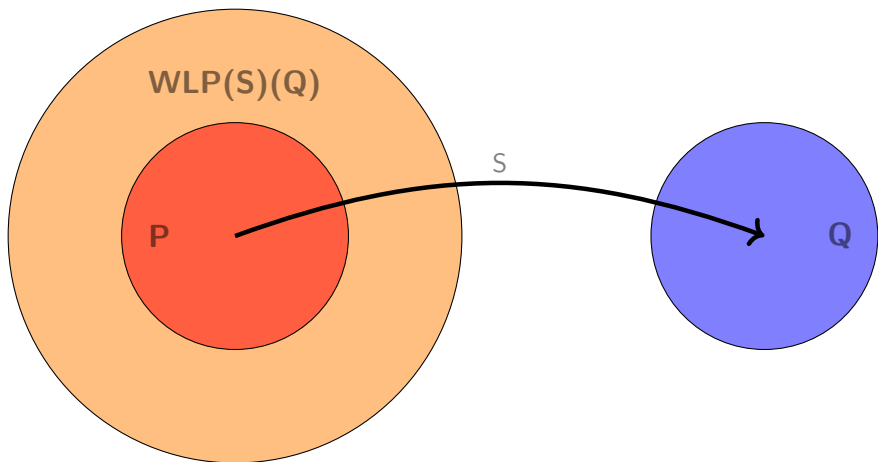




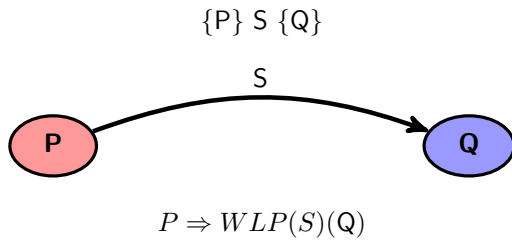
- ① Programs as Predicate Transformers
- ② Mechanizing the contract checking
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$$P \Rightarrow WLP(S)(Q)$$

Computing  $WLP(S)(Q)$  ?



A program  $P$  *satisfies* a contract  $(x, \text{pre}, \text{post})$  :

- ▶  $P$  transforms a variable  $x$  from an initial value  $x_0$  and produces a final value  $x_f$  :  $x_0 \xrightarrow{P} x_f$
- ▶  $x_0$  satisfies pre :  $\text{pre}(x_0)$  and  $x_f$  satisfies post :  $\text{post}(x_0, x_f)$
- ▶  $\text{pre}(x_0) \wedge x_0 \xrightarrow{P} x_f \Rightarrow \text{post}(x_0, x_f)$
- ▶  $\mathbb{D}$  is the domain of  $x$  for RTE (No Run Time Errors) .

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i

```
variables  $x : \mathbb{D}$   
requires  $\text{pre}(x_0)$   
ensures  $\text{post}(x_0, x_f)$   
[  
  begin  
     $0 : P_0(x_0, x)$   
    S  
     $f : P_f(x_0, x)$   
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- ▶ For any pair  $\ell, \ell'$  such that  $\ell \longrightarrow \ell'$ , we verify that for any values  $x, x' \in \text{MEMORY}$   
$$\left( \begin{array}{l} P_\ell(x_0, x) \\ \wedge \text{cond}_{\ell, \ell'}(x) \wedge x' = f_{\ell, \ell'}(x) \end{array} \right) \Rightarrow P_{\ell'}(x_0, x')$$
- ▶ For any pair  $m, n$  such that  $m \longrightarrow n$ , we verify that  $\forall x, x' \in \text{MEMORY}$  :  
 $\text{pre}(x_0) \wedge P_m(x_0, x) \Rightarrow \text{DOM}(m, n)(x)$

- ▶  $pre(x_0) \wedge x = x_0 \Rightarrow P_0(x_0, x)$
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- ▶ For any pair  $m, n$  such that  $m \longrightarrow n$ , we verify that  $\forall x, x' \in \text{MEMORY} : pre(x_0) \wedge P_m(x_0, x) \Rightarrow \mathbf{DOM}(m, n)(x)$

### Example $\mathbf{DOM}(m, n)(x)$

$DOM(\ell_0, \ell_1)(u) = u \in \text{minint}.. \text{maxint} \wedge 5 \in \text{minint}.. \text{maxint} \wedge u+5 \in$

$\text{minint}.. \text{maxint}$  where

$$\begin{array}{l} \ell_0 : P_{\ell_0}(u); \\ u := u+5; \\ \ell_1 : P_{\ell_0}(u); \end{array}$$

- ▶ A program  $P$  *produces* results or outputs from inputs according to a (operational or denotational) semantics
  - STATES is the set of states of  $P$  :  $STATES = x \rightarrow \mathbb{Z}$  where  $x$  designate variables of  $P$ .
  - $s_0$  et  $s_f$  two states of STATES :  $\mathcal{D}(P)(s_0) = s_f$  means that  $P$  is executed from the memory state  $s_0$  and produces a final state  $s_f$ .
  - For any current state  $s$  of  $P$ ,  $s(x) = x$  for expressing the value of  $x$  in state  $s$  :

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  - For any current state  $s$  of  $P$ ,  $s(x) = x$  for expressing the value of  $x$  in state  $s$  :

$$s_0(x) = x_0, s_f(x) = x_f, s'(x) = x'$$

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```
variables x :  $\mathbb{D}$ 
requires  $\text{pre}(x_0)$ 
ensures  $\text{post}(x_0, x_f)$ 
[
  begin
    0 :  $P_0(x_0, x)$ 
    S
    f :  $P_f(x_0, x)$ 
  end
```

- ▶  $\text{pre}(x_0) \wedge x = x_0 \Rightarrow P_0(x_0, x)$
- ▶  $\text{pre}(x_0) \wedge P_f(x_0, x) \Rightarrow \text{post}(x_0, x)$
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requires  $x_0 \geq 0$ ;  
ensures  $x_f = x_0 + 2$ ;  
variables  $X$

```
begin  
  int X = x0;  
  0 : x = x0  
  X = X + 2;  
  1 : x = x0 + 2  
end
```

- ▶  $x_0 \geq 0 \wedge x = x_0 \Rightarrow x = x_0$
- ▶  $x = x_0 + 2 \Rightarrow x = x_0 + 2$
- ▶ conditions de vérification  $0 \longrightarrow 1$  :  
 $x = x_0 \wedge x' = x + 2 \Rightarrow x' = x_0 + 2$
- ▶  $(x_0 \geq 0, x == x_0, x != x_0)$
- ▶  $(x == x_0 + 2, x != x_0 + 2)$
- ▶  $(x == x_0, x \neq x + 2, x \neq x_0 + 2)$

### Listing 1 – z3 en Python

```
from numbers import Real  
from z3 import *  
x = Real('x')  
xp = Real('xp')  
x0 = Real('x0')  
s = Solver()  
s.add(x0 >= 0, x == x0, x != x0)  
print(s.check())  
s.add(x == x0 + 2, x != x0 + 2)  
print(s.check())  
s.add(x == x0, xp == x + 2, xp != x0 + 2)  
print(s.check())
```



►  $\forall x_0, x_f. \text{pre}(x_0) \wedge x_0 \xrightarrow{P} x_f \Rightarrow \text{post}(x_0, x_f)$

- ▶  $\forall x_0, x_f. \text{pre}(x_0) \wedge x_0 \xrightarrow{P} x_f \Rightarrow \text{post}(x_0, x_f)$
- ▶  $\forall x_0, x. \text{pre}(x_0) \Rightarrow (x_0 \xrightarrow{P} x \Rightarrow \text{post}(x_0, x))$

- ▶  $\forall x_0, x_f. \text{pre}(x_0) \wedge x_0 \xrightarrow{P} x_f \Rightarrow \text{post}(x_0, x_f)$
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- ▶  $\forall x_0. \text{pre}(x_0) \Rightarrow \forall x. x_0 \xrightarrow{P} x \Rightarrow \text{post}(x_0, x)$

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- ▶  $\forall x_0. \text{pre}(x_0) \Rightarrow \{P\} \text{post}(x_0, x)$

- ▶  $\forall x_0, x_f. \text{pre}(x_0) \wedge x_0 \xrightarrow{P} x_f \Rightarrow \text{post}(x_0, x_f)$
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### Weakest Liberal Precondition of S for P

$$\{S\}P(x) \stackrel{def}{=} \forall x_f. x \xrightarrow{S} x_f \Rightarrow P(x_f)$$

- ▶  $\forall x_0, x_f. \text{pre}(x_0) \wedge x_0 \xrightarrow{P} x_f \Rightarrow \text{post}(x_0, x_f)$
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### Weakest Liberal Precondition of S for P

$$\{S\}P(x) \stackrel{\text{def}}{=} \forall x_f. x \xrightarrow{S} x_f \Rightarrow P(x_f)$$

- ▶  $\{x := e\}P(x) = P[x \mapsto e]$
- ▶  $\{\text{if } b(x) \text{ then } S1 \text{ else } S2\}P(x) =$   
 $b(x) \wedge \{S1\}P(x) \vee \text{not } b(x) \wedge \{S2\}P(x)$

- ▶  $WLP(S)(P(x))$  is another notation for  $\{S\}P(x)$ .
- ▶  $\{\text{while } b(x) \text{ do } S \text{ end}\}P(x) = \{w\}(P(x))$

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- ▶  $\{\text{while } b(x) \text{ do } S \text{ end}\}P(x) = \{w\}(P(x))$
- ▶  $\{\text{if } b(x) \text{ then } S; w \text{ else } skip \}P(x) =$

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- ▶  $b(x) \wedge \{S; w\}P(x) \vee \text{not } b(x) \wedge \{\text{skip}\}P(x) =$
- ▶  $b(x) \wedge \{S\}(\{w\}(P(x))) \vee \text{not } b(x) \wedge P(x) = \{w\}(P(x))$
- ▶  $F(\{w\})(P(x)) = \{w\}(P(x))$

### Examples

- ▶  $\{\text{while } x > 0 \text{ do } x := x-1 \text{ end}\}(x = 0) = x \geq 0$
- ▶  $\{\text{while } x > 0 \text{ do } x := x+1 \text{ end}\}(x = 0) = x \geq 0$
- ▶  $\{\text{while } x > 0 \text{ do } x := x+1 \text{ end}\}(x \leq 0) = x \in \mathbb{Z}$



### Computing WLP function

- ▶  $\forall x_0. \text{pre}(x_0) \Rightarrow \{P\}\text{post}(x_0, x)$
- ▶  $\forall x_0. x = x_0 \wedge \text{pre}(x_0) \Rightarrow \{P\}\text{post}(x_0, x)$
- ▶ Hoare Triple :  $\{pre(x_0) \wedge x = x_0\}P\{post(x_0, x)\}$

.....

☒ Definition(Axiomes et règles d'inférence)

- ▶ Axiome d'affectation :  $\{P(e/x)\} \mathbf{X} := \mathbf{E}(\mathbf{X}) \{P\}$ .
  - ▶ Axiome du saut :  $\{P\} \mathbf{skip} \{P\}$ .
  - ▶ Règle de composition : Si  $\{P\} \mathbf{S}_1 \{R\}$  et  $\{R\} \mathbf{S}_2 \{Q\}$ , alors  $\{P\} \mathbf{S}_1 ; \mathbf{S}_2 \{Q\}$ .
  - ▶ Si  $\{P \wedge B\} \mathbf{S}_1 \{Q\}$  et  $\{P \wedge \neg B\} \mathbf{S}_2 \{Q\}$ , alors  $\{P\} \mathbf{if\ B\ then\ S_1\ then\ S_2\ fi} \{Q\}$ .
  - ▶ Si  $\{P \wedge B\} \mathbf{S} \{P\}$ , alors  $\{P\} \mathbf{while\ B\ do\ S\ od} \{P \wedge \neg B\}$ .
  - ▶ Règle de renforcement/affaiblissement : Si  $P' \Rightarrow P$ ,  $\{P\} \mathbf{S} \{Q\}$ ,  $Q \Rightarrow Q'$ , alors  $\{P'\} \mathbf{S} \{Q'\}$ .
- .....

Exemple de preuve  $\{x = 1\} \mathbf{Z} := \mathbf{X}; \mathbf{X} := \mathbf{Y}; \mathbf{Y} := \mathbf{Z} \{y = 1\}$

- ▶ (1)  $x = 1 \Rightarrow (z = 1)[x/z]$  (propriété logique)
- ▶ (2)  $\{(z = 1)[x/z]\} \mathbf{Z} := \mathbf{X} \{z = 1\}$  (axiome d'affectation)
- ▶ (3)  $\{x = 1\} \mathbf{Z} := \mathbf{X} \{z = 1\}$  (Règle de renforcement/affaiblissement avec (1) et (2))
- ▶ (4)  $z = 1 \Rightarrow (z = 1)[y/x]$  (propriété logique)
- ▶ (5)  $\{(z = 1)[y/x]\} \mathbf{X} := \mathbf{Y} \{z = 1\}$  (axiome d'affectation)
- ▶ (6)  $\{z = 1\} \mathbf{X} := \mathbf{Y} \{z = 1\}$  (Règle de renforcement/affaiblissement avec (4) et (5))
- ▶ (7)  $z = 1 \Rightarrow (y = 1)[z/y]$  (propriété logique)
- ▶ (8)  $\{(z = 1)[x/z]\} \mathbf{Y} := \mathbf{Z} \{y = 1\}$  (axiome d'affectation)
- ▶ (9)  $\{z = 1\} \mathbf{Y} := \mathbf{Z} \{y = 1\}$  (Règle de renforcement/affaiblissement avec (7) et (8))
- ▶ (10)  $\{x = 1\} \mathbf{Z} := \mathbf{X}; \mathbf{X} := \mathbf{Y}; \{z = 1\}$  (Règle de composition avec 3 et 6)
- ▶ (11)  $\{x = 1\} \mathbf{Z} := \mathbf{X}; \mathbf{X} := \mathbf{Y}; \mathbf{Y} := \mathbf{Z} \{y = 1\}$  (Règle de composition avec 11 et 9)

.....

### ⊗ Définition

$\{P\}\mathbf{S}\{Q\}$  est défini par  $\forall s, t \in STATES : P(s) \wedge \mathcal{D}(S)(s) = t \Rightarrow Q(t)$

.....

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☺ PropertyCorrection du système axiomatique des programmes commentés

- ▶ S'il existe une preuve construite avec les règles précédentes de  $\{P\}\mathbf{S}\{Q\}$ , alors  $\{P\}\mathbf{S}\{Q\}$  est valide.
- ▶ Si  $\{P'\}\mathbf{S}\{Q'\}$  est valide et si le langage d'assertions est suffisamment expressif, alors il existe une preuve construite avec les règles précédentes de  $\{P\}\mathbf{S}\{Q\}$ .

.....

### ⊗ Définition

Un langage d'assertions est la donnée d'un ensemble de prédicats et d'opérateurs de composition comme la disjonction et la conjonction ; il est muni d'une relation d'ordre partielle appelée implication. On le notera  $(\text{PRED}, \Rightarrow, \mathbf{false}, \mathbf{true}, \wedge, \vee) : (\text{PRED}, \Rightarrow, \mathbf{false}, \mathbf{true}, \wedge, \vee)$  est un treillis complet.

- ▶  $\{P\}\mathbf{S}\{Q\}$
- ▶  $\forall s, t \in STATES : P(s) \wedge \mathcal{D}(S)(s) = t \Rightarrow Q(t)$
- ▶  $\forall s \in STATES : P(s) \Rightarrow (\forall t \in STATES : \mathcal{D}(S)(s) = t \Rightarrow Q(t))$

.....

Définition de wlp

$$wlp(S)(Q) \stackrel{def}{=} (\forall t \in STATES : \mathcal{D}(S)(s) = t \Rightarrow Q(t))$$

.....

$$wlp(S)(Q) \equiv \overline{(\exists t \in STATES : \mathcal{D}(S)(s) = t \wedge \overline{Q}(t))}$$

.....

Lien entre wp et wlp

- ▶  $loop(S) \equiv \overline{(\exists t \in STATES : \mathcal{D}(S)(s) = t)}$  (ensemble des états qui ne permettent pas à S de terminer)
  - ▶  $wp(S)(Q) \equiv wlp(S)(Q) \wedge \overline{loop(S)}$
- .....

.....

☒ Definition

$$WLP(S)(P) = \nu \lambda X. ((B \wedge wlp(BS)(X)) \vee (\neg B \wedge P))$$

.....

.....

☺ Property

► Si  $P \Rightarrow Q$ , then  $wlp(S)(P) \Rightarrow wlp(S)(Q)$ .

---

.....

⊠ Definition triplets de Hoare

$$\{P\}\mathbf{S}\{Q\} \stackrel{def}{=} P \Rightarrow wlp(S)(Q)$$

.....

.....

## ⊠ Definition triplets de Hoare

$$\{P\}S\{Q\} \stackrel{def}{=} P \Rightarrow wlp(S)(Q)$$

.....

.....

## ⊠ Definition (Axiomes et règles d'inférence)

- ▶ Axiome d'affectation :  $\{P(e/x)\}X := E(X)\{P\}$ .
  - ▶ Axiome du saut :  $\{P\}\text{skip}\{P\}$ .
  - ▶ Règle de composition : Si  $\{P\}S_1\{R\}$  et  $\{R\}S_2\{Q\}$ , alors  $\{P\}S_1;S_2\{Q\}$ .
  - ▶ Si  $\{P \wedge B\}S_1\{Q\}$  et  $\{P \wedge \neg B\}S_2\{Q\}$ , alors  $\{P\}\text{if } B \text{ then } S_1 \text{ then } S_2 \text{ fi}\{Q\}$ .
  - ▶ Si  $\{P \wedge B\}S\{P\}$ , alors  $\{P\}\text{while } B \text{ do } S \text{ od}\{P \wedge \neg B\}$ .
  - ▶ Règle de renforcement/affaiblissement : Si  $P' \Rightarrow P$ ,  $\{P\}S\{Q\}$ ,  $Q \Rightarrow Q'$ , alors  $\{P'\}S\{Q'\}$ .
- .....



- ▶  $\{P\}\mathbf{S}\{Q\}$
- ▶  $\forall s \in STATES. P(s) \Rightarrow wlp(S)(Q)(s)$
- ▶  $\forall s \in STATES. P(s) \Rightarrow (\forall t \in STATES : \mathcal{D}(S)(s) = t \Rightarrow Q(t))$
- ▶  $\forall s, t \in STATES. P(s) \wedge \mathcal{D}(S)(s) = t \Rightarrow Q(t)$
- ▶ Correction : Si on a construit une preuve de  $\{P\}\mathbf{S}\{Q\}$  avec les règles de la logique de Hoare, alors  $P \Rightarrow wlp(S)(Q)$
- ▶ Complétude sémantique : Si  $P \Rightarrow wlp(S)(Q)$ , alors on peut construire une preuve de  $\{P\}\mathbf{S}\{Q\}$  avec les règles de la logique de Hoare si on peut exprimer  $wlp(S)(P)$  dans le langage d'assertions.

### Listing 2 – difference of two numbers

```
#include <limits.h>
/*@ requires a-b >= INT_MIN && a-b <= INT_MAX;
    assigns \nothing;
    ensures \result == (a - b);
*/
static int difference(int a, int b) {
    return a-b;
}
```

- ▶  $INT\_MIN$  (resp.  $INT\_MAX$ ) is the smallest codable integer (resp. greatest codable integer).
- ▶  $a_0 - b_0 \geq INT\_MIN \wedge a_0 - b_0 \leq INT\_MAX \wedge a = a_0 \wedge b = b_0 \Rightarrow [\backslash result = a - b](\backslash result = (a - b))$

### Listing 3 – incrément de nombre

```
/*@ requires x0 >= 0;  
    assigns \nothing;  
    ensures \result == x0+2;  
    @*/
```

```
int exemple(int x0) {  
    int x=x0;  
    //@ assert x == x0;  
    x = x + 2;  
    //@ assert x == x0+2;  
    return x;  
}
```

requires  $x_0 \geq 0$ ;  
ensures  $x_f = x_0 + 2$ ;  
variables  $x$

```
begin  
  int x = x0;  
  0 : x = x0  
  x := x + 2;  
  1 : x = x0 + 2  
end
```

Conditions de vérification  $0 \rightarrow 1$  :

- ▶  $x = x_0 \wedge x' = x + 2 \Rightarrow x' = x_0 + 2$
- ▶  $x = x_0 \Rightarrow (x' = x + 2 \Rightarrow x' = x_0 + 2)$
- ▶  $x = x_0 \Rightarrow (x + 2 = x_0 + 2)$
- ▶  $wp(x := x + 2)(x = x_0 + 2) = (x + 2 = x_0 + 2)$
- ▶  $x = x_0 \wedge x_0 \geq 0 \Rightarrow wp(x := x + 2)(x = x_0 + 2)$
- ▶  $x = x_0 \wedge x_0 \geq 0 \Rightarrow x + 2 = x_0 + 2$
- ▶  $x = x_0 \wedge x_0 \geq 0 \Rightarrow x_0 + 2 = x_0 + 2$

- ▶  $x_0 \geq 0 \wedge x = x_0 \Rightarrow x = x_0$
- ▶  $x = x_0 + 2 \Rightarrow x = x_0 + 2$
- ▶  $x = x_0 \Rightarrow wp(x := x + 2)(x = x_0 + 2)$



calcul de  $wp(X := X + 2)(x = x_0 + 2)$

### Listing 4 – incrément de nombre

```
/*@ requires x0 >= 0;  
    assigns \nothing;  
    ensures \result == x0+1;  
    @*/
```

```
int exemple(int x0) {  
    int x=x0;  
    //@ assert x == x0;  
    x = x + 2;  
    //@ assert x == x0+2;  
    return x;  
    //@ assert \result == x0+2;  
}
```

### Listing 5 – incrément de nombre

```
/*@ requires x0 >= 0;
   assigns \nothing;
   ensures \result == x0+1;
  @*/
```

```
int example(int x0) {  
    //@ assert x0 == x0;  
    //@ assert x0+2 == x0+2;  
    int x=x0;  
    //@ assert x == x0;  
    //@ assert x+2 == x0+2;  
    x = x + 2;  
    //@ assert x== x0+2;  
    return x;  
}
```

## Opérateur WP

Soit STATES l'ensemble des états sur l'ensemble  $X$  des variables. Soit  $S$  une instruction de programme sur  $X$ . Soit  $A$  une partie de STATES.

$s \in WP(S)(A)$ , si la condition suivante est vérifiée :

$$\left( \begin{array}{l} \forall t \in STATES : \mathcal{D}(S)(s) = t \Rightarrow t \in A \\ \wedge \\ \exists t \in STATES : \mathcal{D}(S)(s) = t \end{array} \right)$$

- ▶  $WP(X := X+1)(A) = \{s \in STATES | s[X \mapsto s(X) \oplus 1] \in A\}$
- ▶  $WP(X := Y+1)(A) = \{s \in STATES | s[X \mapsto s(Y) \oplus 1] \in A\}$
- ▶  $WP(\text{while } X > 0 \text{ do } X := X-1 \text{ od})(A) = \{s \in STATES | (s(X) \leq 0) \vee (s(X) \in A \wedge s(X) < 0)\}$
- ▶  $WP(\text{while } x > 0 \text{ do } x := x+1 \text{ od})(A) = \{s \in STATES | (s(X) \in A \wedge s(X) \leq 0)\}$
- ▶  $WP(\text{while } x > 0 \text{ do } x := x+1 \text{ od})(\emptyset) = \emptyset$
- ▶  $WP(\text{while } x > 0 \text{ do } x := x+1 \text{ od})(STATES) = \{s \in STATES | s(X) \leq 0\}$

### Propriétés

- ▶  $WP$  est une fonction monotone pour l'inclusion d'ensembles de STATES.
- ▶  $WP(S)(\emptyset) = \emptyset$
- ▶  $WP(S)(A \cap B) = WP(S)(A) \cap WP(S)(B)$
- ▶  $WP(S)(A) \cup WP(S)(B) \subseteq WP(S)(A \cup B)$
- ▶ Si  $S$  est déterministe,  $WP(S)(A \cup B) = WP(S)(A) \cup WP(S)(B)$
  
- ▶  $WP$  est un opérateur avec le profil suivant  
pour toute instruction  $S$  du langage de programmation,  
$$WP(S) \in \mathcal{P}(\text{STATES}) \rightarrow \mathcal{P}(\text{STATES})$$
- ▶  $(\mathcal{P}(\text{STATES}), \subseteq)$  est un treillis complet.
- ▶  $(Pred, \Rightarrow)$  est une structure où
  - (1)  $Pred$  est une *extension* du langage d'expressions booléennes
  - (2)  $Pred$  est une *intension* introduite comme un langage d'assertions
  - $\Rightarrow$  est l'implication
  - $s \in A$  correspond une assertion  $P$  vraie en  $s$  notée  $P(s)$ .



- ▶  $S$  est une instruction de STATS.
- ▶  $T$  est le type ou les types des variables et  $D$  est la constante ou les constantes Définie(s).
- ▶  $P$  est un prédicat du langage Pred
- ▶  $X$  est une variable de programme
- ▶  $E(X, D)$  (resp.  $B(X, D)$ ) est une expression arithmétique (resp. booléenne) dépendant de  $X$  et de  $D$ .
- ▶  $x$  est la valeur de  $X$  ( $X$  contient la valeur  $x$ ).
- ▶  $e(x, d)$  (resp.  $b(x, d)$ ) est l'expression arithmétique (resp. booléenne) du langage Pred associée à l'expression  $E(X, D)$  (resp.  $B(X, D)$ ) du langage des expressions arithmétiques (resp. booléennes) du langage de programmation Prog
- ▶  $b(x, d)$  est l'expression arithmétique du langage Pred associée à l'expression  $E(X, D)$  du langage des expressions arithmétiques du langage de programmation Prog

## Définition structurelle des transformateurs de prédicats

S	$wp(S)(P)$
$X := E(X, D)$	$P[e(x, d)/x]$
SKIP	$P$
$S_1; S_2$	$wp(S_1)(wp(S_2)(P))$
IF $B$ $S_1$ ELSE $S_2$ FI	$(B \Rightarrow wp(S_1)(P)) \wedge (\neg B \Rightarrow wp(S_2)(P))$
WHILE $B$ DO $S$ OD	$\mu.(\lambda X. (B \Rightarrow wp(S)(X)) \wedge (\neg B \Rightarrow P))$

- ▶  $wp(X := X+5)(x \geq 8) \stackrel{def}{=} x+5 \geq 8 \wedge x \geq 3$
- ▶  $wp(\text{WHILE } x > 1 \text{ DO } X := X+1 \text{ OD})(x = 4) = FALSE$
- ▶  $wp(\text{WHILE } x > 1 \text{ DO } X := X+1 \text{ OD})(x = 0) = x = 0$

.....

⊠ Definition triplets de Hoare Correction Totale

$$[P]\mathbf{S}[Q] \stackrel{def}{=} P \Rightarrow wp(S)(Q)$$

.....

.....

### ⊠ Definition triplets de Hoare Correction Totale

$$[P]\mathbf{S}[Q] \stackrel{def}{=} P \Rightarrow wp(S)(Q)$$

.....

.....

### ⊠ Definition (Axiomes et règles d'inférence)

- ▶ Axiome d'affectation :  $[P(e/x)]\mathbf{X} := \mathbf{E}(\mathbf{X})[P]$ .
  - ▶ Axiome du saut :  $[P]\mathbf{skip}[P]$ .
  - ▶ Règle de composition : Si  $[P]\mathbf{S}_1[R]$  et  $[R]\mathbf{S}_2[Q]$ , alors  $[P]\mathbf{S}_1 ; \mathbf{S}_2[Q]$ .
  - ▶ Si  $[P \wedge B]\mathbf{S}_1[Q]$  et  $[P \wedge \neg B]\mathbf{S}_2[Q]$ , alors  $[P]\mathbf{if\ B\ then\ S_1\ then\ S_2\ fi}[Q]$ .
  - ▶ Si  $[P(n+1)]\mathbf{S}[P(n)]$ ,  $P(n+1) \Rightarrow b$ ,  $P(0) \Rightarrow \neg b$ , alors  $[\exists n \in \mathbb{N}. P(n)]\mathbf{while\ B\ do\ S\ od}[P(0)]$ .
  - ▶ Règle de renforcement/affaiblissement : Si  $P' \Rightarrow P$ ,  $[P]\mathbf{S}[Q]$ ,  $Q \Rightarrow Q'$ , alors  $[P']\mathbf{S}[Q']$ .
- .....

### Correction

:  
Si  $[P]\mathbf{S}[Q]$  est dérivé selon les règles ci-dessus, alors  $P \wp(S) \wp Q$ .

- ▶  $[P(e/x)]\mathbf{X} := \mathbf{E}(\mathbf{X})[P]$  est valide :  $wp(X := E)(P)/x = P(e/x)$ .
- ▶  $[\exists n \in \mathbb{N}. P(n)]\mathbf{while\ B\ do\ S\ od}[P(0)]$  : si  $s$  est un état de  $P(n)$  alors au bout de  $n$  boucles on atteint un état  $s_f$  tel que  $P(0)$  est vrai en  $s_f$ .

## Complétude

:

Si  $P \Rightarrow wp(S)(Q)$ , alors il existe une preuve de  $[P]\mathbf{S}[Q]$  construites avec les règles ci-dessus,

- ▶  $P \Rightarrow wp(X := E(X))(Q) : P \Rightarrow Q(e/x)$  et  $[Q(e/x)]\mathbf{X} := \mathbf{E}(\mathbf{X})[Q]$  constituent une preuve.
- ▶  $P \Rightarrow wp(\text{while})(Q) :$ 
  - On construit la suite de  $P(n)$  en définissant  $P(n) = W_n$ .
  - On vérifie que cela vérifie la règle du while.

## Verification of contract (I)

A program  $P$  *satisfies* a contract  $(pre, post)$  :

- ▶  $P$  transforms a variable  $x$  from an initial value  $x_0$  and produces a final value  $x_f$  :  $x_0 \xrightarrow{P} x_f$
- ▶  $x_0$  satisfait  $pre$  :  $pre(x_0)$  and  $x_f$  satisfait  $post$  :  $post(x_0, x_f)$
- ▶  $pre(x_0) \wedge x_0 \xrightarrow{P} x_f \Rightarrow post(x_0, x_f)$

requires  $pre(x_0)$

ensures  $post(x_0, x_f)$

variables  $X$

begin

$0 : P_0(x_0, x)$

instruction<sub>0</sub>

...

$i : P_i(x_0, x)$

...

instruction <sub>$f-1$</sub>

$f : P_f(x_0, x)$

end

▶  $pre(x_0) \wedge x = x_0 \Rightarrow P_0(x_0, x)$

▶  $pre(x_0) \wedge P_f(x_0, x) \Rightarrow post(x_0, x)$

▶ For each pair  $\ell, \ell'$   
such that  $\ell \longrightarrow \ell'$ , one checks that  
for any value  $x, x' \in \text{MEMORY}$

$$\left( \begin{array}{l} pre(x_0) \wedge P_\ell(x_0, x) \\ \wedge cond_{\ell, \ell'}(x) \wedge x' = f_{\ell, \ell'}(x) \\ \Rightarrow P_{\ell'}(x_0, x') \end{array} \right),$$





## Verification du contrat (II)

A program  $P$  *satisfies* a contract  $(\text{pre}, \text{post})$  :

- ▶ P transforms a variable  $x$  from an initial value  $x_0$  and produces a final value  $x_f$  :  $x_0 \xrightarrow{P} x_f$
- ▶  $x_0$  satisfait pre :  $\text{pre}(x_0)$  and  $x_f$  satisfait post :  $\text{post}(x_0, x_f)$
- ▶  $\text{pre}(x_0) \wedge x_0 \xrightarrow{P} x_f \Rightarrow \text{post}(x_0, x_f)$

```
requires  $pre(x_0)$ 
ensures  $post(x_0, x_f)$ 
variables  $X$ 
```

```

begin
0 :  $P_0(x_0, x)$ 
instruction0
...
i :  $P_i(x_0, x)$ 
...
instructionf-1
f :  $P_f(x_0, x)$ 
end

```

- ▶  $\forall x_f, x_0. \text{pre}(x_0) \wedge x_0 \xrightarrow{P} x_f \Rightarrow \text{post}(x_0, x_f)$
- ▶  $\forall x_f, x_0. \text{pre}(x_0) \Rightarrow (x_0 \xrightarrow{P} x_f \Rightarrow \text{post}(x_0, x_f))$
- ▶  $\forall x_0. \text{pre}(x_0) \Rightarrow \forall x_f. (x_0 \xrightarrow{P} x_f \Rightarrow \text{post}(x_0, x_f))$
- ▶  $\forall x_0. \text{pre}(x_0) \Rightarrow \forall x. (x_0 \xrightarrow{P} x \Rightarrow \text{post}(x_0, x))$
- ▶  $\forall x_0. \text{pre}(x_0) \Rightarrow WLP(P)(\text{post}(x_0, x))$

Un programme  $P$  *satisfies* a contract  $(\text{pre}, \text{post})$  :

- ▶  $P$  transforms a variable  $x$  from an initial value  $x_0$  and produces a final value  $x_f$  :  $x_0 \xrightarrow{P} x_f$
- ▶  $x_0$  satisfies  $\text{pre}$  :  $\text{pre}(x_0)$  and  $x_f$  satisfies  $\text{post}$  :  $\text{post}(x_0, x_f)$
- ▶  $\text{pre}(x_0) \wedge x_0 \xrightarrow{P} x_f \Rightarrow \text{post}(x_0, x_f)$
- ▶  $\forall x_0. \text{pre}(x_0) \Rightarrow WLP(P)(\text{post}(x_0, x))$

Un programme  $P$  *satisfies* a contract  $(\text{pre}, \text{post})$  :

- ▶  $P$  transforms a variable  $x$  from an initial value  $x_0$  and produces a final value  $x_f$  :  $x_0 \xrightarrow{P} x_f$
- ▶  $x_0$  satisfies  $\text{pre}$  :  $\text{pre}(x_0)$  and  $x_f$  satisfies  $\text{post}$  :  $\text{post}(x_0, x_f)$
- ▶  $\text{pre}(x_0) \wedge x_0 \xrightarrow{P} x_f \Rightarrow \text{post}(x_0, x_f)$
- ▶  $\forall x_0. \text{pre}(x_0) \Rightarrow WLP(P)(\text{post}(x_0, x))$
- ▶ WLP is not computable ...
- ▶ Using Hoare logic in the WLP computing as suggested by Rustan Leino. de WLP.

## Verification of contract (III)

A program  $P$  *satisfies* a contract (pre,post) :

- ▶  $P$  transforms a variable  $x$  from an initial value  $x_0$  and produces a final value  $x_f$  :  $x_0 \xrightarrow{P} x_f$
- ▶  $x_0$  satisfait pre :  $\text{pre}(x_0)$  and  $x_f$  satisfait post :  $\text{post}(x_0, x_f)$
- ▶  $\text{pre}(x_0) \wedge x_0 \xrightarrow{P} x_f \Rightarrow \text{post}(x_0, x_f)$

```
requires  $\text{pre}(x_0)$ 
ensures  $\text{post}(x_0, x_f)$ 
variables  $X$ 
begin
  /·@assert  $P_0(x_0, x)$ ·/
  T;
  /·@loop invariant  $I(x_0, x)$ ·/
  while  $B(x)$  do
    S
  od
  /·@assert  $P_f(x_0, x)$ ·/
end
```

- ▶  $x = x_0 \wedge \text{pre}(x_0) \Rightarrow P_0(x_0, x)$
- ▶  $\text{pre}(x_0) \wedge P_0(x_0, x) \Rightarrow \text{WLP}(T)(I(x_0, x))$
- ▶  $I(x_0, x) \wedge B(x) \Rightarrow \text{WLP}(S)(I(x_0, x))$
- ▶  $I(x_0, x) \wedge \neg B(x) \Rightarrow P_f(x_0, x)$

```
requires  $pre(x_0)$   
ensures  $post(x_0, x_f)$   
variables  $X$   
[ begin  
  /·@assert  $P_0(x_0, x)$ ·/  
  S1;  
  S2;  
  /·@assert  $P_f(x_0, x)$ ·/  
end
```

- ▶  $x =$   
 $x_0 \wedge pre(x_0) \Rightarrow P_0(x_0, x)$
- ▶  $P_0(x_0, x) \Rightarrow$   
 $WLP(S1; S2)(P_f(x_0, x))$

```
requires  $pre(x_0)$ 
ensures  $post(x_0, x_f)$ 
variables  $X$ 
begin
  /·@assert  $P_0(x_0, x)$ ·/
  if  $B(x)$  do
     $S1$ 
  else
     $S2$ 
  elfi
  /·@assert  $P_f(x_0, x)$ ·/
end
```

$$x = x_0 \wedge pre(x_0) \Rightarrow P_0(x_0, x)$$

$$\begin{aligned} P_0(x_0, x) \Rightarrow \\ B(x) \wedge WLP(S1)(P_f(x_0, x)) \\ \vee \\ \neg B(x) \wedge WLP(S2)(P_f(x_0, x)) \end{aligned}$$