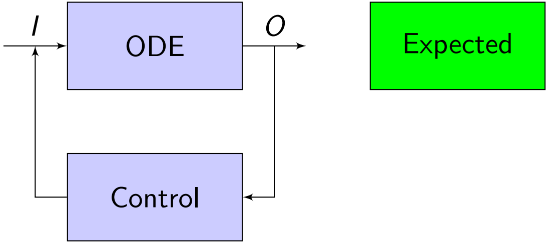
So far...



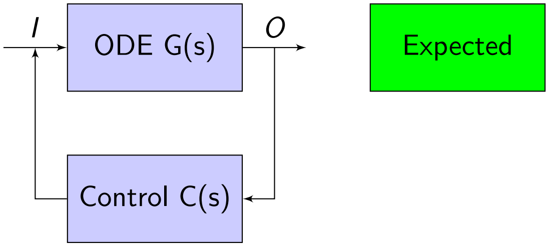
A Standard Problem in Modern Control



A Standard Problem in Modern Control



System Design Based on Laplace Method

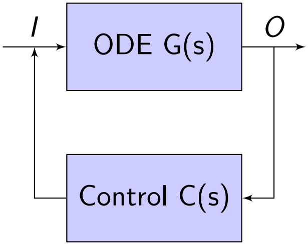


Benefits:

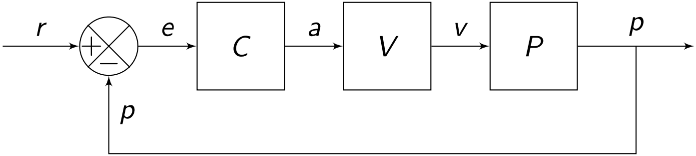
- Simplify design (Block Diagram Algebra)
- Reason design (e.g. stability analysis)

Block Diagram

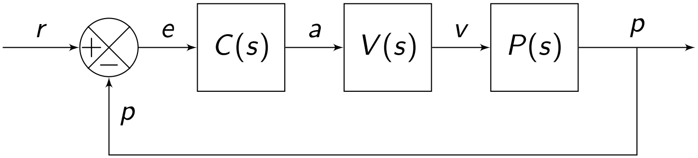
A graphical representation of a system broken down into multiple sub-systems (blocks) as well as connectors (arrows) showing signal flow between sub-systems.



Ex: Car Position Tracking

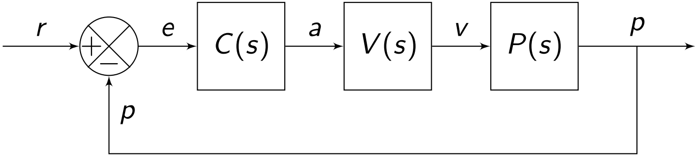


Block Diagram Components

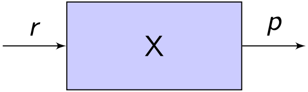


- Blocks represent systems
- Arrows represent signal flow
- System dynamics given by s-domain functions
- Summing node, $o = i_1 \pm i_2$
- Branch point, $i = o_1 = o_2$

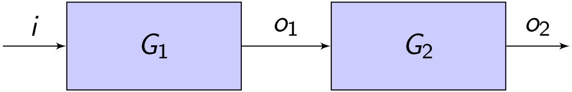
Block Diagram Algebra



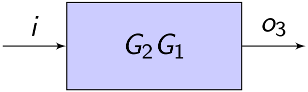
into:



1. Combine in Series

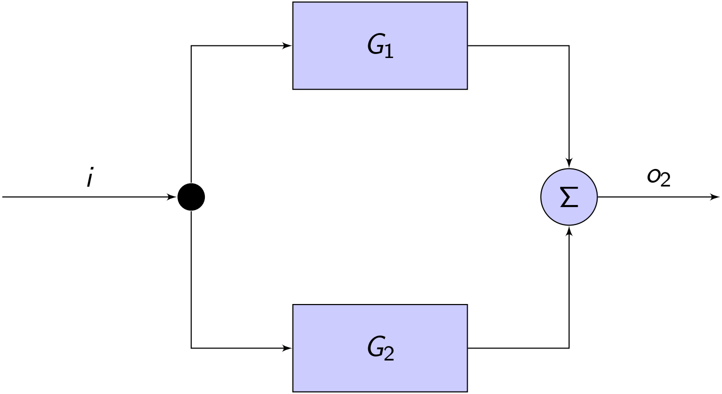


is equivalent to:

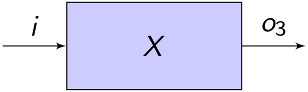


- $o_2 = G_2 o_1 = G_2 G_1 i$ (convolution)
- $o_3 = G_2 G_1 i$ (convolution)
- $o_2 = o_3$

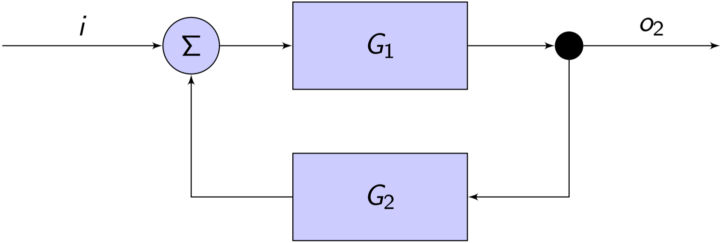
Exercise: Eliminate a Summing Node



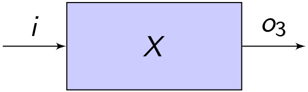
What is equivalent to the above?



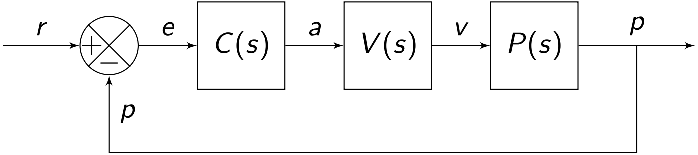
Exercise: Eliminate a Feedback Loop



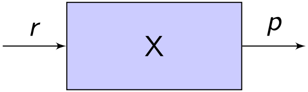
What is equivalent to the above?



Exercise: Reduction of Car Position Tracking Design



What is equivalent to the above?



Goals

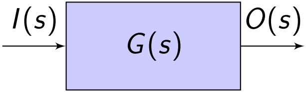
The Laplace Transform

The Inverse Laplace Transform

Block Diagram Algebra

Analysis

After Block Diagram Algebra ...



- We reduce a complex design into a single algebra expression $G(s)$
- If we fix the input $I(s)$, the $O(s) = G(s)I(s)$
- What kind of analysis we can do on $O(s)$?

Stability Analysis

- Give $O(s) = \sum_{i=1}^{n} \frac{C_i}{s-p_i}$
- When $s = p_i$, s is called a pole of $O(s)$
- $O(s)$ is stable if all its poles have negative real parts
  - Because: $\mathcal{L}^{-1}\{\frac{C_i}{s-p_i}\} = e^{p_i t}...$

Example

Stability Analysis of an Stable System

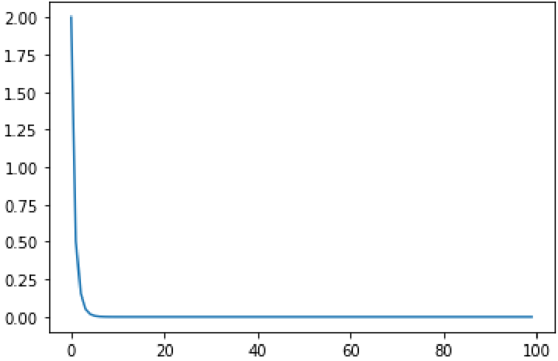
Q: Why $O(s) = \frac{2s+3}{s^2+3s+2}$ is stable?

Example

Stability Analysis of an Stable System

$O(s) = \frac{2s+3}{s^2+3s+2} = \frac{1}{s+1} + \frac{1}{s+2}$

- poles: -1, -2
- Stable
- plot $o(t) = e^{-t} + e^{-2t}$ to confirm:



Example

Stability Analysis of an Unstable System

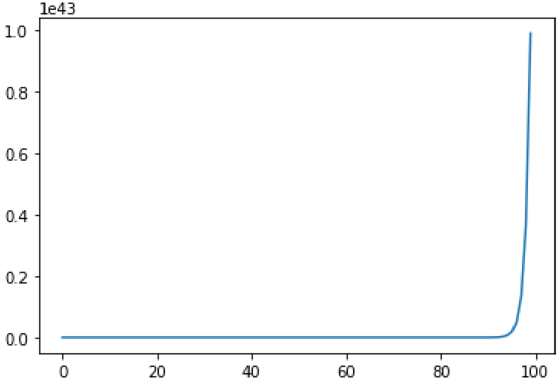
Q: Why $O(s) = \frac{2s+1}{s^2+s-2}$ is unstable?

Example

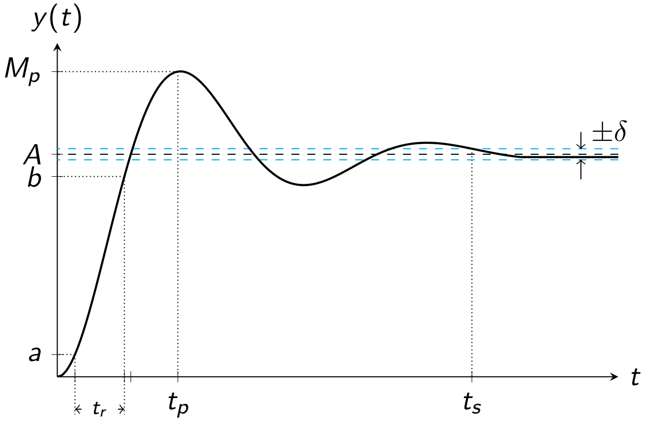
Stability Analysis of an Unstable System

$O(s) = \frac{2s+1}{s^2+s-2} = \frac{1}{s-1} + \frac{1}{s+2}$

- poles: 1, -2
- Unstable
- plot $o(t) = e^t + e^{-2t}$ to confirm:



Estimating Time-domain Metrics



DC Gain

- The DC gain is the ratio of the system steady state to the system input.
- System steady state can be estimated by the **final value theorem**: if all poles of $O(s)$ are in the left half of the complex-plane, then $\lim_{t \to \infty} o(t) = \lim_{s \to 0} sO(s)$

Example

Estimating DC Gain by Final Value Theorem

Q: What is the DC Gain of $O(s) = \frac{2s+1}{s^2+s-2}$?

Example

Estimating DC Gain by Final Value Theorem

$O(s) = \frac{2s+1}{s^2+s-2} = \frac{1}{s-1} + \frac{1}{s+2}$

- The system has poles at 1 and -2
- Unstable, no steady state

Example

Estimating DC Gain by Final Value Theorem

Q: What is the DC Gain of $O(s) = G(s)I(s) = \frac{3s+6}{s^2+2s+1}\frac{1}{s}$?
where:

- Designed system $G(s) = \frac{3s+6}{s^2+2s+1}$
- Step input of amplitude 1: $I(s) = \frac{1}{s}$

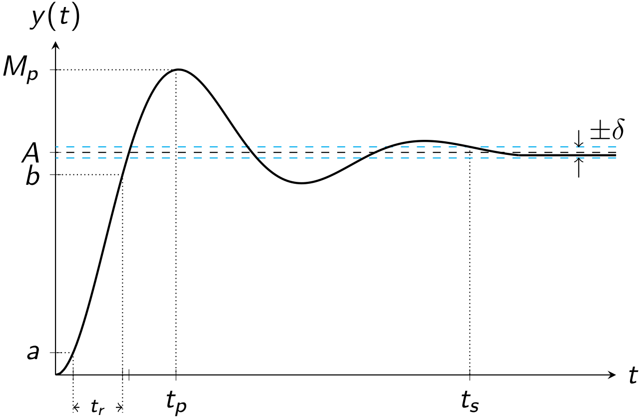
Example

Estimating DC Gain by Final Value Theorem

$$O(s) = G(s)I(s) = \frac{3s+6}{s^2+2s+1}\frac{1}{s} = \frac{3s+6}{s(s^2+2s+1)} = \frac{3s+6}{s(s+1)^2}$$

- The system has poles at 0 and -1
- Stable
- By final value theorem, the system steady state: $\lim_{t\to\infty} o(t) = \lim_{s\to 0} sO(s) = s\frac{3s+6}{s(s+1)^2} = 6$
- DC gain $= \frac{SteadyState}{Input} = \frac{6}{1} = 6$

Other Time-domain Metrics



Other Time-domain Metrics

- Overshoot (M_p) is the max amount of the system output exceeds its steady state divides steady state.
- Peak time (t_p) is the first time that takes the system to reach the overshoot point.
- Rise time (t_r) is the time that takes the system output from the set point a to reach a new set point b for the first time.
- The settling time (t_s) is the time taken the system output to reach and remain within a certain tolerance δ of its steady state value.

Estimating Time-domain Metrics for 2nd-order Systems

Let $O(s) = G(s)I(s)$, if:

- $I(s) = \frac{A}{s}$
- $G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
    - It is called a 2nd-order system, since the highest order of s in the denominator is 2
    - ζ is called damping ratio of the system
    - ω_n is called undamped natural frequency of the system

Then,

- $M_p \approx e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$, where $0 \leq \zeta < 1$
- $t_p \approx \frac{\pi}{\omega_d}$, where $\omega_d = \omega_n\sqrt{1-\zeta^2}$
- $t_r \approx \frac{2.16\zeta+0.60}{\omega_n}$
- $t_s \approx -\frac{ln(\delta)}{\zeta\omega_n}$

Example

Estimating Time-domain Metrics for a 2nd-order System

Q: Estimating Time-domain Metrics for $G(s) = \frac{1}{s^2+s+1}$, excited with $I(s) = \frac{1}{s}$

Example

Estimating Time-domain Metrics for a 2nd-order System

$G(s) = \frac{1}{s^2+s+1}$, excited with $I(s) = \frac{1}{s}$

- Mapping to 2nd-order form of $G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$, $\omega_n = 1, \zeta = 0.5$
- $M_p \approx 16.3\%$
- $t_p \approx 3.62s$
- $t_r \approx 1.68s$
- $t_s \approx 6s$
- Let us compare our estimations with the actual plot in `Matlab`

2nd-Order Systems

Let $O(s) = G(s)I(s)$,

- $I(s) = \frac{A}{s}$
- $G(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$
- The time-domain function
  $o(t) = A - Ae^{-\sigma t}\sqrt{1+\frac{\sigma^2}{\omega_d^2}}\left(cos(\omega_d t) + \frac{\sigma}{\omega_d}sin(\omega_d t)\right)$
  - $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and
  - $\sigma = \zeta\omega_n$
- Time-domain metrics estimations are derived based on $o(t)$.

High-Order System Reduction

- Recall that any proper functions can be expressed as $F(s) = \sum_{i=1}^{n} \frac{C_i}{s-p_i}$ by partial fraction expansion, whose corresponding time-domain function is $f(t) = \sum_{i=1}^{n} C_i e^{p_i t}$.
- p_i in $F(s)$ controls how fast the term $e^{p_i t}$ in $f(t)$ reaches to its steady state. **Keep only dominant poles whose real parts are close to** 0.
- C_i in $F(s)$ controls the weight of $e^{p_i t}$ in $f(t)$. **Keep only weighted poles**.
- Rule of thumb: maintain significant dynamics; if in doubt, plot.

Example

High-Order System Reduction

- $F_1(s) = \frac{1}{s+10} + \frac{1}{s+1} \approx ?$
- $F_2(s) = \frac{10}{s+2} + \frac{0.01}{s+1} \approx ?$
- $F_3(s) = \frac{1}{s-10} + \frac{1}{s+1} \approx ?$