Cours MALG & MOVEX

MALG

Analyse des programes

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Summary

Summary

Problem of Software Safety : Ariane 5 – Explosion after launch (1996)

- The Ariane 5 rocket explodes 37 seconds after launch.
- Loss of the rocket and its valuable cargo (500 million euros).
- Software error : Incorrect type conversion : a floating point number (64 bits) was converted to an integer (16 bits) without checking that the value fit in the space.
- Result : overflow =¿ exception =¿ guidance system shutdown =¿ disorientation =¿ self-destruction.
- The software came from Ariane 4 and was reused without modification, even though the speeds were much higher in Ariane 5.

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Ariane 501

A 64-bit floating point number (representing the horizontal speed of the rocket relative to the launch pad) was converted to a 16-bit integer. Unfortunately, the number in question was larger than 32768, the largest integer that can be encoded in 16 bits (WORD), and the conversion was incorrect.

Therac-25 – Radiotherapy overdose (1985-87)

- Several patients received massive doses of radiation (up to 100 times the intended dose).
- At least six patients were seriously irradiated, with several deaths.
- Software error : Race conditions between two internal processes : in some cases, a mechanical safety feature was disabled but the software thought it was active.
- Insufficient safeguards to check the actual status of the device.
- The software assumed that certain errors were 'impossible'. No alarm was raised.

Patriot incident – Gulf War (1991)

- A battery of Patriot missiles failed to intercept an Iraqi Scud missile.
- Consequences : 28 American soldiers killed in Dhahran, Saudi Arabia.
- Software error* : Rounding error in floating point. The system used a time counter that accumulated a small error every second.
- After 100 hours of operation without restarting, the error had accumulated =¿ incorrect calculation of the missile's position =¿ interception failure.
- This precision bug caused an error of 0.34 seconds . . . enough to miss a high-speed missile.

Static Program Analysis

- Objectives of static program analysis
  - to prove properties about the run-time behaviour of a program
  - in a fully automatic way ie without interaction
  - without actually executing the program
- Applications
  - code optimisation
  - error detection (array out of bound access, null pointers)
  - proof support for generation of invariant

iOS bug : Emoji crashes Messages (2018)

- What happened : In 2018, a simple sequence of characters (a Telugu character, an Indian language) sent via iMessage or other apps such as WhatsApp, Facebook Messenger, etc., caused the app to crash or even completely freeze the iPhone.
- Where ? iPhone (iOS 11.2 to 11.2.5), iPad, Mac (in some apps too)
- Technical error : The iOS system did not know how to display a Telugu character correctly. The text rendering engine (CoreText) attempted to draw it, but poor memory management caused the system to crash or freeze completely.
- Consequences : The Messages app could not be opened (it crashed on launch). In some cases, the iPhone would restart in a loop. The conversation had to be deleted from another device or via iCloud.
- Apple fixed the bug in iOS 11.2.6, which was released as an emergency update shortly afterwards.
- Everyone uses Messages or Messenger. A single character could crash an iPhone : it was used as a joke on social media. It shows how a small detail (a text character !) can have a disproportionate effect on complex systems.

Analysing software-based systems

- Ensuring safety of produced system
- Technique 1 : Testing but incomplete and non exhautive
- Technbique 2 : Verification by model checling but specific and not applicable to any case and explosion of states number
- Technique 3 : Verification by using proof tools as Frama-c or Dafny or JML but limits due to the undecidability of program verification

Summary

- Role of approximation as limiting the complexity, finiteness of systems, restriction over specifications and programs
- Approximations may be formalised by abstract interpretation a mathematical framework for analysing programs.

Foundational Ideas of Abstract Interpretation

- A Theory described in works of Patrick Cousot (1976 $\longrightarrow$ now), Father of the Abstract Interpretation : analysis of large codes for embeded software A380 with Astrée.
- ... and Radhia Cousot is the second Almate of the Abstract Interpretation Company.
- A Comprehensive Web Site is maintanined by Professor Patrick Cousot at ENS Paris.
- *abstract interpretation relies on an idea of discrete approximation which consists in replacing the reasoning on a concrete exact semantics by a computation on an abstract approximate semantics.*
- A theory unifying abstract and concrete objects with respect to a given semantics
- A theory providing a way to *transfer* statements from a concrete (complex) to an abstract (simpler) semantcical framework.
- A theory formalizing the approximated analyse of programs and allowing to compare relative precision of analyses

AI or Abstract Interpretation

- Static Analysis *computes* approximations
- Abstract Interpretation (AI) provides a mathematical framework for relating *approximations*
- Properties of programs are generally non computable :
  - the halting problem is undecidable
  - Model checking is computing over finite structures
  - Proof assistant may be useful for proving partial correctness or total correctness by applying induction pricniles (see Event B)
  - AI provides another solution by trabfering results from a *concret* framework to an *abstract* structure

Static Analysis of Program Properties

- $\mathcal{CS}(P)$ is the concrete semantics of a program P : the set of reachable states of P.
- $\mathcal{AS}(P)$ is the approximation of $\mathcal{CS}(P)$: $\mathcal{CS}(P) \subseteq \mathcal{AS}(P)$.
- $\mathcal{CS}(P)$ is generally not computable and we will seek for *computable* approximation or abstract semantics $\mathcal{AS}(P)$.
- Problems : $\mathcal{AS}(P)$ may *loose* the expression of properties.

- φ is a program property stating the possible bugs or arrors which we want to avoid.
- $\mathcal{CS}(P)$ is the concrete semantics of a program P : the set of reachable states of P.
- $\mathcal{AS}(P)$ is the approximation of $\mathcal{CS}(P)$: $\mathcal{CS}(P) \subseteq \mathcal{AS}(P)$.
- Case 1 : $\mathcal{CS}(P) \cap \varphi = \varnothing$ and $\mathcal{AS}(P) \cap \varphi = \varnothing$
- Case 2 : $\mathcal{CS}(P) \cap \varphi \neq \varnothing$ and $\mathcal{AS}(P) \cap \varphi \neq \varnothing$
- Case 3 : $\mathcal{CS}(P) \cap \varphi = \varnothing$ and $\mathcal{AS}(P) \cap \varphi \neq \varnothing$

- Case 1 : $\mathcal{CS}(P) \cap \varphi = \varnothing$ and $\mathcal{AS}(P) \cap \varphi = \varnothing$:
  - P is safe with respect to φ and no error specified by φ is possible for P.
  - Checking is computable on the approximation
- Case 2 : $\mathcal{CS}(P) \cap \varphi \neq \varnothing$ and $\mathcal{AS}(P) \cap \varphi \neq \varnothing$:
  - An error is detected on the approximation and on the concrete semantics.
  - P is unsafe with respect to φ
  - and an error is detected by the analyser.
- Case 3 : $\mathcal{CS}(P) \cap \varphi = \varnothing$ and $\mathcal{AS}(P) \cap \varphi \neq \varnothing$:
  - P is safe with respect to φ
  - but an error is detected by the analyser
  - A false alarm is provided by the analyzer
  - Approximation is over-approximating P with respect to φ
  - The analysis should be refined

Summary

The Interproc Analyzer

- a web interface to the Interproc analyzer connected to the APRON Abstract Domain Library
- Analysis of programs uting different *abstract domains*
- `http://pop-art.inrialpes.fr/interproc/`
- developed by Antoine Miné and his team.

Example 1 : Increment of a value

```
proc incr (x:int) returns (y:int)
begin
  y = x+1;
end

var i:int;
begin
  i = 0;
  while (i<=10) do
    i = incr(i);
  done;
end
```

Example 1 : Increment of a value

```
proc incr (x : int) returns (y : int) var ;
begin
  /* (L3 C5) [|x>=0; -x+10>=0|] */
  y = x + 1; /* (L4 C10)
                [|x>=0; -x+10>=0; y-1>=0; -y+11>=0|] */
end

var i : int;
begin
  /* (L8 C5) top */
  i = 0; /* (L9 C8) [|i>=0; -i+11>=0|] */
  while i <= 10 do
    /* (L10 C18) [|i>=0; -i+10>=0|] */
    i = incr(i); /* (L11 C16)
                    [|i-1>=0; -i+11>=0|] */
  done; /* (L12 C7) [|i-11=0|] */
end
```

Example 2 : Division

```
var Q : int, R : int, X : int, Y : int;
begin

  Q = 0;
  R = Y;

     while R >= Y do
       Q = Q + 1;
       R = R - Y;
     done;

end
```

Example 2 : Division

```
Annotated program after forward analysis
var Q : int, R : int, X : int, Y : int;
begin
  /* (L3 C5) top */
  Q = 0; /* (L5 C8) [|Q=0|] */
  R = Y; /* (L6 C8) [|Q>=0|] */
  while R >= Y do
    /* (L8 C20) [|Q>=0|] */
    Q = Q + 1; /* (L9 C17) [|Q-1>=0|] */
    R = R - Y; /* (L10 C17) [|Q-1>=0|] */
  done; /* (L11 C10) [|Q>=0|] */
end
```

Example 3 : modified division

```
var Q : int, R : int, X : int, Y : int;
begin

  Q = 0;
  R = Y;
  if Y > 0 then
     while R >= Y do
       Q = Q + 1;
       R = R - Y;
     done;
  else
    skip;
  endif;
end
```

Example 3 : modified division

Annotated program after forward analysis

```
Annotated program after forward analysis
var Q : int, R : int, X : int, Y : int;
begin
  /* (L3 C5) top */
  Q = 0; /* (L5 C8) [|Q=0|] */
  R = Y; /* (L6 C8) [|Q=0|] */
  if Y > 0 then
     /* (L7 C15) [|Q>=0; Y-1>=0|] */
     while R >= Y do
       /* (L8 C20) [|Q>=0; R-1>=0; Y-1>=0|] */
       Q = Q + 1; /* (L9 C17)
                     [|Q-1>=0; R-1>=0; Y-1>=0|] */
       R = R - Y; /* (L10 C17)
                     [|Q-1>=0; Y-1>=0|] */
     done; /* (L11 C10) [|Q>=0; Y-1>=0|] */
  else
    /* (L12 C6) [|Q=0; -Y>=0|] */
    skip; /* (L13 C9) [|Q=0; -Y>=0|] */
  endif; /* (L14 C8) [|Q>=0|] */
end
```

Summary

Static analysis

Static program analysis analyses computer software without actually executing programs :

- absence of run time errors
- detecting variables used before initialisation.

- Data flow analysis
- Abstract interpretation
- Use of property-preserving abstractions
- Programs are interpreted in abstractions

Sign analysis

- Sign analysis is used to determine the sign of variables
- x is an integer variable and has the following possible *abstract* states :
  - $x > 0$
  - $x \geq 0$
  - $x = 0$
  - $x < 0$
  - $x \leq 0$
  - $x \neq 0$

Simple example

ℓ_0 :
$y := -11;$
ℓ_0 :
IF $x < y$ **THEN**
  ℓ_1 :
  $z := y;$
  ℓ_2 :
ELSE
  ℓ_3 :
  $z := x;$
  ℓ_4 :
FI
ℓ_5 :

Simple example

ℓ_0 :
$y := -11;$
$\ell_0 : y < 0$
IF $x < y$ **THEN**
  ℓ_1 :
  $z := y;$
  ℓ_2 :
ELSE
  ℓ_3 :
  $z := x;$
  ℓ_4 :
FI
ℓ_5 :

Simple example

ℓ_0 :
$y := -11;$
$\ell_0 : y < 0$
IF $x < y$ **THEN**
  $\ell_1 : y < 0 \quad x < 0$
  $z := y;$
  ℓ_2 :
ELSE
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  $z := x;$
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FI
ℓ_5 :

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$y := -11;$
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IF $x < y$ **THEN**
  $\ell_1 : y < 0 \quad x < 0$
  $z := y;$
  $\ell_2 : y < 0 \quad x < 0 \quad z < 0$
ELSE
  $\ell_3 : y < 0 \quad x \in \mathbb{Z}$
  $z := x;$
  ℓ_4 :
FI
ℓ_5 :

Simple example

ℓ_0 :
$y := -11;$
$\ell_0 : y < 0$
IF $x < y$ **THEN**
  $\ell_1 : y < 0 \quad x < 0$
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  $\ell_2 : y < 0 \quad x < 0 \quad z < 0$
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  $z := x;$
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Simple example

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  $z := x;$
  $\ell_4 : y < 0 \quad x \in \mathbb{Z}$
FI
$\ell_5 : y < 0 \quad x \in \mathbb{Z} \quad z \in \mathbb{Z}$

Simple example

ℓ_0 :
$y := -11;$
$\ell_0 : y < 0$
IF $x < y$ **THEN**
  $\ell_1 : y < 0 \quad x < 0$
  $z := x;$
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  $\ell_3 : y < 0 \quad x \in \mathbb{Z}$
  $z := x;$
  $\ell_4 : y < 0 \quad x \in \mathbb{Z}$
FI
$\ell_5 : y < 0 \quad x \in \mathbb{Z} \quad z \in \mathbb{Z}$

Result

$y < 0 \quad x \in \mathbb{Z} \quad z < 0$ means that $z < 0$ is an information resulting from the analysis over abstract domain of signs.

Verification by computing set of reachable states

- $\mathcal{MS}$ is $(Th(s,c), x, \textsc{Vals}, \textsc{Init}(x), \{r_0, \ldots, r_n\})$
- $\textsc{Next} \stackrel{def}{=} r_0 \vee \ldots \vee r_n$.
- S is a safety property, when
  $\forall x_0, x \in \textsc{Vals}. Init(x_0) \wedge \textsc{Next}^\star(x_0, x) \Rightarrow x \in S$.
- S is a safety property for $\mathcal{MS}$ if, and only if, $\textsc{reachable}(\mathcal{MS}) \subseteq \mathcal{S}$

Characterisation of $\textsc{reachable}(\mathcal{MS}) \subseteq \mathcal{S}$ as a fixed-point

$(\mathcal{P}(\textsc{Vals}), \subseteq, \varnothing, \cup, \cap)$ is a complete lattice and
$F \in \mathcal{P}(\textsc{Vals}) \longrightarrow \mathcal{P}(\textsc{Vals})$ is defined as :
$F(X) = \{x | x \in \textsc{Vals} \wedge \textsc{Init}(x)\} \cup X \cup \longrightarrow [X]$ and satisfies the following properties :

- F is a monotonic function.
- $\textsc{reachable}(\mathcal{MS}) = \mu\mathcal{F}$
- μF is defined as follows :
  - $F^0 = \varnothing$
  - $F^{i+1} = F(F^i)$, $\forall i \in \mathbb{N}$
  - $\mu F = Sup\{F^i | i \in \mathbb{N}\}$
  - For any safety property S, $\mu F \subseteq S$.

Computing the least fixed-point over a finite lattice

INPUT $F \in T \longrightarrow T$
OUTPUT $result = \mu.F$
VARIABLES $x, y \in T, i \in \mathbb{N}$
$\ell_0 : \{x, y \in T\}$
$x := \bot;$
$y := \bot;$
$i := 0;$
$\ell_{11} : \{x, y \in T \wedge x = F^i \wedge y = \bigcup_{k=0;k=i} F^k \wedge i \leq Card(T) \wedge i = 0\};$
WHILE $i \leq Card(T)$
$\ell_1 : \{x, y \in T \wedge x = F^i \wedge y = \bigcup_{k=0;k=i} F^k \wedge i \leq Card(T)\};$
$x := F(x);$
$\ell_2 : \{x, y \in T \wedge x = F^{i+1} \wedge y = \bigcup_{k=0;k=i} F^k \wedge i \leq Card(T)\};$
$y := x \sqcup y;$
$\ell_3 : \{x, y \in T \wedge x = F^{i+1} \wedge y = \bigcup_{k=0;k=i+1} F^k \wedge i \leq Card(T)\};$
$i := i+1;$
$\ell_4 : \{x, y \in T \wedge x = F^i \wedge y = \bigcup_{k=0;k=i} F^k \wedge i \leq Card(T)+1\};$
OD;
$\ell_5 : \{x, y \in T \wedge x = F^i \wedge y = \bigcup_{k=0;k=i} F^k \wedge i = Card(T)+1\};$
$result := y;$
$\ell_6 : \{x, y \in T \wedge x = F^i \wedge y = \bigcup_{k=0;k=i} F^k \wedge i = Card(T)+1 \wedge result = y\};$

Verification in action

- Identify the safety property S to check.
- Run the algorithm for computing μF.
- Check that $\mu F \subseteq S$ or $\overline{S} \cap \mu F = \varnothing$.
- Check that $BUG \cap \mu F = \varnothing$, when BUG is a set of states that you identify as *bad states*.

Problem

- The general case is either infinite or large ... approximations of μF.
- Computing over abstract finite domain
- How to compute when it is not decidable ?
- Develop a framework for defining sound abstractions of software systems under analysis.

Summary

Main concepts

- **Syntax of programs** ($P \in PL$) defines the class of programs for applying the analysis@
- **Semantics** ($[\![P]\!]$) for each program ($P \in PL$) which are defined by fixed-point equations and are organised through a hierarchy of semantics : trace semantics, wp semantics, denotational semantics, relational semantics ...
-

Summary

Standard, Collecting and Abstract Semantics

- Abstract interpretation of programs is an approximation of programs semantics
- Correctness proof of the abstract interpretation requires the existence of the standard semantics describing the possible behaviours of programs during their execution.
- The class of properties of program executions is defined by a collecting semantics or static semantics.
- The collecting semantics can be an instrumented version of the standard semantics to gather information about programs executions.
- or the standard semantics reduced to essentials in order to ignore irrelevant details about program execution.
- The collecting semantics provides a sound and relatively complete proof method for the considered class of properties.
- It can be used subsequently as a reference semantics for proving the correctness of all other approximate semantics for that class of properties.
- The abstract semantics usually considers effectively computable properties of programs.
- The soundness of this abstract semantics is proved with respect to the collecting semantics.

Collecting semantics

- programs properties are expressed as sets of finite partial execution traces.
- forward collecting semantics : traces are finite sequences of states, starting with an initial state and such that two consecutive states on the trace satisfy $\longrightarrow$.
- backward collecting semantics : traces are finite sequences of states, starting with a final state and such that two consecutive states on the trace satisfy $\longrightarrow^{-1}$.
- the descendant states of the initial states are the collecting semantics.
- The abstraction consists in approximating a set of forward traces by the set of states which occur on any one of these traces.

Examples

- Computation Traces of Program
- Transitive Closure of the program transition relation
- Set of states

The collecting semantics is the semantics which is interesting our analysis and we will consider as collecting semantics the set of states.

Summary of the technique

- *Collecting semantics*
  - Static analysis of a program states a property of program executions defined by a *standard* semantics.
  - Defining a so-called *collecting* semantics defining the strongest static property of interest
  - Collecting semantics defines the class of static analysis, which approximates it
  - State properties are subsets of $\mathbb{I}\times\mathbb{I}\times\mathbb{I}\times\mathbb{I}$ and abstract interpretation executes programs on thse properties
- Approximation
  - Spaces of values should be restricted to computable entities
  - Over-approximation of concrete properties

Summary

Small Programming Language

$$
\begin{array}{lcll}
Expr & ::= & v & v \in \mathbb{Z} \\
 & | & ? & \\
 & | & x & x \in \mathbb{V} \\
 & | & Expr\ op\ Expr & op \in \{+, -, \times, /\} \\
 & & & \\
cond & ::= & Expr\ relop\ Expr & relop \in \{<, \leq, >, \geq, =, \neq \\
 & | & \textbf{not}\ cond & \\
 & | & cond\ \textbf{and}\ cond & \\
 & & & \\
stmt & ::= & \ell[x := Expr] & \ell \in \mathbb{C} \\
 & | & \ell[skip] & \\
 & | & \textbf{if}\ \ell[cond]\ \textbf{then}\ stmt\ \textbf{else}\ stmt\ \textbf{end if} & \\
 & | & \textbf{while}\ \ell[cond]\ \textbf{do}\ \ stmt\ \textbf{end do} & \\
 & | & stmt; stmt &
\end{array}
$$

Two examples of annotated programs

$\ell_0[X := 0];$
$\ell_1[Y := Y+X];$
$\ell_2[skip]$
$\ell_3[X := Y];$

$\ell_0[Q := 0];$
$\ell_1[R := X];$
IF $\ell_5[Y > 0]$
  WHILE $\ell_2[R \geq Y]$
    $\ell_3[Q := Q+1];$
    $\ell_4[R := R-Y]$
  ENDWHILE
ELSE
  $\ell_6[skip]$
ENDIF

Defining the semantics of the small programming language

- Semantic Domains

$$Mem \stackrel{def}{=} \mathbb{V} \longrightarrow \mathbb{Z}$$

$$States \stackrel{def}{=} \mathbb{C} \times Mem$$

- Semantics for Expressions

$\mathcal{E}[\![v]\!](m) \in \mathcal{P}(\mathbb{Z})$, $e \in Expr$,$m \in Mem$, $x \in \mathbb{V}$, $op \in \{+, -, \times, /\}$

$$\mathcal{E}[\![v]\!](m) \stackrel{def}{=} \{v\}$$

$$\mathcal{E}[\![?]\!](m) \stackrel{def}{=} \mathbb{Z}$$

$$\mathcal{E}[\![x]\!](m) \stackrel{def}{=} \{m(x)\}$$

$$\mathcal{E}[\![e_1\ op\ e_2]\!](m) \stackrel{def}{=} \{v | \exists ve_1, ve_2. \left(\begin{array}{l} ve_1 \in \mathcal{E}[\![e_1]\!](m) \\ ve_2 \in \mathcal{E}[\![e_2]\!](m) \\ v = ve_1\ o\ ve_2 \end{array} \right) \}$$

Defining the semantics of the small programming language

- Semantics for conditions
  $\mathcal{C}[\![cond]\!](m) \in \mathcal{P}(\mathbb{B})$, $cond \in Cond$,$m \in Mem$, $x \in \mathbb{V}$,
  $op \in \{+, -, \times, /\}$

$$tt \in \mathcal{C}[\![e_1\ relop\ e_2]\!](m) \stackrel{def}{=} \exists v_1, v2. \left(\begin{array}{l} v_1 \in \mathcal{E}[\![e_1]\!](m) \\ v_2 \in \mathcal{E}[\![\ e_2]\!](m) \\ v_1\ relop\ v2 \end{array} \right)$$

$$ff \in \mathcal{C}[\![e_1\ relop\ e_2]\!](m) \stackrel{def}{=} \exists v_1, v2. \left(\begin{array}{l} v_1 \in \mathcal{E}[\![e_1]\!](m) \\ v_2 \in \mathcal{E}[\![\ e_2]\!](m) \\ \neg(v_1\ relop\ v2) \end{array} \right)$$

$$be_1\ \wedge\ be_2 \in \mathcal{C}[\![be_1\ \mathbf{and}\ be_2]\!](m) \stackrel{def}{=} and \left(\begin{array}{l} be_1 \in \mathcal{C}[\![be_1]\!](m) \\ be_2 \in \mathcal{C}[\![be_2]\!](m) \end{array} \right)$$

- $(x := e, m) \longrightarrow m[x \mapsto v]$, **where** $v \in \mathcal{E}[\![e]\!](m)$
- $(skip, m) \longrightarrow m$
- **If** $(S_1, m) \longrightarrow m'$, **then** $(S_1; S_2, m) \longrightarrow (S_2, m')$.
- **If** $tt \in \mathcal{C}[\![be]\!]$, **then** $(\textbf{if } be \textbf{ then } S_1 \textbf{ else } S_2 \textbf{ end if}, m) \longrightarrow (S_1, m)$.
- **If** $ff \in \mathcal{C}[\![be]\!]$, **then**
  $(\textbf{if } be \textbf{ then } S_1 \textbf{ else } S_2 \textbf{ end if}, m) \longrightarrow (S_2, m)$.
- **If** $tt \in \mathcal{C}[\![be]\!]$, **then**
  $(\textbf{while } be \textbf{ do } S \textbf{ end do}, m) \longrightarrow (S; \textbf{while } be \textbf{ do } S \textbf{ end do}, m)$.
- **If** $ff \in \mathcal{C}[\![be]\!]$, **then** $(\textbf{while } be \textbf{ do } S \textbf{ end do}, m) \longrightarrow m$.

Generating Control Flowchart Graph from Program

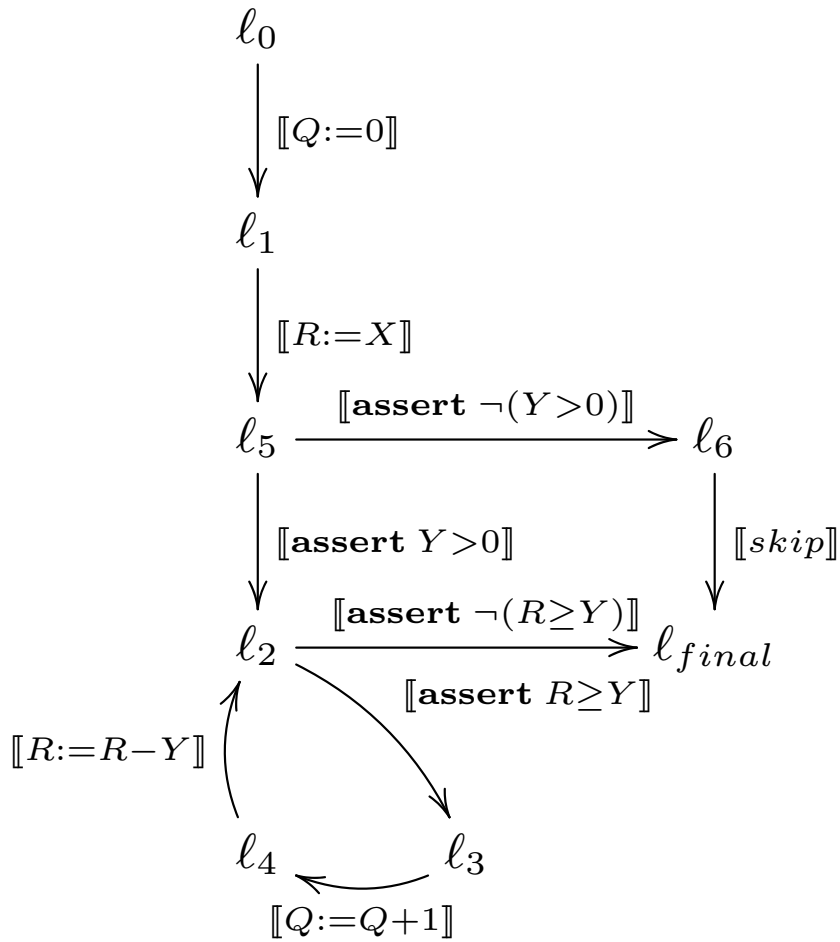
- A *control flow graph* is generated from the program under consideration namely P.
- A control flow graph $\mathcal{CFG}[\![P]\!]$ is defined by nodes ($l \in \mathcal{C}$) which are program control points of P, $\mathcal{C}ontrol[\![P]\!]$ and by labelled edges with actions ($\mathcal{A}ctions[\![P]\!]$) defined by the following rules :

$$\begin{array}{lll} actions & ::= & v := exp \\ & | & skip \\ & | & \textbf{assert } be \end{array}$$

- A *control flow graph* is effectively defined by :
  - $\ell_{init} \in \mathcal{C}ontrol[\![P]\!]$: the entry point
  - $\ell_{end} \in \mathcal{C}ontrol[\![P]\!]$: the exit point
  - $\mathcal{E}dges[\![P]\!] \subseteq \mathcal{C}ontrol[\![P]\!] \times \mathcal{A}ctions[\![P]\!] \times \mathcal{C}ontrol[\![P]\!]$
- $\mathcal{CFG}[\![P]\!] = (\ell_{init}, \mathcal{E}dges[\![P]\!], \ell_{end})$

From program to flowchart

$\ell_0[Q := 0];$
$\ell_1[R := X];$
IF $\ell_5[Y > 0]$
  WHILE $\ell_2[R \geq Y]$
    $\ell_3[Q := Q{+}1];$
    $\ell_4[R := R{-}Y]$
  ENDWHILE
ELSE
  $\ell_6[skip]$
ENDIF



Small-step Semantics for Control Flowcharts

- $Mem \stackrel{def}{=} \mathbb{V} \longrightarrow \mathbb{Z}$
- Semantics of actions : $\stackrel{a}{\longrightarrow} \subseteq Mem \times Mem$

  $m \stackrel{x:=e}{\longrightarrow} m[x \mapsto v]$ **if** *there is a value* $v \in \mathcal{E}[\![e]\!](m)$

  $m \stackrel{skip}{\longrightarrow} m$

  $m \stackrel{\textbf{assert}\ be}{\longrightarrow} m]$ $\qquad\qquad$ $if\ tt \in \mathcal{C}[\![be]\!](m)$
- Semantics for $\mathcal{CFG}[\![P]\!]$: $\stackrel{P}{\longrightarrow} \subseteq States \times States$
  - **If** $m \stackrel{a}{\longrightarrow} m'$ and $(\ell_1, a, \ell_2) \in \mathcal{E}dges[\![P]\!]$, then $(\ell_1, m) \stackrel{P}{\longrightarrow} (\ell_2, m')$
  - The set of initial states is $\{\ell_{init}\} \times Mem$
  - The set of reachable states for P is denoted $\textsc{reachable}(P)$ and defined by $[\![P]\!] = \{s | \exists s_0 \in \{\ell_{init} \times Mem : s_0 \stackrel{P}{\underset{\star}{\longrightarrow}} s\}$.

Collecting Semantics for Programs

- Defining for each control point ℓ of P the set of reachables values :

$$[\![P]\!]^{coll}_{\ell} = \{s | s \in States \wedge s \in [\![P]\!] \wedge \exists m \in Mem : s = (\ell, m)\}$$

- Characterizing $[\![P]\!]^{coll}_{\ell}$: it satisfies the system of equations

$$\forall \ell \in \mathcal{C}(P).X_{\ell} = X^{init}_{\ell} \cup \bigcup_{(\ell_1,a,\ell)\in \mathcal{E}dges[\![P]\!]} [\![a]\!](X_{\ell_1}) \qquad (1)$$

- Let $a \in \mathcal{A}ctions[\![P]\!]$ and $x \subseteq Mem$.

$$[\![a]\!](x) = \{e | e \in States \wedge \exists f.f \in x \wedge f \xrightarrow{a} e\}$$

- $\forall \ell \in \mathcal{C}(P). \left(\begin{array}{l} \ell = \ell_{init} \Rightarrow X^{init}_{\ell} = Mem \\ \ell \neq \ell_{init} \Rightarrow X^{init}_{\ell} = \varnothing \end{array} \right)$

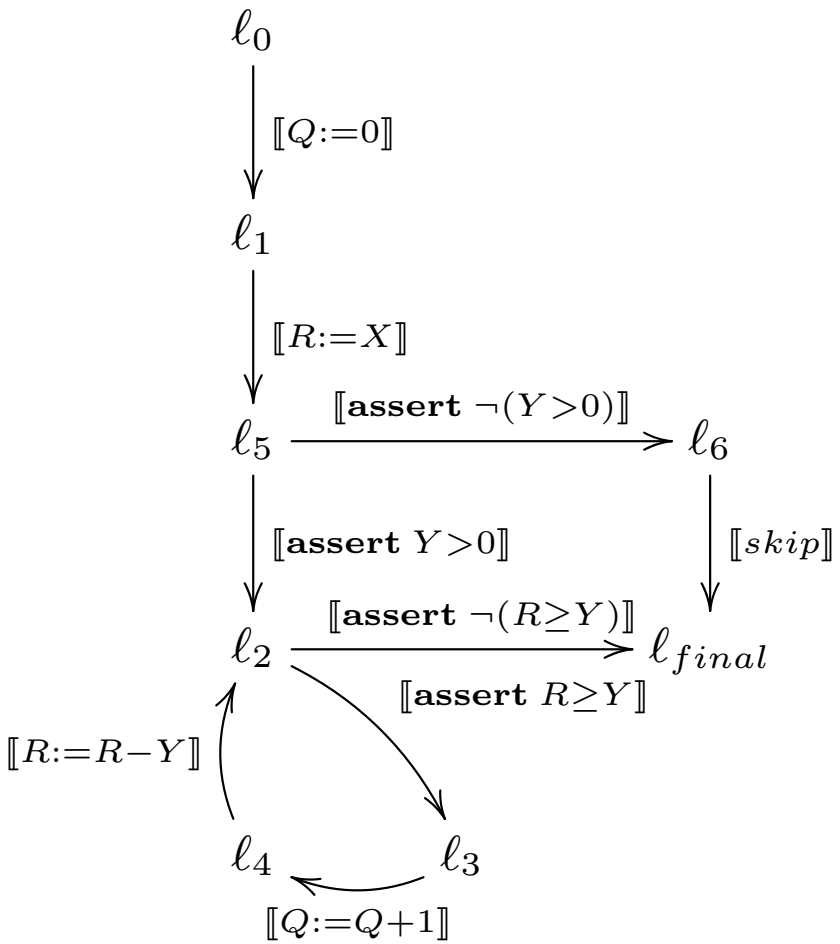
Collecting Semantics for Programs

☺ Théorème Let F the function defined as follows :

- n is the cardinality of $\mathcal{C}(P)$.
- $F \in \mathcal{P}(States)^n \longrightarrow \mathcal{P}(States)^n$
- If $X \in \mathcal{P}(States)^n$, then $F(X) = (\ldots, F_\ell(X), \ldots)$
- $\forall \ell \in \mathcal{C}(P).F_\ell(X) = X_\ell^{init} \cup \bigcup_{(\ell_1,a,\ell)\in \mathcal{E}dges[\![P]\!]} [\![a]\!](X_{\ell_1})$

The function F is monotonic over the complete lattice $(\mathcal{P}(States)^n, \subseteq)$ and has a least fixed-point μF defining the collecting semantics.

From flowchart to equational system



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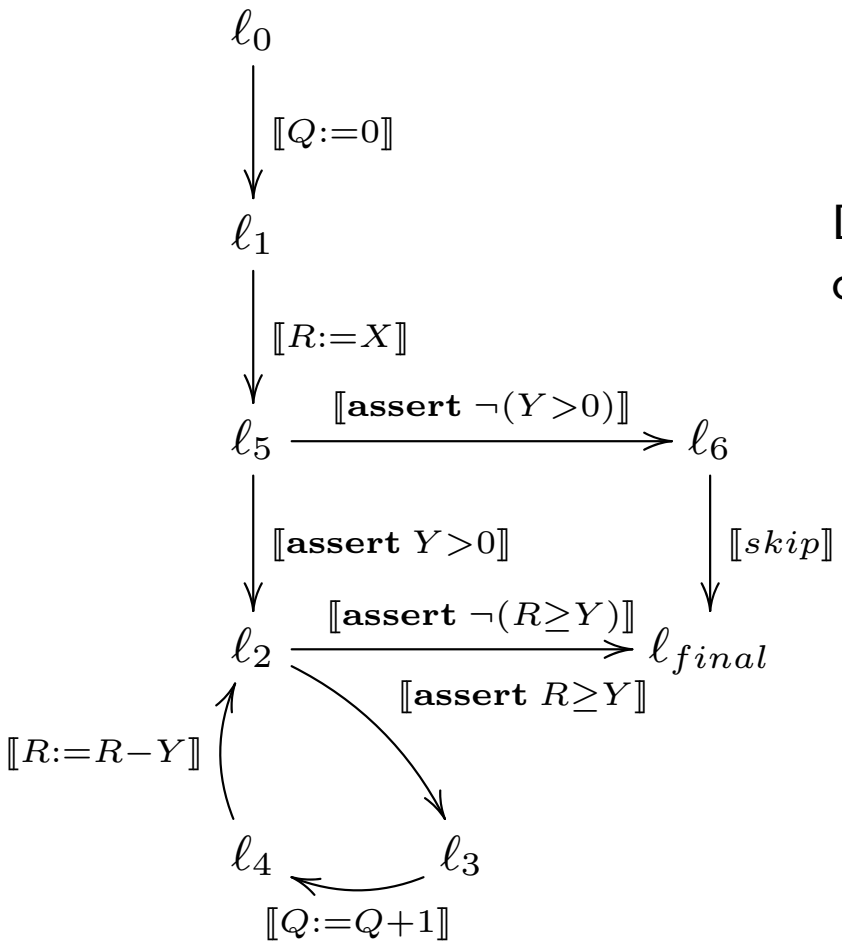
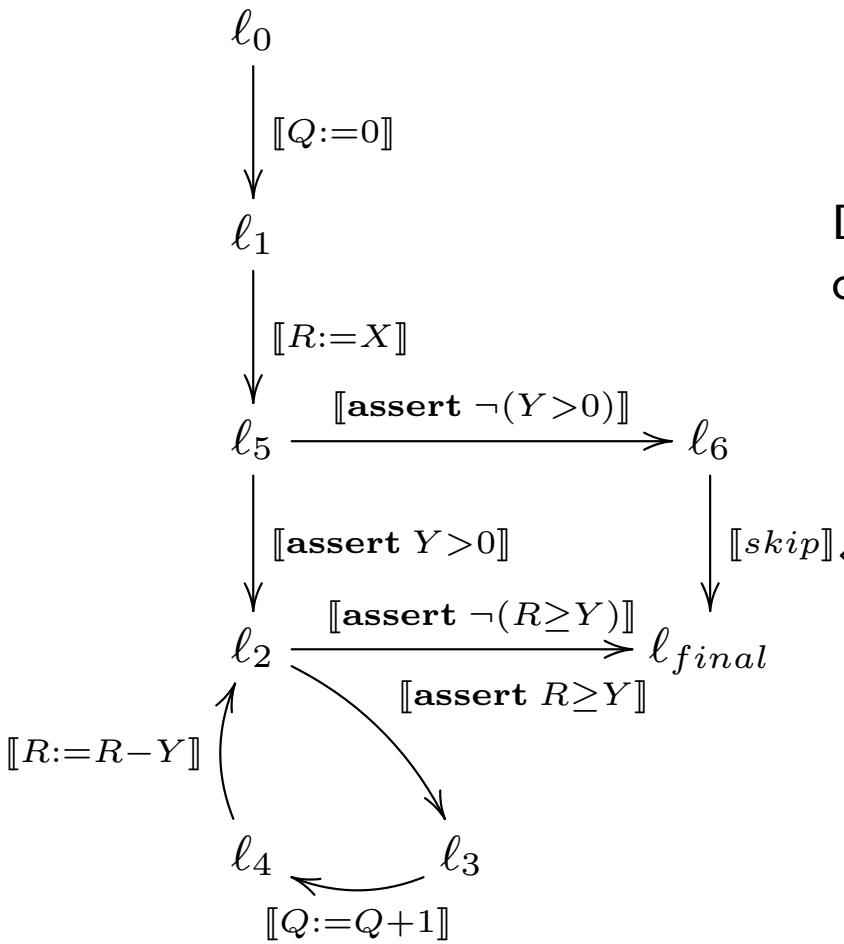


Defining equational systems for the collecting semantics :

From flowchart to equational system



Defining equational systems for the collecting semantics :

$$\begin{cases} X_0 = Mem \\ X_1 = [\![Q := 0]\!](X_0) \\ X_5 = [\![R := X]\!](X_1) \\ X_2 = [\![assert\ (Y > 0)]\!](X_5) \cup [\![R := R - Y]\!](X_4) \\ X_3 = [\![assert\ R \geq Y]\!](X_2) \\ X_4 = [\![Q := Q + 1]\!](X_3) \\ X_6 = [\![assert\ \neg(Y > 0)]\!](X_5) \\ X_7 = X_6 \cup [\![assert\ \neg(R \geq Y)]\!](X_2) \end{cases}$$

Solving the equational system

- The collecting semantics is the least fixed-point of the system of equations, which exists by fixed-point theorems.
- Questions :
  - How to compute the solution ?
  - Computing over finite structures, when it is possible....
  - Using an approximation of fixed-points ?
  - What is an approximation ?
  - What is an abstraction ?
  - What is the best abstraction ?

Next step

Defining a framework for computing lfp solution of these equational systems in any case.

Example for computing reachable states (I)

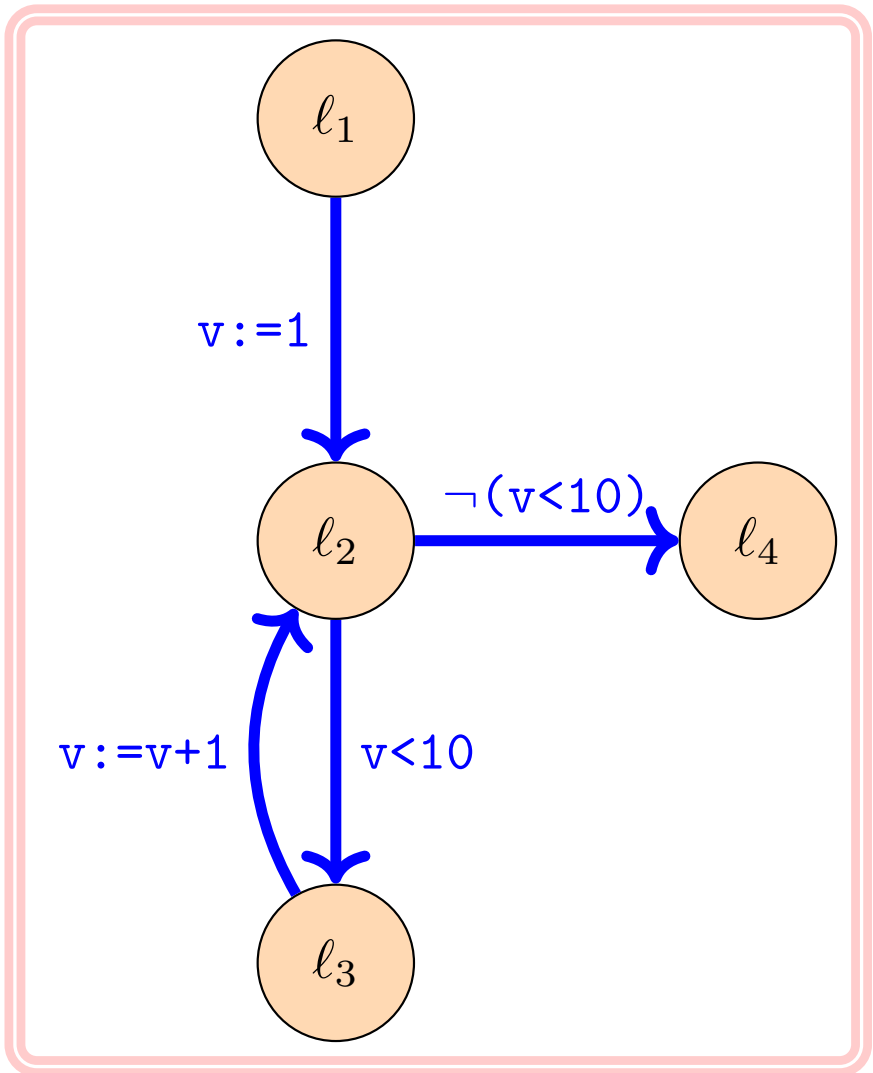
```
begin
v := 1;
while v < 10 {
   v := v+1;
};
end
```

Example for computing reachable states (I)

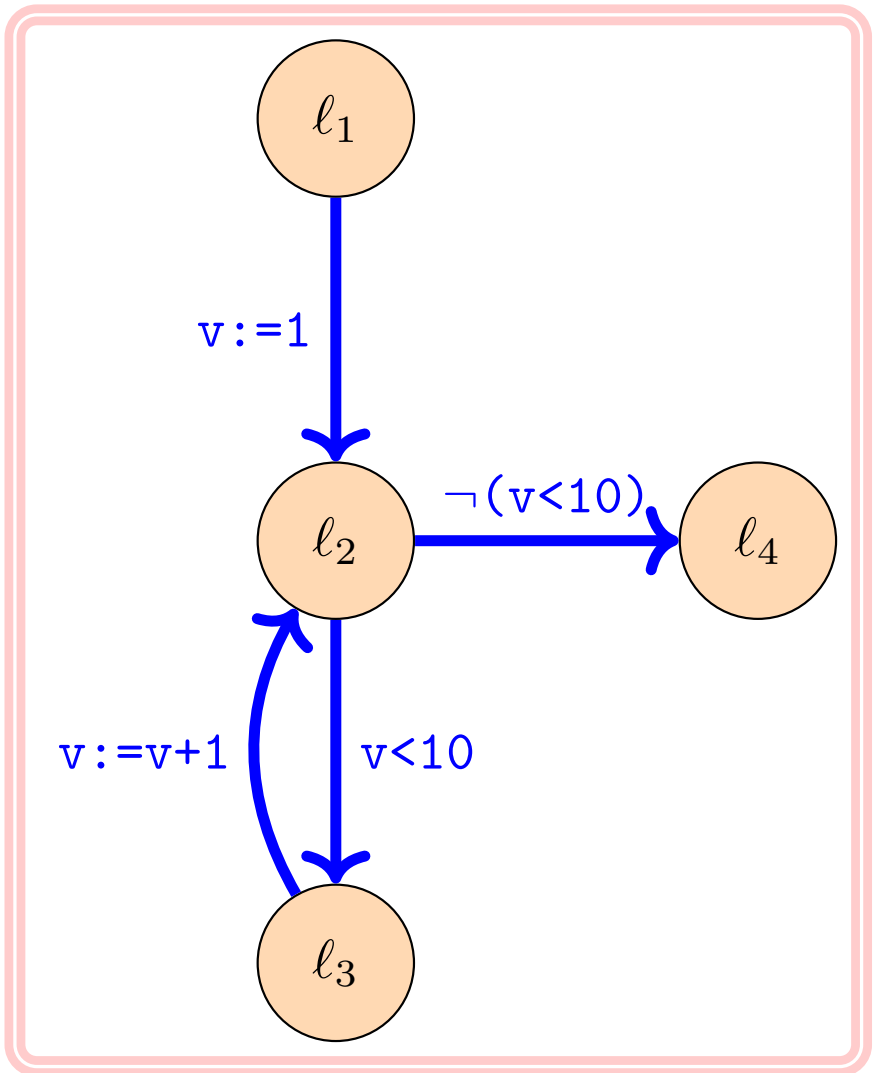
```
begin
v := 1;
while v < 10 {
    v := v+1;
};
end
```

```
begin
ℓ1[v := 1];
while ℓ2[v < 10] {
    ℓ3[v := v+1];
};
ℓ4 :
end
```

Example for computing reachable states(I)



Example for computing reachable states(I)



- System of equations over $(\mathcal{P}(\mathbb{Z}, \subseteq)$
  - $X_1 = \mathbb{Z}$
  - $X_2 = \{1\} \cup \{v | v \in \mathbb{Z} \wedge v-1 \in X_3\}$
  - $X_3 = \{v | v \in X_2 \wedge v < 10\ \}$
  - $X_4 = \{v | v \in X_2 \wedge v \geq 10\ \}$
- Reachability
  - $X_1 = \mathbb{Z}$
  - $X_2 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
  - $X_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
  - $X_4 = \{10\}$

Summary

Finding Sound Abstractions for Computing

- $n \in \mathbb{N}$: abstraction for n is defined as $modulo(n, 9)$.
- $\mathbb{A}$ is the set of possible abstract values $\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}, \bar{8}$
- $n \bar{\star} m = \bar{n} \bar{\star} \bar{m}$
- $25 \star 25 = 625$, $\bar{25} \bar{\star} \bar{25} = \bar{625}$, $\bar{7} \bar{\star} \bar{7} = \bar{4}$, $\overline{7 \star 7} = \bar{4}$, $\overline{49} = \bar{4}$, $\bar{4} = \bar{4}$,
- But $25 \star 25 \neq 265$ and $\overline{25 \star 25} = \overline{265}$

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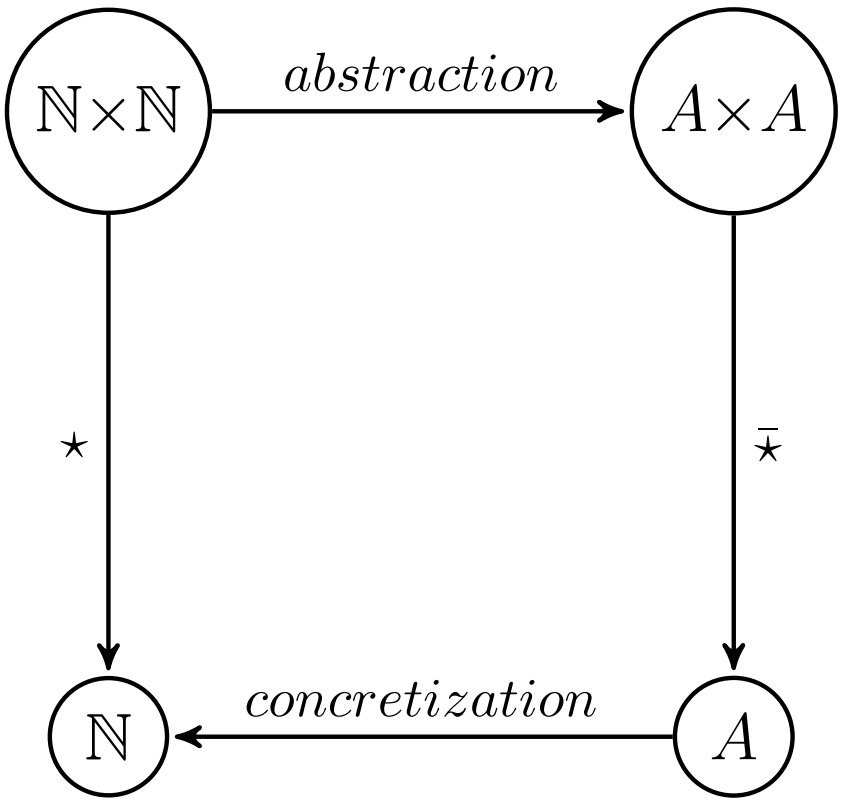

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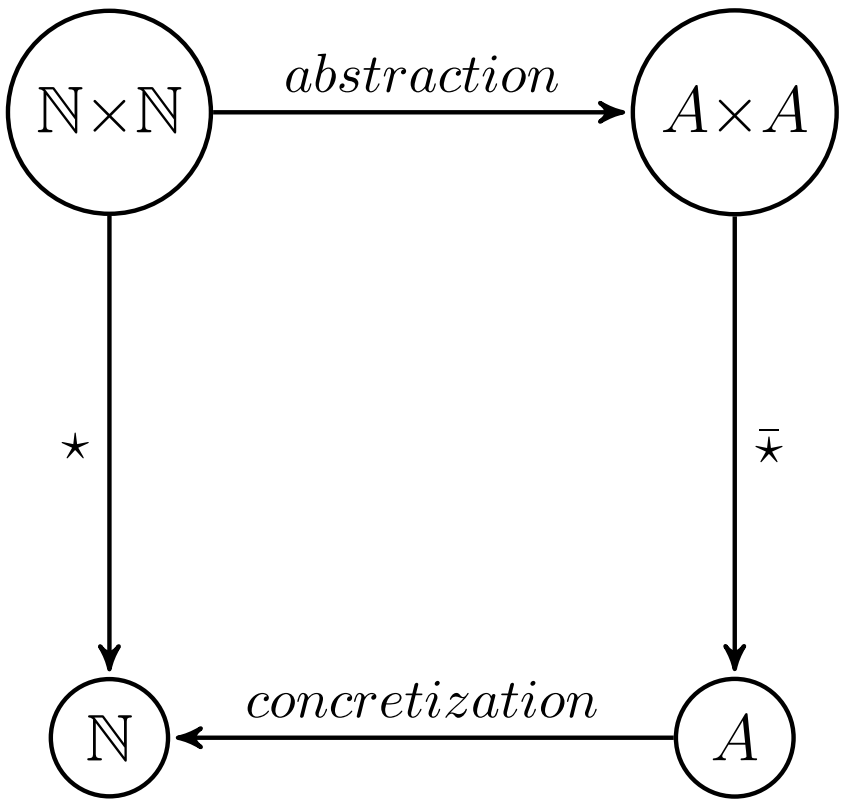

☺ Rule
If $\overline{a \star b} \neq \bar{a} \bar{\star} \bar{b}$,
then the operation is not correct.

Finding Sound Abstractions for Computing

- A number $z \in \mathbb{Z}$ is soundly approximated by an abstract value $abstract(z) \in Signs$.
- 2 is approximated by $pos : \{2\} \subseteq concrete(pos)$
- $\{2, 8\}$ is approximated by $pos : \{2, 8\} \subseteq concrete(pos)$
- -2 is approximated by $neg : \{2\} \subseteq concrete(neg)$
- 0 is approximated by $zero : \{0\} \subseteq concrete(zero)$
- $\{-2, -8\}$ is approximated by $pos : \{-2, -8\} \subseteq concrete(neg)$
- $\{-2, 2, 8\}$ is approximated by $pos : \{-2, 2, 8\} \subseteq concrete(nonzero)$

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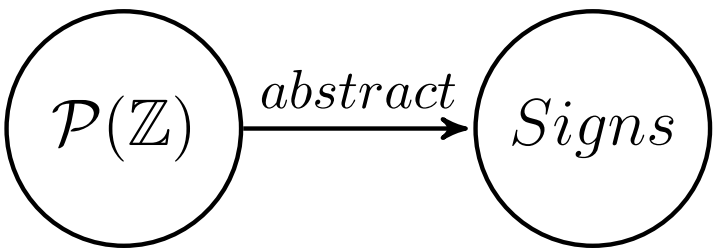

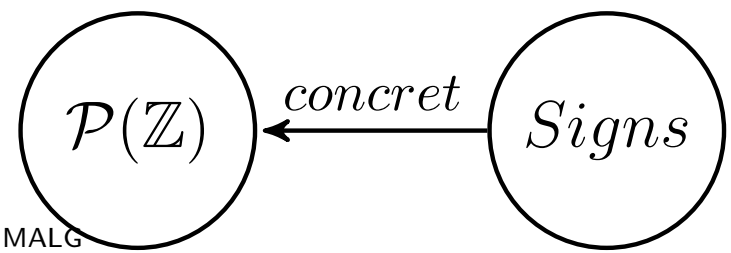

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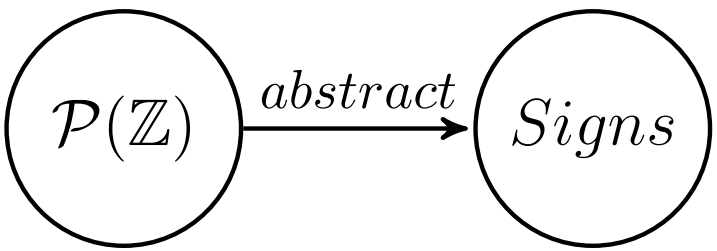
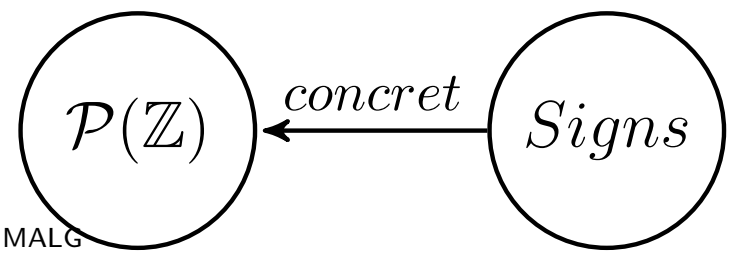


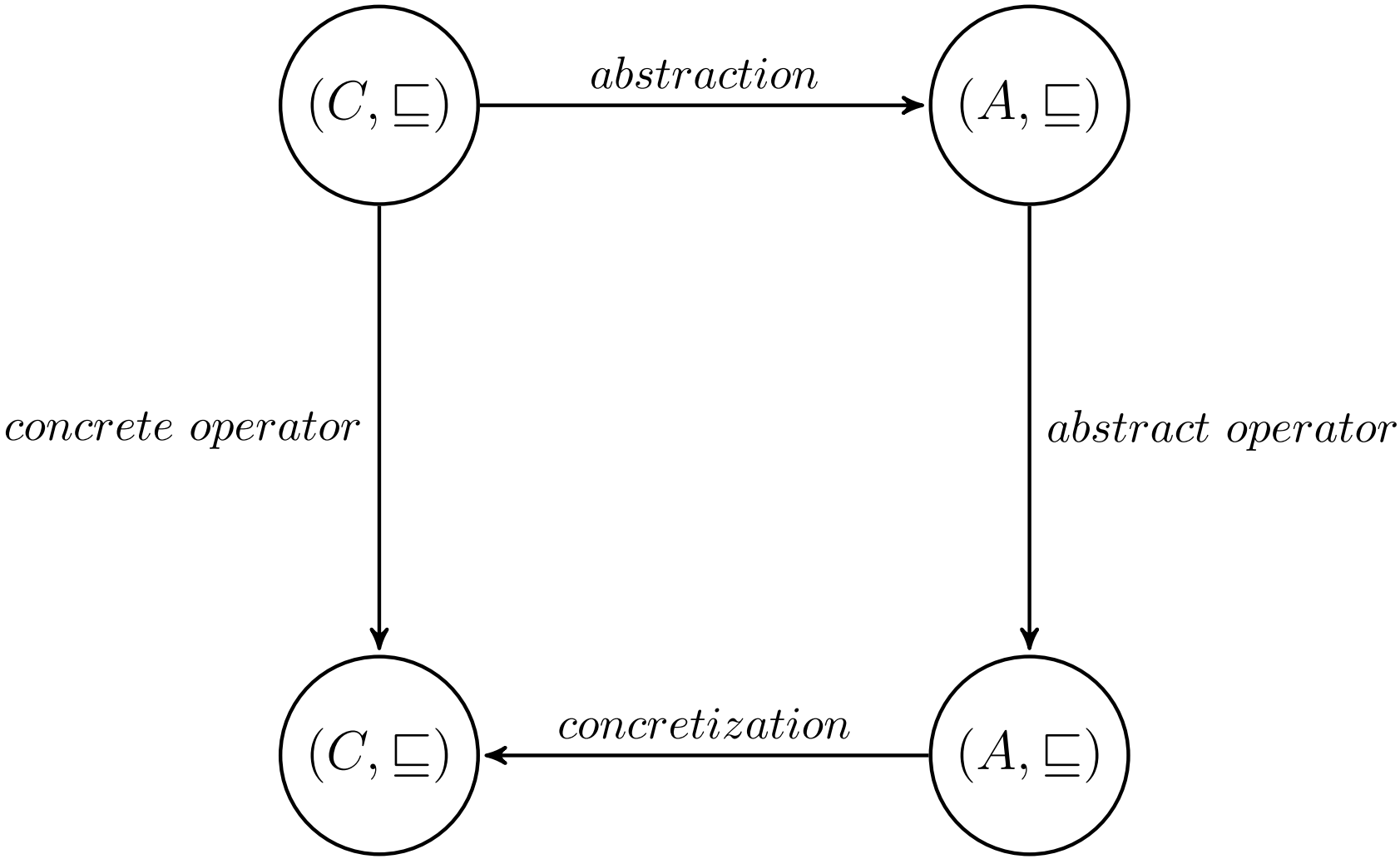


- $concret(pos) = \{z | z \in \mathbb{Z} \wedge z > 0\}$
- $concret(neg) = \{z | z \in \mathbb{Z} \wedge z < 0\}$
- $concret(non) = \varnothing$
- $concret(nonzero) = \{z | z \in \mathbb{Z} \wedge z \neq 0\}$

Finding Sound Abstractions for Computing

- A number $z \in \mathbb{Z}$ is soundly approximated by an abstract value $abstract(z) \in Signs$.
- 2 is approximated by pos :
  - $\{2\} \subseteq concrete(pos)$
  - $abstract(\{2\}) = pos$
- $\{2, 8\}$ is approximated by pos :
  - $\{2, 8\} \subseteq concrete(pos)$
  - $abstract(\{2, 8\}) = pos$
- $\{-2, 2, 8\}$ is approximated by pos :
  - $\{-2, 2, 8\} \subseteq concrete(nonzero)$
    $abstract(\{-2, 2, 8\}) = nonzero$

Finding Sound Abstractions for Computing



Summary

Defining good abstractions

- Let $(A, \sqsubseteq, \sqcup, \sqcap)$ be a complete lattice. Let Q a subset of A. Q is a Moore family, if for each part Q' of Q, $\sqcap\, Q' \in Q$.
- **Property** : Let $(A, \sqsubseteq, \sqcup, \sqcap)$ be a complete lattice and $B \subseteq A$.
  1. If, for any $p \in A$, $\{q \in B | p \sqsubseteq q\}$ has a least element, then B is Moore family.
  2. If B is Moore family, then for any $p \in A$, $\{q \in B | p \sqsubseteq q\}$ has a least element.

If B is a Moore family, then it is a good abstraction.

Defining good abstractions

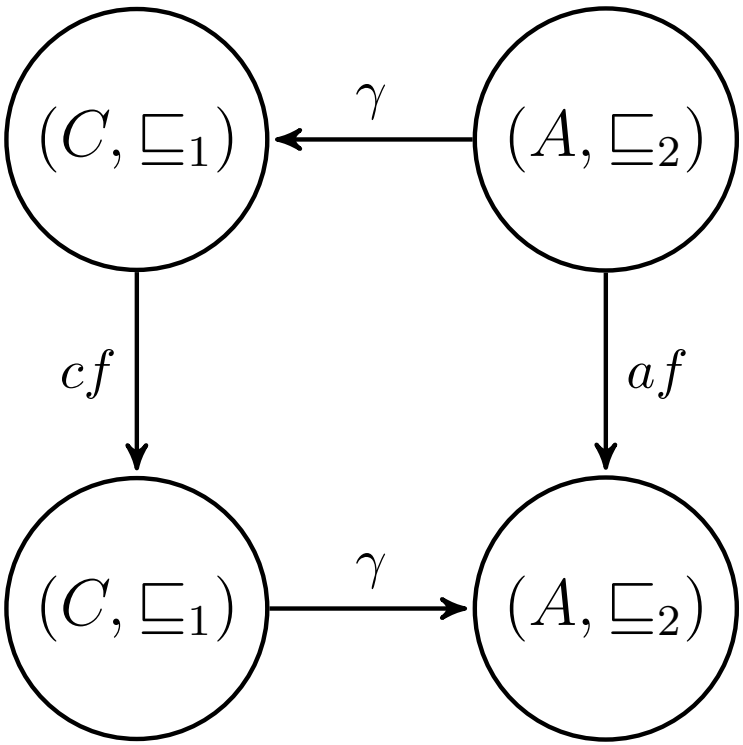
- Let $(A, \sqsubseteq, \sqcup, \sqcap)$ be a complete lattice.
- $\rho \in A \longrightarrow A$ is a upper closure operator, if it satisfies the following properties :
  - ρ is monotonic : $\forall x, y \in A. x \sqsubseteq y \Rightarrow \rho(x) \sqsubseteq \rho(y)$.
  - ρ is extensive : $\forall x \in A. x \sqsubseteq \rho(x)$.
  - ρ is idempotent : $\forall x \in A. \rho(x) = \rho(\rho(x))$.
- **Property** : Let $(A, \sqsubseteq, \sqcup, \sqcap)$ be a complete lattice. and $B \subseteq A$.

  B is a Moore family if, and only if, there exists a closure operatorρ such that $B = \rho(A)$
- B is a *good* abstraction of C, if it satisfies $B = \rho(C)$ where ρ is a closure operator.

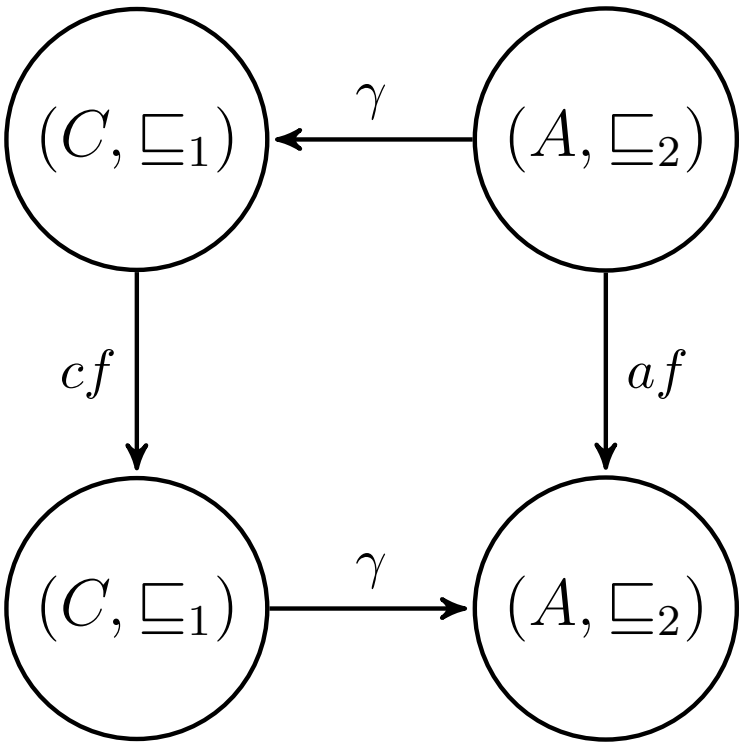
Defining good abstractions

- Two complete lattices $(C, \sqsubseteq_1, \sqcup_1, \sqcap_1)$ and $(A, \sqsubseteq_2, \sqcup_2, \sqcap_2)$ are supposed to be given.
- Two functions α and γ are supposed to be defined as follows :
  - $\alpha \in C \longrightarrow A$
  - $\gamma \in A \longrightarrow C$
- The pair (α, γ) is a Galois connection, if it satisfies the following property : $\forall x_1 \in C, x_2 \in A. \alpha(x_1) \sqsubseteq_2 x_2 \Leftrightarrow x_1 \sqsubseteq_1 \gamma(x_2)$
- A complete lattice A is a good abstraction of L, when there is a Galois connection between A and L.

Finding Sound Abstractions for Computing

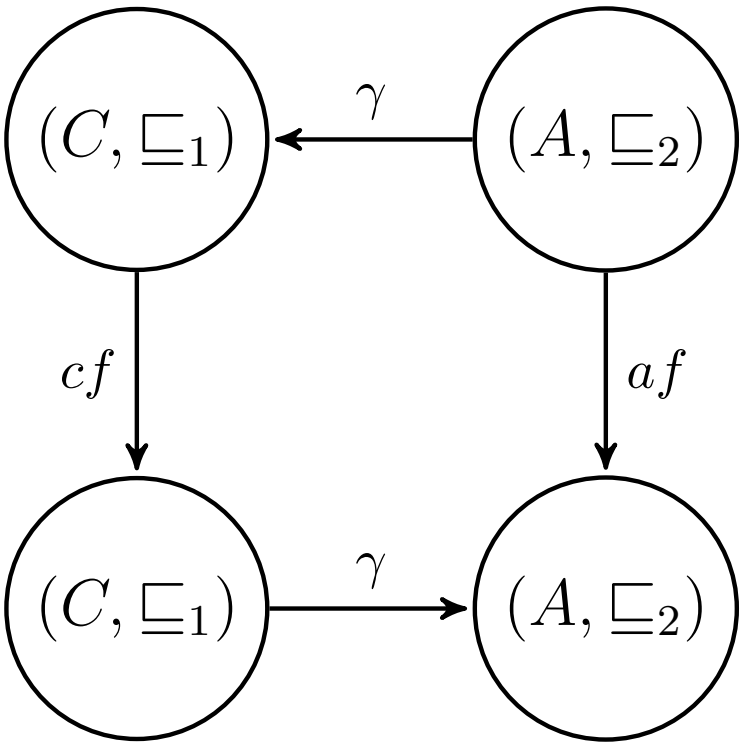


Finding Sound Abstractions for Computing



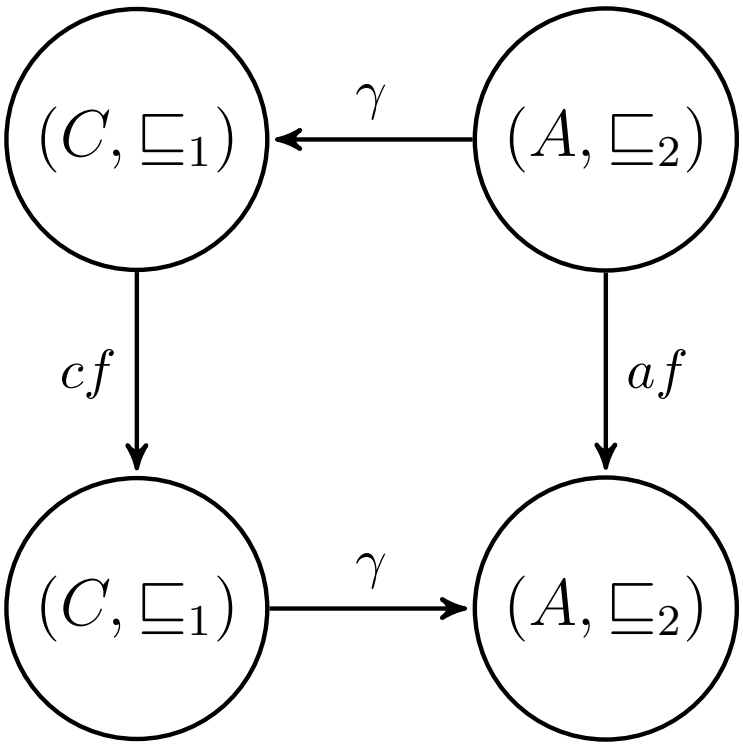
- a is a sound abstraction of c, if $c \sqsubseteq_1 \gamma(a)$.

Finding Sound Abstractions for Computing



- a is a sound abstraction of c, if $c \sqsubseteq_1 \gamma(a)$.
- functional operator : af is a sound abstraction of cf, if $\forall a \in A.cf(\gamma(a)) \sqsubseteq_1 \gamma(af(a))$

Finding Sound Abstractions for Computing



- a is a sound abstraction of c, if $c \sqsubseteq_1 \gamma(a)$.
- functional operator : af is a sound abstraction of cf, if $\forall a \in A.cf(\gamma(a)) \sqsubseteq_1 \gamma(af(a))$
- relational operator : ar is a sound abstraction of cr, if $\forall a \in A.cr(\gamma(a_1), \ldots, \gamma(a_n)) \sqsubseteq_1 \gamma(ac(a_1, \ldots, a_n)))$

Galois Connections

The pair (α, γ) is a Galois connection, if it satisfies the following property : $\forall x_1 \in L, x_2 \in L.\alpha(x_1) \sqsubseteq' x_2 \Leftrightarrow x_1 \sqsubseteq \gamma(x_2)$

Notation : $L \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} L'$

Properties of a Galois connection $L \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} L'$

- α and γ are monotonic over the lattices.
- $\mathrm{id}(L) \subseteq \gamma \circ \alpha$: $\gamma \circ \alpha$ is extensive.
- $\alpha \circ \gamma \subseteq \mathrm{id}(L')$: $\alpha \circ \gamma$ is retractive.
- $\alpha \circ \gamma \circ \alpha = \alpha$ and $\gamma \circ \alpha \circ \gamma = \gamma$
- $\alpha(x) = \bigsqcap'\{y \in L' | x \sqsubseteq \gamma(y)\}$
- $\gamma(y) = \bigsqcup\{x \in L | \alpha(x) \sqsubseteq' y\}$

Properties

- $\gamma \circ \alpha \circ \gamma \circ \alpha = \gamma \circ \alpha$
- $\alpha \circ \gamma \circ \alpha \circ \gamma = \alpha \circ \gamma$
- We assume that $\{(\alpha_i, \gamma_i) | i \in \{1 \dots n\}\}$ is a family of Galois connections :

  $$L_1 \underset{\alpha_1}{\overset{\gamma_1}{\leftrightarrows}} L_2 \underset{\alpha_2}{\overset{\gamma_2}{\leftrightarrows}} \dots L_{n-1} \underset{\alpha_{n-1}}{\overset{\gamma_{n-1}}{\leftrightarrows}} L_n$$

  Then $(\alpha_1; \dots; \alpha_i; \dots; \alpha_{n-1}, \gamma_{n-1}; \dots \gamma_i; \dots; \gamma_1)$ is a Galois connection. or equivalently
  $L_1 \underset{\alpha_{n-1} \circ \dots \circ \alpha_i \circ \dots \circ \alpha_1}{\overset{\gamma_1 \circ \dots \gamma_i \circ \dots \circ \gamma_{n-1}}{\leftrightarrows}} L_n$ is a Galois connection.
- We assume that $\{(\alpha_i, \gamma_i) | i \in \{1, 2\}\}$ two Galois connections : $\alpha_1 = \alpha_2$ if, and only if, $\gamma_1 = \gamma_2$

Summary

Examples (I)

- We consider a transition system (S, I, t) where S is the set of states, I is the set of initial states and t is a binary relation over S.
- A property P of the transition system is a subset of S : $P \subseteq S$.
- P holds in $s \in S$, when $s \in P$.
- Four operators over properties can be defined as follows :
  - $\mathsf{pre}[t]P \stackrel{def}{=} \{s | s \in S \wedge \exists s'.((s, s') \in t \wedge s' \in P)\}$
  - $\widetilde{\mathsf{pre}}\ [t]P \stackrel{def}{=} \{s | s \in S \wedge \forall s'.((s, s') \in t \Rightarrow s' \in P)\}$
  - $\mathsf{post}[t]P \stackrel{def}{=} \{s | s \in S \wedge \exists s'.((s', s) \in t \wedge s' \in P)\}$
  - $\widetilde{\mathsf{post}}\ [t]P \stackrel{def}{=} \{s | s \in S \wedge \forall s'.((s', s) \in t \Rightarrow s' \in P)\}$
- Duality of operators :
  1. $\widetilde{\mathsf{pre}}\ [t]\neg P = \neg\mathsf{pre}[t]P$
  2. $\widetilde{\mathsf{post}}\ [t]\neg P = \neg\mathsf{post}[t]P$
- Galois connections over $\mathcal{P}$, the set of subsets of S :

$$(\mathcal{P}, \subseteq) \underset{\mathsf{pre}[t]}{\overset{\widetilde{\mathsf{post}}[t]}{\leftrightarrows}} (\mathcal{P}, \subseteq) \qquad (\mathcal{P}, \subseteq) \underset{\mathsf{post}[t]}{\overset{\widetilde{\mathsf{pre}}[t]}{\leftrightarrows}} (\mathcal{P}, \subseteq)$$

Examples (II)

- Let two sets $\mathcal{L}$ standing for labels et $\mathcal{M}$ standing for memories.
- First step :
  - $\sqsubseteq$ is the partial ordering over functions using the subset relationship over function graphs : $f \sqsubseteq g$ means that $\mathbb{G}raph(f) \subseteq \mathbb{G}raph(g)$.
  - $\alpha_1 = \lambda P.\lambda l.\{m | (l,m) \in P\}$
  - $\gamma_1 = \lambda Q.\{(l,m) | l \in \mathcal{L} \wedge m \in Q(l)\}$
  - $(\mathcal{P}(\mathcal{L} \times \mathcal{M}), \subseteq) \underset{\alpha_1}{\overset{\gamma_1}{\leftrightarrows}} (\mathcal{L} \longrightarrow \mathcal{P}(\mathcal{M}), \subseteq)$ is a Galois connection
- Second step :
  - Let two sets $Pred$, set of predicates, and $\mathcal{M}$, a set of memories.
  - The relationship between both sets is stating as follows : For any given predicate p and any given memory m, p holds in m.
  - We define $B(p) = \{m | m \in \mathcal{M} \wedge p(m)\}$, set of predicates in which p holdsd.
  - Next we define :
    - $\alpha_2 = \lambda Q.\{p | p \in Pred \wedge Q \subseteq B(p)\}$
    - $\gamma_2 = \lambda P.\cap\{B(p) | p \in P\}$
  - $(\mathcal{P}(\mathcal{M}), \subseteq) \underset{\alpha_2}{\overset{\gamma_2}{\leftrightarrows}} (\mathcal{P}(Pred), \Rightarrow)$ is a Galois connection.

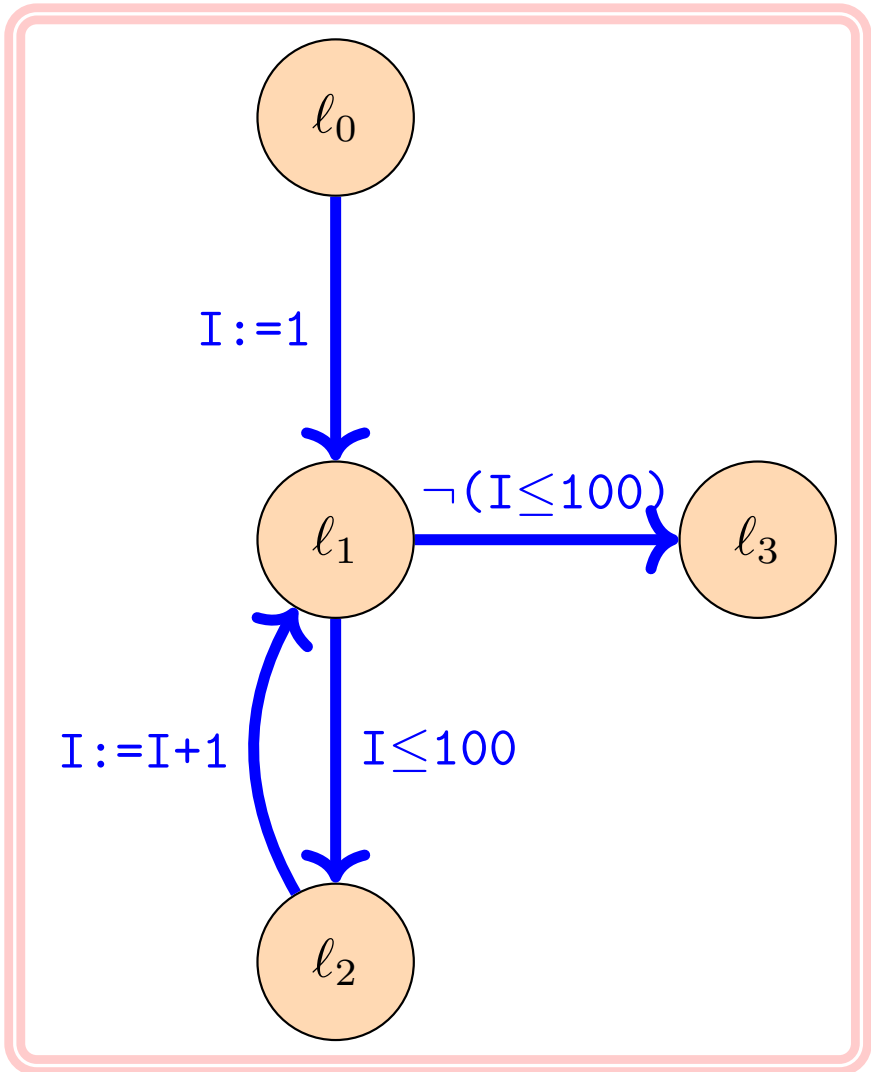
Examples (II)

- Third step
  - $\alpha_3 = \lambda \ell.\alpha_2(Q_\ell) : Q \subseteq_1 Q' \stackrel{def}{=} \forall \ell \in \mathcal{L}.Q_\ell \subseteq {Q'}_\ell$.
  - $\gamma_3 = \lambda \ell.\gamma_2(P\ell) : P \Rightarrow_1 P' \stackrel{def}{=} \forall \ell \in \mathcal{L}.P_\ell \Rightarrow {P'}_\ell$.
  - $(\mathcal{L} \longrightarrow \mathcal{P}(\mathcal{M}), \subseteq_1) \underset{\alpha_3}{\overset{\gamma_3}{\leftrightarrows}} (\mathcal{L} \longrightarrow \mathcal{P}(Pred), \Rightarrow_1)$ is a Galois connection.

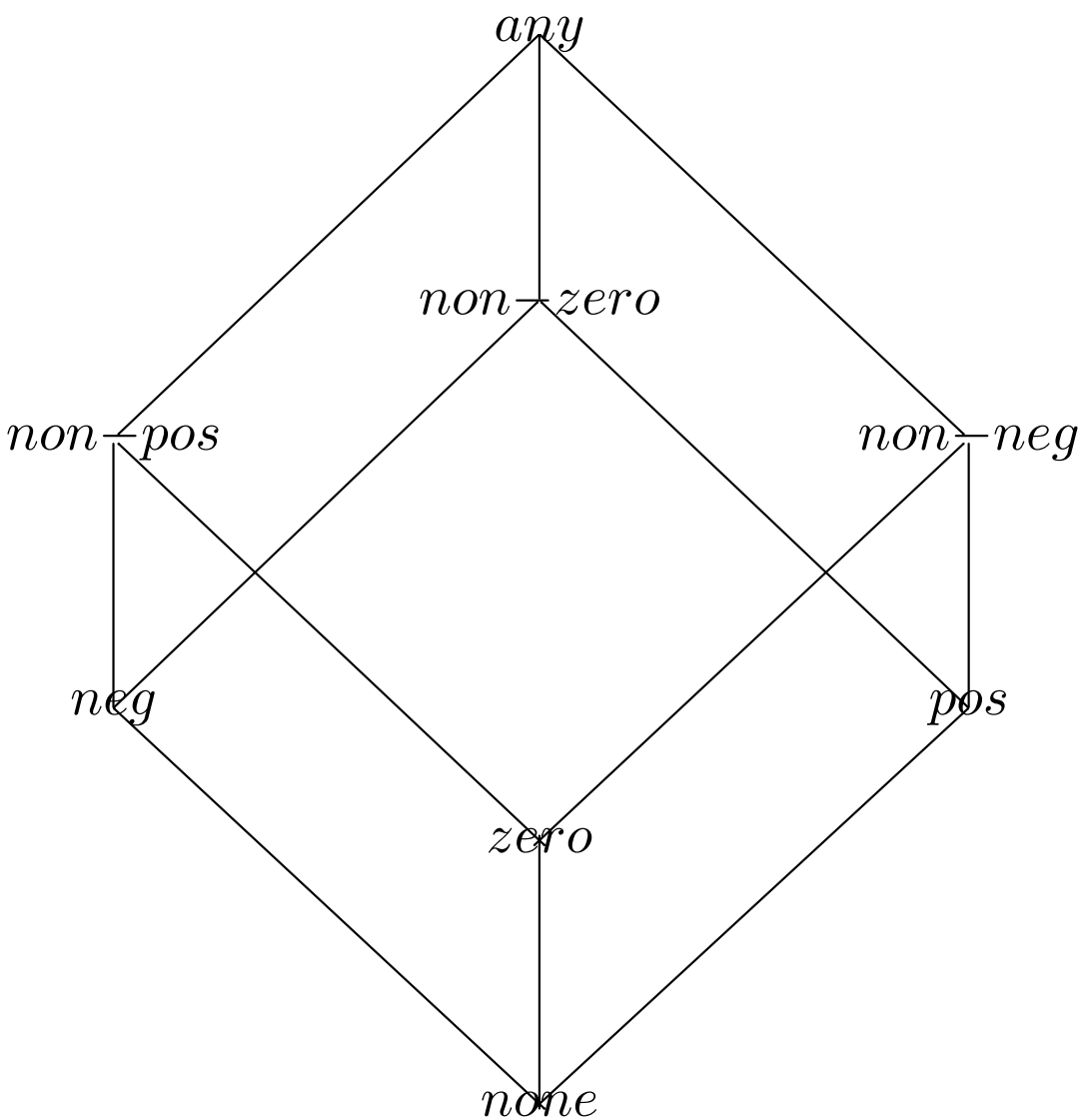
Summary

Examples of Abstractions

$\ell_0[I := 1];$
while $\ell_1[I \leq 100]$ **do**
  $\ell_2[I := I{+}1];$
end while
$\ell_3[skip]$



Domain of Signs



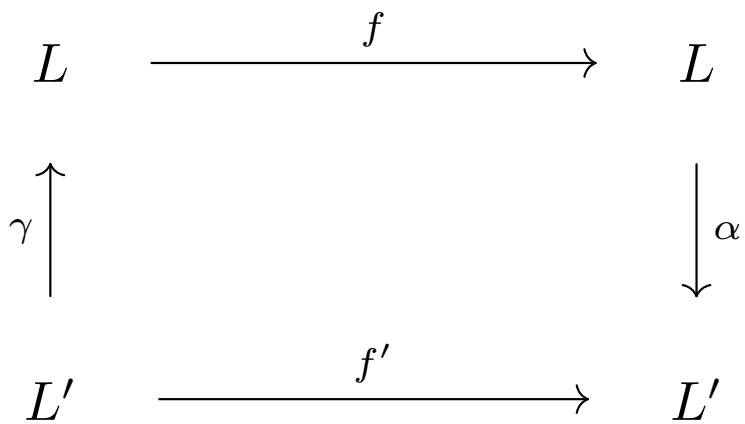
- Defining an abstraction for integers
  $\alpha \in \mathcal{P}(\mathbb{Z}) \longrightarrow \mathbb{S}igns$
- $$\begin{cases} z & \alpha(z) \\ z < 0 & neg \\ z > 0 & pos \\ z = 0 & zero \end{cases}$$

Composing Galois Connections

- Abstraction by projection :
  $(\mathcal{P}(Var \longrightarrow \mathbb{Z}), \subseteq) \underset{\alpha_\pi}{\overset{\gamma_\pi}{\leftrightarrows}} (Var \longrightarrow \mathcal{P}(\mathbb{Z}), \subseteq)$
- Abstraction of signs
  $(Var \longrightarrow \mathcal{P}(\mathbb{Z}), \subseteq) \underset{\alpha_{sign}}{\overset{\gamma_{sign}}{\leftrightarrows}} (Var \longrightarrow \mathbb{S}igns), \subseteq)$
- Composition of abstractions :
  $(\mathcal{P}(Var \longrightarrow \mathbb{Z}), \subseteq) \underset{\alpha_{sign} \circ \alpha_\pi}{\overset{\gamma_\pi \circ \gamma_{sign}}{\leftrightarrows}} (Var \longrightarrow \mathbb{S}igns), \subseteq)$
- $\alpha = \alpha_{sign} \circ \alpha_\pi$ and $\gamma = \gamma_\pi \circ \gamma_{sign}$

Best approximation of a function

- L is the concrete domain and L' is the abstract model :



$$f' = \alpha \circ f \circ \gamma \tag{2}$$

f' is the best approximation of f

Best evaluation and abstract evaluation

- Concrete states : $cv \in Var \longrightarrow \mathcal{P}(\mathbb{Z})$: if if X is in Var, then $cv(X) \in \mathcal{P}(\mathbb{Z})$.
- Abstract states : $av \in Var \longrightarrow \mathbb{S}igns$: if X is in Var, then $av(X) \in \mathbb{S}igns$.
- (α, γ) is extended as :
  (α_1, γ_1) entre $(Var \longrightarrow \mathcal{P}(\mathbb{Z}), \subseteq)$ et $(Var \longrightarrow \mathbb{S}igns, \sqsubseteq)$. En particulier, $\alpha_1(cv) = av$ et, pour tout X de Var, $av(X) = \alpha(cv(X))$; $\gamma_1(av) = cv$ et, pour tout X de Var, $cv(X) = \gamma(av(X))$.
- Any expression e can be evaluated on each domain :
  - concrete domain : $States = Var \longrightarrow \mathcal{P}(\mathbb{Z})$:
    $[\![e]\!] \in (Var \longrightarrow \mathcal{P}(\mathbb{Z})) \longrightarrow \mathcal{P}(\mathbb{Z})$ and $[\![e]\!](cv)$
  - abstract domain : $AStates = Var \longrightarrow \mathbb{S}igns$:
    $[\![e]\!]_a \in (Var \longrightarrow \mathbb{S}igns) \longrightarrow \mathbb{S}igns$ and $[\![e]\!]_a(av)$.

Domain of signs

- The best abstraction is simply dedined as follows :
  $[\![e]\!]_{best}(av) = \alpha \circ [\![e]\!] \circ \gamma_1(av)$.
- Applying the best approximation for assignment :
  $$[\![x := e]\!]_{best}(av) = \begin{cases} av(y), y \neq x \\ [\![e]\!]_{best}(av) \end{cases}$$
- $(\mathcal{P}(Var \longrightarrow \mathbb{Z}), \subseteq)$:
  $A, B \in \mathcal{P}(\mathbb{Z}) : A+B = \{a+b | a \in A \wedge b \in B\}$
- $(Var \longrightarrow \mathbb{S}igns), \subseteq)$:
  $x, y \in \mathbb{S}igns : x \oplus y = \alpha(\gamma(x)+\gamma(y))$
- examples :
  - $pos \oplus neg = \alpha(\gamma(pos)+\gamma(neg)) = \alpha((1, +\infty)+(-\infty, -1)) = \alpha((-\infty, +\infty)) = any$
  - $pos \oplus zero = \alpha(\gamma(pos)+\gamma(zero)) = \alpha((1, +\infty)+(0)) = \alpha((1, +\infty)) = pos$
  - Building a table for the abstract operation $\oplus$.

Forward analysis in the domain of signs : first way

- Applying the analysis on the example

$$\ell_0[X := 1];$$
$$\ell_1[Y := 5];$$
$$\ell_2[X := X+1];$$
$$\ell_3[Y := Y-1];$$
$$\ell_4[X := Y+X];$$
$$\ell_{final}[skip];$$

ℓ	X	Y
ℓ_0	any	any
ℓ_1	pos	any
ℓ_2	pos	pos
ℓ_3	pos	pos
ℓ_4	pos	$non-neg$
ℓ_{final}	$non-neg$	$non-neg$

- ℓ_3 to ℓ_4 : abstract value of Y is pos and by γ, we obtain $(1, +\infty)$ a,d now we can compute in concrete domain $\mathbb{Z}$ $(1, +\infty)+(-1) = (0, +\infty)$. By reapplying α we obtain $non-neg$.
- Computations may be not computable and one should use techniques for accelarating the convergence like widening.
- Computing is still costly : computing now in the abstraction and defining a sound approximation of f.

Forward analysis in the domain of signs : second way

- Evaluation is using the *best* approximation :
  $[\![e]\!]_{best}(av) = \alpha \circ [\![e]\!] \circ \gamma_1(av)$

Forward analysis in the domain of signs : second way

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- Idea : approximation of the *best* approximation :

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  $av \in Var \longrightarrow \mathbb{S}igns$:

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- Abstract semantics is defined as follows :
  $av \in Var \longrightarrow \mathbb{S}igns$:
  - $[\![const]\!]_a(av) = \alpha(\{c\})$
  - $[\![x]\!]_a(av) = av(x)$
  - $[\![e_1{+}e_2]\!]_a(av) = [\![e_1]\!]_a(av) \oplus [\![e_2]\!]_a(av)$
  - $[\![e_1{+}e_2]\!]_a(av) = [\![e_1]\!]_a(av) \otimes [\![e_2]\!]_a(av)$

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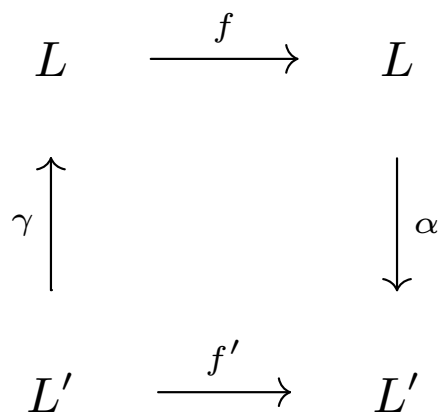
  - $[\![e_1{+}e_2]\!]_a(av) = [\![e_1]\!]_a(av) \otimes [\![e_2]\!]_a(av)$
- $\ell[X := E] : [\![E]\!]_a$ in av ou encore $[\![E]\!]_a(av)$:
  $[\![Y{+}X{+}6]\!]_a(av) = [\![Y]\!]_a(av) +_a [\![X]\!]_a(av) +_a [\![6]\!]_a(av)$.
  - $[\![Y{-}1]\!]_a(av) = [\![Y]\!]_a(av) \oplus [\![-1]\!]_a(av)_a = pos \oplus neg = any$
  - $[\![Y{-}1]\!]_{best}(av) = \alpha \circ [\![Y{-}1]\!] \circ \gamma_1(av) == \alpha([\![Y{-}1]\!](\gamma_1(av))) = \alpha([\![Y{-}1]\!](\{Y \mapsto (1, +\infty)\}) = \alpha((1{+}\infty){+}(-1)) = \alpha((0, +\infty)) = non{-}neg$

Sound approximations of f with respect to a Galois connection

A sound approximation of f with respect to a Galois connection f' satisfies the following property :

$$\forall x \in L, y \in L'.\alpha(x) \sqsubseteq y \Rightarrow \alpha(f(x)) \sqsubseteq f'(y)$$



The four statements are equivalent

- f' is a sound approximation of f with respect to a Galois connection
- $\alpha \circ f \sqsubseteq' f' \circ \alpha$
- $\alpha \circ f \circ \gamma \sqsubseteq' f'$
- $f \circ \gamma \sqsubseteq' \gamma \circ f'$

Defining an abstract semantics of expressions

- $[\![e]\!]_{best}(av) = \alpha \circ [\![e]\!] \circ \gamma_1(av)$ provide the best abstraction but is costly.
- Another solution is to define an abstract semantics for expressions : $[\![e]\!]_a$ such that for any av, $[\![e]\!]_{best}(av) \sqsubseteq [\![e]\!]_a(av)$.
- $av \in Var \longrightarrow \mathbb{S}igns$:
  - $[\![const]\!]_a(v) = \alpha(\{c\})$
  - $[\![x]\!]_a(v) = v(x)$
  - $[\![e_1{+}e_2]\!]_a(v) = [\![e_1]\!]_a(v) \oplus [\![e_2]\!]_a(v)$
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- $[\![Y-1]\!]_a(av) = [\![Y]\!]_a(av) \oplus [\![-1]\!](av)_a = pos \oplus neg = may$
- $[\![Y-1]\!]_{best}(av) = \alpha_1 \circ [\![Y-1]\!] \circ \gamma_1(av) == \alpha_1([\![Y-1]\!](\gamma_1(av))) = \alpha_1([\![Y-1]\!](\{Y \mapsto (1, +\infty)\}) = \alpha_1((1+\infty)+(-1)) = \alpha_1((0, +\infty)) = non-neg$

Forward analysis in the domain of signs using the approximation

- Applying the analysis on the example

$$\ell_0[X := 1];$$
$$\ell_1[Y := 5];$$
$$\ell_2[X := X+1];$$
$$\ell_3[Y := Y-1];$$
$$\ell_4[X := Y+X];$$
$$\ell_{final}[skip];$$

ℓ	X	Y
ℓ_0	any	any
ℓ_1	pos	any
ℓ_2	pos	pos
ℓ_3	pos	pos
ℓ_4	pos	any
ℓ_{final}	any	any

- The new analysis is less precise but more efficient since we compute in the domain of signs.

Summary

Abstract Domain of Intervals

- $\mathbb{I}(\mathbb{Z}) = \{\bot\} \cup \{[l,u] | l \in \mathbb{Z} \cup \{-\infty\}, u \in \mathbb{Z} \cup \{\infty\}, l \leq u\}$
- $[l_1,u_1] \sqsubseteq [l_2,u_2]$ si, et seulement si, $l2 \leq l1$ et $u_1 \leq u_2$.
- $(\mathbb{I}(\mathbb{Z}), \sqsubseteq)$ est une structure partiellement ordonnée.
- 1. $[l_1,u_1] \sqcup [l_2,u_2] = [min(l_1,l_2), max(u_1,u_2)]$
  2. $[l_1,u_1] \sqcap [l_2,u_2] = \begin{cases} [max(l_1,l_2), min(u_1,u_2)] \\ \bot, si\ max(l_1,l_2) > min(u_1,u_2) \end{cases}$
- $(\mathbb{I}(\mathbb{Z}), \sqcup)$ is a complete lattice.
- 1. $\alpha(X) = \begin{cases} [min(X), max(X)] \\ \bot, si\ X = \varnothing \end{cases}$
  2. $\gamma([l,u]) = [l..u]$ et $\gamma(\bot] = \varnothing$
- (α, γ) is a Galois connexion.
- 1. $i_1 \oplus i_2 = [l_1 + l_2, u_1 + u_2]$
  2. $i_1 \ominus i_2 = [l_1 - u_2, u_1 - l_2]$
  3. $i_1 \otimes i_2 = [min(l_1 \cdot l_2, l_1 \cdot u_2, u_1 \cdot l_2, u_1 \cdot u_2, max(l_1 \cdot l_2, l_1 \cdot u_2, u_1 \cdot l_2, u_1 \cdot u_2)]$
  4. $i_1 \oslash i_2 =$
     $[min(l_1/l_2, l_1/u_2, u_1/l_2, u_1/u_2, max(l_1/l_2, l_1/u_2, u_1/l_2, u_1/u_2)]$

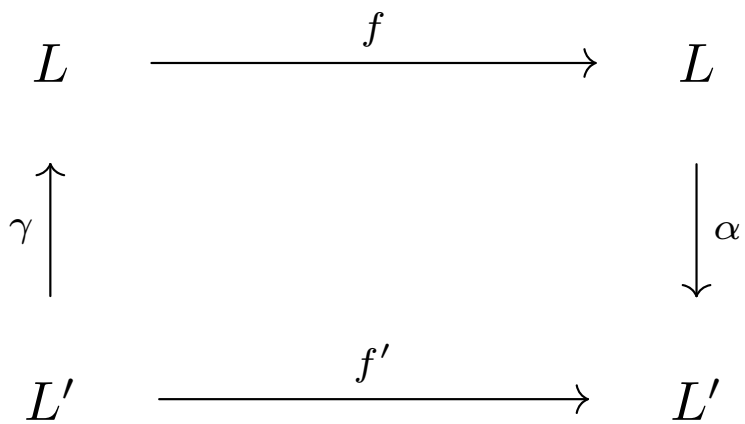
Summary

Operators for approximating fixed-point computations

- Computing *collecting semantics* is generally undecidable :
  - S is a safety property for $\mathcal{MS}$ if, and only if, $\text{REACHABLE}(\mathcal{MS}) \subseteq \mathcal{S}$.
  - Finding a sound approximation of $\text{REACHABLE}(\mathcal{MS})$, denoted $\alpha(\text{REACHABLE}(\mathcal{MS}))$, and satisfying $\gamma(\alpha(\text{REACHABLE}(\mathcal{MS}))) \subseteq \mathcal{S}$.
  - $\text{REACHABLE}(\mathcal{MS}) \subseteq \gamma(\alpha(\text{REACHABLE}(\mathcal{MS})))$ and $\gamma(\alpha(\text{REACHABLE}(\mathcal{MS}))) \subseteq \mathcal{S}$.
- Abstract domains can be finite as the domain of Signs but the domain of intervals is infinite : computing $\text{REACHABLE}(\mathcal{MS})$ remains undecidable but we can approximate its computation.
- Abstract domains can be infinite : we have to accelarate the computations of fixed-points in the case of loops for instance : widening and narrowing.

Best approximation of a function

- L is the concrete domain and L' is the abstract model :



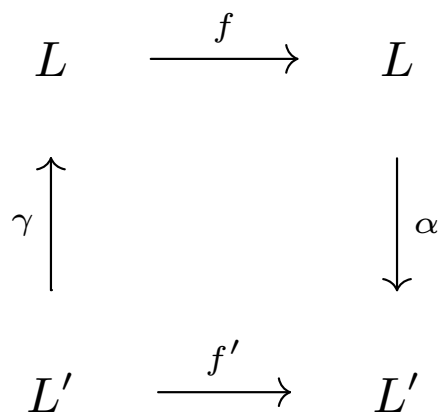
$$f' = \alpha \circ f \circ \gamma \tag{3}$$

f' is the best approximation of f

Sound approximations of f with respect to a Galois connection

A sound approximation of f with respect to a Galois connection f' satisfies the following property :

$$\forall x \in L, y \in L'.\alpha(x) \sqsubseteq y \Rightarrow \alpha(f(x)) \sqsubseteq f'(y)$$



The four statements are equivalent

- f' is a sound approximation of f with respect to a Galois connection
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- $f \circ \gamma \sqsubseteq' \gamma \circ f'$

Defining an abstract semantics of expressions

- $[\![e]\!]_{best}(av) = \alpha \circ [\![e]\!] \circ \gamma_1(av)$ provides the best abstraction but is costly.
- Another solution is to define an abstract semantics for expressions : hide $[\![e]\!]_a$ such that for any av, $[\![e]\!]_{best}(av) \sqsubseteq [\![e]\!]_a(av)$.
- $av \in Var \longrightarrow \mathbb{S}igns$:
  - $[\![const]\!]_a(v) = \alpha(\{c\})$
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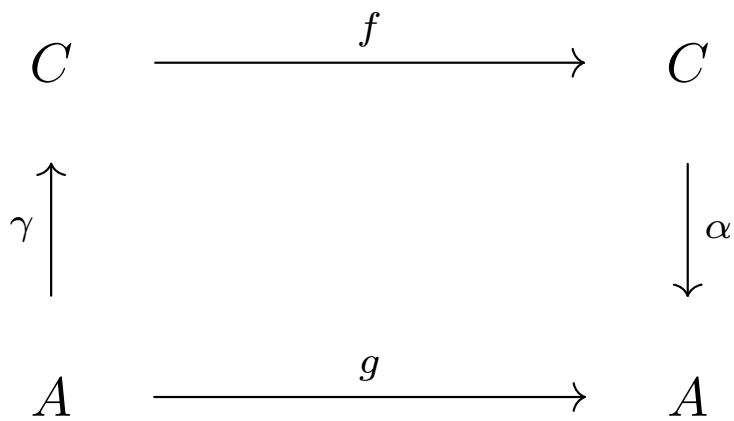
  - $[\![e_1{+}e_2]\!]_a(v) = [\![e_1]\!]_a(v) \oplus [\![e_2]\!]_a(v)$
  - $[\![e_1{+}e_2]\!]_a(v) = [\![e_1]\!]_a(v) \otimes [\![e_2]\!]_a(v)$

Approximation of a function f

- Suppose that $C \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} A$ is a Galois connection
- a function $f \in C \to C$: to find a function g



- f is monotone
- $g = R(\alpha, \gamma, f)$ and $f \sqsubseteq \gamma \circ g \circ \alpha$
- $f \sqsubseteq \gamma \circ g \circ \alpha$ or equivalently $\alpha \circ f \circ \gamma \sqsubseteq g$
- $g = \alpha \circ f \circ \gamma$ is the *best* approximation.

Sound approximation of a function f

Definition of a sound approximation of a function f

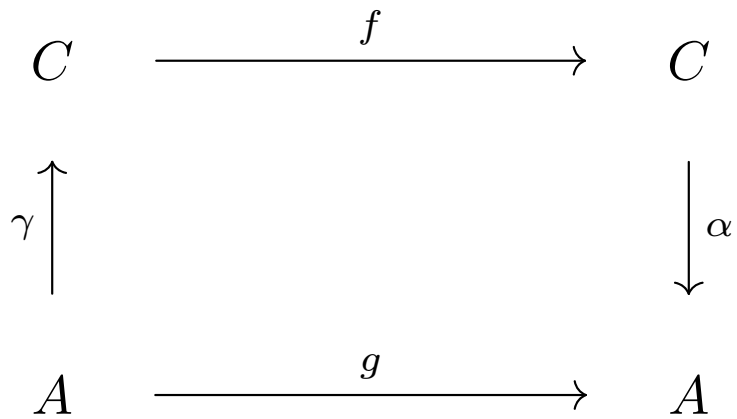
A function $g \in A \longrightarrow A$ is a sound approximation of a function $f \in C \longrightarrow C$, if it satisfies the following condition :
$\forall c \in C : \forall a \in A : \alpha(c) \sqsubseteq a \Rightarrow \alpha(f(c)) \sqsubseteq g(a)$

Properties

Suppose that $C \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} A$ is a Galois connection.
The four statements are equivalent

1. g is a sound approximation of f with respect to a Galois connection
2. $\alpha \circ f \sqsubseteq g \circ \alpha$
3. $\alpha \circ f \circ \gamma \sqsubseteq g$
4. $f \circ \gamma \sqsubseteq \gamma \circ g$
5. $f \sqsubseteq \gamma \circ g \circ \alpha$

Fixpoint Abstraction



Best abstraction

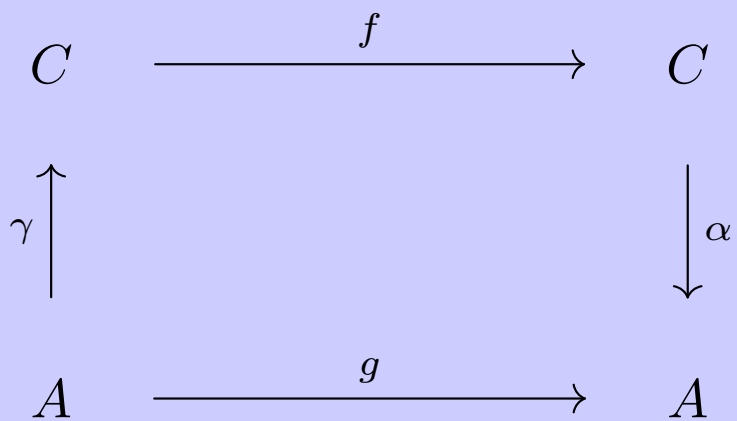
Suppose that :

- $C \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} A$ is a Galois connection.
- $f \in C \longrightarrow C$ is monotonous
- $g = \alpha \circ f \circ \gamma$

Then $lfp(f) \sqsubseteq \gamma(lfp(g))$ and $\alpha(lfp(f)) \sqsubseteq lfp(g)$
or equivalently rewritten as $\mu f \sqsubseteq \gamma(\mu g)$ and $\alpha(\mu f) \sqsubseteq \mu g$

First theorem

- Suppose that $C \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} A$ is a Galois connection
- Two functions $f \in C \to C$ and $g \in A \to A$:



- f and g are monotone
- $\alpha \circ f = g \circ \alpha$.

Then $\alpha(\mu.f) = \mu.g$.

Proof

1. $\mu g \sqsubseteq \alpha(\mu f)$
   - $f(\mu f) = \mu f$ (fixed-point property)
   - $\alpha(f(\mu f)) = \alpha(\mu f)$ (applying the relation over f and g)
   - $\alpha(f(\mu f)) = g(\alpha(\mu f)) = \alpha(\mu f)$
   - $\alpha(\mu f)$ is a fixed-point of g and $\mu g \sqsubseteq \alpha(\mu f)$
2. $\alpha(\mu f) \sqsubseteq \mu g$
   - Consider y a fixed-point of g : $g(y) = y$ and $\mu g \sqsubseteq y$.
   - $\gamma(y)$ is a fixed-point of f
   - $\mu f \sqsubseteq \gamma(y)$
   - $\alpha(\mu f) \sqsubseteq y$
   - $\alpha(\mu f) \sqsubseteq \mu g$

Second theorem

- Suppose that $C \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} A$ is a Galois connection
- Two functions $f \in C \to C$ and $g \in A \to A$:

$$\begin{array}{ccc} C & \xrightarrow{\quad f \quad} & C \\ {\scriptstyle\gamma}\uparrow & & \downarrow{\scriptstyle\alpha} \\ A & \xrightarrow{\quad g \quad} & A \end{array}$$

- f and g are monotone
- $\alpha \circ f \sqsubseteq g \circ \alpha$.

Then $\alpha(\mu f) \sqsubseteq \mu g$.

Example of computation

- $\alpha \in \mathcal{P}(\mathbb{Z}) \rightarrow \mathbb{S}igns : \begin{cases} z & \alpha(z) \\ z < 0 & neg \\ z > 0 & pos \\ z = 0 & zero \end{cases}$
- $f \in \mathcal{P}(\mathbb{Z}) \rightarrow \mathcal{P}(\mathbb{Z})$ where $f(X) = \{0\} \cup \{x+2 | x \in \mathbb{Z} \wedge x \in X\}$
- $g = \alpha \circ f \circ \gamma$
- $f^0 = \varnothing$, $f^1 = \{0\}$, $f^2 = \{0, 2\}$, $\ldots$
- $g(\bot) = \bot$, $g^1 = \alpha \circ f \circ \gamma(\bot) = [0, \infty[$, $g^2 = [0, \infty[$, $\ldots$ and $\forall i \geq 2 : g^i = [0, \infty[$.
- $\mu.g = [0, \infty[$

Widening

Definition

$\bigtriangledown$ is a widening operator over $(L, \sqsubseteq)$ ($\bigtriangledown \in L \times L \to L$)

- For any x and y in L : $x \sqcup y \sqsubseteq x \bigtriangledown y$
- For any sequence $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq x_3 \ldots \sqsubseteq x_i \sqsubseteq x_{i+1} \ldots$, the sequence $\{y_i | i \in \mathbb{N}\}$
  - $y_0 = x_0$
  - $y_{i+1} = y_i \bigtriangledown x_{i+1}$

  stabilizes after a finite amount of time.

Widening

Theorem

If $\bigtriangledown$ is a widening operator over $(L, \sqsubseteq)$ ($\bigtriangledown \in L \times L \to L$), then the ascending sequence $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq x_3 \ldots \sqsubseteq x_i \sqsubseteq x_{i+1} \ldots$ defined by :

- $x_0 = \bot$
- $x_{i+1} = x_i \bigtriangledown f(x_i)$

is eventually stationary and its limit satisfies $lfp(f) \sqsubseteq\sqsubseteq \{x_i | i \in \mathbb{N}\}$ stabilizes after a finite amount of time.

- Using $\bigtriangledown$ instead of $\sqsubseteq$ for computing approximation of upper bound.

Defining Widening Operator

Intervals

- $\bot \bigtriangledown \bot = \bot$
- $\bot \bigtriangledown (l, u) = (l, u) \bigtriangledown \bot = (l, u)$
- $(l1, u1) \bigtriangledown (l2, u2) = \left(\left(\begin{array}{l} -\infty \ if \ l2 < l1 \\ l1 \end{array} \right), \left(\begin{array}{l} \infty \ if \ u2 > u1 \\ u1 \end{array} \right) \right)$

Examples of widening

- $\mathbb{I}(\mathbb{Z}) = \{\bot\} \cup \{[l,u] | l \in \mathbb{Z} \cup \{-\infty\}, u \in \mathbb{Z} \cup \{\infty\}, l \leq u\}$
- $(\mathbb{I}(\mathbb{Z}), \sqsubseteq)$ est une structure partiellement ordonnée.
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Approximation of a fixed-point operator

Let us assume that $(L, \sqsubseteq)$ is a complete lattice and f is a monotonic function defined from L to L.

Theorem

If $\bigtriangledown \in L \times L \rightarrow L$ is a widening operator, then the sequence $\{x_i | i \in \mathbb{N}\}$ defined by

- $x_0 = \bot$
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is eventually stationary and its limit satisfies $lfp(f) \sqsubseteq \bigcup \{x_i | i \in \mathbb{N}\}$

Sound approximation of a function f

Definition of a sound approximation of a function f

A function $g \in A \longrightarrow A$ is a sound approximation of a function $f \in C \longrightarrow C$, if it satisfies the following condition :
$\forall c \in C : \forall a \in A : \alpha(c) \sqsubseteq a \Rightarrow \alpha(f(c)) \sqsubseteq g(a)$

Properties

Suppose that $C \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} A$ is a Galois connection.
The four statements are equivalent

1. g is a sound approximation of f with respect to a Galois connection
2. $\alpha \circ f \sqsubseteq g \circ \alpha$
3. $\alpha \circ f \circ \gamma \sqsubseteq g$
4. $f \circ \gamma \sqsubseteq \gamma \circ g$
5. $f \sqsubseteq \gamma \circ g \circ \alpha$

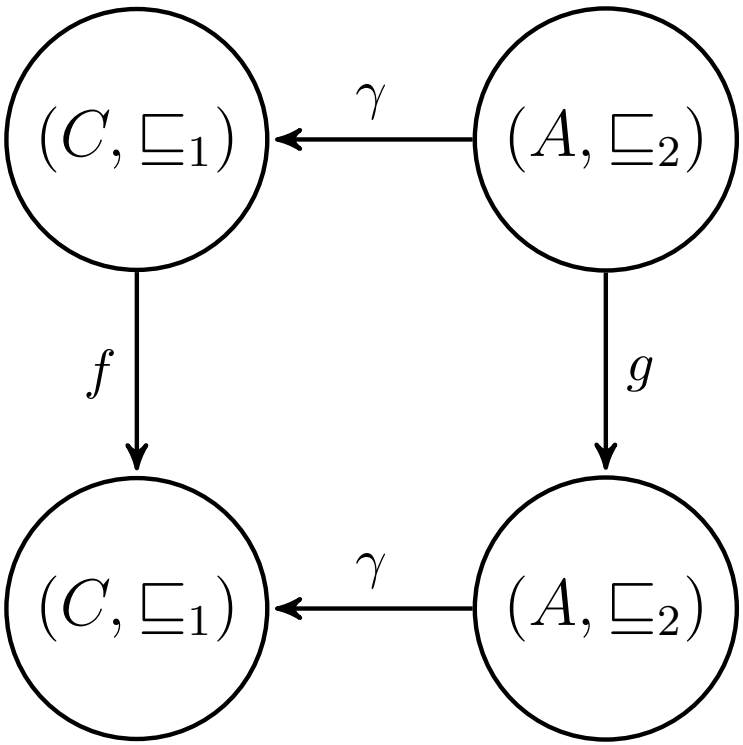
Example of a sound approximation of the invariant of a system

- C is the set of concrete states : $cv \in Var \longrightarrow \mathcal{P}(\mathbb{Z})$: if X is in Var, then $cv(X) \in \mathcal{P}(\mathbb{Z})$.
- A is the set of abstract states : $av \in Var \longrightarrow \mathbb{S}igns$: if X is in Var, then $av(X) \in \mathbb{S}igns$.
- (α, γ) is extended as :
  (α_1, γ_1) entre $(Var \longrightarrow \mathcal{P}(\mathbb{Z}), \subseteq)$ et $(Var \longrightarrow \mathbb{S}igns, \sqsubseteq)$. En particulier, $\alpha_1(cv) = av$ et, pour tout X de Var, $av(X) = \alpha(cv(X))$; $\gamma_1(av) = cv$ et, pour tout X de Var, $cv(X) = \gamma(av(X))$.

Computing the set of computing states of a transition system TS

- $Init \subseteq C$ is the set of initial states.
- $\textsc{Next}$ defines the transition over concrete states
- $\textsc{reachable}(TS) = \{u | u \in C \wedge (\exists x_0 . x_0 \in C \wedge (x_0 \in Init) \wedge \textsc{Next}^\star(x_0, x))\}$
- pour toute partie U de Σ, $U = \mathrm{FP}(U)$
- pour toute partie U de Σ, $\mathrm{FP}(U) = Init_{\mathrm{S}} \cup \longrightarrow [U]$

Sound approximation of fixed-point



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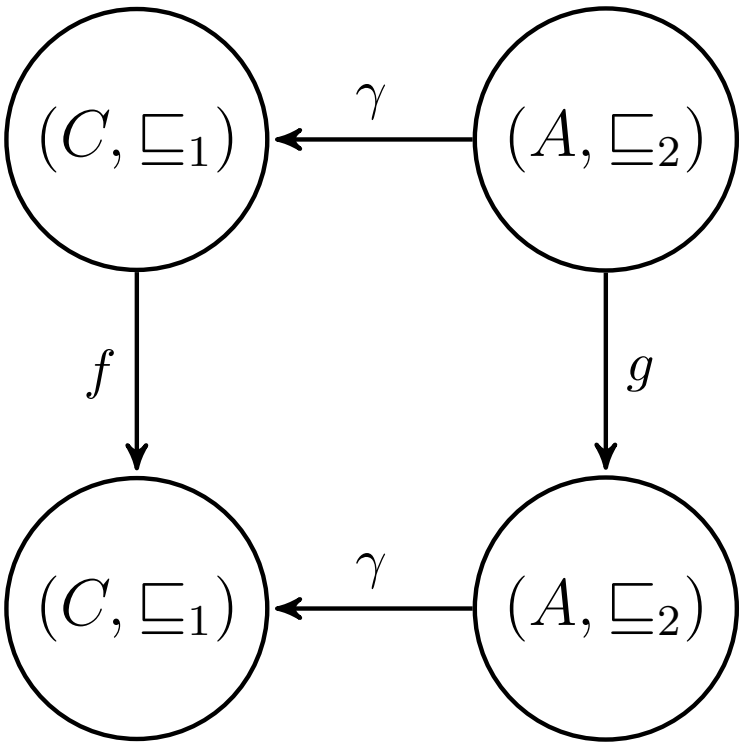

First Theorem

- Suppose that $C \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} A$ is a Galois connection
- Two functions $f \in C \to C$ and $g \in A \to A$:
- f and g are monotone
- $\alpha \circ f = g \circ \alpha$.

Then $\alpha(\mu.f) = \mu.g$.

Proof

1. $\mu g \sqsubseteq \alpha(\mu f)$
   - $f(\mu f) = \mu f$ (fixed-point property)
   - $\alpha(f(\mu f)) = \alpha(\mu f)$ (applying the relation over f and g)
   - $\alpha(f(\mu f)) = g(\alpha(\mu f)) = \alpha(\mu f)$
   - $\alpha(\mu f)$ is a fixed-point of g and $\mu g \sqsubseteq \alpha(\mu f)$
2. $\alpha(\mu f) \sqsubseteq \mu g$
   - Consider y a fixed-point of g : $g(y) = y$ and $\mu g \sqsubseteq y$.
   - $\gamma(y)$ is a fixed-point of f
   - $\mu f \sqsubseteq \gamma(y)$
   - $\alpha(\mu f) \sqsubseteq y$
   - $\alpha(\mu f) \sqsubseteq \mu g$

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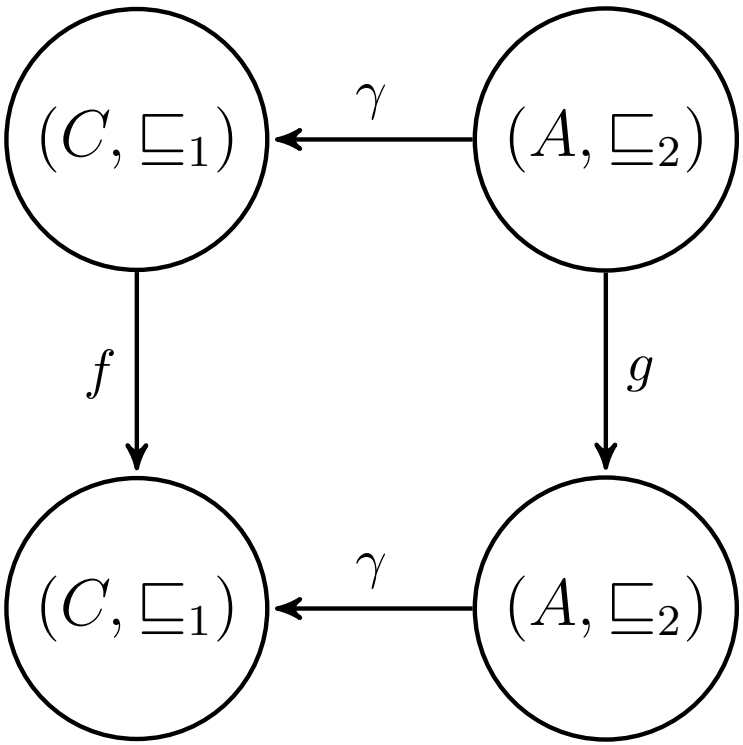

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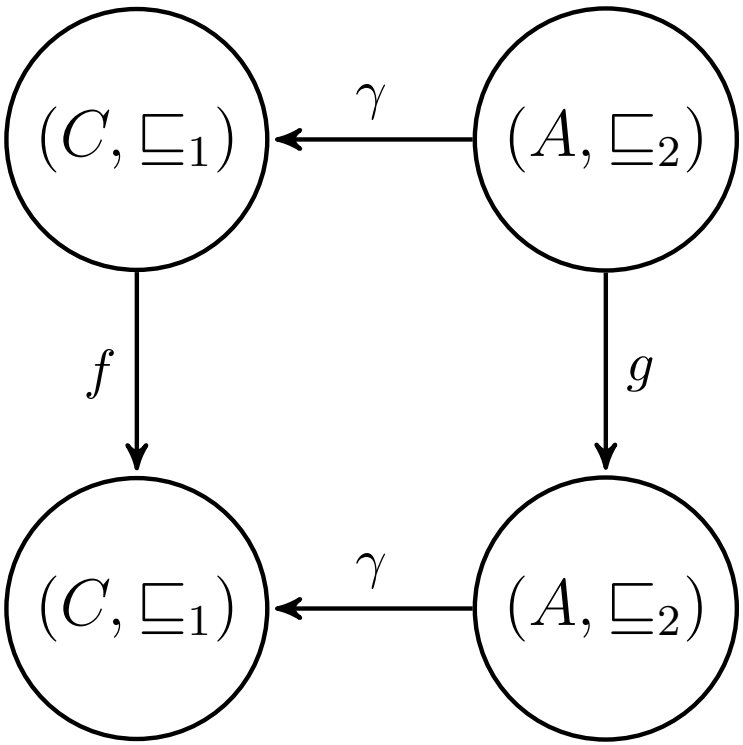

Second Theorem

- Suppose that $C \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} A$ is a Galois connection
- Two functions $f \in C \to C$ and $g \in A \to A$:
- f and g are monotone
- $\alpha \circ f \sqsubseteq g \circ \alpha$.

Then $\alpha(\mu f) \sqsubseteq \mu g$.

Example of computation

- $\alpha \in \mathcal{P}(\mathbb{Z}) \rightarrow \mathbb{S}igns : \begin{cases} z & \alpha(z) \\ z < 0 & neg \\ z > 0 & pos \\ z = 0 & zero \end{cases}$
- $f \in \mathcal{P}(\mathbb{Z}) \rightarrow \mathcal{P}(\mathbb{Z})$ where $f(X) = \{0\} \cup \{x+2 | x \in \mathbb{Z} \wedge x \in X\}$
- $g = \alpha \circ f \circ \gamma$
- $f^0 = \varnothing$, $f^1 = \{0\}$, $f^2 = \{0, 2\}$, $\ldots$
- $g(\bot) = \bot$, $g^1 = \alpha \circ f \circ \gamma(\bot) = [0, \infty[$, $g^2 = [0, \infty[$, $\ldots$ and $\forall i \geq 2 : g^i = [0, \infty[$.
- $\mu.g = [0, \infty[$

Summary

Widening

Definition

$\bigtriangledown$ is a widening operator over $(L, \sqsubseteq)$ ($\bigtriangledown \in L \times L \to L$)

- For any x and y in L : $x \sqcup y \sqsubseteq x \bigtriangledown y$
- For any sequence $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq x_3 \ldots \sqsubseteq x_i \sqsubseteq x_{i+1} \ldots$, the sequence $\{y_i | i \in \mathbb{N}\}$
  - $y_0 = x_0$
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  stabilizes after a finite amount of time.

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If $\bigtriangledown$ is a widening operator over $(L, \sqsubseteq)$ ($\bigtriangledown \in L \times L \rightarrow L$), then the ascending sequence $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq x_3 \ldots \sqsubseteq x_i \sqsubseteq x_{i+1} \ldots$ defined by :

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is eventually stationary and its limit satisfies $lfp(f) \sqsubseteq\sqsubseteq \{x_i | i \in \mathbb{N}\}$ stabilizes after a finite amount of time.

- Using $\bigtriangledown$ instead of $\sqsubseteq$ for computing approximation of upper bound.

Defining Widening Operator

Intervals

- $\bot \bigtriangledown \bot = \bot$
- $\bot \bigtriangledown (l, u) = (l, u) \bigtriangledown \bot = (l, u)$
- $(l1, u1) \bigtriangledown (l2, u2) = \left(\begin{pmatrix} -\infty \ if \ l2 < l1 \\ l1 \end{pmatrix}, \begin{pmatrix} \infty \ if \ u2 > u1 \\ u1 \end{pmatrix} \right)$

Examples of widening

- $\mathbb{I}(\mathbb{Z}) = \{\bot\} \cup \{[l,u] | l \in \mathbb{Z} \cup \{-\infty\}, u \in \mathbb{Z} \cup \{\infty\}, l \leq u\}$
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Approximation of a fixed-point operator

Let us assume that $(L, \sqsubseteq)$ is a complete lattice and f is a monotonic function defined from L to L.

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Narrowing

Definition

$\triangle$ is a narrowing operator over $(L, \sqsubseteq)$ $(\triangle \in L \times L \to L)$

- For any x and y in L : $x \sqsubseteq y \Rightarrow x \sqsubseteq \triangle(x, y) \sqsubseteq y$
- For any sequence $x_0 \sqsupseteq x_1 \sqsupseteq x_2 \sqsupseteq x_3 \ldots \sqsupseteq x_i \sqsupseteq x_{i+1} \ldots$, the sequence $\{y_i | i \in \mathbb{N}\}$
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Simple example

ℓ_0 :
$y := -11;$
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IF $x < y$ **THEN**
  ℓ_1 :
  $z := y;$
  ℓ_2 :
ELSE
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Result : $y < 0 \quad x \in \mathbb{Z} \quad z < 0$ means that $z < 0$ is an information resulting from the analysis over abstract domain of Signs.

Verification by computing set of reachable states

- $\mathcal{MS}$ is $(Th(s,c), x, \textsc{Vals}, \textsc{Init}(x), \{r_0, \ldots, r_n\})$
- $\textsc{Next} \stackrel{def}{=} r_0 \vee \ldots \vee r_n$.
- S is a safety property, when
  $\forall y, x \in \textsc{Vals}. Init(y) \wedge \textsc{Next}^\star(y, x) \Rightarrow x \in S$.
- $(\mathcal{P}(\textsc{Vals}), \subseteq, \varnothing, \cup, \cap)$ is a complete lattice.
- μF is defined as follows :
  - $F^0 = \varnothing$
  - $F^{i+1} = F(F_i)$, $\forall i \in \mathbb{N}$
  - $\mu F = Sup\{F^i | i \in \mathbb{N}\}$
  - For any safety property S, $\mu F \subseteq S$.

Computing the least fixed-point over a finite lattice

INPUT $tf \in T \longrightarrow T$
OUTPUT $result = \mu.f$
VARIABLES $x, y \in T, i \in \mathbb{N}$
$\ell_0 : \{x, y \in T\}$
$x := \bot;$
$y := \bot;$
$i := 0;$
$\ell_{11} : \{x, y \in T \wedge x = F^i \wedge y = \bigcup_{k=0;k=i} F^k \wedge i \leq Card(T) \wedge i = 0\};$
WHILE $i \leq Card(T)$
  $\ell_1 : \{x, y \in T \wedge x = F^i \wedge y = \bigcup_{k=0;k=i} F^k \wedge i \leq Card(T)\};$
  $x := f(x);$
  $\ell_2 : \{x, y \in T \wedge x = F^{i+1} \wedge y = \bigcup_{k=0;k=i} F^k \wedge i \leq Card(T)\};$
  $y := x \sqcup y;$
  $\ell_3 : \{x, y \in T \wedge x = F^{i+1} \wedge y = \bigcup_{k=0;k=i+1} F^k \wedge i \leq Card(T)\};$
  $i := i+1;$
  $\ell_4 : \{x, y \in T \wedge x = F^i \wedge y = \bigcup_{k=0;k=i} F^k \wedge i \leq Card(T)+1\};$
OD;
$\ell_5 : \{x, y \in T \wedge x = F^i \wedge y = \bigcup_{k=0;k=i} F^k \wedge i = Card(T)+1\};$
$result := y;$
$\ell_6 : \{x, y \in T \wedge x = F^i \wedge y = \bigcup_{k=0;k=i} F^k \wedge i = Card(T)+1 \wedge result = y\};$

Conclusion and Comments

- Abstract interpretation is a general framework for defining sound approximation of the semantics of computer programs, based on monotonic functions over ordered sets, especially lattices.
- Main concrete application is formal static analysis, the automatic extraction of information about the possible executions of computer programs.
- When defining an abstract domain, it can be finite (diomain of signs) or infinite (domain of intervals) : it means that we have to manage undecidability questions for computing fixed-points.
- interproc is a tool that can be used for analysing recursive programs and for playing with abstract interpretation.

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Result

$y < 0 \quad x \in \mathbb{Z} \quad z < 0$ means that $z < 0$ is an information

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  $\forall y, x \in \textsc{Vals}. Init(y) \wedge \textsc{Next}^\star(y, x) \Rightarrow x \in S$.
- $(\mathcal{P}(\textsc{Vals}), \subseteq, \varnothing, \cup, \cap)$ is a complete lattice.
- μF is defined as follows :
  - $F^0 = \varnothing$
  - $F^{i+1} = F(F_i)$, $\forall i \in \mathbb{N}$
  - $\mu F = Sup\{F^i | i \in \mathbb{N}\}$
  - For any safety property S, $\mu F \subseteq S$.

i

Computing the least fixed-point over a finite lattice

INPUT $tf \in T \longrightarrow T$
OUTPUT $result = \mu.f$
VARIABLES $x, y \in T, i \in \mathbb{N}$
$\ell_0 : \{x, y \in T\}$
$x := \bot;$
$y := \bot;$
$i := 0;$
$\ell_{11} : \{x, y \in T \wedge x = F^i \wedge y = \bigcup_{k=0;k=i} F^k \wedge i \leq Card(T) \wedge i = 0\};$
WHILE $i \leq Card(T)$
  $\ell_1 : \{x, y \in T \wedge x = F^i \wedge y = \bigcup_{k=0;k=i} F^k \wedge i \leq Card(T)\};$
  $x := f(x);$
  $\ell_2 : \{x, y \in T \wedge x = F^{i+1} \wedge y = \bigcup_{k=0;k=i} F^k \wedge i \leq Card(T)\};$
  $y := x \sqcup y;$
  $\ell_3 : \{x, y \in T \wedge x = F^{i+1} \wedge y = \bigcup_{k=0;k=i+1} F^k \wedge i \leq Card(T)\};$
  $i := i+1;$
  $\ell_4 : \{x, y \in T \wedge x = F^i \wedge y = \bigcup_{k=0;k=i} F^k \wedge i \leq Card(T)+1\};$
OD;
$\ell_5 : \{x, y \in T \wedge x = F^i \wedge y = \bigcup_{k=0;k=i} F^k \wedge i = Card(T)+1\};$
$result := y;$
$\ell_6 : \{x, y \in T \wedge x = F^i \wedge y = \bigcup_{k=0;k=i} F^k \wedge i = Card(T)+1 \wedge result = y\};$

Summary

Conclusion and Comments

- Abstract interpretation is a general framework for defining sound approximation of the semantics of computer programs, based on monotonic functions over ordered sets, especially lattices.
- Main concrete application is formal static analysis, the automatic extraction of information about the possible executions of computer programs.
- When defining an abstract domain, it can be finite (diomain of signs) or infinite (domain of intervals) : it means that we have to manage undecidability questions for computing fixed-points.
- interproc is a tool that can be used for analysing recursive programs and for playing with abstract interpretation.

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$$\mathcal{D}, \mathcal{S} \quad \text{SATISFAIT} \quad \mathcal{R}$$

- $\mathcal{R}$: pre/post.
- $\mathcal{D}$: entiers, réels, ...
- $\mathcal{S}$: code, procédure, programme, ...

Sommaire des résultats

$$\mathcal{D}, \textsc{Alg} \quad \textsc{satisfait} \quad \begin{cases} \textbf{pre}(\textsc{Alg})(v) \\ \textbf{post}(\textsc{Alg})(v_0, v) \end{cases}$$

$$\frac{\mathcal{D}}{\dfrac{\textbf{pre}(\textsc{Alg})(v)}{\textbf{post}(\textsc{Alg})(v_0, v)}}{\textsc{Alg}}$$

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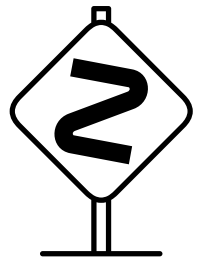

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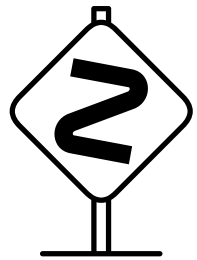

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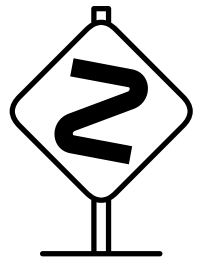

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Sommaire des résultats

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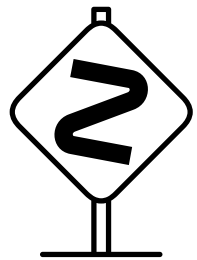

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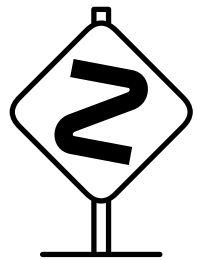

Vérification de conditions de vérification

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- Vérification des conditions de vérification avec un outil de preuve formelle QED
- Vérification des conditions de vérification avec un outil de preuve formelle Alt-Ergo