Cours MALG & MOVEX

Analyse des programmes (II)

Dominique Méry
Telecom Nancy, Université de Lorraine

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Plan

Sommaire

Current Summary

Position of the problem to solve

- Analysing programs with respect to safety properties
- Computing invariants of a program
- Problem : computing invariants is undecidable
- Idea : developing techniques of abstractions for simplifying computations using abstract interpretation frameworks.

Current Summary

Syntax for a Small Programming Language

$$
\begin{array}{lcll}
expr & ::= & v & v \in \mathbb{Z} \\
 & | & ? & \\
 & | & x & x \in \mathbb{V} \\
 & | & expr\ op\ expr & op \in \{+, -, \times, /\} \\
 & & & \\
cond & ::= & expr\ relop\ Expr & relop \in \{<, \leq, >, \geq, =, \\
 & | & \textbf{not}\ cond & \\
 & | & cond\ \textbf{and}\ cond & \\
 & & & \\
stmt & ::= & \ell[x := expr] & \ell \in \mathbb{C} \\
 & | & \ell[skip] & \\
 & | & \textbf{if}\ \ell[cond]\ \textbf{then}\ stmt\ \textbf{else}\ stmt\ \textbf{end if} & \\
 & | & \textbf{while}\ \ell[cond]\ \textbf{do}\ \ stmt\ \textbf{end do} & \\
 & | & stmt; stmt & \\
actions & ::= & x := exp & \\
 & | & skip & \\
 & | & \textsf{assert}\ cond &
\end{array}
$$

- $\mathbb{C}$: set of labels for programs.
- $Mem = V \to \mathbb{Z}$: set of memory states for variables V.
- $\mathcal{E} \in expr \to (Mem \to \mathcal{P}(\mathbb{Z}))$: $\mathcal{E}(e)(s)$ is the set of possible values of e in $s \in Mem$
- $\mathcal{C} \in cond \to (Mem \to \mathcal{P}(\mathbb{B}))$: $\mathcal{C}(cond)(m)$ is the set of possible values of $cond$ in $s \in Mem$.

Semantics for expressions

$\mathcal{E}[\![v]\!](m) \in \mathcal{P}(\mathbb{Z})$, $e \in Expr$,$m \in Mem$, $x \in \mathbb{V}$, $op \in \{+,-,\times,/\}$

$$\begin{array}{lcl}
\mathcal{E}[\![v]\!](m) & \stackrel{def}{=} & \{v\} \\
\mathcal{E}[\![?]\!](m) & \stackrel{def}{=} & \mathbb{Z} \\
\mathcal{E}[\![x]\!](m) & \stackrel{def}{=} & \{m(x)\} \\
\mathcal{E}[\![e_1\ op\ e_2]\!](m) & \stackrel{def}{=} & \{v|\exists ve_1, ve_2. \left(\begin{array}{l} ve_1 \in \mathcal{E}[\![e_1]\!](m) \\ ve_2 \in \mathcal{E}[\![e_2]\!](m) \\ v = ve_1\ o\ ve_2 \end{array}\right)\}
\end{array}$$

Semantics for conditions

$\mathcal{C}[\![cond]\!](m) \in \mathcal{P}(\mathbb{B})$,$cond \in Cond$,$m \in Mem$, $x \in \mathbb{V}$, $op \in \{+,-,\times,/\}$

$$\begin{array}{lcl}
tt \in \mathcal{C}[\![e_1\ relop\ e_2]\!](m) & \stackrel{def}{=} & \exists v_1, v2. \left(\begin{array}{l} v_1 \in \mathcal{E}[\![e_1]\!](m) \\ v_2 \in \mathcal{E}[\![\ e_2]\!](m) \\ v_1\ relop\ v2 \end{array}\right) \\
ff \in \mathcal{C}[\![e_1\ relop\ e_2]\!](m) & \stackrel{def}{=} & \exists v_1, v2. \left(\begin{array}{l} v_1 \in \mathcal{E}[\![e_1]\!](m) \\ v_2 \in \mathcal{E}[\![\ e_2]\!](m) \\ \neg(v_1\ relop\ v2) \end{array}\right) \\
be_1\ \wedge\ be_2 \in \mathcal{C}[\![be_1\ \mathbf{and}\ be_2]\!](m) & \stackrel{def}{=} & and \left(\begin{array}{l} be_1 \in \mathcal{C}[\![be_1]\!](m) \\ be_2 \in \mathcal{C}[\![be_2]\!](m) \end{array}\right)
\end{array}$$

Generating Control Flowchart Graph from Program

- A *control flow graph* is generated from the program under consideration namely P.
- A control flow graph $\mathcal{CFG}[\![P]\!]$ is defined by nodes ($l \in \mathcal{C}$) which are program control points of P, $\mathcal{C}ontrol[\![P]\!]$ and by labelled edges with actions ($\mathcal{A}ctions[\![P]\!]$) defined by the following rules :

$$
\begin{array}{lcl}
actions & ::= & v := exp \\
 & | & skip \\
 & | & \textbf{assert}\ be
\end{array}
$$

- A *control flow graph* is effectively defined by :
  - $\ell_{init} \in \mathcal{C}ontrol[\![P]\!]$: the entry point
  - $\ell_{end} \in \mathcal{C}ontrol[\![P]\!]$: the exit point
  - $\mathcal{E}dges[\![P]\!] \subseteq \mathcal{C}ontrol[\![P]\!] \times \mathcal{A}ctions[\![P]\!] \times \mathcal{C}ontrol[\![P]\!]$
- $\mathcal{CFG}[\![P]\!] = (\ell_{init}, \mathcal{E}dges[\![P]\!], \ell_{end})$

Small-step Semantics for Control Flowcharts

- $Mem \stackrel{def}{=} \mathbb{V} \longrightarrow \mathbb{Z}$
- Semantics of actions : $\xrightarrow{a} \subseteq Mem \times Mem$
  $m \xrightarrow{x:=e} m[x \mapsto v]$ **if** *there is a value* $v \in \mathcal{E}[\![e]\!](m)$
  $m \xrightarrow{skip} m$
  $m \xrightarrow{\textbf{assert}\ be} m]$ $if\ tt \in \mathcal{C}[\![be]\!](m)$
- Semantics for $\mathcal{CFG}[\![P]\!]$: $\xrightarrow{P} \subseteq States \times States$
  - **If** $m \xrightarrow{a} m'$ and $(\ell_1, a, \ell_2) \in \mathcal{E}dges[\![P]\!]$, then $(\ell_1, m) \xrightarrow{P} (\ell_2, m')$
  - The set of initial states is $\{\ell_{init}\} \times Mem$
  - The set of reachable states for P is denoted $\text{REACHABLE}(P)$ and defined by $[\![P]\!] = \{s | \exists s_0 \in \{\ell_{init} \times Mem : s_0 \xrightarrow[\star]{P} s\}$.

Collecting Semantics for Programs

- Defining for each control point ℓ of P the set of reachables values :
$$[\![P]\!]^{coll}_{\ell} = \{s | s \in States \wedge s \in [\![P]\!] \wedge \exists m \in Mem : s = (\ell, m)\}$$
- Characterizing $[\![P]\!]^{coll}_{\ell}$: it satisfies the system of equations

$$\forall \ell \in \mathcal{C}(P). X_{\ell} = X^{init}_{\ell} \cup \bigcup_{(\ell_1, a, \ell) \in \mathcal{E}dges[\![P]\!]} [\![a]\!](X_{\ell_1}) \tag{1}$$

- Let $a \in \mathcal{A}ctions[\![P]\!]$ and $x \subseteq Mem$.
$$[\![a]\!](x) = \{e | e \in States \wedge \exists f . f \in x \wedge f \xrightarrow{a} e\}$$
- $\forall \ell \in \mathcal{C}(P). \left(\begin{array}{l} \ell = \ell_{init} \Rightarrow X^{init}_{\ell} = Mem \\ \ell \neq \ell_{init} \Rightarrow X^{init}_{\ell} = \varnothing \end{array} \right)$

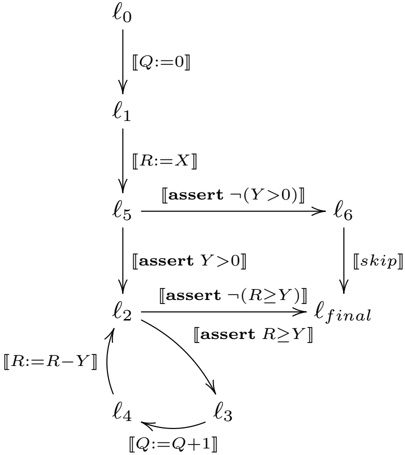
. . . .

☺ Théorème Let F the function defined as follows :

- n is the cardinality of $\mathcal{C}(P)$.
- $F \in \mathcal{P}(States)^n \longrightarrow \mathcal{P}(States)^n$
- If $X \in \mathcal{P}(States)^n$, then $F(X) = (\ldots, F_\ell(X), \ldots)$
- $\forall \ell \in \mathcal{C}(P).F_\ell(X) = X_\ell^{init} \cup \bigcup_{(\ell_1,a,\ell) \in \mathcal{E}dges[\![P]\!]} [\![a]\!](X_{\ell_1})$

The function F is monotonic over the complete lattice $(\mathcal{P}(States)^n, \subseteq)$ and has a least fixed-point μF defining the collecting semantics.

From flowchart to equational system



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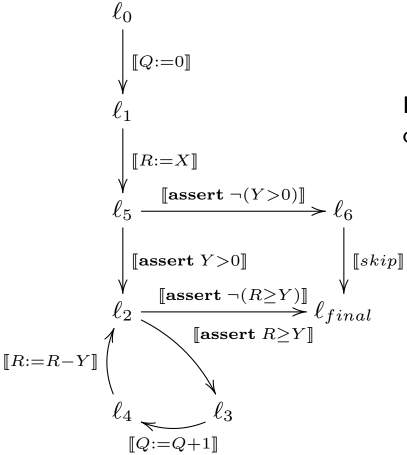
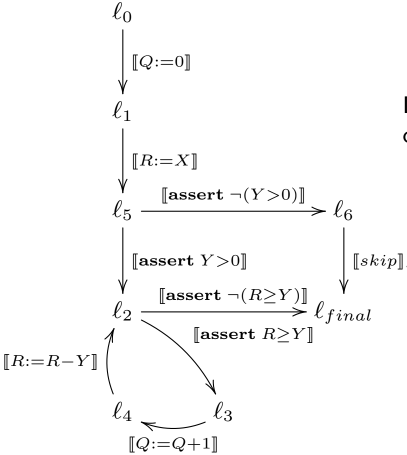


Defining equational systems for the collecting semantics :

From flowchart to equational system



Defining equational systems for the collecting semantics :

$$\begin{cases} X_0 = Mem \\ X_1 = [\![Q := 0]\!](X_0) \\ X_5 = [\![R := X]\!](X_1) \\ X_2 = [\![assert\ (Y > 0)]\!](X_5) \cup [\![R := R{-}Y]\!](X_4) \\ X_3 = [\![assert\ R \geq Y]\!](X_2) \\ X_4 = [\![Q := Q{+}1]\!](X_3) \\ X_6 = [\![assert\ \neg(Y > 0)]\!](X_5) \\ X_7 = X_6 \cup [\![assert\ \neg(R \geq Y)]\!](X_2) \end{cases}$$

Finding Sound Abstractions for Computing

$$\begin{tikzcd}[column sep=huge, row sep=huge]
(C, \sqsubseteq) \arrow[r, "abstraction"] \arrow[d, "concrete\ operator"'] & (A, \sqsubseteq) \arrow[d, "abstract\ operator"] \\
(C, \sqsubseteq) & (A, \sqsubseteq) \arrow[l, "concretization"']
\end{tikzcd}$$

Current Summary

Current Subsection Summary

- Two complete lattices $(C, \sqsubseteq_1, \sqcup_1, \sqcap_1)$ and $(A, \sqsubseteq_2, \sqcup_2, \sqcap_2)$ are supposed to be given.
- Two functions α and γ are supposed to be defined as follows :
  - $\alpha \in C \longrightarrow A$
  - $\gamma \in A \longrightarrow C$
- The pair (α, γ) is a Galois connection, if it satisfies the following property : $\forall x_1 \in C, x_2 \in A. \alpha(x_1) \sqsubseteq_2 x_2 \Leftrightarrow x_1 \sqsubseteq_1 \gamma(x_2)$
- A complete lattice A is a good abstraction of L, when there is a Galois connection between A and L.

Finding Sound Abstractions for Computing

$$
\begin{tikzcd}
(C, \sqsubseteq_1) \arrow[d, "cf"'] & (A, \sqsubseteq_2) \arrow[l, "\gamma"'] \arrow[d, "af"] \\
(C, \sqsubseteq_1) & (A, \sqsubseteq_2) \arrow[l, "\gamma"']
\end{tikzcd}
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\end{tikzcd}$$

- a is a sound abstraction of c, if $c \sqsubseteq_1 \gamma(a)$.

Finding Sound Abstractions for Computing

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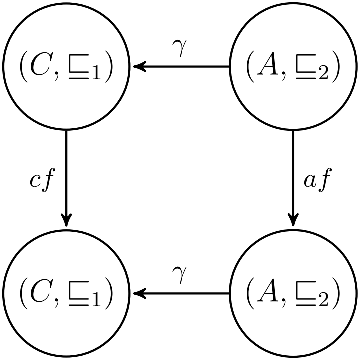
$$

- a is a sound abstraction of c, if $c \sqsubseteq_1 \gamma(a)$.
- functional operator : af is a sound abstraction of cf, if $\forall a \in A.cf(\gamma(a)) \sqsubseteq_1 \gamma(af(a))$

Finding Sound Abstractions for Computing



- a is a sound abstraction of c, if $c \sqsubseteq_1 \gamma(a)$.
- functional operator : af is a sound abstraction of cf, if $\forall a \in A.cf(\gamma(a)) \sqsubseteq_1 \gamma(af(a))$
- relational operator : ar is a sound abstraction of cr, if $\forall a \in A.cr(\gamma(a_1), \ldots, \gamma(a_n)) \sqsubseteq_1 \gamma(ac(a_1, \ldots, a_n)))$

Galois Connections

The pair (α, γ) is a Galois connection, if it satisfies the following property : $\forall x_1 \in L, x_2 \in L.\alpha(x_1) \sqsubseteq' x_2 \Leftrightarrow x_1 \sqsubseteq \gamma(x_2)$
Notation : $L \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} L'$

Properties of a Galois connection $L \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} L'$

- α and γ are monotonic over the lattices.
- $\mathrm{id}(L) \subseteq \gamma \circ \alpha$: $\gamma \circ \alpha$ is extensive.
- $\alpha \circ \gamma \subseteq \mathrm{id}(L')$: $\alpha \circ \gamma$ is retractive.
- $\alpha \circ \gamma \circ \alpha = \alpha$ and $\gamma \circ \alpha \circ \gamma = \gamma$
- $\alpha(x) = \bigcap' \{y \in L' | x \sqsubseteq \gamma(y)\}$
- $\gamma(y) = \bigcup \{x \in L | \alpha(x) \sqsubseteq' y\}$

Properties

- $\gamma \circ \alpha \circ \gamma \circ \alpha = \gamma \circ \alpha$
- $\alpha \circ \gamma \circ \alpha \circ \gamma = \alpha \circ \gamma$
- We assume that $\{(\alpha_i, \gamma_i) | i \in \{1 \ldots n\}\}$ is a family of Galois connections :

  $$L_1 \underset{\alpha_1}{\overset{\gamma_1}{\leftrightarrows}} L_2 \underset{\alpha_2}{\overset{\gamma_2}{\leftrightarrows}} \ldots L_{n-1} \underset{\alpha_{n-1}}{\overset{\gamma_{n-1}}{\leftrightarrows}} L_n$$

  Then $(\alpha_1; \ldots ; \alpha_i; \ldots ; \alpha_{n-1}, \gamma_{n-1}; \ldots \gamma_i; \ldots ; \gamma_1)$ is a Galois connection. or equivalently
  $L_1 \underset{\alpha_{n-1} \circ \ldots \circ \alpha_i \circ \ldots \circ \alpha_1}{\overset{\gamma_1 \circ \ldots \gamma_i \circ \ldots \circ \gamma_{n-1}}{\leftrightarrows}}$ is a Galois connection.
- We assume that $\{(\alpha_i, \gamma_i) | i \in \{1, 2\}\}$ two Galois connections : $\alpha_1 = \alpha_2$ if, and only if, $\gamma_1 = \gamma_2$

Current Subsection Summary

Examples

- We consider a transition system (S, I, t) where S is the set of states, I is the set of initial states and t is a binary relation over S.
- A property P of the transition system is a subset of S : $P \subseteq S$.
- P holds in $s \in S$, when $s \in P$.
- Four operators over properties can be defined as follows :
  - $\mathsf{pre}[t]P \stackrel{def}{=} \{s | s \in S \wedge \exists s'.((s, s') \in t \wedge s' \in P)\}$
  - $\widetilde{\mathsf{pre}}\,[t]P \stackrel{def}{=} \{s | s \in S \wedge \forall s'.((s, s') \in t \Rightarrow s' \in P)\}$
  - $\mathsf{post}[t]P \stackrel{def}{=} \{s | s \in S \wedge \exists s'.((s', s) \in t \wedge s' \in P)\}$
  - $\widetilde{\mathsf{post}}\,[t]P \stackrel{def}{=} \{s | s \in S \wedge \forall s'.((s', s) \in t \Rightarrow s' \in P)\}$
- Duality of operators :
  1. $\widetilde{\mathsf{pre}}\,[t]\neg P = \neg \mathsf{pre}[t]P$
  2. $\widetilde{\mathsf{post}}\,[t]\neg P = \neg \mathsf{post}[t]P$
- Galois connections over $\mathcal{P}$, the set of subsets of S :

$$(\mathcal{P}, \subseteq) \underset{\mathsf{pre}[t]}{\overset{\widetilde{\mathsf{post}}[t]}{\leftrightarrows}} (\mathcal{P}, \subseteq) \qquad (\mathcal{P}, \subseteq) \underset{\mathsf{post}[t]}{\overset{\widetilde{\mathsf{pre}}[t]}{\leftrightarrows}} (\mathcal{P}, \subseteq)$$

Examples

- Let two sets $\mathcal{L}$ standing for labels et $\mathcal{M}$ standing for memories.
- First step :
  - $\sqsubseteq$ is the partial ordering over functions using the subset relationship over function graphs : $f \sqsubseteq g$ means that $\mathbb{G}raph(f) \subseteq \mathbb{G}raph(g)$.
  - $\alpha_1 = \lambda P.\lambda l.\{m|(l,m) \in P\}$
  - $\gamma_1 = \lambda Q.\{(l,m)|l \in \mathcal{L} \wedge m \in Q(l)\}$
  - $(\mathcal{P}(\mathcal{L}\times\mathcal{M}), \subseteq) \underset{\alpha_1}{\overset{\gamma_1}{\leftrightarrows}} (\mathcal{L} \longrightarrow \mathcal{P}(\mathcal{M}), \subseteq)$ is a Galois connection
- Second step :
  - Let two sets $Pred$, set of predicates, and $\mathcal{M}$, a set of memories.
  - The relationship between both sets is stating as follows : For any given predicate p and any given memory m, p holds in m.
  - We define $B(p) = \{m|m \in \mathcal{M} \wedge p(m)\}$, set of predicates in which p holdsd.
  - Next we define :
    - $\alpha_2 = \lambda Q.\{p|p \in Pred \wedge Q \subseteq B(p)\}$
    - $\gamma_2 = \lambda P.\cap\{B(p)|p \in P\}$
  - $(\mathcal{P}(\mathcal{M}), \subseteq) \underset{\alpha_2}{\overset{\gamma_2}{\leftrightarrows}} (\mathcal{P}(Pred), \Rightarrow)$ is a Galois connection.

Examples

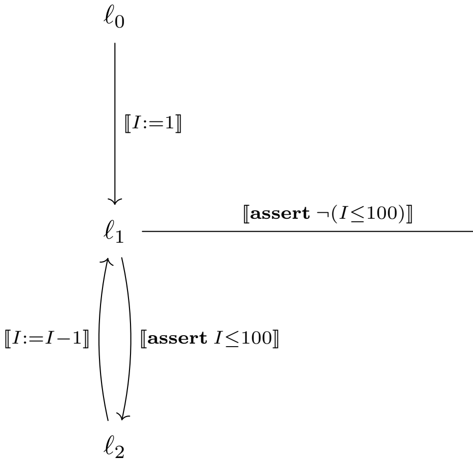
- Third step
  - $\alpha_3 = \lambda \ell . \alpha_2(Q_\ell) : Q \subseteq_1 Q' \stackrel{def}{=} \forall \ell \in \mathcal{L}. Q_\ell \subseteq {Q'}_\ell$.
  - $\gamma_3 = \lambda \ell . \gamma_2(P\ell) : P \Rightarrow_1 P' \stackrel{def}{=} \forall \ell \in \mathcal{L}. P_\ell \Rightarrow {P'}_\ell$.
  - $(\mathcal{L} \longrightarrow \mathcal{P}(\mathcal{M}), \subseteq_1) \underset{\alpha_3}{\overset{\gamma_3}{\longleftrightarrow}} (\mathcal{L} \longrightarrow \mathcal{P}(Pred), \Rightarrow_1)$ is a Galois connection.

Current Summary

Current Subsection Summary

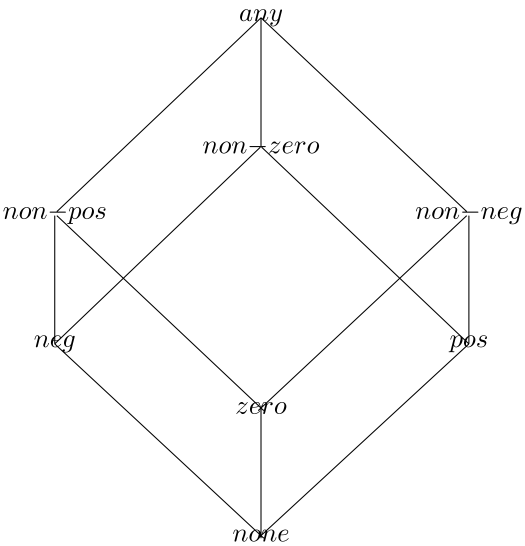
Examples of Abstractions

$\ell_0[I := 1];$
while $\ell_1[I \leq 100]$ **do**
  $\ell_2[I := I+1];$
end while
$\ell_{final}[skip]$



Domain of Sign

s



- Defining an abstraction for integers $\alpha \in \mathcal{P}(\mathbb{Z}) \longrightarrow \mathbb{S}igns$
- $\begin{cases} z & \alpha(z) \\ z < 0 & neg \\ z > 0 & pos \\ z = 0 & zero \end{cases}$

Composing Galois Connections

- Abstraction by projection :
  $(\mathcal{P}(Var \longrightarrow \mathbb{Z}), \subseteq) \underset{\alpha_\pi}{\overset{\gamma_\pi}{\leftrightarrows}} (Var \longrightarrow \mathcal{P}(\mathbb{Z}), \subseteq)$
- Abstraction of signs
  $(Var \longrightarrow \mathcal{P}(\mathbb{Z}), \subseteq) \underset{\alpha_{sign}}{\overset{\gamma_{sign}}{\leftrightarrows}} (Var \longrightarrow \mathbb{S}igns), \subseteq)$
- Composition of abstractions :
  $(\mathcal{P}(Var \longrightarrow \mathbb{Z}), \subseteq) \underset{\alpha_{sign} \circ \alpha_\pi}{\overset{\gamma_\pi \circ \gamma_{sign}}{\leftrightarrows}} (Var \longrightarrow \mathbb{S}igns), \subseteq)$
- $\alpha = \alpha_{sign} \circ \alpha_\pi$ and $\gamma = \gamma_\pi \circ \gamma_{sign}$

Best approximation of a function

- L is the concrete domain and L' is the abstract model :

$$\begin{tikzcd} L \arrow[r, "f"] & L \arrow[d, "\alpha"] \\ L' \arrow[u, "\gamma"] \arrow[r, "f'"] & L' \end{tikzcd}$$

$$f' = \alpha \circ f \circ \gamma \tag{2}$$

f' is the best approximation of f

Best evaluation and abstract evaluation

- Concrete states : $cv \in Var \longrightarrow \mathcal{P}(\mathbb{Z})$: if X is in Var, then $cv(X) \in \mathcal{P}(\mathbb{Z})$.
- Abstract states : $av \in Var \longrightarrow \mathbb{S}igns$: if X is in Var, then $av(X) \in \mathbb{S}igns$.
- (α, γ) is extended as : (α_1, γ_1) entre $(Var \longrightarrow \mathcal{P}(\mathbb{Z}), \subseteq)$ et $(Var \longrightarrow \mathbb{S}igns, \sqsubseteq)$. En particulier, $\alpha_1(cv) = av$ et, pour tout X de Var, $av(X) = \alpha(cv(X))$; $\gamma_1(av) = cv$ et, pour tout X de Var, $cv(X) = \gamma(av(X))$.
- Any expression e can be evaluated on each domain :
  - concrete domain : $States = Var \longrightarrow \mathcal{P}(\mathbb{Z})$: $[\![e]\!] \in (Var \longrightarrow \mathcal{P}(\mathbb{Z})) \longrightarrow \mathcal{P}(\mathbb{Z})$ and $[\![e]\!](cv)$
  - abstract domain : $AStates = Var \longrightarrow \mathbb{S}igns$: $[\![e]\!]_a \in (Var \longrightarrow \mathbb{S}igns) \longrightarrow \mathbb{S}igns$ and $[\![e]\!]_a(av)$.

Domain of signs

- The best abstraction is simply dedined as follows : $[\![e]\!]_{best}(av) = \alpha \circ [\![e]\!] \circ \gamma_1(av)$.
- Applying the best approximation for assignment : $[\![x := e]\!]_{best}(av) = \begin{cases} av(y), y \neq x \\ [\![e]\!]_{best}(av) \end{cases}$
- $(\mathcal{P}(Var \longrightarrow \mathbb{Z}), \subseteq)$: $A, B \in \mathcal{P}(\mathbb{Z}) : A{+}B = \{a{+}b | a \in A \wedge b \in B\}$
- $(Var \longrightarrow \mathbb{S}igns), \subseteq)$: $x, y \in \mathbb{S}igns : x \oplus y = \alpha(\gamma(x) {+} \gamma(y))$
- examples :
  - $pos \oplus neg = \alpha(\gamma(pos) {+} \gamma(neg)) = \alpha((1, +\infty) {+} (-\infty, -1)) = \alpha((-\infty, +\infty)) = any$
  - $pos \oplus zero = \alpha(\gamma(pos) {+} \gamma(zero)) = \alpha((1, +\infty) {+} (0)) = \alpha((1, +\infty)) = pos$
  - Building a table for the abstract operation $\oplus$.

Forward analysis in the domain of signs : first way

- Applying the analysis on the example

$$\ell_0[X := 1];$$
$$\ell_1[Y := 5];$$
$$\ell_2[X := X{+}1];$$
$$\ell_3[Y := Y{-}1];$$
$$\ell_4[X := Y{+}X];$$
$$\ell_{final}[skip];$$

ℓ	X	Y
ℓ_0	any	any
ℓ_1	pos	any
ℓ_2	pos	pos
ℓ_3	pos	pos
ℓ_4	pos	$non{-}neg$
ℓ_{final}	$non{-}neg$	$non{-}neg$

- ℓ_3 to ℓ_4 : abstract value of Y is pos and by γ, we obtain $(1, +\infty)$ a,d now we can compute in concrete domain $\mathbb{Z}$ $(1, +\infty){+}(-1) = (0, +\infty)$. By reapplying α we obtain $non{-}neg$.
- Computations may be not computable and one should use techniques for accelarating the convergence like widening.
- Computing is still costly : computing now in the abstraction and defining a sound approximation of f.

Forward analysis in the domain of signs : second way

- Evaluation is using the *best* approximation :
  $[\![e]\!]_{best}(av) = \alpha \circ [\![e]\!] \circ \gamma_1(av)$

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- Idea : approximation of the *best* approximation :$[\![e]\!]_a$ and, for any av abstract state, $[\![e]\!]_{best}(av) \sqsubseteq [\![e]\!]_a(av)$.

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- Abstract semantics is defined as follows :
  $av \in Var \longrightarrow \mathbb{S}igns$:

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- Abstract semantics is defined as follows :
  $av \in Var \longrightarrow \mathbb{S}igns$:
  - $[\![const]\!]_a(av) = \alpha(\{c\})$
  - $[\![x]\!]_a(av) = av(x)$
  - $[\![e_1{+}e_2]\!]_a(av) = [\![e_1]\!]_a(av) \oplus [\![e_2]\!]_a(av)$
  - $[\![e_1{+}e_2]\!]_a(av) = [\![e_1]\!]_a(av) \otimes [\![e_2]\!]_a(av)$

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  $[\![e]\!]_{best}(av) = \alpha \circ [\![e]\!] \circ \gamma_1(av)$
- Computing over the concrete domain is remaining complex
- Idea : approximation of the *best* approximation :$[\![e]\!]_a$ and, for any av abstract state, $[\![e]\!]_{best}(av) \sqsubseteq [\![e]\!]_a(av)$.
- Abstract semantics is defined as follows :
  $av \in Var \longrightarrow \mathbb{S}igns$:
  - $[\![const]\!]_a(av) = \alpha(\{c\})$
  - $[\![x]\!]_a(av) = av(x)$
  - $[\![e_1{+}e_2]\!]_a(av) = [\![e_1]\!]_a(av) \oplus [\![e_2]\!]_a(av)$
  - $[\![e_1{+}e_2]\!]_a(av) = [\![e_1]\!]_a(av) \otimes [\![e_2]\!]_a(av)$
- $\ell[X := E]$: $[\![E]\!]_a$ in av ou encore $[\![E]\!]_a(av)$:
  $[\![Y{+}X{+}6]\!]_a(av) = [\![Y]\!]_a(av) +_a [\![X]\!]_a(av) +_a [\![6]\!]_a(av)$.
  - $[\![Y{-}1]\!]_a(av) = [\![Y]\!]_a(av) \oplus [\![-1]\!]_a(av)_a = pos \oplus neg = any$
  - $[\![Y{-}1]\!]_{best}(av) = \alpha \circ [\![Y{-}1]\!] \circ \gamma_1(av) == \alpha([\![Y{-}1]\!](\gamma_1(av))) = \alpha([\![Y{-}1]\!](\{Y \mapsto (1,+\infty)\}) = \alpha((1{+}\infty){+}(-1)) = \alpha((0,+\infty)) = non{-}neg$

Sound approximations of f with respect to a Galois connection

A sound approximation of f with respect to a Galois connection f' satisfies the following property :

$$\forall x \in L, y \in L'.\alpha(x) \sqsubseteq y \Rightarrow \alpha(f(x)) \sqsubseteq f'(y)$$

$$\begin{tikzcd} L \arrow[r, "f"] & L \arrow[d, "\alpha"] \\ L' \arrow[u, "\gamma"] \arrow[r, "f'"] & L' \end{tikzcd}$$

The four statements are equivalent

- f' is a sound approximation of f with respect to a Galois connection
- $\alpha \circ f \sqsubseteq' f' \circ \alpha$
- $\alpha \circ f \circ \gamma \sqsubseteq' f'$
- $f \circ \gamma \sqsubseteq' \gamma \circ f'$

Defining an abstract semantics of expressions

- $[\![e]\!]_{best}(av) = \alpha \circ [\![e]\!] \circ \gamma_1(av)$ provide the best abstraction but is costly.
- Another solution is to define an abstract semantics for expressions : $[\![e]\!]_a$ such that for any av, $[\![e]\!]_{best}(av) \sqsubseteq [\![e]\!]_a(av)$.
- $av \in Var \longrightarrow \mathbb{S}igns$:
  - $[\![const]\!]_a(v) = \alpha(\{c\})$
  - $[\![x]\!]_a(v) = v(x)$
  - $[\![e_1+e_2]\!]_a(v) = [\![e_1]\!]_a(v) \oplus [\![e_2]\!]_a(v)$
  - $[\![e_1+e_2]\!]_a(v) = [\![e_1]\!]_a(v) \otimes [\![e_2]\!]_a(v)$

$\ell[X := E]$

- $[\![Y - 1]\!]_a(av) = [\![Y]\!]_a(av) \oplus [\![-1]\!](av)_a = pos \oplus neg = may$
- $[\![Y - 1]\!]_{best}(av) = \alpha_1 \circ [\![Y - 1]\!] \circ \gamma_1(av) == \alpha_1([\![Y - 1]\!](\gamma_1(av))) = \alpha_1([\![Y - 1]\!](\{Y \mapsto (1, +\infty)\}) = \alpha_1((1 + \infty) + (-1)) = \alpha_1((0, +\infty)) = non{-}neg$

Current Summary

Forward analysis in the domain of signs using the approximation

- Applying the analysis on the example

$\ell_0[X := 1];$
$\ell_1[Y := 5];$
$\ell_2[X := X+1];$
$\ell_3[Y := Y-1];$
$\ell_4[X := Y+X];$
$\ell_{final}[skip];$

ℓ	X	Y
ℓ_0	any	any
ℓ_1	pos	any
ℓ_2	pos	pos
ℓ_3	pos	pos
ℓ_4	pos	any
ℓ_{final}	any	any

- The new analysis is less precise but more efficient since we compute in the domain of signs.

Current Subsection Summary

Abstract Domain of Intervals

- $\mathbb{I}(\mathbb{Z}) = \{\perp\} \cup \{[l,u] | l \in \mathbb{Z} \cup \{-\infty\}, u \in \mathbb{Z} \cup \{\infty\}, l \leq u\}$
- $[l_1,u_1] \sqsubseteq [l_2,u_2]$ si, et seulement si, $l2 \leq l1$ et $u_1 \leq u_2$.
- $(\mathbb{I}(\mathbb{Z}), \sqsubseteq)$ est une structure partiellement ordonnée.
-
  1. $[l_1,u_1] \sqcup [l_2,u_2] = [min(l_1,l_2), max(u_1,u_2)]$
  2. $[l_1,u_1] \sqcap [l_2,u_2] = \begin{cases} [max(l_1,l_2), min(u_1,u_2)] \\ \perp, si\ max(l_1,l_2) > min(u_1,u_2) \end{cases}$
- $(\mathbb{I}(\mathbb{Z}), \sqcup)$ is a complete lattice.
-
  1. $\alpha(X) = \begin{cases} [min(X), max(X)] \\ \perp, si\ X = \varnothing \end{cases}$
  2. $\gamma([l,u]) = [l..u]$ et $\gamma(\perp] = \varnothing$
- (α, γ) is a Galois connexion.
-
  1. $i_1 \oplus i_2 = [l_1+l_2, u_1+u_2]$
  2. $i_1 \ominus i_2 = [l_1-u_2, u_1-l_2]$
  3. $i_1 \otimes i_2 = [min(l_1 \cdot l_2, l_1 \cdot u_2, u_1 \cdot l_2, u_1 \cdot u_2, max(l_1 \cdot l_2, l_1 \cdot u_2, u_1 \cdot l_2, u_1 \cdot u_2)]$
  4. $i_1 \oslash i_2 =$
     $[min(l_1/l_2, l_1/u_2, u_1/l_2, u_1/u_2, max(l_1/l_2, l_1/u_2, u_1/l_2, u_1/u_2)]$

Current Summary

Sound approximation of a function f

Definition of a sound approximation of a function f

A function $g \in A \longrightarrow A$ is a sound approximation of a function $f \in C \longrightarrow C$, if it satisfies the following condition :
$\forall c \in C : \forall a \in A : \alpha(c) \sqsubseteq a \Rightarrow \alpha(f(c)) \sqsubseteq g(a)$

Properties

Suppose that $C \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} A$ is a Galois connection.
The four statements are equivalent

1. g is a sound approximation of f with respect to a Galois connection
2. $\alpha \circ f \sqsubseteq g \circ \alpha$
3. $\alpha \circ f \circ \gamma \sqsubseteq g$
4. $f \circ \gamma \sqsubseteq \gamma \circ g$
5. $f \sqsubseteq \gamma \circ g \circ \alpha$