









$$P \Rightarrow WLP(S)(Q)$$

Computing $WLP(S)(Q)$?

A program P *satisfies* a contract $(x, \text{pre}, \text{post})$:

- ▶ P transforms a variable x from an initial value x_0 and produces a final value x_f : $x_0 \xrightarrow{P} x_f$
- ▶ x_0 satisfies pre : $\text{pre}(x_0)$ and x_f satisfies post : $\text{post}(x_0, x_f)$
- ▶ $\text{pre}(x_0) \wedge x_0 \xrightarrow{P} x_f \Rightarrow \text{post}(x_0, x_f)$
- ▶ \mathbb{D} is the domain of x for RTE (No Run Time Errors) .

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i

```
variables  $x : \mathbb{D}$ 
requires  $\text{pre}(x_0)$ 
ensures  $\text{post}(x_0, x_f)$ 
[
  begin
     $0 : P_0(x_0, x)$ 
    S
     $f : P_f(x_0, x)$ 
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```


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- ▶ $\text{pre}(x_0) \wedge P_f(x_0, x) \Rightarrow \text{post}(x_0, x)$
- ▶ For any pair ℓ, ℓ' such that $\ell \longrightarrow \ell'$, we verify that for any values $x, x' \in \text{MEMORY}$
$$\left(\begin{array}{l} P_\ell(x_0, x) \\ \wedge \text{cond}_{\ell, \ell'}(x) \wedge x' = f_{\ell, \ell'}(x) \end{array} \right) \Rightarrow P_{\ell'}(x_0, x')$$
- ▶ For any pair m, n such that $m \longrightarrow n$, we verify that $\forall x, x' \in \text{MEMORY}$:
 $\text{pre}(x_0) \wedge P_m(x_0, x) \Rightarrow \text{DOM}(m, n)(x)$

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Example $\mathbf{DOM}(m, n)(x)$

$DOM(\ell_0, \ell_1)(u) = u \in \text{minint}.. \text{maxint} \wedge 5 \in \text{minint}.. \text{maxint} \wedge u+5 \in$

$\text{minint}.. \text{maxint}$ where

$$\begin{array}{l} \ell_0 : P_{\ell_0}(u); \\ u := u+5; \\ \ell_1 : P_{\ell_0}(u); \end{array}$$

- ▶ A program P *produces* results or outputs from inputs according to a (operational or denotational) semantics
 - STATES is the set of states of P : $STATES = x \rightarrow \mathbb{Z}$ where x designate variables of P .
 - s_0 et s_f two states of STATES : $\mathcal{D}(P)(s_0) = s_f$ means that P is executed from the memory state s_0 and produces a final state s_f .
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requires $x_0 \geq 0$;
ensures $x_f = x_0 + 2$;
variables X

```
begin  
  int X = x0;  
  0 : x = x0  
  X = X + 2;  
  1 : x = x0 + 2  
end
```

- ▶ $x_0 \geq 0 \wedge x = x_0 \Rightarrow x = x_0$
- ▶ $x = x_0 + 2 \Rightarrow x = x_0 + 2$
- ▶ conditions de vérification $0 \longrightarrow 1$:
 $x = x_0 \wedge x' = x + 2 \Rightarrow x' = x_0 + 2$
- ▶ $(x_0 \geq 0, x == x_0, x != x_0)$
- ▶ $(x == x_0 + 2, x != x_0 + 2)$
- ▶ $(x == x_0, x \neq x + 2, x \neq x_0 + 2)$

Listing 1 – z3 en Python

```
from numbers import Real  
from z3 import *  
x = Real('x')  
xp = Real('xp')  
x0 = Real('x0')  
s = Solver()  
s.add(x0 >= 0, x == x0, x != x0)  
print(s.check())  
s.add(x == x0 + 2, x != x0 + 2)  
print(s.check())  
s.add(x == x0, xp == x + 2, xp != x0 + 2)  
print(s.check())
```


► $\forall x_0, x_f. \text{pre}(x_0) \wedge x_0 \xrightarrow{P} x_f \Rightarrow \text{post}(x_0, x_f)$

- ▶ $\forall x_0, x_f. \text{pre}(x_0) \wedge x_0 \xrightarrow{P} x_f \Rightarrow \text{post}(x_0, x_f)$
- ▶ $\forall x_0, x. \text{pre}(x_0) \Rightarrow (x_0 \xrightarrow{P} x \Rightarrow \text{post}(x_0, x))$

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- ▶ $\forall x_0. \text{pre}(x_0) \Rightarrow \{P\} \text{post}(x_0, x)$

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Weakest Liberal Precondition of S for P

$$\{S\}P(x) \stackrel{\text{def}}{=} \forall x_f. x \xrightarrow{S} x_f \Rightarrow P(x_f)$$

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Weakest Liberal Precondition of S for P

$$\{S\}P(x) \stackrel{def}{=} \forall x_f. x \xrightarrow{S} x_f \Rightarrow P(x_f)$$

- ▶ $\{x := e\}P(x) = P[x \mapsto e]$
- ▶ $\{\text{if } b(x) \text{ then } S1 \text{ else } S2\}P(x) =$
 $b(x) \wedge \{S1\}P(x) \vee \text{not } b(x) \wedge \{S2\}P(x)$

- ▶ $WLP(S)(P(x))$ is another notation for $\{S\}P(x)$.
- ▶ $\{\text{while } b(x) \text{ do } S \text{ end}\}P(x) = \{w\}(P(x))$

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- ▶ $b(x) \wedge \{S\}(\{w\}(P(x))) \vee \text{not } b(x) \wedge P(x) = \{w\}(P(x))$
- ▶ $F(\{w\})(P(x)) = \{w\}(P(x))$

Examples

- ▶ $\{\text{while } x > 0 \text{ do } x := x-1 \text{ end}\}(x = 0) = x \geq 0$
- ▶ $\{\text{while } x > 0 \text{ do } x := x+1 \text{ end}\}(x = 0) = x \geq 0$
- ▶ $\{\text{while } x > 0 \text{ do } x := x+1 \text{ end}\}(x \leq 0) = x \in \mathbb{Z}$

Computing WLP function

- ▶ $\forall x_0. \text{pre}(x_0) \Rightarrow \{P\}\text{post}(x_0, x)$
- ▶ $\forall x_0. x = x_0 \wedge \text{pre}(x_0) \Rightarrow \{P\}\text{post}(x_0, x)$
- ▶ Hoare Triple : $\{pre(x_0) \wedge x = x_0\}P\{post(x_0, x)\}$

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☒ Definition(Axiomes et règles d'inférence)

- ▶ Axiome d'affectation : $\{P(e/x)\} \mathbf{X} := \mathbf{E}(\mathbf{X}) \{P\}$.
 - ▶ Axiome du saut : $\{P\} \mathbf{skip} \{P\}$.
 - ▶ Règle de composition : Si $\{P\} \mathbf{S}_1 \{R\}$ et $\{R\} \mathbf{S}_2 \{Q\}$, alors $\{P\} \mathbf{S}_1 ; \mathbf{S}_2 \{Q\}$.
 - ▶ Si $\{P \wedge B\} \mathbf{S}_1 \{Q\}$ et $\{P \wedge \neg B\} \mathbf{S}_2 \{Q\}$, alors $\{P\} \mathbf{if\ B\ then\ S_1\ then\ S_2\ fi} \{Q\}$.
 - ▶ Si $\{P \wedge B\} \mathbf{S} \{P\}$, alors $\{P\} \mathbf{while\ B\ do\ S\ od} \{P \wedge \neg B\}$.
 - ▶ Règle de renforcement/affaiblissement : Si $P' \Rightarrow P$, $\{P\} \mathbf{S} \{Q\}$, $Q \Rightarrow Q'$, alors $\{P'\} \mathbf{S} \{Q'\}$.
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Exemple de preuve $\{x = 1\} \mathbf{Z} := \mathbf{X}; \mathbf{X} := \mathbf{Y}; \mathbf{Y} := \mathbf{Z} \{y = 1\}$

- ▶ (1) $x = 1 \Rightarrow (z = 1)[x/z]$ (propriété logique)
- ▶ (2) $\{(z = 1)[x/z]\} \mathbf{Z} := \mathbf{X} \{z = 1\}$ (axiome d'affectation)
- ▶ (3) $\{x = 1\} \mathbf{Z} := \mathbf{X} \{z = 1\}$ (Règle de renforcement/affaiblissement avec (1) et (2))
- ▶ (4) $z = 1 \Rightarrow (z = 1)[y/x]$ (propriété logique)
- ▶ (5) $\{(z = 1)[y/x]\} \mathbf{X} := \mathbf{Y} \{z = 1\}$ (axiome d'affectation)
- ▶ (6) $\{z = 1\} \mathbf{X} := \mathbf{Y} \{z = 1\}$ (Règle de renforcement/affaiblissement avec (4) et (5))
- ▶ (7) $z = 1 \Rightarrow (y = 1)[z/y]$ (propriété logique)
- ▶ (8) $\{(z = 1)[x/z]\} \mathbf{Y} := \mathbf{Z} \{y = 1\}$ (axiome d'affectation)
- ▶ (9) $\{z = 1\} \mathbf{Y} := \mathbf{Z} \{y = 1\}$ (Règle de renforcement/affaiblissement avec (7) et (8))
- ▶ (10) $\{x = 1\} \mathbf{Z} := \mathbf{X}; \mathbf{X} := \mathbf{Y}; \{z = 1\}$ (Règle de composition avec 3 et 6)
- ▶ (11) $\{x = 1\} \mathbf{Z} := \mathbf{X}; \mathbf{X} := \mathbf{Y}; \mathbf{Y} := \mathbf{Z} \{y = 1\}$ (Règle de composition avec 11 et 9)

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☒ Definition

$\{P\}\mathbf{S}\{Q\}$ est défini par $\forall s, t \in STATES : P(s) \wedge \mathcal{D}(S)(s) = t \Rightarrow Q(t)$

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☺ Property Correction du système axiomatique des programmes commentés

- ▶ S'il existe une preuve construite avec les règles précédentes de $\{P\}\mathbf{S}\{Q\}$, alors $\{P\}\mathbf{S}\{Q\}$ est valide.
- ▶ Si $\{P'\}\mathbf{S}\{Q'\}$ est valide et si le langage d'assertions est suffisamment expressif, alors il existe une preuve construite avec les règles précédentes de $\{P\}\mathbf{S}\{Q\}$.

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☒ Definition

Un langage d'assertions est la donnée d'un ensemble de prédicats et d'opérateurs de composition comme la disjonction et la conjonction ; il est muni d'une relation d'ordre partielle appelée implication. On le notera $(\text{PRED}, \Rightarrow, \mathbf{false}, \mathbf{true}, \wedge, \vee) : (\text{PRED}, \Rightarrow, \mathbf{false}, \mathbf{true}, \wedge, \vee)$ est un treillis complet.

- ▶ $\{P\}S\{Q\}$
- ▶ $\forall s, t \in STATES : P(s) \wedge \mathcal{D}(S)(s) = t \Rightarrow Q(t)$
- ▶ $\forall s \in STATES : P(s) \Rightarrow (\forall t \in STATES : \mathcal{D}(S)(s) = t \Rightarrow Q(t))$

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Définition de wlp

$$wlp(S)(Q) \stackrel{def}{=} (\forall t \in STATES : \mathcal{D}(S)(s) = t \Rightarrow Q(t))$$

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$$wlp(S)(Q) \equiv \overline{(\exists t \in STATES : \mathcal{D}(S)(s) = t \wedge \overline{Q}(t))}$$

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Lien entre wp et wlp

- ▶ $loop(S) \equiv \overline{(\exists t \in STATES : \mathcal{D}(S)(s) = t)}$ (ensemble des états qui ne permettent pas à S de terminer)
 - ▶ $wp(S)(Q) \equiv wlp(S)(Q) \wedge \overline{loop(S)}$
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☒ Definition

$$WLP(S)(P) = \nu \lambda X. ((B \wedge wlp(BS)(X)) \vee (\neg B \wedge P))$$

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☺ Property

► Si $P \Rightarrow Q$, then $wlp(S)(P) \Rightarrow wlp(S)(Q)$.

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⊠ Definition triplets de Hoare

$$\{P\}\mathbf{S}\{Q\} \stackrel{def}{=} P \Rightarrow wlp(S)(Q)$$

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-

- ▶ $\{P\}\mathbf{S}\{Q\}$
- ▶ $\forall s \in STATES. P(s) \Rightarrow wlp(S)(Q)(s)$
- ▶ $\forall s \in STATES. P(s) \Rightarrow (\forall t \in STATES : \mathcal{D}(S)(s) = t \Rightarrow Q(t))$
- ▶ $\forall s, t \in STATES. P(s) \wedge \mathcal{D}(S)(s) = t \Rightarrow Q(t)$
- ▶ Correction : Si on a construit une preuve de $\{P\}\mathbf{S}\{Q\}$ avec les règles de la logique de Hoare, alors $P \Rightarrow wlp(S)(Q)$
- ▶ Complétude sémantique : Si $P \Rightarrow wlp(S)(Q)$, alors on peut construire une preuve de $\{P\}\mathbf{S}\{Q\}$ avec les règles de la logique de Hoare si on peut exprimer $wlp(S)(P)$ dans le langage d'assertions.

Listing 2 – difference of two numbers

```
#include <limits.h>
/*@ requires a-b >= INT_MIN && a-b <= INT_MAX;
    assigns \nothing;
    ensures \result == (a - b);
*/
static int difference(int a, int b) {
    return a-b;
}
```

- ▶ INT_MIN (resp. INT_MAX) is the smallest codable integer (resp. greatest codable integer).
- ▶ $a_0 - b_0 \geq INT_MIN \wedge a_0 - b_0 \leq INT_MAX \wedge a = a_0 \wedge b = b_0 \Rightarrow [\backslash result = a - b](\backslash result = (a - b))$

Listing 3 – incrément de nombre

```
/*@ requires x0 >= 0;  
    assigns \nothing;  
    ensures \result == x0+2;  
    @*/
```

```
int exemple(int x0) {  
    int x=x0;  
    //@ assert x == x0;  
    x = x + 2;  
    //@ assert x == x0+2;  
    return x;  
}
```

requires $x_0 \geq 0$;
ensures $x_f = x_0 + 2$;
variables x

```
begin  
  int x = x0;  
  0 : x = x0  
  x := x + 2;  
  1 : x = x0 + 2  
end
```

Conditions de vérification $0 \rightarrow 1$:

- ▶ $x = x_0 \wedge x' = x + 2 \Rightarrow x' = x_0 + 2$
- ▶ $x = x_0 \Rightarrow (x' = x + 2 \Rightarrow x' = x_0 + 2)$
- ▶ $x = x_0 \Rightarrow (x + 2 = x_0 + 2)$
- ▶ $wp(x := x + 2)(x = x_0 + 2) = (x + 2 = x_0 + 2)$
- ▶ $x = x_0 \wedge x_0 \geq 0 \Rightarrow wp(x := x + 2)(x = x_0 + 2)$
- ▶ $x = x_0 \wedge x_0 \geq 0 \Rightarrow x + 2 = x_0 + 2$
- ▶ $x = x_0 \wedge x_0 \geq 0 \Rightarrow x_0 + 2 = x_0 + 2$

- ▶ $x_0 \geq 0 \wedge x = x_0 \Rightarrow x = x_0$
- ▶ $x = x_0 + 2 \Rightarrow x = x_0 + 2$
- ▶ $x = x_0 \Rightarrow wp(x := x + 2)(x = x_0 + 2)$



calcul de $wp(X := X + 2)(x = x_0 + 2)$

Listing 4 – incrément de nombre

```
/*@ requires x0 >= 0;  
   assigns \nothing;  
   ensures \result == x0+1;  
  @*/  
  
int exemple(int x0) {  
    int x=x0;  
    //@ assert x == x0;  
    x = x + 2;  
    //@ assert x == x0+2;  
    return x;  
    //@ assert \result == x0+2;  
}
```

Listing 5 – incrément de nombre

```
/*@ requires x0 >= 0;  
    assigns \nothing;  
    ensures \result == x0;  
    @*/
```

```
int exemple(int x0) {  
    int x=x0;  
    //@ assert x == x0+1;  
    x = x + 2;  
    //@ assert x == x0+2;  
    return x;  
  
}
```

Opérateur WP

Soit $STATES$ l'ensemble des états sur l'ensemble X des variables. Soit S une instruction de programme sur X . Soit A une partie de $STATES$.

$s \in WP(S)(A)$, si la condition suivante est vérifiée :

$$\left(\begin{array}{l} \forall t \in STATES : \mathcal{D}(S)(s) = t \Rightarrow t \in A \\ \wedge \\ \exists t \in STATES : \mathcal{D}(S)(s) = t \end{array} \right)$$

- ▶ $WP(X := X+1)(A) = \{s \in STATES \mid s[X \mapsto s(X) \oplus 1] \in A\}$
- ▶ $WP(X := Y+1)(A) = \{s \in STATES \mid s[X \mapsto s(Y) \oplus 1] \in A\}$
- ▶ $WP(\text{while } X > 0 \text{ do } X := X-1 \text{ od})(A) = \{s \in STATES \mid (s(X) \leq 0) \vee (s(X) \in A \wedge s(X) < 0)\}$
- ▶ $WP(\text{while } x > 0 \text{ do } x := x+1 \text{ od})(A) = \{s \in STATES \mid (s(X) \in A \wedge s(X) \leq 0)\}$
- ▶ $WP(\text{while } x > 0 \text{ do } x := x+1 \text{ od})(\emptyset) = \emptyset$
- ▶ $WP(\text{while } x > 0 \text{ do } x := x+1 \text{ od})(STATES) = \{s \in STATES \mid s(X) < 0\}$

Propriétés

- ▶ WP est une fonction monotone pour l'inclusion d'ensembles de STATES.
- ▶ $WP(S)(\emptyset) = \emptyset$
- ▶ $WP(S)(A \cap B) = WP(S)(A) \cap WP(S)(B)$
- ▶ $WP(S)(A) \cup WP(S)(B) \subseteq WP(S)(A \cup B)$
- ▶ Si S est déterministe, $WP(S)(A \cup B) = WP(S)(A) \cup WP(S)(B)$

- ▶ WP est un opérateur avec le profil suivant
pour toute instruction S du langage de programmation,
$$WP(S) \in \mathcal{P}(STATES) \leftrightarrow \mathcal{P}(STATES)$$
- ▶ $(\mathcal{P}(STATES), \subseteq)$ est un treillis complet.
- ▶ $(Pred, \Rightarrow)$ est une structure où
 - (1) $Pred$ est une *extension* du langage d'expressions booléennes
 - (2) $Pred$ est une *intension* introduite comme un langage d'assertions
 - \Rightarrow est l'implication
 - $s \in A$ correspond une assertion P vraie en s notée $P(s)$.

- ▶ S est une instruction de STATS.
- ▶ T est le type ou les types des variables et D est la constante ou les constantes Définie(s).
- ▶ P est un prédicat du langage Pred
- ▶ X est une variable de programme
- ▶ $E(X, D)$ (resp. $B(X, D)$) est une expression arithmétique (resp. booléenne) dépendant de X et de D .
- ▶ x est la valeur de X (X contient la valeur x).
- ▶ $e(x, d)$ (resp. $b(x, d)$) est l'expression arithmétique (resp. booléenne) du langage Pred associée à l'expression $E(X, D)$ (resp. $B(X, D)$) du langage des expressions arithmétiques (resp. booléennes) du langage de programmation Prog
- ▶ $b(x, d)$ est l'expression arithmétique du langage Pred associée à l'expression $E(X, D)$ du langage des expressions arithmétiques du langage de programmation Prog

S	$wp(S)(P)$
$X := E(X, D)$	$P[e(x, d)/x]$
SKIP	P
$S_1; S_2$	$wp(S_1)(wp(S_2)(P))$
IF B S_1 ELSE S_2 FI	$(B \Rightarrow wp(S_1)(P)) \wedge (\neg B \Rightarrow wp(S_2)(P))$
WHILE B DO S OD	$\mu.(\lambda X. (B \Rightarrow wp(S)(X)) \wedge (\neg B \Rightarrow P))$

- ▶ $wp(X := X+5)(x \geq 8) \stackrel{def}{=} x+5 \geq 8 \wedge x \geq 3$
- ▶ $wp(\text{WHILE } x > 1 \text{ DO } X := X+1 \text{ OD})(x = 4) = FALSE$
- ▶ $wp(\text{WHILE } x > 1 \text{ DO } X := X+1 \text{ OD})(x = 0) = x = 0$

.....
☒ Definition triplets de Hoare Correction Totale

$$[P]\mathbf{S}[Q] \stackrel{def}{=} P \Rightarrow wp(S)(Q)$$

.....

.....

⊠ Definition triplets de Hoare Correction Totale

$$[P]\mathbf{S}[Q] \stackrel{def}{=} P \Rightarrow wp(S)(Q)$$

.....

.....

⊠ Definition (Axiomes et règles d'inférence)

- ▶ Axiome d'affectation : $[P(e/x)]\mathbf{X} := \mathbf{E}(\mathbf{X})[P]$.
 - ▶ Axiome du saut : $[P]\mathbf{skip}[P]$.
 - ▶ Règle de composition : Si $[P]\mathbf{S}_1[R]$ et $[R]\mathbf{S}_2[Q]$, alors $[P]\mathbf{S}_1 ; \mathbf{S}_2[Q]$.
 - ▶ Si $[P \wedge B]\mathbf{S}_1[Q]$ et $[P \wedge \neg B]\mathbf{S}_2[Q]$, alors $[P]\mathbf{if\ B\ then\ S_1\ then\ S_2\ fi}[Q]$.
 - ▶ Si $[P(n+1)]\mathbf{S}[P(n)]$, $P(n+1) \Rightarrow b$, $P(0) \Rightarrow \neg b$, alors $[\exists n \in \mathbb{N}. P(n)]\mathbf{while\ B\ do\ S\ od}[P(0)]$.
 - ▶ Règle de renforcement/affaiblissement : Si $P' \Rightarrow P$, $[P]\mathbf{S}[Q]$, $Q \Rightarrow Q'$, alors $[P']\mathbf{S}[Q']$.
-

Correction

:
Si $[P]\mathbf{S}[Q]$ est dérivé selon les règles ci-dessus, alors $P \wp(S) \wp Q$.

- ▶ $[P(e/x)]\mathbf{X} := \mathbf{E}(\mathbf{X})[P]$ est valide : $wp(X := E)(P)/x = P(e/x)$.
- ▶ $[\exists n \in \mathbb{N}. P(n)]\mathbf{while\ B\ do\ S\ od}[P(0)]$: si s est un état de $P(n)$ alors au bout de n boucles on atteint un état s_f tel que $P(0)$ est vrai en s_f .

Complétude

:

Si $P \Rightarrow wp(S)(Q)$, alors il existe une preuve de $[P]\mathbf{S}[Q]$ construites avec les règles ci-dessus,

- ▶ $P \Rightarrow wp(X := E(X))(Q) : P \Rightarrow Q(e/x)$ et $[Q(e/x)]\mathbf{X} := \mathbf{E}(\mathbf{X})[Q]$ constituent une preuve.
- ▶ $P \Rightarrow wp(\text{while})(Q) :$
 - On construit la suite de $P(n)$ en définissant $P(n) = W_n$.
 - On vérifie que cela vérifie la règle du while.

Verification of contract (I)

A program P satisfies a contract $(pre, post)$:

- ▶ P transforms a variable x from an initial value x_0 and produces a final value x_f : $x_0 \xrightarrow{P} x_f$
- ▶ x_0 satisfait pre : $pre(x_0)$ and x_f satisfait $post$: $post(x_0, x_f)$
- ▶ $pre(x_0) \wedge x_0 \xrightarrow{P} x_f \Rightarrow post(x_0, x_f)$

requires $pre(x_0)$

ensures $post(x_0, x_f)$

variables X

begin

$0 : P_0(x_0, x)$

instruction₀

...

$i : P_i(x_0, x)$

...

instruction _{$f-1$}

$f : P_f(x_0, x)$

end

▶ $pre(x_0) \wedge x = x_0 \Rightarrow P_0(x_0, x)$

▶ $pre(x_0) \wedge P_f(x_0, x) \Rightarrow post(x_0, x)$

▶ For each pair ℓ, ℓ'
such that $\ell \longrightarrow \ell'$, one checks that
for any value $x, x' \in \text{MEMORY}$

$$\left(\begin{array}{l} pre(x_0) \wedge P_\ell(x_0, x) \\ \wedge cond_{\ell, \ell'}(x) \wedge x' = f_{\ell, \ell'}(x) \\ \Rightarrow P_{\ell'}(x_0, x') \end{array} \right),$$

Verification du contrat (II)

A program P *satisfies* a contract $(\text{pre}, \text{post})$:

- ▶ P transforms a variable x from an initial value x_0 and produces a final value x_f : $x_0 \xrightarrow{P} x_f$
- ▶ x_0 satisfait pre : $\text{pre}(x_0)$ and x_f satisfait post : $\text{post}(x_0, x_f)$
- ▶ $\text{pre}(x_0) \wedge x_0 \xrightarrow{P} x_f \Rightarrow \text{post}(x_0, x_f)$

```
requires  $pre(x_0)$ 
ensures  $post(x_0, x_f)$ 
variables  $X$ 
```

begin

$$0 : P_0(x_0, x)$$
instruction₀

• • •

$$i : P_i(x_0, x)$$

• • •

$$\text{instruction}_{f-1}$$
$$f : P_f(x_0, x)$$

end

- ▶ $\forall x_f, x_0. \text{pre}(x_0) \wedge x_0 \xrightarrow{P} x_f \Rightarrow \text{post}(x_0, x_f)$
- ▶ $\forall x_f, x_0. \text{pre}(x_0) \Rightarrow (x_0 \xrightarrow{P} x_f \Rightarrow \text{post}(x_0, x_f))$
- ▶ $\forall x_0. \text{pre}(x_0) \Rightarrow \forall x_f. (x_0 \xrightarrow{P} x_f \Rightarrow \text{post}(x_0, x_f))$
- ▶ $\forall x_0. \text{pre}(x_0) \Rightarrow \forall x. (x_0 \xrightarrow{P} x \Rightarrow \text{post}(x_0, x))$
- ▶ $\forall x_0. \text{pre}(x_0) \Rightarrow WLP(P)(\text{post}(x_0, x))$

Un programme P *satisfies* a contract $(\text{pre}, \text{post})$:

- ▶ P transforms a variable x from an initial value x_0 and produces a final value x_f : $x_0 \xrightarrow{P} x_f$
- ▶ x_0 satisfies pre : $\text{pre}(x_0)$ and x_f satisfies post : $\text{post}(x_0, x_f)$
- ▶ $\text{pre}(x_0) \wedge x_0 \xrightarrow{P} x_f \Rightarrow \text{post}(x_0, x_f)$
- ▶ $\forall x_0. \text{pre}(x_0) \Rightarrow WLP(P)(\text{post}(x_0, x))$

Un programme P *satisfies* a contract $(\text{pre}, \text{post})$:

- ▶ P transforms a variable x from an initial value x_0 and produces a final value x_f : $x_0 \xrightarrow{P} x_f$
- ▶ x_0 satisfies pre : $\text{pre}(x_0)$ and x_f satisfies post : $\text{post}(x_0, x_f)$
- ▶ $\text{pre}(x_0) \wedge x_0 \xrightarrow{P} x_f \Rightarrow \text{post}(x_0, x_f)$
- ▶ $\forall x_0. \text{pre}(x_0) \Rightarrow WLP(P)(\text{post}(x_0, x))$
- ▶ WLP is not computable ...
- ▶ Using Hoare logic in the WLP computing as suggested by Rustan Leino. de WLP.

Verification of contract (III)

A program P *satisfies* a contract $(\text{pre}, \text{post})$:

- ▶ P transforms a variable x from an initial value x_0 and produces a final value x_f : $x_0 \xrightarrow{P} x_f$
- ▶ x_0 satisfait pre : $\text{pre}(x_0)$ and x_f satisfait post : $\text{post}(x_0, x_f)$
- ▶ $\text{pre}(x_0) \wedge x_0 \xrightarrow{P} x_f \Rightarrow \text{post}(x_0, x_f)$

```

requires  $pre(x_0)$ 
ensures  $post(x_0, x_f)$ 
variables  $X$ 
[
  begin
  /·@assert  $P_0(x_0, x)$ ·/
   $T$ ;
  /·@loop invariant  $I(x_0, x)$ ·/
  while  $B(x)$  do
     $S$ 
  od
  /·@assert  $P_f(x_0, x)$ ·/
  end

```

- ▶ $x =$
 $x_0 \wedge \text{pre}(x_0) \Rightarrow P_0(x_0, x)$
- ▶ $\text{pre}(x_0) \wedge P_0(x_0, x) \Rightarrow$
 $WLP(T)(I(x_0, x))$
- ▶ $I(x_0, x) \wedge B(x) \Rightarrow$
 $WLP(S)(I(x_0, x))$
- ▶ $I(x_0, x) \wedge \neg B(x) \Rightarrow$
 $P_f(x_0, x)$

```
requires  $pre(x_0)$   
ensures  $post(x_0, x_f)$   
variables  $X$   
[ begin  
  /·@assert  $P_0(x_0, x)$ ·/  
  S1;  
  S2;  
  /·@assert  $P_f(x_0, x)$ ·/  
end
```

- ▶ $x =$
 $x_0 \wedge \text{pre}(x_0) \Rightarrow P_0(x_0, x)$
- ▶ $P_0(x_0, x) \Rightarrow$
 $WLP(S1; S2)(P_f(x_0, x))$

```
requires  $pre(x_0)$ 
ensures  $post(x_0, x_f)$ 
variables  $X$ 
begin
  /·@assert  $P_0(x_0, x)$ ·/
  if  $B(x)$  do
     $S1$ 
  else
     $S2$ 
  elfi
  /·@assert  $P_f(x_0, x)$ ·/
end
```

$$x = x_0 \wedge pre(x_0) \Rightarrow P_0(x_0, x)$$

$$\begin{aligned} P_0(x_0, x) \Rightarrow \\ B(x) \wedge WLP(S1)(P_f(x_0, x)) \\ \vee \\ \neg B(x) \wedge WLP(S2)(P_f(x_0, x)) \end{aligned}$$