DESIGN OF CORRECT BY CONSTRUCTION SEQUENTIAL ALGORITHMS

PREPRINT

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December 20, 2024

ABSTRACT

These notes describe two techniques for deriving an algorithm from an event evolution. The first is proposed by Jean-Raymond Abrial, who recombines the events of the last model by refinement. obtained by refinement. The second is based on the expression of calls by events and allows the development of recursive programs. We give some development examples to illustrate these techniques.

Keywords Correct by construction · refinement · modelling · safety · partial correctness · total correctness · induction · refinement calculus

[*]Supported by the ANR Project EBRP-EventB-Rodin-Plus (ANR-19-CE25-0010) and by the ANR Project DISCONT (ANR-17-CE25-0005)

Contents

1 Introduction

The development of correct sequential algorithms or sequential programs from specifications (Dijkstra 1976) is a scientific theme linked to that of the verification of programs or algorithms(Turing 1949 ; Floyd 1967 ; Hoare 1969). The fundamental question can be summarised in the form of a symbolic relation $\mathsf{D}, \mathsf{A} \Rightarrow \mathsf{C}$ where D (resp. A, C) is the problem domain (resp. the algorithm, the contract). In this relation, we assume that the problem domain D is known and may be, for example, $\mathbb{Z}$ the domain of integers, and we will be interested in problems requiring properties on integers. A problem is a general expression to designate the calculation of a value from data or the search for a value in a set of data. The A algorithm is an algorithmic expression for expressing assignment statements, conditional statements and bounded or unbounded iterations. Finally, C is a contract expression in the form of two elements a pre-condition $\mathsf{pre}(v_0)$ and a post-condition $\mathsf{post}(v_0, v_f)$ relating the initial value v_0 of a flexible variable v to its final value v_f. Solving the problem consists in expressing it in the form of a contract and ensuring that for any initial value v_0 satisfying $\mathsf{pre}(v_0)$, there exists a value v_f satisfying $\mathsf{post}(v_0, v_f)$. On the other hand, it is important that the final value of v_f corresponds to a calculation of an algorithm A in the classical sense of computability (Rogers 1967). The relation can therefore be rewritten in the following form:$\forall v_0, v_f \in \mathsf{D}.\mathsf{pre}(v_0) \wedge v_0 \xrightarrow{\text{A}} v_f \Rightarrow \mathsf{post}(v_0, v_f)$ and we obtain the expression for the partial correctness of the A algorithm in relation to the contract $\mathsf{C}(v, \mathsf{pre}(v_0), \mathsf{post}(v_0, v_f))$ on the domain D. The relation $\xrightarrow{\text{A}}$ expresses the calculation of A and we can add a second expression which plays the role of the termination of A: $\forall v_0 \in \mathsf{D}.\mathsf{pre}(v_0) \Rightarrow \exists v_f \in \mathsf{D}.v_0 \xrightarrow{\text{A}} v_f$. The relation $\xrightarrow{\text{A}}$ has the right property of determinism in our case of classical sequential algorithms. The two translations produce a synthetic expression of the following form:

$$\forall v_0 \in \mathsf{D}.\mathsf{pre}(v_0) \Rightarrow \begin{pmatrix} \forall v_f \in \mathsf{D}.v_0 \xrightarrow{\text{A}} v_f \Rightarrow \mathsf{post}(v_0, v_f) \\ \exists v_f \in \mathsf{D}.v_0 \xrightarrow{\text{A}} v_f \end{pmatrix}$$

which we rewrite with the weakest-precondition (wp) calculus as follows: $\forall v_0 \in \mathsf{D}.v = v_0 \wedge \mathsf{pre}(v_0) \Rightarrow wp(A)(\mathsf{post}(v0, v))$

which we rewrite with Hoare triples as follows: $\{v = v_0 \wedge \mathsf{pre}(v_0)\}\mathsf{A}\{\mathsf{post}(v_0, v)\}$. Note that the operator wp expresses the total correctness of the statement and leads to the Hoaree logic for total correctness.

This discussion led us to give meaning to the correctness of an algorithm by considering its partial correctness as well as its termination. Hoare logic most often expresses partial correctness and in all rigour it would be necessary to use two notations, one expressing partial correctness and the other total correctness, but the objective here is not to verify an algorithm A and therefore to verify a list of verification conditions as Floyd method indicates, but to find an algorithm A which satisfies this expression $\forall v_0 \in \mathsf{D}.v = v_0 \wedge \mathsf{pre}(v_0) \Rightarrow wp(A)(\mathsf{post}(v0, v))$. The problem is therefore to construct an algorithm A enabling the contract C to be fulfilled in the domain D. In the *a posteriori* approach to correcting algorithms, we propose a solution for A and then apply the list of verification conditions. This technique also consists of applying the verification conditions without having clearly stated the contract C. Semantic analysis techniques can thus be developed with the abstract interpretation (Cousot 2021) and this is based on semantic techniques that simplify the life of the programmer who obtains analysis feedback. Correctness by construction is a technique which starts with an abstract algorithm $\mathsf{A0}$ which fulfils the contract in a verified way and which is progressively enriched with increasingly complex control structures while observing the property of correctness with respect to the contract C. This progressive strategy of adding elements to make the result more precise is guided by the refinement relationship between the algorithms. Each transformation or refinement step guarantees that the resulting algorithm is correct. C. Morgan (Morgan 1990) develops the refinement calculus which makes it possible to progressively and correctly transform one algorithm into another algorithm by guaranteeing that the final algorithm is correct with respect to the first algorithm which is a pre/post specification considered as an algorithmic action of the form $v : |(pre, post)$. $v : |(pre, post)$ designates an algorithmic statement I whose effect is to modify v by respecting the contract defined by pre and post. Thus, the strategy consists of constructing a sequence of algorithms A0,..., Ai, ..., An with the properties:

- $\mathsf{A0}$ is the expression of the contract: $\mathsf{D}, \mathsf{A0} \Rightarrow \mathsf{A0}$
- for all i in $0..n-1$, the algorithm Ai refines $\mathsf{Ai}-1$: $\mathsf{D}, \mathsf{Ai} \Rightarrow \mathsf{C}$ and $\mathsf{D}, \mathsf{Ai}-1 \Rightarrow \mathsf{C}$.
- An is the algorithm satisfying the contract: $\mathsf{D}, \mathsf{An} \Rightarrow \mathsf{C}$.

More recently, Derrick G. Kourie and Bruce W. Watson (Kourie and Watson 2012) follow this strategy and implement the correction-by-construction paradigm on classical examples of classical programming problems. This approach is equipped with Key to enable the rules applied to be validated using a tool. the rules applied. We can identify in these two calculations of the fact that the contract becomes an algorithmic statement corresponding to a generalised

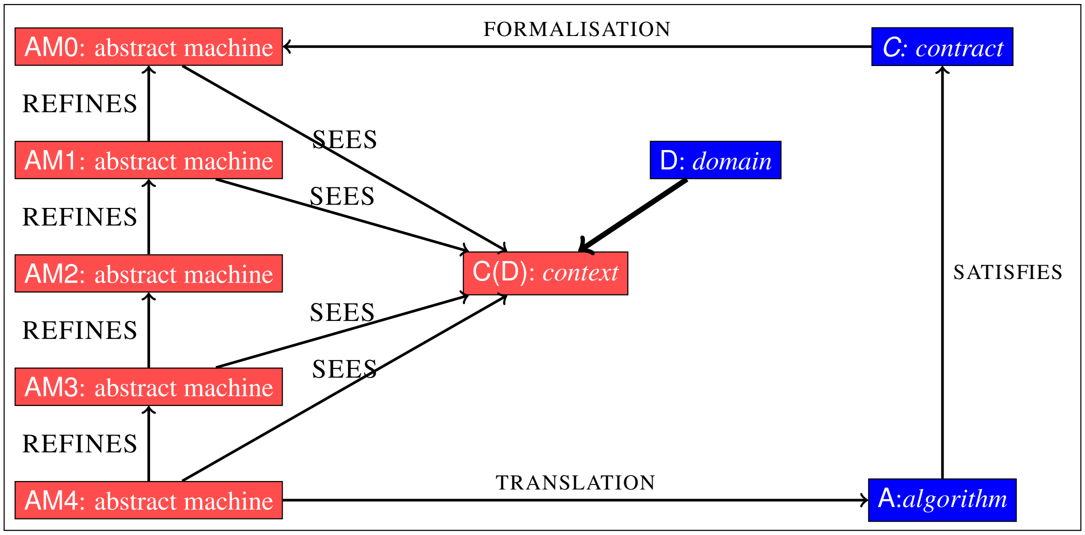


Figure 1: Correctness by construction in Event-B

algorithmic structure. In fact, as we can see, a contract can express the halting of Turing machines (Rogers 1967) in a language of assertions which is still fairly abstract, but this does not mean that we have solved the halting problem, but that we have extended the algorithmic language with a statement magic enabling it to be solved. This amounts to extending the space of solutions and then choosing what corresponds to the theory of computability. C. Morgan reminded us that all second degree equations have solutions in fields of complexes, but that the method of solving in the set of reals retains only the real solutions. The specification statement v : [pre, post] is a valid statement in this algorithmic language. Such a specification statement can be expressed in the Event-B language.

This approach to developing correct by construction algorithms is quite simply implemented in the Event-B language and in fact equipped by the Rodin environment. Figure 1 describes the general idea. This idea consists of translating the contract as a *event* which observes the calculation described by the contract. The contract expresses the *what* but carries out the calculation as an event observation. This event is placed in a first abstract machine AM0, which uses the mathematical elements extracted from the problem domain D and expressed in the context Event-B C(D). The development of an algorithm consists in the gradual enrichment of the extracted machines AM0, . . . AMn, by expressing the computations necessary to systematically translate the last machine into an algorithmic form so as to guarantee the correctness of the A algorithm thanks to the correctness of the transformations provided by the refinement. It should be noted that the correctness concerns partial correctness and termination, and that the abstract machines are models containing variables that will not be implemented in the algorithm produced.

We will present two techniques that implement this development pattern:

- the inductive pattern based on transformations of abstract machines by Jean-Raymond Abrial (Abrial 2010, Chapter 15).
- the recursive pattern based on the relation call,textitevent that we have developed (Méry 2009 ; Cheng et al. 2016).

We will present these two methods by highlighting case studies of classical sequential algorithms.

2 Design of a Iterative Sequential Algorithm

2.1 Problem 1 : Calculating the sum of a vector v of integer values

First, we define the contract of calculating the sum of the elements of the vector v_0. The algorithm we are looking for is called SUM.

variables n, v, r
definitions $pre(n_0, v_0, r_0) \stackrel{def}{=} \begin{cases} n_0 \in \mathbb{N} \wedge n_0 \neq 0 \\ v_0 \in 1..n_0 \rightarrow \mathbb{Z} \\ r_0 \in \mathbb{Z} \wedge i_0 \in \mathbb{Z} \end{cases}$
requires $\left(\begin{array}{l} n_0 \in \mathbb{N} \wedge n_0 \neq 0 \\ v_0 \in 1..n_0 \rightarrow \mathbb{Z} \end{array} \right.$ ensures $\left(\begin{array}{l} r_f = \sum_{k=1}^{k=n_0} v_0(k) \\ n_f = n_0 \\ v_f = v_0 \end{array} \right.$

The domain of the problem to be solved is that of the integers $\mathbb{Z}$ and the contract states that the value of the result is the sum of the integers in the sequence v. This mathematical expression is not directly expressible in the mathematical language of Event-B and we define a sequence u characterising the values of the partial sums. The context associated with our $\mathsf{C}(\mathbb{Z})$ Event-B model is defined by enumerating the *requires* hypotheses and defining u.
First, we need to express the summation r of the sequence v_0 in the language of Event-B ; this formulation is immediate in mathematical terms: $r = \sum_{k=1}^{k=n_0} v_0(k)$. As the notation for summing a finite sequence of values is not provided in the basic elements of the language, we must *define* this notion in a context *c0* which will contain the data of the problem and the notations defined specifically for this case.
Thus, the *data* n_0 and v_0 are defined as being respectively a non-zero natural integer (axioms pre1,pre2) and a function v_0 of domain $1..n_0$ and codomain $\mathbb{Z}$ (axiom pre3). THe prefix pre intends to mean that the axioms are requirements. The aim is to define the theory in which we will describe our data.

Secondly, we introduce a sequence u of integer values corresponding to the partial sums $\sum_{k=1}^{k=i} v_0(k)$. To do this, the idea is to define the partial summations using an inductive definition inductive definition, which technically requires us to be sure of the *well definedness* of this sequence u. The sequence u is therefore defined as follows:

- u is a total function of $0..n0$ in $\mathbb{Z}$ (axiom axm1).
- Initially, the summation starts with 0 and $u(0) = 0$ (axiom axm2).
- For values of i less than n_0, the value of $u(i)$ is defined from that of $u(i-1)$ and $v_0(i)$ (axiom axm3).

Axioms are given in the context of *c0* and constitute a theory which will be useful for proving the properties of the models we will develop later.

CONTEXT $S0$

CONSTANTS $n0\ v0\ u$

AXIOMS
$@pre1\ n0 \ \in\ \mathbb{N}$
$@pre2\ n0 \ \neq\ 0$
$@pre3\ v0 \ \in\ 1..n0 \rightarrow \mathbb{Z}$
$@axm1\ u \ \in\ 0..n0 \ \rightarrow\ \mathbb{Z}$
$@axm2\ u(0) = 0$
$@axm3\ \forall k.\ k \ \in\ 1..n0 \ \Rightarrow\ u(k) = u(k-1) + v0(k)$
end

Each axiom is validated by a set of proof obligations (WD) to ensure the well-definedness of axioms. In We have therefore defined the mathematical framework of the problem and we will now define the problem of summing the sequence v_0.

2.1.1 Specification of the problem to solve

```
MACHINE  S1  SEES  S0

 VARIABLES  r v n

 INVARIANTS
  @inv1 r  ∈  ℤ
  @inv2 n  ∈  ℤ
  @inv3 v  ∈  1..n → ℤ
  @read − values n = n0  ∧  v = v0

 EVENTS
  EVENT  INITIALISATION
     then
    @act1 r  :∈  ℤ
    @act2 n  :=  n0
    @act3 v  :=  v0
   end

  EVENT  final
     then
    @act1 r  :=  u(n)
   end

  anticipated EVENT  keep
     then
    @act1 r, n, v  :|
        ( r′  ∈  ℤ  ∧  n′  ∈  ℤ
         ∧  v′  ∈  1..n′  →  ℤ
         ∧  n′ = n0  ∧  v′ = v0  ∧  )
   end
 end
```

The problem is therefore to calculate the value of the sum of the elements of the sequence v. We define a *S1* machine which is an abstract machine expressing through the final event the *postcondition* $r = u(n)$. In fact, the new value of the variable r will be $u(n)$, when the event Event final has been observed. The initial value of r is arbitrary at initialisation. Finally, the variable r must satisfy the very simple invariant $inv1 \ : \ r \in \mathbb{Z}$; this information constitutes a typing of the variable r. The event Event final is therefore simply an assignment of the value $u(n)$ to r.
We can express it as a HOARE triple:

$$\left\{\begin{array}{l} n = n_0 \\ \wedge v = v_0 \\ \wedge n_0 > 0 \\ \wedge v_0 \in 1 \mathinner{..} n_0 \rightarrow \mathbb{N} \end{array}\right\} \text{SUM} \left\{ \begin{array}{l} r = u(n_0) n_0 > 0 \end{array} \right\}.$$

Note that the data is *visible* from the context *S0*. The problem is therefore to find an algorithm that calculates the value $u(n)$ and stores it in r. A second event called keep can be also added to simulate some hidden activity before the observation of the event final. Note that the event is *anticipated*.
We have therefore described the problem domain to be solved and we have formulated what we want to calculate. The next step is to inventing a *method of calculation* and this requires a *idea of solution* and the use of refinement.

2.1.2 Refining to compute inductively

We have defined the specification of the problem for calculating the sum of the elements of a sequence v_0 and we now need to find a way to *calculate* the value of the sequence u at term n_0. The assignment $r := u(n_0)$ is an expression mixing a variable r and a mathematical value $u(n_0)$. A trivial and inefficient solution is well known: store the values of the sequence u in an array uu and translate the assignment into the form $r := uu(n)$ where uu verifies the following property $\forall k.k \in dom(uu) \Rightarrow uu(k) = u(k)$ and this property constitutes an element of the invariant inv4. The idea is therefore to use the variable uu ($uu \in 0 \mathinner{..} n_0 \nrightarrow \mathbb{Z}$) to control the calculation and its progress. Progression is ensured by the event step, which decreases the quantity $n - i$ and therefore ensures that the process converges.

```
MACHINE S2
  REFINES S1
  SEES S0

 VARIABLES r i uu n v

 INVARIANTS
  @inv1 i ∈ 0..n
  @inv2 uu ∈ 0..n ⇸ ℤ
  @inv3 dom(uu) = 0..i
  @inv4 ∀ k. k ∈ dom(uu)
       ⇒ uu(k) = u(k)
  theorem @inoutdata1 v = v0
  theorem @inputdata2 n = n0

  VARIANT n − i
```

```
EVENT INIT
    then
   @act1 r :∈ ℤ
   @act3 i := 0
   @act4 uu := { 0 ↦ 0}
   @act5 n := n0
   @act6 v := v0
 end

EVENT final
 REFINES final
    where
   @grd1 n ∈ dom(uu)
    then
   @act1 r := uu(n)
 end
```

```
convergent EVENT step
  REFINES keep
    where
   @grd1 n ∉ dom(uu)
    then
   @act1 i := i + 1
   @act2 uu(i + 1) := uu(i) + v(i + 1)
  end

END
```

The S2 machine therefore describes a process which progressively fills the uu table and therefore retains all the intermediate results. The proof obligations are fairly easy to prove insofar as we have *prepared* the work of the proof assistant. We will give the details of the statistics in a table at the end of the development. It is quite clear that the variable uu is in fact a witness or a trace of the intermediate values and that this variable can therefore be hidden in this model which will have to be refined. Before hiding this variable, we will set aside the value that we need to keep $uu(i)$.

2.1.3 Focus on the value to be preserved

The following refinement S3 will lead to the introduction of a new variable cu which will retain the last current value $uu(i)$. We therefore operate a *superposition* (Chandy and Misra 1988) on the S2 machine. The idea is therefore that this model refines or simulates the model S2 and this also means that the properties of the refined machines remain verified by the new machine S3 insofar as the proof obligations are all verified.

```
MACHINE S3 REFINES S2 SEES S0

 VARIABLES r i uu cu n v

 INVARIANTS
  @inv1 cu ∈ ℤ
  @inv2 cu = uu(i)

 EVENTS
  EVENT INITIALISATION
      then
     @act1 r :∈ ℤ
     @act2 i := 0
     @act3 uu := { 0 ↦ 0}
     @act4 cu := 0
     @act6 n := n0
     @act7 v := v0
   end
```

```
EVENT final REFINES final
     where
   @grd1 i = n
     then
   @act1 r := cu
  end

 EVENT step REFINES step
     where
   @grd1 i < n
     then
   @act1 i := i + 1
   @act2 uu(i + 1) := uu(i) + v(i + 1)
   @act3 cu := cu + v(i + 1)
  end
```

This machine is very expressive and provides a lot of informations to ensure that the machine is suitable for the problem expressed in the S2 machine, which is refined by this S3 machine. A important issue is that the new guards are closer to an implementation: $n \in dom(uu)$ (resp. $n \notin dom(uu)$) is substituted by $i = n$ (resp. $i < n$). It is even clearer that this S3 machine is expensive in terms of variables and the refinement allows us to keep only the variables that are useful for the calculation. In what follows, we will make the model more algorithmic and retain only those variables in the concrete model that are sufficient for the calculation.

2.1.4 Obtaining an algorithmic machine

In this final step, we refine the S3 machine into a S4 machine and hide the uu variable from the abstract S3 machine. Thus, the S4 machine includes the variables r, n, v, cu and i and we will also note that it satisfies safety properties called theorems in the S4 machine. These properties are proved from the properties of previous refined machines. We have thus obtained a machine comprising an initialisation and two events:

- The event final is observed when the value of i is n and, in this case, the variable cu contains the value $u(n)$. The invariant guarantees that the value of cu is $u(n)$.

- The event step000 is observed, when the value of i is less than n. This also means that, as long as this value is less than n, the event can be observed and the traces generated from these events therefore correspond to an iteration algorithmic structure.

MACHINE $S4$ REFINES $S3$ SEES $S0$

VARIABLES $r\ i\ cu\ n\ v$

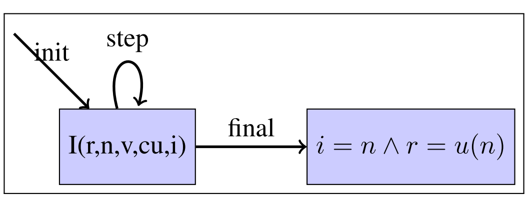
INVARIANTS
$theorem\ @inv1\ cu = u(i)$
$theorem\ @inv2\ i \leq n$

EVENTS
EVENT $INITIALISATION$
then
$@act1\ r\ :\in\ \mathbb{Z}$
$@act2\ i\ :=\ 0$
$@act4\ cu\ :=\ 0$
$@act5\ v\ :=\ v0$
$@act6\ n\ :=\ n0$
end

EVENT $final$ REFINES $final$
where
$@grd1\ i = n$
then
$@act1\ r\ :=\ cu$
end

EVENT $step000$ REFINES $step$
where
$@grd1\ i\ <\ n$
then
$@act1\ i\ :=\ i + 1$
$@act3\ cu\ :=\ cu + v(i + 1)$
end

The Rodin project archive abk-summation corresponds to this development by refinement, taking care to use the calculation method defined by the u sequence. The following diagram describes a view of the events observed as a function of the value of i.



$r :\in \mathbb{Z} \parallel i := 0 \parallel cu := 0 \parallel v := v0 \parallel n := n0$
while $i < n$ do
$i := i + 1 \parallel cu := cu + v(i + 1)$
od;
$r := cu$;

The components of the project abk-summation are constructed using the u sequence as a guide, taking care to obtain conditions that can be expressed in an algorithmic language. In our case, the condition $n \notin dom(uu)$ (resp. $n \in dom(uu)$) is refined by the condition $i < n$ (resp.$i = n$). Note that the diagram on the left corresponds to the algorithm on the right. These transformations can be defined more clearly and are implemented in the EB2ALGO (Singh 2024) plugin which produces the above algorithm from the project abk-summation. Jean-Raymond Abrial (Abrial 2010, Chapter 15) suggests progressive transformation rules to be applied on events like S4 and we give a more complete treatment of these transformation rules implemented by EB2ALGO (Singh 2024).

variables n, v, r

definitions

$$pre(n_0, v_0, r_0) \stackrel{def}{=} \begin{cases} n_0 \in \mathbb{N} \land n_0 \neq 0 \\ v_0 \in 1..n_0 \to \mathbb{Z} \\ r_0 \in \mathbb{Z} \land i_0 \in \mathbb{Z} \end{cases}$$

requires $\left(\begin{array}{l} n_0 \in \mathbb{N} \land n_0 \neq 0 \\ v_0 \in 1..n_0 \to \mathbb{Z} \end{array} \right.$

ensures $\left(\begin{array}{l} r_f = \sum_{k=1}^{k=n_0} v_0(k) \\ n_f = n_0 \\ v_f = v_0 \end{array} \right.$

```
int SUM(int n0, v0, r0, i0)
variables
   int r, i = 0, cu = 0, v = v0, n = n0;
while i < n do
   cu := cu + v(i + 1);
   i := i + 1;
od;
r := cu;
return(r);
```

2.1.5 Comments on the methodology

A specific methodology was employed in the selection of the variables. The uu variable is used for the storage of the values calculated from the u sequence, with the convention being to link the u sequence and the uu variable obtained by doubling the name of u. Obviously, we don't want to store all the intermediate values, just the ones used in the induction step. So the variable cu acts as a cursor to the value of uu that is useful in the induction step. uu is a model variable that is no longer necessary to retain for the algorithm. However, it has made the proof work easier, so it should be retained. Hiding uu provides a truly algorithmic view. It is also possible to obtain the termination of this algorithm with minimal effort, thanks to the variant which indicates that the event step leads to the decreasing and convergence of this algorithm. The name final is only imposed by the plugin EB2ALGO (Singh 2024) and the use of Jean-Raymond Abrial (Abrial 2010, Chapter 15). Note that abstract machines implicitly contain the event skip and that each new event refines the previous level event skip. Another strategy would have been to introduce into the machine S1 an event keep which simulates the loop by anticipation. The archive abk-summation gives a version using this artifice and illustrates the use of an event *anticipated*.

2.2 Transformations of machines Event-B into sequential algorithms

We take the conclusions of this simple problem and add the extra elements the reader need to develop iterative sequential algorithms. The plugin EB2ALGO implements the transformations of Jean-Raymond Abrial (Abrial 2010, Chapitre 15). We apply two transformations to the abstract machines obtained at the end of the refinement process, which we called ALGO and it simplifies the calculation process by hiding model variables. We recall the algorithmic language used by Jean-Raymond Abrial (Abrial 2010, Chapter 15)).

Definition 1 (The Pidgin Programming Language (Abrial 2010, Chapitre 15))

Statements of the language are:

- $variable\_list := expression\_list$
- statement; statement
- if *condition* then *statement* else *statement* end
- if *condition* then *statement* elseif ... else *statement* end
- while *condition* do *statement* end

A program can be *broken down* into a set of events, which are then triggered according to the values of the variables. This decomposition leads to the use of a composition of events. The idea is straightforward, but it must adhere to strict conventions to use the EB2ALGO (Singh 2024) plugin. We give these conditions which are implemented by the plugin and which must be respected, when designing the development.

Let us consider two events, which we will merge into an algorithmic expression:

EVENT e	EVENT f		EVENT $merge(e,f)$
WHEN P Q THEN S END	WHEN P $\neg Q$ THEN T END	$\rightsquigarrow$	WHEN P THEN WHILE Q DO S OD; T; END

We must specify the conditions of application. The event e appears as new or non-anticipated, therefore convergent at a lower level of refinement than that of f. We can be sure that there is a variant which terminates the loop. We must also assume that P is invariant to the event e. The event merge(e,f) appears at the same level as the event f. It is possible that P is not present.

			event $merge(e,f)$

WHEN
P
Q
THEN
S
END
	WHEN
	P
	$\neg Q$
	THEN
	T
	END
		$\rightsquigarrow$	WHEN
			P
			THEN
			IF Q THEN
			S
			ELSE;
			T;
			FI;
			END

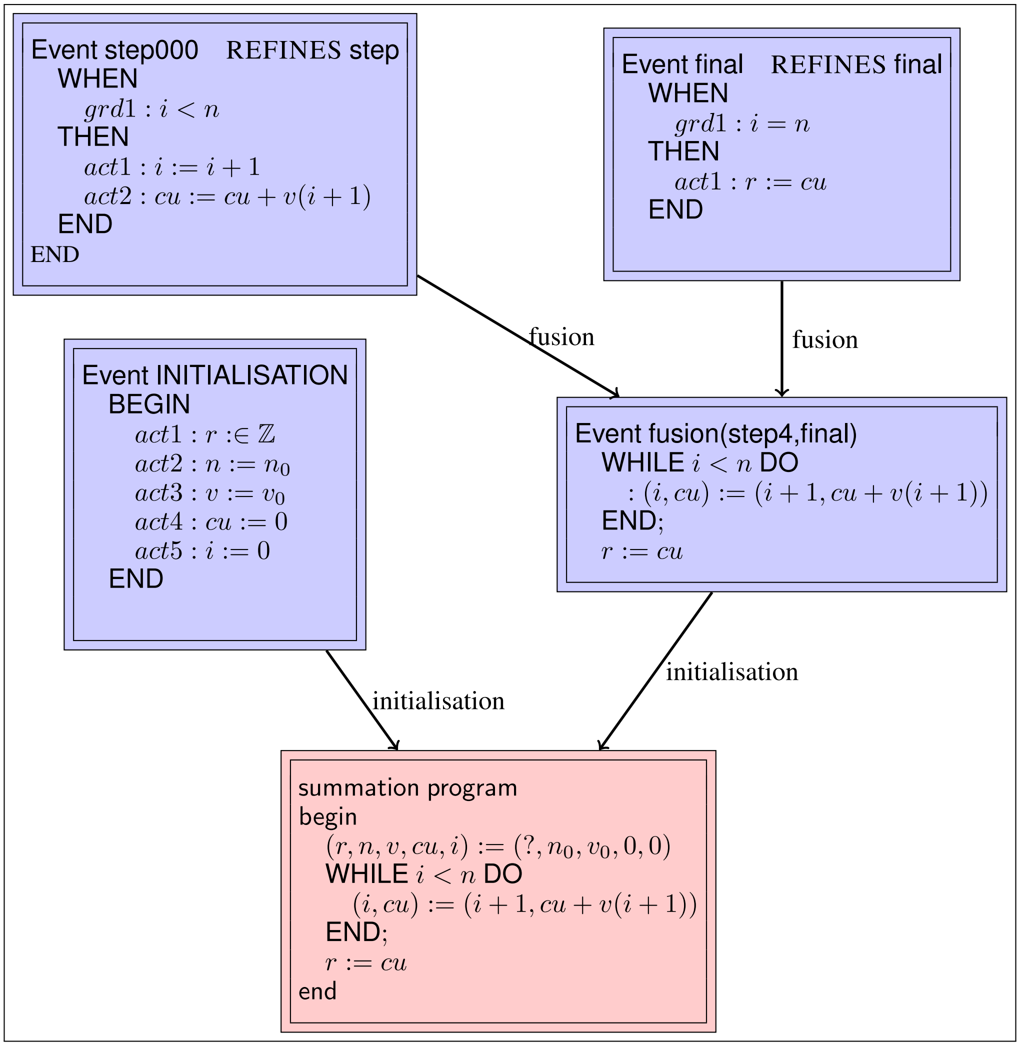
This transformation should be applied, when the two events have been introduced at the same time. The event merge(e,f) must appear at the same level as the component. The guard P may not be present.

These two transformations can be used to design a plugin that produces a program in the language mentioned. The initial event is always called final and corresponds to the specification. Then the refinement process guides the design phase and it is also important to express that the new events that are introduced must decrease by a variant that ensures the convergence of the process described by the events.

Let us take the example we've already dealt with and apply the transformations.

```
MACHINE S4 REFINES S3 SEES S0

 VARIABLES r i cu n v

 INVARIANTS
  theorem @inv1 cu = u(i)
  theorem @inv2 i ≤ n

 EVENTS
  EVENT INITIALISATION
     then
   @act1 r :∈ ℤ
   @act2 i := 0
   @act4 cu := 0
   @act5 v := v0
   @act6 n := n0
  end
```

```
EVENT final REFINES final
   where
  @grd1 i = n
   then
  @act1 r := cu
 end

EVENT step000 REFINES step
   where
  @grd1 i < n
   then
  @act1 i := i + 1
  @act3 cu := cu + v(i + 1)
 end
```



We have given an example of how to apply the merge transformation for iteration and we refer the user to the plugin that implements these transformations.

3 Examples of development

In this section, we illustrate the method using a few classic examples. Other examples have been produced by the Event-B community in particular EB2ALGO (Singh 2024) et Jean-Raymond Abrial (Abrial 2010, Chapter 15)).

3.1 Problem 2: Computing the function power 3 $\lambda x.x^3$ using only additions

The objective is to calculate the function power 3 ($\lambda x.x^3$) of a positive or zero integer using only addition operations. The method is to define a sequence z corresponding to the cubes of positive integers. The analysis that leads to the sequences defining z is not provided here, but the sequence is correct.

variables x, r
definitions $pre(x_0, r_0) \stackrel{def}{=} \begin{cases} x_0 \in \mathbb{N} \\ r_0 \in \mathbb{Z} \end{cases}$
requires $\begin{pmatrix} n_0 \in \mathbb{N} \\ r_0 \in \mathbb{Z} \end{pmatrix}$ ensures $\begin{pmatrix} r_f = x_0 * x_0 * x_0 \\ x_f = x_0 \end{pmatrix}$

Calculating the function power 3 ($\lambda x.x^3$) using only addition is based on the following sequences:

- $z_0 = 0$ et $\forall n \in \mathbb{N} : z_{n+1} = z_n + v_n + w_n$
- $v_0 = 0$ et $\forall n \in \mathbb{N} : v_{n+1} = v_n + t_n$
- $t_0 = 3$ et $\forall n \in \mathbb{N} : t_{n+1} = t_n + 6$
- $w_0 = 1$ et $\forall n \in \mathbb{N} : w_{n+1} = w_n + 3$
- $u_0 = 0$ et $\forall n \in \mathbb{N} : u_{n+1} = u_n + 1$

The first step is to show that the sequence z defines the sequence of cubes of different integers and therefore gives an inductive way of calculating the cube of a positive integer $x0$ using only addition alone. The domain of the problem to be solved is that of the integers $\mathbb{Z}$ and the contract expresses that the value of the result is the cube of the positive integer x_0.

The context P3-0 expresses the sequences defining the values to be calculated to produce a value corresponding to the cube of x_0. The important result is to show the following theorem with the Rodin platform.

Property 1 (Soundness of the sequence z)

$\forall k.k \in \mathbb{N} \Rightarrow z(k) = k * k * k$

This property guarantees that calculating the terms of the z sequence allows the value to be calculated using auxiliary sequences and only addition operations. These elements are given in the context P3-0. The contract can then be written in the form of a machine. P3-1 contains a single event, FINAL, whose action is `z := z(x0)`.

The method consists in refining the P3-1 machine into a P3-2 machine and introducing an iteration variable i covering the interval $0..x_0$ and variables for each sequence uu, vv, ww, tt, zz whose role is to store the values of the sequences to calculate $z(x_0)$. A new event is introduced to update the variables uu, vv, ww, tt, zz and i. The invariant expresses the link between the mathematical values of the sequences u, v, w, t, z and the values stored and calculated in the variables uu, vv, ww, tt, zz. The invariant is fairly easy to determine, but we probably need to provide more relationships between the different sequences, so we come back to those sequences which have expressions that only mention i.

Property 2 (Properties of sequences u,v,w,t)

- $\forall k \in \mathbb{N} : v_k = 3 * k * k$
- $\forall k \in \mathbb{N} : w_k = 3 * k + 1$
- $\forall k \in \mathbb{N} : u_k = k$
- $\forall k \in \mathbb{N} : t_k = 3 * k + 3$

The properties are proved in the context of P3-0 and will be used in the refinement of the P3-2 machine. We introduce variables that point to the elements of the sequences that are sufficient for the computation. The refinement of P3-2 into P3-3 amounts to adding a variable for each sequence cu, cv, cw, ct, cz and these variables verify the following invariant property:

$$
\begin{array}{l}
inv1 : 0 \leq i \wedge i \leq x0 \wedge cv = vv(i) \\
inv2 : cw = ww(i) \\
inv3 : cz = zz(i) \\
inv4 : ct = tt(i) \\
inv5 : cu = uu(i)
\end{array}
$$

In addition, the events final and step are refined by making the guards *verifiable*. An expression of the form $x0 \in dom(zz)$ or $x0 \notin dom(zz)$ is difficult to translate efficiently into an algorithmic language. Thus, the new guard $i < x0$ implies $x0 \notin dom(zz)$ and the new guard $i = x0$ implies $x0 \in dom(zz)$. The resulting machine is therefore more deterministic and more approximate to an algorithmic expression. The proofs are automatic.

We finish by hiding the variables uu, vv, ww, tt, zz in a refinement P3-4 of the machine P3-3. It remains to use the property of sequences that we have proved in the context. The proof effort made in the context of P3-4 pays off when it comes to expressing the invariant properties constituting the loop invariant of the iterative algorithm produced from this machine.

The variable cu is useless, since it contains i. We have translated the P3-4 machine in the form of an ACSL algorithm in the listing 1 verified by the Frama-c (Baudin et al. 2021) application automatically. We obtained a cross-check of the algorithm obtained by translation from the P3-i machines.

Listing 1: ACSL power3.c

```
/*@  requires  0 <= x;
     ensures \result == x*x*x;
*/
int power3(int x)
{int  r, cz,cv,cu,cw,ct, i;
  cz=0;cv=0;cw=1;ct=3;i=0;
      /*@
        @ loop invariant ct == 6*i +3;
        @ loop invariant cv== 3*i*i;
        @ loop invariant cw == 3*i+1;
        @ loop invariant cz == i*i*i;
        @ loop invariant i  <= x;
        @ loop assigns ct,cz,i,cv,cw,r; */
  while (i<x)
        {
         cz=cz+cv+cw;
         cv=cv+ct;
         ct=ct+6;
         cw=cw+3;
         i=i+1;}
  r=cz;return(r);
}
```

The bb-power3 project contains all the machines used to develop this algorithm.

3.2 Problem 3: Searching a value in an array

The problem is to find the occurrence of a value x in an array t of dimension n. There are no constraints on the array or the search technique.

variables t, n, x, r

definitions

$$pre(x_0,, r_0) \stackrel{def}{=} \begin{cases} x_0 \in \mathbb{V} \\ t_0 : 1..n_0 \rightarrow \mathbb{V} \\ r_0 \in \mathbb{Z} \times BOOL \end{cases}$$

$$\text{requires} \begin{pmatrix} x_0 \in \mathbb{V} \\ t_0 : 1..n_0 \rightarrow \mathbb{V} \\ r_0 \in \mathbb{Z} \times BOOL \end{pmatrix}$$

$$\text{ensures} \begin{pmatrix} t_f = t_0 \wedge n_f = n_0 \\ x \notin ran(t) \Rightarrow r_f = (0, FALSE) \\ x \in ran(t) \Rightarrow prj2(r_f) = FALSE \Rightarrow t_f(prj1(r_j)) = x \end{pmatrix}$$

This problem must be reformulated in the form of a sequence that expresses for a value $i \in 0..n$ whether the array t contains the value x between 1 and n. As soon as the value is found in i, $u(j)$ is equal to x and the value found is i. If the value x is not in the table, then the sequence u is equal to the pair $(0, FALSE)$. We will define the sequence u in the context of S-0 and use the same methodology.

The context S0 contains definitions of the data t_0, n_0, x_0 and the axioms defining the sequence u whose value $u(n_0)$ is the expected solution.

CONTEXT $S0$

$sets\ V$

CONSTANTS $n\ t\ x\ u$

AXIOMS
$@axm1\ n \in \mathbb{N}$
$@axm2\ t \in 1..n \rightarrow V$
$@axm3\ x \in V$
$@axm4\ u \in 0..n \rightarrow \mathbb{Z} \times BOOL$
$@axm5\ u(0) = 0 \mapsto FALSE$
$@axm6\ \forall i\ .\ i \in 0..n-1 \wedge t(i+1) = x \wedge prj2(u(i)) = FALSE \Rightarrow u(i+1) = i+1 \mapsto TRUE$
$@axm7\ \forall i\ .\ i \in 0..n-1 \wedge t(i+1) \neq x \Rightarrow u(i+1) = u(i)$
$@axm8\ \forall i\ .\ i \in 0..n-1 \wedge t(i+1) = x \wedge prj2(u(i)) = TRUE \Rightarrow u(i+1) = u(i)$
$theorem\ @th1\ \forall i\ .\ i \in 1..n \wedge prj2(u(i)) = FALSE \Rightarrow (\forall k.\ k \in 1..i \Rightarrow t(k) \neq x)$
$theorem\ @th2\ \forall i\ .\ i \in 1..n \wedge prj2(u(i)) = TRUE \Rightarrow (\exists k.\ k \in 1..i \Rightarrow t(k) = x)$
$theorem\ @th3\ \forall i\ .\ i \in 1..n \wedge prj2(u(i)) = TRUE \Rightarrow (t(prj1(u(i))) = x)$
end

The machine S1 expresses that the desired value is $u(n_0)$ and in fact expresses the contract for this problem. Unlike the Unlike the previous cases, the S1 machine is refined into a S-2 machine which introduces the variables i and uu which define the inductive calculation scheme and which introduces two events step1 and step2 which simulate a conditional instruction according to the axioms defining u. The process continues with the introduction of the value of the sequence uu useful for the calculation, i.e. cu in the S3 machine. The S4 machine hides the uu variable and produces a fairly classic algorithm.

```
r := (0, FALSE) || k := 0 || cuu := (0, FALSE)
while k < n do
   if t(k + 1) = x then
      if prj2(cuu) = FALSE then
         k := k + 1 || cuu := (k + 1, TRUE)
      else
         k := k + 1 || cuu := (k + 1, TRUE)
      elsek := k + 1||cuu := cuuelsek := k + 1||cuu := cuuod;
```

The project db-searching contains all the machines used to develop this algorithm.

3.3 Problem 4: Computing a primitive recursive function

Recursive primitive functions correspond to bounded iterations and a recursive primitive function is constructed from a scheme using two recursive primitive functions to define a function whose value we want to construct the algorithm that calculates its value. Examples of such functions are addition, multiplication, division, primality, ...

variables x, n, r
services $H \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ $G \in \mathbb{N} \rightarrow \mathbb{N}$ $F \in \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ $F = PRIMREC(G, H)$
requires $\left(\begin{array}{l} x_0 \in \mathbb{N} \\ n_0 \in \mathbb{N} \end{array} \right.$ ensures $\left(\begin{array}{l} x_f = x_0 \wedge n_f = n_0 \\ r_f = F(x_0, n_0) \end{array} \right.$

The problem is to calculate the function F defined using two other functions G and H which are calculated by algorithms. algorithms. F is defined by the following equation: for any natural natural value x,y, $\left\{ \begin{array}{l} F(x,0) = G(x) \\ F(x;y+1) = H(x,y,F(x,y)) \end{array} \right.$
The function F is defined using G and H, which are themselves defined by the same calculability schemes. This function F is itself a sequence which is specialised in v. We obtain the following algorithm.

```
r :∈ ℤ ∥ k := 0 ∥ cv := G(x)
while k < x do
k := k + 1 ∥
cv := H(x, k, cv)
od;
r := cv
```

The eb-primitiverecursive archive contains all the machines used to develop this algorithm.

Jean-Raymond Abrial (Abrial 2010, Chapter 15) gives a list of programs built using this methodology and provides the Rodin[2]archives. It is important to note that Jean-Raymond Abrial's examples begin with a machine with an event final modelling the calculation corresponding to the problem solved, but also an event progress whose status *anticipated* means that events will appear to model one or more loops. This event models the implicit and hidden presence of intermediate calculations which must respect the invariant of the abstraction level. This event *conserves* or *preserves* the calculation and its invariant; we sometimes call it keep as opposed to an event which is always present called skip. Finally, it is quite clear that the translation still requires an intervention to produce an executable program as we have shown with the 1 algorithm and this allows us to check that the translation is correct since Frama-C is used to check the contract obtained. The loop invariant is derived from the Event-B development. Jean-Raymond Abrial uses Hoare triples to express the specification of the problem to be solved and we prefer to use contracts that are available in programming languages such as ACSL/Frama-C or Spec#/Boogie.

4 Design of a Recursive Sequential Algorithm

In this section, we will apply the simple idea of analysing the diagram in figure 1 to develop a recursive program or algorithm. We will also promote one-shot refinement. In the previous section, we refined as long as necessary, especially as long as we obtained a set of events corresponding to computable elements. We have produced the EB2RC plugin (Cheng et al. 2016) , which implements this idea.

4.1 The "Call as Event" Idea

The refinement-based design of iterative sequential algorithms uses a sequence of values in a domain $\mathcal{D}$ and the computation process is based on the recording of the values of the sequence. In the case of the call-as-event paradigm, the pattern is based on the link between the occurrence of an event and a call of a function or procedure or method satisfying the pre-condition and post-condition respectively at the call point and the return point. The context C0 defines the sequence of values and the definition of the sequence is used as a guide for the shape of events. The definitions of sequence are reformulated by a diagram which is simulating the different cases when the procedure under development is called namely $P(x, r)$.

The diagram is derived from the Event-B model called ALGOREC and is a finite state diagram. It includes a liveness proof very close to the proof lattices of Owicki and Lamport (Owicki and Lamport 1982). We use special names for events in the diagram: P(0,r): $x = 0$ stands for the event observed when the procedure P(x,r) is called with x=0; P(x-1,tr): $x \neq 0$ models the observation of the recursive call of P; P(x,r): $x \neq 0$ stands for the event observed when the procedure P(x,r) is called with $x \neq 0$. P(0,r): $x = 0$ and P(x,r): $x \neq 0$ are refining the event computing which is observed when the procedure P is called.

[2]The link `https://web-archive.southampton.ac.uk/deploy-eprints.ecs.soton.ac.uk/122/index.html` gives a list of sequential program developments with a tutorial detailing how to refine and what to transform.

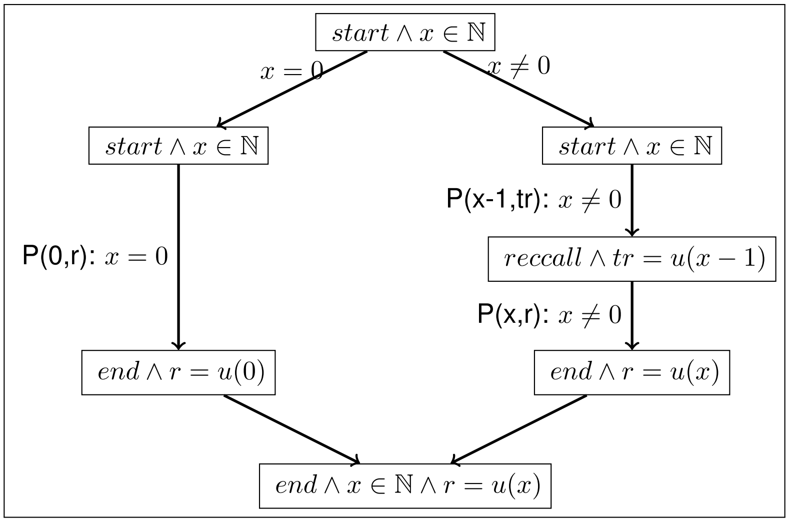


Figure 2: Organisation of the computation in a recursive solution using assertion diagram

MACHINE $ALGOREC$
REFINES $PREPOST$
SEES $C0$
VARIABLES
  r, pc, tr
INVARIANTS
    $art : pc \in L$
    $inv1 : tr \in D$
    $inv2 : pc = callrec \Rightarrow tr = v(x-1)$
    $inv3 : pc = end \Rightarrow r = v(x)$

The refinement is an organisation of the inductive definition using a control variable *pc*. The control variable *pc* is organising the different steps of the computations simulated by the events. The invariant is derived directly from the definitions of the intermediate values. Proof obligations are simple to prove. It remains to prove that the values of the sequence v correspond to the required value in the post-condition.

Event P(x,r):x=0
  REFINES computing
  WHEN
    $grd1 : x = 0$
    $grd2 : pc = start$
  THEN
    $act1 : r := d0$
    $act2 : pc := end$
  END

Event P(x-1,tr):x/=0
  WHEN
    $grd1 : pc = start$
    $grd2 : x \neq 0$
  THEN
    $act1 : tr := v(x-1)$
    $act2 : pc := callrec$
  END

Event P(x,r):x/=0
  REFINES computing
  WHEN
    $grd1 : pc = callrec$
  THEN
    $act1 : r := f(tr)$
    $act2 : pc := end$
  END

The machine is simulating the organisation of the computations following two cases according to the figure 2. The first case is the path on the left part of the diagram and is when x is 0 and the second case if when x is not 0.

The first path is a three steps path and is labelled by the condition $x = 0$ an d the event P(0,r):x=0. The event P(x,r):x=0 is assigning the value $d0$ to r according to the definition of $u(0)$. It refines the event computing in the abstraction. The third step is an implication leading to the postcondition.

The second path is a four steps path and is labelled by the condition $x \neq 0$, then the event P(x-1,r):$x \neq 0$ is modelling the recursive call of the same procedure. Finally the event P(,r):$x \neq 0$ is refining the event computing. The call as event paradigm is applied when one considers that one event is defining the specification of the recursive call and the user is giving the name of the call to indicate that the event should be translated into a call. The EB2RC plugin (Cheng et al. 2016) generates automatically a C-like program.

The model ALGOREC is simple to checked. Proof obligations are simple, because the recursive call is hiding the previous values stored in the variable vv of the iterative paradigm. The prover is much more efficient.

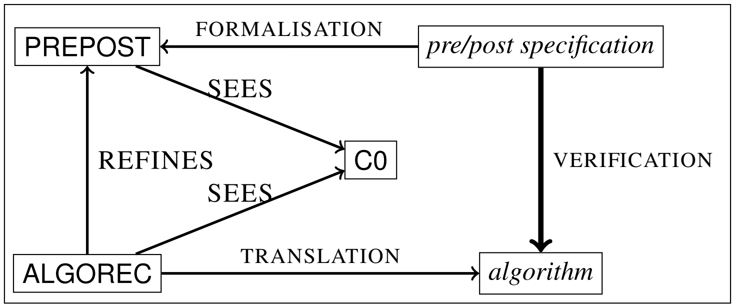


Figure 3: The recursive pattern

The recursive pattern is linked to a diagram which is helping to structure the solution. We have labelled arrows by guards or by events. The diagram helps to structure the analysis based on the inductive definitions. Following this pattern, we have developed the ERB2RC plugin based on the identification of three possible events. When a pre/post specification is stated, the program to build can be expressed by a simple event expressing the relationship between input and output and it provides a way to express pre/post specification as events. The first model is a very abstract model containing the pre/post events.

Since the refinement-based process requires an idea for introducing more concrete events. A very simple and powerful way to refine is to introduce a more concrete model which is based on an inductive definition of outputs with respect to the input.

A first consequence is that the concrete model is containing events which are computing the same function but corresponding to a recursive call expressed as events (Event rec%PROC(h(x),y)%P(y)). The event Event rec%PROC(h(x),y)%P(y) is simply simulating the recursive call of the same function and this expression makes the proofs easier. The invariant is defined in a simpler way by analysing the inductive structure and a control variable is introduced for structuring the inductive computation. We have identified three possible events to use in the concrete model:

Event
  e
where
  $\ell = \ell_1$
  $g_{\ell_1,\ell_2}(x)$
then
  $\ell := \ell_2$
  $x := f_{\ell_1,\ell_2}(x)$
end

Event
  rec%PROC(h(x),y)%P(y)
any y
where
  $\ell = \ell_1$
  $g_{\ell_1,\ell_2}(x, y)$
then
  $\ell := \ell_2$
  $x := f_{\ell_1,\ell_2}(x, y)$
end

Event
  call%APROC(h(x),y)%P(y)
any y
where
  $\ell = \ell_1$
  $g_{\ell_1,\ell_2}(x, y)$
then
  $\ell := \ell_2$
  $x := f_{\ell_1,\ell_2}(x, y)$
end

4.2 Applying the call as event technique

We will illustrate this method of designing recursive programs using a few relevant examples.

4.2.1 Problem 1: Computing the power 2 ($\lambda x.x^2$)

Applying the recursive pattern is made easier by the first steps of the iterative pattern. In fact, the context C0 and the machine PREPOST are the starting points of the iterative pattern as well as the recursive pattern. We use the computation of the function x^2 and we obtained the following refinement of PREPOST. Fig. 2 is the diagram analysing the way to solve the computation of the value of $u(x)$ following the call-as-event paradigm.

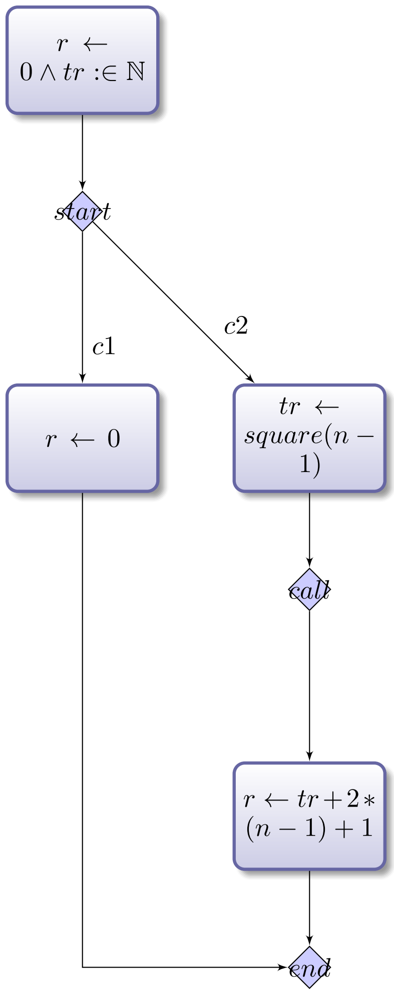
```
MACHINE square // square(n; r)
  REFINES specquare  SEES control0

 VARIABLES r l tr

 INVARIANTS
  @inv1 r ∈ ℕ
  @inv2 l = end ⇒ r = n ∗ n
  @inv3 l = call ⇒ n ≠ 0
  @inv4 l = call ⇒ tr = (n − 1) ∗ (n − 1)
  @inv5 l ∈ C
  @inv6 tr ∈ ℕ
  @inv7 l = end ⇒ r = n ∗ n
  @inv8 l = end ∧ n ≠ 0
        ⇒ tr = (n − 1) ∗ (n − 1) ∧ r = tr + 2 ∗ (n − 1) + 1
  theorem @inv9 l = call ⇒ n ∗ n = tr + 2 ∗ (n − 1) + 1
```

```
EVENTS
  EVENT INITIALISATION
     then
    @act1 r := 0
    @act2 l := start
    @act3 tr :∈ ℕ
  end

  EVENT square(n; r)@n = 0
   REFINES square(n; r)
     where
    @grd1 l = start
    @grd2 n = 0
     then
    @act1 l := end
    @act2 r := 0
    end
```

```
 EVENT square(n; r)@n/ = 0 REFINES square(n; r)
     where
    @grd1 l = call
     then
    @act1 r := tr + 2 ∗ (n − 1) + 1
    @act2 l := end
   end

  EVENT rec@square(n − 1; tr)@n ≠ 0
     where
    @grd1 l = start
    @grd2 n ≠ 0
     then
    @act1 l := call
    @act2 tr := (n − 1) ∗ (n − 1)
   end
 end
```

The variable l is modelling the control in the diagram. We introduce control points corresponding to assertions in the labels of the diagram as $C = \{start, end, callrec\}$. Three events are defined and the invariant is written very easily and proofs are derived automatically. The event rec%square(n-1;tr) is the key event modelling the recursive call. In the current example, we have modified the machine by using directly the fact that $v(n) = n * n$ and normally we had to use the sequence following the recursive pattern and then we had to derive the theorem $v(n) = n * n$.

Proofs are simpler and invariants are easier to extract from the inductive definitions. From the contexts and the machines we constructed, respecting the rules and choosing a name for the elements that allow us to produce an algorithm using the EB2RC plugin, we obtain a recursive algorithm that meets the problem specification and we obtain a diagram that shows the different steps in this algorithm.



The diagram is produced with tikz and has annotations defined by this list.

c1 $n = 0$

c2 $n \neq 0$

The algorithm produced is given below and is very simple to produce from the model.

```
procedure square(n; r)
begin
r ← 0
tr :∈ ℕ
if n = 0
      r ← 0
elsif n ≠ 0
      tr ← square(n − 1)
      r ← tr + 2 ∗ (n − 1) + 1
endif
end
```

The invariant states simple and obvious properties related to control points. Theorem 9 is particularly worth examining. It is easy to derive because it corresponds to the recursive call. All the calls and all the details are swept under the carpet, leaving only the last call. This shows the importance and interest of recursion.

MACHINE $square$ // $square(n;r)$
REFINES $specquare$ SEES $control0$

VARIABLES $r\ l\ tr$

INVARIANTS
$@inv1\ r \in \mathbb{N}$
$@inv2\ l = end \Rightarrow r = n * n$
$@inv3\ l = call \Rightarrow n \neq 0$
$@inv4\ l = call \Rightarrow tr = (n-1)*(n-1)$
$@inv5\ l \in C$
$@inv6\ tr \in \mathbb{N}$
$@inv7\ l = end \Rightarrow r = n * n$
$@inv8\ l = end \wedge n \neq 0$
$\quad \Rightarrow tr = (n-1)*(n-1) \wedge r = tr + 2*(n-1)+1$
$theorem\ @inv9\ l = call \Rightarrow n * n = tr + 2*(n-1)+1$

4.2.2 Problem 2: Binary search in an array

We solve the problem of *searching for a value in a table*. The input parameters of the *binsearch* procedure are: a sorted array t; the bounds of the array within which the algorithm should search (lo and hi); and the value for which the algorithm should search (val). Output parameters are $result$ and a boolean flag ok that indicates if $t(result) = val$. The procedures pre and post conditions are presented as follow:

contract $binsearch(t, val, lo, hi, ok, result)$

requires $\left(\begin{array}{l} t \in 0..t.Length \longrightarrow \mathbb{N} \\ \forall k.k \in lo..hi-1 \Rightarrow t(k) \leq t(k+1) \\ val \in \mathbb{N} \\ l, h \in 0..t.Length \\ lo \leq hi \end{array} \right)$

ensures $\left(\begin{array}{l} ok = TRUE \Rightarrow \ t(result) = val \\ ok = FALSE \Rightarrow (\forall i.i \in lo..hi \Rightarrow t(i) \neq val \end{array} \right)$

EVENT $find$
any j
where
$@grd1\ j \in lo..hi$
$@grd2\ t(j) = key$
then
$@act1\ ok := TRUE$
$@act2\ i := j$
end

EVENT $fail$
where
$@grd1\ \forall k\ k \in lo..hi \Rightarrow t(k) \neq key$
then
$@act1\ ok := FALSE$
end

The array t is sorted with respect to the ordering over integers and a simple inductive analysis is applied leading to a binary search strategy.
The specification is first expressed by two events corresponding to the two possible cases: either a key exists in the array t containing the value val, or there is no such key. These two events correspond to the two possible resulting *calls* to the procedure $binsearch(t, val, lo, hi; ok, result)$:

- Event find is $binsearch(t, val, lo, hi; ok, result)$ where $ok = TRUE$
- Event fail is $binsearch(t, val, lo, hi; ok, result)$ where $ok = FALSE$

These two events form the machine called *binsearch1* which is refined to obtain *binsearch2* (corresponding to PROCESS of Figure 7). In addition to these events, the events of this refined machine contains a new control label, l, which *simulates* how the binary search is achieved.

The refinement of binsearch1 is interesting for showing an invariant derived from the properties of the decomposition following the analysis of the searching process. The invariant is explicitly considering the role of the index *middle* and it is a clear statement of the property. We say that the recursivity is putting the dust under the carpet and it makes simpler to write and to read the solution.

MACHINE $binsearch2$ REFINES $binsearch1$ SEES $binsearch0$

VARIABLES $i\ ok\ l\ mi$

INVARIANTS
$@inv1\ i \in 1..n$
$@inv2\ l \in LOC$
$@inv3\ dom(t) = 1..n$
$@inv4\ mi \in 1..n$
$@inv5\ l = middle \Rightarrow lo < hi \wedge mi \in lo..hi$
$@inv6\ l = middle \wedge key < t(mi) \Rightarrow (\forall k\ k \in mi..hi \Rightarrow t(k) \neq key)$
$@inv7\ l = middle \wedge key > t(mi) \Rightarrow (\forall k\ k \in lo..mi \Rightarrow t(k) \neq key)$
$@inv8\ l = end \wedge ok = TRUE \Rightarrow i \in lo..hi \wedge t(i) = key$
$@inv9\ l = end \wedge ok = FALSE \Rightarrow (\forall k\ k \in lo..hi \Rightarrow t(k) \neq key)$
$theorem\ @inv10\ lo..hi \subseteq 1..n$
$theorem\ @inv11\ (\exists j\ l = middle \wedge j \in mi+1..hi \wedge key > t(mi) \wedge t(j) = key \wedge mi+1 \leq hi)$
$\Rightarrow (mi+1 \leq hi)$

The first events are the INITIALISATION event, the two events m1 and m2 corresponding the the case $mo = hi$ and the split event which is corresponding to the case $mo \neq hi$. The events are written the case analysis.

EVENT $INITIALISATION$
then
$@act1\ i :\in\ 1..n$
$@act2\ ok\ := FALSE$
$@act3\ l\ :=\ start$
$@act4\ mi\ :\in\ 1..n$
end

EVENT $m1$ REFINES $find$
where
$@grd1\ l = start$
$@grd2\ lo = hi$
$@grd3\ t(lo) = key$
$with$
$@j\ j = lo$
then
$@act1\ l\ :=\ end$
$@act2\ ok\ :=\ TRUE$
$@act3\ i\ :=\ lo$
end

EVENT $m2$ REFINES $fail$
where
$@grd1\ l = start$
$@grd2\ lo = hi$
$@grd3\ t(lo)\ \neq\ key$
then
$@act1\ l\ :=\ end$
$@act2\ ok\ :=\ FALSE$
end

EVENT $split$
where
$@grd1\ l = start$
$@grd2\ lo\ <\ hi$
then
$@act1\ l\ :=\ middle$
$@act2\ mi\ :=\ (lo\ +\ hi)2$
end

The split event directs the analysis and divides the exploration space into three parts of the indexes. Figure 4.2.2, the m3 event considers the case where the mi index is the one where the searched value is found. Then the other two events correspond to the segment between mi+1 and hi and are in fact recursive calls to the procedure under construction. These two events refine find and fail respectively. We need to add two more events corresponding to the segment $lo, mi - 1$ segment to complete the model.

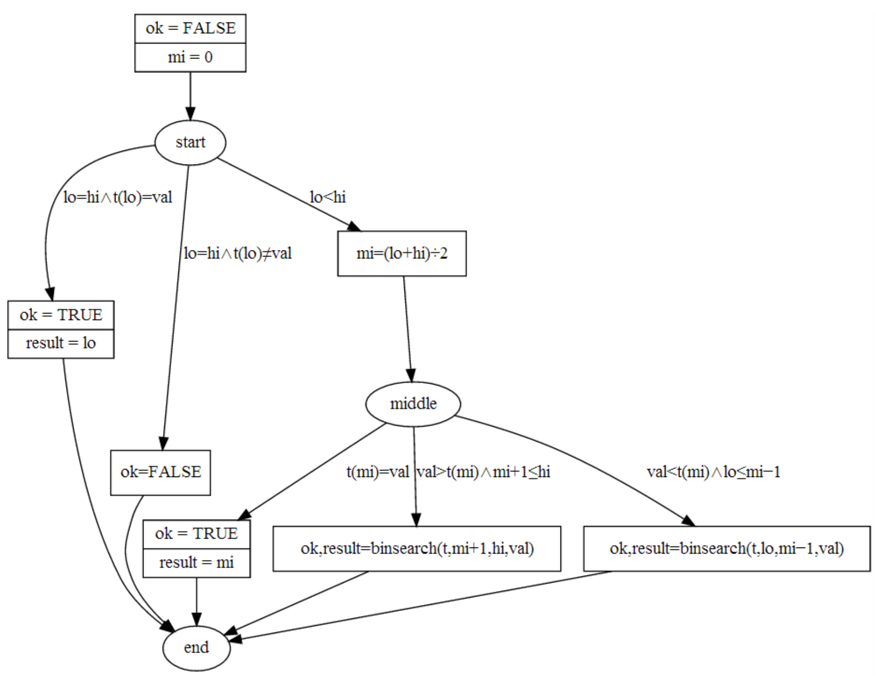


Figure 4: Visualized Representation of the Binary Search Algorithm

Model	Total	Auto	Manual	Reviewed	Undischarged	% auto
binsearch1	5	5	0	0	0	100 %
binsearch2	71	63	8	0	0	78 %

Table 1: Proof effort of our refinement approach for the binary search case study

A textual representation of the binary search algorithm is constructed by the EB2RC. The produced algorithm (as shown in Algorithm **??**) has been compared to the algorithm produced by hand by the authors. The two algorithms are identical up to a slight reformatting.

The proof effort of our refinement approach for the Binary Search case study is illustrated in Table 1. The first abstract model is proved automatically and the second concrete model is automatic in 78 % of its proof obligations.

The integrated development framework takes this into consideration. As shown in Fig. 7, we suggest to translate every recursive algorithm ALGORITHM into a partially annotated and iterative OPTIMISED ALGORITHM to be verified within the Spec# Programming System. In (Méry and Monahan 2013), we have proposed and proved a sound translation procedure from ALGORITHM to OPTIMISED ALGORITHM to perform this task. For example, the iterative version of the binary search algorithm in Spec# is shown in Fig. 5.

By sending this program to Spec#, Spec# reports the program as verified. No user interaction is required in this verification as all assertions required (preconditions, postconditions and loop invariants) have been generated as part of the refinement and transformation of the initial abstract specification into the final iterative algorithm. The automatic verification of the final Spec# program is available online at `http://www.rise4fun.com/SpecSharp/kyKW`.

Recursive Algorithm binsearch(t,lo,hi,val;ok,result) generated by EB2RC
begin
$ok := FALSE; mi := 0;$
if $lo = hi \wedge t(lo) = val$ then
  $ok := TRUE;$
  $result := lo;$
elseif $lo = hi \wedge t(lo) \neq val$ then
  $ok := FALSE;$
elseif $lo < hi$ then
  $mi := (lo + hi) \div 2;$
  if $t(mi) = val$ then
    $ok := TRUE;$
    $result := mi;$
  elsif $val > t(mi) \wedge mi + 1 \leq hi$ then
    $ok, result := binsearch(t, mi + 1, hi, val);$
  elsif $val < t(mi) \wedge lo \leq mi - 1$ then
    $ok, result := binsearch(t, lo, mi - 1, val);$
end

4.3 Comments on the call as event idea

In the example of subsubsection4.2.1, we do not use the event like call%APROC(h(x),y)%P(y) but the event is clearly a call for another procedure or function. For instance, when a sorting algorithm is developed, you may need an auxiliary operation for scanning a list of values to get the index of the minimum. It means that we have a way to define a library of models and to use correct-by-construction procedures or functions. In (Cheng et al. 2016) , we detail the tool and the way to define a library of *correct-by-construction programs*. The EB2RC plugin is used on this project and we obtain two files: one containing the algorithm and another containing the diagram built from the Event-B model.

5 Final comments

We have presented a use of the Event-B language in the derivation of sequential programs or algorithms. The first technique uses the sequences of values leading to a given term, which constitutes the required solution. This is the term that constitutes the desired solution. The calculation of the square or the cube is carried out by defining a sequence in

```
class BS {
 int BinarySearch(int[] t, int val, int lo, int hi, bool ok)
  requires 0 <= lo && lo < t.Length && 0 <= hi && hi < t.Length;
  requires lo <= hi && 0 < t.Length;
  requires forall{int i in(0:t.Length),int j in (i:t.Length);t[i]<=t[j]};
  ensures -1 <= result && result < t.Length;
  ensures (0 <= result && result < t.Length)==> t[result] == val;
  ensures result == -1 ==> forall {int i in (lo..hi); t[i] != val};
 { int mi = (lo + hi) / 2;
   while (!(lo == hi && t[lo] == val) || ( lo == hi && t[lo] != val)
                 || (lo < hi && (mi == (lo + hi) /2) && t[mi] == val))
    invariant 0 <= lo && lo < t.Length && 0 <= hi && hi < t.Length;
    invariant 0 <= mi && mi < t.Length;
    invariant (val < t[mi]) ==> forall {int i in (mi..hi); t[i] != val};
    invariant (val > t[mi]) ==> forall {int i in (lo..mi); t[i] != val};
   { mi = (lo + hi) / 2;
     if ((mi+1 <= hi) && (val > t[mi])) lo = mi +1;
     else if ((lo <= mi-1) && (val < t[mi])) hi = mi - 1;
   }
   if ((lo == hi) && (t[lo] == val)) {ok = true; return lo;}
   else{
    if ((lo == hi) && (t[lo] != val)) {ok = false; return -1;}
    else if ((lo < hi) && (t[mi] == val)) {ok = true; return mi;}
    else {ok = false; return -1;}
   }
 }
}
```

Figure 5: Binary Search C# program corresponding to the generated iterative procedure.

which one of the terms is the value of the square or the cube. Refinement is a very general relationship on a set of also very general models. The discipline of refinement begins with a technique that encourages us to experiment with the model variables, reducing them to those used exclusively for calculation. This approach helps us grasp the concepts of model and ghost variables, which are crucial in certain proof tools. In our case, we introduce them and then hide them in the abstract models and they make it easier to prove and state invariants. The method based on refinement is very close to the *programming from specifications* technique proposed by Carroll Morgan (Morgan 1990). The second technique is simpler because it relies on inductive definitions that are interpreted in the recursive paradigm. The objective is to save refinement and develop at a level of refinement. The resulting program is recursive and must be derecursed and simplified. However, it is relatively easy to produce using our event conventions. In this case, we noted that the proofs were relatively simpler to derive. Both techniques rely on experimental plugins but could be combined. Figure 7 illustrates an environment for co-ordinating the various techniques for relatively complex sequential systems, but a new formal IDE integrating these techniques and close to the diagram in Figure n7 needs to be developed. Very naturally, the development of concurrent, parallel or distributed algrorithms or programs and the chapter *Design of Correct by Construction Distributed Algorithms* will provide some elements.

¡

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EVENT $m3$ REFINES $find$
where
$@grd1\ l = middle$
$@grd3\ t(mi) = key$
$with$
$@j\ j = mi$
then
$@act1\ l\ := \ end$
$@act2\ ok\ := \ TRUE$
$@act3\ i\ := \ mi$
end

EVENT $rec@search(t, val, mi + 1, hi, ok, result)@ok = TRUE$ REFINES $find$
any j
where
$@grd1\ l = middle$
$@grd2\ key\ >\ t(mi)$
$@grd3\ j\ \in\ mi + 1..hi$
$@grd4\ t(j) = key$
$@grd5\ mi + 1\ \leq\ hi$
then
$@act1\ i\ :=\ j$
$@act2\ ok\ :=\ TRUE$
end

EVENT $rec@search(t, val, mi + 1, hi, ok, result)@ok = FALSE$ REFINES $fail$
where
$@grd1\ l = middle$
$@grd2\ key\ >\ t(mi)$
$@grd4\ \forall\ j\ j\ \in\ mi + 1..hi\ \Rightarrow\ t(j)\ \neq\ key$
$@grd5\ mi + 1 \leq hi$
then
$@act3\ l\ :=\ end$
$@act2\ ok\ :=\ FALSE$
end

Figure 6: Events for recursive calls
fig:

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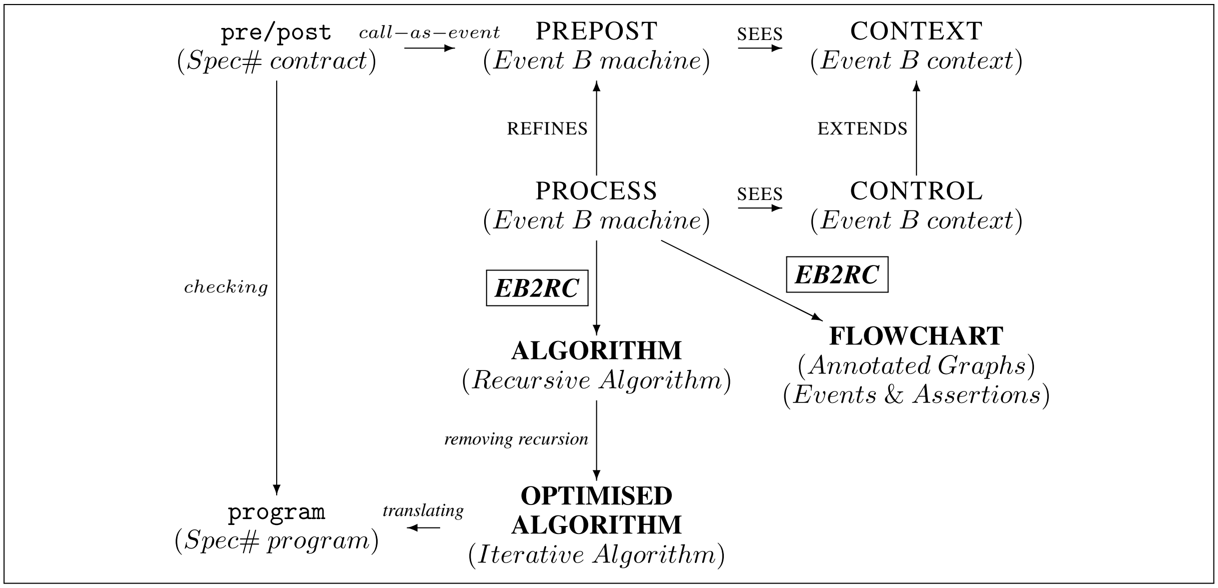

Figure 7: An overview of our integrated development framework to combine program refinement with program verification ((Cheng et al. 2016))

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