Cours MVSI
Modélisation et Vérification des Systèmes Informatiques

Vérification mécanisée de contrats (I)

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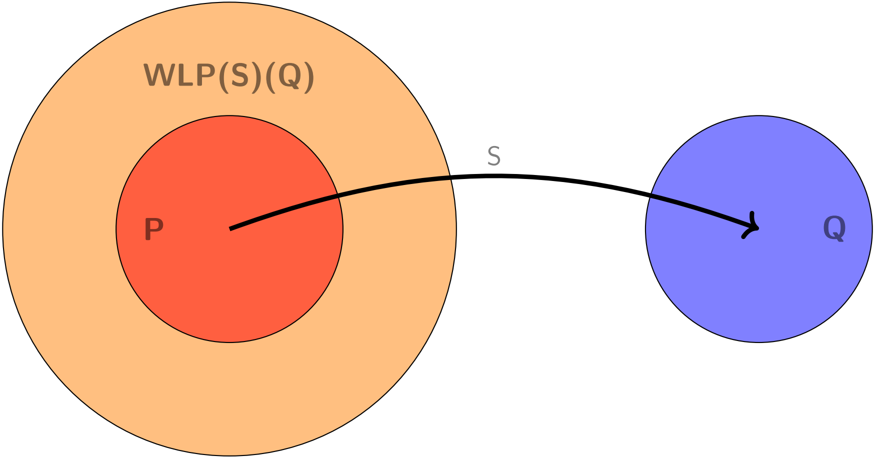
Plan

Sommaire

Asserted Program {P} S {Q}



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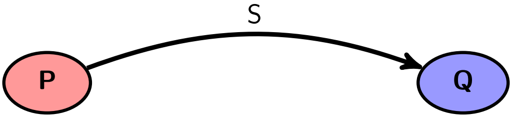
$$\{P\}\ S\ \{Q\}$$



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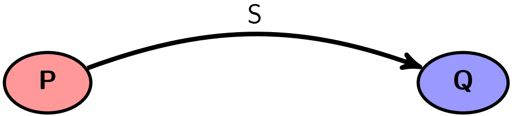

$$P \Rightarrow WLP(S)(\mathsf{Q})$$

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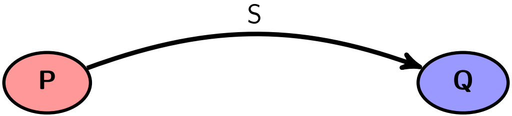

$$P \Rightarrow WLP(S)(\mathsf{Q})$$

Computing WLP(S)(Q) ?

Method for verifying partial correctness and RTE

A program P *satisfies* a contract (x,pre,post) :

- P transforms a variable x from an iinitial value x_0 and produces a final value x_f : $x_0 \xrightarrow{\mathsf{P}} x_f$
- x_0 satisfies pre : pre(x_0) and x_f satisfies post : post(x_0, x_f)
- pre$(x_0) \wedge x_0 \xrightarrow{\mathsf{P}} x_f \Rightarrow$ post(x_0, x_f)
- $\mathbb{D}$ is the domain of x for RTE (No Run Time Errors) .

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variables x : $\mathbb{D}$
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ensures $post(x_0, x_f)$

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- $pre(x_0) \wedge x = x_0 \Rightarrow P_0(x_0, x)$
- $pre(x_0) \wedge P_f(x_0, x) \Rightarrow post(x_0, x)$
- For any pair ℓ, ℓ' such that $\ell \longrightarrow \ell'$, we verify that for any values $x, x' \in \textsc{Memory}$
$$\left(\begin{array}{l} \left(\begin{array}{l} P_\ell(x_0, x)) \\ \wedge cond_{\ell,\ell'}(x) \wedge x' = f_{\ell,\ell'}(x) \end{array} \right) \\ \Rightarrow P_{\ell'}(x_0, x') \end{array} \right),$$
- For any pair m, n such that $m \longrightarrow n$, we verify that $\forall x, x' \in \textsc{Memory}$:
$pre(x_0) \wedge P_m(x_0, x) \Rightarrow$ **DOM**$(m, n)(x)$

Checking verificatioon conditions

- $pre(x_0) \wedge x = x_0 \Rightarrow P_0(x_0, x)$
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Example **DOM**(m,n)(x)

$DOM(\ell_0, \ell_1)(u) = u \in minint..maxint \wedge 5 \in minint..maxint \wedge u{+}5 \in minint..maxint$ where $\boxed{\begin{array}{l} \ell_0 : P_{\ell_0}(u); \\ \text{u := u+5;} \\ \ell_1 : P_{\ell_0}(u); \end{array}}$

Intuition

- A program P *produces* results or outputs from inputs according to a (operational or denotational) semantics
  - STATES is the set of states of P : $STATES = \mathsf{x} \rightarrow \mathbb{Z}$ where x designate variables of P.
  - s_0 et s_f two states of STATES : $\mathcal{D}(P)(s_0) = s_f$ means that P is executed from the memory state s_0 and produces a final state s_f.
  - For any current state s of P, $s(\mathsf{x}) = x$ for expressing the value of x in state s :

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  - pre$(x_0) \wedge x_0 \xrightarrow{\mathsf{P}} x_f \Rightarrow$ post(x_0, x_f)

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```
variables x : 𝔻
requires pre(x0)
ensures post(x0, xf)
      begin
      0 : P0(x0, x)
      S
      f : Pf(x0, x)
      end
```

$$
\begin{array}{l}
\text{variables x} : \mathbb{D} \\
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\text{ensures } post(x_0, x_f) \\
\qquad \left[\begin{array}{l} \text{begin} \\ 0 : P_0(x_0, x) \\ \text{S} \\ f : P_f(x_0, x) \\ \text{end} \end{array} \right.
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Checking a contract using the solver Z3

requires $x0 \geq 0$;
ensures $x_f = x0{+}2$;
variables X

$$\left[\begin{array}{l}\text{begin}\\ intX = x0;\\ 0 : x = x0\\ X = X{+}2;\\ 1 : x = x0{+}2\\ \text{end}\end{array}\right.$$

- $x0 \geq 0 \wedge x = x_0 \Rightarrow x = x0$
- $x = x0{+}2 \Rightarrow x = x0{+}2$
- conditions de vérification $0 \longrightarrow 1$:
  $x = x0 \wedge x' = x{+}2 \Rightarrow x' = x0{+}2$
- $(x0 >= 0, x == x0, x! = x0)$
- $(x == x0{+}2, x! = x0{+}2)$
- $(x == x0, xp == x{+}2, xp! = x0{+}2)$

Listing 1 – z3 en Python

```python
from numbers import Real
from z3 import *
x = Real('x')
xp = Real('xp')
x0 = Real('x0')
s = Solver()
s.add(x0 >= 0, x == x0, x != x0)
print(s.check())
s.add(x ==  x0+2, x != x0+2)
print(s.check())
s.add(x == x0,   xp == x + 2,  xp != x0+2)
print(s.check())
```

Transformation of predicates

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- $\forall x_0, x_f.\mathsf{pre}(x_0) \wedge x_0 \xrightarrow{\mathsf{P}} x_f \Rightarrow \mathsf{post}(x_0, x_f)$

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- $\forall x_0, x.\mathsf{pre}(x_0) \Rightarrow (x_0 \xrightarrow{\mathsf{P}} x \Rightarrow \mathsf{post}(x_0, x))$

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Weakest Liberal Precondition of S for P

$$\{S\}P(x) \stackrel{def}{=} \forall x_f.x \xrightarrow{\mathsf{S}} x_f \Rightarrow \mathsf{P}(x_f)$$

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Weakest Liberal Precondition of S for P

$$\{S\}P(x) \stackrel{def}{=} \forall x_f.x \xrightarrow{\mathsf{S}} x_f \Rightarrow \mathsf{P}(x_f)$$

- $\{x := e\}P(x) = P[x \mapsto e]$
- $\{\text{if } b(x) \text{ then } S1 \text{ else } S2\ \}P(x) =$
  $b(x) \wedge \{S1\}P(x) \vee \text{ not } b(x)\ \{S2\}P(x)$

Weakest Liberal Precondition of S for P

- $WLP(S)(P(x))$ is another notation for $\{S\}P(x)$.
- $\{\text{while } b(x) \text{ do } S \text{ end}\}P(x) = \{w\}(P(x))$

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- $\{\text{if } b(x) \text{ then } S; w \text{ else } skip\ \}P(x) =$
- $b(x) \wedge \{S; w\}P(x) \vee$ not $b(x) \ \wedge \ \{skip\}P(x) =$
- $b(x) \wedge \{S\}(\{w\}(P(x))) \vee$ not $b(x) \wedge P(x) = \{w\}(P(x))$
- $F(\{w\})(P(x)) = \{w\}(P(x))$

Examples

- $\{\text{while } x > 0 \text{ do } x := x-1 \text{ end}\}(x = 0) = x \geq 0$
- $\{\text{while } x > 0 \text{ do } x := x+1 \text{ end}\}(x = 0) = x \geq 0$
- $\{\text{while } x > 0 \text{ do } x := x+1 \text{ end}\}(x \leq 0) = x \in \mathbb{Z}$

Expressing a contract using predicate transformer

Computing WLP function

- $\forall x_0.\mathsf{pre}(x_0) \Rightarrow \{P\}\mathsf{post}(x_0, x)$
- $\forall x_0.x = x_0 \wedge \mathsf{pre}(x_0) \Rightarrow \{P\}\mathsf{post}(x_0, x)$
- Hoare Triple : $\{pre(x_0) \wedge x = x_0\}P\{post(x_0, x)\}$

⊠ Definition(Axiomes et règles d'inférence)

- Axiome d'affectation : $\{P(e/x)\}$**X :=E(X)**$\{P\}$.
- Axiome du saut : $\{P\}$**skip**$\{P\}$.
- Règle de composition : Si $\{P\}\mathbf{S}_1\{R\}$ et $\{R\}\mathbf{S}_2\{Q\}$, alors $\{P\}\mathbf{S}_1 ;\mathbf{S}_2\{Q\}$.
- Si $\{P \wedge B\}\mathbf{S}_1\{Q\}$ et $\{P \wedge \neg B\}\mathbf{S}_2\{Q\}$, alors $\{P\}$**if B then** $\mathbf{S}_1$ **then** $\mathbf{S}_2$ **fi**$\{Q\}$.
- Si $\{P \wedge B\}\mathbf{S}\{P\}$, alors $\{P\}$**while B do S od**$\{P \wedge \neg B\}$.
- Règle de renforcement/affaiblissement : Si $P' \Rightarrow P$, $\{P\}\mathbf{S}\{Q\}$, $Q \Rightarrow Q'$, alors $\{P'\}\mathbf{S}\{Q'\}$.

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Exemple de preuve $\{x = 1\}$**Z :=X ;X :=Y ;Y :=Z**$\{y = 1\}$

- (1) $x = 1 \Rightarrow (z = 1)[x/z]$ (propriété logique)
- (2) $\{(z = 1)[x/z]\}$**Z :=X**$\{z = 1\}$ (axiome d'affectation)
- (3) $\{x = 1\}$**Z :=X**$\{z = 1\}$ (Règle de renforcement/affaiblissement avec (1) et (2))
- (4) $z = 1 \Rightarrow (z = 1)[y/x]$ (propriété logique)
- (5) $\{(z = 1)[y/x]\}$**X :=Y**$\{z = 1\}$ (axiome d'affectation)
- (6) $\{z = 1\}$**X :=Y**$\{z = 1\}$ (Règle de renforcement/affaiblissement avec (4) et (5))
- (7) $z = 1 \Rightarrow (y = 1)[z/y]$ (propriété logique)
- (8) $\{(z = 1)[x/z]\}$**Y :=Z**$\{y = 1\}$ (axiome d'affectation)
- (9) $\{z = 1\}$**Y :=Z**$\{y = 1\}$ (Règle de renforcement/affaiblissement avec (7) et (8))
- (10) $\{x = 1\}$**Z :=X ;X :=Y ;**$\{z = 1\}$ (Règle de composition avec 3 et 6)
- (11) $\{x = 1\}$**Z :=X ;X :=Y ;Y :=Z**$\{y = 1\}$ (Règle de composition avec 11 et 9)

Sémantique des triplets de Hoare

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⊠ Definition

$\{P\}\mathbf{S}\{Q\}$ est défini par $\forall s,t \in STATES : P(s) \wedge \mathcal{D}(S)(s) = t \Rightarrow Q(t)$

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☺ Property Correction du système axiomatique des programmes commentés

- S'il existe une preuve construite avec les règles précédentes de $\{P\}\mathbf{S}\{Q\}$, alors $\{P\}\mathbf{S}\{Q\}$ est valide.
- Si $\{P'\}\mathbf{S}\{Q'\}$ est valide et si le langage d'assertions est suffisamment expressif, alors il existe une preuve construite avec les règles précédentes de $\{P\}\mathbf{S}\{Q\}$.

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⊠ Definition

Un langage d'assertions est la donnée d'un ensemble de prédicats et d'opérateurs de composition comme la disjonction et la conjonction ; il est muni d'une relation d'ordre partielle appelée implication. On le notera $(\textsc{Pred}, \Rightarrow, \mathbf{false}, \mathbf{true}, \wedge, \vee)$: $(\textsc{Pred}, \Rightarrow, \mathbf{false}, \mathbf{true}, \wedge, \vee)$ est un treillis complet.

Introduction de wlp

- $\{P\}\mathbf{S}\{Q\}$
- $\forall s,t \in STATES : P(s) \wedge \mathcal{D}(S)(s) = t \Rightarrow Q(t)$
- $\forall s \in STATES : P(s) \Rightarrow (\forall t \in STATES : \mathcal{D}(S)(s) = t \Rightarrow Q(t))$

....

Définition de wlp

$$wlp(S)(Q) \stackrel{def}{=} (\forall t \in STATES : \mathcal{D}(S)(s) = t \Rightarrow Q(t))$$

....

$$wlp(S)(Q) \equiv \overline{(\exists t \in STATES : \mathcal{D}(S)(s) = t \wedge \overline{Q}(t))}$$

....

....

Lien entre wp et wlp

- $loop(S) \equiv \overline{(\exists t \in STATES : \mathcal{D}(S)(s) = t}$ (ensemble des états qui ne permettent pas à S de terminer)
- $wp(S)(Q) \equiv wlp(S)(Q) \wedge \overline{loop(S)}$

....

Définition de wlp

☒ Definition

$WLP(S)(P) = \nu \lambda X.((B \wedge wlp(BS)(X)) \vee (\neg B \wedge P))$

☺ Property

- Si $P \Rightarrow Q$, then $wlp(S)(P) \Rightarrow wlp(S)(Q)$.

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☒ Definitiontriplets de Hoare

$\{P\}\mathbf{S}\{Q\} \stackrel{def}{=} P \Rightarrow wlp(S)(Q)$

. . . .

Axiomatisation de la Logique de Hoare

⊠ Definitiontriplets de Hoare

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- Règle de renforcement/affaiblissement : Si $P' \Rightarrow P$, $\{P\}\mathbf{S}\{Q\}$, $Q \Rightarrow Q'$, alors $\{P'\}\mathbf{S}\{Q'\}$.

Vérification

- $\{P\}\mathbf{S}\{Q\}$
- $\forall s \in STATES.P(s) \Rightarrow wlp(S)(Q)(s)$
- $\forall s \in STATES.P(s) \Rightarrow (\forall t \in STATES : \mathcal{D}(S)(s) = t \Rightarrow Q(t))$
- $\forall s, t \in STATES.P(s) \wedge \mathcal{D}(S)(s) = t \Rightarrow Q(t))$
- Correction : Si on a construit une preuve de $\{P\}\mathbf{S}\{Q\}$ avec les règles de la logique de Hoare, alors $P \Rightarrow wlp(S)(Q)$
- Complétude sémantique : Si $P \Rightarrow wlp(S)(Q)$, alors on peut construire une preuve de $\{P\}\mathbf{S}\{Q\}$ avec les règles de la logique de Hoare si on peut exprimer $wlp(S)(P)$ dans le langae d'assertions.

Example 1

Listing 2 – difference of two numbers

```
#include <limits.h>
/*@ requires a-b >= INT_MIN && a-b <= INT_MAX;
    assigns \nothing;
    ensures \result == (a - b);
*/
static int difference(int a, int b) {
  return a-b;
}
```

- `INT_MIN` (resp. `INT_MAX`) is the smallest codable integer (resp. greatest codable integer).
- $a0{-}b0 \geq INT\_MIN \wedge a0{-}b0 \leq INT\_MAX \wedge a = a0 \wedge b = b0 \Rightarrow [\backslash result = a{-}b](\backslash result = (a{-}b))$

Example 2

Listing 3 – incrément de nombre

```
/*@ requires x0 >= 0;
    assigns \nothing;
    ensures \result == x0+2;
  @*/

int  exemple(int x0) {
  int x=x0;
  //@ assert x == x0;
  x = x + 2;
//@ assert  x== x0+2;
return x;
}
```

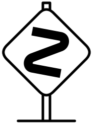
requires $x0 \geq 0$;
ensures $x_f = x0+2$;
variables x

$$\left[\begin{array}{l} \text{begin} \\ int\text{x} = x0; \\ 0 : x = x0 \\ \text{x := x+2;} \\ 1 : x = x0+2 \\ \text{end} \end{array}\right.$$

Conditions de vérification $0 \longrightarrow 1$:

- $x = x0 \wedge x' = x+2 \Rightarrow x' = x0+2$
- $x = x0 \Rightarrow (x' = x+2 \Rightarrow x' = x0+2)$
- $x = x0 \Rightarrow (x+2 = x0+2)$
- $wp(x := x+2)(x = x0+2) = (x+2 = x0+2)$
- $x = x0 \wedge x0 \geq 0 \Rightarrow wp(x := x+2)(x = x0+2)$
- $x = x0 \wedge x0 \geq 0 \Rightarrow x+2 = x0+2$
- $x = x0 \wedge x0 \geq 0 \Rightarrow x0+2 = x0+2$

- $x0 \geq 0 \wedge x = x_0 \Rightarrow x = x0$
- $x = x0+2 \Rightarrow x = x0+2$
- $x = x0 \Rightarrow wp(x := x+2)(x = x0+2)$



calcul de $wp(X := X+2)(x = x0+2)$

Exemple avec wp de ACSL/Frama-c avant wp

Listing 4 – incrément de nombre

```
/*@ requires x0 >= 0;
    assigns \nothing;
    ensures \result == x0+1;
  @*/

int  exemple(int x0) {
  int x=x0;
  //@ assert x == x0;
  x = x + 2;
//@ assert  x== x0+2;
return x;
//@ assert  \result== x0+2;
}
```

Exemple avec wp de ACSL/Frama-c après wp

Listing 5 – incrément de nombre

```
/*@ requires x0 >= 0;
    assigns \nothing;
    ensures \result == x0;
  @*/

int  exemple(int x0) {
  int x=x0;
//@ assert x == x0+1;
  x = x + 2;
//@ assert  x== x0+2;
return x;

}
```

Opérateur WP

Soit STATES l'ensemble des états sur l'ensemble X des variables. Soit S une instruction de programme sur X. Soit A une partie de STATES.
$s \in WP(S)(A)$, si la condition suivante est vérifiée :

$$\left(\begin{array}{l} \forall t \in STATES : \mathcal{D}(S)(s) = t \Rightarrow t \in A \\ \wedge \\ \exists t \in STATES : \mathcal{D}(S)(s) = t \end{array}\right)$$

- $WP(X := X{+}1)(A) = \{s \in STATES | s[X \mapsto s(X) \oplus 1] \in A\}$
- $WP(X := Y{+}1)(A) = \{s \in STATES | s[X \mapsto s(Y) \oplus 1] \in A\}$
- $WP(while\ X > 0\ do\ X := X{-}1\ od)(A) = \{s \in STATES | (s(X) \leq 0) \vee (s(X) \in A \wedge s(X) < 0)\}$
- $WP(while\ x > 0\ do\ x := x{+}1\ od)(A) = \{s \in STATES | (s(X) \in A \wedge s(X) \leq 0)\}$
- $WP(while\ x > 0\ do\ x := x{+}1\ od)(\varnothing) = \varnothing$
- $WP(while\ x > 0\ do\ x := x{+}1\ od)(STATES) = \{s \in STATES | s(X) \leq 0\}$

Propriétés de WP

Propriétés

- WP est une fonction monotone pour l'inclusion d'ensembles de STATES.
- $WP(S)(\varnothing) = \varnothing$
- $WP(S)(A \cap B) = WP(S)(A) \cap WP(S)(B)$
- $WP(S)(A) \cup WP(S)(B) \subseteq WP(S)(A \cup B)$
- Si S est déterministe, $WP(S)(A \cup B) = WP(S)(A) \cup WP(S)(B)$

- WP est un opérateur avec le profil suivant

  pour toute instruction S du langage de programmation,
  $$WP(S) \in \mathcal{P}(STATES) \nrightarrow \mathcal{P}(STATES)$$
- $(\mathcal{P}(STATES), \subseteq)$ est un treillis complet.
- $(Pred, \Rightarrow)$ est une structure où
  - (1) $Pred$ est une *extension* du langage d'expressions booléennes
  - (2) $Pred$ est une *intension* introduite comme un langage d'assertions
  - $\Rightarrow$ est l'implication
  - $s \in A$ correspond une assertion P vraie en s notée $P(s)$.

Définition structurelle des transformateurs de prédicats

- S est une instruction de STATS.
- T est le type ou les types des variables et D est la constante ou les constantes Définie(s).
- P est un prédicat du langage Pred
- X est une variable de programme
- $E(X,D)$ (resp. $B(X,D)$) est une expression arithmétique (resp. booléenne) dépendant de X et de D.
- x est la valeur de X (X contient la valeur x).
- $e(x,d)$ (resp. $b(x,d)$) est l'expression arithmétique (resp. booléenne) du langage Pred associée à l'expression $E(X,D)$ (resp. $B(X,D)$) du langage des expressions arithmétiques (resp. booléennes) du langage de programmation Prog
- $b(x,d)$ est l'expression arithmétique du langage Pred associée à l'expression $E(X,D)$ du langage des expressions arithmétiques du langage de programmation Prog

Définition structurelle des transformateurs de prédicats

S	wp(S)(P)
X :=E(X,D)	$P[e(x,d)/x]$
SKIP	P
$\mathrm{S}_1;\mathrm{S}_2$	$wp(\mathrm{S}_1)(wp(\mathrm{S}_2)(P))$
IF B S_1 ELSE S_2 FI	$(B \Rightarrow wp(S_1)(P)) \wedge (\neg B \Rightarrow wp(S_2)(P))$
WHILE B DO S OD	$\mu.(\lambda X.(B \Rightarrow wp(S)(X)) \wedge (\neg B \Rightarrow P))$

- $wp(X := X{+}5)(x \geq 8) \stackrel{def}{=} x{+}5 \geq 8 \wedge x \geq 3$
- $wp(\text{WHILE } x > 1 \text{ DO } X := X{+}1 \text{ OD})(x = 4) = FALSE$
- $wp(\text{WHILE } x > 1 \text{ DO } X := X{+}1 \text{ OD})(x = 0) = x = 0$

Logique de Hoare Correction Totale

⊠ Definitiontriplets de Hoare Correction Totale

$$[P]\mathbf{S}[Q] \stackrel{def}{=} P \Rightarrow wp(S)(Q)$$

. . . .

☒ Definitiontriplets de Hoare Correction Totale

$[P]\mathbf{S}[Q] \stackrel{def}{=} P \Rightarrow wp(S)(Q)$

. . . .

. . . .

☒ Definition(Axiomes et règles d'inférence)

- Axiome d'affectation : $[P(e/x)]$**X :=E(X)**$[P]$.
- Axiome du saut : $[P]$**skip**$[P]$.
- Règle de composition : Si $[P]\mathbf{S}_1[R]$ et $[R]\mathbf{S}_2[Q]$, alors $[P]\mathbf{S}_1\ \mathbf{;}\mathbf{S}_2[Q]$.
- Si $[P \wedge B]\mathbf{S}_1[Q]$ et $[P \wedge \neg B]\mathbf{S}_2[Q]$, alors $[P]$**if B then** $\mathbf{S}_1$ **then** $\mathbf{S}_2$ **fi**$[Q]$.
- Si $[P(n{+}1)]\mathbf{S}[P(n)]$, $P(n{+}1) \Rightarrow b$, $P(0) \Rightarrow \neg b$, alors $[\exists n \in \mathbb{N}.P(n)]$**while B do S od**$[P(0)]$.
- Règle de renforcement/affaiblissement : Si $P' \Rightarrow P$, $[P]\mathbf{S}[Q]$, $Q \Rightarrow Q'$, alors $[P']\mathbf{S}[Q']$.

. . . .

Correction

:
Si $[P]$**S**$[Q]$ est dérivé selon les règles ci-dessus, alors $P\wp(S)5Q)$.

- $[P(e/x)]$**X :=E(X)**$[P]$ est valide : $wp(X := E)(P)/x = P(e/x)$.
- $[\exists n \in \mathbb{N}.P(n)]$**while B do S od**$[P(0)]$: si s est un état de P(n) alors au bout de n boucles on atteint un état s_f tel que P(0) est vrai en s_f.

Complétude

:
Si $P \Rightarrow wp(S)(Q)$, alors il existe une preuve de $[P]$**S**$[Q]$ construites avec les règles ci-dessus,

- $P \Rightarrow wp(X := E(X))(Q)$: $P \Rightarrow Q(e/x)$ et $[Q(e/x)]$**X :=E(X)**$[Q]$ constituent une preuve.
- $P \Rightarrow wp(while)(Q)$:
  - On construit la suite de $P(n)$ en définissant $P(n) = W_n$.
  - On vérifie que cela vérifie la règle du while.

Verification of contract (I)

A program P *satisfies* a contract (pre,post) :

- P transforms a variable x from an initial value x_0 and produces a final value x_f : $x_0 \xrightarrow{\mathsf{P}} x_f$
- x_0 satisfait pre : pre(x_0) and x_f satisfait post : post(x_0, x_f)
- pre$(x_0) \wedge x_0 \xrightarrow{\mathsf{P}} x_f \Rightarrow$ post(x_0, x_f)

requires $pre(x_0)$
ensures $post(x_0, x_f)$
variables X

begin
$0 : P_0(x_0, x)$
$\mathtt{instruction}_0$
$\ldots$
$i : P_i(x_0, x)$
$\ldots$
$\mathtt{instruction}_{f-1}$
$f : P_f(x_0, x)$
end

- $pre(x_0) \wedge x = x_0 \Rightarrow P_0(x_0, x)$
- $pre(x_0) \wedge P_f(x_0, x) \Rightarrow post(x_0, x)$
- For each pair ℓ, ℓ' such that $\ell \longrightarrow \ell'$, one checks that foa any value $x, x' \in \textsc{Memory}$

$$\left(\begin{array}{l} \left(\begin{array}{l} pre(x_0) \wedge P_\ell(x_0, x)) \\ \wedge cond_{\ell,\ell'}(x) \wedge x' = f_{\ell,\ell'}(x) \end{array} \right) \\ \Rightarrow P_{\ell'}(x_0, x') \end{array} \right),$$

Verification du contract (II)

A program P *satisfies* a contract (pre,post) :

- P transforms a variable x from an initial value x_0 and produces a final value x_f : $x_0 \xrightarrow{\mathsf{P}} x_f$
- x_0 satisfait pre : $\mathsf{pre}(x_0)$ and x_f satisfait post : $\mathsf{post}(x_0, x_f)$
- $\mathsf{pre}(x_0) \wedge x_0 \xrightarrow{\mathsf{P}} x_f \Rightarrow \mathsf{post}(x_0, x_f)$

requires $pre(x_0)$
ensures $post(x_0, x_f)$
variables X

begin
$0 : P_0(x_0, x)$
`instruction`$_0$
...
$i : P_i(x_0, x)$
...
`instruction`$_{f-1}$
$f : P_f(x_0, x)$
end

- $\forall x_f, x_0.\mathsf{pre}(x_0) \wedge x_0 \xrightarrow{\mathsf{P}} x_f \Rightarrow \mathsf{post}(x_0, x_f)$
- $\forall x_f, x_0.\mathsf{pre}(x_0) \Rightarrow (x_0 \xrightarrow{\mathsf{P}} x_f \Rightarrow \mathsf{post}(x_0, x_f))$
- $\forall x_0.\mathsf{pre}(x_0) \Rightarrow \forall x_f.(x_0 \xrightarrow{\mathsf{P}} x_f \Rightarrow \mathsf{post}(x_0, x_f))$
- $\forall x_0.\mathsf{pre}(x_0) \Rightarrow \forall x.(x_0 \xrightarrow{\mathsf{P}} x \Rightarrow \mathsf{post}(x_0, x))$
- $\forall x_0.\mathsf{pre}(x_0) \Rightarrow WLP(P)(\mathsf{post}(x_0, x))$
-

Partial Correctness by computing WLPs

Un programme P *satisfies* a contract (pre,post) :

- P transforms a variable x from an initial value x_0 and produces a final value x_f : $x_0 \xrightarrow{\mathsf{P}} x_f$
- x_0 satisfies pre : $\mathsf{pre}(x_0)$ and x_f satisfies post : $\mathsf{post}(x_0, x_f)$
- $\mathsf{pre}(x_0) \wedge x_0 \xrightarrow{\mathsf{P}} x_f \Rightarrow \mathsf{post}(x_0, x_f)$
- $\forall x_0.\mathsf{pre}(x_0) \Rightarrow WLP(P)(\mathsf{post}(x_0, x))$

Partial Correctness by computing WLPs

Un programme P *satisfies* a contract (pre,post) :

- P transforms a variable x from an initial value x_0 and produces a final value x_f : $x_0 \xrightarrow{\mathsf{P}} x_f$
- x_0 satisfies pre : pre(x_0) and x_f satisfies post : post(x_0, x_f)
- pre$(x_0) \wedge x_0 \xrightarrow{\mathsf{P}} x_f \Rightarrow$ post(x_0, x_f)
- $\forall x_0$.pre$(x_0) \Rightarrow WLP(P)($post$(x_0, x))$
- WLP is not computable . . .
- Using Hoare logic in the WLP computing as suggested by Rustan Leino. de WLP.

Verification of contract (III)

A program P *satisfies* a contract (pre,post) :

- P transforms a variable x from an initial value x_0 and produces a final value x_f : $x_0 \xrightarrow{\text{P}} x_f$
- x_0 satisfait pre : $\text{pre}(x_0)$ and x_f satisfait post : $\text{post}(x_0, x_f)$
- $\text{pre}(x_0) \wedge x_0 \xrightarrow{\text{P}} x_f \Rightarrow \text{post}(x_0, x_f)$

requires $pre(x_0)$
ensures $post(x_0, x_f)$
variables X

```
begin
/·@assert P0(x0, x)·/
T;
/·@loop invariant I(x0, x)·/
while B(x) do
   S
od
/·@assert Pf(x0, x)·/
end
```

- $x = x_0 \wedge \text{pre}(x_0) \Rightarrow P_0(x_0, x)$
- $\text{pre}(x_0) \wedge P_0(x_0, x) \Rightarrow WLP(T)(I(x_0, x))$
- $I(x_0, x) \wedge B(x) \Rightarrow WLP(S)(I(x_0, x))$
- $I(x_0, x) \wedge \neg B(x) \Rightarrow P_f(x_0, x)$

Verification of contract (IV)

requires $pre(x_0)$
ensures $post(x_0, x_f)$
variables X

begin
$/\cdot$@assert $P_0(x_0, x)\cdot/$
$S1;$
$S2;$
$/\cdot$@assert $P_f(x_0, x)\cdot/$
end

- $x = x_0 \wedge \mathsf{pre}(x_0) \Rightarrow P_0(x_0, x)$
- $P_0(x_0, x) \Rightarrow WLP(S1; S2)(P_f(x_0, x))$

Verification of contract (V)

requires $pre(x_0)$
ensures $post(x_0, x_f)$
variables X

```
begin
/·@assert P0(x0, x)·/
if B(x) do
   S1
else
   S2
elfi
/·@assert Pf(x0, x)·/
end
```

$$x = x_0 \wedge \mathsf{pre}(x_0) \Rightarrow P_0(x_0, x)$$

$$\boxed{\begin{array}{l} P_0(x_0, x) \Rightarrow \\ \quad B(x) \wedge WLP(S1)(P_f(x_0, x)) \\ \quad \vee \\ \quad \neg B(x) \wedge WLP(S2)(P_f(x_0, x)) \end{array}}$$