Modelling Software-based Systems

Lecture 3

Correctness by Construction with the Modelling Language Event-B using the Refinement

Master Informatique

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General Summary

Current Summary

Correctness by Construction

- Correctness by Construction is a method of building software -based systems with **demonstrable correctness** for security- and safety-critical applications.
- Correctness by Construction advocates a **step-wise refinement** process from specification to code using tools for checking and transforming models.
- Correctness by Construction is an approach to software/system construction
  - starting with an abstract model of the problem.
  - progressively adding details in a step-wise and checked fashion.
  - each step guarantees and proves the correctness of the new concrete model with respect to requirements

The Cleanroom Method as CbC

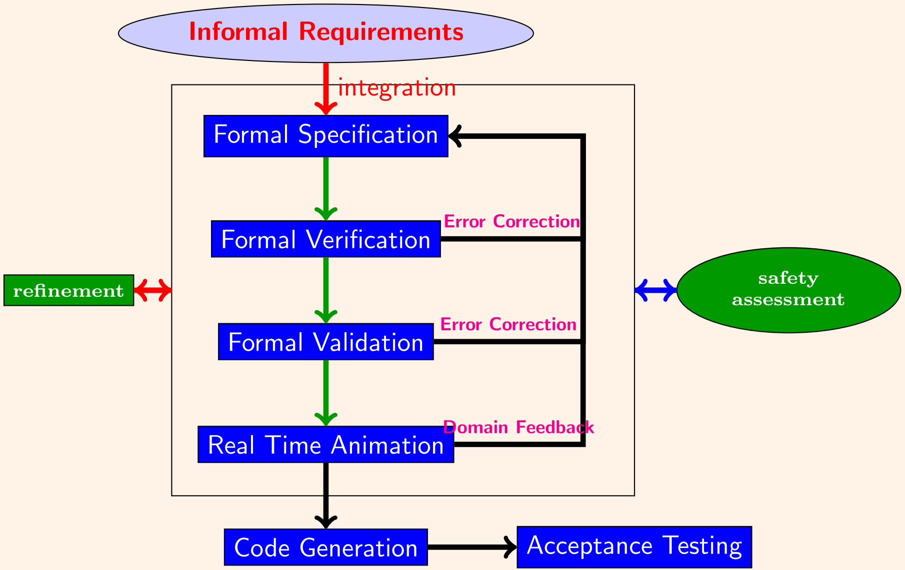
- The **Cleanroom** method, developed by Harlan Mills and his colleagues at IBM and elsewhere, attempts to do for software what cleanroom fabrication does for semiconductors : to achieve quality by keeping defects out during fabrication.
- In semiconductors, **dirt** or **dust** that is allowed to **contaminate** a chip as it is being made cannot possibly be removed later.
- But we try to do the equivalent when we write programs that are full of bugs, and then attempt to remove them all using debugging.

The Cleanroom Method as CbC

The Cleanroom method, then, uses a number of techniques to develop software carefully, in a well-controlled way, so as to avoid or eliminate as many defects as possible before the software is ever executed. Elements of the method are :

- specification of all components of the software at all levels ;
- stepwise refinement using constructs called "box structures" ;
- verification of all components by the development team ;
- statistical quality control by independent certification testing ;
- no unit testing, no execution at all prior to certification testing.

Critical System Development Life-Cycle Methodology



Overview of Methodology

- Informal Requirements : Restricted form of natural language.
- Formal Specification : Modeling language like Event-B , Z, ASM, VDM, TLA+...
- Formal Verification : Theorem Prover Tools like PVS, Z3, SAT, SMT Solver...
- Formal Validation : Model Checker Tools like ProB, UPPAAL , SPIN, SMV ...
- Real-time Animation : **Our proposed approach ... Real-Time Animator ...**
- Code Generation : **Our proposed approach ... EB2ALL : EB2C, EB2C++, EB2J, EB2C# ...**
- Acceptance Testing : Failure Mode, Effects and Critically analysis(FMEA and FMEA), System Hazard Analyses(SHA)

Case Studies

- *Colin Boyd and Anish Mathuria. Protocols Authentication and Key Establisment. Springer 2003.*
- *C. C. Marquezan and L. Z. Granville. Self-* and P2P for Network Management - Design Principles and Case Studies. Springer Briefs in Computer Science. Springer, 2012.*
- *Pacemaker Challenge Contribution*

Current Summary

Problems for Modelling systems

- Systems are generally very complex
- Invariant should be strong enough for proving safety properties
- Problems for modelling : finding suitable mathematical structures, listing events or actions of the system, proving proof obligations, ...

Solution : refining models

- To understand more and more the system
- To distribute the complexity of the system
- To distribute the difficulties of the proof
- To improve explanations
- Validation (step by step)
- Refinement (invariant & behavior)

refinement between two events (I)

definition

Let x be the abstract variable (or list of variables) and $I(s,c,x)$ the abstract invariant, y the concrete variable (or list of variables) and $J(s,c,x,y)$ the concrete invariant.
Let c be a concrete event observing the variable y and a an event observing the variable x and preserving $I(s,c,x)$.
Event c refines event a with respect to x, $I(s,c,x)$, y and $J(s,c,x,y)$, if

$$AX(s,c) \vdash I(s,c,x) \wedge J(s,c,x,y) \Rightarrow [c](\neg[a](\neg J(s,c,x,y)))$$

Abstract event refined by a concrete event

$\stackrel{def}{=}$

$$a \stackrel{def}{=} \left\{ \begin{array}{l} \text{ANY } u \text{ WHERE} \\ \quad G(u,s,c,x) \\ \text{THEN} \\ \quad x :| ABAP(u,s,c,x,x') \\ \text{END} \end{array} \right.$$

$$c \stackrel{def}{=} \left\{ \begin{array}{l} \text{ANY } v \text{ WHERE} \\ \quad H(v,s,c,y) \\ \quad \text{WITNESS} \\ \quad u : WP(u,s,c,v,y) \\ \quad x' : WV(v,s,c,y',x') \\ \text{THEN} \\ \quad y :| CBAP(v,s,c,y,y') \\ \text{END} \end{array} \right.$$

The two events a and c are normalised by a relationship called BA(e)(s,c,x,x'), which simplifies the notations used.
The two events a and c are equivalent to events of the following normalized form :

- a is equivalent to
  begin $x :| (\exists u.G(u,s,c,x) \wedge ABAP(u,s,c,x,x'))$ end
- c is equivalent to
  begin $y :| (\exists v.H(v,s,c,y) \wedge CBAP(v,s,c,y,y'))$ end

Explanations for the refinement

(Hypothesis)
(1) $AX(s,c) \vdash I(s,c,x) \wedge J(s,c,x,y) \Rightarrow [c](\neg[a](\neg J(s,c,x,y)))$
equivalent to
(Definition of $[a]$: $[a](\neg J(s,c,x,y)) \equiv$
$\forall x'.(\exists u.G(u,s,c,x) \wedge ABAP(u,s,c,x,x')) \Rightarrow \neg J(s,c,x',y)))$
(2) $AX(s,c) \vdash I(s,c,x) \wedge J(s,c,x,y) \Rightarrow [c](\neg(\forall x'.(\exists u.G(u,s,c,x) \wedge$
$ABAP(u,s,c,x,x')) \Rightarrow \neg J(s,c,x',y))$
equivalent to
(Transformation by simplification of logical connectives)
(3) $AX(s,c) \vdash I(s,c,x) \wedge J(s,c,x,y) \Rightarrow [c](\exists x'.(\exists u.G(u,s,c,x) \wedge$
$ABAP(u,s,c,x,x')) \wedge J(s,c,x',y))$
equivalent to
(Definition of $[c]$)

(4) $AX(s,c) \vdash$
$I(s,c,x) \wedge J(s,c,x,y) \Rightarrow (\forall y'.(\exists v.H(v,s,c,x) \wedge CBAP(v,s,c,y,y')) \Rightarrow$
$((\exists x'.(\exists u.G(u,s,c,x) \wedge ABAP(u,s,c,x,x')) \wedge J(s,c,x',y')))$
equivalent to
(Transformation by quantifier elimination $\forall$)
(5) $AX(s,c) \vdash$
$I(s,c,x) \wedge J(s,c,x,y) \Rightarrow (\exists v.H(v,s,c,y) \wedge CBAP(v,s,c,y,y')) \Rightarrow$
$((\exists x'.(\exists u.G(u,s,c,x) \wedge ABAP(u,s,c,x,x')) \wedge J(s,c,x',y')))$
equivalent to
(Transformation by elimination of connector $\wedge$)
(6) $AX(s,c) \vdash$
$I(s,c,x) \wedge J(s,c,x,y) \wedge (\exists v.H(v,s,c,y) \wedge CBAP(v,s,c,y,y')) \Rightarrow$
$((\exists x'.(\exists u.G(u,s,c,x) \wedge ABAP(u,s,c,x,x')) \Rightarrow J(s,c,x',y')))$

equivalent to
(Transformation by elimination of quantifier $\exists$)
(7)
$AX(s,c) \vdash I(s,c,x) \wedge J(s,c,x,y) \wedge H(v,s,c,y) \wedge CBAP(v,s,c,y,y') \Rightarrow$
$((\exists x'.(\exists u.G(u,s,c,x) \wedge ABAP(u,s,c,x,x')) \wedge J(s,c,x',y')))$
equivalent to
(Transformation by property of quantifier $\exists$)
(8)
$AX(s,c) \vdash I(s,c,x) \wedge J(s,c,x,y) \wedge H(v,s,c,y) \wedge CBAP(v,s,c,y,y') \Rightarrow$
$((\exists x'.((\exists u.G(u,s,c,x) \wedge ABAP(u,s,c,x,x')) \wedge J(s,c,x',y')))$

equivalent to
(Transformation by elimination of $\wedge$)
(9)

1. $AX(s,c) \vdash I(s,c,x) \wedge J(s,c,x,y) \wedge H(v,s,c,y) \wedge CBAP(v,s,c,y,y') \Rightarrow (((\exists u.G(u,s,c,x))$
2. $AX(s,c) \vdash$
   $I(s,c,x) \wedge J(s,c,x,y) \wedge H(v,s,c,x) \wedge CBAP(v,s,c,y,y') \Rightarrow ((\exists x'.\exists u.(ABAP(u,s,c,x,x')) \wedge J(s,c,x',y')))$

property refinement between events (II)

Let x be the abstract variable (or list of variables) and $I(s,c,x)$ the abstract invariant, y the concrete variable (or list of variables) and $J(s,c,x,y)$ the concrete invariant. the concrete invariant.
Let c be a concrete event observing the variable y and a an event observing the variable x and preserving $I(s,c,x)$.
Event c refines event a with respect to x, $I(s,c,x)$, y and $J(s,c,x,y)$ *if, and only if,*

1. (GRD) $AX(s,c) \vdash I(s,c,x) \wedge J(s,c,x,y) \wedge H(v,s,c,x) \wedge CBAP(v,s,c,y,y') \Rightarrow \exists u.G(u,s,c,x)$
2. (SIM) $AX(s,c) \vdash I(s,c,x) \wedge J(s,c,x,y) \wedge H(v,s,c,x) \wedge CBAP(v,s,c,y,y') \Rightarrow ((\exists x'.\exists u.ABAP(u,s,c,x,x') \wedge J(s,c,x',y')))$

Proof obligations for Event-B refinement)

- (INIT) $AX(s,c), CInit(s,c,y') \vdash \exists x'.(AInit(s,c,x') \wedge J(s,c,x',y')$
- (GRD)
  $AX(s,c), I(s,c,x), J(s,c,x,y), H(v,s,c,x), CBAP(v,s,c,y,y') \vdash$
  $(((\exists u.G(u,s,c,x))$
- (GRD-WIT)
  $AX(s,c), I(s,c,x), J(s,c,x,y), H(v,s,c,x), CBAP(v,s,c,y,y'),$
  $WP(u,s,c,v,y) \vdash G(u,s,c,x)$
- (SIM) $AX(s,c), I(s,c,x), J(s,c,x,y), H(v,s,c,x), CBAP(v,s,c,y,y') \vdash$
  $((\exists x'.(\exists u.ABAP(u,s,c,x,x')) \wedge J(s,c,x',y')))$
- (SIM-WIT)
  $AX(s,c), I(s,c,x), J(s,c,x,y), H(v,s,c,x), CBAP(v,s,c,y,y'),$
  $WP(u,s,c,v,y), WV(v,s,c,y,x') \vdash ABAP(u,s,c,x,x')) \wedge J(s,c,x',y')$
- (WFIS-P)
  $AX(s,c) \vdash I(s,c,x) \wedge J(s,c,x,y) \wedge H(v,s,c,x) \wedge CBAP(v,s,c,y,y') \vdash$
  $\exists u.WP(u,s,c,v,y)$
- (WFIS-V)
  $AX(s,c) \vdash I(s,c,x) \wedge J(s,c,x,y) \wedge H(v,s,c,x) \wedge CBAP(v,s,c,y,y') \vdash$
  $\exists x'.WV(v,s,c,y,x')$
- (TH) $AX(s,c) \vdash I(s,c,x) \wedge J(s,c,x,y) \vdash SAFE_1(s,c,x,y)$

```
MACHINE CM    REFINES  AM
SEES  E
VARIABLES  y
INVARIANTS
  jnv_1 : J_1(s,c,x,y)
  ...
  jnv_r : J_r(s,c,x,y)
THEOREMS
  th_1 : SAFE_1(s,c,x,y)
  ...
  th_n : SAFE_n(s,c,x,y)
VARIANTS
  var_1 : varexp_1(s,c,y)
  ...
  var_t : varexp_t(s,c,y)
EVENTS
  EVENT initialisation
    BEGIN
      y :|(CInit(s,c,y'))
    END
  ...
  EVENT c  REFINES a
    ANY v WHERE
      H(v,s,c,y)
    WITNESS
    u : WP(u,s,c,v,y)
    x' : WV(v,s,c,y',x')
    THEN
      y :|CBAP(v,s,c,y,y')
  END
  ...
END
```

- The machine CM is a model describing a set of events E(CM) modifying the y variable declared in the clause VARIABLES.
- A clause REFINES indicates that the CM machine refines a AM machine and E(AM) is the set of abstract events in AM .
- A particular event defines the initialisation of variable y according to the relationship $CInit(s,c,y')$.
- The property " Event c refines event a with respect to x, $I(s,c,x)$, y and $J(s,c,x,y)$" is denoted by the expression c refines a. Events a and c are attached to two machines AM and CM ; the invariant attached to each event is the invariant of its machine.

- A clause INVARIANTS describes the inductive invariant invariant $J(s, c, x, y)$ that this machine is assumed to respect provided that the associated verification conditions are shown to be valid in the theory induced by the context E mentioned by the clause SEES. $J(s, c, x, y)$ is the gluing invariant linking the variable y to the variable x.
- The clause THEOREMS introduces the list of safety properties derived in the theory. These properties relate to the variables y and x and must be proved valid. It is possible to add theorems about sets and constants ; this can help the proofs to be carried out during the verification process.
- To conclude this description, we would like to add that events can carry very important information for the proof process, in particular for proposing witnesses during event refinement. Furthermore, each event has a status (ordinary, convergent, anticipated) which is important in the production of verification conditions. The clause VARIANTS is linked to events of convergent and anticipated status. The event c (concrete) explicitly refines an event a of the AM machine.

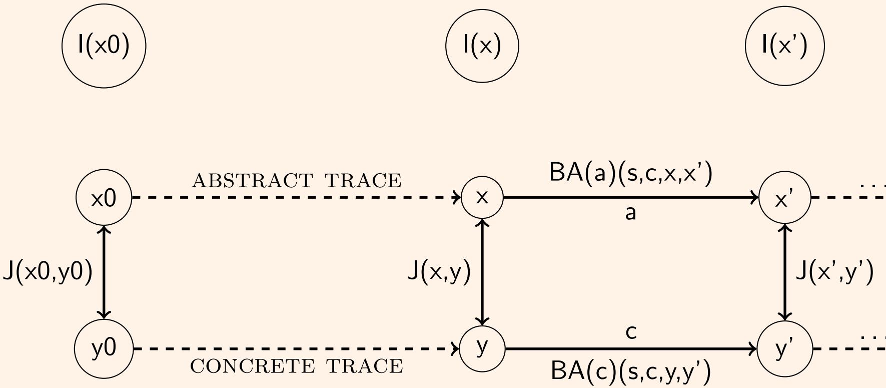
Event-B machine refinement REFINES

definition

The machine CM refines the machine AM , if any event c of CM refines an event a of AM :
$\forall$c.c $\in$ E(CM) $\Rightarrow$ $\exists$a.a $\in$ E(AM) $\wedge$ e refines a.

- Each machine has an event skip which does not modify the machine's variables.
- A concrete event c can refine an event skip whose effect is not to modify x in the abstract machine AM.
- The invariant of AM is $I(s,c,x)$ and that the initialisation of AM is $AInit(s,c,x')$.
- The proof witnesses are used to give properties of the parameter u and the variable x which have disappeared in the machine CM but for which the user must give an expression according to the state of CM .

Refinement between two machines



Current Summary

Example of a clock

- A machine M1 models hours or a machine M1 reports observations of hours

Example of a clock

- A machine M1 models hours or a machine M1 reports observations of hours
- and a machine M2 reports hours and minutes.
- A very special case of refinement called *superposition* and the proof is fairly straightforward.

CONTEXT C

CONSTANTS $H\ M$

AXIOMS

$@axm1\ H = 0..23$

$@axm2\ M = 0..59$

end

```
MACHINE  M1  SEES  C

  VARIABLES  h

  INVARIANTS
   @inv1 h ∈ H

  EVENTS
   EVENT  INITIALISATION
       then
     @act1 h :∈ H
   end

   EVENT  h1
       where
     @grd1 h < 23
       then
     @act1 h := h + 1
   end

   EVENT  h2
       where
     @grd1 h = 23
       then
     @act1 h := 0
   end
 end
```

MACHINE $M2$
REFINES $M1$
SEES C

VARIABLES $h\ m$

INVARIANTS
$@inv1\ m \in M$
$theorem\ @inv2\ h \in H$

EVENTS
EVENT $INITIALISATION$
then
$@act1\ h :\in H$
$@act2\ m :\in M$
end

EVENT $h1m1$
where
$@grd1\ h < 23$
$@grd2\ m < 59$
then
$@act2\ m := m+1$
end

EVENT $h1m2$ REFINES $h1$
where
$@grd1\ h < 23$
$@grd2\ m = 59$
then
$@act1\ h := h+1$
$@act2\ m := 0$
end

EVENT $h2m1$ REFINES $h2$
where
$@grd1\ h = 23$
$@grd2\ m = 59$
then
$@act1\ h := 0$
$@act2\ m := 0$
end

EVENT $h2m2$
where
$@grd1\ h = 23$
$@grd2\ m < 59$
then
$@act1\ m := m+1$
end
end

Current Summary

Refinement of a model by another one (I)

$$
\begin{tikzcd}
I(x) & x \arrow[r, "ae(x{,}x')"] \arrow[d, leftrightarrow] & x' \arrow[d, leftrightarrow] & I(x') \\
J(x,y) & y \arrow[r, "ce(y{,}y')"] & y' & J(x',y')
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & I(x) & AM \arrow[r, "\text{SEES}"] & AC & \mathcal{T}h_i \\
1 & J(x,y) & CM \arrow[u, "\text{REFINES}"] \arrow[r, "\text{SEES}"] & CC \arrow[u, "\text{EXTENDS}"'] & \mathcal{T}h_{i+1}
\end{tikzcd}
$$

Refinement of a model by another one (II)

$$
\begin{tikzcd}
I(x) & x \arrow[r, "x=x'"] \arrow[d, leftrightarrow] & x' \arrow[d, leftrightarrow] & I(x') \\
J(x,y) & y \arrow[r, "ce(y{,}y')"] & y' & J(x',y')
\end{tikzcd}
$$

$$
\begin{tikzcd}
I(x) & & x & & I(x) \\
J(x,y) & y \arrow[ur] \arrow[rr, "ce(y{,}y')"] & & y' \arrow[ul] & J(x,y')
\end{tikzcd}
$$

Proof obligations for refinement

(REF1) : refinement of initial conditions

$$\textsc{InitC}(y) \;\Rightarrow\; \exists x * (\textsc{Init}(x) \;\;\wedge\;\; \mathrm{J}(x,y)) \quad :$$

The initial condition of the refinement model imply that there exists an abstract value in the abstract model such that that value satisfies the initial conditions of the abstract one and implies the new invariant of the refinement model.

Proof obligations for refinement

(REF2) : refinement of events

$$\mathrm{I}(x) \ \wedge \ \mathrm{J}(x,y) \ \wedge \ ce(y,y') \Rightarrow \exists x'.(ae(x,x') \ \wedge \ \mathrm{J}(x',y')) \quad :$$

The invariant in the refinement model is preserved by the refined event and the activation of the refined event triggers the corresponding abstract event.

Proof obligations for refinement

(REF3) : refinement of stuttering steps

$$\mathrm{I}(x) \;\wedge\; \mathrm{J}(x,y) \;\wedge\; ce(y,y') \Rightarrow \mathrm{J}(x,y') \quad :$$

The invariant in the refinement model is preserved by the refined event but the event of the refinement model is a new event which was not visible in the abstract model ; the new event refines $skip$.

Proof obligations for refinement

(REF4) : Refinement does not introduce more blocking states

$$\mathrm{I}(x) \;\; \wedge \;\; \mathrm{J}(x,y) \;\; \wedge \;\; (G_1(x) \vee \ldots \vee G_n(x)) \;\Rightarrow\; H_1(y) \vee \ldots \vee H_k(y) \quad :$$

The guards of events in the refinement model are strengthened and we have to prove that the refinement model is not more blocked than the abstract.

Proof obligations for refinement

(REF5) : Well-definedness of variant

$$\mathrm{I}(x) \ \wedge \ \mathrm{J}(x,y)) \ \Rightarrow \ V(y) \in \mathbb{N}$$

Proof obligations for refinement

(REF6) : Well behaviour of new events

$$\mathrm{I}(x) \;\wedge\; \mathrm{J}(x,y) \;\wedge\; ce(y,y') \;\Rightarrow\; V(y') < V(y) \;\;:$$

New events should not block forever abstract ones.

Proof obligations for refinement

(REF7) : Feasibility of refined events

$$\Gamma(s,c) \vdash I(x) \wedge J(x,y) \wedge grd(E) \Rightarrow \exists y' \cdot P(y,y')$$

Current Summary

The factorial model

CONTEXT
  $fonctions$
CONSTANTS $factorial, n$
AXIOMS

$$n \in \mathbb{N} \land factorial \in \mathbb{N} \leftrightarrow \mathbb{N} \land 0 \mapsto 1 \in factorial \land$$

$$\forall (i, fn).(i \mapsto fn \in factorial \Rightarrow i+1 \mapsto (i+1) * fi \in factorial) \land$$

$$\forall f \cdot \left(\begin{array}{l} f \in \mathbb{N} \leftrightarrow \mathbb{N} \land \\ 0 \mapsto 1 \in f \land \\ \forall (n, fn).(n \mapsto fn \in f \Rightarrow n+1 \mapsto (n+1) \times fn \in f) \\ \Longrightarrow \\ factorial \subseteq f \end{array} \right)$$

END

The factorial pre/post specification

MACHINE $B-prepostok$ SEES $A-functions$

VARIABLES $r\ ok$

INVARIANTS
$@inv1\ ok \in BOOL$
$@inv2\ r \in \mathbb{Z}$
$@inv3\ ok = TRUE \Rightarrow r = factorial(n)$

EVENTS
EVENT $INITIALISATION$
then
$@act1\ r :\in \mathbb{Z}$
$@act2\ ok := FALSE$
end

EVENT $final$
where
$@grd1\ ok = FALSE$
then
$@act1\ r := factorial(n)$
$@act2\ ok := TRUE$
end
end

The factorial context

CONTEXT $A - functions$

CONSTANTS $factorial\ n$

AXIOMS

$@axm1\ n \in \mathbb{N}$

$@axm2\ factorial \in \mathbb{N} \to \mathbb{N}$

$@axm3\ 0 \mapsto 1 \in factorial$

$@axm4\ \forall\ a. \forall\ b\ .\ a \in \mathbb{N} \land b \in \mathbb{N} \land a \mapsto b \in factorial$
$\quad \Rightarrow a+1 \mapsto (a+1)*b \in factorial$

$@axm5\ \forall\ f\ .\ f \in \mathbb{N} \to \mathbb{N}$
$\quad \land\ 0 \mapsto 1 \in f$
$\quad \land\ (\forall\ a, b\ .\ a \in \mathbb{N} \land b \in \mathbb{N} \land a \mapsto b \in f$
$\quad \Rightarrow a+1 \mapsto (a+1)*b \in f)$
$\quad \Rightarrow factorial \subseteq f$

$theorem\ @th1\ factorial \in \mathbb{N} \to \mathbb{N}$

$theorem\ @th2\ factorial(0) = 1$

$theorem\ @th3\ \forall\ u\ .\ u \in \mathbb{N} \land u \neq 0 \Rightarrow factorial(u) = u * factorial(u-1)$

$@axm6\ n > 3$

end

The factorial model

$$
\begin{array}{l}
\textsf{MACHINE} \\
\quad specification \\
\textsf{SEES}\ fonctions \\
\textsf{VARIABLES} \\
\quad resultat \\
\textsf{INVARIANT} \\
\quad resultat \in \mathbb{N} \\
\textsf{THEOREMS} \\
\quad factorial \in \mathbb{N} \longrightarrow \mathbb{N}\ ; \\
\quad factorial(0) = 1\ ; \\
\quad \forall n.(n \in \mathbb{N} \Rightarrow factorial(n+1) = (n+1) \times factorial(n)) \\
\textsf{INITIALISATION} \\
\quad resultat :\in \mathbb{N} \\
\textsf{EVENTS} \\
\quad computing1 = \textsf{BEGIN}\ resultat := factorial(n)\ \textsf{END} \\
\textsf{END}
\end{array}
$$

Refining *specification* by *computation*

MACHINE $computation$
REFINES $specification$
SEES $fonctions$
VARIABLES $resultat, fac, x$
INVARIANTS
  $inv1 : fac \in \mathbb{N} \nrightarrow \mathbb{N}$
  $inv2 : dom(fac) \subseteq 0 \,..\, n$
  $inv4 : dom(fac) \neq \varnothing$
  $inv5 : \forall i.i \in dom(fac) \Rightarrow fac(i) = factorial(i)$
  $inv3 : x \in dom(fac)$
  $inv6 : dom(fac) = 0 \,..\, x$
EVENTS
EVENT INITIALISATION
  BEGIN
    $act1 : resultat :\in \mathbb{N}$
    $act2 : fac := \{0 \mapsto 1\}$
    $act3 : x := 0$
  END
EVENT computing2 REFINES EVENT computing1
  WHEN
    $grd1 : n \in dom(fac)$
  THEN
    $act1 : resultat := fac(n)$
  END
END

Refining $specification$ by $computation$

$$
\begin{array}{l}
\textbf{MACHINE}\ \ computation\\
\textbf{REFINES}\ \ specification\\
\textbf{SEES}\ \ fonctions\\
\textbf{VARIABLES}\ \ resultat, fac, x\\
\textbf{INVARIANTS}\\
\quad inv1 : fac \in \mathbb{N} \pfun \mathbb{N}\\
\quad inv2 : dom(fac) \subseteq 0\,..\,n\\
\quad inv4 : dom(fac) \neq \varnothing\\
\quad inv5 : \forall i.i \in dom(fac) \Rightarrow fac(i) = factorial(i)\\
\quad inv3 : x \in dom(fac)\\
\quad inv6 : dom(fac) = 0\,..\,x\\
\textbf{EVENTS}\\
\textbf{EVENT}\ \mathsf{event2}\\
\quad \textbf{WHEN}\\
\quad\quad grd11 : x \in dom(fac)\\
\quad\quad grd12 : x+1 \notin dom(fac)\\
\quad\quad grd13 : n \notin dom(fac)\\
\quad \textbf{THEN}\\
\quad\quad act11 : fac(x+1) := (x+1) * fac(x)\\
\quad\quad act1 : x := x+1\\
\quad \textbf{END}\\
\textbf{END}
\end{array}
$$

Refining *computation* by *algorithm*

EVENTS
EVENT INITIALISATION
BEGIN
$act1 : resultat :\in \mathbb{N}$
$act2 : fac := \{0 \mapsto 1\}$
$act3 : cfac := 0$
$act4 : vfac := 1$
$act5 : x := 0$
END
EVENT computing3 REFINES EVENT computing2
WHEN $grd2 : cfac = n$
THEN $act1 : resultat := vfac$
END
EVENT event3 REFINES EVENT event2
WHEN $grd1 : cfac \neq n$
THEN
$act1 : vfac := (cfac + 1) * vfac$
$act2 : cfac := cfac + 1$
$act3 : fac(cfac + 1) := (cfac + 1) * fac(cfac)$
$act4 : x := x + 1$
END
END

Refining *computation* by *algorithm*

MACHINE $algorithm$ **REFINES** $computation$
SEES $fonctions$
VARIABLES $resultat, vfac, \quad cfac, fac, x$
INVARIANTS

$inv1 : vfac \in \mathbb{N}$
$inv2 : cfac \in \mathbb{N}$
$inv3 : cfac \leq n$
$inv4 : cfac \geq 0$
$inv6 : cfac \in dom(fac)$
$inv5 : vfac = fac(cfac)$
$inv7 : cfac + 1 \notin dom(fac)$
$inv8 : dom(fac) = 0 \,..\, cfac$
$inv9 : x = cfac$

Refining $algorithm$ by $simplealgorithm$

MACHINE $simplealgorithm$ REFINES $algorithm$
SEES $fonctions$
VARIABLES $resultat, vfac, cfac$
THEOREMS $thm1 : vfac = factorial(cfac)$
EVENTS
EVENT INITIALISATION
  BEGIN
    $act1 : resultat :\in \mathbb{N}$
    $act3 : cfac := 0$
    $act4 : vfac := 1$
  END
EVENT computing4 REFINES EVENT computing3
  WHEN
    $grd2 : cfac = n$
  THEN
    $act1 : resultat := vfac$
  END
EVENT event4 REFINES EVENT event3
  WHEN
    $grd1 : cfac \neq n$
  THEN
    $act1 : vfac := (cfac + 1) * vfac$
    $act2 : cfac := cfac + 1$
  END
END

Current Summary

Simple Form of an Event

- An event of the simple form is denoted by :

$$\begin{array}{l} < event\_name > \ \widehat{=} \\ \quad \text{WHEN} \\ \quad\quad < condition > \\ \quad \text{THEN} \\ \quad\quad < action > \\ \quad \text{END} \end{array}$$

where
- $< event\_name >$ is an identifier
- $< condition >$ is the firing condition of the event
- $< action >$ is a generalized substitution (parallel "assignment")

Non-deterministic Form of an Event

- An event of the non-deterministic form is denoted by :

$$
\begin{array}{l}
< event\_name > \;\widehat{=} \\
\quad \text{ANY} \;\; < variable > \;\; \text{WHERE} \\
\qquad < condition > \\
\quad \text{THEN} \\
\qquad < action > \\
\quad \text{END}
\end{array}
$$

where

- $< event\_name >$ is an identifier
- $< variable >$ is a (list of) variable(s)
- $< condition >$ is the firing condition of the event
- $< action >$ is a generalized substitution (parallel "assignment")

Shape of a Generalized Substitution

A generalized substitution can be
- Simple assignment : $x := E$
- Generalized assignment : $x : P(x, x')$
- Set assignment : $x :\in S$

'
- Parallel composition : $\begin{array}{l} T \\ \cdots \\ U \end{array}$

Invariant Preservation Verification (0)

$$\begin{array}{l}\text{INVARIANT} \quad \wedge \quad \text{GUARD} \\ \Longrightarrow \\ \text{ACTION } \textcolor{red}{\text{establishes}} \text{ INVARIANT}\end{array}$$

Invariant Preservation Verification (1)

- Given an event of the simple form :

$$\begin{array}{l} \textcolor{green}{\text{EVENT}}\ \mathsf{EVENT} \quad \widehat{=} \\ \quad \textbf{WHEN} \\ \qquad G(x) \\ \quad \textbf{THEN} \\ \qquad x := E(x) \\ \quad \textbf{END} \end{array}$$

and invariant $I(x)$ to be preserved, the statement to prove is :

$$\boxed{I(x) \ \wedge \ G(x) \ \Longrightarrow \ I(E(x))}$$

Invariant Preservation Verification (2)

- Given an event of the simple form :

$$
\begin{array}{l}
\textcolor{green}{\mathrm{EVENT}}\ \mathsf{EVENT} \quad \widehat{=} \\
\quad \textcolor{blue}{\mathrm{WHEN}} \\
\qquad G(x) \\
\quad \textcolor{blue}{\mathrm{THEN}} \\
\qquad x : | P(x, x') \\
\quad \textcolor{blue}{\mathrm{END}}
\end{array}
$$

and invariant $I(x)$ to be preserved, the statement to prove is :

$$\boxed{I(x) \ \wedge \ G(x) \ \wedge \ P(x, x') \ \Longrightarrow \ I(x')}$$

Invariant Preservation Verification (3)

- Given an event of the simple form :

$$
\begin{array}{l}
\textcolor{green}{\mathrm{EVENT}}\ \mathsf{EVENT} \quad \widehat{=} \\
\quad \textcolor{blue}{\mathrm{WHEN}} \\
\qquad G(x) \\
\quad \textcolor{blue}{\mathrm{THEN}} \\
\qquad x :\in S(x) \\
\quad \textcolor{blue}{\mathrm{END}}
\end{array}
$$

and invariant $I(x)$ to be preserved, the statement to prove is :

$$\boxed{I(x) \quad \wedge \quad G(x) \quad \wedge \quad x' \in S(x) \quad \Longrightarrow \quad I(x')}$$

Invariant Preservation Verification (4)

- Given an event of the non-deterministic form :

$$
\begin{array}{l}
\textsf{EVENT}\ \text{EVENT} \ \widehat{=} \\
\quad \textsf{ANY}\ v\ \textsf{WHERE} \\
\quad\quad G(x,v) \\
\quad \textsf{THEN} \\
\quad\quad x := E(x,v) \\
\quad \textsf{END}
\end{array}
$$

and invariant $I(x)$ to be preserved, the statement to prove is :

$$\boxed{I(x) \ \wedge \ G(x,v) \ \Longrightarrow \ I(E(x,v))}$$

Refinement Technique (1)

- Abstract models works with variables x, and concrete one with y
- A gluing invariant $J(x, y)$ links both sets of vrbls
- Each abstract event is refined by concrete one (see below)

Refinement Technique (2)

- Some new events may appear : they refine “skip”
- Concrete events must not block more often than the abstract ones
- The set of new event alone must always block eventually

- Given an abstract and a corresponding concrete event

$$\begin{array}{l} \text{EVENT } \mathsf{ea} \ \widehat{=} \\ \quad \text{WHEN} \\ \qquad G(x) \\ \quad \text{THEN} \\ \qquad x := E(x) \\ \quad \text{END} \end{array} \qquad \begin{array}{l} \text{EVENT } \mathsf{ec} \ \widehat{=} \\ \quad \text{WHEN} \\ \qquad H(y) \\ \quad \text{THEN} \\ \qquad y := F(y) \\ \quad \text{END} \end{array}$$

and invariants $I(x)$ and $J(x,y)$, the statement to prove is :

$$\boxed{I(x) \ \wedge \ J(x,y) \ \wedge \ H(y) \implies G(x) \ \wedge \ J(E(x),F(y))}$$

Correct Refinement Verification (2)

- Given an abstract and a corresponding concrete event

EVENT ea $\widehat{=}$ ANY v WHERE $G(x,v)$ THEN $x := E(x,v)$ END	EVENT ec $\widehat{=}$ ANY w WHERE $H(y,w)$ THEN $y := F(y,w)$ END

$$
\begin{array}{l}
I(x) \quad \wedge \quad J(x,y) \quad \wedge \quad H(y,w) \\
\Longrightarrow \\
\exists v \cdot (\, G(x,v) \quad \wedge \quad J(E(x,v), F(y,w))\,)
\end{array}
$$

- Given a NEW event

$$\begin{array}{l} \textbf{EVENT}\ \mathsf{EVENT}\ \ \widehat{=} \\ \quad \textbf{WHEN} \\ \quad\quad H(y) \\ \quad \textbf{THEN} \\ \quad\quad y := F(y) \\ \quad \textbf{END} \end{array}$$

and invariants $I(x)$ and $J(x, y)$, the statement to prove is :

$$\boxed{I(x) \ \wedge \ J(x,y) \ \wedge \ H(y) \ \Longrightarrow \ J(x, F(y))}$$

Current Summary

General form of proof obligations for an event e

- *INIT/I/INV* : $C(s,c), INIT(c,s,x) \vdash I(c,s,x)$
- *e/I/INV* : $C(s,c), I(c,s,x), G(c,s,t,x), P(c,s,t,x,x') \vdash I(c,s,x')$
- *e/act/FIS* : $C(s,c), I(c,s,x), G(c,s,t,x) \vdash$
- *e/act/WD* : $C(s,c), I(c,s,x), G(c,s,t,x) \vdash \exists x'.P(c,s,t,x,x')$

Well-definedness of an Axiom	m / WD	m is the axiom name
Well-definedness of a Derived Axiom	m / WD	m is the axiom name
Derived Axiom	m / THM	m is the axiom name
Well-definedness of an Invariant	v / WD	v is the invariant name
Well-definedness of a Derived Invariant	m / WD	m is the invariant name
Well-definedness of an event Guard	t / d / WD	t is the event name d is the action name
Well-definedness of an event Action	t / d / WD	t is the event name d is the action name
Feasibility of a non-det. event Action	t / d / FIS	t is the event name d is the action name
Derived Invariant	m / THM	m is the invariant name
Invariant Establishment	INIT. / v / INV	v is the invariant name
Invariant Preservation	t / v / INV	t is the event name v is the invariant name

Current Summary

```
WHEN
  P
  Q
THEN
  S
END
```

```
WHEN
  P
  ¬Q
THEN
  T
END
```

are merged into

```
WHEN
  P
THEN
  WHILE  Q  DO
    S
  END;
  T
END
```

Side Conditions :

- P must be invariant under S.
- The first event must have been introduced at one refinement step below the second one.
- Special Case : If P is missing the resulting "event" has no guard

WHEN P Q THEN S END	WHEN P $\neg Q$ THEN T END	*are merged into*	WHEN P THEN IF Q THEN S ELSE T END; END

Side Conditions :

- The disjunctive negation of the previous side conditions
- Special Case : If P is missing the resulting "event" has no guard

Applying the rule for the while

EVENT computing4 REFINES EVENT computing3
  WHEN
    $grd2 : cfac = n$
  THEN
    $act1 : resultat := vfac$
  END

EVENT event4 REFINES EVENT event3
  WHEN
    $grd1 : cfac \neq n$
  THEN
    $act1 : vfac := (cfac + 1) * vfac$
    $act2 : cfac := cfac + 1$
  END
END

EVENT computing4 EVENT event4
  WHILE $cfac \neq n$ DO
    $act1 : vfac := (cfac + 1) * vfac$
    $act2 : cfac := cfac + 1$
  END;
END

Applying the INITIALISATION rule

EVENT INITIALISATION
  BEGIN
    $act1 : resultat :\in \mathbb{N}$
    $act3 : cfac := 0$
    $act4 : vfac := 1$
  END

$init$
  $resultat :\in \mathbb{N};$
  $cfac := 0;$
  $vfac := 1;$

Deriving an algorithm

precondition : $n \in \mathbb{N}$
postcondition : $result = factorial(n)$
local variables : $vfac, cfac \in \mathbb{N}$

$cfac := 0; vfac := 1; result :\in \mathbb{N};$
while $cfac \neq n$ **do**
  Invariant : $vfac = fac(cfac)$
  $vfac := (cfac + 1) * vfac; cfac := cfac + 1;$
;
$result := vfac;$

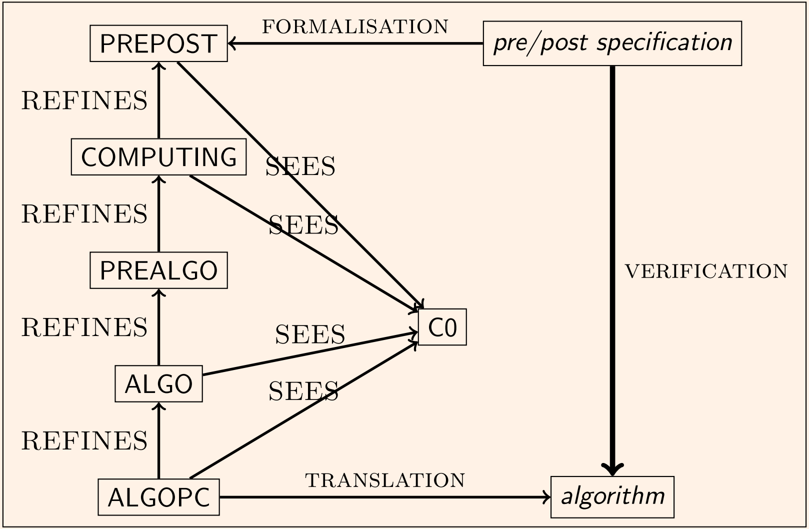
Current Summary

Conclusion

- Refinement helps in discovering invariants
- Refinement helps in proving invariants
- The choice of the *good* abstraction is not very simple and is a challenge by itself

Current Summary

The Iterative Pattern



General context C0

CONTEXT $C0$
SETS
U
CONSTANTS
$x, v, d0, f, D$
AXIOMS
$axm1 : x \in \mathbb{N}$
$axm25 : D \subseteq U$
$axm24 : f \in D \rightarrow D$
$axm23 : d0 \in D$
$axm2 : v \in \mathbb{N} \rightarrow D$
$axm3 : v(0) = d0$
$axm4 : \forall n \cdot n \in \mathbb{N} \Rightarrow v(n+1) = f(v(n))$
$th1 : Q(d_0, d) \equiv (d = v(x))$

- the sequence v expresses the post-condition $Q(d_0, d)$ with the precondition $P(d_0)$.
- $Q(d_0, d)$ is equivalent to $d = v(x)$.
- The theorem $th1$ should be proved in the context C0. he

General PREPOST Machine

```
MACHINE PREPOST
SEES C0
VARIABLES
  r
INVARIANTS
  inv1 : r ∈ D
EVENTS
INITIALISATION
  BEGIN
    act1 : r :∈ D
  END
EVENT computing
  BEGIN
    act1 : r := v(x)
  END
END
```

- The theorem $th1$ is validating the definition of the result r to compute.
- The event computing is expressing the *contract* of the given problem.
- it by a very simple problem that is the computation of the function n^2 using the addition operator.

First Refinement COMPUTING : Inductive Computation

EVENT INITIALISATION
  BEGIN
    $act1 : r :\in D$
    $act3 : vv := \{0 \mapsto d0\}$
    $act5 : k := 0$
  END

INITIALISATION is initializing the variables with respect to the initial values of the sequences of the context.

First Refinement COMPUTING : Inductive Computation

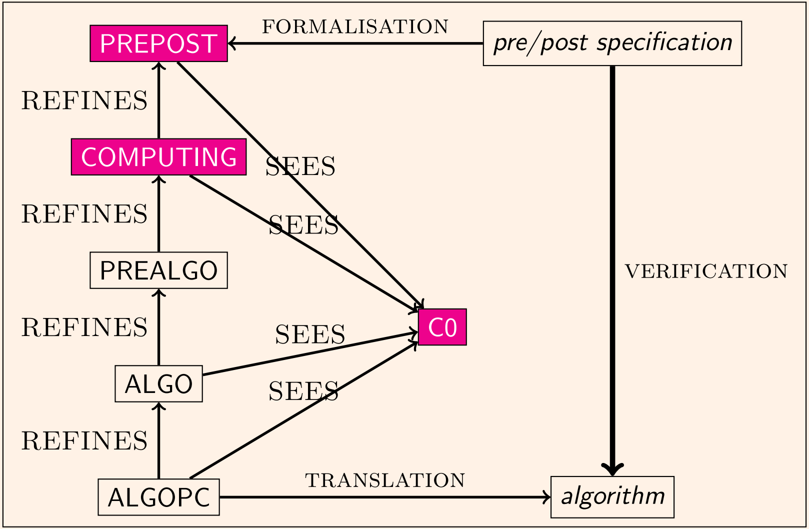
EVENT computing
  REFINES computing,
  WHEN
    $grd1 : x \in dom(vv)$
  THEN
    $act1 : r := vv(x)$
  END
END

computing is imply observing that the result is computed *simulating* the sequence vv.

EVENT step
  WHEN
    $grd1 : x \notin dom(vv)$
  THEN
    $act2 : vv(k+1) := f(vv(k))$
    $act4 : k := k+1$
  END

step is *simulating* the computation of the values of the sequence vv as a model computation.

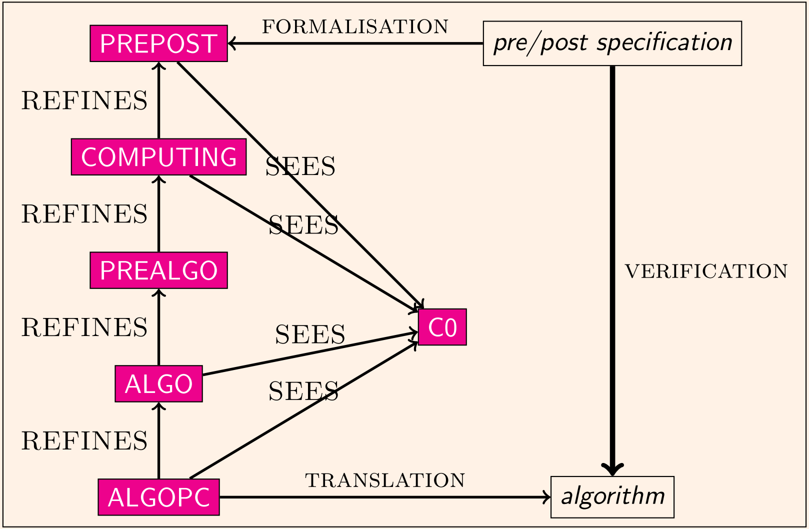
The Iterative Pattern



Completing the machines

- PREALGO : adding new variables for pointing out the necessary values to store cvv
- ALGO : hiding the model variables storing the unnecessary values of sequence vv
- ALGOPC ; adding control variable c

The Iterative Pattern



Listing 1 – Function derived from pattern for the sequence v

```
type(D)   f(int  x)
{int   r,k,cv,or,ok,ocv;
  r=0;k=0;cv=0;or=0;ok=k;ocv=cv;
  while  (k<x)
        {
          ok=k;ocv=cv;
          k=ok+1;
          cv=f(ocv);
        }
  r=cv;   return(r);}
```

Translation of Event-B Models

Comments

- The produced algorithm can be now checked using another proof environment as for instance Frama-C.
- The inductive property of the invariant is clearly verified and is easily derived from the Event-B machines.
- The verification is not required, since the system is correct by construction but it is a checking of the process itself
- the project called ITERATIVE-PATTERN ;
- the project is the pattern itself
- The invariants of the Event-B models can be reused in the verification using Frama-C, for instance, and the verification of the resulting algorithm is a confirmation of the translation.

Translation of Event-B Models

Listing 2 – Function derived from pattern power3

```
#include <limits.h>
/*@  requires  0 <= x;
     requires x*x*x <= INT_MAX ;
     ensures \result ==x*x*x;
*/
int power3(int x)
{int  r,ocz,cz,cv,cu,ocv,cw,ocw,ct,oct,ocu,k,ok;
  cz=0;cv=0;cw=1;ct=3;cu=0; ocw=cw;ocz=cz;
  oct=ct;ocv=cv;ocu=cu;k=0;ok=k;
      /*@ loop invariant cz == k*k*k;
        @ loop invariant cu  == k;
        @ loop invariant cv+ct==3*(cu+1)*(cu+1);
        @ loop invariant cz+cv+cw==3*(cu+1)*(cu+1)*(cu+1);
        @ loop invariant cv== 3*cu*cu;
        @ loop invariant cw == 3*cu+1;
        @ loop invariant k  <= x;
        @ loop assigns ct,oct,cu,ocu,cz,ocz,k,cv,cw,r,ok;
        @ loop assigns ocv,ocw;*/
  while (k<x)
        {
          ocz=cz;ok=k;ocv=cv;ocw=cw;oct=ct;ocu=cu;
          cz=ocz+ocv+ocw;
          cv=ocv+oct;
          ct=oct+6;
          cw=ocw+3;
          cu=ocu+1;
          k=ok+1;}
  r=cz;return(r);}
```

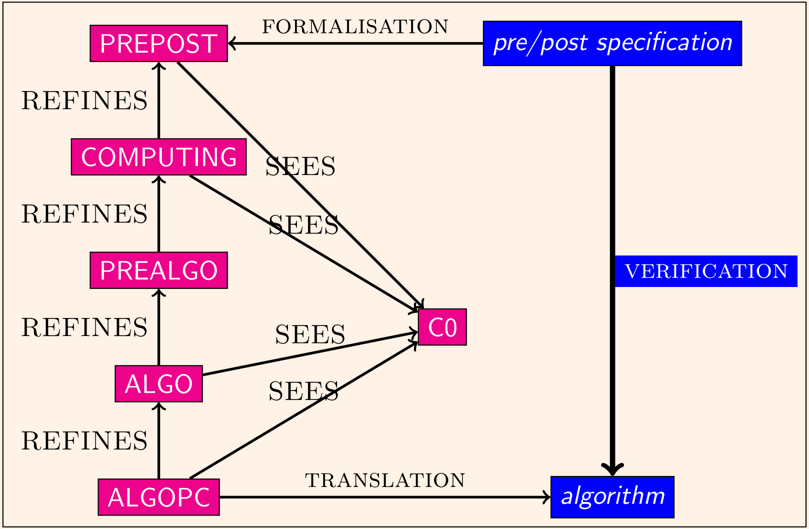
Summary for proof obligations

Name	Total	Automatic	Interactive
ex-induction	40	36	4
C0	2	0	2
PREPOST	4	4	0
COMPUTING	16	14	2
PREALGO	9	9	0
ALGO	6	6	0
ALGOPC	3	3	0

Summary

- The loop invariant is inductive but Frama-C does not prove it completely.
- Not the case with the RODIN platform which is able to discharge the whole set of proof obligations.
- However, the Event-B model is using auxiliary knowledge over sequences used for defining the computing process.
- The most difficult theorem is to prove that $\forall n \in \mathbb{N} : z_n = n * n * n$.

The Iterative Pattern



Current Summary

Summary on refinement

- Refining means making models more deterministic
- Refining means adding new variable and new events
- Refining is simulating
- Refining preserves safety properties of the refined model.
- The very abstract model is crucial.
- The process should be incremental to make proofs easier for the proof tool.
- Problem : Preserving the liveness properties