Probability Theory

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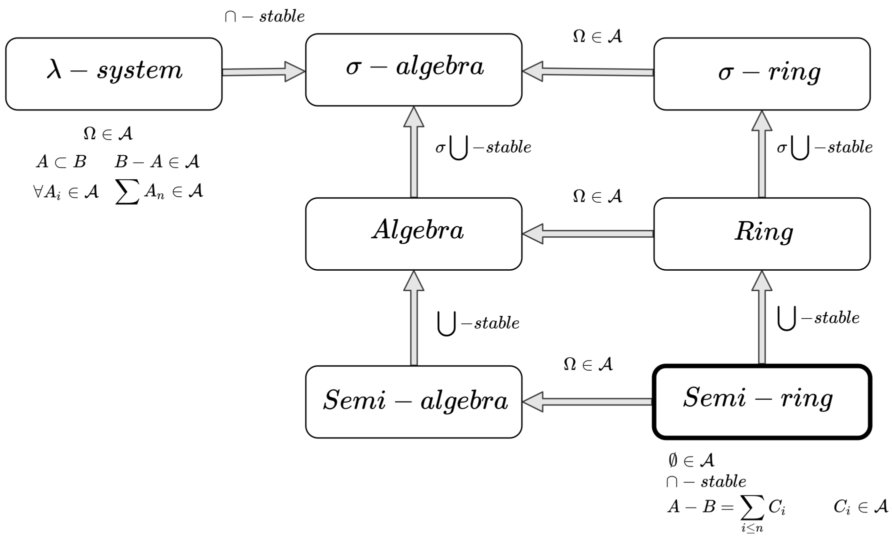
1 Set Theory



2 Measures

2.1 Definitions

(X, S) is a **measurable space** if S is a sigma algebra and $\bigcup S = X$
(X, S, μ) is a **measure space** if (X, S) is a measurable space and μ a measure defined on S
E is a **measurable set** if $E \in S$.
$\mu : \xi \to \mathbb{R}$ s.t ξ is a class of sets of X is a **set function**
A set function $\mu : \xi \to [0, \infty]$ defined on a semi-ring ξ is :

- **Content** if additive
- A **premeasure** if countably additive
- A **measure** if is a premeasure on a σ-algebra
- A **probability measure** if is a measure with $\mu(\Omega) = 1$

A and B respectively positive and negative with respect to μ are a **Hahn decomposition** of X if $X = A \cup B$ and $A \cap B = \emptyset$

2.2 Properties of set functions

- Finitly additive: $E_1 \dots E_n \in \xi \implies \mu(\sum_{i=1}^n E_i) = \sum_{i=1}^n \mu(E_i)$
- Countably additive: $E_1 \dots \in \xi \implies \mu(\sum_{i=1}^\infty E_i) = \sum_{i=1}^\infty \mu(E_i)$
- Finite: $E \in \xi \implies |\mu(E)| < \infty$
- Continuous from below

$$\left.\begin{array}{l} \star\ E \in \xi \\ \star\ \forall i\ E_i \in \xi \\ \star\ (E_n)_{n \in \mathbb{N}} \uparrow E \end{array}\right\} \lim_{n \to \infty} \mu(E_n) = \mu(E)$$

- Continuous from above

$$\left.\begin{array}{l}\star\ E \in \xi \\ \star\ \forall i\ E_i \in \xi \\ \star\ (E_n)_{n\in\mathbb{N}} \downarrow E \\ \star\ \exists m \text{ s.t } |\mu(E_m)| < \infty\end{array}\right\} \lim_{n\to\infty} \mu(E_n) = \mu(E)$$

3 Random Elements

3.1 Definitions

$(\Omega, \mathcal{F}, P)$ is a **Probability space**
A **Random variable** is a measurable function $X : \Omega \to \mathbb{R}$
A **Random element** is a measurable transformation $X : \Omega \to Y$ with $(Y, \mathcal{S})$ measurable space.
The **Distribution** of X is the induced measure:

$$\begin{aligned} P_X : \sigma(Y) &\longmapsto [0,1] \\ B &\longmapsto P\{\omega : X(\omega) \in B\} && \text{(written also } P\{X \in B\}) \\ B &\longmapsto P\{\omega \in X^{-1}(B)\} && \text{(written also } PX^{-1}(B)) \end{aligned}$$

The **Distribution function** of X is defined as:

$$\begin{aligned} F : \mathbb{R} &\longmapsto \mathbb{R} \\ x &\longmapsto P_X(-\infty, x) \end{aligned}$$

The **Expectation** of a random variable X is $E(X) = \int_\Omega X(\omega) dP(\omega)$

3.2 Sum of two Random variables

Let X, Y be two random variables. Let $\phi(x_1, x_2) = x_1 + x_2$. We have the distribution:

$$\begin{aligned} P\{\omega : X(\omega) + Y(\omega) \in B\} &= P\{X + Y \in B\} \\ &= P\{\phi(X, Y) \in B\} \\ &= P\{(X+Y) \in \phi^{-1}B\} \\ &= \int \mathbb{1}_{\phi^{-1}B}(x, y)\ dP(X^{-1} \times Y^{-1}) \\ &= \iint \mathbb{1}_{\phi^{-1}B}(x, y)\ dPX^{-1}\ dPY^{-1} \\ &= \int PX^{-1}(B - y) dPY^{-1} = \int PY^{-1}(B - x) dPX^{-1} \\ &= \int P_X(B - y) dP_Y = \int P_Y(B - x) dP_X \\ &= \int_{-\infty}^{\infty} P_X(B - y) dP_Y(y) = \int_{-\infty}^{\infty} P_Y(B - x) dP_X(x) \end{aligned}$$

We derive the distribution function:

$$
\begin{aligned}
P\{X+Y<z\} = F_{X+Y}(z) &= \int_{-\infty}^{\infty} P_Y((-\infty, z]-x)dP_X(x) = \int_{-\infty}^{\infty} P_X((-\infty, z]-y)dP_Y(y) \\
&= \int_{-\infty}^{\infty} F_Y(z-x)dP_X(x) = \int_{-\infty}^{\infty} F_X(z-y)dP_Y(y)
\end{aligned}
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